ON THE MATHEMATICAL WORKS OF STEPHEN SHATZ

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**Theorem 1.** (S. S. S. 1962) Let $k$ be a non-archimedean locally compact field. Let $G$ be an artinian commutative group scheme over $k$, and let $G^D$ be the Cartier dual of $G$. Then the cup product pairing

$$H^r_{fppf}(k, G) \times H^r_{fppf}(k, G^D) \longrightarrow H^2_{fppf}(k, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}$$

is a perfect duality pairing between locally compact groups, for every $r \in \mathbb{Z}$. In particular $H^r_{fppf}(k, G) = (0)$ if $r \geq 3$.

Note: In the above $\mathbb{G}_m = \text{Spec}(k[T, T^{-1}])$ is the multiplicative group scheme over $k$, and $H^r_{fppf}(k, G) := H^r(\text{Spec}(k), G)$ is the $r$-th cohomology with coefficient $G$ for the fppf (flat and of finite presentation) topology of the spectrum of $k$. 
Theorem 2. (S. S. S. 1966) Let \((R, \mathfrak{m})\) be a noetherian complete local ring of characteristic \(p > 0\), i.e. \(p \cdot 1_R = 0\) in \(R\). Assume either that \(\mathfrak{m} \neq 0\), or that \(R\) is a field which is not perfect.

(i) The cohomological dimension of the flat fppf topology of \(\text{Spec}(R)\) is \(\infty\).

(ii) Assume moreover that the residue field \(k\) is separably closed. Then for every integer \(n > 0\), there exists a sheaf \(F_n\) of abelian groups for the flat fppf topology of \(\text{Spec}(R)\), such that

\[
H^i_{\text{fppf}}(\text{Spec}(R), F_n) = (0) \quad \forall i \neq 0, n,
\]

and

\[
H^n_{\text{fppf}}(\text{Spec}(R), F_n) = R/R^p.
\]
Theorem 3. (S. S. S. 1964) Let $k$ be a field of characteristic $p > 0$. Let $G$ be a commutative group scheme of finite type over $k$. Then

$$(H^i_{\text{fppf}}(\text{Spec}(k), G)[p^\infty] = (0) \quad \forall i > 2,$$

where $H^i_{\text{fppf}}(\text{Spec}(k), G)[p^\infty]$ is the $p$-primary part of torsion group $H^i_{\text{fppf}}(\text{Spec}(k), G)$. 
1977 Compositio paper

The set-up.
– Let $X$ be a projective smooth algebraic variety over an algebraically closed field $k$, and let $S$ be a noetherian scheme over $k$.
– Let $\mathcal{L}$ be an ample line bundle on $X$ Let $\mathcal{V} \to X \times S$ be a family of algebraic vector bundles on $X$ parametrized by $S$.
– For each $s \in S$, let $\text{HNP}_\mathcal{L}(V_s)$ be the polygon attached to the Harder–Narasimhan filtration on $V_s$ w.r.t. $\mathcal{L}$.

Each $\text{HNP}_\mathcal{L}(V_s)$ is a convex polygon with integer vertices on the first quadrant, connecting $(0,0)$ to $(\text{rk}(V_s), \deg \mathcal{L}(V_s))$, the latter end point being independent of $s \in S$).

Convexity here means: $\text{HNP}_\mathcal{L}(V_s)$ lies above the line segment from $(0,0)$ to $(\text{rk}(V_s), \deg \mathcal{L}(V_s))$, and the region bounded in between is convex. Equivalently, $\text{HNP}_\mathcal{L}(V_s)$ is the graph of a concave function.
The Shatz stratification

Theorem 4. (S. S. S. 1977)

(i) The function \( s \mapsto \text{HNP}_L(V_s) \) is upper semi-continuous on \( S \), for the Zariski topology on \( S \) and the discrete poset consisting of all convex polygons from \((0,0)\) to \((\text{rk}(V_s), \deg_L(V_s))\) with integer vertices. In other words, the HN polygon *rises* under specialization.

(ii) The function \( s \mapsto \text{HNP}_L(V_s) \) induces a stratification on \( S \) by locally closed subsets, indexed by the set of all convex polygons from \((0,0)\) to \((\text{rk}(V_s), \deg_L(V_s))\) with integer vertices. (Some strata may by empty.)

Remark. Applied to “the moduli space of vector bundles on \( X \) of a given rank and degree”, this theorem gives a stratification of these moduli spaces.
Arithmetic duality for local fields

**Theorem 1.** (S. S. S. 1962) Let $k$ be a non-archimedean locally compact field. Let $G$ be an artinian commutative group scheme over $k$, and let $G^D$ be the Cartier dual of $G$. Then the cup product pairing

$$H^r_{fppf}(k, G) \times H^r_{fppf}(k, G^D) \longrightarrow H^2_{fppf}(k, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}$$

is a perfect duality pairing between locally compact groups, for every $r \in \mathbb{Z}$. In particular $H^r_{fppf}(k, G) = (0)$ if $r \geq 3$. 

Cohomological dimension of the small fppf site

**Theorem 2.** (S. S. S. 1966) Let \((R, \mathfrak{m})\) be a noetherian complete local ring of characteristic \(p > 0\), i.e. \(p \cdot 1_R = 0\) in \(R\). Assume either that \(\mathfrak{m} \neq 0\), or that \(R\) is a field which is *not perfect*. Then the cohomological dimension of the small flat site \(\text{Spec}(R)_{\text{fppf}}\) is \(\infty\). Furthermore if \(k\) is separably closed, then \(\forall n > 0, \exists\) an fppf abelian sheaf \(F_n\) on \(\text{Spec}(R)\), s.t. \(H^n_{\text{fppf}}(\text{Spec}(R), F_n) = R/R^p\) and \(H^i_{\text{fppf}}(\text{Spec}(R), F_n) = (0)\) \(\forall i \neq 0, n\).

**Theorem 3.** (S. S. S. 1964) Let \(k\) be a field of char. \(p > 0\). Let \(G\) be a commutative group scheme of finite type over \(k\). Then

\[
H^i_{\text{fppf}}(\text{Spec}(k), G)[p^\infty] = (0) \quad \forall i > 2.
\]
Historical context: homological algebra and number theory

1. Galois cohomology and class field theory (T. Takagi 1920, E. Artin 1927, Chevalley’s algebraic approach with adeles and ideles 1940)


1b. Artin–Tate Seminar 1951/52.


1d. J. Tate, Duality theorems in Galois cohomology over number fields, ICM 1962
Historical context: homological algebra and algebraic geometry


Historical context: flatness and algebraic geometry

3a. The notion of flatness for a module or an algebra over a commutative ring $R$ was introduced in the appendix of Serre’s GAGA.

3b. Serre’s GAGA principle was generalized by Grothendieck to the context of coherent sheaves on formal schemes in *Géométrie formelle et géométrie algébrique*, Séminaire Bourbaki, n° 182, May 1959.

Historical background: Grothendieck topology


Note: Grothendieck visited Harvard in 1961/62, gave a course which contains a large portion of which is now EGA IV. The main results on étale cohomology in SGA 3 part III was obtained by him and Michael Artin between September 1962 and March 1983. The published version of Steve’s 1962 thesis was mentioned by Grothendieck in the preface of (the second edition of) SGA 4 (with a typo “Schatz”).
Theorem 4. (S. S. S. 1977) Let $X$ be a smooth projective variety over an algebraically closed field $k$. Let $\mathcal{L}$ be an ample line bundle on $X$, and let $\mathcal{V} \to X \times S$ be a family of vector bundles on $X$ parametrized by $S$.

(i) The function $s \mapsto \text{HNP}_{\mathcal{L}}(V_s)$ is upper semi-continuous on $S$, for the Zariski topology on $S$ and the discrete poset consisting of all convex polygons from $(0,0)$ to $(\text{rk}(V_s), \deg_{\mathcal{L}}(V_s))$ with integer vertices. In other words, the HN polygon rises under specialization.

(ii) The function $s \mapsto \text{HNP}_{\mathcal{L}}(V_s)$ induces a stratification on $S$ by locally closed subsets, indexed by the set of all convex polygons from $(0,0)$ to $(\text{rk}(V_s), \deg_{\mathcal{L}}(V_s))$ with integer vertices. (Some strata may by empty.)
**Definition.** Let $X$ be a $d$-dimensional projective smooth variety over an algebraically closed field $k$, and let $\mathcal{L}$ be an ample line bundle on $X$. For every torsion-free coherent sheaf of $\mathcal{O}_X$-module $E$, define the $\mathcal{L}$-slope $\mu_{\mathcal{L}}(E)$ of $E$ by

$$\mu_{\mathcal{L}}(E) = \frac{\deg_{\mathcal{L}}(E)}{\text{rk}(E)} = \frac{c_1(E) \cdot \mathcal{L}^{d-1}}{\text{rk}(E)}.$$
Harder–Narasimhan flag of a vector bundle

**Definition.** Let $V$ be a vector bundle on $X$. The **Harder–Narasimhan flag** on $V$ is the unique flag

$$V = V_m \supsetneq V_{m-1} \supsetneq \cdots \supsetneq V_1 \supsetneq V_0 = (0)$$

of torsion free coherent sheaves of $\mathcal{O}_X$-submodules of $V$ such that

(i) $\mu_L(V_j/E_{j-1}) > \mu_L(V_{j+1}/V_j)$ for $j = 1, \ldots, m - 1$, and

(ii) $V_j/V_{j-1}$ is semistable for all $j = 1, \ldots, m$ in the sense that

$$\mu_L(F) \leq \mu_L(V_j/V_{j-1})$$

for every torsion free coherent $\mathcal{O}_X$-submodule $F \subseteq V_j/V_{j-1}$. 
The slope polygon of a vector bundle

**Definition.** Let \( V = V_m \supsetneq V_{m-1} \supsetneq \cdots \supsetneq V_1 \supsetneq V_0 = (0) \) be the Harder–Narasimhan flag of a vector bundle \( V \) on \( X \). The slope polygon \( \text{HNP}_\mathcal{L}(V) \) of \( V \) is the polygon on \( \mathbb{R}^2 \) with vertices

\[
(0, 0), (\text{rk}(V_1), \deg_\mathcal{L}(V_1)), \ldots, (\text{rk}(V_m), \deg_\mathcal{L}(V_m)).
\]

Note. The \( \mathcal{L} \)-slopes of \( \text{HNP}_\mathcal{L}(V) \) between vertices are

\[
\frac{\deg_\mathcal{L}(V_1)}{\text{rk}(V_1)} = \mu_\mathcal{L}(V_1/V_0) > \cdots > \mu_\mathcal{L}(V_m/V_{m-1}) = \frac{\deg_\mathcal{L}(V_m)}{\text{rk}(V_m)}.
\]