On the mathematical works of Stephen Shatz

Ching-Li Chai

Department of Mathematics
University Pennsylvania

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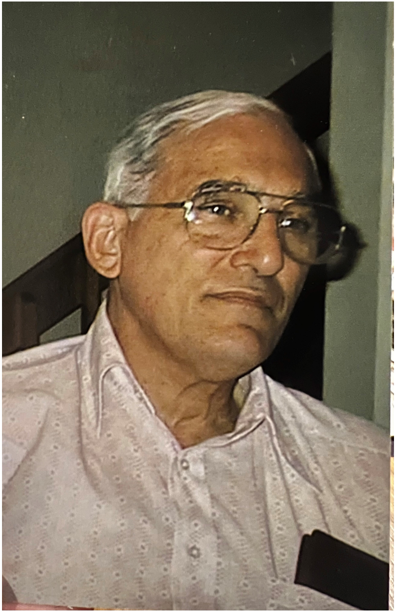

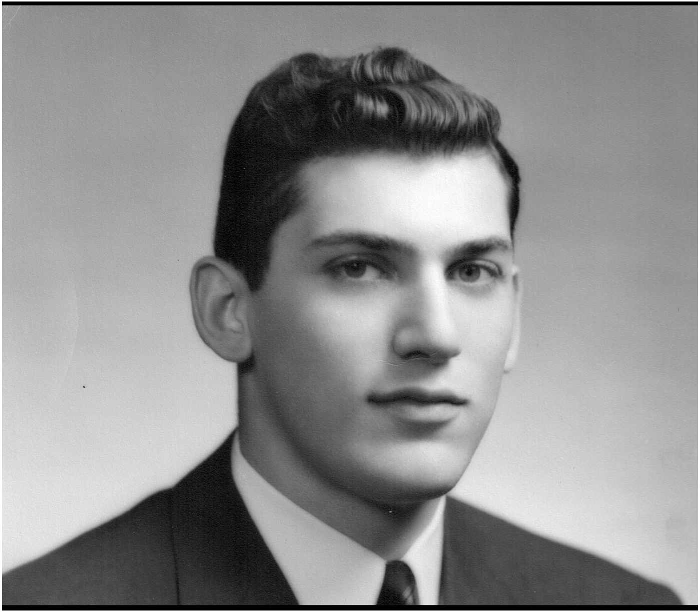

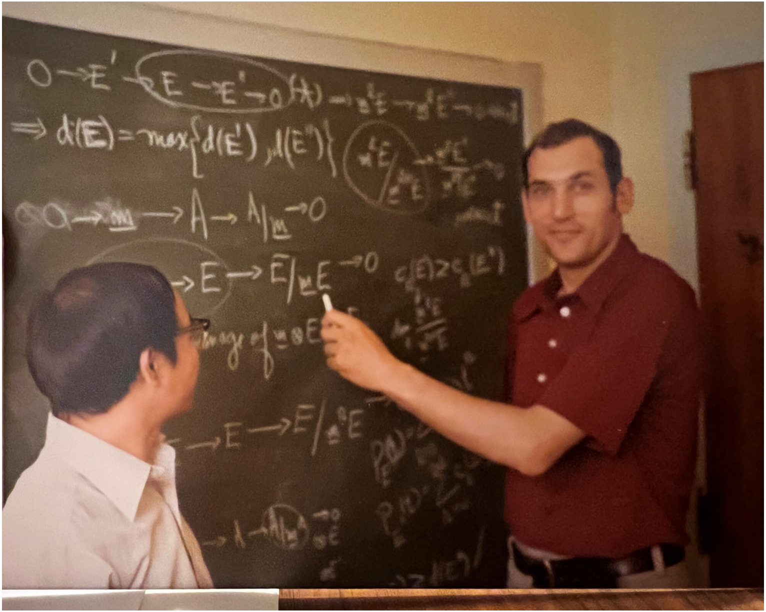

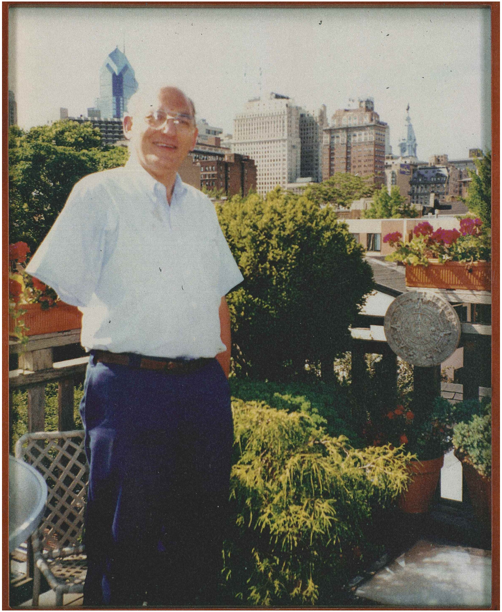

References

1. S. S. Shatz, *Cohomology of Artinian group schemes over local fields*, Ann. Math. **79** (1964), 411–449, received January 29, 1963.

2. S. S. Shatz, *The cohomological dimension of certain Grothendieck topologies*, Ann. Math. **83** (1966), 572–595.

3. S. S. Shatz, *The decomposition and specialization of algebraic families of vector bundles*, Compositio Math. **35** (1977), 163–187.

S. S. Shatz, *The mathematical circle around Tate in the late fifties and early sixties*, Pure Appl. Math. Quarterly 5 (2009), 1429–1433.

1962 Harvard Thesis = 1964 Annals paper

Theorem 1. (S. S. S. 1962) Let k be a non-archimedean locally compact field. Let G be an artinian commutative group scheme over k, and let G^D be the Cartier dual of G. Then the cup product pairing

$$H^r_{\mathrm{fppf}}(k,G) \times H^{2-r}_{\mathrm{fppf}}(k,G^D) \longrightarrow H^{2-r}_{\mathrm{fppf}}(k,\mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}$$

is a perfect duality pairing between locally compact groups, for every $r \in \mathbb{Z}$. In particular $H^r_{\mathrm{fppf}}(k,G) = (0)$ if $r \geq 3$.

Note: In the above $\mathbb{G}_m = \mathrm{Spec}(k[T,T^{-1}])$ is the multiplicative group scheme over k, and $H^r_{\mathrm{fppf}}(k,G) := H^r(\mathrm{Spec}(k),G)$ is the r-th cohomology with coefficient G for the fppf (flat and of finite presentation) topology of the spectrum of k.

1966 Annals paper

Theorem 2. (S. S. S. 1966) Let $(R, \mathfrak{m})$ be a noetherian complete local ring of characteristic $p > 0$, i.e. $p \cdot 1_R = 0$ in R. Assume either that $\mathfrak{m} \neq 0$, or that R is a field which is *not perfect*.

(i) The cohomological dimension of the flat fppf topology of $\mathrm{Spec}(R)$ is ∞.

(ii) Assume moreover that the residue field k is separably closed. Then for every integer $n > 0$, there exists a sheaf F_n of abelian groups for the flat fppf topology of $\mathrm{Spec}(R)$, such that

$$H^i_{\mathrm{fppf}}(\mathrm{Spec}(R), F_n) = (0) \quad \forall\, i \neq 0, n,$$

and

$$H^n_{\mathrm{fppf}}(\mathrm{Spec}(R), F_n) = R/R^p.$$

1964 Annals paper continued

Theorem 3. (S. S. S. 1964) Let k be a field of characteristic $p > 0$. Let G be a commutative group scheme of finite type over k. Then

$$(H^i_{\mathrm{fppf}}(\mathrm{Spec}(k), G)[p^\infty] = (0) \quad \forall\, i > 2,$$

where $H^i_{\mathrm{fppf}}(\mathrm{Spec}(k), G)[p^\infty]$ is the p-primary part of torsion group $H^i_{\mathrm{fppf}}(\mathrm{Spec}(k), G)$.

1977 Compositio paper

The set-up.
– Let X be a projective smooth algebraic variety over an algebraically closed field k, and let S be a noetherian scheme over k.

– Let $\mathscr{L}$ be an ample line bundle on X Let $\mathscr{V} \to X \times S$ be a family of algebraic vector bundles on X parametrized by S.

– For each $s \in S$, let $\mathrm{HNP}_{\mathscr{L}}(V_s)$ be the polygon attached to the Harder–Narasimhan filtration on V_s w.r.t. $\mathscr{L}$.

Each $\mathrm{HNP}_{\mathscr{L}}(V_s)$ is a convex polygon with integer vertices on the first quadrant, connecting $(0,0)$ to $(\mathrm{rk}(V_s), \deg_{\mathscr{L}}(V_s))$, the latter end point being independent of $s \in S$).
Convexity here means: $\mathrm{HNP}_{\mathscr{L}}(V_s)$ lies above the line segment from $(0,0)$ to $(\mathrm{rk}(V_s), \deg_{\mathscr{L}}(V_s))$, and the region bounded in between is convex. Equivalently, $\mathrm{HNP}_{\mathscr{L}}(V_s)$ is the graph of a *concave* function.

The Shatz stratification

Theorem 4. (S. S. S. 1977)

(i) The function $s \mapsto \mathrm{HNP}_{\mathscr{L}}(V_s)$ is *upper semi-continuous* on S, for the Zariski topology on S and the discrete poset consisting of all convex polygons from $(0,0)$ to $(\mathrm{rk}(V_s), \deg_{\mathscr{L}}(V_s))$ with integer vertices. In other words, the HN polygon *rises* under specialization.

(ii) The function $s \mapsto \mathrm{HNP}_{\mathscr{L}}(V_s)$ induces a stratification on S by locally closed subsets, indexed by the set of all convex polygons from $(0,0)$ to $(\mathrm{rk}(V_s), \deg_{\mathscr{L}}(V_s))$ with integer vertices. (Some strata may by empty.)

Remark. Applied to "the moduli space of vector bundles on X of a given rank and degree", this theorem gives a stratification of these moduli spaces.

Arithmetic duality for local fields

Theorem 1. (S. S. S. 1962) Let k be a non-archimedean locally compact field. Let G be an artinian commutative group scheme over k, and let G^D be the Cartier dual of G. Then the cup product pairing

$$H^r_{\mathrm{fppf}}(k,G) \times H^{2-r}_{\mathrm{fppf}}(k,G^D) \longrightarrow H^2_{\mathrm{fppf}}(k,\mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}$$

is a perfect duality pairing between locally compact groups, for every $r \in \mathbb{Z}$. In particular $H^r_{\mathrm{fppf}}(k,G) = (0)$ if $r \geq 3$.

Cohomological dimension of the small fppf site

Theorem 2. (S. S. S. 1966) Let $(R, \mathfrak{m})$ be a noetherian complete local ring of characteristic $p > 0$, i.e. $p \cdot 1_R = 0$ in R. Assume either that $\mathfrak{m} \neq 0$, or that R is a field which is *not perfect*. Then the cohomological dimension of the small flat site $\mathrm{Spec}(R)_{\mathrm{fppf}}$ is ∞. Furthermore if k is separably closed, then $\forall n > 0$, $\exists$ an fppf abelian sheaf F_n on $\mathrm{Spec}(R)$, s.t. $H^n_{\mathrm{fppf}}(\mathrm{Spec}(R), F_n) = R/R^p$ and $H^i_{\mathrm{fppf}}(\mathrm{Spec}(R), F_n) = (0)$ $\forall i \neq 0, n$.

Theorem 3. (S. S. S. 1964) Let k be a field of char. $p > 0$. Let G be a commutative group scheme of finite type over k. Then

$$H^i_{\mathrm{fppf}}(\mathrm{Spec}(k), G)[p^\infty] = (0) \quad \forall i > 2.$$

Historical context: homological algebra and number theory

1. Galois cohomology and class field theory (T. Takagi 1920, E. Artin 1927, Chevalley's algebraic approach with adeles and ideles 1940)

1a. G. Hochschild and T. Nakayama, *Cohomology in class field theory*, Ann. Math. **55** (1952), 348–366.

1b. Artin–Tate Seminar 1951/52.

1c. J. Tate, *The higher dimensional cohomology groups of class field theory*, Ann. Math. **56**, 1952, 294-297.

1d. J. Tate, Duality theorems in Galois cohomology over number fields, ICM 1962

Historical context: homological algebra and algebraic geometry

2a. J.-P. Serre, *Faisceaux algébriques cohérents*, *Ann. Math.* **61** (1955), 197–278.

2b. J.-P. Serre, *Géométrie algébrique et géometrie analytique*, Ann. Inst. Fourier **6** (1956), 1–42.

2c. A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tohoku Math. J. **2** (1957), 119–221.

Historical context: flatness and algebraic geometry

3a. The notion of *flatness* for a module or an algebra over a commutative ring R was introduced in the appendix of Serre's GAGA.

3b. Serre's GAGA principle was generalized by Grothendieck to the context of *coherent sheaves on formal schemes* in *Géométrie formelle et gémétrie algébrique*, Séminaire Bourbaki, n^0 182, May 1959.

3c. A new vision and a prelude to Grothendieck's theory of topos: A. Grothendieck, *Technique de descente et théorème d'existence en géometrie alg'ebriques* I–VI, Séminaire Bourbaki 1959/60–1961/62, December 1959–May 1962.

Historical background: Grothendieck topology

4a. M. Artin, *Grothendieck topologies*, Harvard seminar notes, spring 1962.

4b. J. Giraud, *Analysis situs*, Séminaire Bourbaki 1962/63, n^0 256, March 1963.

4c. J.-L. Verdier, exposé I–IV of SGA4 (1963–64), Lecture Notes in Math. 269, on the theory of topos.

Note: Grothendieck visited Harvard in 1961/62, gave a course which contains a large portion of which is now EGA IV. The main results on etale cohomology in SGA 3 part III was obtained by him and Michael Artin between September 1962 and March 1983. The published version of Steve's 1962 thesis was mentioned by Grothendieck in the preface of (the second edition of) SGA 4 (with a typo "Schatz").

Shatz stratification of moduli of vector bundles

Theorem 4. (S. S. S. 1977) Let X be a smooth projective variety over an algebraically closed field k. Let $\mathscr{L}$ be an ample line bundle on X, and let $\mathscr{V} \to X \times S$ be a family of vector bundles on X parametrized by S.

(i) The function $s \mapsto \mathrm{HNP}_{\mathscr{L}}(V_s)$ is *upper semi-continuous* on S, for the Zariski topology on S and the discrete poset consisting of all convex polygons from $(0,0)$ to $(\mathrm{rk}(V_s), \deg_{\mathscr{L}}(V_s))$ with integer vertices. In other words, the HN polygon *rises* under specialization.

(ii) The function $s \mapsto \mathrm{HNP}_{\mathscr{L}}(V_s)$ induces a stratification on S by locally closed subsets, indexed by the set of all convex polygons from $(0,0)$ to $(\mathrm{rk}(V_s), \deg_{\mathscr{L}}(V_s))$ with integer vertices. (Some strata may by empty.)

Slope of a vector bundle

Definition. Let X be a d-dimensional projective smooth variety over an algebraically closed field k, and let $\mathscr{L}$ be an ample line bundle on X. For every torsion-free coherent sheaf of $\mathscr{O}_X$-module E, define the $\mathscr{L}$*-slope* $\mu_{\mathscr{L}}(E)$ of E by

$$\mu_{\mathscr{L}}(E) = \frac{\deg_{\mathscr{L}}(E)}{\mathrm{rk}(E)} = \frac{c_1(E)\cdot \mathscr{L}^{d-1}}{\mathrm{rk}(E)}.$$

Harder–Narasimhan flag of a vector bundle

Definition. Let V be a vector bundle on X. The *Harder–Narasimhan flag* on V is the *unique* flag

$$V = V_m \supsetneqq V_{m-1} \supsetneqq \cdots \supsetneqq V_1 \supsetneqq V_0 = (0)$$

of torsion free coherent sheaves of $\mathscr{O}_X$-submodules of V such that
(i) $\mu_{\mathscr{L}}(V_j/E_{j-1}) > \mu_{\mathscr{L}}(V_{j+1}/V_j)$ for $j = 1, \ldots m-1$, and
(ii) V_j/V_{j-1} is *semistable* for all $j = 1, \ldots, m$ in the sense that

$$\mu_{\mathscr{L}}(F) \leq \mu_{\mathscr{L}}(V_j/V_{j-1})$$

for every torsion free coherent $\mathscr{O}_X$-submodule $F \subseteq V_j/V_{j-1}$.

The slope polygon of a vector bundle

Definition. Let $V = V_m \supsetneq V_{m-1} \supsetneq \cdots \supsetneq V_1 \supsetneq V_0 = (0)$ be the Harder–Narasimhan flag of a vector bundle V on X. The slope polygon $\mathrm{HNP}_{\mathscr{L}}(V)$ of V is the polygon on $\mathbb{R}^2$ with vertices

$$(0,0), (\mathrm{rk}(V_1), \deg_{\mathscr{L}}(V_1)), \ldots, (\mathrm{rk}(V_m), \deg_{\mathscr{L}}(V_m)).$$

Note. The $\mathscr{L}$-slopes of $\mathrm{HNP}_{\mathscr{L}}(V)$ between vertices are $\frac{\deg_{\mathscr{L}}(V_1)}{\mathrm{rk}(V_1)} = \mu_{\mathscr{L}}(V_1/V_0) > \cdots > \mu_{\mathscr{L}}(V_m/V_{m-1}) = \frac{\deg_{\mathscr{L}}(V_m)}{\mathrm{rk}(V_m)}$.