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- Party A claims $100
- Party B claims $50
Inheritance

- Two parties are fighting over inheritance (say $100)
  - Party A claims $100
  - Party B claims $50
- How should one split the $100?
Two Methods

Two Methods:

- Equal division
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- Equal division
  - Both parties get $50
Two Methods

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- **Equal division**
  - Both parties get $50
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- **Proportional Division**
  - Party A gets $67
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  - Parties get the proportion of what they claimed to the sums of all claims

Resolution depends on social customs
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- Can divide inheritance, chores, business profits, Berlin, cake, …
Fair Division

- Let $u_i(X_j)$ be the value that person $i$ assigns to $j$’s division
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- Fairness can be measured by:
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- Problem: Someone getting everything, and everyone else getting nothing, is Pareto optimal
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Three Wives

A rule in the Talmud:

- Man is married to three women

If estate worth is 100, each wife receives 33\frac{1}{3} (this agrees with equal division)

If estate worth is 300, the wives receive 50, 100, 150 (this agrees with proportional division)

If estate worth is 200, the wives receive 50, 75, 75 (???)

Is there a coherent rule that outlines these cases?

Solved by game theorists in 1985
Three Wives

A rule in the Talmud:

- Man is married to three women
- Upon husband’s death, each wife is to receive 100, 200, 300 (zuz)

Problem: estate is not worth 600

- If estate worth is 100, each wife receives 33.33 (this agrees with equal division)
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Equal Division of Contested Sums

- Trying to fairly divide amongst two people
Equal Division of Contested Sums

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- Idea:
  - Give everyone their uncontested amounts
Equal Division of Contested Sums

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- Idea:
  - Give everyone their uncontested amounts
  - Split contested amount in half
Equal Division of Contested Sums

- Two parties are trying to split 100

  - Party A receives 75
  - Party B receives 25
Equal Division of Contested Sums

- Two parties are trying to split 100
- Party A claims 100

A receives 75
B receives 25
Equal Division of Contested Sums

- Two parties are trying to split 100
- Party A claims 100
- Party B claims 50
Equal Division of Contested Sums

- Two parties are trying to split 100
- Party A claims 100
- Party B claims 50
- Using equal division of contested sums, how much does each party receive?

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Equal Division of Contested Sums

- Two parties are trying to split inheritance

  - If estate is worth 80, how much does each party receive?
    - Party A receives 40
    - Party B receives 40

  - If estate is worth 125, how much does each party receive?
    - Party A receives 50
    - Party B receives 75

  - If estate is worth 200, how much does each party receive?
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  The allotment of any two wives is split using the above rule!

  (three things to check in each case)

  Need method for when there are more than two parties
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Algorithm

Want algorithm for the equal division of contested sums between \( n \) claims

- Order claims from lowest to highest (from 1 to \( n \))
Algorithm

Want algorithm for the equal division of contested sums between $n$ claims

- Order claims from lowest to highest (from 1 to $n$)
- Divide estate equally until 1 receives half of their claim

Give $n$ money until their loss equals $n-1$'s loss (loss is money owed minus money given)

Give $n-1$ and $n$ money until their loss equals $n-2$'s loss

Proceed until all losses are equal
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Equal Division of Contested Sums

- Suppose the estate is worth 550
Equal Division of Contested Sums

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- How much do the parties receive using the algorithm?
Equal Division of Contested Sums

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- How much do the parties receive using the algorithm?
  - $83\frac{1}{3}$, $183\frac{1}{3}$, and $283\frac{1}{3}$