# STRATIFICATIONS ON MODULI SPACES OF ABELIAN VARIETIES IN POSITIVE CHARACTERISTIC

Jeffrey D. Achter

A Dissertation in Mathematics

Presented to the Faculties of the University of Pennsylvania in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

1998

______________________________
Supervisor of Dissertation

______________________________
Graduate Group Chairperson

## Acknowledgments

This thesis would not have been written without Ching-Li Chai, whom I thank for patiently introducing me to abelian varieties and awaiting this work. Chia-Fu Yu has been no less patient, and I thank him for both general discussions and technical remarks.

The University of Pennsylvania has been a warm place to learn and do mathematics. Let me thank the faculty in general and Steve Shatz in particular for support and guidance.

I started work on this project in autumn of 1995 at Harvard University, whose hospitality I enjoyed. I particularly benefited from conversations with Johan de Jong.

I am grateful to Scott Pauls and Rachel Pries for their friendship and support, both mathematical and otherwise; and Eileen Anderson, Geoff Pike and Todd Sinai for theirs, mainly otherwise.

Final thanks are due to my parents, Kathy and Gene Achter, and my brother, Mike, for their love and encouragement.

ABSTRACT

# STRATIFICATIONS ON MODULI SPACES OF ABELIAN VARIETIES IN POSITIVE CHARACTERISTIC

Jeffrey D. Achter

Ching-Li Chai

Over a field of positive characteristic $p$, we consider moduli spaces of polarized abelian varieties equipped with an action by a ring unramified at $p$. Using deformation theory, we show that ordinary points are dense in each of the following situations: the polarization is separable; the polarization is mildly inseparable, and the ring of endomorphisms is a totally real number field; or the polarization is arbitrary, and the ring is a real quadratic field acting on abelian fourfolds. We introduce a new invariant which measures the extent to which a polarized Dieudonné module admits an isotropic splitting lifting the Hodge filtration, and use it to explain the singularities arising from mildly inseparable polarizations.

# Contents

The rich arithmetic of abelian varieties comes from playing the group law against the underlying geometry. While to some extent any abelian variety behaves like a classical complex projective torus, intriguing features arise in positive characteristic. Apparently simple questions about the structure of the torsion group reveal decisively new phemonena.

Consider an abelian variety $X$ of dimension $g$ over an algebraically closed field $k$ of positive characteristic $p$. There is a number $\rho$ between zero and $g$, the $p$-rank of $X$, such that the number of $p$-torsion points in the abelian group $X(k)$ is $p^\rho$. When $\rho$ is as large as possible, the abelian variety is said to be ordinary.

The classification by $p$-rank induces a stratification on any family of abelian varieties. This dissertation explores this stratification on moduli spaces of abelian varieties with a given endomorphism ring.

The problems considered here arise in two distinct but closely related lines of inquiry. On one hand, Deuring shows that the generic elliptic curve is ordinary [Deu]. Mumford announces [Mum], and Norman and Oort prove [N-O], the obvious generalization of this statement to higher dimension: ordinary points are dense in the moduli space of polarized abelian varieties.

On the other hand, moduli spaces of PEL type – those parametrizing abelian varieties with certain polarization, endomorphism and level-structure data – are important spaces in their own right. Roughly speaking, when the characteristic of the ground field is relatively prime to the moduli problem, the resulting space is smooth. When the characteristic resonates with the moduli functor, things get interesting and the spaces get singular. The singularities of such spaces have attracted considerable attention. For results along these lines see, e.g., [dJ] and [Nor] for moduli spaces with inseparable polarizations; [D-P] and [R-Z] for endomorphism rings ramified at $p$; and [C-N], [D-R] and [KaMa] for $p$ level structure.

With this context, I set out to understand the locus parametrizing ordinary abelian schemes in moduli spaces of PEL type. The main results in this direction are 3.3, 4.2 and 5.1. Along the way, we are able to say something about the geometry of these spaces.

This paper is organized in the following way. The first section gives the precise definition of the moduli stacks in question. The second section collects a number of results on the deformation theory of abelian varieties. As we avail ourselves of techniques from Kodaira-Spencer, Dieudonné and crystalline theories, we present a utilitarian review of their main theorems. Subsequently we extend these techniques to the deformation of an abelian variety with given endomorphism structure.

The final three sections show that ordinary points are dense in moduli spaces of PEL type under varying hypotheses. Section three shows directly that ordinary points are dense in any such smooth space. Section four proves a similar result for spaces with singularities coming from mildly inseparable polarizations. We also give a complete description of the singularities which arise. The final section uses slightly different techniques to examine a slightly different class of spaces; at the expense of serious restrictions on the type of endomorphism ring, we allow arbitrarily inseparable polarizations.

# 1 Moduli spaces

This thesis investigates moduli spaces for polarized abelian varieties equipped with endomorphisms. These spaces are defined in the following way.

Let $O_B$ be an order in a finite-dimensional $\mathbb{Q}$-algebra $B$ with positive involution $*$. Let $E$ be the reflex field of $B$, essentially the field of traces of elements of $B$ on the representation space Lie$(X)$ below, and let $\Delta$ be the product of all primes of $E$ lying over primes in $\mathbb{Q}$ which ramify in $B$ or $E$. For natural numbers $g$ and $d$ we denote by $\widetilde{\mathcal{A}}_{g,d}^{O_B}$ the category of triples $(X/S, \iota, \lambda)$ where

i. $X \to S \to \operatorname{Spec} O_E[\frac{1}{\Delta}]$ is an abelian scheme of relative dimension $g$.

ii. $O_B \overset{\iota}{\hookrightarrow} \operatorname{End}(X)$ is a ring homomorphism taking 1 to id $X$, so that Lie$(X)$ is a free $O_B \otimes O_S$-module.

iii. $X \overset{\lambda}{\to} X^\vee$ is a polarization of degree $d^2$, taking the given involution on $O_B$ to the Rosati involution of End$(X)$.

Recall that $X \to S$ is an abelian scheme if it is a smooth proper group scheme with [geometrically] connected fibers.

Fix an algebraically closed field $O_E[\frac{1}{\Delta}] \to k$ of characteristic $p > 0$. Denote the reduction of the global moduli space modulo $p$ by

$$\mathcal{A}_{g,d}^{O_B} \overset{\text{def}}{=} \widetilde{\mathcal{A}}_{g,d}^{O_B} \times_{\operatorname{Spec} O_E[\frac{1}{\Delta}]} \operatorname{Spec} k.$$

**Remark 1.1** The demanded compatibilities in (ii) and (iii) are quite reasonable requests of our moduli space. The freeness constraint in (ii) expresses one instance of Kottwitz's “determinantal condition” [Kott]. While modifying this condition still yields a reasonable moduli space, any other such condition forbids the existence of ordinary points. Moreover, $\mathrm{Lie}(X)$ is always free over $O_B \otimes O_S$ if $d$ is invertible on $S$.

It may be worth making (iii)'s meaning explicit, too. An ample line bundle $\mathcal{L}$ on an abelian variety $X$ over a field $k$ induces an isogeny $\phi_{\mathcal{L}} : X \to X^\vee \stackrel{\text{def}}{=} \mathrm{Pic}^0(X)$, $x \mapsto \mathcal{L} \otimes T_x^* \mathcal{L}^{-1}$. An isogeny arising in this way is called a polarization of $X/k$. If $X$ is an abelian scheme over $S$, then a polarization of $X$ is a map $\lambda : X \to X^\vee$ which is a polarization of abelian varieties at every geometric point of $S$. The degree of a polarization is simply its degree as an isogeny, that is, the rank of its kernel. Any polarization induces a Rosati involution on $\mathrm{End}(X) \otimes \mathbb{Q}$, defined by $\alpha^\dagger = \lambda^{-1} \circ \alpha^\vee \circ \lambda$. We insist that, for any $b \in O_B$, $\iota(b^*) = \iota(b)^\dagger$.

The functor $(X/S, \iota, \lambda) \mapsto S$ clearly reveals $\widetilde{\mathcal{A}}_{g,d}^{O_B}$ as a fibered category over $\mathsf{Sch}_{O_E[\frac{1}{\Delta}]}$.

**Theorem 1.2** The category $\widetilde{\mathcal{A}}_{g,d}^{O_B}$ is an algebraic stack over $O_E[\frac{1}{\Delta}]$.

**Proof** The sketch in Theorème 1.20 of [Rap], which treats the case where $O_B$ is a totally real field of dimension $g$, is a standard exegesis of Artin's method which works for general $B$. A standard class-number argument, which I learned from Chia-Fu Yu, shows that the forgetful functor $\widetilde{\mathcal{A}}_{g,d}^{O_B} \xrightarrow{\phi} \widetilde{\mathcal{A}}_{g,d}$ is quasifinite. Indeed, one can directly prove that, for any pair of orders in $\mathbb{Q}$-algebras with positive involutions, $\mathrm{Hom}((O_{B_1}, *_1), (O_{B_2}, *_2))$ is finite. Moreover, a rigidity statement on homomorphisms of abelian varieties ([F-C], I.2.7) shows $\phi$ is proper, too. So $\widetilde{\mathcal{A}}_{g,d}^{O_B}$ is finite over $\widetilde{\mathcal{A}}_{g,d}$, itself well known to be an algebraic stack. Thus, $\widetilde{\mathcal{A}}_{g,d}^{O_K}$ is an algebraic stack.

**Remark 1.3** Given the introductory remarks on problems of PEL type, the reader may reasonably wonder at the absence of level structure in these moduli problems. Level data has been omitted here, as such structure has no effect on the local arguments used throughout. Indeed, all results proved for $\widetilde{\mathcal{A}}_{g,d}^{O_B}$ are true for moduli spaces of polarized $O_B$-abelian varieties with given prime-to-$p$ level structure. If the level structure is sufficiently fine, the associated fine moduli space is actually a scheme; this may afford some small psychological comfort to the reader.

## 2 How to deform an abelian variety

For the most part, we will attempt to understand $\mathcal{A}_{g,d}^{O_B}$ by studying its local behavior. Roughly speaking, the local ring at a $k$-point of this space represents the deformation space of some triple $(X/k, \iota, \lambda)$. There are several approaches to deforming an abelian variety in positive characteristic; we describe three such here. Notwithstanding the compatibilities they must obviously share, each methodology renders different information accessible. Hopefully, applying these varied techniques simultaneously will let us apprehend "the" deformation theory of a polarized $O_B$-abelian variety.

Kodaira and Spencer studied deformations of complex analytic spaces. Grothendieck realized their techniques work in the algebraic category, too. Although it is difficult to use these methods for detailed local computations, they quickly yield coarse information on the relevant deformation spaces' structure.

A theorem of Serre and Tate assures us that to deform an abelian variety in characteristic $p$ is to deform its $p$-divisible group. The Cartier-Dieudonné theory associates to [deformations of] a $p$-divisible group its Dieudonné module, a $\sigma$-linear algebraic object. As this module is defined even for a deformation over an arbitrary ring, Dieudonné theory allows us [in principle] to explicitly compute the universal deformation space of an abelian variety.

The third approach to deforming an abelian variety relies on crystalline cohomology. Over the complex numbers, to deform an abelian variety is to deform its associated Hodge structure. The crystalline cohomology provides a positive-characteristic analogue for this procedure. While amenable to computation, it is only available for deformations over divided power extensions of $k$. In practice, this means crystalline cohomology only lets us compute the leading terms of the local equations at a point $(X, \iota, \lambda)$. We will exhibit a large class of spaces, however, in which these leading terms are enough to determine the local geometry of $\mathcal{A}_{g,d}^{O_B}$.

In this section we provide rough introductions for each of these approaches. While we provide precise references for the results we quote, a true history is beyond our purview and these references are not necessarily to the original sources. The review of each method is supplemented with new results adapting the deformation theory to the presence of endomorphism rings.

## 2.1 Kodaira-Spencer theory

Kodaira and Spencer studied the deformations of a complex analytic manifold $Z$. Such an object may be described as a topological space equipped with analytic charts $\{D^n \overset{\phi_\alpha}{\hookrightarrow} Z\}$ and compatibilities $\phi_{\alpha\beta}$. To deform the complex structure of $Z$ is to deform the overlap data. Thus, first-order deformations of $Z$ are parametrized by $H^1(Z, \mathcal{T}_Z)$ [KS].

A similar argument works for a general algebraic variety over a given field, and in particular for an abelian variety $X/k$. To preserve the analogy with the complex case, define the equicharacteristic deformation functors as follows. Let $\mathsf{Art}_p(k)$ be the category of Artin local $k$-algebras of characteristic $p$. Then $\mathrm{Def}(X)$ is the covariant functor $\mathsf{Art}_p(k) \to \mathsf{Set}$ taking $R$ to the set of all pairs $(\widetilde{X}/R, \phi)$, where $\widetilde{X}/R$ is an abelian scheme and $\widetilde{X} \times_{\operatorname{Spec} R} \operatorname{Spec} k \xrightarrow{\phi} X$ is an isomorphism. The subfunctors $\mathrm{Def}(X, \lambda)$ and $\mathrm{Def}(X, \iota, \lambda)$ are defined analogously.

An algebraic version of Kodaira and Spencer's argument, written quite explicitly in [Oort], shows that

$$\begin{aligned} \mathrm{Def}(X)(k[\epsilon]/(\epsilon^2)) &= H^1(X, \mathcal{T}_X) && \text{Kodaira-Spencer} \\ &\cong H^1(X, \mathcal{O}_X) \otimes_k T_e X && X \text{ an abelian variety} \\ &\cong T_e X \otimes_k T_e X^\vee \end{aligned}$$

Here, $T_e X$ is the tangent space of $X$ at its identity element. In fact, formal deformations of an abelian variety are unobstructed, and the deformation functor is pro-represented by a power series ring.

$$\begin{aligned} \mathrm{Def}(X) &\cong \widehat{\mathrm{Symm}}^\bullet (T_e X^\vee \otimes_k T_e X) \\ &\cong k[\![t_{11}, \cdots, t_{gg}]\!] \end{aligned}$$

However, this isn't quite the deformation functor of interest to the algebraic geometer. Just as we might ask for deformations of $Z/\mathbb{C}$ which support *algebraic* as well as *analytic* structures, we seek formal deformations which are actually algebraizable. An existence theorem of Grothendieck says that it suffices to consider deformations of $X$ and a polarization $\lambda$. Roughly speaking, the obstruction to lifting a polarization lives in $H^2(X, \mathcal{O}_X) \cong H^1(X, \mathcal{O}_X) \wedge_k H^1(X, \mathcal{O}_X)$. More precisely, there are power series $a_1, \cdots, a_{\binom{g}{2}}$ so that

$$\mathrm{Def}(X, \lambda) \cong \frac{k[\![t_{11}, \cdots, t_{gg}]\!]}{(a_1, \cdots, a_{\binom{g}{2}})}.$$

We can extend this line of analysis to the study of $\mathrm{Def}(X, \iota, \lambda)$. The first-order deformations are now parametrized by $T_e X \otimes_{k \otimes \mathcal{O}_B} T_e X^\vee$, and the obstruction to lifting a polarization lives in $H^1(X, \mathcal{O}_X) \wedge_{k \otimes \mathcal{O}_B} H^1(X, \mathcal{O}_X)$.

**Theorem 2.1** Let $s = \dim_k T_eX \otimes_{k\otimes O_B} T_eX^\vee$ and $c = \dim_k(T_eX^\vee \wedge_{k\otimes O_B} T_eX^\vee)$. Then there are power series $a_1, \cdots, a_c$ so that

$$\mathrm{Def}(X, \iota, \lambda) \cong \frac{k[\![t_1, \cdots, t_s]\!]}{(a_1, \cdots, a_c)}.$$

**Proof** The proof is quite similar to that of theorem 2.3.3 of [Oort], which proves the analogous result for $\mathrm{Def}(X, \lambda)$. I thank Chia-Fu Yu for valuable discussions on the details of the argument. Clearly, $\mathrm{Def}(X, \iota)$ is a smooth, pro-representable subfunctor of $\mathrm{Def}(X)$. Moreover, either using general arguments from Kodaira-Spencer theory or [the dual of] the Dieudonné-theoretic description of $\mathrm{Def}(X)$ in [Nor], we see that $\mathrm{Def}(X, \iota)(k[\epsilon]/(\epsilon^2)) \cong \mathrm{Hom}_{k\otimes O_B}((T_eX)^\vee, T_eX^\vee) = T_eX \otimes_{k\otimes O_B} T_eX^\vee$, and

$$\begin{aligned} \mathrm{Def}(X, \iota) &\cong \widehat{\mathrm{Symm}}^\bullet_k(T_eX \otimes_{k\otimes O_B} T_eX^\vee) \\ &\cong k[\![t_1, \cdots, t_s]\!] \stackrel{\mathrm{def}}{=} \mathcal{D} \end{aligned}$$

It remains to compute the closed subfunctor $\mathrm{Def}(X, \iota, \lambda)$ of $\mathrm{Def}(X, \iota)$, necessarily represented by $\mathcal{D}/\mathfrak{a}$ for some ideal $\mathfrak{a}$. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{D}$. By the Artin-Rees lemma, there is an $n > 0$ so that $\mathfrak{m}^n \cap \mathfrak{a} = \mathfrak{m}^n \cap \mathfrak{m}\mathfrak{a}$, and thus

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \dfrac{\mathfrak{a}}{\mathfrak{m}\mathfrak{a}} & \longrightarrow & \dfrac{\mathcal{D}}{\mathfrak{m}\mathfrak{a} + \mathfrak{m}^n} & \longrightarrow & \dfrac{\mathcal{D}}{\mathfrak{a} + \mathfrak{m}^n} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & R' & \longrightarrow & 0 \end{array}$$

is exact. Note that $I$ is an ideal with $I^2 = \mathfrak{m}I = (0)$. The canonical surjections

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathcal{D}/\mathfrak{a} \\ \downarrow & & \downarrow \\ R & \longrightarrow & R' \end{array}$$

give an $O_B$-abelian scheme $(X/R, \iota)$, a polarized $O_B$-abelian scheme $(X'/R', \iota', \lambda')$, and an isomorphism $(X, \iota) \otimes_R R' \cong (X', \iota')$. We work with the first de Rham homology $H_1^{\mathrm{dR}}(X)$ and $H_1^{\mathrm{dR}}(X')$ in order to apply the discussion of endomorphisms in section 2.2. The former is a free filtered $R$-module $\mathrm{Fil}(X) = (T_e X^\vee)^\vee \subset H_1^{\mathrm{dR}}(X)$ equipped with an eigenspace decomposition $\mathrm{Fil}(X) = \oplus \mathrm{Fil}^i(X)$, $H_1^{\mathrm{dR}}(X) = \oplus H_1^{\mathrm{dR}}(X)^i$, and $H_1^{\mathrm{dR}}(X')$ is an analogous object over $R'$. Of course, $\oplus \mathrm{Fil}^i(X) \subset \oplus H_1^{\mathrm{dR}}(X)^i$ lifts $\oplus \mathrm{Fil}^i(X') \subset H_1^{\mathrm{dR}}(X')^i$. The polarization $\lambda'$ induces a bilinear form on $H_1^{\mathrm{dR}}(X')$, for which $\mathrm{Fil}(X')$ is a Lagrangian subspace. The polarization lifts to $X/R$ if and only if $\mathrm{Fil}(X)$ is isotropic under the induced form. Choose a basis $\{b^i_j\}$ for each eigenspace $H_1^{\mathrm{dR}}(X)^i$, and set

$$\widetilde{b}^i_{jl} = \langle b^i_j, b^i_l \rangle$$

if the center of $B$, acting on $\mathrm{Fil}^i$, is fixed by the involution of $B$, and $\widetilde{b}^i_{jl} = \langle b^i_j, b^{\bar{i}}_l \rangle$ otherwise; see remark 2.8.

Since $\langle \mathrm{Fil}(X'), \mathrm{Fil}(X') \rangle = (0) \subset R'$, $\widetilde{b}^i_{jl} \in I$. For each $i, j, l$ let $b^i_{jl}$ be a lift of $\widetilde{b}^i_{jl}$ to $\mathfrak{a}$; these are the $a_1, \cdots, a_c$ promised in the theorem. Finally, let $\mathfrak{b} \subset \mathcal{D}$ be the ideal generated by all $b^i_{jl}$.

In fact, $\mathfrak{b} = \mathfrak{a}$. To see this, let $R'' = \frac{\mathcal{D}}{\mathfrak{b} + \mathfrak{m}\mathfrak{a} + \mathfrak{m}^n}$. By the construction of $R''$, $\langle \mathrm{Fil}(X''), \mathrm{Fil}(X'') \rangle = (0) \subset R''$. Consequently the polarization $\lambda'$ lifts to $R''$ and there is a map $\mathcal{D}/\mathfrak{a} \to R''$. Thus, $\mathfrak{a} \subset \mathfrak{b} + \mathfrak{m}\mathfrak{a} + \mathfrak{m}^n$, and $\mathfrak{b} = \mathfrak{a}$.$\diamond$

Note that this gives a quick lower bound on the dimension of each component of $\mathcal{A}_{g,d}^{O_B}$. The following simple observation will be of decisive importance later.

**Corollary 2.2** Let $K$ be a totally real field of dimension $f = [K : \mathbb{Q}]$, and let $r = \frac{g}{f}$. Then the dimension of each component of $\mathcal{A}_{g,d}^{O_K}$ is at least $f \cdot \frac{r(r+1)}{2}$.

**Proof** Indeed, $\dim_{O_K \otimes k} T_e X = \dim_{O_K \otimes k} T_e X^\vee = r$, so $s = \dim_k(T_e X \otimes_{O_K \otimes k} T_e X^\vee) = fr^2$; and $c = \dim_k(T_e X^\vee \wedge_{k \otimes O_K} T_e X^\vee) = f\frac{r(r-1)}{2}$. This gives a lower bound of $s - c = f\frac{r(r+1)}{2}$ on the dimension of the local ring at any point of $\mathcal{A}_{g,d}^{O_K}$, and thus on the dimension of each component.$\diamond$

**Remark 2.3** This lower bound is achieved at $(X/k, \iota, \lambda) \in \mathcal{A}_{g,d}^{O_K}(k)$ exactly when the moduli space is a local complete intersection at $(X/k, \iota, \lambda)$.

### 2.2 Cartier-Dieudonné theory

As always, let $X$ be a $g$-dimensional abelian variety over an algebraically closed field $k$ of characteristic $p > 0$. If $(n, p) = 1$, then the $n$-torsion of $X$ looks like that of a complex abelian variety; $X[n] \stackrel{\text{def}}{=} \ker(X \stackrel{[n]}{\to} X) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$. If one asks for $p$-torsion, however, all one knows *a priori* is that $X[p]$ is a finite flat group scheme of rank $p^{2g}$. In fact, $X[p]$ is an extension of an étale group scheme by a non-trivial connected group scheme

$$0 \longrightarrow X[p]^0 \longrightarrow X[p] \longrightarrow X[p]^{\text{ét}} \longrightarrow 0$$

It is the étale quotient which contributes to the physical $p$-torsion; $X[p](k) = X[p]^{\text{ét}}(k) \cong (\mathbb{Z}/p\mathbb{Z})^\rho$, where $0 \leq \rho \leq g$ is the $p$-rank of $X$. While this sort of variation may be disconcerting to the complex algebraic geometer, all information about $X$ is somehow encoded in the $p$-power torsion. Let us try to make this vague notion precise.

The $p$-power torsion behaves in a coherent way; there is a similar diagram for each $X[p^n]$, and $X[p^n](k) \cong (\mathbb{Z}/p^n\mathbb{Z})^\rho$. One is quickly led to consider the ind-group scheme

$$X[p^\infty] \stackrel{\text{def}}{=} \varinjlim_{n} X[p^n],$$

the $p$-divisible [or Barsotti-Tate] group of $X$. The importance of understanding $X[p^\infty]$ is seen in the following theorem of Serre and Tate.

**Proposition** [Serre-Tate] For $R \in \operatorname{Ob} \mathsf{Art}_p(k)$, let $\mathsf{AbSch}(R)$ be the category of abelian schemes over $R$ and let $\mathsf{BT}(R)$ be the category of triples $(X, G, \epsilon)$, where $X$ is an abelian variety over $k$, $G$ is a $p$-divisible group over $R$, and $\epsilon : G \times_{\operatorname{Spec} R} \operatorname{Spec} k \to X[p^\infty]$ is an isomorphism of $p$-divisible groups over $k$. Then

$$\begin{array}{ccc} \mathsf{AbSch}(R) & \longrightarrow & \mathsf{BT}(R) \\ \widetilde{X} & \longmapsto & (\widetilde{X} \times \operatorname{Spec} k, \widetilde{X}[p^\infty], \text{natural } \epsilon) \end{array}$$

is an equivalence of categories.

In short, $\operatorname{Def}(X) \cong \operatorname{Def}(X[p^\infty])$. While first announced in section 6 of [LST], the reader might profitably consult V.2.3 of [Mess] and 1.2.1 of [Katz] for an exposition of this important theorem.

There is a crystalline Dieudonné theory which works quite well for deforming $p$-divisible groups, at least over divided power extensions of the base. Still, the classical covariant Dieudonné theory reveals information inaccessible by crystalline techniques; we try to explain this method now. We sketch the global and detail the local Cartier-Dieudonné theory, which are efficiently documented in chapters III and IV, respectively, of [Zink]. All this and more may also be found in [Haz].

Let $G$ be a [commutative] formal group over $S = \operatorname{Spec} R$. This means that $G$ is a functor $\mathsf{Art}(R) \to \mathsf{AbGp}$. The example to have in mind is the neutral component of a $p$-divisible group, especially $X[p^\infty]^0$. In fact, we will often take $G$ to be $X[p^\infty]'$, the part which is neither étale nor toroidal [dual to an étale group].

In any event Dieudonné and his successors try to find an object which linearizes the formal group as a Lie algebra linearizes a Lie group. They are led to consider $G(R[\![T]\!]) = \lim_{\overleftarrow{n}} G(R[\![T]\!]/(T^n))$, which may be thought of as the group of formal curves through the identity element of $G$. Let $\Lambda$ be the formal group $(R, \mathfrak{m}) \mapsto \widehat{\mathbb{G}}_m(\mathfrak{m})$. There is a canonical isomorphism of abelian groups $G(R[\![T]\!]) \cong \operatorname{Hom}(\Lambda, G)$. Written this way, it is clear that $G(R[\![T]\!])$ is a left module over $\operatorname{Cart}(R) \stackrel{\text{def}}{=} (\operatorname{End} \Lambda)^{\text{opp}}$. Moreover, the association $G \mapsto G(R[\![T]\!])$ gives a fully faithful embedding into a certain category of Cartier modules.

Now assume that, as in all applications in this paper, $R$ has positive characteristic $p$. All the useful information is encoded in a subgroup of $G(R[\![T]\!])$, the group of all $p$-typical curves. Let $\mathbb{D}_*(G)$ be the group of $p$-typical curves. It is canonically isomorphic to $\operatorname{Hom}(\widehat{W}, G)$, where $\widehat{W}$ is the formal group of Witt vectors. We again see that $\mathbb{D}_*(G)$ is a module over the local Cartier ring $\operatorname{Cart}_p(R) \stackrel{\text{def}}{=} (\operatorname{End} \widehat{W})^{\text{opp}}$ (see [Zink] 4.17). As a set, $\operatorname{Cart}_p(R)$ consists of all sums $\sum_{r,s \geq 0} V^r [x_{rs}] F^s$, where $x_{rs} \in R$ and, for each $r$, almost all $x_{rs}$ are zero. These sums are subject to the following relations.

$$\begin{aligned} [1] \stackrel{\text{def}}{=} V^0[1]F^0 &= 1_{\mathrm{Cart}_p(R)} & FV &= p \cdot 1_{\mathrm{Cart}_p(R)} \\ [x]V &= V[x^p] & F[x] &= [x^p]F \\ [x][y] &= [xy] & [x+y] &= [x]+[y]+\textstyle\sum_{n\geq 1} V^n[a_n(x,y)]F^n \end{aligned}$$

where the $a_n$ are certain universal polynomials. Moreover, there is an embedding of the Witt vectors into the local Cartier ring

$$W(R) \hookrightarrow \mathrm{Cart}_p(R)$$

$$x = (x_n) \longmapsto \sum_n V^n[x_n]F^n$$

A Dieudonné module over $R$ is a $V$-adically separated and complete $\mathrm{Cart}_p(R)$-module $M$ such that $V : M \to M$ is injective and $M/VM$ is a free, finite $R$-module. When $R$ is a perfect field $k$, the local Cartier ring is

$$\frac{W(k)[F][[V]]}{(FV - p, VaF - a^\tau, Fa - a^\sigma F, Va^\sigma - aV)}$$

where $\sigma$ and $\tau$ are the Frobenius and Verschiebung of $W(k)$. A Dieudonné module may then be thought of as a free $W(k)$-module of rank height($G$), equipped with $\sigma$- and $\tau$-linear operators $F$ and $V$ satisfying certain identities; and $M/VM$ is canonically the tangent space of $G$.

**Proposition** The functor $G \mapsto \mathrm{Hom}(\widehat{W}, G)$ is an equivalence between the category of smooth, commutative formal groups over $R$ and the category of Dieudonné modules over $R$.

See 4.23 of [Zink] for the proof of this fundamental theorem.

The covariant Dieudonné theory makes it easy to write down deformations of a formal group, and thus of an abelian variety, over an arbitrary base ring. Roughly speaking, one can twist the $F$ action by a nilpotent endomorphism and thereby produce a family of deformations. Let $\nu : M \to M$ be a nilpotent endomorphism; set $\mu = \mathrm{id} + \epsilon\nu : \widetilde{M} \to \widetilde{M}$, where $\widetilde{M} = M \otimes_{\mathrm{Cart}_p(k)} \mathrm{Cart}_p(k[\![\epsilon]\!])$. Set $\widetilde{F} = \mu \circ F$, a twisted form of the original Frobenius. One can adapt the Verschiebung as well so that $(\widetilde{M}, \widetilde{F})$ is a Dieudonné module. See section 1 of [Nor] for details, or [C-N] for a coordinate-free formulation.

Let $M = \mathbb{D}_*(X) \stackrel{\mathrm{def}}{=} \mathbb{D}_*(X[p^\infty])$ be the Dieudonné module of a polarized abelian variety over $k$, which we temporarily assume has $p$-rank zero. Thus, $M$ is a free, rank $2g = 2\dim X$ module over $W(k)$. By functoriality, a polarization $X \stackrel{\lambda}{\to} X^\vee$ induces a homomorphism of Dieudonné modules $\mathbb{D}_*(X) \stackrel{\lambda_*}{\to} \mathbb{D}_*(X^\vee)$. Moreover $\mathbb{D}_*(X^\vee)$ is, up to Tate twist, the $W(k)$-linear dual of $\mathbb{D}_*(X)$. Carefully following through dualities (as in [Mum] or [Nor]; see also section 5.1 of [BBM]) shows $\lambda$ induces a $W(k)$-linear pairing $\langle\cdot,\cdot\rangle : M \times M \to W(k)$ such that

- $\langle m, m'\rangle = -\langle m', m\rangle$.
- $\langle Fm, m'\rangle = \langle m, Vm'\rangle^\sigma$.

This is the Dieudonné-theoretic analogue of the Riemann form of a polarized complex abelian variety.

If $O_B$ acts on $X$, then $O_B \otimes \mathbb{Z}_p$ acts on $M$. Idempotents of this ring, $O_B \otimes \mathbb{Z}_p \cong \oplus_j \mathrm{Mat}_{s_j}(W(\mathbb{F}_{p^{f_j}}))$, give a decomposition of $M$ as a Dieudonné module; $M = \oplus M_j$. Thus, for most purposes it suffices to assume that $O_B \otimes \mathbb{Z}_p \cong W(\mathbb{F}_{p^f})$, i.e., that $B$ is a number field inert at $p$. The only number fields which arise are totally imaginary or totally real, depending on whether or not they are fixed by the involution [Mum2].

In fact, for the rest of this subsection assume $p$ really is inert in a totally real field $K$; at the end we indicate how to adapt these considerations to a totally imaginary field $L$ inert at $p$. Let $f = [K : \mathbb{Q}]$ and $r = \frac{g}{f}$, the relative dimension of $X$ over $K$. The action of $O_K$ on $M$ becomes particularly easy to describe. There is a canonical structure of $W(k)$-module on $M$, and $O_K$ acts on $M$ via embeddings $O_K \hookrightarrow W(k)$. For convenience's sake, identify $\mathrm{Hom}(O_K, W(k))$ with $\mathbb{Z}/f\mathbb{Z}$ by fixing one such map $O_K \otimes \mathbb{Z}_p \cong W(\mathbb{F}_{p^f}) \overset{\sigma_0}{\hookrightarrow} W(k)$. Let $\sigma$ be the Frobenius of $W(\mathbb{F}_{p^f})$ and set $\sigma_i = \sigma_0 \circ \sigma^i : W(\mathbb{F}_{p^f}) \hookrightarrow W(k)$ for $1 \leq i \leq f-1$. Let $M^i$ be the eigenspace where $O_K$ acts via $\sigma_i$. There is a decomposition of $M$ as a $W(k)$-module,

$$M = \oplus_{i \in \mathbb{Z}/f\mathbb{Z}} M^i.$$

This is *not* a direct sum of Dieudonné modules. Indeed, the Frobenius and Verschiebung operators interweave the $M^i$. For any element $m^i$ of $M^i$,

$$\begin{aligned} F(\alpha m^i) &= F(\sigma_i(\alpha) m^i) \\ &= \sigma_i(\alpha)^\sigma F m^i \\ &= \sigma_{i+1}(\alpha) F m^i. \end{aligned}$$

Thus, $FM^i \subseteq M^{i+1}$ with the expected arithmetic for indices. Similarly, $VM^i \subseteq VM^{i-1}$.

Let $\langle \cdot, \cdot \rangle$ be the nondegenerate alternating form on $M$ induced by a polarization $\lambda$. It turns out that $\langle \cdot, \cdot \rangle|_{M^i}$ is nondegenerate for each $i$. Recall that, as $K$ is totally real, the Rosati involution is actually trivial on elements of $K$. For any $\alpha \in O_K$ and $m^i \in M^i, m^j \in M^j$, on one hand $\langle \alpha m^i, m^j\rangle = \langle \sigma_i(\alpha) m^i, m^j\rangle = \sigma_i(\alpha)\langle m^i, m^j\rangle$; on the other, $\langle \alpha m^i, m^j\rangle = \langle m^i, \alpha m^j\rangle = \langle m^i, \sigma_j(\alpha) m^j\rangle = \sigma_j(\alpha)\langle m^i, m^j\rangle$. If $i \neq j$, then choosing any $\alpha$ with $\sigma_i(\alpha) \neq \sigma_j(\alpha)$ shows that $\langle m^i, m^j\rangle = 0$.

It is possible – and, for the explicit deformation theory which follows, desirable – to choose bases for $M$ which clearly expose the behavior of $F$, $V$ and $\langle \cdot, \cdot\rangle$.

**Lemma 2.4** Let $(M, \iota, \langle \cdot, \cdot\rangle)$ be a quasipolarized Dieudonné module equipped with an action by $O_K$. Let $M^i$ be the eigenspace corresponding to $\sigma_i$ as above. There are $W(k)$-bases $\{e^i_1, \cdots, e^i_{2r}\}$ and $\{f^i_1, \cdots, f^i_{2r}\}$ for $M^i$ such that

- $Fe^i_j \in \{f^{i+1}_j, pf^{i+1}_j\}$.
- If $W(k)(Fe^i_{r+j})$ is a direct summand then so is $W(k)(Fe^i_j)$.
- $\langle e^i_j, e^{i'}_{j'}\rangle \neq 0 \iff i = i', |j - j'| = r$.
- $\langle e^i_j, e^i_{r+j}\rangle = p^{\delta^i_j}$ for some $\delta^i_j \in \mathbb{Z}_{\geq 0}$.

**Proof** In view of the previous computations, the proof is essentially a careful meditation on the elementary divisors lemma. It may be worth pointing out that, in the absence of an $O_K$-action, this recovers the "displayed module" of [Nor].

Fix $i \in \mathbb{Z}/f\mathbb{Z}$. By the freeness hypothesis, $(M/FM)^i \cong \mathrm{Lie}(X^\vee)^i \cong k^r$. By, say, the elementary divisors lemma there are bases $e^i_{1j}$ and $f^i_{1j}$ so that $Fe^i_{1j} \in \{f^i_{1j}, pf^i_{1j}\}$. Let's agree to say that $F$ acts unimodularly, or with index zero, on $e^i_{1j}$ if $W(k)Fe^i_{1j}$ is a direct summand; and with index one if $Fe^i_{1j} = pf^i_{1j}$. [In general, the index of an element $x \in M$ is the largest $n \in \mathbb{Z}$ with $x \in p^n M$; and if $T$ is a $[\sigma^{\pm 1}$-$]$linear operator on $M$, declare that $T$ acts with index (ind $Tx$ − ind $x$) on $x$.] The idea is simply to diagonalize this basis with respect to the symplectic form.

Order the $e^i_{1j}$ and $f^i_{1j}$ so that $\mathrm{ord}_p \langle e^i_{11}, e^i_{1,r+1}\rangle$ is minimal among all $\mathrm{ord}_p \langle e^i_{1j}, e^i_{1l}\rangle$. We may further choose these first elements so that $\mathrm{ind}\, Fe^i_{11} + \mathrm{ind}\, Fe^i_{1,r+1}$ is *maximal* over all $i, j$ with $\mathrm{ord}_p \langle e^i_{1j}, e^i_{1j}\rangle$ minimal. Start orthogonalizing, by setting

$$e^i_{2j} = \begin{cases} e^i_{1j} & j = 1, r+1 \\ e^i_{1j} + \frac{\langle e^i_{1j}, e^i_{1,r+1}\rangle}{\langle e^i_{11}, e^i_{1,r+1}\rangle} e^i_{11} + \frac{\langle e^i_{1j}, e^i_{11}\rangle}{\langle e^i_{11}, e^i_{1,r+1}\rangle} e^i_{1,r+1} & j \neq 1, r+1 \end{cases}.$$

Note that, by the minimality of $\mathrm{ord}_p \langle e^i_{11}, e^i_{1,r+1}\rangle$, the division is permissible over $W(k)$. Moreover, for $j \neq 1, r+1$, $\langle e^i_{2j}, e^i_{21}\rangle = \langle e^i_{2j}, e^i_{2,r+1}\rangle = 0$. Indeed,

$$\begin{aligned} \langle e^i_{21}, e^i_{2j}\rangle &= \langle e^i_{11}, e^i_{1j}\rangle + \frac{\langle e^i_{1j}, e^i_{11}\rangle}{\langle e^i_{11}, e^i_{1,r+1}\rangle}\langle e^i_{11}, e^i_{1,r+1}\rangle \\ &= 0 \end{aligned}$$

and the same computation works for $\langle e^i_{2j}, e^i_{2,r+1}\rangle$.

The only issue is whether $e^i_{21}, \cdots, e^i_{2,2r}$ is still a good basis for describing the action of $F$. Fix a $j \neq 1, r+1$.

If $Fe^i_{1j}$ is unimodular, then $Fe^i_{2j}$ is clearly such, too. If $Fe^i_{1j}$ has index one, however, there is still a small amount of verification to be done. Ideally, $Fe^i_{2j}$ should have index one as well.

If $\mathrm{ord}_p \langle e^i_{1j}, e^i_{11}\rangle$ and $\mathrm{ord}_p \langle e^i_{1j}, e^i_{1,r+1}\rangle$ are strictly greater than $\mathrm{ord}_p \langle e^i_{11}, e^i_{1,r+1}\rangle$, then there's no problem, as $e^i_{2j} - e^i_{1j} \in pM$.

The situation to worry about is the following; $Fe^i_{1j} = pf^i_{1j}$, $Fe^i_{11} = f^i_{11}$, $\mathrm{ord}_p \langle e^i_{1j}, e^i_{1,r+1}\rangle = \mathrm{ord}_p \langle e^i_{11}, e^i_{1,r+1}\rangle$. But then $\mathrm{ind}\, Fe^i_{1j} + \mathrm{ind}\, Fe^i_{1,r+1} > \mathrm{ind}\, Fe^i_{11} + \mathrm{ind}\, Fe^i_{1,r+1}$, contradicting the second assumption on $e^i_{11}$ and $e^i_{1,r+1}$.

Now iterate this procedure, ultimately constructing $e^i_j = e^i_{rj}$, to finish; and the $f^i_j$ are determined by the $e^i_j$.$\diamond$

Call any such choice of bases a normal form for $(M, \iota, \lambda_*)$. Empirically, it is *much* easier to write down deformations of Dieudonné modules which enjoy a certain property.

(*) There is a normal form such that, for each $i \in \mathbb{Z}/f\mathbb{Z}$ and $1 \leq j \leq r$, $Fe^i_j = f^{i+1}_j$ and $Fe^i_{r+j} = pf^{i+1}_{r+j}$.

Suppose such exists. Define certain direct summands of $M$ in terms of the given normal form.

$$\begin{aligned} Q^i &= \oplus_{j=1}^r W(k)e^i_j & Q &= \oplus_{i\in\mathbb{Z}/f\mathbb{Z}} Q^i \\ P^i &= \oplus_{j=1}^r W(k)e^i_{r+j} & P &= \oplus_{i\in\mathbb{Z}/f\mathbb{Z}} P^i \end{aligned}$$

By the definition of normal form, these summands satisfy the following conditions:

i. $M^i = P^i \oplus Q^i$, hence $M = P \oplus Q$.

ii. $\langle P, P\rangle = \langle Q, Q\rangle = (0) \subset W(k)$.

iii. $P \bmod pM = VM/pM$.

In fact, such summands characterize the sort of Dieudonné module we're after:

**Lemma 2.5** $M$ satisfies (*) if and only if there are $P^i, Q^i \subseteq M$ satisfying (i) through (iii).

**Proof** If $M$ satisfies (*), then the $P^i$ and $Q^i$ obviously satisfy (i) through (iii), by the definition of normal form.

Conversely, suppose we are given such $P^i$ and $Q^i$. Start with arbitrary $W(k)$-bases for $P^i$ and $Q^i$, and diagonalize as in the proof of lemma 2.4. Since $\langle P, P \rangle = \langle Q, Q \rangle = 0$, the algorithm will merely produce new bases for $P$ and $Q$. By (iii), $F$ acts with index one on $P$. So it must act with index zero on $Q$, and (*) is satisfied.$\diamond$

Call an abelian variety whose Dieudonné module satisfies (*) nice. With a slight abuse of nomenclature, say the $W(k)$-summand $M^i$ is nice if there are $P^i$ and $Q^i$ as above.

In view of the commutative diagram associated to any such decomposition,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & P & \longrightarrow & M = P \oplus Q & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & VM/pM & \longrightarrow & M/pM & \longrightarrow & M/VM & \longrightarrow & 0 \\
 & & \downarrow = & & \downarrow = & & \downarrow = & & \\
0 & \longrightarrow & H^1(X, O_X)^\vee & \longrightarrow & H_1^{\mathrm{dR}}(X) & \longrightarrow & \mathrm{Lie}(X) & \longrightarrow & 0
\end{array}
$$

this condition may be reasonably paraphrased as demanding that the Hodge filtration admit an isotropic lifting.

**Remark 2.6** If $X$ is separably polarized, then $\mathbb{D}_*(X)$ is nice. For suppose not. Then there are $i$ and $j$ so that $F$ acts with index zero on $e^i_j$ and $e^i_{r+j}$. On one hand, $\langle Fe^i_j, Fe^i_{r+j} \rangle = \langle e^i_j, VFe^i_{r+j} \rangle^\sigma = p$; on the other hand, $\langle Fe^i_j, Fe^i_{r+j} \rangle = \langle e^{i+1}_j, e^{i+1}_{r+j} \rangle$. Thus, $p|d$, contradicting the hypothesis of separability.$\diamond$

There is a nice rank $n(X) = (n_0, \cdots, n_{f-1})$ which measures the defect of $(X, \iota, \lambda)$ from nice. Set

$$n_i = \max \#\{j | 1 \leq j \leq r, Fe^i_j = f^{i+1}_j, Fe^i_{r+j} = pf^{i+1}_{r+j}\},$$

where the maximum is taken over all possible normal forms for $M$. Note that $X$ is nice if and only if each $n_i = r$.

**Remark 2.7** The following remarks, while not logically necessary in the sequel, may help give the reader some idea of the lay of the land.

One might reasonably ask whether it is necessary to consider *all* possible normal forms for $M$ to determine its nice rank; it is certainly distasteful. In the special case where all elementary divisors of $M$ are 1 or $p$, it suffices to consider a single normal form. Indeed, suppose $M$ is such and that there is some normal form which is not visibly nice. Then are are $i \in \mathbb{Z}/f\mathbb{Z}$ and $1 \leq j \leq r$ with $\langle e^i_j, e^i_{r+j} \rangle = 1$, $F\{e^i_j, e^i_{r+j}\} = \{f^i_j, f^i_{r+j}\}$. Define $P^i$ and $Q^i$ as above, although $\langle P^i, Q^i \rangle \supsetneq (0)$. Any apparent improvement to the nice rank must come from finding $x, y \in pP^i + Q^i$, not both in $Q^i$, so that $\langle e^i_j + y, e^i_{r+j} + x \rangle = 0$, i.e., $\langle x, y \rangle = 1$. Given the constraints on the elementary divisors, this is impossible.

A similar argument shows the same claim when the relative dimension $r$ is two. Unfortunately, it fails for arbitrary polarized $O_K$-abelian varieties; this may help explain why we only use this notion in studying mildly inseparable polarizations.

In contrast with the $p$-rank, the nice rank depends on the integral structure of $(X, \iota, \lambda)$; it is not preserved by isogenies. Still, this rank induces a reasonable stratification on $\mathcal{A}^{O_K}_{g,d}$. Nice is an open condition; we sketch a proof. Suppose $(X, \iota, \lambda)$ is nice. This is equivalent to the existence of $W(k)$-submodules $Q, R \subset M$ so that $\langle Q, Q\rangle = \langle R, R\rangle = (0)$; $\dim_k FQ \bmod pM = \mathrm{rk}_{W(k)} Q = g$; and $\dim_k VR \bmod pM = \mathrm{rk}_{W(k)} R = g$ [simply take $R = V^{-1}P$]. Consider any deformation of this polarized Dieudonné module. Working only with the polarization, there are submodules $\widetilde{Q}, \widetilde{R}$, lifting $Q$ and $R$, so that $\langle \widetilde{Q}, \widetilde{Q}\rangle = \langle \widetilde{R}, \widetilde{R}\rangle = (0)$. Since "having full rank under $F \bmod p$ or $V \bmod p$" is an open condition, the generic lifts $\widetilde{Q}, \widetilde{R}$ still have $\dim_k F\widetilde{Q} \bmod pM = \dim_k V\widetilde{R} \bmod pM = g$. Now, the deformation of $M$ also changes the action of $\widetilde{F}$ and $\widetilde{V}$; but if $F\widetilde{Q} \bmod pM$ has full rank, then so must the generic $\widetilde{F}\widetilde{Q} \bmod pM$. This argument works for any suitable summands, not just ones of full rank. Thus, if $\mathbb{N}^f$ is equipped with the product partial order, then the function $(X, \iota, \lambda) \mapsto n(X)$ is a lower semicontinuous function on $\mathcal{A}_{g,d}^{O_K}$.

**Remark 2.8** The fields of endomorphisms with which we are concerned are those which are totally real or totally imaginary. While the foregoing discussion deals with the former case, it is not difficult to adapt it to the latter situation. Indeed, let $O_L$ be the ring of integers of a totally imaginary field $L$ of degree $2f$ over $\mathbb{Q}$. It is a quadratic extension of a totally real subfield $K$, the field fixed by complex conjugation. Suppose $O_L$ acts on a polarized abelian variety $(X, \lambda)$. An $O_L$-abelian variety is in particular an $O_K$-abelian variety, so its Dieudonné module $M = \mathbb{D}_*(X)$ admits a decomposition as before: $M = \oplus_{i \in \mathbb{Z}/f\mathbb{Z}} M^i$. The eigenspace decomposition of $M$ as an $O_L$-module necessarily refines this decomposition. Specifically, fix $\tau_0 : O_L \hookrightarrow W(k)$ extending $\sigma_0$; let $\tau_i = \sigma_0 \circ \sigma^i$ and $\overline{\tau}_i = \sigma_0 \circ \sigma^{i+f}$. Note that $\overline{\tau}_i|_{O_K} = \tau_i|_{O_K}$. The action of $O_L$ on $M$ is then given by

$$M^i = M^{(\tau_i)} \oplus M^{(\overline{\tau}_i)}.$$

Bearing in mind that the Rosati involution on $O_L$ is complex conjugation, one quickly shows that $\langle\cdot,\cdot\rangle : M^{(\tau_i)} \times M^{(\overline{\tau}_i)} \to W(k)$ is nondegenerate. The proof of lemma 2.4 can be modified to take advantage of this structure, producing good bases with $\{e^i_1, \cdots, e^i_r\} \subset M^{(\tau_i)}$ and $\{e^i_{r+1}, \cdots, e^i_{2r}\} \subset M^{(\overline{\tau}_i)}$.

## 2.3 Crystalline cohomology

Crystalline Dieudonné theory provides an excellent tool for studying $p$-divisible groups, and thus abelian varieties, in positive characteristic. Let us start by recalling the basic concepts, most of which are documented in chapters IV and V of [Gr] and I of [BBM].

Let $(S, I, \gamma)$ be a triple consisting of a scheme $S$, an ideal $I \subset O_S$, and a divided-power structure $\gamma$ on $S$. The [Zariski] crystalline site on $S$ $S_{\text{cris}}$ is the category of triples $(U \to S, U \hookrightarrow T, \delta)$ where $U \to S$ is a [Zariski] open set, $U \hookrightarrow T$ is a nilpotent immersion, and $\delta$ is a divided power structure on the ideal sheaf of $U$ in $T$ compatible with $\gamma$. One defines a sheaf on the crystalline site in the expected way; the structure sheaf $O_{S,\text{cris}}$ is given by $O_{S,\text{cris}}(U \to S, U \hookrightarrow T, \delta) = \Gamma(T, O_T)$. An $O_{S,\text{cris}}$-module $\mathcal{F}$ is a crystal if it enjoys a certain strong extension property: for every morphism $(u,t) : (U', T', \delta') \to (U, T, \delta)$ in $S_{\text{cris}}$, the canonical restriction map of $O_{T'}$-modules

$$t^* \mathcal{F}_{(U,T,\delta)} = \mathcal{F}_{(U,T,\delta)} \otimes_{O_T} O_{T'} \xrightarrow{\rho_{(u,t)} \otimes \text{id}} \mathcal{F}(U', T', \delta')$$

is an isomorphism.

In the special case where $S = \operatorname{Spec} k$ is the spectrum of an algebraically closed field of positive characteristic, $S_{\text{cris}}$ consists of Artin local $k$-algebras $(R, \mathfrak{m})$ with a divided power structure on $\mathfrak{m}$. A crystal $\mathcal{F}$ of $O_{S,\mathrm{cris}}$-modules gives an $R$-module $\mathcal{F}(R)$ for each such $R$; and any PD [puissances divisées] morphism of algebras $R \rightharpoonup R'$ induces a canonical isomorphism of $R'$-modules:

$$\mathcal{F}(R) \otimes_R R' \xrightarrow{\cong} \mathcal{F}(R')$$

To a $p$-divisible group $G/S$ one can associate a Dieudonné crystal $\mathbb{D}^*(G)$. This is a crystal of $O_{S,\mathrm{cris}}$-modules equipped with the usual $\sigma$- and anti-$\sigma$-linear operators $F$ and $V$, respectively.

Let $\underline{\omega}_G$ be the space of invariant differentials on the formal group $G$. There is an exact sequence of free $O_S$-modules

$$0 \longrightarrow \underline{\omega}_G \longrightarrow \mathbb{D}^*(G)(S) \longrightarrow \mathrm{Lie}(G^\vee) \longrightarrow 0$$

Any lift of $G$ to a PD extension $S'$ of $S$ determines, and is in fact determined by, such a filtration; there is an equivalence between $\mathrm{Def}(G)(S')$, the $p$-divisible groups over $S'$ lifting $G$, and locally direct factors $\mathrm{Fil}(S) \subset \mathbb{D}^*(G)(S')$ lifting the above filtration. This is actually an equivalence of categories; $\phi \in \mathrm{Hom}_S(G, H)$ extends an element of $\mathrm{Hom}_{S'}(G', H')$ if and only if it induces a map of *filtered* Dieudonné crystals.

If $G = X[p^\infty]$ is the $p$-divisible group of an abelian variety $X \xrightarrow{\pi} S = \mathrm{Spec}\, k$, then $\mathbb{D}^*(G) = H^1_{\mathrm{cris}}(X) = R^1\pi_{*,\mathrm{cris}} O_X$. In fact, $H^1_{\mathrm{cris}}(X)(k) \cong H^1_{\mathrm{dR}}(X)$, and the filtration on $\mathbb{D}^*(G)(S)$ is none other than the Hodge filtration:

$$0 \longrightarrow H^0(X, \Omega^1_X) \longrightarrow H^1_{\mathrm{dR}}(X) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow 0$$

In view of the Serre-Tate theorem, to deform $X$ to a PD extension $R/k$ is to give a direct summand $\mathrm{Fil}(X)(R) \subset H^1_{\mathrm{cris}}(X)(R) \cong H^1_{\mathrm{cris}}(X)(k) \otimes R$ lifting the Hodge filtration.

We will often prefer to work with the linear dual $H_1^{\mathrm{cris}}(X)$ and its [dual] Hodge filtration, as this exposes the connection between the crystalline theory and the covariant Dieudonné theory. Indeed, let $M = \mathbb{D}_*(X[p^\infty])$. There are canonical isomorphisms $M/pM = H_1^{\mathrm{dR}}(X) = H_1^{\mathrm{cris}}(X)(k)$ and $M/VM \cong \mathrm{Lie}(X)$ (see [BBM]). Dualizing the Hodge filtration yields

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Lie}(X^\vee)^\vee & \longrightarrow & H_1^{\mathrm{dR}}(X) & \longrightarrow & \mathrm{Lie}(X) & \longrightarrow & 0 \\
 & & \downarrow = & & \downarrow = & & \downarrow = & & \\
0 & \longrightarrow & \mathrm{Fil}(M/pM) = VM/pM & \longrightarrow & M/pM & \longrightarrow & M/VM & \longrightarrow & 0
\end{array}$$

Let $M^* = \mathbb{D}_*(X^\vee)$; up to a Tate twist, it is the $W(k)$-linear dual of the free $W(k)$-module $M$. Clearly, $\mathrm{Fil}(M^*/pM^*)$ may be computed in the same way. Alternately, observe that $\mathrm{Fil}(M^*/pM^*) = \mathrm{Lie}(X^{\vee\vee})^\vee = \mathrm{Hom}(M/VM, k) = \{e^* \in M^* : (VM, e^*) = (0)\}$.

# 3 Nice deforms to ordinary

I first encountered niceness [nicety?] as a convenient hypothesis for the following result.

**Lemma 3.1** Let $K$ be a totally real field. Suppose $(X, \iota, \lambda) \in \mathcal{A}_{g,d}^{O_K}(k)$ is nice but not ordinary. Then $(X, \iota, \lambda)$ admits an infinitesimal deformation to a polarized $O_K$-abelian variety with strictly bigger $p$-rank.

**Proof** We use the covariant Dieudonné theory described in 2.2. The Serre-Tate theory assures us we may work directly with the $p$-divisible group $X[p^\infty] = X[p^\infty]' \oplus X[p^\infty]^{\text{tor}} \oplus X[p^\infty]^{\text{ét}}$, where $X[p^\infty]'$ is the local-local part of $X[p^\infty]$ which keeps $X$ from being ordinary. By, say, the classification of $p$-divisible groups [Man], this decomposition is stable under the $O_K$-action, so we may study $X[p^\infty]'$ and its Dieudonné module $M$. As explained in 2.2, we may assume that $K$ is actually inert at $p$.

We will produce a nontrivial deformation of $X[p^\infty]$ to a family of $p$-divisible groups over $k[\![\underline{\epsilon}]\!]$. The quasipolarization will be preserved; by the Serre-Tate theory, this gives a polarized formal abelian scheme $(\widetilde{X}/k[\![\underline{\epsilon}]\!], \widetilde{\iota}, \widetilde{\lambda})$. By [EGA] III.5.4.5, this algebraizes to an honest abelian scheme.

Choose a normal basis for $M$ as in lemma 2.4. Define a nilpotent endomorphism $\nu$ of $M$ by

$$\nu(e_j^i) = \begin{cases} 0 & 1 \leq j \leq r \\ e_{j-r}^i & r+1 \leq j \leq 2r \end{cases}$$

Let $\widetilde{M} = \otimes W(k[\![\underline{\epsilon}]\!])$, and let $\epsilon$ be the Teichmüller lift of $\underline{\epsilon}$. Following the dicussion in 2.2, use $\mu \stackrel{\text{def}}{=} \text{id} + \epsilon\nu \in \text{End}(\widetilde{M})$ to produce a deformation $(\widetilde{M}, \widetilde{F})$ of $(M, F)$, and thus of $X$. As $\nu$ preserves the blocks $M^i$, and $\langle \mu(x), \mu(y) \rangle = \langle x, y \rangle$ for all $x, y \in M = M \otimes 1 \subset \widetilde{M}$, $\iota$ and $\lambda$

extend to $\widetilde{M}$. Hence, this gives a deformation $(\widetilde{X}, \widetilde{\iota}, \widetilde{\lambda})$ of $(X, \iota, \lambda)$. It is worth remarking that it is exactly the nice condition which makes it so easy to produce deformations which preserve the quasipolarization.

In order to show that the $p$-rank has increased under our deformation, it's certainly enough to produce some $x \in \widetilde{M}$ and $l \in \mathbb{N}$ with $\widetilde{F}^l x = \gamma x$, $\gamma \in W(k((\underline{\epsilon})))^\times$. [This is, of course, the same as showing that $\widetilde{F}$ is not nilpotent on $\widetilde{M}/\widetilde{V}\widetilde{M}$.] It is in fact slightly more convenient, and for the purposes of computing the $p$-rank harmless, to verify this for a geometric generic point of the formal deformation. So let $\widetilde{k} = k((\underline{\epsilon}))^{\text{perf}}$, and base change to $\widetilde{k}$.

Consider, say, $e_1^1$; $F$ acts unimodularly on it, so

$$\begin{aligned}
\widetilde{F}e_1^1 &= \mu(f_1^2) \\
&= \sum_{j=1}^{r} a_{j1}^2 e_j^2 + \sum_{j=r+1}^{2r} (e_{j1}^2 + \epsilon e_{j-r,1}^2) \\
&= \sum_{j=1}^{r} (a_{j1}^2 + \epsilon a_{r+j,1}^2) e_j^2 + \sum_{j=r+1}^{2r} a_{j1}^2 e_j^2
\end{aligned}$$

Now, there is some $1 \leq j \leq r$ such that $a_{j1}^2 + \epsilon a_{r+j,1}^2 \in W(\widetilde{k})^\times$; otherwise, $p | a_{j1}^2$ for *all* $j$, and $(a_{jk}^2)$ would be singular. So $\widetilde{F}e_1^1 \in W(\widetilde{k})^\times e_j^2 \oplus W(\widetilde{k}) e_{l\, l \neq j}^2$, and $\widetilde{F}$ acts unimodularly on $e_j^2$. Continuing in this way, and remembering that there are only finitely many $e_j^i$, we produce some $e_j^i$ on which $\widetilde{F}$ doesn't act nilpotently.$\diamond$

A similar statement should be true for complex multiplications as well. Nonetheless, we content ourselves with proving the statement necessary for the current applications.

**Lemma 3.2** Let $L$ be a CM field and $p \nmid d$. Any non-ordinary $(X, \iota, \lambda) \in \mathcal{A}_{g,d}^{O_L}(k)$ admits an infinitesimal deformation to a polarized $O_L$-abelian variety with strictly bigger $p$-rank.

**Proof** Not surprisingly, the proof is similar to that of 3.1; we merely indicate the necessary modifications. Thus, assume that $L$ is inert at $p$, and let $M$ be the Dieudonné-module of the local-local part of $X[p^\infty]$. As usual, let $K$ be the totally real quadratic subfield of $L$, so that $s \stackrel{\text{def}}{=} \frac{g}{[L:\mathbb{Q}]}$, $r = \frac{g}{[K:\mathbb{Q}]} = 2s$. Since the polarization is separable, remark 2.6 lets us construct bases as in 2.8 with

$$
\begin{aligned}
Fe_j^i &= \begin{cases} f_j^i & 1 \le j \le s \text{ or } r+s+1 \le j \le 2r \\ pf_j^i & s+1 \le j \le r \text{ or } r+1 \le s \le r+s \end{cases} \\
\langle e_j^i, e_l^i \rangle &= \begin{cases} 1 & l-j=r \\ -1 & j-l=r \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Once again, we deform the Dieudonné module by using a certain nilpotent endomorphism of $M$:

$$
\begin{aligned}
\nu(e_{s+j}^i) &= e_j^i \\
\nu(e_{r+j}^i) &= -e_{r+s+j}^i \quad \text{for } i \in \mathbb{Z}/f\mathbb{Z}, 1 \le j \le s. \\
\nu(e_j^i) = \nu(e_{r+s+j}^i) &= 0
\end{aligned}
$$

Define $\widetilde{M}$ as before and let $\mu = \mathrm{id} + \epsilon\nu$. Then $\mu(M^{(\tau_i)}) \subseteq M^{(\tau_i)}$, $\langle \mu(x), \mu(y) \rangle = \langle x, y \rangle$, and the induced deformation of $M$ really is a deformation of $X$ as a polarized $O_L$-abelian variety. A verification exactly like that in the proof of 3.1 shows that the $p$-rank improves.$\diamond$

**Corollary 3.3** Suppose $p \nmid d$. Then ordinary points are dense in $\mathcal{A}_{g,d}^{O_B}$.

**Proof** Indeed, by remark 2.6, every point of $\mathcal{A}_{g,d}^{O_B}$ is nice. Using the standard reduction to the situation where $B$ is a number field, 3.1 and 3.2 show how to infinitesimally deform such a point so its $p$-rank improves; iterate this to produce an infinitesimal deformation whose generic fiber is ordinary.$\diamond$

**Remark 3.4** Norman and Oort show that ordinary points are dense in the Siegel moduli space $\mathcal{A}_{g,d}$. They start by showing that, for the generic non-ordinary $(X, \lambda) \in \mathcal{A}_{g,d}$, $a(X) \stackrel{\text{def}}{=} \dim_k \operatorname{Hom}(\alpha_p, X)$ is one. Norman and Oort then choose a good basis for the Dieudonné module of such an abelian variety, and write down a deformation with strictly larger $p$-rank. In the present terminology, much of their computation comes down to showing that, if $a(X) = 1$, then $X$ must be nice.

# 4 Mild inseparability

When a low power of $p$ divides $d$, the moduli spaces tend to be singular but not unmanageably so. We consider here a class of such spaces.

As always, let $K$ be a totally real field unramified at $p$ of degree $f = [K : \mathbb{Q}]$. Throughout this section assume $d = p^f m$ with $m$ prime to $p$. Moreover, since we make vital use of crystalline techniques, *assume throughout this section that $p > 3$.*

Crystalline cohomology supplies a good description of the local geometry of $\mathcal{A}_{g,d}^{O_K}$. This information will be exploited to prove that nice points, and thus ordinary points, are dense in $\mathcal{A}_{g,d}^{O_K}$. We start by computing an infinitesimal neighborhood of $(X/k, \iota, \lambda)$ in $\mathcal{A}_{g,d}^{O_K}$.

We have seen in 2.3 that to deform $X/k$ to a PD extension $R$ of $k$ is to give an admissible filtration of $H_1^{\text{cris}}(X)(R)$. An action $\iota$ extends to $\widetilde{X}$ if the filtration is $O_K$-linear, in the sense that

$$\text{Fil}(X)(R) = \oplus_{i \in \mathbb{Z}/f\mathbb{Z}} \text{Fil}(X)(R)^i \subset \oplus_{i \in \mathbb{Z}/f\mathbb{Z}} H_1^{\text{cris}}(X)(R)^i = H_1^{\text{cris}}(X)(R).$$

Because of this, we may compute deformations to PD rings by examining each eigenspace $H_1^{\text{cris}}(X)^i$ separately. This program is carried out below. The reader is invited to perform these computations for herself, possibly after glancing briefly at the exposition given here.

As usual, let $M = \mathbb{D}_*(X) = \oplus M^i$. Recall that there is a canonical isomorphism $(VM/pM \subset M/pM) \cong \text{Fil}(X)(k) \subset H_1^{\text{cris}}(X)(k))$.

$\boxed{M^i \text{ nice}}$ Suppose $M^i$ is nice. Using 2.4, we may choose a normal form for $M^i$; $M^i = W(k)\{x_1^i, \cdots, x_r^i, y_1^i, \cdots, y_r^i\}$, where

$$\begin{aligned} Fx^i_j &\in M^{i+1} - pM^{i+1} \\ Fy^i_j &\in pM^{i+1} \\ \langle x^i_j, x^i_l \rangle &= 0 \\ \langle y^i_j, y^i_l \rangle &= 0 \end{aligned} \qquad \langle x^i_j, y^i_l \rangle = \begin{cases} 1 & j = l < r \\ p & j = l = r \\ 0 & j \neq l \end{cases}$$

The filtration on $M^i$ is given by $\mathrm{Fil}(M^i/pM^i) = (VM/pM)^i = k\{y^i_1, \cdots, y^i_r\}$. According to identifications in section 2.3, $\mathrm{Fil}(M^{i*}/pM^{i*}) = (M/VM)^{i^\vee} = k\{x^{i*}_1, \cdots, x^{i*}_r\}$.

Up to order $p-1$, the formal moduli space for the filtered vector space $\mathrm{Fil}(M^i/pM^i) \subset M^i/pM^i$ is $R^i = k[\![\alpha^i_{jl}]\!]_{1 \le j,l \le r}/(\alpha^i_{jl})^p$. The universal filtration, of course, is

$$\widetilde{\mathrm{Fil}}(M/pM)^i = \mathrm{span}\langle y^i_j + \sum_{l=1}^{r} \alpha^i_{jl} x^i_l \rangle.$$

Similarly, the local moduli space for the filtration on the first homology of the dual abelian variety is $k[\![\beta_{jl}]\!]/(\beta_{jl})^p$, and the filtration which lives over it is

$$\widetilde{\mathrm{Fil}}(M^*/pM^*)^i = \mathrm{span}\langle x^{i*}_j + \sum_{l=1}^{r} \beta_{jl} y^{i*}_l \rangle.$$

In the present setting these two moduli spaces should be somehow linked; to any algebraic deformation of $X$ corresponds a deformation of $X^\vee$, so that $X^\vee$ truly does remain the dual abelian scheme. The important condition is that

$$\langle \widetilde{\mathrm{Fil}}(M/pM)^i, \widetilde{\mathrm{Fil}}(M^*/pM^*)^i \rangle = (0).$$

This imposes certain relations, e.g.,

$$
\begin{aligned}
0 &= \langle y_j^i + \sum \alpha_{jl}^i x_l^i, x_{j'}^{i*} + \sum_{l'} \beta_{j'l'} y_{l'}^{i*} \rangle \\
&= \beta_{j'j} + \alpha_{jj'}^i \\
\beta_{j'j} &= -\alpha_{jj'}^i
\end{aligned}
$$

So make these identifications systematically and write

$$\widetilde{\mathrm{Fil}}(M^*/pM^*)^i = \mathrm{span}\langle x_j^{i*} - \sum_{l=1}^{r} \alpha_{lj}^i y_l^{i*} \rangle.$$

The polarization $X \xrightarrow{\lambda} X^\vee$ induces $M \xrightarrow{\lambda_*} M^*$ and $H_1^{\mathrm{cris}}(X) \xrightarrow{\lambda_*} H_1^{\mathrm{cris}}(X^\vee)$. If the deformation is algebraic, it must be a map of filtered crystals; $\lambda_*(\mathrm{Fil}(X)^i) \subseteq \mathrm{Fil}(X^\vee)^i$.

For $1 \leq j \leq r-1$, $\lambda_*(y_j^i + \sum \alpha_{jl}^i x_l^i) = -x_j^{i*} + \sum_{l=1}^{r-1} \alpha_{jl}^i y_l^{i*} + p\alpha_{jr}^i y_r^{i*}$. If this is to lie in $\widetilde{\mathrm{Fil}}(X)^i$, then

$$-x_j^{i*} + \sum_{l=1}^{r-1} \alpha_{jl}^i y_l^{i*} + p\alpha_{jr}^i y_r^{i*} = -x_j^{i*} + \sum_{l=1}^{r} \alpha_{lj}^i y_l^{i*}.$$

Equate coefficients of $y_l^{i*}$ to find that

$$
\begin{aligned}
&\boxed{\alpha_{jl}^i = \alpha_{lj}^i} \quad 1 \leq j < l \leq r-1 \\
&\boxed{\alpha_{rj}^i = 0} \quad 1 \leq j \leq r-1
\end{aligned}
$$

Similarly, $\lambda_*(y_r^i + \sum \alpha_{rl}^i x_l^i) = \sum_{l=1}^{r-1} \alpha_{rl}^i y_l^{i*} \bmod p$, which again forces $\alpha_{rl}^i = 0$ for $1 \leq l \leq r-1$.

This gives the leading terms of certain local equations for the moduli space at $(X/k, \iota, \lambda)$. We will see shortly that these represent all the equations.

$\boxed{M^i \text{ not nice}}$ Not surprisingly, a similar methodology computes local equations for the non-nice eigenspaces. This time, choose a normal form for $M^i = W(k)\{x_1^i, \cdots, x_r^i, y_1^i, \cdots, y_r^i\}$, where

$$\begin{aligned}
Fx_j^i &\in M^{i+1} - pM^{i+1} \\
Fy_j^i &\in pM^{i+1} \\
\langle x_1^i, x_2^i \rangle &= 1 \\
\langle y_1^i, y_2^i \rangle &= p \\
\langle x_j^i, y_j^i \rangle &= 1 \quad (3 \leq j \leq r)
\end{aligned}$$

and all other elements pair to zero. Again, the universal filtrations are given by

$$\begin{aligned}
\widetilde{\mathrm{Fil}}(\mathrm{X})^i &= \mathrm{span}\langle y_j^i + \sum_{l=1}^{r} \alpha_{jl}^i x_l^i \rangle \\
\widetilde{\mathrm{Fil}}(X^\vee)^i &= \mathrm{span}\langle x_j^{i*} - \sum_{l=1}^{r} \alpha_{lj}^i y_l^{i*} \rangle
\end{aligned}$$

and we must find the conditions ensuring that $\lambda_*(\widetilde{\mathrm{Fil}}(X)^i) \subseteq \widetilde{\mathrm{Fil}}(X^\vee)^i$. We find that

$$\begin{aligned}
\lambda_*(y_1^i + \sum_{l=1}^{r} \alpha_{1l}^i x_l^i) &= py_2^{i*} + \alpha_{11}^i x_2^{i*} - \alpha_{12}^i x_1^{i*} + \sum_{l=3}^{r} \alpha_{1l}^i y_l^{i*} \\
&= \alpha_{11}^i (x_2^{i*} - \sum_{l=1}^{r} \alpha_{l2}^i y_l^{i*}) - \alpha_{12}^i (x_1^{i*} - \sum_{l=1}^{r} \alpha_{l1}^i y_l^{i*}) \\
&= -\alpha_{12}^i x_1^{i*} + \alpha_{11}^i x_2^{i*} + \sum_{l=1}^{r} (\alpha_{12}^i \alpha_{l1}^i - \alpha_{11}^i \alpha_{l2}^i) y_l^{i*}
\end{aligned}$$

No relation comes from the coefficient of $y_1^{i*}$, but

$$\boxed{\alpha^i_{12}\alpha^i_{21} - \alpha^i_{11}\alpha^i_{22} = 0} \quad l = 2$$

$$\alpha^i_{1l} = \alpha^i_{12}\alpha^i_{l1} - \alpha^i_{11}\alpha^i_{l2} \quad 3 \leq l \leq r$$

Already, we see that the defect from nice introduces singularities to the moduli space. Similarly, examining $\lambda_*(y^i_2 + \sum_{l=1}^r \alpha^i_{2l}x^i_l)$ yields the additional relation

$$\alpha^i_{2l} = \alpha^i_{22}\alpha^i_{l1} - \alpha^i_{21}\alpha^i_{l2} \quad 3 \leq l \leq r$$

When $3 \leq j \leq r$, the natural constraint is

$$\begin{aligned}
\lambda_*(y^i_j + \sum_{l=1}^r \alpha^i_{jl}x^i_l) &= -x^{i*}_j + \alpha^i_{j1}x^{i*}_2 - \alpha^i_{j2}x^{i*}_1 + \sum_{l=3}^r \alpha^i_{jl}y^{i*}_l \\
&= -(x^{i*}_j + \sum_{l=1}^r \alpha^i_{lj}y^{i*}_l) + \alpha^i_{j1}(x^{i*}_2 - \sum_{l=1}^r \alpha^i_{l2}y^{i*}_l) - \alpha^i_{j2}(x^{i*}_1 - \sum_{l=1}^r \alpha^i_{l1}y^{i*}_l)
\end{aligned}$$

Thus, we get local equations

$$\boxed{\alpha^i_{lj} = \alpha^i_{j1}\alpha^i_{l2} - \alpha^i_{j2}\alpha^i_{l1}} \qquad 1 \leq l < 3 \leq j \leq r$$

$$\boxed{\alpha^i_{jl} = \alpha^i_{lj} - \alpha^i_{j1}\alpha^i_{l2} + \alpha^i_{j2}\alpha^i_{l1}} \quad 3 \leq l < j \leq r$$

We resume the general discussion. Note that, whether or not $M^i$ is nice, the crystalline cohomology sees $s \stackrel{\text{def}}{=} \frac{r(r-1)}{2}$ local equations for each $i \in \mathbb{Z}/f\mathbb{Z}$. A priori, it is possible that there are other equations of sufficiently high leading degree that we cannot detect them using crystalline techniques. However, in view of the lower bound 2.2, we know that we have seen the avatars of all equations for the local moduli space.

Indeed, let $R^i = k[[\alpha^i_{jl}]]$ with maximal ideal $\mathfrak{m}^i$; let $I^i \subset R^i$ be the ideal generated by the boxed relations $f^i_1, \cdots, f^i_s$ above. We have shown that

**Lemma 4.1**

$$\widehat{O}_{\mathcal{A}_{g,d}^{O_K},(X,\iota,\lambda)}/\mathfrak{m}^p_{(X,\iota,\lambda)} = \otimes_{i\in\mathbb{Z}/f\mathbb{Z}} \frac{R^i}{(I^i,(\mathfrak{m}^i)^p)}.$$

Much of the structure of $\mathcal{A}_{g,d}^{O_K}$ is encoded in this lemma, which we now exploit.

**Theorem 4.2** Let $K$ be a totally real field unramified at $p$, and suppose $\mathrm{ord}_p\, d = [K:\mathbb{Q}]$. Then ordinary points are dense in $\mathcal{A}_{g,d}^{O_K}$.

**Proof** This may be proved with a direct calculation in covariant Dieudonné theory, guided by the crystalline computation. Indeed, this is the proof the author originally used. However, an easier argument is available.

Although it seems likely that the nice rank completely determines the isomorphism class of every completed local ring of $\mathcal{A}_{g,d}^{O_K}$ are integral, we content ourselves here with extracting the minimal amount of information necessary to deduce the theorem. Start with the preliminary observation that any smooth point of $(\mathcal{A}_{g,d}^{O_K})_{\mathrm{red}}$ is nice. Indeed, let $J$ be the local ring at a non-nice point $(X,\iota,\lambda)$. Lemma 4.1 gives the initial forms of elements $f^i_j \in k[[\alpha^i_{jl}]]$ presenting $J$, one of which has the form

$$\alpha^i_{12}\alpha^i_{21} - \alpha^i_{11}\alpha^i_{22} + \text{higher order terms} = 0.$$

It is conceivable that there are additional relations in $\mathrm{rad}(J)$; but if they were linear or quadratic, then the dimension of $O_{\mathcal{A}_{g,d}^{O_K},(X,\iota,\lambda)}$ would drop below the lower bound guaranteed by 2.2. Thus, any non-nice point is singular in $(\mathcal{A}_{g,d}^{O_K})_{\mathrm{red}}$.

This means that the nice locus, which is always open, is actually dense in [every component of] $\mathcal{A}_{g,d}^{O_K}$. Now use theorem 3.1 to see that the ordinary locus is consequently open and dense.$\diamond$

**Remark 4.3** In fact, the leading terms computed above tell us a little bit more about the [formal] local structure of $\mathcal{A}_{g,d}^{O_K}$. At any fixed $k$-point, lemma 4.1 provides the initial forms of $f\frac{r(r-1)}{2}$ equations. The tangent cone is the spectrum of a quotient of $k[[\alpha^i_{jl}]]/(I^i)$, the algebra defined by these initial forms. However, since this ring already has the minimum allowed dimension, it must actually be the ring of functions of the tangent cone. In particular, the local ring is a local complete intersection, and it is an integral domain since the tangent cone is one, too.

# 5 Arbitrary inseparability

So far, only mild singularities have been allowed in the spaces considered. It is quite difficult to understand the geometry of $\mathcal{A}_{g,d}^{O_B}$ when $d$ is highly divisible by $p$. The spaces tend to be nonreduced, so that the crystalline theory tells us little about them. We introduce now an arbitrary degree of inseparability, and attempt to describe the structure of the associated moduli space. Unfortunately, at present the only proof we know of the density of the ordinary locus works in the presence of serious restrictions on the ring of endomorphism. Still, the current result highlights certain techniques which may prove of independent interest.

**Theorem 5.1** Suppose $K$ is a totally real field, all of whose residue degrees at $p$ are 1 or 2; $g = 2[K : \mathbb{Q}]$; and $d \in \mathbb{N}$ is arbitrary. Then ordinary points are dense in $\mathcal{A}_{g,d}^{O_K}$.

**Proof** Using idempotents in the usual fashion, we may assume that either $K = \mathbb{Q}$ and $g = 2$, or $K$ is a real quadratic field inert at $p$ and $g = 4$. The former case is handled by [N-O], and we thus restrict our attention to abelian fourfolds equipped with an action by a real quadratic field.

We want to show that any point in $\mathcal{A}_{g,d}^{O_K}$, if it is not itself ordinary, admits a deformation to an abelian variety with strictly greater $p$-rank. Let $(X, \iota, \lambda) \in \mathcal{A}_{g,d}^{O_K}(k)$ be such a point. Because of the $O_K$-action, the $p$-rank $\rho(X)$ is constrained; $\rho(X) \in \{0, 2, 4\}$. If $\rho(X) = 4$, there's nothing to do.

If $\rho(X) = 2$, we quickly arrive at a Hilbert-Blumenthal type deformation problem in the following way. The $p$-divisible group $X[p^\infty]$ splits as a sum $X[p^\infty] = X[p^\infty]^{\text{ét,tor}} \oplus X[p^\infty]'$, where each summand is a height four $p$-divisible group equipped with an $O_K$-action. The local-local part $X[p^\infty]'$ easily deforms to an ordinary $O_K$-$p$-divisible group; either a direct deformation theory computation – $\mathbb{D}_*(X[p^\infty]')$ is nice, so 3.1 applies – or an appeal to [D-Ri] suffices. Thus, $(X, \iota, \lambda)$ deforms to ordinary.

Finally, if $\rho(X) = 0$, the following argument from dimension counts lets us conclude the existence of an ordinary deformation. For suppose no such exists; then there is a component $(X, \iota, \lambda) \in Z \subset \mathcal{A}_{g,d}^{O_K}$ with $\rho(Z) = 0$. On one hand, by 2.2, $\dim_k Z \geq 6$. On the other hand, as explained in 1.2, the forgetful morphism $\phi : \mathcal{A}_{g,d}^{O_K} \to \mathcal{A}_{g,d}$ is quasifinite. Thus, $\dim Z = \dim \phi(Z) \geq 6$. But $\phi(Z) \subseteq \mathcal{A}_{g,d}(\rho = 0)$, and the latter has dimension six by theorem 4.1 of [N-O]. So $\dim Z = \dim \mathcal{A}_{g,d}(\rho = 0) = 6$, and a big piece of the $p$-rank zero locus in the Siegel moduli space parametrizes abelian varieties with an $O_K$-action. I claim that this is absurd.

Indeed, the existence [or non-existence] of such a component of $\mathcal{A}_{g,d}^{O_K}$ depends only on the structure of $K \otimes \mathbb{Q}_p$. Let $\{K_i\}$ be an infinite family of distinct real quadratic fields inert at $p$. By supposition, for each $i$ there is a six-dimensional subscheme $Z_i \subset \mathcal{A}_{g,d}^{O_{K_i}}$ with $\rho(Z_i) = 0$. Since $\mathcal{A}_{g,d}(\rho = 0)$ is of finite type, there are $i_1, \cdots, i_h$ and a component $Y$ of $\mathcal{A}_{g,d}(\rho = 0)$ so that $\phi_{i_j}(Z_{i_j}) \subseteq Y$, and the compositum $K'$ of the $K_{i_j}$ has degree $> g$ over $\mathbb{Q}$. Take any $(X', \lambda') \in \cap_{j=1}^{h} \phi_{i_j}(Z_{i_j}) \neq \emptyset$; it has multiplication by a totally real field of dimension $> g$, which is absurd (e.g., [Mum2], p.178). So no component $Z \subset \mathcal{A}_{g,d}^{O_K}$ has $\rho(Z) = 0$, and ordinary points are dense in $\mathcal{A}_{g,d}^{O_K}$.$\diamond$

## References


[EGA] A. Grothendieck and J. Dieudonné, *Éléments de Geometrie Algebrique*, Publ. Math. de l'IHES **4, 8, 11, 17, 20, 24** and **32** (1960-67).

[BBM] P. Berthelot, L. Breen, W. Messing, *Théorie de Dieudonné cristalline II*, Lecture Notes in Mathematics **930**, Springer-Verlag (1982).

[C-N] C.-L. Chai and P. Norman, "Singularities of the $\Gamma_0(p)$-level Structure", *J. Alg. Geom.* **1** (1992), 251-278.

[D-P] P. Deligne and G. Pappas, "Singularités des espaces de modules de Hilbert, en les caractéristiques divisant le discriminant", *Comp. Math.* **90** (1994), 59-79.

[D-R] P. Deligne and M. Rapoport, "Modules des courbes elliptiques", in *Modular functions of one variable II*, Lecture Notes in Mathematics **349** (1973), ???.

[D-Ri] P. Deligne and K. Ribet, "Values of abelian $L$-functions at negative integers over totally real fields", *Inv. Math.* **59** (1980), 227-286.

[Deu] M. Deuring, "Die Typen der Multiplikatorenringe elliptischer Funktionen körper", *Abhanlungen aus dem Math. Sem. der Universität Hamburg* **14** (1941), 197-272.

[F-C] G. Faltings and C.-L. Chai, *Degeneration of Abelian Varieties*, Ergebnisse der Mathematik und ihrer Grezgebiete, 3. Folge Band 22, Springer-Verlag 1990.

[Gr] A. Grothendieck, *Groupes de Barsotti-Tate et cristaux de Dieudonné*, Seminaire de Mathématiques Supérieures **45**, Presses de l'Université de Montréal, 1974.

[Haz] M. Hazewinkel, *Formal groups and applications*, Academic Press (1978).

[dJ] A.J. de Jong, "The moduli spaces of polarized abelian varieties", *Math. Ann.* **295** (1993), 485-503.

[Katz] N. Katz, "Serre-Tate local moduli", in *Surfaces algébriques*, ed. J. Giraud, L. Illusie and M. Raynaud, Lecture Notes in Mathematics **868**, Springer-Verlag (1981), 138-202.

[KaMa] N. Katz and B. Mazur, *Arithmetic moduli of elliptic curves*, Princeton University Press (1985).

[KS] K. Kodaira and D.C. Spencer, "On deformations of complex analytic structures I, II", *Ann. of Math.* **2** (1958), 328-466.

[Kott] R. Kottwitz, "Points on some Shimura varieties over finite fields", *J. Amer. Math. Soc.* **5** (1992), 373-444.

[LST] J. Lubin, J.P. Serre and J. Tate, "Elliptic curves and formal groups", Woods Hole 1964 (mimeographed notes).

[Man] Yu. Manin, "Theory of commutative formal groups over fields of finite characteristic", *Uspehi Mat. Nauk* **18** (1963) pp.3-90.

[Mess] W. Messing, *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes*, Lecture Notes in Mathematics **264**, Springer-Verlag (1972).

[Mum] D. Mumford, "Bi-extensions of formal groups", *Algebraic Geometry*, Oxford University Press, London, 1968.

[Mum2] D. Mumford, *Abelian Varieties*, Oxford University Press (1974).

[Nor] P. Norman, “An algorithm for computing local moduli of abelian varieties”, *Ann. of Math.* **101** (1975), 499-509.

[N-O] P. Norman and F. Oort, “Moduli of abelian varieties”, *Ann. of Math.* **112** (1980), 413-429.

[Oort] F. Oort, “Finite group schemes, local moduli for abelian varieties, and lifting problems”, *Algebraic Geometry*, Oslo 1970, 223-254.

[Rap] M. Rapoport, “Compactifications de l’espace de modules de Hilbert-Blumenthal”, *Comp. Math.* **36** (1978), 255-335.

[R-Z] M. Rapoport and Th. Zink, *Period Spaces for p-divisible Groups*, Annals of Mathematics Studies **141**, Princeton University Press, 1996.

[Zink] Th. Zink, *Cartiertheorie kommutativer formaler gruppen*, Teubner 1984.