ON HYPER-SYMMETRIC ABELIAN VARIETIES

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Abstract. Motivated by Oort's Hecke-orbit conjecture, Chai introduced hyper-symmetric points in the study of fine structures of modular varieties in positive characteristics. We prove a necessary and sufficient condition to determine which Newton polygon stratum of PEL-type contains at least one such point.

1. Introduction

This work is to extend the study of hyper-symmetric abelian varieties initiated by Chai-Oort [1]. The notion is motivated by the Hecke-orbit conjecture.

For the reduction of a PEL-type Shimura variety, the conjecture claims that every orbit under the Hecke correspondences is Zariski dense in the leaf containing it. In positive characterisitic p, the decomposition of a Shimura variety into leaves is a refinement of the decomposition into disjoint union of Newton polygon strata. A leaf is a smooth quasi-affine scheme over $\overline{\mathbb{F}}_p$. Its completion at a closed point is a successive fibration whose fibres are torsors under certain Barsotti-Tate groups. The resulting canonical coordinates, a terminology of Chai, provides the basic tool for understanding its structure.

Fix an integer $g \geq 1$ and a prime number p. Consider the Siegel modular variety $\mathcal{A}_g$ in characteristic p. Denote by $\mathcal{C}(x)$ the leaf passing through a closed point x. By applying the local stabilizer principle at a *hyper-symmetric* point x, Chai [3] first gave a very simple proof that the p-adic monodromy of $\mathcal{C}(x)$ is big. Later, in their solution of the Hecke-orbit conjecture for $\mathcal{A}_g$, Chai and Oort used the technique of hyper-symmetric points to deduce the irreducibility of a non-supersingular leaf from the irreducibility of a non-supersingular Newton polygon stratum, see [2]. Note that although hyper-symmetric points distribute scarcely, at least one such point exists in every leaf [1].

Here we are mainly interested in the existence of hyper-symmetric points of PEL-type. Let us fix a positive simple algebra $(\Gamma, *)$, finite dimensional over $\mathbb{Q}$. Following Chai-Oort [1], we have the definition:

Definition 1.1. A Γ-linear polarized abelian variety (Y, λ) over an algebraically closed field k of characteristic p is Γ-*hyper-symmetric*, if the natural map

$$\mathrm{End}^0_\Gamma(Y) \otimes_{\mathbb{Q}} \mathbb{Q}_p \to \mathrm{End}_\Gamma(H^1(Y))$$

is a bijection.

For simplicity we denote by $H^1(Y)$ the isocrystal $H^1_{\mathrm{crys}}(Y/W(k))\otimes_{\mathbb{Z}}\mathbb{Q}$. The goal of this paper is to answer the following question:

Question. Does every Newton polygon stratum contain a hyper-symmetric point?

The answer to the question in general is *no*; a Newton polygon stratum must satisfy certain conditions to contain a Γ-hyper-symmetric point. See (5.3) for an example when Γ is a real quadratic field split at p, and (5.7) when Γ is a division algebra over a CM-field and the Γ-linear isocrystal M only has slopes $0, 1$.

In the main theorem (5.1), we characterize isocrystals of the form $H^1(Y)$ for Γ-hyper-symmetric abelian varieties Y as the underlying isocrystals of *partitioned isocrystals with supersingular restriction* (S).

Consider a typical situation. Let $Y = Y'\otimes_{\mathbb{F}_{p^a}}\overline{\mathbb{F}}_p$ be a Γ-simple hyper-symmetric abelian variety over $\overline{\mathbb{F}}_p$, where Y' is a Γ-simple abelian variety over a finite field $\mathbb{F}_{p^a}$. By the theory of Honda-Tate, up to isogeny, Y' is completely characterized by its Frobenius endomorphism $\pi_{Y'}$. Let F be the center of Γ. Assume that $\mathbb{F}_{p^a}$ is sufficiently large. We show in (3.4) that Y is Γ-hyper-symmetric if and only if the extension $F(\pi_{Y'})/F$ is totally split everywhere above p, that is,

$$F(\pi_{Y'})\otimes_F F_v \simeq F_v\times\cdots\times F_v,$$

for every prime v of F above p. Thus Y is Γ-hyper-symmetric if and only if it is F-hyper-symmetric.

Denote by T_Γ the set of finite prime-to-p places ℓ of F where Γ is ramified. To Y, one can associate its isocrystal $H^1(Y)$ as well as a family of partitions $P = (P_\ell)$ of the integer $N = [F(\pi_{Y'}) : F]$ indexed by $\ell \in T_\Gamma$. For each $\ell\in T_\Gamma$, P_ℓ is given by

$$P_\ell(\ell') = [F(\pi_{Y'})_{\ell'} : F_\ell]$$

with ℓ' ranging over the places of $F(\pi_{Y'})$ above ℓ. The pair $(H^1(Y), P)$ is the *partitioned isocrystal* attached to Y. In particular, we denote by s_Γ the pair attached to the unique Γ-simple supersingular abelian variety up to isogeny over $\overline{\mathbb{F}}_p$, see (4.19).

To study the pair $(H^1(Y), P)$, it is more convenient to consider Y as an F-linear abelian variety equipped with a Γ-action. Write $\rho : \Gamma \to \mathrm{End}_F(H^1(Y))$ for the ring homomorphism defining the Γ-action induced by functoriality on its isocrystal $H^1(Y)$. In essence, the definition (4.11) of partitioned isocrystals is a purely combinatorial formulation of the conditions that Y is F-hyper-symmetric and ρ factors through the endomorphism algebra $\mathrm{End}^0_F(Y)$ of the F-linear abelian variety Y.

The introduction of supersingular restriction (S) (4.20) has its origin in the following example. Assume that F is a totally real number field. If a Γ-linear isocrystal M contains a slope $1/2$ component at some place v of F above p, but not all, then there is *no* Γ-hyper-symmetric abelian variety Y such that $H^1(Y)$

is isomorphic to M. In the proof of the main theorem (5.1), we treat specially supersingular abelian varieties and isocrystals containing slope $1/2$ components.

Given any pair $y = (M, P)$ satisfying the supersingular restriction (S) and containing no s_Γ component, the construction of a Γ-hyper-symmetric abelian variety Y realizing y goes as follows. Let N be the integer such that $P = (P_\ell)_{\ell \in T_\Gamma}$ is a family of partitions of N. The Hilbert irreducibility theorem [4] enables us to find a suitable CM extension K/F of degree N, so that the family of partitions $(P_{K/F,\ \ell})_{\ell \in T_\Gamma}$ given by

$$P_{K/F,\ \ell}(\ell') := [K_{\ell'} : F_\ell], \quad \forall\ \ell' \mid \ell$$

concide with (P_ℓ). Then a simple formula (7.1) gives directly a p^a-Weil number π for a certain integer $a \geq 1$, such that $K = F(\pi)$ and the slopes of M at a place v of F above p are equal to $\lambda_w = \mathrm{ord}_w(\pi)/\mathrm{ord}_w(p^a)$, for $w|v$. Let Y' be the unique abelian variety up to Γ-isogeny corresponding to π. For some integer e, $(Y')^e \otimes_{\mathbb{F}_{p^a}} \overline{\mathbb{F}}_p$ equipped with a suitable polarization is a desired Γ-hyper-symmetric abelian variety.

The organization of this paper is as follows. In section 2 we set up the notations and review the fundamentals of isocrystals with extra structures, Dieudonné's theorem on the classification of isocrystals and the Honda-Tate theory. In section 3, we show that every Γ-hyper-symmetric abelian variety is isogenous to an abelian variety defined over $\overline{\mathbb{F}}_p$ (3.2). Then we prove a criterion of hyper-symmetry in terms of endomorphism algebras (3.4). In the next section, we define partitions and partitioned isocrystals. The main theorem (5.1) is stated in section 5. Several examples are provided to illustrate how to determine which data of slopes are realizable by hyper-symmetric abelian varieties. The proof of (5.1) is divided into two parts. The "only-if" part, in section 6, shows that to every Γ-hyper-symmetric abelian variety Y, one can associate a partitioned isocrystal y. We prove that y satisfies the supersingular restriction (S). A key ingredient of the proof is that the characteristic polynomial of the Frobenius endomorphism of $H^1(Y_{\mathbb{F}_{p^a}})$ has rational coefficients. In section 7 we prove the inverse, the "if" part.

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2. Notations and Generalities

Let p be a prime number fixed once and for all.

2.1. Let Γ be a positive simple algebra, finite dimensional over the field of rational numbers. We fix a positive involution $*$ on Γ. Let F be the center of Γ; F is either a totally real number field or a CM field. Let $v_1, \cdots, v_t$ be the places of F above p. We have

$$\Gamma \otimes_{\mathbb{Q}} \mathbb{Q}_p = \Gamma_{v_1} \times \cdots \times \Gamma_{v_t}.$$

Let T_Γ denote the following set

$$T_\Gamma = \{\ell \in \mathrm{Spec}(\mathcal{O}_F) |\ \ell \nmid p, \ell \neq (0), \mathrm{inv}_\ell(\Gamma) \neq 0\}.$$

2.2. Recall the computation of Brauer invariants. Let K be a finite extension of $\mathbb{Q}_p$. Let A be a central simple K-algebra of dimension d^2. By Hasse, A contains a d-dimensional unramified extension L/K such that for an element $u \in A$, the vectors $1, u, \cdots, u^{d-1}$ form an L-basis of A, and

$$\begin{cases} ua = \sigma(a)u, & \forall a \in L \\ u^d = \alpha \in L \end{cases}$$

where $\sigma \in \mathrm{Gal}(L/K)$ is the Frobenius automorphism of L/K. Then we define the Brauer invariant $\mathrm{inv}_K(A) \in \mathrm{Br}(K) \simeq \mathbb{Q}/\mathbb{Z}$ as

$$\mathrm{inv}_K(A) = -\mathrm{ord}_L(\alpha)/d,$$

where ord_L is the normalized valuation of L, i.e. $\mathrm{ord}_L(\pi) = 1$, for a uniformizer $\pi \in \mathcal{O}_L$.

2.3. If k is a perfect field of characteristic p, we denote by $W(k)$ the ring of Witt vectors of k. Let $K(k)$ be the fraction field of $W(k)$. The Frobenius automorphism of k induces by functoriality an automorphism σ of $W(k)$, namely,

$$\sigma(a_0, a_1, \cdots) = (a_0^p, a_1^p, \cdots)$$

for all $a_0, a_1, \cdots \in k$.

2.4. An *isocrystal* over k is a finite dimensional $K(k)$-vector space M equipped with a σ-linear automorphism Φ. A morphism $f : (M, \Phi) \to (M', \Phi')$ is a $K(k)$-linear map $f : M \to M'$ such that $f\Phi = \Phi' f$. Isocrystals over k form an abelian category.

2.5. Let $\overline{k}$ be an algebraic closure of k, a perfect field of characteristic p. We have the fundamental theorem of Dieudonné, cf. Kottwitz [8]:

(1) The category of isocrystals over $\overline{k}$ is semi-simple.

(2) A set of representatives of simple objects $\mathbb{E}_r$ can be given as follows,

$$\mathbb{E}_r = (K(\overline{k})[T]/(T^b - p^a), T)$$

where $r = a/b$ is a rational number with $(a, b) = 1$, $b > 0$. The endomorphism ring of $\mathbb{E}_r$ is a central division algebra over $\mathbb{Q}_p$ with Brauer invariant $-r \in \mathbb{Q}/\mathbb{Z}$.

(3) Every isocrystal M over k admits a unique decomposition

$$M = \bigoplus_{r \in \mathbb{Q}} M(r)$$

where $M(r)$ is the largest sub-isocrystal of *slope* r, i.e.

$$M(r) \otimes_{K(k)} K(\overline{k}) \simeq \mathbb{E}_r^{m_r}$$

for an integer m_r.

The rational numbers occurred in the decomposition $M = \bigoplus_{r\in\mathbb{Q}} M(r)$ are called the *slopes* of M. If all slopes are non-negative, the isocrystal is *effective*.

2.6. A *polarization of weight* 1 or simply a *polarization* of an isocrystal M is a symplectic form $\psi : M \times M \to K(k)$ such that

$$\psi(\Phi x, \Phi y) = p\sigma(\psi(x, y))$$

for all $x, y \in M$. The slopes of a polarized isocrystal, arranged in increasing order, are symmetric with respect to 1/2.

2.7. Let Γ be as in (2.1). A Γ-*linear isocrystal* over k is an isocrystal (M, Φ) over k together with a ring homomorphism $i : \Gamma \to \mathrm{End}(M, \Phi)$. The following variant of Dieudonné's theorem is proven in Kottwitz [8],

(1) The category of Γ-linear isocrystals over $\overline{k}$ is semi-simple. It is equivalent to the direct product of $\mathcal{C}_v$, the Γ_v-linear isocrystals over $\overline{k}$.

(2) For each place v of F above p, the simple objects of $\mathcal{C}_v$ are parametrized by $r \in \mathbb{Q}$, whose endomorphism ring is a central division algebra over F_v, with Hasse invariant $-[F_v : \mathbb{Q}_p]r - \mathrm{inv}_v(\Gamma)$ in the Brauer group $\mathrm{Br}(F_v)$.

If M is a Γ-linear isocrystal, and $M = M_{v_1} \times \cdots \times M_{v_t}$ is the decomposition defined in (1), we call the slopes of M_v the *slopes of* M *at* v and define the *multiplicity* of a slope r at v by

$$\mathrm{mult}_{M_v}(r) = \dim_{K(k)} M_v(r)/([F_v : \mathbb{Q}_p][\Gamma : F]^{1/2})$$

2.8. A Γ-*linear polarized isocrystal* is a quadruple (M, Φ, i, ψ), where (M, Φ) is an isocrystal, $i : \Gamma \to \mathrm{End}(M, \Phi)$ is a ring homomorphism, and ψ is a polarization on M such that

$$\psi(\gamma x, y) = \psi(x, \gamma^* y)$$

for all $\gamma \in \Gamma, x, y \in M$. If F is a totally real number field, the slopes of M at each place v of F above p, arranged in increasing order, are symmetric about 1/2. If F is a CM field, the slopes at v and $\overline{v}$ collected together, arranged in increasing order, are symmetric with respect to 1/2.

2.9. Recall that a morphism of abelian varieties $f : X \to X'$ is an *isogeny* if it is surjective with a finite kernel. Let X be an abelian variety over a finite field $k = \mathbb{F}_{p^a}$. The relative Frobenius morphism

$$F_{X/k} : X \to X^{(p)}$$

is an isogeny. We call $\pi_X = F^a_{X/k}$ the *Frobenius endomorphism* of X. If X is a *simple* abelian variety, the Frobenius endomorphism π_X is a p^a-*Weil number*, that is, an algebraic integer π such that for every complex imbedding $\iota : \mathbb{Q}(\pi) \hookrightarrow \mathbb{C}$, one has

$$\mid \iota(\pi) \mid = p^{a/2}.$$

Here is a basic result, due to Honda-Tate [11]:

(1) The map $X \mapsto \pi_X$ defines a bijection from the isogeny classes of simple abelian varieties over k to the conjugacy classes of p^a-Weil numbers.

(2) The endomorphism algebra $\mathrm{End}^0(X_\pi)$ of a simple abelian variety X_π corresponding to π is a central division algebra over $\mathbb{Q}(\pi)$. One has

$$2.\dim(X_\pi) = [\mathbb{Q}(\pi) : \mathbb{Q}][\mathrm{End}^0(X_\pi) : \mathbb{Q}(\pi)]^{1/2} .$$

(a) If $a \in 2\mathbb{Z}$, and $\pi = p^{a/2}$, then X_π is a supersingular elliptic curve, whose endomorphism algebra is $D_{p,\infty}$, the quaternion division algebra over $\mathbb{Q}$, ramified exactly at p and the infinity.

(b) If $a \in \mathbb{Z} - 2\mathbb{Z}$, and $\pi = p^{a/2}$, then $X_\pi \otimes_k k'$ is isogenous to the product of two supersingular elliptic curves, where k' is the unique quadratic extension of k.

(c) If π is totally imaginary, the division algebra $D = \mathrm{End}^0(X_\pi)$ is unramified away from p. For a place w of $\mathbb{Q}(\pi)$ above p, the local invariant of D at w is

$$\mathrm{inv}_w(D) = -\mathrm{ord}_w(\pi)/\mathrm{ord}_w(p^a).$$

2.10. A Γ-*linear polarized abelian variety* is a triple (Y, λ, i) consisting of a polarized abelian variety (Y, λ) and a ring homomorphim $i : \Gamma \to \mathrm{End}^0(Y)$. We require that i is compatible with the involution $*$ and the Rosati involution on $\mathrm{End}^0(Y)$ associated to the polarization λ. The category of Γ-linear polarized abelian varieties up to isogeny is semi-simple. In particular, any such abelian variety Y admits a Γ-*isotypic* decomposition,

$$Y \sim_{\Gamma\text{-isog}} Y_1^{e_1} \times \cdots \times Y_r^{e_r}$$

where each Y_i is Γ-simple and for $i \neq j$, Y_i and Y_j are not Γ-isogenous. For each i, there exist a simple abelian variety X_i and an integer e_i, such that $Y_i \sim_{\text{isog}} X_i^{e_i}$. We say Y_i is of *type* X_i.

2.11. Let Y be a Γ-simple abelian variety of type X, i.e. $Y \sim_{\text{isog}} X^e$, for an integer e. Let Z_0, Z be the center of $\mathrm{End}^0(X)$ and $\mathrm{End}^0_\Gamma(Y)$, respectively. There is the following relation [8],

$$e.[\mathrm{End}^0(X) : Z_0]^{1/2}[Z_0 : \mathbb{Q}] = [\Gamma : F]^{1/2}[\mathrm{End}^0_\Gamma(Y) : Z]^{1/2}[Z : \mathbb{Q}].$$

One deduces that the $\mathbb{Q}$-dimension of any maximal étale sub-algebra of $\mathrm{End}^0(Y)$ is equal to $[\Gamma : F]^{1/2}$ times the $\mathbb{Q}$-dimension of any maximal étale sub-algebra of $\mathrm{End}^0_\Gamma(Y)$.

2.12. Let $k = \mathbb{F}_{p^a}$ be a finite field. Kottwitz [8] proved a variant of the theorem of Honda-Tate:

(1) The map $Y \mapsto \pi_Y$ is a bijection from the set of isogeny classes of Γ-simple abelian varieties over k to the F-conjugacy classes of p^a-Weil numbers.

(2) The endomorphism algebra $\mathrm{End}^0_\Gamma(Y_\pi)$ of a Γ-simple abelian variety Y_π corresponding to π is a central division algebra over $F(\pi)$. Let X_π be a simple abelian variety up to isogeny corresponding to π as in (2.9); Y_π is of type X_π. Let $D = \mathrm{End}^0(X_\pi)$, $C = \mathrm{End}^0_\Gamma(Y_\pi)$. Then one has the equality

$$[C] = [D \otimes_{\mathbb{Q}(\pi)} F(\pi)] - [\Gamma \otimes_F F(\pi)]$$

in the Brauer group of $F(\pi)$, and

$$2.\dim(Y_\pi) = [F(\pi) : \mathbb{Q}][\Gamma : F]^{1/2}[C : F(\pi)]^{1/2}.$$

3. A Criterion of Hyper-Symmetry

Let Y be a Γ-linear polarized abelian variety over an algebraically closed field k of characteristic p, and let $Y \sim_{\Gamma\text{-isog}} Y_1^{e_1} \times \cdots \times Y_r^{e_r}$ be the Γ-isotypic decomposition of Y, cf. (2.10). For the rest, $H^1(Y)$ stands for the first crystalline cohomology of Y, $H^1_{\mathrm{crys}}(Y/W(k)) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Lemma 3.1. *The abelian variety Y is Γ-hyper-symmetric if and only if each Y_i is Γ-hyper-symmetric and for any place v of F above p, for different i, j, Y_i and Y_j have no common slopes at v.*

Proof. This is clear. □

Proposition 3.2. *If Y is Γ-hyper-symmetric, there exists a Γ-hyper-symmetric abelian variety Y' over $\overline{\mathbb{F}}_p$ such that $Y' \otimes_{\overline{\mathbb{F}}_p} k$ is Γ-isogenous to Y.*

We first prove a weaker result.

Corollary 3.3. *There is a Γ-hyper-symmetric abelian variety Y' over $\overline{\mathbb{F}}_p$ such that the isocrystal $H^1(Y' \otimes_{\overline{\mathbb{F}}_p} k)$ is isomorphic to $H^1(Y)$.*

Proof. There is a Γ-linear polarized abelian variety Y_K over a finitely generated subfield K such that $Y_K \otimes_K k$ is isomorphic to Y and $\mathrm{End}(Y_K) = \mathrm{End}(Y)$.

Choose a scheme S, irreducible, smooth, of finite type over the prime field, so that, if η denotes the generic point of S, $k(\eta) = K$. We may and do assume that Y_K extends to an abelian scheme $\mathcal{Y}$ over S.

By a theorem of Grothendieck-Katz [6], the function assigning any point x of S the Newton polygon of the isocrystal $H^1(\mathcal{Y}_x)$ is constructible. Let S' be the open subset consisting of points x with the generic Newton polygon, i.e. the same Newton polygon with that of $H^1(Y)$. As S' is regular, the canonical homomorphism $\mathrm{End}(\mathcal{Y}_{S'}) \to \mathrm{End}(Y_K)$ is an isomorphism. So there is a well defined specialization map $sp : \mathrm{End}(Y_K) \to \mathrm{End}(\mathcal{Y}_t)$ for any point $t \in S'$. By the rigidity lemma 6.1

[9], sp is injective. Let t be a closed point of S' and $\mathcal{Y}_{\bar{t}} = \mathcal{Y}_t \otimes_{k(t)} \overline{k(t)}$. As Y is Γ-hyper-symmetric, $\mathrm{End}^0_\Gamma(Y_K) \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $\mathrm{End}_\Gamma(H^1(\mathcal{Y}_{\bar{t}}))$ have the same dimension. Thus the composite map

$$\mathrm{End}^0_\Gamma(Y_K) \otimes_{\mathbb{Q}} \mathbb{Q}_p \hookrightarrow \mathrm{End}^0_\Gamma(\mathcal{Y}_{\bar{t}}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \hookrightarrow \mathrm{End}_\Gamma(H^1(\mathcal{Y}_{\bar{t}}))$$

is bijective. It follows that $\mathcal{Y}_{\bar{t}}$ is a desired Γ-hyper-symmetric abelian variety over $\overline{k(t)} \simeq \overline{\mathbb{F}}_p$. □

Proof. of (3.2). Recall that by Grothendieck [10], an abelian variety Y over an algebraically closed field k of characteristic p is isogenous to an abelian variety defined over $\overline{\mathbb{F}}_p$ if and only if Y has sufficiently many complex multiplication, i.e. any maximal étale sub-algebra of $\mathrm{End}^0(Y)$ has dimension $2.\dim(Y)$ over $\mathbb{Q}$.

We only need to show that Y has sufficiently many complex multiplication. Without loss of generality we assume that Y is Γ-simple of type X, namely, X is simple and $Y \sim_{\mathrm{isog}} X^e$ for an integer e. Let Z_0, Z denote respectively the center of $\mathrm{End}^0(X)$ and $\mathrm{End}^0_\Gamma(Y)$. The dimension r of any maximal étale sub-algebra of $\mathrm{End}^0(Y)$ is

$$e.[\mathrm{End}^0(X) : Z_0]^{1/2}[Z_0 : \mathbb{Q}],$$

thus by (2.11), is equal to

$$[\Gamma : F]^{1/2}[\mathrm{End}^0_\Gamma(Y) : Z]^{1/2}[Z : \mathbb{Q}] = [\Gamma : F]^{1/2}[\mathrm{End}_\Gamma(H^1(Y)) : E]^{1/2}[E : \mathbb{Q}_p],$$

since Y is Γ-hyper-symmetric. In the above, E denotes the center of $\mathrm{End}_\Gamma(H^1(Y))$.

Let Y' be an abelian variety over $\overline{\mathbb{F}}_p$ as in Corollary (3.3). Similarly, the dimension r' of any maximal étale sub-algebra of $\mathrm{End}^0(Y')$ is equal to

$$[\Gamma : F]^{1/2}[\mathrm{End}_\Gamma(H^1(Y')) : E']^{1/2}[E' : \mathbb{Q}_p],$$

where E' is the center of $\mathrm{End}_\Gamma(H^1(Y'))$.

By the choice of Y', r and r' are equal. As any abelian variety over $\overline{\mathbb{F}}_p$ has sufficiently many complex multiplication (2.9), we have $r = r' = 2.\dim(Y')$. This finishes the proof. □

In the following we prove a criterion of Γ-hyper-symmetry in terms of the center Z of $\mathrm{End}^0_\Gamma(Y)$.

Proposition 3.4. *A Γ-linear polarized abelian variety Y over $\overline{\mathbb{F}}_p$ is Γ-hyper-symmetric if and only if the F_v-algebra $Z \otimes_F F_v$ is completely decomposed, i.e., $Z \otimes_F F_v \simeq F_v \times \cdots \times F_v$, for every place v of F above p.*

Proof. Let Y' be a Γ-linear polarized abelian variety over a finite field $\mathbb{F}_{p^a}$, such that $Y' \otimes_{\mathbb{F}_{p^a}} \overline{\mathbb{F}}_p \simeq Y$ and $\mathrm{End}(Y') = \mathrm{End}(Y)$. The center Z can be identified with $F(\pi)$, the sub-algebra generated by the Frobenius endomorphism of Y'. By Tate [11], over $\mathbb{F}_{p^a}$, the map

$$\mathrm{End}^0(Y') \otimes_{\mathbb{Q}} \mathbb{Q}_p \to \mathrm{End}(H^1(Y'))$$

is bijective.

Hence, the condition for Y to be Γ-hyper-symmetric is equivalent to

$$\mathrm{End}_\Gamma(H^1(Y')) = \mathrm{End}_\Gamma(H^1(Y)).$$

Let $M' := H^1(Y')$, and $M' = \bigoplus_{v|p} M'_v$ be the decomposition defined in (2.7). The isocrystal M'_v is Γ_v-linear and has a decomposition into isotypic components,

$$M'_v = \bigoplus_{r\in\mathbb{Q}} M'_v(r).$$

With these decompositions, the condition for Y to be Γ-hyper-symmetric is equivalent to

$$\mathrm{End}_{\Gamma_v}(M'_v(r)) = \mathrm{End}_{\Gamma_v}(M'_v(r) \otimes_{K(\mathbb{F}_{p^a})} K(\overline{\mathbb{F}}_p)),$$

for any $v|p$, and $r \in \mathbb{Q}$.

On the left hand side, the center of $\mathrm{End}_{\Gamma_v}(M'_v(r))$ is $F_v(\pi_{v,r})$, where $\pi_{v,r}$ stands for the endomorphism $\pi|M'_v(r)$. On the right hand side, the center is isomorphic to a direct product $F_v \times \cdots \times F_v$ with the number of factors equal to the number of Γ_v-simple components of $M'_v(r) \otimes_{K(\mathbb{F}_{p^a})} K(\overline{\mathbb{F}}_p)$.

Therefore, if Y is Γ-hyper-symmetric, the F-algebra $Z = F(\pi)$ is completely decomposed at every place v of F above p. Conversely, if Z/F is completely decomposed everywhere above p, any Γ-linear endomorphism f of the isocrystal $(H^1(Y), \Phi)$ commutes with the operator $\pi^{-1}\Phi^a$, and thus stabilizes the invariant sub-space of $\pi^{-1}\Phi^a$, i.e. $H^1(Y')$. Hence $f \in \mathrm{End}_\Gamma(H^1(Y'))$. This implies that Y is Γ-hyper-symmetric. □

4. Partitions and Partitioned Isocrystals

Definition 4.1. Let N be a positive integer. A *partition* of N *with support in a finite set* I is a function $P : I \to \mathbb{Z}_{>0}$, such that $\sum_{i\in I} P(i) = N$.

Definition 4.2. Let $f : X \to S$ be a surjective map of sets such that for all $s \in S$, $f^{-1}(s)$ is finite. An S-*partition* of N *with support in the fibres of* f is a function $P : X \to \mathbb{Z}_{>0}$ such that for each $s \in S$, $P \mid f^{-1}(s)$ is a partition of N with support in $f^{-1}(s)$.

$$\begin{array}{ccc} X & \xrightarrow{P} & \mathbb{Z}_{>0} \\ {\scriptstyle f}\downarrow & & \\ S & & \end{array}$$

Definition 4.3. Let P be an S-partition of N with structural map $f : X \to S$. For any map $g : S' \to S$, the *pull-back partition* $g^*(P) = P \circ p$ is an S'-partition of N, where $p : X \times_S S' \to X$ is the projection.

Definition 4.4. Let P_i be an S_i-partition of N, $i = 1, 2$. We say that P_1 is *equivalent* to P_2 if there exist a bijection $u : S_1 \to S_2$ and a u-isomorphism $g : X_1 \to X_2$ such that $P_1 = P_2 \circ g$.

Definition 4.5. Consider S-partitions P_i of N_i, $i = 1, 2$. Let $f_i : X_i \to \mathbb{Z}_{>0}$ be the structural maps. The *sum* $P_1 \oplus P_2$ is the following S-partition P of $N_1 + N_2$,

$$\begin{array}{ccc} X_1 \coprod X_2 & \xrightarrow{P} & \mathbb{Z}_{>0} \\ {\scriptstyle f}\downarrow & & \\ S & & \end{array}$$

where $P|X_i = P_i$, and $f|X_i = f_i$, $i = 1, 2$.

Example 4.6. Let S be a scheme, $f : X \to S$ a finite étale cover of rank N. We define an S-partition $P : X \to \mathbb{Z}_{>0}$ of N associated to f by

$$P(x) = [k(x) : k(f(x))], \quad \forall x \in X.$$

Example 4.7. Let F be a number field, K/F a finite field extension of degree N. Let $S = \mathrm{Spec}(\mathcal{O}_F)$, $I = \mathrm{Spec}(\mathcal{O}_K)$, and $f : I \to S$ the structural morphism. Consider the function $P_{K/F} : I \to \mathbb{Z}_{>0}$ defined as

$$P_{K/F}(w) = \begin{cases} [K_w : F_{f(w)}], & \text{if } w \text{ is a finite prime} \\ N, & \text{if } w = (0) \end{cases}$$

This $P_{K/F}$ defines an S-partition of N. The most interesting case is $K = F(\pi_Y)$, the field generated by the Frobenius endomorphism π_Y of a Γ-simple non-supersingular abelian variety Y over a finite field k (2.12). We study this example in more detail.

(a). F is totally real, K is a CM extension.

One has $[K_w : F_{f(w)}] = [K_{\overline{w}} : F_{f(\overline{w})}]$, and $[K_w : F_{f(w)}]$ is an even integer if $w = \overline{w}$. Recall that T_Γ (2.1) denotes the set of finite prime-to-p places ℓ of F where Γ is ramified. The restriction $P_{K/F}|T_\Gamma$ (4.3) is equivalent to a T_Γ-partition $\{P_\ell : [1, d_\ell] \to \mathbb{Z}_{>0} |\ \ell \in T_\Gamma\}$ of $N = [K : F]$, which satisfies the following property

$$\begin{cases} P_\ell(2i-1) = P_\ell(2i), & \text{for } i \in [1, c_1(\ell)] \\ P_\ell(i) \text{ is even}, & \text{for } i \in [2c_1(\ell)+1, d_\ell] \end{cases}$$

where $d_\ell = \mathrm{Card}(f^{-1}(\ell))$, $2c_1(\ell) = \mathrm{Card}(\{w \in f^{-1}(\ell) |\ w \neq \overline{w}\})$.

(b). F is a CM field, K is a CM extension.

One has $[K_w : F_{f(w)}] = [K_{\overline{w}} : F_{f(\overline{w})}]$. The restriction $P_{K/F}|T_\Gamma$ is equivalent to

$$\{P_\ell : [1, d_\ell] \to \mathbb{Z}_{>0} |\ \ell \in T_\Gamma\}$$

which satisfies the property

$$\begin{cases} P_\ell(2i-1) = P_\ell(2i), & \text{if } \ell = \overline{\ell}, i \in [1, c_1(\ell)] \\ P_\ell(i) = P_{\overline{\ell}}(i), & \text{if } \ell \nmid \overline{\ell} \end{cases}$$

where $d_\ell = \mathrm{Card}(f^{-1}(\ell))$. If $\ell = \overline{\ell}$, $2c_1(\ell) := \mathrm{Card}(\{w \in f^{-1}(\ell) |\ w \neq \overline{w}\})$.

Definition 4.8. A T_Γ-partition P of an integer N is said to be *of* CM-*type* or a CM-*type partition* if it is equivalent to the pull-back partition $P_{K/F}|T_\Gamma$ for a CM field K of degree N over F.

Partitions of CM-type can be characterized as follows.

Proposition 4.9. *A T_Γ-partition $P = \{P_\ell ; \ell \in T_\Gamma\}$ of an integer N is of* CM-*type if and only if it satisfies the properties in* (4.7) (a) *or* (b)*.*

For a proof, we need the following lemma.

Lemma 4.10. *Let D be a number field, T a set of maximal ideals in $\mathcal{O}_D$. For any T-partition $R : I \to \mathbb{Z}_{>0}$ of an integer N with support in the fibres of $u : I \to T$,*

$$\begin{array}{ccc} I & \xrightarrow{R} & \mathbb{Z}_{>0} \\ {\scriptstyle u}\downarrow & & \\ T & & \end{array}$$

there is a finite étale cover $f_t : X_t \to \mathrm{Spec}(\mathcal{O}_{D_t})$ of rank N, such that the partition associated to f_t restricted to $\{t\}$ is equivalent to $R|u^{-1}(t)$, for every $t \in T$.

Proof. Here D_t denotes a local field, the completion of D with respect to the t-adic absolute value. For each $i \in I$, $t = u(i)$, let X_i be the unique connected étale cover of $\mathrm{Spec}(\mathcal{O}_{D_t})$ of rank $R(i)$. The desired scheme X_t can be chosen as

$$X_t = \coprod_{i \in u^{-1}(t)} X_i,$$

for $t \in T$. □

Proof. of (4.9). It remains to prove the "if"-part of the Proposition (4.9). Let P be a given T_Γ-partition of N satisfying the conditions of (4.7) (a) or (b).

(a). Assume first that F is a totally real number field. We define a T_Γ-partition R of the integer $N/2$,

$$R_\ell(j) = \begin{cases} P_\ell(2j), & j \in [1, c_1(\ell)] \\ P_\ell(j + c_1(\ell))/2, & j \in [c_1(\ell) + 1, d_\ell - c_1(\ell)]. \end{cases}$$

For each $\ell \in T_\Gamma$, let

$$X_\ell = \coprod_{j \in [1, d_\ell]} X_j$$

be the étale cover of $\mathrm{Spec}(\mathcal{O}_{F_\ell})$ constructed in Lemma (4.10) corresponding to the partition R. Then by Proposition (7.5), there exists a totally real extension E

of F of degree $N/2$, such that X_ℓ is isomorphic to the spectrum of $\mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_{F_\ell}$. Define a scheme Y_ℓ over X_ℓ,

$$Y_\ell := \coprod_{j\in[1,c_1(\ell)]} (X_j \coprod X_j) \coprod_{j\in[c_1(\ell)+1,d_\ell-c_1(\ell)]} Y_j$$

where, for $j \in [c_1(\ell)+1, d_\ell - c_1(\ell)]$, Y_j denotes the unique connected étale cover of X_j of rank 2. We apply weak approximation to get a CM quadratic extension K of E, so that for each $\ell \in T_\Gamma$, Y_ℓ is isomorphic to the spectrum of the ring $\mathcal{O}_K \otimes_{\mathcal{O}_F} \mathcal{O}_{F_\ell}$. One verifies that K is a desired solution.

(b). Next assume that F is totally imaginary. Let F_0 be its maximal totally real subfield, and T_0 be the image of T_Γ under the morphism $\mathrm{Spec}(\mathcal{O}_F) \to \mathrm{Spec}(\mathcal{O}_{F_0})$. From the T_Γ-partition P we construct a T_0-partition of R of the same integer N as follows. If $\ell_0 = \ell\overline{\ell}$ is split in F,

$$R_{\ell_0}(j) := P_\ell(j), \quad j \in [1, d_\ell].$$

If ℓ_0 is inert or ramified in F, $\ell_0 = \ell|F_0$, $\ell \in T_\Gamma$,

$$R_{\ell_0}(j) := \begin{cases} 2.P_\ell(2j), & j \in [1, c_1(\ell)] \\ P_\ell(j + c_1(\ell)), & j \in [c_1(\ell)+1, d_\ell - c_1(\ell)] \end{cases}$$

By Proposition (7.5), for a suitable totally real extension E/F_0 of degree N, one has

(i) if $\ell_0 = \ell\overline{\ell}$ is split,

$$E \otimes_{F_0} (F_0)_{\ell_0} \simeq \prod_{j\in[1,d_\ell]} E_j,$$

where E_j is the unique unramified extension of $(F_0)_{\ell_0}$ of degree $R_{\ell_0}(j)$.

(ii) if $\ell_0 = \ell|F_0$ is inert or ramified in F,

$$E \otimes_{F_0} (F_0)_{\ell_0} \simeq \prod_{j\in[1,d_\ell-c_1(\ell)]} E_j$$

where E_j is the unique unramified extension of F_ℓ of degree $R_{\ell_0}(j)/2$, for $j \in [1, c_1(\ell)]$, and is an extension of $(F_0)_{\ell_0}$ of degree $R_{\ell_0}(j)$ linearly disjoint with F_ℓ, for $j \in [c_1(\ell)+1, d_\ell - c_1(\ell)]$.

Form the tensor product $K := E \otimes_{F_0} F$. One checks that the T_Γ-partition $P_{K/F}|T_\Gamma$ is equivalent to P. □

Definition 4.11. A Γ-*linear polarized simply partitioned isocrystal* x is a pair (M, P) consisting of a polarized Γ-linear isocrystal M and a T_Γ-partition of an integer $N(x)$, $P : I \to \mathbb{Z}_{>0}$, with support in the fibres of $f : I \to T_\Gamma$,

$$\begin{CD} I @>P>> \mathbb{Z}_{>0} \\ @VfVV \\ T_\Gamma \end{CD}$$

which satisfies the following conditions:

(SPI1) There exists a constant $n(x)$ such that for every place v of F above p, the Γ_v-linear isocrystal M_v has $N(x)$ isotypic components, and the multiplicity (2.7) of each component is equal to $n(x)$.

(SPI2) For every $\ell' \in I$, $n(x).\mathrm{inv}_{f(\ell')}(\Gamma)P(\ell') = 0$ in $\mathbb{Q}/\mathbb{Z}$.

We shorten Γ-linear polarized simply partitioned isocrystal to *simply partitioned isocrystal* if this causes no confusion. We call M the underlying isocrystal, P the defining partition of $x = (M, P)$. The dimension, slopes, multiplicity $n(x)$, Newton polygons, and polarizations of x will be understood to be those of M.

Definition 4.12. Two simply partitioned isocrystals x, y are said to be *equivalent* if their isocrystals are isomorphic and their partitions are equivalent (4.4).

Definition 4.13. Let $x = (M, P)$ be a simply partitioned isocrystal. For any non-negative integer a, we define the *scalar multiple* $a.x$ to be (M^a, P); $a.x$ is a simply partitioned isocrystal. If $a \geq 1$, then $N(a.x) = N(x)$, $n(a.x) = a.n(x)$. If there exist an integer $a > 1$ and a simply partitioned isocrystal y such that $x = a.y$, then x is called *divisible*.

Definition 4.14. There is a partially defined *sum* operation on the set of simply partitioned isocrystals. Suppose that the simply partitioned isocrystals $x_i = (M_i, P_i)$, $i = 1, 2$, satisfy the following assumptions:

(1) Their multiplicities are equal $n(x_1) = n(x_2)$.
(2) For any place v of F above p, $(M_1)_v$ and $(M_2)_v$ have no common slopes.

Then we define the *sum* $x_1 + x_2$ to be the pair $(M_1 \oplus M_2, P_1 \oplus P_2)$, see (4.5); $x_1 + x_2$ is a simply partitioned isocrystal.

One verifies that if $x_1 + x_2$ is defined, then $x_2 + x_1$ is also defined and

$$x_1 + x_2 = x_2 + x_1.$$

If $x_1 + x_2$ and $(x_1 + x_2) + x_3$ are both defined, then $x_2 + x_3$ and $x_1 + (x_2 + x_3)$ are also defined, and the associativity holds, i.e.

$$(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3).$$

Definition 4.15. A Γ-*linear polarized partitioned isocrystal* is a finite collection of simply partitioned isocrystals $x = \{x_a; a \in A\}$, such that the following conditions are satisfied.

(PI1) For each pair $a, b \in A$, and each place v of F above p, $(x_a)_v$ and $(x_b)_v$ have no common slopes.

(PI2) The multiplicities $n(x_a)$ are distinct, for $a \in A$.

We call x a *partitioned isocrystal* if no confusion arises. Each x_a is called a *component* of x. The direct sum of the underlying isocrystals of x_a, $M = \bigoplus_{a \in A} M_a$, is called the *underlying isocrystal* of x.

Definition 4.16. Two partitioned isocrystals $x = \{x_a; a \in A\}$ and $y = \{y_b; b \in B\}$ are *equivalent* if there exists a bijection $u : A \to B$ such that each x_a is equivalent to $y_{u(a)}$.

Up to equivalence, every partitioned isocrystal $x = \{x_a; a \in A\}$ can be naturally indexed by the multiplicities of its simple components, cf. (PI2) (4.15).

Definition 4.17. Let $x = \{x_a; a \in A\}$ be a partitioned isocrystal (4.15). For any non-negative integer h, we define the *scalar multiple* $h.x$ to be $\{h.x_a; a \in A\}$. A partitioned isocrystal is *divisible* if $x = h.y$ for some integer $h > 1$ and a partitioned isocrystal y, cf. (4.13).

Definition 4.18. The sum operation defined for simply partitioned isocrystals can be extended to partitioned isocrystals. Given two partitioned isocrystals $x = \{x_a; a \in A\}$, $y = \{y_b; b \in B\}$ satisfying the following restriction,

(N) For each pair $a \in A$, $b \in B$, and for each place v of F above p, x_a and y_b have no common slopes at v.

we define their *joint*, $s = x \bigvee y$, another partitioned isocrystal, as follows. Let C be the finite set of positive integers c such that either x or y or both has a component whose multiplicity is c. This set C will parametrize the components of s. In other words, we have

$$s = \{s_c; c \in C\}$$

(i) If exactly one of the x, y has a component with multiplicity c, say $n(x_a) = c$, one defines s_c to be x_a.

(ii) If both x and y have components, say x_a, y_b, such that $n(x_a) = n(y_b) = c$, one defines s_c to be the sum $x_a + y_b$ (4.14).

Whenever it is defined, the joint operation is clearly commutative and associative up to canonical equivalence.

Definition 4.19. A simply partitioned isocrystal s_Γ. We define s_Γ to be the simply partitioned isocrystal $(H^1(A), P)$ associated to the unique Γ-simple supersingular abelian variety A up to isogeny over $\overline{\mathbb{F}}_p$. The partition P is the unique

T_Γ-partition of 1, i.e. $P(\ell) = 1$, for any $\ell \in T_\Gamma$.

$$\begin{array}{ccc} T_\Gamma & \xrightarrow{P} & \mathbb{Z}_{>0} \\ {\scriptstyle id}\downarrow & & \\ T_\Gamma & & \end{array}$$

At every place v of F above p, s_Γ is isotypic of slope 1/2 and its multiplicity $n(s_\Gamma)$ is equal to the order e_Γ of the class $[D_{p,\infty} \otimes_{\mathbb{Q}} F] - [\Gamma]$ in $\mathrm{Br}(F)$, see (6.1).

Definition 4.20. Partitioned Isocrystal with (S)-Restriction. A partitioned isocrystal $x = \{x_a; a \in A\}$ is said to satisfy the *supersingular restriction* (S) if there exist an integer $h \geq 0$ and a partitioned isocrystal $y = \{y_b; b \in B\}$ such that

(S1) $x = h.s_\Gamma \bigvee y$,

(S2) if F is totally real, y contains no slope 1/2 part,

(S3) the partition P_b of each component $y_b = (M_b, P_b)$ is of CM-type (4.8).

For simplicity we call x an (S)-*restricted* partitioned isocrystal.

Remarks 4.21. (a). When $h \geq 1$, the condition (S1) implies that for every place v of F above p, y has no slope 1/2 component at v, see (4.18).

(b). The condition (S3) is a purely combinatorial condition, see the characterization of CM-type partitions in (4.9).

5. Main Theorem and Examples

For the rest of the paper, all abelian varieties and isocrystals are defined over $\overline{\mathbb{F}}_p$.

Now we formulate our criterion for a Γ-linear polarized isocrystal to be realizable by a Γ-hyper-symmetric abelian variety.

Theorem 5.1. *An effective Γ-linear polarized isocrystal M is isomorphic to the Dieudonné isocrystal $H^1(Y)$ of a Γ-hyper-symmetric abelian variety Y if and only if M underlies an* (S)*-restricted partitioned isocrystal.*

The theorem will be proven in the next two sections. Here we apply it to some examples of simple algebras Γ for which we work out explicitly the slopes and multiplicities of the Γ-hyper-symmetric abelian varieties. Note that the *multiplicity* is defined in (2.7).

Example 5.2. (Siegel) $\Gamma = \mathbb{Q}$. As T_Γ is empty, the supersingular restriction (S) is reduced to (S1) and (S2). A non-divisible simply partitioned isocrystal without slope 1/2 component is called *balanced* in the terminology of Chai-Oort [1]. In general, any simply partitioned isocrystal x can be expressed uniquely as

$$x = h.s_\Gamma + m.y$$

with integers $h, m \geq 0$ and a balanced isocrystal y. One deduces that any Newton polygon of the form

$$\rho_0.(1/2) + \sum_{i\in[1,t]} (\rho_i.(\lambda_i) + \rho_i.(1-\lambda_i))$$

can be realized by a hyper-symmetric abelian variety, where $\lambda_i \in [0, 1/2)$ are pairwise distinct slopes, $\rho_0 = \mathrm{mult}(1/2)$, $\rho_i = \mathrm{mult}(\lambda_i)$ are multiplicities. This example recovers the Proposition (2.5) of Chai-Oort [1].

Example 5.3. Let F be a real quadratic field split at p, $p = v_1 v_2$. The following slope data

$$\begin{cases} 2.(1/2), & \text{at } v_1 \\ 1.(0) + 1.(1), & \text{at } v_2 \end{cases}$$

admit *no* hyper-symmetric point.

Example 5.4. Let $\Gamma = F$ be a totally real field of degree d over $\mathbb{Q}$. The restriction (S) is reduced to (S1) and (S2).

The isocrystal s_F is isotypic of slope $1/2$ at every place v of F. The multiplicity is $n(s_F) = e_F$, the order of the class $[D_{p,\infty} \otimes_{\mathbb{Q}} F]$ in the Brauer group of F, cf. (6.1).

Any simply partitioned isocrystal y without slope $1/2$ component can be decomposed as a finite sum

$$y = y_1 + \cdots + y_n,$$

where each y_i has two isotypic components at every place $v|p$.

Let z be one of the y_i's , and let $\{\lambda_v, 1-\lambda_v\}$ be the two slopes of z at v. Then the multiplicity $n(z)$ is a common multiple of the denominators of $[F_v : \mathbb{Q}_p]\lambda_v$, where v runs over the places of F above p.

As a consequence, an F-linear polarized isocrytal M of dimension $2d$ over $K(\overline{\mathbb{F}}_p)$ is realizable by an F-hyper-symmetric abelian variety over $\overline{\mathbb{F}}_p$ if and only if the slopes of M has exactly one of the following two patterns:

(i) At every place $v|p$, there is only one slope $1/2$ with multiplicity 2.

(ii) At every place $v|p$, there are two slopes $\{\lambda_v, 1-\lambda_v\}$, each of multiplicity 1. These λ_v are such that $[F_v : \mathbb{Q}_p]\lambda_v \in \mathbb{Z}$.

Example 5.5. Let $\Gamma = F$ be a CM field, $[F : \mathbb{Q}] = 2d$. The restriction (S) is reduced to (S1).

The isocrystal s_F is isotypic of slope $1/2$ at every place v of F above p. The multiplicity is $n(s_F) = e_F$, the order of the class $[D_{p,\infty} \otimes_{\mathbb{Q}} F]$ in the Brauer group of F.

Any (S)-restricted simply partitioned isocrystal y is decomposed as a finite sum

$$y = y_1 + \cdots + y_n,$$

where each y_i has either one or two isotypic components. More explicitly, for a fixed $z = y_i$,

(i) if z has one isotypic component at every place $v|p$, the slopes are such that $\lambda_v + \lambda_{\overline{v}} = 1$. In particular, $\lambda_v = 1/2$, if $v = \overline{v}$. The multiplicity $n(z)$ is a multiple of the common denominator of $[F_v : \mathbb{Q}_p]\lambda_v$, for $v|p$.

(ii) if z has two isotypic components at every place $v|p$,

(a) if $v = \overline{v}$, the slopes are $\{\lambda_v, 1 - \lambda_v\}$, with $\lambda_v \in [0, 1/2)$.

(b) if $v \neq \overline{v}$, the slopes are either

$$\begin{cases} \lambda_v,\ 1-\lambda_v, & \text{at } v \\ \lambda_{\overline{v}},\ 1-\lambda_{\overline{v}}, & \text{at } \overline{v} \end{cases}$$

or

$$\begin{cases} \mu_v,\ \ \nu_v, & \text{at } v \\ 1-\mu_v,\ 1-\nu_v, & \text{at } \overline{v} \end{cases}$$

with $\lambda_v,\ \lambda_{\overline{v}} \in [0, 1/2)$, $\mu_v \neq \nu_v \in [0, 1]$.

Example 5.6. Let Γ be a definite quaternion division algebra over $\mathbb{Q}$. We assume that Γ is ramified exactly at the infinity and a prime ℓ different from p. Hence $T_\Gamma = \{\ell\}$ and $\mathrm{inv}_\ell(\Gamma) = 1/2$.

The partitioned isocrystal s_Γ is isotypic of slope $1/2$ with multiplicity $n(s_\Gamma) = 2$, because the order e_Γ of the class $[D_{p,\infty}] - [\Gamma]$ in the Brauer group of $\mathbb{Q}$ is 2.

Let y be a simply partitioned isocrystal without slope $1/2$ component. Let

$$P_\ell : [1, d_\ell] \to \mathbb{Z}_{>0}$$

be the defining partition of y. The condition (SPI2) says that

$$n(y).P_\ell(i).1/2 \in \mathbb{Z}, \quad \text{for all } i \in [1, d_\ell].$$

If y is (S)-restricted, then by (S3), its partition is of the following form

$$\begin{cases} P_\ell(2i-1) = P_\ell(2i), & i \in [1, c_1(\ell)] \\ P_\ell(i) \text{ is even}, & i \in [2c_1(\ell)+1, d_\ell] \end{cases}$$

for some integer $c_1(\ell) \in \mathbb{Z}_{\geq 0}$.

Now let M be any effective Γ-linear polarized isocrystal satisfying the condition (SPI1) and without slope $1/2$ component. We claim that M underlies an (S)-restricted simply partitioned isocrystal y. In fact, one can choose $y = (M, P_l)$, where $d_\ell = 1$, $P_\ell(1) = N(y)$, and $N(y)$ is the number of isotypic components of M. Note that $N(y)$ is an even integer because M is polarized and has no slope $1/2$ component.

With this choice of partition P_l, the simply partitioned isocrystal y decomposes as a finite sum

$$y = y_1 + \cdots + y_m,$$

where each y_i has exactly two isotypic components with slopes $\{\lambda_i, 1-\lambda_i\}$. The multiplicity $n(y)$ is a multiple of the common denominator of the λ_i's.

For example, let us work out the slopes and multiplicities of all (S)-restricted partitioned isocrystals of dimension 12 over $K(\overline{\mathbb{F}}_p)$. There are exactly five Newton polygons which are realizable by 6-dimensional Γ-hyper-symmetric abelian varieties:

a. $3.(1/2)$.
b. $1.(0) + 1.(1) + 2.(1/2)$.
c. $2.(0) + 2.(1) + 1.(1/2)$.
d. $3.(0) + 3.(1)$.
e. $1.(1/3) + 1.(2/3)$.

The above notation, for example, $1.(0)+1.(1)+2.(1/2)$ means that the slopes are $\{0, 1, 1/2\}$, with multiplicities $\{1, 1, 2\}$, respectively.

Example 5.7. Let F be a CM field, and Γ be a positive central division algebra over F. We make the following assumptions on Γ,

(i) $[F:\mathbb{Q}] = 4$; $[F_{v_1} : \mathbb{Q}_p] = 2$, $[F_{v_2} : \mathbb{Q}_p] = [F_{\overline{v_2}} : \mathbb{Q}_p] = 1$, $v_1, v_2, \overline{v_2}$ are above p.

(ii) Γ is ramified exactly at v_1 and a finite prime-to-p place ℓ, $\ell = \overline{\ell}$; $\mathrm{inv}_{v_1}(\Gamma) = 1/3$, $\mathrm{inv}_\ell(\Gamma) = 2/3$.

The Brauer class $c = [D_{p,\infty} \otimes_{\mathbb{Q}} F] - [\Gamma] \in \mathrm{Br}(F)$ has local invariants

$$\mathrm{inv}_\nu(c) = \begin{cases} -1/3, & \text{if } \nu = v_1 \\ -1/2, & \text{if } \nu = v_2, \overline{v_2} \\ -2/3, & \text{if } \nu = \ell \\ 0, & \text{otherwise} \end{cases}$$

Hence the order of c, as well as the multiplicity $n(s_\Gamma)$, is equal to 6.

Let y be a simply partitioned isocrystal. Let $N(y)$ be the number of isotypic components, $n(y)$ the multiplicity of y at each place $v \in \{v_1, v_2, \overline{v_2}\}$. Denote by P_ℓ the defining partition of y

$$P_\ell : [1, d_\ell] \to \mathbb{Z}_{>0}.$$

In this case, the condition (SPI2) says that

$$n(y)P_\ell(i).2/3 \in \mathbb{Z}, \quad \text{for all } i \in [1, d_\ell].$$

If y is (S)-restricted, then by (4.9), its partition P_l satisfies the condition

$$P_\ell(2i-1) = P_\ell(2i), \quad \forall\, i \in [1, c_1(\ell)],$$

for some integer $c_1(\ell)$, with $0 \le 2c_1(\ell) \le d_\ell$.

We give another example of Newton polygon which admits *no* hyper-symmetric point.

$$\xi = \begin{cases} 1.(0) + 1.(1), & \text{at } v_1 \\ 1.(0) + 1.(1), & \text{at } v_2 \\ 1.(0) + 1.(1), & \text{at } \overline{v_2} \end{cases}$$

Note that if M has ξ as Newton polygon, then

$$\dim_{K(\overline{\mathbb{F}}_p)}(M_{v_1}) = 12, \quad \dim_{K(\overline{\mathbb{F}}_p)}(M_{v_2}) = \dim_{K(\overline{\mathbb{F}}_p)}(M_{\overline{v_2}}) = 6.$$

At each place $v \in \{v_1, v_2, \overline{v_2}\}$, M has $N = 2$ isotypic components, the multiplicity of every isotypic component is $n = 1$. But there is no partition P_ℓ of $N = 2$, such that $n.P_\ell(i).2/3 \in \mathbb{Z}$.

Now we compute the Newton polygons of all (S)-restricted partitioned isocrystals of dimension 72 over $K(\overline{\mathbb{F}}_p)$. By (S1), we can write $x = h.s_\Gamma \bigvee y$. Note that the dimension of s_Γ is 72. One has either $x = s_\Gamma$ or $x = y$. Consider the case $x = y$ and write

$$y = \{y_b; b \in B\},$$

where y_b are the simple components of y. Comparing the dimensions of y_b and y, one has

$$72 = [\Gamma : F]^{1/2}[F : \mathbb{Q}] \sum_{b \in B} N(y_b)n(y_b),$$

where $N(y_b)$ denotes the number of isotypic components, $n(y_b)$ the multiplicity, of y_b at each place of F above p. Since $[\Gamma : F]^{1/2} = 3$, $[F : \mathbb{Q}] = 4$, this equation is reduced to

$$6 = \sum_{b \in B} N(y_b)n(y_b).$$

One verifies that this condition forces that y is simply partitioned, $N(y) = 2$, and $n(y) = 3$. Here we list all the realizable Newton polygons as follows.

(i) The slopes at v_1 are one of:

$$\begin{cases} 0, \ 1 \\ 1/3, \ 2/3 \\ 1/6, \ 5/6 \end{cases}$$

(ii) The slopes at $v_2, \overline{v_2}$, in this order, are one of:

$$\begin{cases} 0, \ \ 1; & 1/3, 2/3 \\ 0, \ \ 1/3; & 1, 2/3 \\ 0, \ \ 2/3; & 1, 1/3 \\ 1, \ \ 1/3; & 0, 2/3 \\ 1, \ \ 2/3; & 0, 1/3 \\ 1/3, 2/3; & 0, 1 \end{cases}$$

6. Proof of the "only-if" part of (5.1)

Given a Γ-hyper-symmetric abelian variety Y, we let $Y \sim_{\Gamma\text{-isog}} Y_1^{e_1} \times \cdots \times Y_r^{e_r}$ be the Γ-isotypic decomposition. By (3.1), for the only-if part, we only need to show that each $H^1(Y_i)$ underlies an (S)-restricted partitioned isocrystal x_i. Indeed, if this is proved, $H^1(Y)$ is isomorphic to the underlying isocrystal of $x = \{e_1.x_1\} \bigvee \cdots \bigvee \{e_r.x_r\}$.

From now on, we assume that Y is Γ-simple. Let $q = p^a$ and $Y_{\mathbb{F}_q}$ be a Γ-linear polarized abelian variety over $\mathbb{F}_q$ such that $Y_{\mathbb{F}_q} \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_p \simeq Y$. Suppose that a is sufficiently divisible. The abelian variety $Y_{\mathbb{F}_q}$ is Γ-simple, therefore, $Y_{\mathbb{F}_q} \sim_{\text{isog}} X^s_{\mathbb{F}_q}$, for some $X_{\mathbb{F}_q}$ simple over $\mathbb{F}_q$. Let π denote the Frobenius endomorphism of $Y_{\mathbb{F}_q}$ as well as that of $X_{\mathbb{F}_q}$. Let $K = F(\pi)$.

Proposition 6.1. *The pair $x = (H^1(Y), P_{K/F}|T_\Gamma)$ associated to the Γ-simple hyper-symmetric abelian variety Y is a simply partitioned isocrystal satisfying the supersingular restriction* (S). *More explicitly,*

(a) *if π is totally real, then $x = s_\Gamma$ is isotypic of slope* $1/2$ *with multiplicity $n(s_\Gamma)$ equal to the order of Brauer class $[D_{p,\infty} \otimes_{\mathbb{Q}} F] - [\Gamma]$ in* $\mathrm{Br}(F)$.

(b) *if π is totally imaginary, then x has $N(x) = [K : F]$ isotypic components at every place v of F above p, the multiplicity $n(x)$ is the order of the class* $[\mathrm{End}^0_\Gamma(Y)]$ *in* $\mathrm{Br}(K)$.

Proof. Let $N = [F(\pi) : F]$ and denote by P the T_Γ-partition $P_{K/F}|T_\Gamma$ of N. Let $C := \mathrm{End}^0_\Gamma(Y_{\mathbb{F}_q})$ and $L_v := F_v \otimes_{\mathbb{Q}_p} K(\mathbb{F}_q)$. Decompose

$$H^1(Y_{\mathbb{F}_q}) = \bigoplus_{v|p} M_v$$

as in (2.7). Each M_v is a free L_v-module, by the next lemma and 11.5 [8]. We consider the characterisitic polynomial $f_v(T)$ of π as an L_v-linear transformation of M_v. Since Y is Γ-hyper-symmetric, by (3.4),

$$f_v(T) = \prod_{w|v} (T - \iota_w(\pi))^{n_w}$$

is a product of linear polynomials, where $\iota_w : F(\pi) \hookrightarrow F_v$ denote the F-embeddings of $F(\pi)$ into F_v indexed by the places w. Thus the characterisitic polynomial $f(T) = \det(T - \pi|H^1(Y_{\mathbb{F}_q}))$ of the $K(\mathbb{F}_q)$-linear endomorphism π can be factored as

$$\prod_v \mathrm{Norm}_{L_v/K(\mathbb{F}_q)} f_v(T) = \prod_v \prod_w \mathrm{Norm}_{F_v/\mathbb{Q}_p}(T - \iota_w(\pi))^{n_w}.$$

Since the $\mathbb{Q}$-embeddings ι_u of $F(\pi)$ into $\overline{\mathbb{Q}}_p$ are one-to-one correspondence with the set of triples $u = (v, w, \tau)$ consisting of a place v of F above p, a place w of

$F(\pi)$ above v, and a $\mathbb{Q}_p$-linear homomorphism $\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_p$, we can rewrite $f(T)$ as

$$f(T) = \prod_u (T - \iota_u(\pi))^{n_w}.$$

By Katz-Messing [7], the polynomial $f(T) \in \mathbb{Z}[T]$, so $n_w = n$ is independent of the place w, and thus, is equal to $2.\dim(Y)/[F(\pi) : \mathbb{Q}]$. Because $f_v(T)$ has N different irreducible factors, i.e. $T - \iota_w(\pi)$, $H^1(Y)$ has N isotypic components at every place v of F above p [8]. By the dimension formula in (2.12), the multiplicity of each isotypic component is equal to

$$[L_v : K(\mathbb{F}_q)]n/([\Gamma : F]^{1/2}[F_v : \mathbb{Q}_p]) = \mathrm{order}([C]).$$

Observe that for every place ℓ' of K above a place $\ell \in T_\Gamma$, the local invariant of C at ℓ' is

$$\mathrm{inv}_{\ell'}(C) = -\mathrm{inv}_\ell(\Gamma)[K_{\ell'} : F_\ell].$$

It certainly follows that $\mathrm{order}([C])\mathrm{inv}_\ell(\Gamma)P(\ell') = 0$ in $\mathbb{Q}/\mathbb{Z}$.

If now $\pi = q^{1/2}$ is a totally real algebraic number, then, since we have assumed that a is sufficiently divisible, $X_{\mathbb{F}_q}$ is a super-singular elliptic curve. The isocrystal $H^1(Y)$ underlies the simply partitioned isocrystal s_Γ (4.19). At every place v of F above p, s_Γ is isotypic of slope $1/2$.

If π is totally imaginary, the field $K = F(\pi)$ is a CM extension of F; so the condition (S3) is a priori satisfied. In case that F is a totally real number field, the slopes of $H^1(Y)$ at a place v of F above p, if arranged in increasing order, are symmetric with respect to $1/2$. As there are $N = [K : F]$ of them, and N is even, $H^1(Y)$ contains no slope $1/2$ component. The proof is now complete. □

The following lemma is certainly well known and an analogous statement for ℓ-adic cohomology can be found in Mumford's book on abelian varieties.

Lemma 6.2. *If X is an abelian variety over a finite field k, the Frobenius endomorphism π acts in a semi-simple way on the isocrystal $H^1(X)$.*

Proof. We may and do assume that X is a simple abelian variety. Let $\pi = s + n$ be the Jordan decomposition of π considered as a linear endomorphism of $H^1(X)$. By Katz-Messing [7], the characterisic polynomial $\det(T - \pi | H^1(X))$ has rational coefficients. Hence we can find a polynomial $f(T) \in \mathbb{Q}[T]$ without constant term, such that the nilponent part $n = f(\pi)$. The image of ℓn, for a sufficiently divisible integer ℓ, is a proper sub-abelian variety of X, thus equal to 0. □

7. Proof of the "if" part of (5.1)

Let $x = h.s_\Gamma \bigvee y$ be an (S)-restricted partitioned isocrystal. This section is devoted to showing that x is realizable by a Γ-hyper-symmetric abelian variety. Here is the first step towards proving the existence theorem.

Proposition 7.1. *Let K be a* CM *field, $\{\lambda_w; w|p\}$ a set of rational numbers contained in the interval $[0,1]$ and indexed by the places w of K above p. Assume that $\lambda_w + \lambda_{\overline{w}} = 1$. Then there exist an integer $a \geq 1$ and a p^a-Weil number π such that*

$$\mathrm{ord}_w(\pi)/\mathrm{ord}_w(p^a) = \lambda_w,$$

for all $w|p$.

Proof. Let E be the maximal totally real subfield of K. For any place v of E above p, we define $\lambda_v := \min\{\lambda_w, \lambda_{\overline{w}}\}$, $v = w|E$. Either v is split, $v = w\overline{w}$, or there is only one prime w above v. In the first case, let $a_w \in \mathcal{O}_K$ be a generator of the ideal w^h; in the latter case, let $a_v \in \mathcal{O}_E$ be a generator of v^h, where h is the ideal class number of K. Consider the factorization

$$p\mathcal{O}_K = \prod_v (w\overline{w})^{e(v|p)} \prod_v v^{e(v|p)},$$

where the first product counts those v split in K/E, the second counts those v inert or ramified in K/E. Raising to the h-th power, one has

$$p^h = \prod_v (a_w\overline{a}_w)^{e(v|p)} \prod_v a_v^{e(v|p)}.u.$$

The element u is a unit of $\mathcal{O}_E$. Now choose a sufficiently divisible positive integer c, and write $\lambda_v = m_v/(m_v + n_v)$, with $c = m_v + n_v$, $m_v, n_v \in \mathbb{Z}$. We then define an algebraic integer π as

$$\pi = \prod_v (a_w^{m_v}\overline{a}_w^{n_v})^{e(v|p)} \prod_v a_v^{ce(v|p)/2}.u^{c/2}.$$

One checks easily that $\pi\overline{\pi} = p^{hc}$ and π is the desired p^{hc}-Weil number. □

In case that K is an extension of F, it is important to know when the Weil number we have just constructed generates K over F.

Proposition 7.2. *Let F be a field, and K/F be a separable field extension of degree n. Assume that the normal hull L of K/F has a Galois group isomorphic to the symmetric group S_n of n letters. Then K/F has no sub-extensions other than F and itself.*

Proof. This is equivalent to the assertion that the stabilizer subgroup S_{n-1} of the letter $1 \in \{1, \cdots, n\}$ is a maximal subgroup of S_n. It suffices to show that any subgroup H properly containing S_{n-1} acts transitively on the letters $\{1, \cdots, n\}$. If $n = 1, 2$, this is clear. Assume that $n \geq 3$. Let τ be an element of H, $\tau(1) = i$, $i \neq 1$. For any $j \in \{1, \cdots, n\}$, different from 1 and i, the permutation $\sigma := (ij)\tau$ in H sends 1 to j. □

Proposition 7.3. (Ekedahl) *Let K be a number field, and $\mathcal{O}_K$ its ring of integers. Let S be a dense open sub-scheme of* $\mathrm{Spec}(\mathcal{O}_K)$*. Let X, Y be two schemes of finite type over S, and let $g : Y \to X$ be a finite étale surjective S-morphism. Suppose* *that* $Y_K := Y \times_S \operatorname{Spec}(K)$ *is geometrically irreducible and* $X_K := X \times_S \operatorname{Spec}(K)$ *satisfies the property of weak approximation. Then the set of* K*-rational points* x *of* X *such that* $g^{-1}(x)$ *is connected satisfies also the property of weak approximation.*

Remark 7.4. Let X be a scheme of finite type over a number field K. Recall that a subset E of $X(K)$ is said to satisfy the property of *weak approximation*, if for any finite number of places $\{v_1, \cdots, v_r\}$ of K, E is dense in the product

$$X(K_{v_1}) \times \cdots \times X(K_{v_r})$$

under the diagonal embedding. The topology on $X(K_v)$ is induced from that of K_v. In particular, the K-scheme X is said to satisfy the property of weak approximation, if $X(K)$ does.

Proposition 7.5. *Let* n *be a positive integer, and* K *a totally real number field. Let* Σ *be a finite set of non-archimedean places of* K*. For each* $\ell \in \Sigma$ *let* K'_ℓ *be a finite étale algebra over* K_ℓ *of rank* n*. Then there is a totally real extension* K'/K *of degree* n*, such that its normal hull has a Galois group isomorphic to the symmetric group* S_n *of* n *letters, and* $K' \otimes_K K_\ell \simeq K'_\ell$*, for all* $\ell \in \Sigma$*.*

Proof. We consider the following situation. Let $S = \operatorname{Spec}(\mathcal{O}_K)$, $X' = S[a_1, \cdots, a_n]$, an S-affine space with coordinates $a_1, \cdots, a_n$. Let Y' be the hyper-surface in $X'[t]$ defined by the equation

$$f = t^n + a_1 t^{n-1} + \cdots + a_n.$$

Let R be the resultant of f and its derivative f'. We denote by X the complement of $\{R = 0\}$ in X' and by $Y := Y' \times_{X'} X$; Y is an étale cover of X of rank n. The scheme X_K, being a non-empty open sub-scheme of an affine space, clearly satisfies the property of weak approximation. The geometric fibre $Y_{\overline{K}} := Y_K \otimes_K \overline{K}$ is affine of ring $\Gamma(\mathcal{O}_{Y_{\overline{K}}}) = (\overline{K}[a_1, \cdots, a_n, t]/(f))_R$. We will prove in the next lemma that $\Gamma(\mathcal{O}_{Y_{\overline{K}}})$ is an integral domain. Now it is ready to apply Ekedahl's Hilbert irreducibility theorem (7.3) according to which, the subset M of the K-rational points x where Y_x is connected, i.e. Y_x is the spectrum of a field extension K' of K of degree n, satisfies the property of weak approximation. Requiring the K_l-algebras $K' \otimes_K K_l$ to be isomorphic to some given étale algebras at finitely many places l of K imposes a weak approximation question on the parameters $a_1, \cdots, a_n \in K$. The condition on the Galois group of the normal hull is a weak approximation property, cf. [5]. The proposition follows by modifying a little the content but not the proof of Ekedahl's theorem [4]. □

Lemma 7.6. *Let* K *be a factorial domain,* $A = K[a_1, \cdots, a_n]$ *a polynomial algebra over* K*. The "generic" polynomial* $f = t^n + a_1 t^{n-1} + \cdots + a_n$ *is irreducible in* $A[t]$*.*

Proof. Let $B = K[b_1, \cdots, b_n]$, where $b_i = a_i/a_n$, for $1 \leq i \leq n-1$, and $b_n = a_n$. As A is a subring of B, it suffices to prove that f is irreducible in $B[t]$. This is so because f is an Eisenstein polynomial in $B[t]$ with respect to the prime a_n. □

Now consider an (S)-restricted partitioned isocrystal $x = h.s_\Gamma \bigvee y$. For proving the "if" part, it suffices to show that each component of y is realizable by a Γ-isotypic hyper-symmetric abelian variety. From now on, we assume that $y = (M, P)$ is a simply partitioned isocrystal. By the supersingular restriction (S), there is a CM extension B/F such that P is equivalent to $P_{B/F}|T_\Gamma$. Let B_0 be the maximal totally real subfield of B. We also let N be the common number of isotypic components of y at all places v of F above p.

These reductions and hypothesis are in force for the rest. Let us now finish the proof of the main theorem (5.1). First, assume that F is a CM field. Let F_0 be the maximal totally real subfield of F.

Proposition 7.7. *Assume that F is a* CM *field. Suppose that $y = (M, P)$ is an* (S)*-restricted simply partitioned isocrystal. Then there exists a Γ-isotypic hyper-symmetric abelian variety Y such that M is Γ-isomorphic to $H^1(Y)$.*

Proof. For each place v of F above p, we define an $(F_0)_{v|F_0}$-algebra $T_{v|F_0}$ of rank N:

$$T_{v|F_0} = \begin{cases} (F_0)_{v|F_0}^N, & \text{if } v \neq \overline{v} \\ (F_0)_{v|F_0} \times F_v^{(N-1)/2}, & \text{if } v = \overline{v}, N \text{ odd} \\ F_v^{N/2}, & \text{if } v = \overline{v}, N \text{ even} \end{cases}$$

It follows from Proposition (7.5) that there is a totally real extension E/F_0 of relative degree N such that its normal hull has a Galois group isomorphic to S_N and that

(1) for each $v|p$, $E \otimes_{F_0} (F_0)_{v|F_0} \simeq T_{v|F_0}$,

(2) for every $\ell \in T_\Gamma$, $E \otimes_{F_0} (F_0)_{\ell|F_0} \simeq B_0 \otimes_{F_0} (F_0)_{\ell|F_0}$.

Consider the CM field $K := E \otimes_{F_0} F$. One has

(i) the normal hull of K/F has a Galois group isomorphic to S_N,

(ii) for each $\ell \in T_\Gamma$, $K \otimes_F F_\ell \simeq B \otimes_F F_\ell$,

(iii) for each place v of F above p, $K \otimes_F F_v \simeq F_v^N$ is totally split.

The property (iii) allows us to index the slopes of y at v as $\{\lambda_w; w|v\}$, where w runs over the places of K above v. One can even arrange that $\lambda_w + \lambda_{\overline{w}} = 1$, since the underlying isocrystal M of y is polarized, cf. (2.7). We apply (7.1) to get an integer $a \geq 1$ and a p^a-Weil number $\pi \in K$, so that

$$\mathrm{ord}_w(\pi)/\mathrm{ord}_w(p^a) = \lambda_w, \quad \text{for all } w|p$$

Note that the field $F(\pi)$ must be equal to K. Indeed, if $N = 1$, this is clear because $F = F(\pi) = K$. If $N > 1$, π is not an element of F, because, otherwise, we would have $\mathrm{ord}_{w_1}(\pi) = \mathrm{ord}_{w_2}(\pi)$, for any two places w_1, w_2 above v. This is absurd in view of the choice of π. By (7.2) and (i), we have $F(\pi) = K$.

According to the theorem of Honda-Tate (2.12), up to isogeny there is a unique Γ-simple abelian variety $Y'_{\mathbb{F}_q}$ defined over $\mathbb{F}_q$, $q = p^a$, corresponding to the p^a-Weil number π. We assume that a is chosen to be sufficiently divisible so that $Y'_{\mathbb{F}_q}$ is absolutely Γ-simple. Let $Y' := Y'_{\mathbb{F}_q} \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_p$. Kottwitz [8] proved that there exists a Γ-linear $\mathbb{Q}$-polarization on Y'. Since the center $F(\pi)$ of $\mathrm{End}^0_\Gamma(Y')$ is totally split at every place $v|p$ of F, the abelian variety Y' is therefore Γ-hyper-symmetric, cf. (3.4).

The pair $y' = (H^1(Y'), P_{K/F}|T_\Gamma)$ is a simply partitioned isocrystal satisfying the supersingular restriction (S) by (6.1). By construction, y' and y have the same slopes at every place v of F above p.

Now we prove that the multiplicity $n(y')$ divides $n(y)$. In fact, $n(y')$ is the order of $[\mathrm{End}^0_\Gamma(Y)]$ in the Brauer group of K, cf. (6.1). Since y satisfies the condition (SPI2) (4.11), one has $n(y).\mathrm{inv}_\ell(\Gamma)[K_{\ell'} : F_\ell] = 0$ in $\mathbb{Q}/\mathbb{Z}$. Look at the local Brauer invariants of $C := \mathrm{End}^0_\Gamma(Y)$

$$\mathrm{inv}_\nu(C) = \begin{cases} -[F_v : \mathbb{Q}_p]\lambda_\nu - \mathrm{inv}_v(\Gamma), & \text{if } \nu \mid v \\ -[K_\nu : F_\ell].\mathrm{inv}_\ell(\Gamma), & \text{if } \nu \nmid p. \end{cases}$$

By Kottwitz 11.5 [8], $n(y).\mathrm{inv}_w(C) = 0$ in $\mathbb{Q}/\mathbb{Z}$, for all w above p. These two equations together show that $n(y')$ divides $n(y)$. Let e be the integer such that $n(y) = e.n(y')$.

It remains to prove that the underlying isocrystals of y and $e.y'$ are isomorphic as *polarized* Γ-linear isocrystals. Indeed, we can modify the polarization on $Y := Y'^e$ so that $e.y'$ with this modified polarization is isomorphic to y. For a proof, let S be the $\mathbb{Q}$-vector space of the symmetric elements in $\mathrm{Hom}^0_\Gamma(Y, Y^*)$, where Y^* denotes the dual abelian variety of Y. As Y is Γ-linear hyper-symmetric, $S \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is isomorphic to the symmetric elements of $\mathrm{Hom}_\Gamma(H^1(Y^*), H^1(Y))$. The space S being dense in $S \otimes_{\mathbb{Q}} \mathbb{Q}_p$, our claim is clearly justified and the proof in the case that F is a CM field is now complete. □

Proposition 7.8. *Assume that F is a totally real number field. And suppose that $y = (M, P)$ is an* (S)*-restricted simply partitioned isocrystal. Then there exists a Γ-isotypic hyper-symmetric abelian variety Y such that $M \simeq H^1(Y)$.*

Proof. As y is Γ-linearly polarized and contains no slope $1/2$ part by (S2), N is an even integer. By Proposition (7.5), there is a totally real extension E/F of degree $N/2$ such that

(1) for each place $v|p$ of F, $E \otimes_F F_v \simeq F_v^{N/2}$,
(2) for each $\ell \in T_\Gamma$, there is an F_ℓ-isomorphism $f_\ell : E \otimes_F F_\ell \simeq B_0 \otimes_F F_\ell$,
(3) the normal hull of E/F has a Galois group isomorphic to $S_{N/2}$.

By the lemma 5.7 [1], there exists a totally imaginary quadratic extension K/E such that

(i) for each place ν of E above p, $K \otimes_E E_\nu \simeq E_\nu \times E_\nu$,
(ii) for each $\ell \in T_\Gamma$, there is an isomorphism $g_\ell : K \otimes_F F_\ell \simeq B \otimes_F F_\ell$ compatible with f_ℓ,
(iii) the field K contains no proper CM sub-extension of F.

The properties (1) and (i) show that K/F is totally split everywhere above v. Thus we can index the slopes of y at v as $\{\lambda_w; w|v\}$ with w running over the places of K above v. Moreover, as y is Γ-linearly polarized, one can even arrange that $\lambda_w + \lambda_{\overline{w}} = 1$, cf. (2.7). Similarly as in the preceding proposition, there is a p^a-Weil number π, for a suitable integer $a \geq 1$, such that $F(\pi) = K$, and

$$\mathrm{ord}_w(\pi)/\mathrm{ord}_w(p^a) = \lambda_w,$$

for all places w of K above p.

We assume that a is sufficiently divisible. The unique Γ-simple abelian variety $Y'_{\mathbb{F}_{p^a}}$ up to isogeny corresponding to π admits a Γ-linear $\mathbb{Q}$-polarization by Kottwitz [8]. Let $Y' := Y'_{\mathbb{F}_{p^a}} \otimes_{\mathbb{F}_{p^a}} \overline{\mathbb{F}}_p$, which is by construction Γ-hyper-symmetric. We then modify, if necessary, the polarization on Y' so that a copy $Y := Y'^e$ realizes y. The argument is the same as that in (7.7). We have proved the Proposition (7.8). □

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