P-ADIC MONODROMY OF THE ORDINARY LOCUS OF PICARD MODULI SCHEME

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A Dissertation

in

Mathematics

Presented to the Faculties of the University of Pennsylvania in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

2005

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Acknowledgments

I could not express my gratitude to my advisor Ching-Li Chai enough. Many years of his guidence, help and encouragement made this work possible.

Also, I would like to thank Steve Shatz, Ted Chinburg, David Harbater and Florian Pop for their help and encouragement as well as mathematical teaching.

My life as graduate student at Penn would not have been much fun were it not be for the fellowship in and outside math of my fellow graduate students, especially, Sukhendu Mehrota, Jimmy Dillies, Laurentiu Maxim, Cherng-tiao Perng, and Shuichiro Takeda.

Finally, I would like to thank my parents for their support, and Ahram Seo for new delight in my life.

ABSTRACT

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Let E be an imaginary quadratic number field, p be a rational prime splitting in $\mathcal{O}_E$ and m, n be distinct natural numbers. The naive p-adic monodromy of the ordinary locus of the good reduction of a Shimura variety of $U(m,n)$ type over $\bar{\mathbb{F}}_p$ is a subgroup of $\mathrm{GL}_m(\mathbb{Z}_p) \times \mathrm{GL}_n(\mathbb{Z}_p)$. In this paper, we prove that for any point in the basic locus of the moduli space, the local monodromy is an open subgroup of $\mathrm{GL}_m(\mathbb{Z}_p) \times \mathrm{GL}_n(\mathbb{Z}_p)$. From this local information, the global p-adic monodromy is shown to be as big as possible, i.e. $\mathrm{GL}_m(\mathbb{Z}_p) \times \mathrm{GL}_n(\mathbb{Z}_p)$.

Contents

Chapter 1

Introduction

Let X be a connected PEL-type modular variety over an algebraically closed field of characteristic p, and $\mathcal{A}$ be its universal abelian scheme. Assume that the open subscheme X^{or} of X whose points correspond to ordinary abelian varieties is nonempty. Let $\mathcal{A}^{\mathrm{or}}$ be the pullback of $\mathcal{A}$ over X^{or}, the maximal étale quotient of its Barsotti-Tate group $\mathcal{A}^{\mathrm{or}}[p^\infty]$ defines a lisse sheaf of free $\mathbb{Z}_p$ modules of rank g (g being the dimension of the abelian variety) and thus provides a representation of $\pi_1(X^{\mathrm{or}})$.

$$\rho \ : \ \pi_1(X^{\mathrm{or}}) \to \mathrm{GL}_g(\mathbb{Z}_p)$$

We are interested in the image of this representation, which we will call the naive (global) p-adic monodromy.

In the Siegel case, this image is well-known to be as big as possible, i.e. $\mathrm{GL}_g(\mathbb{Z}_p)$. Faltings-Chai proved this as an application of the minimal compactification [12]. In fact, they proved that the local monodromy at 0-dimensional cusp is already quite big, i.e. $\mathrm{SL}_g(\mathbb{Z}_p)$. Ekedahl also proved the same result, analyzing deformations [11]. In this paper, we adopt this latter strategy.

In general, for Shimura varieties of PEL type with nonempty ordinary locus, the naive global p-adic monodromy is expected to have a simple group theoretic description in terms of the group G of the Shimura datum. This conjectured description implies, among other things, that the naive p-adic monodromy is reductive ([6], conjecture (7.4)(ii)). The results of this paper confirm this expectation for a certain class of (PEL-type) Shimura varieties, including the Picard modular variety of $U(m,n)$ type provided that the prime p splits in $\mathcal{O}_E$.

On the other hand, often a substantial part of global monodormy already appears in local monodromy. For example, Igusa considered a supersingular point s of a certain modular curve X over an algebraically closed field k of characteristic $p > 0$ and showed that the local monodromy at that point is already as big as possible, i.e. $\mathbb{Z}_p^\times$. In the higher dimensional Siegel case, other than the cusp used by Faltings and Chai mentioned above, a superspecial point still works equally well ([7], [11]). Recall that superspecial abelian variety is an abelian variety which after an extension to an algebraically closed field, becomes isomorphic to the product E^g of copies of a supersingular elliptic curve E.

In this paper, we study local monodromies of certain PEL-type Shimura varieties which do not contain supersingular points. Therefore, we need to consider other points which would do the same job as supersingular points did for Siegel moduli scheme as far as the local monodromy is concerned.

On the other hand, there have been some evidences that the local monodromy of formal Lie group at a point with different generic and closed slopes is big, for exmaple [5]. Note that in the Siegel case, the biggest slope change occurs from a supersingular abelian variety to an ordinary abelian variety; in the Newton polygon stratification of Siegel modular variety, the supersingular strautm is minimal and ordinary stratum is maxiaml. Every other slope stratum is between these. In this paper, we also consider the minimal stratum (the basic locus) in the Newton polygon stratification of some classes of Shimura varieties and show that the local monodromies at these points are quite big as far as some obvious global restriction can allow. Along this line, the result of this paper also can be regarded as another evidence for this phenomenon.

Here is a brief sketch of the structure of the paper. In the first section of the chapter 2, we first give a set-up for the Shimura varieties that we consider. In the next two sections we describe a particular point in the basic locus of Picard moduli scheme that will serve for the computation of the local monodromy. In the last section, we state the main theorem about the local monodromy and its corollary about the global monodromy. In chapter 3, we prove the main Theorem about the local monodromies at the points defined in Chapter 2 using deformation theoretic arguments. This consists of two parts. Firstly, we show that the local monomdromy contains the derived group of the target and secondly we prove the determinant is an open subgroup of $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$. In the last section, we derive the corollary using another point in the moduli space.

Chapter 2

Shimura variety of $U(m,n)$ type and Picard moduli scheme

Let G be a quasi-split group over $\mathbb{Q}$ of type $U(m,n)$, split by an imaginary quadratic field E. The Shimura variety $\widetilde{X}$ defined by the Shimura datum associated to (G,h) parametrizes $(m+n)$-dimensional polarized abelian varieties A having an endomorphism by $\mathcal{O}_E$ with level structures such that the action of $\mathcal{O}_E$ on Lie(A) has type (m,n). For a scheme S over the localization $\mathcal{O}_E \otimes \mathbb{Z}_{(p)}$ of $\mathcal{O}_E$ and an abelian scheme over S equipped with an endomorphism by $\mathcal{O}_E$, the Lie algebra $\mathrm{Lie}(A/S)$ has two $\mathcal{O}_E \otimes \mathbb{Z}_{(p)}$-module structures, one via the base scheme $\mathcal{O}_E \otimes \mathbb{Z}_{(p)}$ and the other via the action by $\mathcal{O}_E$. But, since p splits in $\mathcal{O}_E$, $\mathrm{Lie}(A/S)$ becomes a direct sum of two factors M_1, M_2, where the two actions of $\mathcal{O}_E$ coincide on M_1 while they differ by conjugation on M_2. After fixing a prime of $\mathcal{O}_E$ over p which also distinguishes the two $\mathcal{O}_E \otimes \mathbb{Z}_{(p)}$ factors, we require M_1 (resp, M_2) to have rank m(resp. n).

Also, we assume there is given a conjugacy class $h : \mathbb{C} \to \mathrm{M}_{m+n}(E)_{\mathbb{R}}$ satisfying the axioms (2.1.1.1-3) of [9].

When p is unramified in $\mathcal{O}_E$, the Shimura variety defined by (G, h) and of sufficiently small level has good reduction at p. In general, when G is quasisplit over $\mathbb{Q}_p$ and split over the unramified extension $E_\wp$ ($\wp$ is a prime over p), G has a hyperspecial subgroup $K_p \subset G(\mathbb{Q}_p)$. Then for a sufficiently small subgroup K^p of $G(\mathbb{A}_f^p)$, the PEL-type Shimura variety $Sh_{K^pK_p}(G, h)$ is known to have a good reduction at p. Moreover, there exists a smooth quasiprojective model over $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ which is also a fine moduli scheme for a suitable PEL-type moduli functor [24], [18].

Also, when p splits, it is well known that the ordinary locus is nonempty (For this, see e.g. [33]).

In this paper, we assume that p splits in $\mathcal{O}_E$ and the conditions of good reduction be satisfied.

Let X^{or} be the ordinary locus of the reduction X mod p and $x \in X(k)(\bar{k} = k)$ be a geometric point.

Let $S = \mathrm{Spf}(R)$ be the equicharacteristic deformation space of $(A_x, \lambda_x, \iota_x)$; S is the formal completion $X^{/x}$ of X at x. Let $A \to \mathrm{Spec}(R)$ be the universal abelian scheme over $\mathrm{Spec}(R)$ which is also the universal deformation of $(A_x, \lambda_x, \iota_x)$ by the Grothendieck algebraization theorem. Then we consider the associated Barsotti-

Tate group $A[p^\infty]$ over $\mathrm{Spec}(R)$. This can also be constructed as follows. Let G be the universal formal deformation over the formal scheme $\mathrm{Spf}(R)$ of the smooth formal group $\widehat{A_x}$. If R is $R = \lim_{\leftarrow} R_i$ as an adic ring, we have a compatible system of smooth formal groups G_i over $\mathrm{Spec}(R_i)$, each of which is the universal formal deformation of $A_x[p^\infty]$ over $\mathrm{Spec}(R_i)$. Then for fixed n, $\lim_{i\to\infty} G_i[p^n]$ becomes a truncated Barsotti-Tate group over $\mathrm{Spec}(R)$ of level n and the inductive system of finite locally free group schemes thus obtained defines a p-divisible group $G[p^\infty]$ over $\mathrm{Spec}(R)$. This defines an equivalence of the category of smooth formal groups over $\mathrm{Spf}(R)$ and the category of connected BT groups over $\mathrm{Spec}(R)$. In the rest of paper, this equivalence will be used without explicit mention.

If we let $A[p^\infty]^{et}_\eta$ be the maximal etale quotient of the generic fiber of $A[p^\infty]$, then we get the assoiciated Galois representation

$$\rho_G : \mathrm{Gal}(K^{sep}/K) \to \mathrm{GL}(T_p(G)) = \mathrm{GL}(A[p^\infty]^{\acute{e}t}_\eta).$$

The local monodromy is the image of this representation.

Note that the splitting $p\mathcal{O}_E \simeq \wp_1 \times \wp_2$ of p induces the splitting of p-divisible group $A_x[p^\infty] \simeq A_x[\wp_1^\infty] \times A_x[\wp_2^\infty]$ and similar splittings for any lifting of $(A_x[p^\infty], \lambda_x, \iota_x)$. Also, the quasi-polarization $\lambda : A[p^\infty] \mapsto A[p^\infty]^t$ induced by a given polarization of A maps $A[\wp_1^\infty]$ to $A[\wp_2^\infty]^t$.

2.1 Picard moduli scheme

We review the construction of the Picard moduli scheme and the basic locus of its closed fiber over a field of finite characteristic. For more details, one can see [20] or [2]

Let E be an imaginary quadratic number field with discriminant D, H the Hilbert class field of E and $\mathcal{O} = \mathcal{O}_E$, $\mathcal{O}_H$ respectively their rings of integers. For positive integers $m < n$, let $V_0 = \mathcal{O}_E^{m+n}$ be the free $\mathcal{O}_E$-module of rank $m+n$ and q_0 be the skew-Hermitian form on the $\mathcal{O}_E$-module V_0 defined by the diagonal matrix whose first m diagonal entries are 1's and the next n diagonal entries are -1's.

Let $\mathrm{GU}(m,n)$ be the algebraic group over $\mathbb{Q}$ whose R-rational points for $\mathbb{Q}$-algebra R are

$$\begin{aligned}\mathrm{GU}(m,n)(R) &= \{g \in \mathrm{GL}_{\mathcal{O}\otimes R}(V_0 \otimes R) : \text{ there exists a } \mu(g) \in R^\times \text{such that} \\ &\quad q_0(gu, gv) = \mu(g) q_0(u,v)\}.\end{aligned}$$

Definition 2.1.1. [2] [20] Let S be a scheme over $\mathcal{O}[1/D]$. One calls $\mathcal{M}$-structure of type (m,n) over S a triple (A, λ, ι) where A is an abelian scheme over S, of relative dimension $m+n$, λ a principal polarization of A, and ι a ring homomorphism from $\mathcal{O}$ to $\mathrm{End}_S(A)$, such that

(1)The group scheme A over S is of type (m,n).

(2)The Rosatti involution associated with ϕ acts as the complex conjugation on $\iota(\mathcal{O})$.

(3)If D is even, there exists an isomorphism of $\mathcal{O}_2 = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_2$-modules equipped with hermitian form $(T_2(A), q) \simeq (V_0, q_0) \otimes_{\mathbb{Z}} \mathbb{Z}_2$.

Definition 2.1.2. [2] [20] A level N-structure over a $\mathcal{M}$-structure of type (m, n) (A, λ, ι) is defined to be a couple (σ, τ), where

(1)$\sigma : V_0 \otimes \mathcal{O}/N \to A[N]$ is an isomorphism of $\mathcal{O}/N$-module schemes over S.

(2)$\tau : \mathbb{Z}/N\mathbb{Z} \to \mu_N$ is an isomorphism of group schemes over S.

(3)$\tau(q(x, y)) = q_0(\sigma(x), \sigma(y))$.

Let $\mathcal{M}_N$ be the category of $\mathcal{M}$-structures of type (m,n) and level N over $\mathcal{O}[1/DN]$-schemes. It is a fibered category in groupoids over $\mathcal{O}[1/DN]$.

The following result is proved in [20] when $(m, n) = (1, 2)$, but it is easily seen to be true in general.

Theorem 2.1.3. *The stack $\mathcal{M}_N$ is an algebraic stack. It is connected, smooth and of relative dimension mn over Spec$(\mathcal{O}[1/DN])$. For $N \geq 3$, $\mathcal{M}_N$ is an algebraic space.*

If N divides N', over Spec$(\mathcal{O}[1/DN'])$ one has a finite and etale forgetting morphism,

$$\mathcal{M}_{N'} \to \mathcal{M}_N$$

which identifies $\mathcal{M}_N$ with the quotient of $\mathcal{M}_{N'}$ by the group $\Gamma(N', N)$, where

$$\Gamma(N', N) = \mathrm{Ker}[\mathrm{GU}(m, n)(\mathbb{Z}/N'\mathbb{Z}) \to \mathrm{GU}(m, n)(\mathbb{Z}/N\mathbb{Z})].$$

It is known that for a rational prime $l \neq 2$, an $\mathcal{M}$-structure (A, ϕ, ι) over a scheme S above $\mathcal{O}[1/Dl]$, $(T_l(A), q) \simeq (V_0, q_0) \otimes_{\mathbb{Z}} \mathbb{Z}_l$.

2.2 Basic locus

The Newton stratification of the Siegel moduli space $\mathcal{A}_g$ which is defined in terms of the formal isogeny type of the Barsotti-Tate group was generalized to the reductions of general Shimura varieties by Kottwitz and many similar properties as in the Siegel case were shown to hold by Rapoport, Richartz and Chai [19] [30] [4].

Here we give a brief review of the theory. For detailed discussion, we refer to [19] [30] [4]. With the connection to Shimura varieties in mind, the base field is assumed to be $\mathbb{Q}_p$. For more general situation, consult loc. cit.

Let K be the fraction field of the ring of p-adic Witt vectors $W(\bar{\mathbb{F}}_p)$ with the Frobenius automorphism σ of $K/\mathbb{Q}_p$, $\bar{K}$ an algebraic closure of K and let $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

Let G be a connected reductive group over $\mathbb{Q}_p$. Let $B(G)$ be the σ-conjugacy classes of elements of $G(K) : x \sim y \Longleftrightarrow x = g \cdot y \cdot \sigma(g)^{-1}$ for some $g \in G(K)$ and let us define the Newton cone $\mathcal{N}(G)$ by $\mathcal{N}(G) = (\mathrm{Int}\ G(K) \setminus \mathrm{Hom}_K(\mathbb{D}, G))^{<\sigma>} \cong (W \setminus X_*(T)_{\mathbb{Q}})^{\Gamma}$, where $\mathbb{D}$ is the pro-algebraic group with character group $\mathbb{Q}$ and W is the Weyl group of G with respect to a fixed maximal torus T of G.

Then Kottwitz defined the Newton map $\bar{\nu}_G : B(G) \to \mathcal{N}(G)$ satisfying certain properties ([19], Sect.4). We will be just contented with describing this map in the case $G = \mathrm{GL}_h$ ([30], Ex. 1.10) because in our case we have $\mathrm{GU}(m,n) \times_{\mathbb{Q}} \mathbb{Q}_p = \mathrm{GL}_{m+n} \times \mathbb{G}_{\mathrm{m}}$. When $G = \mathrm{GL}_h$, $B(G)$ is the set of all isomorphism classes of σ-K-spaces of height h (i.e. h-dimensional K-vector spaces together with σ-linear bijection) and the Newton map sends a σ-K-space of height h to its usual Newton polygon determined by the decomposition into isotypical components according to the Dieudonne-Manin classification. Also, in this case $G = \mathrm{GL}_h$ the natural partial ordering on the Weyl chamber C (i.e. for $x,\ y \in C,\ x \succeq y \iff x - y \in C^\vee$) becomes the usual ordering on Newton polygons with same end points.

Let $B(G)_{\mathrm{basic}}$ be the set of σ-conjugacy classes of the elements $b \in G(K)$ whose associated homomorphisms $\nu_b \in \mathrm{Hom}_K(\mathbb{D}, G)$ factor through the center of G. There exists a functorial isomorphism $\gamma : B(\cdot)_{\mathrm{basic}} \rightarrow \pi_1(\cdot)_\Gamma$, where for a connected reductive group G over $\mathbb{Q}_p$, $\pi_1(G)$ is the common value of the Galois modules $\pi_1(G,T) = X_*(T)/\sum_{\alpha\in\Phi(G,T)} \mathbb{Z}\alpha^\vee$ for various pairs (T,B) consisting of a maximal torus T and a Borel subgroup B defined over $\overline{\mathbb{Q}}_p$.

Now, we explain the connection of this theory with good reduction of Shimura varieties. Let $(\mathbf{G},\mathbf{X})$ be a PEL Shimura data, in particular $\mathbf{G}$ is a connected reductive group and $\mathbf{X}$ is a $\mathbf{G}(\mathbb{R})$-conjugacy class of a $\mathbb{R}$-group homomorphism $h : \mathbb{C}^\times \rightarrow \mathbf{G}_{\mathbb{R}}$ satisfying the axioms (2.1.1.1-3) of [9]. Assume that $\mathbf{G}$ is quasisplit over $\mathbb{Q}_p$ and splits over an unramified extension of $\mathbb{Q}_p$. This is the most well known candidate of groups for which the associated Shimura varieties are conjectured to have good reduction over places above p. Let $\mathbf{K}_p$ be a hyperspecial maximal compact subgroup of $\mathbf{G}(\mathbb{Q}_p)$ and let $\mathrm{Sh}_{\mathbf{K}_p}(\mathbf{G}, \mathbf{X})$ the tower of Shimura varieties attached to these data. Assume that $\mathrm{Sh}_{\mathbf{K}_p}(\mathbf{G}, \mathbf{X})$ has good reduction at a place v of the Shimura reflex field above p.

Then we apply the above discussion to the connected reductive group $G = \mathbf{G} \times_{\mathbb{Q}} \mathbb{Q}_p$ over $\mathbb{Q}_p$. Let μ be a minuscule dominant coweight of a maximal torus T over $\mathbb{Q}_p$ with respect to a $\mathbb{Q}_p$-rational Borel subgroup B of G with $B \supseteq T$, such that the G-conjugacy class of μ corresponds to $\mathbf{X}$.

Definition 2.2.1. Let $b_0 \in B(G)_{\text{basic}} \cong \pi(G)_\Gamma$ be the basic element in $B(G)$ corresponding to the image of μ in $\pi(G)_\Gamma$. Then $\mathcal{S}_{\text{basic}}$ is defined to be the locus in the reduction of $\mathrm{Sh}_{\mathbf{K}_p}(\mathbf{G}, \mathbf{X})$ at v consisting of points whose associated σ-K-space has type b_0.

Remark 2.2.2. (i) It was shown by Rapoport and Richartz that $\mathcal{S}_{\text{basic}}$ is Zariski-closed ([30], Thm. 3.6). In general, if for $b \in B(G)$, we define $\mathcal{S}_b$ to be the subset of the reduction of $\mathrm{Sh}_{\mathbf{K}_p}(\mathbf{G}, \mathbf{X})$ at v consisting of points whose associated σ-K-space has type b, they showed that $\mathcal{S}_b$ is locally closed ([30], Thm. 3.6) and thus we obtain a generalized Newton stratification. For more about this stratification, we refer to the Bourbaki article by Rapoport [29] and the references therein.

(ii) Chai gave a conjectural group theoretic description of the set of all Newton points that are expected to appear in the good reduction of Shimura varieties ([4], Remark 4.5) and showed that this set is a catenary poset, in particular it has a unique maximal (minimal) element. $\mathcal{S}_{\text{basic}}$ is then the minimal stratum which is expected to appear in the good reduction of Shimura variety. There is also a purely group theoretic description of the maximal stratum which is expected to appear in the same situation. In the Siegel case the unique maximal (resp. minimal) stratum corresponds to ordinary (resp. supersingular) abelian varieties.

(iii) Chai also gave a Lie-theoretic formula for the codimensions of generalized Newton strata (loc. cit., Question 7.6). These were verified in the Siegel case [22], [10].

For the following proposition, we use the setup and notations of Chapter 2. In particular, $\mathcal{M}_N$ is the moduli scheme over $\mathcal{O}[1/DN]$ of $\mathcal{M}$-structures of type (m,n) and level N (for big enough N). For p such that $p \nmid DN$, let $\overline{\mathcal{M}}_N$ be the reduction of $\mathcal{M}_N$ at a place v of $\mathcal{O}$ over p.

Proposition 2.2.3. *When p splits in $\mathcal{O}_E$, the reduction $\overline{\mathcal{M}}_N$ of the Picard moduli scheme of type $U(m,n)(m < n)$ has nonempty basic locus.*

Note that in our reduction case of Picard moduli scheme the reductive group $G = \mathrm{GU}(m,n) \times_{\mathbb{Q}} \mathbb{Q}_p = \mathrm{GL}_{m+n} \times \mathbb{G}_{\mathrm{m}}$ over $\mathbb{Q}_p$ already splits over $\mathbb{Q}_p$ because of the assumption p being split in $\mathcal{O}_E$ and we can take μ to be the unique minuscule dominant cocharacter coming from the Shimura data because the reflex field is E and $E_{\wp} \cong \mathbb{Q}_p$.

Let S be the maximal torus of $\mathrm{GL}_{2(m+n),\mathbb{Q}_p}$ consisting of diagonal matrices and let B be the Borel subgroup of $\mathrm{GL}_{2(m+n),\mathbb{Q}_p}$ consisting of upper triangular matrices. The splitting $E \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \mathbb{Q}_p \oplus \mathbb{Q}_p$ gives rise to an inclusion $\mathrm{GU}(m,n) \times_{\mathbb{Q}} \mathbb{Q}_p \hookrightarrow$

$\mathrm{GL}_{2(m+n),\mathbb{Q}_p}$ such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GU}(m,n) \times_{\mathbb{Q}} \mathbb{Q}_p \mapsto A \in \mathrm{GL}_{m+n,\mathbb{Q}_p}, \quad A, B, C, D \in \mathrm{M}_{(m+n)}(\mathbb{Q}_p)$$

induces isomorphism from $\mathrm{SU}(m,n) \times_{\mathbb{Q}} \mathbb{Q}_p$ to $\mathrm{GL}_{m+n,\mathbb{Q}_p}$. Also under this inclusion, the subtorus

$$T = \{\mathrm{diag}(d_1, \cdots, d_{2(m+n)}) \mid \quad d_i \ d_{2(m+n)+1-i} = \mathrm{const} \quad \text{for all i} = 1, \cdots, m+n \ \}$$

of S becomes a maximal torus and $B \cap \mathrm{GU}(m,n) \times_{\mathbb{Q}} \mathbb{Q}_p$ is a Borel subgroup of $\mathrm{GU}(m,n) \times_{\mathbb{Q}} \mathbb{Q}_p$ containing T. Therefore we have

$$X_*(T) = \{ \sum_{i=1}^{2(m+n)} x_i e_i \mid \quad x_i + x_{2(m+n)+1-i} = \mathrm{const} \ \},$$

where $\{e_i : i = 1, \cdots, 2(m+n)\}$ is the standard basis of $X_*(S)$. The minuscule dominant coweight μ of $G = \mathrm{GU}(m,n) \times_{\mathbb{Q}} \mathbb{Q}_p$ corresponding to the type $U(m,n)$ is $\mu = e_1 + \cdots + e_m + e_{m+n+1} + \cdots + e_{m+2n}$ and its corresponding basic element $b_0 \in B(G)_{\mathrm{basic}}$ has the usual Newton slopes $(\frac{m}{m+n}, \frac{n}{m+n})$ with same height ([30], Prop.1.12). Therefore it suffices to prove the following claim.

There exists a principally polarized AV over $\bar{\mathbb{F}}_p$ with an $\mathcal{O}_E$ action of type $U(m,n)$ which has the slopes $(\frac{m}{m+n}, \frac{n}{m+n})$ with the same height $m+n$.

Let $p\mathcal{O}_E = \wp \ \bar{\wp}$ be the splitting of $p\mathcal{O}_E$ into two distinct prime ideals in $\mathcal{O}_E$. Then for the class number h of E, there exist elements α, β of $\mathcal{O}_E$ such that $\wp^h = (\alpha), \bar{\wp}^h = (\beta)$ and by multiplying a unit to β, we may assume that $p^h = \alpha \cdot \beta$, from which it easily follows that $\beta = \bar{\alpha}$, since $(\beta) = (\bar{\alpha})$. Let $\pi = \alpha^m \beta^n$ and $q = p^{h(m+n)}$.

Then π is a Weil q-number, i.e. $\pi\bar{\pi} = q$. Moreover, $\mathbb{Q}(\pi) = E$, because otherwise $\beta^{n-m} = p^{-hm}\pi \in \mathbb{Z} = \mathbb{Q} \cap \mathcal{O}_E$ and so should be a power of p, contradicting the assumption that p is unramified. Hence by Honda-Tate theory, there exists a simple abelian variety A over $\mathbb{F}_q$ such that $\mathrm{End}_{\mathbb{F}_q}(A)$ is a central division algebra over $\mathbb{Q}(\pi) = E$. Moreover, by construction, we have $v(\pi) = m = v(q^{\frac{m}{m+n}})$ for the valuation v of E such that $v(\wp) = 1, v(\bar{\wp}) = 0$.

By changing in the isogeny class we may assume that $\mathrm{End}_{\mathbb{F}_q}(A)$ contains $\mathcal{O}_E$. Also, there exists a $\mathcal{O}_E$-linear polarization $\lambda : A \to A^t$, i.e. via $\iota : \mathcal{O}_E \to \mathrm{End}_{\mathbb{F}_q}(A)$, the Rosati involution associated with the polarization λ induces the complex conjugation on $\mathcal{O}_E$ ([18], Lemma 9.2). So by another isogeny, we obtain a principal polarization $\lambda : A \times \bar{\mathbb{F}}_p \to A^t \times \bar{\mathbb{F}}_p$ with prescribed properties ([25], 23). ■

Remark 2.2.4. The aforementioned conjecture ([4] Question 7.6) of Chai about the codimensions of generalized Newton stratra was also verified in our Picard case (p being split) by Oort ([29], Thm 5.3).

2.3 Statement of the main theorem

We continue to use the setup and notations of Chapter 2 for the Picard moduli scheme. Moreover, we assume that p splits over $\mathcal{O}_E$.

Theorem 2.3.1. *Let $x = (A_x, \lambda_x, \iota_x)$ be a point in the basic locus of $\overline{\mathcal{M}}_N$ and let* *us assume that $p \geq 5$. Then the image of the local monodromy*

$$\rho_G : \mathrm{Gal}(K^{sep}/K) \to \mathrm{GL}_{m+n}(\mathbb{Z}_p)$$

contains $\mathrm{SL}_m(\mathbb{Z}_p) \times \mathrm{SL}_n(\mathbb{Z}_p)$ and its determinant is an open subgroup of $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$. In particular, the local monodromy is an open subgroup of $\mathrm{GL}_m(\mathbb{Z}_p) \times \mathrm{GL}_n(\mathbb{Z}_p)$.

Note that the Lie condition forces the universal deformation of $x = (A_x, \lambda_x, \iota_x)$ to decompose into the product of two formal groups of respective dimension m, n lifting the product $A[\wp_1^\infty] \times A[\wp_1^\infty]$, hence the image of the local monodromy naturally lands in $\mathrm{GL}_m(\mathbb{Z}_p) \times \mathrm{GL}_n(\mathbb{Z}_p)$.

Corollary 2.3.2. *The naive global p-adic monodromy group of any connected component of $\overline{\mathcal{M}}_N$ is $\mathrm{GL}_m(\mathbb{Z}_p) \times \mathrm{GL}_n(\mathbb{Z}_p)$. In particular, the Zariski closure of the global p-adic monodromy group is a connected reductive algebraic group.*

Chapter 3

Proof of the main theorem and the corollary

We first note that it suffices to prove the main theorem 2.3.1 for any one point in the basic locus. Indeed, since any two Barsotti-Tate groups with given extra structure in the basic locus are isogenous, the local monodromy at a point in the basic locus is an open subgroup of the target if and only if it is thus at any other point.

In the following, we describe the points in the basic locus that will serve for the direct computation of local monodrmoy, in terms of its p-divisible group (equivalently its (covariant) Dieudonne module) (Lemma 3.0.3).

First, we collect general facts about the Dieudonne module of a general polarized abelian variety (A, λ, ι) equipped with endomorphism by $\mathcal{O}_E$ whose p-divisible group $A[p^\infty]$ is connected, assuming only that p is split in $\mathcal{O}_E$.

Let $\sigma_i : \mathcal{O}_E \hookrightarrow \mathbb{Z}_p \subset W(k)$ $(i = 1, 2)$ be the embeddings of $\mathcal{O}_E$. If M is the covariant Dieudonne module of $A[p^\infty] = \hat{A}$, then we have the decomposition $M = M_1 \oplus M_2$ as the direct sum of $\mathcal{O}_E$-eigenspaces M_1, M_2, where $\mathcal{O}_E$ acts on M_i via σ_i i.e, $\iota(\alpha)(m_i) = \sigma_i(\alpha)m_i,\ m_i \in M_i\ i = 1, 2$: M_i is the Dieudonne module of $A[\wp_i^\infty]$. Also, M_i's are isotropic spaces for the pairing $< \cdot, \cdot >$ defined by the given polarization λ, i.e. $< M_i, M_i >= 0$ for $i = 1, 2$. Indeed, if $\alpha \in \mathcal{O}_E,\ x,\ y \in M_1$, $< \sigma_1(\alpha)x, y >=< \iota(\alpha)x, y >=< x, \iota(\overline{\alpha})y >=< x, \sigma_2(\alpha)y >$, implying $< x, y >= 0$. So the quasi-polarization λ gives the quasi-isogenies $\lambda : M_1 \simeq M_2^t$ & $M_2 \simeq M_1^t$.

Lemma 3.0.3. *There exists a principally polarized abelian variety $x = (A_x, \lambda_x, \iota_x)$ over $\bar{\mathbb{F}}_p$ with an action of $\mathcal{O}_E$ of type (m, n) with the following associated covariant Dieudonne module (M, λ_x, ι_x);*

(1)$M = M(A_x[p^\infty]) = M_1 \oplus M_2$, where $M_1 = W(k)[F, V]/W(k)[F, V](F^m - V^n), M_2 = W(k)[F, V]/W(k)[F, V](F^n - V^m)$ with $\mathcal{O}_E$ acting on M_i via σ_i i.e, $\iota(\alpha)(m_i) = \sigma_i(\alpha)m_i,\ m_i \in M_i\ (i = 1, 2)$. In other words, $A_x[\wp_1^\infty] = G_{m,n}$ and $A_x[\wp_1^\infty] = G_{n,m}$ as introduced in [23].

(2)The principal quasi-polarization induces the canonical isomorphism between $G_{n,m}$ and its Serre dual $G_{m,n}^t$.

In view of the proposition 2.2.3, there exists a principally polarized abelian variety (A, λ, ι) over $\bar{\mathbb{F}}_p$ in the basic locus whose covariant Dieudonne module N satisfies

(i) $N \otimes B(\bar{\mathbb{F}}_p)$ is generated over $B(\bar{\mathbb{F}}_p)[F, V]$ by two elements v_1, v_2 such that $(F^m -$

$V^n)v_1 = (F^n - V^m)v_2 = 0$ $(B(\bar{\mathbb{F}}_p) = \mathrm{Frac}W(\bar{\mathbb{F}}_p))$,

(ii)$\mathcal{O}_E$ acting on v_i via σ_i i.e., $\iota(\alpha)(v_i) = \sigma_i(\alpha)v_i,\ (i = 1, 2)$

(iii)The principal quasi-polarization induces a perfect paring between the Diedonne submodules generated by v_1 and v_2.

Then there exist positive integers d_i such that $p^{d_i}v_i \in N$. So there exists an injective homomorphism of (covariant) Dieudonne modules $M \to N : e_i \mapsto p^{d_i}v_i$, where e_i $(i = 1, 2)$ is a generator of M_i.

Recall [3] that for a p-divisible smooth formal group G over a perfect field k of characteristic p, denoting the contravariant(resp. covariant) Dieudonne module of G by $M^*(G)$(resp. $M(G)$), there exists a funtorial σ-linear isomorphism of left $W(k)[F, V]$-modules

$$M^*(G^t) = (M^*(G))^t := \mathrm{Hom}_W(M^*(G), W) \xrightarrow{\sim} M(G).$$

Accordingly, we have an exact sequence of left $W(\bar{\mathbb{F}}_p)[F, V]$-modules

$$0 \to M' \to N' \to N'/M' \to 0,$$

where M'(resp. N') is the left $W(\bar{\mathbb{F}}_p)[F, V]$-module such that $M' \otimes_{(W,\sigma)} W \cong M$ (resp. $N' \otimes_{(W,\sigma)} W \cong N$), and there exists a finite flat group scheme H over $\bar{\mathbb{F}}_p$ whose dual H^t has the contravariant Dieudonne module $M^*(H^t) = N'/M'$. Hence, if B is the abelian variety whose dual B^t is the quotient A^t/H^t, we have $M^*(B^t) = M'$

and $M(B) = M$.

$$0 \to H \to B \to A \to 0,$$

$$0 \to H^t \to A^t \to B^t \to 0,$$

$$0 \to M^*(B^t) \to M^*(A^t) \to M^*(H^t) \to 0.$$

From the functoriality of the isomorphisms involved, it is clear that B has an action by $\mathcal{O}_E$. Also, there exists a polarization $\lambda : B \to B^t$ which induces a principal quasipolarization on $B[p^\infty]$. Then by a separable isogeny, one can find a principally polarized abelian variety with an $\mathcal{O}_E$-action whose covariant Dieudonne module is the given one. $\mathcal{O}_E$-linear ■

We compute the local monodromy of the Picard moduli scheme at the point described in Lemma 3.0.3 and show that the local monodromy at this point is an open subgroup of $\mathrm{GL}_m(\mathbb{Z}_p) \times \mathrm{GL}_n(\mathbb{Z}_p)$. The proof of this fact (hence Thm 2.3.1) consists of two parts. In the first part, we show that the image contains the derived group $\mathrm{SL}_m(\mathbb{Z}_p) \times \mathrm{SL}_n(\mathbb{Z}_p)$ and then we prove that the determinant of the local monodromy is an open subgroup of $Z_p^\times \times \mathbb{Z}_p^\times$.

Both of these parts will follow from some group theories once we have the information modulo suitable powers of p. For the first part, the computation modulo p is enough, but for the second part, it is more subtle and it turns out that the exact powers of p that we need depends more on the precise shape of the Dieudonne module of the particular point that is used.

3.1 Some group theories

If one is only interested in global monodromy, somtimes the modulo p information is enough to conclude the best one can hope for. For example, in [11], Ekedahl used Igusa' theorem alluded to in the introduction, to conclude that the global monodromies of certain moduli spaces are as big as possible (In particular, his argument gives another proof of the same statement about the global naive p-adic monodromy of Siegel moduli scheme). But we can still deduce quite strong conclusion, once we know that the first layer (i.e. mod p) image of the local monodromy is already quite big. In many cases, it is due to following fact.

Lemma 3.1.1. *Let $\{n_i\} : 1 \leq i \leq r\}$ be a set of natural numbers greater than 1. Let p be a prime number bigger than 3 and let X be a closed subgroup of $\prod_i \mathrm{GL}_{n_i}(\mathbb{Z}_p)$. If the image of X under reduction mod p contains $\prod_i \mathrm{SL}_{n_i}(\mathbb{F}_p)$, then X contains $\prod_i \mathrm{SL}_{n_i}(\mathbb{Z}_p)$.*

In the case of single $\mathrm{SL}_n(\mathbb{Z}_p)(p \geq 5)$, this is due to Serre ([31], IV-23 Lemma 3). He proved that a closed subgroup X of $\mathrm{SL}_n(\mathbb{Z}_p)(p \geq 5)$ which maps onto $\mathrm{SL}_n(\mathbb{F}_p)$ must be $\mathrm{SL}_n(\mathbb{Z}_p)$. In a more geneal case as stated above as well, this is probably well known. Though there is other argument which might work in more general setting, we decided to present Serre's original proof since it also proves our lemma with no extra work.

Proof. We prove by induction on n that the image of X under reduction modulo p^n contains $\prod_i \mathrm{SL}_{n_i}(\mathbb{Z}/p^n\mathbb{Z})$. By our condition, this is true for $n = 1$. Assume it is true for n, and let us prove it for $n+1$. From the exact sequence

$$0 \to \mathrm{K}_n = \mathrm{Ker}\pi_n \to \prod_i \mathrm{SL}_{n_i}(\mathbb{Z}/p^{n+1}\mathbb{Z}) \to \prod_i \mathrm{SL}_{n_i}(\mathbb{Z}/p^n\mathbb{Z}) \to 0,$$

it is enough to show that the image of X contains $\mathrm{K}_n = \mathrm{Ker}\pi_n$, in other words, for any $s = (s_i)_i \in \prod_i \mathrm{SL}_{n_i}(\mathbb{F}_p)$ which is congruent to $1 \mod p^n$, there exists $x \in X$ with $x \equiv s \mod p^{n+1}$. Write $s_i = 1 + p^n u_i$; since $\det(s_i) = 1$, one has $Tr(u_i) \equiv 0 \mod p$. Then one can show that any such u_i is congruent mod p to a sum of matrices u_i with $u_i^2 = 0$. Hence, we may assume that $u_i^2 = 0$. By induction hypothesis applied to $(1 + p^{n-1}u_i)_i \in K_{n-1}$, there exists a single $y \in X$ such that $y \equiv (1 + p^{n-1}u_i)_i \mod p^n$, i.e. $y = (1 + p^{n-1}u_i)_i + p^n v$, where v has coefficients in $\mathbb{Z}_p$. If we put $x = y^p$, then we have $x \equiv (1 + p^n u_i)_i \mod p^{n+1}$ which can be checked componentwise as done in the Serre's original argument. ■

For the second part, we need another lemma which is essentially due to R. Pink.

Lemma 3.1.2. *Let $\mathcal{O}_\lambda$ be the ring of integers in a finite extension of $\mathbb{Q}_l$ and let $\{n_i\} : 1 \leq i \leq r\}$ be a set of natural numbers. Let H be a closed subgroup of $\prod_i \mathrm{GL}_{n_i}(\mathcal{O}_\lambda)$. Then there exists an integer ν, depending on K, with the following property. for any closed subgroup H of K, $H = K$ if (and only if) H and K have the same image in $\prod_i \mathrm{GL}_{n_i}(\mathcal{O}_\lambda/l^\nu\mathcal{O}_\lambda)$. In case $n_i = 1$, one can take $\nu = 2$.*

The proof as presented in ([17], Key Lemma 8.18.3) is seen to carry over in this case. Also in the special case $n_i = 1$ one can verify the claim $\nu = 2$ from the proof.

In view of Lemma 3.1.1, the following takes care of the first step to the proof of the main theorem Thm. 2.3.1, namely the claim about the derived group.

Proposition 3.1.3. *Let* $x = (A_x, \lambda_x, \iota_x)$ *be as in Lemma 3.0.3 and let's assume that* $p \geq 5$*. Then the image of the local monodromy under mod* p *reduction*

$$\bar{\rho}_G = \pi \circ \rho_G : \mathrm{Gal}(K^{sep}/K) \to \mathrm{GL}_m(\mathbb{Z}_p) \times \mathrm{GL}_n(\mathbb{Z}_p) \xrightarrow{\pi} \mathrm{GL}_m(\mathbb{F}_p) \times \mathrm{GL}_n(\mathbb{F}_p)$$

contains $\mathrm{SL}_m(\mathbb{F}_p) \times \mathrm{SL}_n(\mathbb{F}_p)$*.*

The main tool will be the Catier-Dieudonne theory. In the next section, we review the theory briefly and give an explicit presentation of the Dieudonne module of the universal deformation of $(\widehat{A}_x, \iota_x)$ in terms of display as presented in [26], [34].

3.2 Deformations of Dieudonne modules

Our strategy for showing the assertion about local monodromy is computing the local monodromies of the restrictions to various formal subschemes of the universal deformation space of $(\widehat{A}_x, \iota_x)$.

For the benefits of readers, we recall the main theorem of Cartier-Dieudonne theory. For more details, we refer to [14].

Theorem 3.2.1. *Let* k *be a commutative* $\mathbb{Z}_{(p)}$*-algebra with* 1*. Then there is a canonical equivalence of categories, between the category of smooth commutative formal groups over* k *and the category of* V*-flat* V*-reduced* $\mathrm{Cart}_p(k)$*-modules.*

Let (A, ι) be the universal deformation over $\mathrm{Spf}(R)$ of the abelian variety $(\widehat{A}_x, \iota_x)$ that is described in Lemma 3.0.3 and let K be the field of fractions of R. Then there is a natural pairing of $\mathrm{Gal}(K^{sep}/K)$-modules

$$\underline{Hom}_K(\widehat{\mathbb{G}}_m, \widehat{A}_\eta) \times \mathrm{T_p}(A^t_\eta) \to \mathbb{Z}_p,$$

where $\widehat{\mathbb{G}}_m$ is the formal multiplicative group and $\widehat{A_\eta}$ (resp, A^t_η) is the formal group (resp, the dual abelian variety) of the generic fiber A_η. Therefore, by the principal polarization of A, the representation $\rho_G : \mathrm{Gal}(K^{sep}/K) \to \mathrm{GL}_{m+n}(\mathbb{Z}_p)$ is dual to the Galois representation attached to the cocharacter group of $\widehat{A}_\eta$,

$$X_* = \underline{Hom}_K(\widehat{\mathbb{G}}_m, \widehat{A}_\eta)$$

By Cartier-Dieudonne theory (Theorem 3.2.1), this is isomorphic to

$$\mathrm{Hom}_{\mathrm{Cart}_p(K^{sep})}(M_0, M),$$

where M_0 is the Dieudonne module of $\widehat{\mathbb{G}}_m$, and M is the Dieudonne module of $A[p^\infty]^\circ_\eta \simeq \widehat{A}_\eta$. So it suffices to show Prop. 3.1.3 for this Galois representation.

Equicharacteristic deformations of a smooth formal group can be described in a fairly simple way when its Dieudonne module is presented in Display.

Definition 3.2.2. ([26], [34]) A $\mathrm{Cart}_p(R)$-module M over arbitrary ring R of characteristic p is displayed if M is given by generators $e_i, i = 1, \cdots, g+h$ and relations

$$\begin{aligned} Fe_i &= \sum a_{ij} e_j \quad i = 1, \cdots, g \\ e_i &= V(\sum a_{ij} e_j) \quad i = g+1, \cdots, g+h \end{aligned}$$

for an invertible matrix (a_{ij}) with entries in $W(R)$.

We will call the matrix (a_{ij}) the display (matrix) of M when the basis is clear in the context. Also note that if the display matrix of M is given in block forms

$$(a_{ij}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then the matrix giving the Frobenius F is $\begin{pmatrix} A & pB \\ C & pD \end{pmatrix}$ and the Hasse-Witt matrix is $A \pmod p$.

Theorem 3.2.3. *([26], [34]) Let G be a smooth formal group over $k = \bar{k}$ whose Dieudonne module has a display matrix (a_{ij}). If R is an Artinian local ring of characteristic p, residue field k, and maximal ideal $\underline{m}$, then there is a one to one correspondence between isomorphism classes of deformations of M over R and maps*

$$\begin{aligned} d : VM/pM &\rightarrow \underline{m} \otimes_k (M/VM), \\ \bar{e}_i &\mapsto \sum \bar{d}_{ij} \otimes \bar{e}_j \ , \end{aligned}$$

where $\bar{e}_i, \bar{e}_j$ are the images of e_i, e_j in VM/pM and M/VM respectively. Set $d_{ij} = (\bar{d}_{ij}, 0, \cdots) \in W(R)$ for $i > g, j \leq g$ and $d_{ij} = 0$ otherwise. The Dieudonne module corresdponding to the map d is defined by

$$\begin{aligned} Fe_i &= \sum (a_{ij}(e_j + \sum d_{jk} e_k)) \quad i = 1, \cdots, g \\ e_i &= V(\sum a_{ij}(e_j + \sum d_{jk} e_k)) \quad i = g+1, \cdots, g+h \end{aligned}$$

Furthermore every Dieudonne module over R which restricts to M over k is isomorphic to exactly one Dieudonne module of these forms.

Now we return to our p-divisible group as defined in Lemma 3.0.3. If we set

$$e_i = \begin{cases} F^{i-1} & (1 \leq i \leq m) \\ V^{m+n-i+1} & (m+1 \leq i \leq m+n) \end{cases}, \quad f_i = \begin{cases} F^{i-1} & (1 \leq i \leq n) \\ V^{m+n-i+1} & (n+1 \leq i \leq n+m) \end{cases}$$

then $\{e_i\}_{1\leq i\leq m+n}$ (resp, $\{f_i\}_{1\leq i\leq m+n}$) form a basis of M_1(resp, M_2). (We keep using the same notations for the images of F and V in the quotients $\mathrm{M}_1 = W(k)[F,V]/W(k)[F,V](F^m - V^n)$, $M_2 = W(k)[F,V]/W(k)[F,V](F^n - V^m)$ as long as there is no fear of confusion).

The display matrices of the Dieudonne modules M_i $(i = 1,2)$ with respect to these bases $\{e_i\}, \{f_i\}$ are the same :

$$D_1 = D_2 = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix},$$

where the lower left $(m+n-1)\times(m+n-1)$ matrix is the identity matrix and the upper right entry is 1. Let us denote by M_2^t the Dieudonne module of the Serre dual $A[\wp_2^\infty]^t$ of $A[\wp_2^\infty]$ with dual basis f_i^t such that $< f_i^t, f_j >= \delta_{ij}$ ([26] p.502-504), then one can easily check that the principal quasi-polarization $\lambda : M_1 \to M_2^t$ is given on M by

$$\lambda(e_i) = \begin{cases} f_{n+i}^t & (1 \leq i \leq m) \\ f_{i-n}^t & (n+1 \leq i \leq n+m) \end{cases}.$$

Next, we construct the universal formal deformation space of (M, λ, ι). Let (M, λ, ι) be the quai-polarized Dieudonne module of the point that was defined in Lemma 3.0.3 equipped with $\mathcal{O}_E$-module structure ι and the quasipolarization attached to λ. Let $(\widetilde{M}, \lambda, \iota)$ be the universal local deformation of (M, λ, ι).

Then, one easily sees that $\widetilde{M}$ is the direct sum $\widetilde{M} = \widetilde{M}_1 \oplus \widetilde{M}_2$ such that

(1) $\widetilde{M}_i$ is a lifting of M_i;

(2) $\mathcal{O}_E$ acts on $\widetilde{M}_i$ via σ_i;

(3) $\lambda : \widetilde{M}_1 \to \widetilde{M}_2^t$.

Proposition 3.2.4. *Let $\widetilde{M}, \widetilde{M}_1, \widetilde{M}_2$ be as above.*

(1) $\widetilde{M}_1$ is given by generators $\{\tilde{e}_i : 1 \leq i \leq m+n\}$ satisfying

$$\begin{aligned} F(\tilde{e}_i) &= \sum_{j=1}^{m+n} a_{ji}\tilde{e}_j \quad (1 \leq i \leq m) \\ \tilde{e}_i &= V(\sum_{j=1}^{m+n} b_{ji}\tilde{e}_j) \quad (m+1 \leq i \leq m+n), \text{ where} \end{aligned}$$

$$(a_{ij}) = \left(\begin{array}{ccc|c|ccc|c} 0 & \cdots & 0 & T_{11} & T_{12} & \cdots & T_{1n} & 1 \\ \hline & & & T_{21} & T_{22} & \cdots & T_{2n} & 0 \\ & I_{m-1} & & \vdots & \vdots & & \vdots & \vdots \\ & & & T_{m1} & T_{m2} & \cdots & T_{mn} & 0 \\ \hline & & & 1 & 0 & \cdots & 0 & 0 \\ & & & & 1 & & & \vdots \\ & & & & & \ddots & & \vdots \\ & & & & & & 1 & 0 \end{array}\right),$$

where $T_{ij} = (t_{ij}, 0, \cdots)$'s are Witt vectors in $W(k[[t_{ij}]])$.

(2) $\widetilde{M}_2$ *is given by generators* $\{\tilde{f}_i : 1 \leq i \leq m+n\}$ *satisfying*

$$\begin{aligned} F(\tilde{f}_i) &= \sum_{j=1}^{m+n} b_{ji}\tilde{f}_j \quad (1 \leq i \leq n) \\ \tilde{f}_i &= V(\sum_{j=1}^{m+n} b_{ji}\tilde{f}_j) \quad (n+1 \leq i \leq m+n), \end{aligned}$$

, where

$$(b_{ij}) = \left(\begin{array}{c|c|ccc|c} & U_{11} & U_{12} & \cdots & U_{1m} & 1 \\ \hline & U_{21} & U_{22} & \cdots & U_{2m} & \\ I_{n-1} & \vdots & \vdots & & \vdots & \\ & U_{n1} & U_{n2} & \cdots & U_{nm} & \\ \hline & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \end{array}\right),$$

$U_{ij} = (u_{ij}, 0, \cdots)$ *are Witt vectors in* $W(k[[u_{ij}]])$.

(3) The principal quasi-polarization λ *lifts to a principal quasi-polarization of* $(\widetilde{M}, \lambda, \iota)$ *when* $T_{i1} = -U_{1i} \quad (1 \leq i \leq m)$.

(1) One just needs to work out Theorem 3.2.3 (also see [26]). This theorem is made more illuminating when one notices that the displayed matrix of the universal deformation $\widetilde{M}_1$ of M_1 with respect to the basis $\{e_i : 1 \leq i \leq m+n\}$ is given by

the product $\left(\begin{array}{c|c} I_m & T \\ \hline & \\ & I_n \\ & \end{array}\right)\left(\begin{array}{cccc} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{array}\right)$. One can easily check that this product gives the display matrix in (1) for the same T.

(2) This is completely analogous to (1).

(3) The theory of biextensions of formal groups by Mumford provides a recipe for the display matrix of the dual of any Dieudonne module in terms of original display ([26] p.502-504, [27] p.420): if M is the Dieudonne module of a local-local p-divisible group G with display matrix a_{ij} (with respect to a basis $\{e_i\}$), then the Dieudonne module M^t of its dual G^t is generated by generators $\{f_i : i = 1, \cdots, n+h\}$ satisfying

$$\begin{aligned} f_i &= V(\sum \alpha'_{ij} f_j), \quad i = 1, \cdots, n \\ F f_i &= \sum \alpha'_{ij} f_j, \qquad i = n+1, \cdots, n+h, \end{aligned}$$

with $(\alpha'_{ij})^{-1} = (a_{ij})^t$.

We apply this to find the dual display $\widetilde{M}_2^t$ of $\widetilde{M}_2$. Observing that the display matrix D_2 of $\widetilde{M}_2$ is obtained from $(b_{ij}) = \left(\begin{array}{c|c} I_n & U \\ \hline & I_m \end{array}\right)$, by moving the first column to the right end, one finds the inverse matrix D_2^{-1} of D_2 is

$$\left(\begin{array}{ccc|c|ccc}
 & 1 & & & -U_{21} & \cdots & -U_{2m} \\
 & & \ddots & & \vdots & & \vdots \\
 & & & 1 & -U_{n1} & \cdots & -U_{nm} \\
\hline
 & & & & 1 & & \\
\hline
 & & & & & \vdots & \\
 & & & & & & 1 \\
\hline
1 & & & & -U_{11} & \cdots & -U_{1m}
\end{array}\right),$$

which is similarly obtained from $\left(\begin{array}{c|c} I_n & -U \\ \hline & I_m \end{array}\right)$, by moving the first row to the bottom. Therefore with respect to the dual basis f_i^t of f_i (i.e. $< f_i^t, f_j >= \delta_{ij}$), one has

$$\begin{aligned}
f_i^t &= V(\sum c_{ij} f_j^t) \quad i = 1, \cdots, n \\
F f_i^t &= \sum c_{ij} f_j^t \quad i = n+1, \cdots, n+m,
\end{aligned}$$

$$
\text{with } (c_{ij}) = \left(\begin{array}{ccc|c|ccc|c}
 & & & & & & & 1 \\ \hline
 & & & & & & & 0 \\
 & I_{n-1} & & & & & & \vdots \\
 & & & & & & & 0 \\ \hline
-U_{21} & \cdots & -U_{n1} & 1 & & & & -U_{11} \\
-U_{22} & & -U_{n2} & & 1 & & & -U_{12} \\
\vdots & & \vdots & & & \ddots & & \vdots \\
-U_{2m} & \cdots & -U_{nm} & & & & 1 & -U_{1m}
\end{array}\right),
$$

If we change the basis

$$
g_i = \begin{cases} f^t_{n+i} & (1 \le i \le m) \\ f^t_{i-n} & (n+1 \le i \le n+m) \end{cases},
$$

then we get a display matrix for

$$
\widetilde{M}^t_2 : \left(\begin{array}{c|c|ccc|c}
 & -U_{11} & -U_{21} & \cdots & -U_{n1} & 1 \\ \hline
 & -U_{12} & -U_{22} & \cdots & -U_{n2} & \\
I_{m-1} & \vdots & \vdots & & \vdots & \\
 & -U_{1m} & -U_{2m} & \cdots & -U_{nm} & \\ \hline
 & 1 & & & & \\
 & & 1 & & & \\
 & & & \ddots & & \\
 & & & & 1 &
\end{array}\right).
$$

Note that the matrix appearing in the middle upper block is $-U^t$. From this (3) follows immediately. ■

If we set the basis $(X_1, \cdots, X_{m+n}, Y_1, \cdots, Y_{m+n})$ to be such that

$$X_i = \begin{cases} \tilde{e}_i & \text{if } 1 \leq i \leq m \\ \tilde{f}_{i-m} & \text{if } m+1 \leq i \leq m+n \end{cases}, \quad Y_i = \begin{cases} \tilde{f}_{i+n} & \text{if } 1 \leq i \leq m \\ \tilde{e}_i & \text{if } m+1 \leq i \leq m+n \end{cases}.,$$

then the Dieudonne display of $\widetilde{M}$ with respect to this basis is given by

$$\widetilde{D} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where in particular

$$A = \left(\begin{array}{c|c|c|c} & T_{11} & & \\ \hline & T_{21} & & \\ I_{m-1} & \vdots & & \\ & T_{m1} & & \\ \hline & & & -T_{11} \\ \hline & & & U_{21} \\ & & I_{n-1} & \vdots \\ & & & U_{n1} \end{array}\right).$$

3.3 Proof of Proposition 3.1.3.

We prove Prop. 3.1.3. It suffices to prove that the image of

$$\bar{\rho} : \mathrm{Gal}(K^{sep}/K) \to X_*/pX_*$$

contains both $\mathrm{SL}_m(\mathbb{F}_p) \times 1$ and $1 \times \mathrm{SL}_n(\mathbb{F}_p)$.

Since the Dieudonné module M_0 of $\widehat{\mathbb{G}}_m$ is generated as a left $\mathrm{Cart}_p(K^{sep})$-module by a single element e_0 with relation $Fe_0 - e_0 = 0$, we have

$$\begin{aligned} X_*/pX_* &\cong \left\{\phi \in \mathrm{Hom}_F(M_0/VM_0, \widetilde{M}/V\widetilde{M})\right\} \\ &= \{e = \phi(e_0) | F(e) = F(\phi(e_0)) = \phi(F(e_0)) = e\}. \end{aligned}$$

$$\begin{aligned} X_*/pX_* &\cong \mathrm{Hom}_F(M_0/VM_0, \widetilde{M}/V\widetilde{M}) \\ &= \left\{ \sum_{i=1}^{m+n} c_i\ X_i \mid (c_1, \cdots, c_{n+m})^t = A\ (c_1^p, \cdots, c_{n+m}^p)^t \right\}, \text{ where} \end{aligned}$$

$$A = \left(\begin{array}{c|c|c|c} & t_{11} & & \\ \hline I_{m-1} & \begin{matrix} t_{21} \\ \vdots \\ t_{m1} \end{matrix} & & \\ \hline & & & -t_{11} \\ \hline & & I_{n-1} & \begin{matrix} u_{21} \\ \vdots \\ u_{n1} \end{matrix} \end{array}\right).$$

Consider the closed formal subschemes

$$\mathcal{Y}\rangle = \mathrm{Spf}(R_i) \hookrightarrow \mathrm{Spf}(R) = \mathrm{Spf}(k[[t_{ij}, u_{ij}]])(i = 1, 2)$$

defined by

$$R_1 = R/(u_{i1} | 2 \le i \le n), R_2 = A/(t_{i1} | 2 \le i \le m).$$

The restrictions of our p-divisible group $A[p^\infty]$ to $\mathrm{Spec}(R_i)$ are still ordinary as clear from the Hasse-Witt matrix and so their maximal étale quotients $A[p^\infty]^{ét}$ restrict to the étale p-divisible groups of same ranks. Therefore to prove Prop. 3.1.3 for the original representation $\bar{\rho}: \mathrm{Gal}(K^{sep}/K) \to X_*/pX_*$, it suffices to consider the local monodromies for these subschemes at the same point $\bar{\rho}_i : G_i \to \mathrm{Spec}(R_i)$ where G_i is the restriction of $A[p^\infty]^{ét}$ to $\mathrm{Spec}(R_i)$.

Proposition 3.3.1. *Let R_i be as above and L_i be their fraction fields. Let $Y_i = \underline{Hom}_L(\widehat{\mathbb{G}}_m, G_i)$. Then the image of $\bar{\rho}_1 : \mathrm{Gal}(L^{sep}/L) \to Y_1/pY_1$ contains $\mathrm{SL}_m(\mathbb{F}_p) \times \{1\}$ and $\bar{\rho}_2 : \mathrm{Gal}(L^{sep}/L) \to Y_2/pY_2$ contains $\{1\} \times \mathrm{SL}_n(\mathbb{F}_p)$.*

We only consider the case $i = 1$. The proof of the other case is the same as that of this case.

On the subscheme $\mathrm{Spec}(R_1)$, the matrix equation in c_i becomes the following two sets of equations in $c_1, \cdots, c_{m+n}$ defined by

$$\begin{pmatrix} c_1 \\ \\ \vdots \\ c_m \end{pmatrix} = \left(\begin{array}{c|c} & t_{11} \\ \hline & t_{21} \\ I_{m-1} & \vdots \\ & t_{m-11} \end{array}\right) \begin{pmatrix} c_1^p \\ \\ \vdots \\ c_m^p \end{pmatrix}, \begin{cases} c_{m+1} = -t_{11} c_{m+1}^{p^n} \\ \\ c_{m+i} = c_m^{p^{i-1}} (1 < i \leq n) \end{cases}$$

The left matrix equation reduces to a nice single equation in c_1 which is an example of "generic vectorial p-polynomial" [1]: all degrees of c_1 occurring are powers of p.

$$\begin{aligned} c_1 &= c_m^p t_{11} \\ c_2 &= c_1^p + c_m^p t_{21} \\ &\vdots \\ c_m &= c_{m-1}^p + c_m^p t_{m-11}. \end{aligned}$$

Note that each of n variables $c_i (i \geq 1)$ is hooked successively to c_m and its previous one c_{i-1}. By taking the p-th power of the second equation and plugging that into c_2^p in the first one, we get $c_2 = t_{11}^p c_m^{p^2} + t_{21} c_m^p$. Repeating this iteration, we get following single equation in c_m.

$$c_m = \sum_{i=1}^{m} t_{m-i+1,1}^{p^{i-1}} c_m^{p^i}$$

For simplicity, let us change the notation $x = c_m, t_i = -t_{i+1,1} (0 \leq i \leq m, t_m = 1)$. The given matrix equations are then

$$\sum_{i=1}^{m} t_{m-i}^{p^{i-1}} x^{p^i} + x = 0, \quad y = t_0 y^{p^n}.$$

The second equation is separable, so its roots generate a separable extension $L' = L(y) = L(t_0^{-1/(p^n-1)})$ of L. Since our equation has no coefficient from the other variables, we can assume that L is the fraction field of $k[[t_0, \cdots, t_{m-1}]]$. Also note that then

$$\begin{aligned} L' = L(t_0^{-1/(p^n-1)}) &= \mathrm{Frac}(k[[t_0, \cdots, t_{m-1}]])(t_0^{-1/(p^n-1)}) \\ &= \mathrm{Frac}(k[[t_0^{1/(p^n-1)}, t_1, \cdots, t_{m-1}]]) \end{aligned}$$

Therefore we are reduced to proving the following statement.

Lemma 3.3.2. *The equation*

$$f(x) = \sum_{i=1}^{m} t_{m-i}^{p^{i-1}} x^{p^i} + x = 0$$

has Galois group $\mathrm{SL}_m(\mathbb{F}_p)$ *over* $L' = \mathrm{Frac}(k[[t_0^{1/(p^n-1)}, \cdots, t_{m-1}]])$.

Let M be the splitting field of $f(x)$ inside L^{sep}. We will first prove that for general $n, m \geq 1$, $f(x)$ has Galois group $\mathrm{GL}_m(\mathbb{F}_p)$ over $L = \mathrm{Frac}(k[[t_0, \cdots, t_{m-1}]])$ (i.e. $\mathrm{Gal}(M/L) = \mathrm{GL}_m(\mathbb{F}_p)$) and then will show that $[L' \cdot M : M] = \frac{p^n-1}{p-1}$, thereby the fact $[L' \cap M : L] = p - 1$. In view of the fact that $\mathrm{Gal}(L' \cdot M/L')$ is a subgroup of $\mathrm{SL}_m(\mathbb{F}_p)$, this will prove the lemma since $\mathrm{Gal}(L' \cdot M/L') = \mathrm{Gal}(M/L' \cap M)$ is then a subgroup of $\mathrm{GL}_m(\mathbb{F}_p)$ of index $p - 1$.

Since $k((t_{m-i}))$ is purely inseparable over $k((t_{m-i}^{p^{i-1}}))$ and L' is separable over L, both of these questions can be answered equally legitimately for $g(x) = \sum_{i=0}^{m} t_{m-i} x^{p^i} = 0$ $(t_m = 1)$

We start with the first claim. This is also more or less well known, ([1], 3). Indeed, if we let $\{s_1, \cdots, s_m\}$ is a $\mathbb{F}_p$-basis of the solution space of this polynomial, we have

$$t_0^{-1} g(x) = \prod_{(a_1, \cdots, a_m) \in \mathbb{F}_p^n} (x - a_1 s_1 - \cdots - a_m s_m)$$

and, since the coefficients are algebraically independent over k, we conclude that the n elements $s_1, \cdots, s_m$ are also algebraically indenpendent over k. Therefore, every $\mathbb{F}_p$-linear automorphism of the $\mathbb{F}_p$-vector space spanned by $\{s_1, \cdots, s_m\}$ defines an automorphism of the splitting field. On the other hand, each such automorphism fixes the ground field $\mathrm{Frac}(k[[t_0, \cdots, t_{m-1}]])$ since each $t_i \in k((s_0, \cdots, s_m))$ is symmetric in $s_0, \cdots, s_m$.

For the second assertion claiming $[L' \cdot M : M] = \frac{p^n-1}{p-1}$, we need to prove that the irreducible polynomial $h(x) = \mathrm{Irred}(t^{-1}, k((s_1, \cdots, s_m)), x)$ has degree $\frac{p^n-1}{p-1}$. We have the following identity

$$\begin{aligned} (t^{-1})^{p^n-1} &= \prod_{(a_1,\cdots,a_m)\in \mathbb{F}_{p^m}-(0)} (a_1 s_1 + \cdots + a_m s_m) \\ &= (\prod_{\alpha \in \mathbb{F}_{p^m}^{\times}} \alpha) \cdot (\prod_{[b_1,\cdots,b_m]\in \mathbb{P}(\mathbb{F}_{p^m})} (b_1 s_1 + \cdots + b_m s_m))^{p-1} \end{aligned}$$

The first equality comes from the coefficients-roots relation of $g(x)$ and in the second product of the second line, the index runs over $\mathbb{P}(\mathbb{F}_{p^m})$ and for each point $P \in \mathbb{P}(\mathbb{F}_{p^m})$, we choose a vector $(b_1, \cdots, b_m) \in \mathbb{F}_{p^m}^{\times} - (0, \cdots, 0)$ representing P (i.e. $P = [b_1, \cdots, b_m] \in \mathbb{P}(\mathbb{F}_{p^m})$). So we have $(t^{-1})^{p^n-1} - (p(s_1, \cdots, s_m))^{p-1} = 0$ for some polynomial $p(s_1, \cdots, s_m)$ in $s_1, \cdots, s_m(k = \bar{k}$. In particular, $p(\underline{s}) = c \cdot \prod_{[b_1,\cdots,b_m]\in\mathbb{P}(\mathbb{F}_{p^m})}(b_1 s_1 + \cdots + b_m s_m)$ for some constant $c \in k$. Since we are in an integral domain, we should have then $(t^{-1})^{\frac{p^n-1}{p-1}} = \beta p(s_1, \cdots, s_m)$ for a $\beta \in \mathbb{F}_p^{\times}$. Let $h(x) = x^{\frac{p^n-1}{p-1}} - \beta p(s_1, \cdots, s_m)$. Since t^{-1} is a root of $h(x)$, the lemma will be proved if we show that $h(x)$ is irreducible.

Since this polynomial is defined over $k[[s_1, \cdots, s_m]]$, it suffices to prove this over a closed subscheme $\mathrm{Spec}(k[[s_1, \cdots, s_m]]/(s_i - r_i(s))) = \mathrm{Spec}(k[[s]])$ for a suitable choice of $r_i(s) \in sk[[s]]$. This can be done, for example, as follows. Let $r_i(s) = s^{e_i}$

such that $\{e_i | 1 \leq i \leq m\}$ form a strictly increasing sequence of natural numbers satisfying

$$\gcd(\frac{p^n - 1}{p - 1}, e_1 p^{m-1} + e_2 p^{m-2} + \cdots + e_m) = 1.$$

Then for this choice of $r_i(s)$, $h(x) = x^{\frac{p^n-1}{p-1}} - \beta p(r_1(s), \cdots, r_m(s))$ is irreducible over $k[[s]]$. Indeed, there are p^{m-1} points in $\mathbb{P}(\mathbb{F}_{p^m})$ having nonzero first coordinate a_1 and for these points $a_1 s_1 + \cdots + a_m s_m = a_1 s^{e_1} + \cdots + a_m s^{e_m}$ has valuation e_1 for a valuation of which s is a uniformizer. Next, in the hyperplane $a_1 = 0$ in $\mathbb{P}(\mathbb{F}_{p^m})$ there are p^{m-2} points whose second coordinates a_2 are non zero and then the corresponding linear factors $a_1 s_1 + \cdots + a_m s_m = a_2 s^{e_2} + \cdots + a_m s^{e_m}$ have valuation e_2. Continuing in an obvious way, we find that the constant term of $p(x)$ has valuation $e_1 p^{m-1} + e_2 p^{m-2} + \cdots + e_m$. Therefore, if e_i's satisfy $\gcd(\frac{p^n-1}{p-1}, e_1 p^{m-1} + e_2 p^{m-2} + \cdots + e_m) = 1$ (this can be always achieved because of e_m), then the Newton polygon of $p(x)$ over $k[[s]]$ becomes a line segment so that $x^{\frac{p^n-1}{p-1}} - \beta p(s^{e_1}, \cdots, s^{e_m})$ becomes irreducible over $k[[s]]$. This finishes the proof of Lemma. ■

Clearly, the same argument will prove the corresponding assertion for the second factor.

Finally, combining the facts about the original (for K, not just for L_i) local monodromy, we conclude that the image of local monodromy in $\mathrm{GL}_m(\mathbb{F}_p) \times \mathrm{GL}_n(\mathbb{F}_p)$ contains $\mathrm{SL}_m(\mathbb{F}_p) \times \mathrm{SL}_n(\mathbb{F}_p)$. (Also, the projection of the image to each factor is surjective as observed previouisly.)

Remark For our choice of the point Lemma 3.0.3, the local monodromy does not become the entire $\mathrm{GL}_m(\mathbb{Z}_p) \times \mathrm{GL}_n(\mathbb{Z}_p)$. In fact, the projection of the local monodromy to the first layer is not already the whole $\mathrm{GL}_m(\mathbb{F}_p) \times \mathrm{GL}_n(\mathbb{F}_p)$. More precisely, we show that

$$\bar{\rho}(\mathrm{Gal}(K^{sep}/K)) \cap \mathrm{GL}_m(\mathbb{F}_p) \times \{1\} \subseteq \mathrm{SL}_m(\mathbb{F}_p) \times \{1\}.$$

Since our representation $\overline{\rho}_G : \mathrm{Gal}(K^{sep}/K) \to X_*/pX_*$ is associated with the pull-back of Lang torsor $\mathrm{GL}_{m+n} \to \mathrm{GL}_{n+m} \ : \ g \mapsto g^{-1} \cdot g^{(p)}$ via

$$\begin{array}{rcl} \mathrm{Spec}(R) = \mathrm{Spec}(k((t_{i1}, u_{1j} | 1 \le i \le m, 2 \le j \le n))) & \xrightarrow{\phi} & \mathrm{GL}_{n+m} \\ \{t_{i1}, u_{1j}\} & \mapsto & A \end{array},$$

the given system of equations $(c_1, \cdots, c_{n+m})^t = A\ (c_1^p, \cdots, c_{n+m}^p)^t$ has solutions in $K^{sep}(K = \mathrm{Frac}(R))$. These solutions form a $\mathbb{F}_p$ vector space of dimension $m+n$ and the representation is the tensor product of two representations of respective dimensions m and n correponding to two block submatrices of A. Let $\{\xi_i = (\xi_{1i}, \cdots, \xi_{mi}) | 1 \le i \le m\}$ be a basis for the solution space corresponding to the upper left block A_{11} of A and let $g = \begin{pmatrix} \xi_{11} & \cdots & \xi_{1m} \\ \vdots & & \vdots \\ \xi_{m1} & \cdots & \xi_{mm} \end{pmatrix}$. Then we have $g = A_{11} g^{(p)}$. So $\det(g) = \det(A_{11}) \det(g)^p$, i.e. $\det(g)^{p-1} = (-1)^m t_{11}^{-1}$. Similarly we have $\det(h)^{p-1} = (-1)^n t_{11}^{-1}$ for the lower right matrix. Here the columns of h are solution vectors. Therefore if $\sigma \in \bar{\rho}(\mathrm{Gal}(K^{sep}/K))$ belongs to $\mathrm{GL}_m(\mathbb{F}_p) \times \{1\}$, namely acts trivially on h, then it will also act trivially on $\det(g)$, which implies that σ is an element of $\mathrm{SL}_m(\mathbb{F}_p) \times \{1\}$. On the other hand, note that the same argument also shows that the projections of $\bar{\rho}(\mathrm{Gal}(K^{sep}/K))$ to each factors are surjective.

3.4 Determinant of the monodromy

In this section, we prove the rest of the proposition 2.3.1, in other words that the image of local monodromy at the point in question is an open subgroup. More precisely, we prove the following.

Proposition 3.4.1. *Let $x = (A_x, \lambda_x, \iota_x)$ be as in Lemma 3.0.3 and assume that $p \geq 5$. In particular, $m < n$. Then the image of the local monodromy under the determinant map*

$$\det \circ \rho_G : \mathrm{Gal}(K^{sep}/K) \to \mathrm{GL}_m(\mathbb{Z}_p) \times \mathrm{GL}_n(\mathbb{Z}_p) \stackrel{\det}{\to} \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$$

contains $(1 + p^n Z_p) \times (1 + p^n Z_p)$.

The idea of proof of this lemma is to analyze the determinant of the local monodromy, modulo high powers of p.

Recall that our representation

$$\rho_G : \mathrm{Gal}(K^{sep}/K) \to \mathrm{GL}_m(\mathbb{Z}_p) \times \mathrm{GL}_n(\mathbb{Z}_p)$$

is given by the action of the Galois group $\mathrm{Gal}(\bar{K}/K)$ on the free $\mathbb{Z}_p$-module $\widetilde{M}^{F-Id}$ of rank $m + n$ which consists of the elements fixed by the Frobenius F. $\widetilde{M}$ is the Dieudonne module of the universal deformation of the point and is a direct sum

$\widetilde{M}^{F-Id} = \widetilde{M}_1^{F-Id} \bigoplus \widetilde{M}_2^{F-Id}$ of two Dieudonne submodules . Let

$$\{v_i = \sum_{j=1}^{m+n} a_{ji}\tilde{e}_j |\ 1 \le i \le m, a_{ji} \in W(\bar{K})\}$$

be a basis of the module $\widetilde{M}_1^{F-Id}$ for the basis $\tilde{e}_j$ of $\widetilde{M}_1$ and let $A = (a_{ij})$ be the $(m+n) \times m$ matrix whose (i,j) entry is a_{ij}.

Then, for an element τ of $\mathrm{Gal}(\bar{K}/K^{\mathrm{perf}})$ and for the basis $\{\tilde{e}_j, \tilde{f}_j\}$ of $\widetilde{M}$, if $(g_1, g_2) = \rho(\tau)$, we have

$$A^\tau = A\ g_1$$

where A^τ is the matrix obtained by applying τ to each entry of A. Hence, if we define a $(m \times m)$ matrix A' by truncating A up to m-th row, we have clearly

$$\det A'^\tau = \det A'\ \det \rho_1(\tau). \tag{3.4.1}$$

Similar statements hold for $\widetilde{M}_2^{F-Id}$ and in the following discussions we deal with $\widetilde{M}_1^{F-Id}$ and $\widetilde{M}_2^{F-Id}$ separately. First, we consider the determinant of the left factor (of rank m). For this, we restrict the p-divisible group $A[\wp_1^\infty]$ to the subscheme $\mathrm{Spec}(R_1) = \mathrm{Spec}(k[[t_{11}, \cdots, t_{m1}]])$ of the universal deformation space and show that the determinant of the local monodromy representation attached to the generic fiber $A_K[\wp_1^\infty]$ contains $1 + p^n\mathbb{Z}_p$. Here K is the field of fractions R_1 and we continue to use the same notations for the new Dieudonne module over K.

Then, each basis vector $v_j = (a_{1,j}, \cdots, a_{m+n,j})^{tr} (1 \le j \le m)$ satisfies $v_j = F(v_j)$, i.e.

$$\begin{pmatrix} a_{1,j} \\ \\ \vdots \\ \\ a_{m,j} \\ a_{m+1,j} \\ \\ \vdots \\ \\ a_{m+n,j} \end{pmatrix} = \left(\begin{array}{ccc|c|ccc|c} 0 & \cdots & 0 & T_{11} & 0 & \cdots & 0 & p \\ \hline & & & T_{21} & 0 & \cdots & 0 & 0 \\ & I_{m-1} & & \vdots & \vdots & & \vdots & \vdots \\ & & & T_{m1} & 0 & \cdots & 0 & 0 \\ \hline & & & 1 & 0 & \cdots & 0 & 0 \\ & & & & p & & & \\ & & & & & \ddots & & \vdots \\ & & & & & & p & 0 \end{array}\right) \begin{pmatrix} a^{\sigma}_{1,j} \\ \\ \vdots \\ \\ a^{\sigma}_{m,j} \\ a^{\sigma}_{m+1,j} \\ \\ \vdots \\ \\ a^{\sigma}_{m+n,j} \end{pmatrix}.$$

Here, $^{\sigma}$ is the Frobenius on the Witt vecotrs in $W(\bar{K})$. So for each j,

$$a_{ij} = \begin{cases} \sum_{l=1}^{i} T_{l1}^{\sigma^{i-l}} a_{m,j}^{\sigma^{i-l+1}} + p^n a_{m,j}^{\sigma^{n+i}} & \text{if } 1 \leq i \leq m \\ p^{i-m-1} a_{m,j}^{\sigma^{i-m}} & \text{if } m+1 \leq i \leq m+n. \end{cases}$$

As before, this becomes a single equation $f(a_{m,j}) = 0$ in $a_{m,j}$, where

$$f(x) = x - T_{m1}x^{\sigma} - T_{m-11}^{\sigma}x^{\sigma^2} - \cdots - T_{11}^{\sigma^{m-1}}x^{\sigma^m} - p^n x^{\sigma^{n+m}}. \tag{3.4.2}$$

Lemma 3.4.2. *For each $1 \leq i \leq m$, there exist elements b_i and $b_{i,n}$ of $W(\bar{K})$ such that*

(i) $a_{m,i} = b_i + p^n b_{i,n}$;

(ii) b_i satisfies $g(b_i) = 0$, where

$$g(x) = f(x) + p^n x^{\sigma^{m+n}} = x - T_{m1}x^{\sigma} - T_{m-11}^{\sigma}x^{\sigma^2} - \cdots - T_{11}^{\sigma^{m-1}}x^{\sigma^m}. \tag{3.4.3}$$

Furthermore, b_i is unique mod p^n.

The uniqueness of b_i mod p^n is obvious.

For the existence, fixing i we inductively show that for each $1 \leq l \leq n$, there exist elements $\tilde{\bar{b}}_{i,j}$ $(0 \leq j \leq l-1)$ and $b_{i,l}$ of $W(\bar{K})$ such that

(a)$a_{m,i} = \tilde{\bar{b}}_{i,0} + p\tilde{\bar{b}}_{i,1} + \cdots + p^{l-1}\tilde{\bar{b}}_{i,l-1} + p^l b_{i,l}$;

(b)$g(\tilde{\bar{b}}_{i,j}) = 0 \quad 1 \leq j \leq l-1$;

(c)$g(b_{i,l}) \equiv 0 \mod p$ if $l < n$.

Then $b_i = \tilde{\bar{b}}_{i,0} + p\tilde{\bar{b}}_{i,1} + \cdots + p^{n-1}\tilde{\bar{b}}_{i,n-1}$ and the same $b_{i,n}$ will satisfy (i) and (ii) of the lemma.

For the first term, we define $\bar{b}_{i,0} \in W(\bar{K})/pW(\bar{K}) = \bar{K}$ by $\bar{b}_{i,0} = \bar{a}_{m,i}$. Then since $\bar{g}(\bar{b}_{i,0}) = \bar{g}(\bar{a}_{m,i}) = \bar{f}(\bar{a}_{m,i}) = 0$ and both $g(x) = 0$ and $\bar{g}(x) = 0$ have rank m solution spaces, there exists a root $\tilde{\bar{b}}_{i,0}$ of $g(x) = 0$ in $W(\bar{K})$ such that the image of $\tilde{\bar{b}}_{i,0}$ in $W(\bar{K})/pW(\bar{K}) = \bar{K}$ is $\bar{b}_{i,0}$. Also there exists an element $b_{i,1}$ of $W(\bar{K})$ such that $a_{m,i} = \tilde{\bar{b}}_{i,0} + pb_{i,1}$. On the other hands, since $a_{m,i}$ is a root of $f(x)$, we have $f(a_{m,i}) = f(\tilde{\bar{b}}_{i,0} + pb_{i,1}) = g(\tilde{\bar{b}}_{i,0}) + pg(b_{i,1}) - p^n(\tilde{\bar{b}}_{i,0}^{\sigma^{n+m}} + pb_{i,1}^{\sigma n+m}) = pg(b_{i,1}) - p^n(\tilde{\bar{b}}_{i,0}^{\sigma^{n+m}} + pb_{i,1}^{\sigma n+m}) = 0$. This implies that $g(b_{i,1}) \equiv 0 \mod p$ and the first step of the induction is done.

Next, assume that for $1 \leq l \leq n-1$ we found $\tilde{\bar{b}}_{i,j}(1 \leq j \leq l-1)$ satisfying (ii) and $b_{i,l} \in W(\bar{K})$ such that $a_{m,i} = \tilde{\bar{b}}_{i,0} + p\tilde{\bar{b}}_{i,1} + \cdots + p^{l-1}\tilde{\bar{b}}_{i,l-1} + p^l b_{i,l}$ and $g(b_{i,l}) \equiv 0$ mod p.

Since $\bar{g}(\bar{b}_{i,l}) = 0$, as before, there exists a zero $\tilde{\bar{b}}_{i,l}$ of $g(x)$ in $W(\bar{K})$ such that the image of $\tilde{\bar{b}}_{i,l}$ in $W(\bar{K})/pW(\bar{K}) = \bar{K}$ is $\bar{b}_{i,l}$. Also there exists an element $b_{i,1+1}$

of $W(\bar{K})$ such that $b_{i,l} = \tilde{\bar{b}}_{i,l} + pb_{i,1+1}$. Clearly, for this choice of $b_{i,l+1}$, we have $a_{m,i} = \tilde{\bar{b}}_{i,0} + p\tilde{\bar{b}}_{i,1} + \cdots + p^l\tilde{\bar{b}}_{i,l} + p^{l+1}b_{i,l+1}$ and so to establish the $(l+1)$-st step of the inductive argument, it remains to show that $g(b_{i,l}) \equiv 0 \mod p$ if $l+1 < n$. But $f(a_{m,i}) = f(\tilde{\bar{b}}_{i,0} + p\tilde{\bar{b}}_{i,1} + \cdots + \tilde{\bar{b}}_{i,l} + p^{l+1}b_{i,l+1}) = \sum_{r=0}^{l} p^r g(\tilde{\bar{b}}_{i,r}) + p^{l+1}g(\tilde{\bar{b}}_{i,l+1}) - p^n(\sum_{r=0}^{l+1} p^r \tilde{\bar{b}}_{i,r}^{\sigma^{m+n}} + p^{l+1}b_{i,l+1}^{\sigma^{m+n}}) = p^{l+1}g(\tilde{\bar{b}}_{i,l+1}) - p^n(\sum_{r=0}^{l+1} p^r \tilde{\bar{b}}_{i,r}^{\sigma^{m+n}} + p^{l+1}b_{i,l+1}^{\sigma^{m+n}}) = 0$. Therefore, by reading modulo p^{l+2}(note that $l+1 < n$), we find $g(\tilde{\bar{b}}_{i,l+1}) \equiv 0 \mod p$.

Lemma 3.4.3. *Let* $\det \circ \rho_1 : \mathrm{Gal}(\bar{K}/K) \to \mathbb{Z}_p^\times$ *be the determinant composed with the first projection of the local monodromy representation. If* K_i *is an extension of* K *satisfying* $\rho_1^{-1}(1 + p^i\mathbb{Z}_p) = \mathrm{Gal}(\bar{K}/K_i)$ $(i \geq 1,\ K_0 = K)$, *then we have*

$$\mathrm{Gal}(K_i/K_{i-1}) = \begin{cases} \mathbb{F}_p^\times & i = 1 \\ 1 & 2 \leq i \leq n \\ \mathbb{F}_p & i = n+1 \end{cases} .$$

From Lemma 3.4.2 (i), $\det A'$ is congruent, mod p^n, to the determinant of the matrix W whose i-th column W_i is

$$W_i = (T_{11}b_i^\sigma,\ T_{11}^\sigma b_i^{\sigma^2} + T_{21}b_i^\sigma,\ \cdots,\ T_{11}^{\sigma^{m-1}}b_i^{\sigma^m} + \cdots + T_{m1}b_i^\sigma)^{tr}.$$

Each column vector $W_i (1 \leq i \leq m)$ satisfies the equation

$$W_i = F' W_i^\sigma, \text{ where } \quad F' = \left(\begin{array}{ccc|c} 0 & \cdots & 0 & T_{11} \\ \hline & & & T_{21} \\ & I_{m-1} & & \vdots \\ & & & T_{m1} \end{array}\right).$$

As for $a_{m,i}$, this is seen by solving $X = F'X^\sigma$ with $X = (x_1, \cdots, x_m)^{tr}$ for $x_m = b_i$. Hence, we have $W = F'W^\sigma$ and $\det W^{(\sigma-1)} = (\det F')^{-1} = ((-1)^{m-1}T_{11})^{-1}$. If $\det W = (x_0, x_1, \cdots) \in W(\bar{K})$, then one has

$$((-1)^{m-1}t_{11})^{-p^j} x_j = x_j^p \quad (j \geq 0).$$

Since $\bar{b}_i (1 \leq i \leq m)$ are linearly independent over $\mathbb{F}_p$ and so $\det W \neq 0$, $x_0 = (-1)^{(m-1)/(p-1)} t_{11}^{1/(p-1)}$ for a $(p-1)$-st root $t_{11}^{1/(p-1)}$ of t_{11} and there exist elements $\alpha_j\,(j \geq 0)$ of $\mathbb{F}_p$ such that $x_j = \alpha_j((-1)^{\frac{m-1}{p-1}} t_{11}^{\frac{1}{(p-1)}})^{-p^j}$. Therefore $\mathrm{Gal}(\bar{K}/K(t_{11}^{1/(p-1)}))$ acts trivially on $\det W$ and the Galois representation

$$\begin{aligned} \rho_1 : \mathrm{Gal}(\bar{K}/K) &\quad\rightarrow\quad \mathbb{Z}_p^\times/(1 + p^n\mathbb{Z}_p) \\ \tau &\quad\longmapsto\quad (\det A')^\tau / \det A' \,(\mathrm{mod}\, p^n) = (\det W)^\tau / \det W \,(\mathrm{mod}\, p^n) \end{aligned}$$

has finite image of cardinality $p - 1$ with Kernel $\mathrm{Gal}(\bar{K}/K(t_{11}^{1/(p-1)}))$. Namely, $K_n = K(t_{11}^{1/(p-1)})$. But, $\det A' \equiv \det W \equiv x_0 = \alpha_0((-1)^{\frac{m-1}{p-1}} t_{11}^{\frac{1}{(p-1)}})^{-1} \mod p$ and so $\mathrm{Gal}(\bar{K}/K) \rightarrow \mathbb{Z}_p^\times/(1 + p\mathbb{Z}_p)$ is surjective and $K_1 = K(t_{11}^{1/(p-1)})$. This proves the claim of Lemma up to $i = n - 1$.

To consider the case $i = n$, we analyze $b_{i,n}$ in Lemma 3.4.2 more closely.

The j-th column A_j of A' is

$$A_j = \begin{pmatrix} T_{11} a_{m,j}^\sigma & & & & + p^n a_{m,j}^{\sigma^{n+1}} \\ T_{11}^\sigma a_{m,j}^{\sigma^2} & + T_{21} a_{m,j}^\sigma & & & + p^n a_{m,j}^{\sigma^{n+2}} \\ \vdots & \vdots & & & \vdots \\ T_{11}^{\sigma^{m-1}} a_{m,j}^{\sigma^m} & + T_{21}^{\sigma^{m-2}} a_{m,j}^{\sigma^{m-1}} & + \cdots & + T_{m1} a_{m,j}^\sigma & + p^n a_{m,j}^{\sigma^{n+m}} \end{pmatrix}.$$

From $a_{m,j} = b_j + p^n b_{j,n}$ (Lemma 3.4.2 (i)), we can write $\det A' = C_n + p^n D_n$, where C_n is the determinant of the matrix whose (i,j) entry c_{ij} is $c_{ij} = g_i(b_j)$, with $g_i(x) = \sum_{l=1}^{i} T_{l1}^{\sigma^{i-l}} x^{\sigma^{i-l+1}}$ and

$$D_n \equiv \sum_{j=1}^{m} \det \begin{pmatrix} g_1(b_1) & \cdots & g_1(b_{j,n}) + b_j^{\sigma^{n+1}} & \cdots & g_1(b_m) \\ \vdots & & \vdots & & \vdots \\ g_m(b_1) & \cdots & g_m(b_{j,n}) + b_j^{\sigma^{n+m}} & \cdots & g_m(b_m) \end{pmatrix} \mod p$$

Then by elementary row operations, it is easily shown that

$$C_n \quad = \quad T_{11}^{1+\sigma+\cdots+\sigma^{m-1}} \det \begin{pmatrix} b_1^{\sigma} & \cdots & b_i^{\sigma} & \cdots & b_m^{\sigma} \\ \vdots & & \vdots & & \vdots \\ b_1^{\sigma^m} & \cdots & b_i^{\sigma^m} & \cdots & b_m^{\sigma^m} \end{pmatrix}.$$

Hence, for $\tau \in \mathrm{Gal}(\bar{K}/K_n)$,

$$\begin{aligned} & (\det A')^{\tau} \cdot (\det A')^{-1} \\ = \;& (C_n + p^n D_n)^{\tau} \cdot (C_n + p^n D_n)^{-1} \\ = \;& C_n + p^n (D_n)^{\tau}) \cdot (C_n(1 + p^n C_n^{-1} D_n))^{-1} \qquad C_n^{\tau} = C_n \\ \equiv \;& (C_n + p^n (D_n)^{\tau}) \cdot C_n^{-1}(1 - p^n C_n^{-1} D_n) \qquad \mod p^{n+1} \\ = \;& (1 + p^n C_n^{-1}(D_n)^{\tau}) \cdot (1 - p^n C_n^{-1} D_n) \\ \equiv \;& 1 + p^n C_n^{-1}((D_n)^{\tau} - D_n) \qquad \mod p^{n+1} \end{aligned}$$

Note that for $\tau \in \mathrm{Gal}(\bar{K}/K_n)$, we have $C_n^{\tau} = C_n$, not just $C_n^{\tau} \equiv C_n \mod p^n$ because $C_n = T_{11}^{-(1+\sigma+\cdots+\sigma^{m-1})} \det W$.

So under the canonical isomorphism $(1 + p^n \mathbb{Z}_p)/(1 + p^{n+1}\mathbb{Z}_p) \cong \mathbb{F}_p$, the restriction (still denoted by ρ_1) of ρ_1 to the subquotient $\mathrm{Gal}(K_{n+1}/K_n)$ becomes

$$\begin{aligned}\rho_1 : \mathrm{Gal}(K_{n+1}/K_n) &\rightarrow (1+p^n\mathbb{Z}_p)/(1+p^{n+1}\mathbb{Z}_p) \cong \mathbb{F}_p \\ \tau &\longmapsto \bar{C}_n^{-1}((\bar{D}_n)^\tau - \bar{D}_n)\end{aligned}$$

On the other hand, from Lemma 3.4.2, we have

$$\begin{aligned} 0 = f(a_{m,j}) &= f(b_j + p^n b_{j,n}) \\ &= g(b_j) + p^n g(b_{j,n}) - p^n(b_j^{\sigma^{m+n}} + p^n b_{j,n}^{\sigma^{m+n}}) \\ &= p^n g(b_{j,n}) - p^n(b_j^{\sigma^{m+n}} + p^n b_{j,n}^{\sigma^{m+n}}) \end{aligned}$$

Reading modulo p^n, we obtain $g(b_{j,n}) - b_j{}^{\sigma^{m+n}} \equiv 0 \mod p$.

In other words, $\bar{b}_{j,n}$ is a zero of polynomial

$$h_j(x) = \bar{g}(x) - \bar{a}_{m,j}^{p^{m+n}} = -\bar{a}_{m,j}^{p^{m+n}} + x - t_{m1}x^p - t_{m-11}^p x^{p^2} - \cdots - t_{11}^{p^{m-1}} x^{p^m}.$$

with coefficients in $K(\bar{a}_{m,j})$. This is of Artin-Schreier type; if α_j is one root, then every other root is $\alpha_j + \sum c_{lj}\bar{a}_{m,l}$ for some $c_{lj} \in \mathbb{F}_p$. Also, for any choice of roots $\{\alpha_j | 1 \le j \le m\}$ for each $h_j(x)$, $\{\alpha_j | 1 \le j \le m\}$ are algebraically independent over k since $\{\bar{a}_{m,j} | 1 \le j \le m\}$ are thus and hence for any $(c_{lj}) \in \mathrm{M}_{m\times m}(\mathbb{F}_p)$, $\tau : \alpha_j \mapsto \alpha_j + \sum c_{lj}\bar{a}_{m,l}$ defines an element of $\mathrm{Aut}(K(\alpha_1, \cdots, \alpha_m))$ which fixes the coefficients of $h_j(x)$'s. Since $K_n = \cdots = K_1 = K(t_{11}^{1/(p-1)})$ is of degree $p-1$ over K,

$$\begin{aligned} &\mathrm{Gal}(K_n(\alpha_1, \cdots, \alpha_m)/K_n(\bar{a}_{m,j}^{p^{m+n}})) \\ =\ &\mathrm{Gal}(K(\alpha_1, \cdots, \alpha_m)/K(\bar{a}_{m,j}^{p^{m+n}})) \\ \cong\ &\mathrm{M}_{m\times m}(\mathbb{F}_p) = \mathfrak{gl}_{m,\mathbb{F}_p}. \end{aligned}$$

If we let $g = \rho(\tau) \in \mathrm{M}_{n\times n}(\mathbb{F}_p)$ such that $\tau(\alpha_j) = \alpha_j + \sum_l g_{l,j}\bar{a}_{m,l}$, we have

$$\begin{aligned}
\bar{D}_n^\tau &= \sum_{j=1}^m \det\begin{pmatrix} \bar{g}_1(\bar{b}_1) & \cdots & \bar{g}_1(\bar{b}_{j,n}) + \bar{b}_j^{p^{n+1}} + \sum_l g_{lj}\bar{g}_1(\bar{b}_l) & \cdots & \bar{g}_1(\bar{b}_m) \\ \vdots & & \vdots & & \vdots \\ \bar{g}_m(\bar{b}_1) & \cdots & \bar{g}_m(\bar{b}_{j,n}) + \bar{b}_j^{p^{n+m}} + \sum_l g_{lj}\bar{g}_m(\bar{b}_l) & \cdots & \bar{g}_m(\bar{b}_m) \end{pmatrix} \\
&= \bar{D}_n + \mathrm{Tr}(g)\bar{C}_n
\end{aligned}$$

This shows that

$$\begin{aligned}
\mathrm{Gal}(K_n(\alpha_1, \cdots, \alpha_m)/K_{n+1}(\bar{a}_{m,i}^{p^{m+n}})) &\cong \mathfrak{sl}_{m,\mathbb{F}_p}, \\
\mathrm{Gal}(K_{n+1}(\bar{a}_{m,i}^{p^{m+n}})/K_n(\bar{a}_{m,i}^{p^{m+n}})) &= \mathbb{F}_p, \quad \text{and} \\
\mathrm{Gal}(K_{n+1}/K_n) &= \mathbb{F}_p.
\end{aligned}$$

Now we consider the second factor and finish the proof of Prop 3.4.1. We observe that by restriction ρ to the subscheme $\mathcal{Y}_\in = \mathrm{Spf}(R_2) \hookrightarrow \mathrm{Spf}(R) = \mathrm{Spf}(k[[t_{ij}, u_{ij}]])$ defined by $R_1 = R/(u_{ij},\ t_{kl}|2 \leq j \leq n, (k,l) \neq (1,1))$, we obtain completely analogous results for the second factor of the original local monodromy representation.

If we let L be the fraction field of R_2 and define L_i similarly, $K(T_{11}^{1/(p-1)}) \cap L(T_{11}^{1/(p-1)}) = k(T_{11}^{1/(p-1)})$ and for the field F

$$\begin{aligned}
F &= \mathrm{Frac}(k[[t_{ij}, u_{rs}]])(t_{11}^{-1/(p-1)}) \\
&= \mathrm{Frac}(k[[t_{11}^{1/(p-1)}, t_{ij}, u_{rs}]]),
\end{aligned}$$

it is easy to see that the two field compositums $F \cdot K^{alg}$ and $F \cdot L^{alg}$ are linearly disjoint over F. Therefore, the determinant of $\rho(\mathrm{Gal}(F^{alg}/F))$ surject onto $(1 +$

$p^{n-1}\mathbb{Z}_p) \times (1 + p^{n-1}\mathbb{Z}_p)/(1 + p^n\mathbb{Z}_p) \times (1 + p^n\mathbb{Z}_p)$ and by Lemma 3.1.2, we are done with the proof of Lemma.

3.5 Proof of Corollary 2.3.2

We prove that the global p-adic monodromy is the entire $\mathrm{GL}_m(\mathbb{Z}_p) \times \mathrm{GL}_n(\mathbb{Z}_p)$.

Lemma 3.5.1. *Let K be a field of characteristic p with an algebraic closure K^{alg}. Let $\phi = (y_o, y_1, \cdots)$ be an invertible element of $W(K)$ with components $y_i \in K$. Then, for $a \in W(K^{alg})^\times$ satisfying $a = \phi\, a^\sigma$, the Galois representation*

$$\begin{aligned} \rho : \mathrm{Gal}(K^{alg}/K) &\rightarrow \mathbb{Z}_p^\times \\ \tau &\mapsto a^\tau \cdot a^{-1} \end{aligned}$$

is surjective if and only if the following two conditions hold

(i) $y_0 \notin K^{p-1}$, where K^{p-1} is the subset of K consisting of $(p-1)$-th powers of elements of K ;

(ii) $X^p - X + y_1 y_0^{-p}$ (equiv. $X^p - y_0^{p-1} X + y_1$) is irreducible over K.

Note that this representation does not depend on the choice of a since any such two differ by an element of $\mathbb{Z}_p^\times$. For proof, we resort again to the lemma 3.1.2 and show that the Galois representation maps surjetively modulo p and p^2 precisely when the conditions (i) and (ii) hold.

We first claim that there exist elements a_0, a_1, b_2 of $W(K^{alg})$ such that $a = a_0 + pa_1 + p^2 b_2$. Indeed, if $a = (x_0, x_1, \cdots)$ with $x_i \in K^{alg}$, let us define $a_0 = (x_0, 0, \cdots)$, $a_1 = (x_1^{1/p}, 0, \cdots)$. Then since $a \equiv a_0 \pmod{p}$, we have $a - a_0 = pa_1'$ for some $a_1' \in W(K^{alg})$. But $a - a_0 = (0, x_1, *, \cdots) = p(x_1^{1/p}, *, \cdots)$, i.e. $a_1' \equiv a_1 \pmod{p}$ and so $a - a_0 - pa_1 \equiv 0 \pmod{p}$, which establishes the claim.

Solving $a = \phi\, a^{\sigma}$ for the first two Witt components, we obtain

$$x_0 = y_0 x_0^p, \quad x_1 = x_1^p y_0^p + y_1 x_0^{p^2}.$$

The condition (i) is clearly equivalent to that $K_1 = K(x_0) = K(y_0^{-1/(p-1)})$ is separable over K of degree $p-1$ and hence to the statement that the Galois representation maps surjectively modulo p. Next, since $a_0^{\tau} = a_0$ for $\tau \in \mathrm{Gal}(K^{alg}/K_1)$, one can easily check that $a^{\tau} \cdot a^{-1} \equiv 1 + p a_0^{-1}(a_1^{\tau} - a_1) \pmod p$ and the Galois representation induced on the second level becomes

$$\begin{aligned} \rho_2 : \mathrm{Gal}(K^{alg}/K_1) &\rightarrow (1 + p\mathbb{Z}_p)/(1 + p^2\mathbb{Z}_p) \cong \mathbb{F}_p \\ \tau &\mapsto \bar{a}_0^{-1}(\bar{a}_1^{\tau} - \bar{a}_1) = x_0^{-1}((x_1^{1/p})^{\tau} - (x_1^{1/p})) \\ &= x_0^{-p}(x_1^{\tau} - x_1) = (y_0^{p/(p-1)} x_1)^{\tau} - (y_0^{p/(p-1)} x_1) \end{aligned}$$

But $x_1 = x_1^p y_0^p + y_1 x_0^{p^2}$ becomes the following Artin-Schreier equation in $y_0^{p/(p-1)} x_1$

$$(y_0^{p/(p-1)} x_1)^p - (y_0^{p/(p-1)} x_1) + y_1 y_0^{-p} = 0.$$

Then the condition (ii) is the necessary and sufficient condition for ρ_2 to be surjective and the conclusion of the lemma follows.

To prove the corollary, by Theorem 2.3.1, it suffices to show that the determinant of the p-adic monodromy is $\mathbb{Z}_p^{\times} \times \mathbb{Z}_p^{\times}$ and we can show this separately for each factor. For this, we will find another point in our moduli space such that the determinant of its local monodromy contains $\mathbb{Z}_p^{\times} \times 1$. Similar argument will work for the other factor.

Let us consider a point $x = (A_x, \lambda_x, \iota_x)$ whose covariant Dieudonne module $M = M_1 \oplus M_2$ has the following display.

$$M_1 \quad : \quad F(e_i) = \begin{cases} e_{i+1} & (1 \leq i < m) \\ e_{i+1} + e_{i+2} & i = m \end{cases}, \; e_i = \begin{cases} V(e_{i+1}) & (m < i \leq m+n) \\ V(e_1) & i = m+n \end{cases},$$
$$M_2 \quad = \quad M_1^t.$$

Note that this Dieudonne module is isogenous to the Dieudonne module of the point defined in Lemma 3.0.3, which guarantees the existence of similarity with the point $x = (A_x, \lambda_x, \iota_x)$.

The display matrix of the universal deformation of the corresponding formal group is

$$\left(\begin{array}{ccc|c|c} 0 & \cdots & 0 & T_{11}+T_{1n} & * \\ \hline & & & T_{21}+T_{2n} & * \\ & I_{m-1} & & \vdots & * \\ & & & T_{m1}+T_{mn} & * \\ \hline * & * & * & * & * \end{array}\right).$$

The determinant ϕ of the $m \times m$ truncated matrix is

$$T_{11} + T_{1n} = (t_{11} + t_{1n}, \frac{1}{p}(t_{11}^p + t_{1n}^p - (t_{11} + t_{1n})^p), *, \cdots).$$

It suffices to show that $y_0 = s + t, y_1 = \frac{1}{p}(s^p + t^p - (s+t)^p)$ satisfies the condition (i) and (ii) for the fraction field K of $R = k[[s,t]](\bar{k} = k)$. (i) is easy to verify. For (ii), since the equation $X^p - y_0^{p-1}X + y_1$ is defined over R, it is enough to show the irreducibility over the quotient $k[[s]] = k[[s,t]]/(t - \bar{c}s)$ of R for $c \in \mathbb{Z}$ such that p^2

does not divide $1 + c^p - (1 + c)^p$. Then the Newton polygon of $X^p - y_0^{p-1}X + y_1$ becomes the straight line joining $(0, p)$ and $(p, 0)$ and thus $X^p - y_0^{p-1}X + y_1$ is irreducible over the discrete valuation ring $k[[s]]$.

The other factor can be dealt with similarly.

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