Local Monodromy for Deformations of One-Dimensional Formal Groups

Ching-Li Chai[1]
University of Pennsylvania

Version 3.1, November 12, 1999

§1. Introduction

Let k be an algebraically closed field of characteristic $p > 0$. Let $R = k[[t]]$ be the formal power series ring over k, and let $K = k((t))$ be the fraction field of R. Let G_R be a smooth one-dimensional formal group over R, such that its fiber G_k has finite height h and its generic fiber G_K of G has height one. Let $G[p^\infty]$ be the Barsotti-Tate group of height h over R attached to G; see 2.5 for the definition of $G[p^\infty]$. Let $\mathrm{T_p(G)}$ be the p-adic Tate module attached to the maximal étale quotient $G[p^\infty]_K^{\text{ét}}$ of $G[p^\infty]_K$. In Katz's terminology $\mathrm{T_p(G)}$ is the "physical" p-adic Tate module, and can be identified with the inverse limit $\varprojlim_n G[p^n](K^{\text{sep}})$. So $\mathrm{T_p(G)}$ is a free $\mathbb{Z}_p$-module of rank $d = h - 1$. Let

$$\rho_G \ : \mathrm{Gal}(K^{\text{sep}}/K) \to \mathrm{GL}(\mathrm{T_p(G)})$$

be the natural representation of the Galois group $\mathrm{Gal}(K^{\text{sep}}/K)$ on $\mathrm{T_p(G)}$.

When $h = 2$, a famous result of Igusa implies that the image of ρ_G is an open subgroup of $\mathbb{Z}_p^\times$. In this article we show that Katz' proof of Igusa's theorem in [Ka, Thm. 4.3], coupled with a limit argument, can be used to prove that the image of the Galois representation ρ_G is quite big.

The main result of this paper says that ρ_G is an irreducible linear representation of $\mathrm{Gal(K^{sep}/K)}$ over $\mathbb{Q}_p$, and the image of its determinant is an open subgroup of $\mathbb{Z}_p^\times$. Moreover the orbit of every non-zero vector in $\mathrm{T_p(G)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is open in the p-adic topology. The upper breaks of the determinant of ρ_G are computed after passing to a finite extension; they are quite similar to the results in [G]. The precise statements can be found in theorems 3.3, 3.5, 4.1, 4.6, 5.6.

In [G] Gross studied the monodromy of one-dimensional formal groups over $k[[t]]$ when the height of the generic fiber is one less than the height of the special fiber, and obtained a very complete answer in that case. But the case where the generic fiber has height one has attracted little attention in the past and does not seem to have been studied in the literature.

In [E] Ekedahl proved that the local monodromy of the maximal étale quotient of the universal deformation of a product of supersingular elliptic curves in the moduli space of g-dimensional principally polarized abelian varieties of characteristic p is equal to $\mathrm{GL}_g(\mathbb{Z}_p)$. A question analogous to the problem treated in this paper, in the context of Ekedahl's theorem, is the following:

> Suppose that A_R is a principally polarized abelian scheme over R, such that its special fiber is isomorphic to a product of supersingular elliptic curves, and its generic fiber is an ordinary abelian variety. Show that the monodromy of the maximal étale quotient of $A[p^\infty]_K$ is "big".

One knows that the image of the monodromy representation is not finite, using the main theorem of de Jong in [dJ]. Otherwise not much is known.

[1]partially supported by grants DMS95-02186 AND DMS 9800609 from the National Science Foundation

For one-dimensional formal groups, there are many open questions about the monodromy representation ρ_G. The most important one is the following "wishful thinking", suggested by the philosophy of motives: The image of ρ_G "should be" an open subgroup of $\mathrm{GL}(\mathrm{T_p(G)})$. Unfortunately we have not been able to supply any solid evidence in its support, even in the first nontrivial case when $h = 3$.

This paper was motivated by the Hecke orbit problem for some $U(d,1)$ Shimura varieties, and also by the related problem about Tate-linear subvarieties in these Shimura varieties. The results in this paper can be used to show that for a good reduction X of a $U(d,1)$ Shimura variety of PEL-type attached to an imaginary quadratic field, and for any point x_0 of X corresponding to an ordinary abelian variety, the Hecke orbit of x_0 is Zariski dense in X.

It is a pleasure to thank J. de Jong for discussions on applications of his theorem in [dJ] to p-adic representations, and on local monodromy resulting from change of slopes. I would like to thank the referee for suggestions which improved the exposition of this paper, and for the second, and better, proof of Theorem 4.1. I would also like to thank T. Chinburg, R. Donagi, B. Gross, D. Harbater, A. Kirillov, S. Shatz, J.-K. Yu, A. Yukie, and W. Ziller for discussions on topics ranging from group theory, prehomogenous vector spaces and computer-aided computation of Galois groups.

§2. Notations and Conventions

(2.1) Let k be an algebraically closed field of characteristic $p > 0$. Let $R = k[[t]]$ be the formal power series ring over k, and let $K = k((t))$ be the fraction field of R.

(2.2) Let G_R be a smooth one-dimensional formal group over R such that its fiber G_k has finite height $h \geq 2$. We assume that the generic fiber G_K of G has height one.

(2.3) Let $G[p^\infty]$ be the Barsotti-Tate group over R attached to G as explained in 2.5. Let $\mathrm{T_p(G)}$ be the p-adic Tate module attached to the maximal étale quotient $G[p^\infty]_K^{\text{ét}}$ of the generic fiber $G[p^\infty]_K$ of $G[p^\infty]$.

(2.4) Let $\rho_G : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{GL}(\mathrm{T_p(G)})$ be the natural representation of the Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on the free $\mathbb{Z}_p$-module $T_p(G)$ of rank $d = h - 1$. The image of ρ_G is a p-adic analytic subgroup H of $\mathrm{GL}(\mathrm{T_p(G)}) \cong \mathrm{GL_d(\mathbb{Z}_p)}$. Let $\mathfrak{h}$ be the Lie algebra of H.

(2.5) The relation between the smooth formal group G_R and the Barsotti-Tate group $G[p^\infty]$ is as follows. Let $G[p^n]_{R/\mathfrak{m}^m}$ be the p^n-torsion subgroup of the base extension of G_R to $R/\mathfrak{m}^m$, where $\mathfrak{m}$ is the maximal ideal of R. Taking the limit as $m \to \infty$, one obtains a finite flat group scheme $G[p^n]_R$ over R. The inductive limit of $G[p^n]_R$ is a Barsotti-Tate group over R, which we call $G[p^\infty]_R$ by abuse of notation. Conversely, from this Barsotti-Tate group $G[p^\infty]_R$ one gets the finite flat group schemes $G[p^n]_{R/\mathfrak{m}^m}$. Taking the limit as $n \to \infty$, one gets a formal group $G_{R/\mathfrak{m}^m}$ over $R/\mathfrak{m}^m$. The limit of $G_{R/\mathfrak{m}^m}$ is the smooth formal group G_R. So G_R and $G[p^n]_R$ determines each other. This kind of going back and forth already appeared in [Ta] and works for all p-divisible formal groups over complete Noetherian local rings with residue characteristic p.

(2.6) Choosing a suitable parameter for G_R, we may and do assume that the group law for G_R is such that $[p]_G$ is represented by a polynomial $[p]_G(x)$ with coefficients in R, of the form

$$[p]_G(x) = A_1(t) \cdot x^p + A_2(t) \cdot x^{p^2} + \cdots + A_{h-1}(t) \cdot x^{p^{h-1}} + x^{p^h}$$

with each $A_j(t) \in t \cdot k[[t]]$, and $A_1(t) \neq 0$. This fact is well-known, and can be justified by [Dr, §4].

The Verschiebung for G is given by the polynomial

$$V(x) = A_1(t) \cdot x + A_2(t) \cdot x^p + \cdots + A_{h-1}(t) \cdot x^{p^{h-2}} + x^{p^{h-1}} .$$

Its p-iterates $V^{(p^i)}$ are

$$V^{(p^i)}(x) = A_1(t)^{p^i} \cdot x + A_2(t)^{p^i} \cdot x^p + \cdots + A_{h-1}^{p^i}(t) \cdot x^{p^{h-2}} + x^{p^{h-1}} .$$

§3. Ramification

(3.1) We proceed as in [Ka, Thm. 4.3]. To find a point of $G[p^N]^{\text{ét}}$, we need to solve the following system of equations

$$\begin{cases} V(y_0) & = & 0 \\ V^{(p)}(y_1) & = & y_0 \\ V^{(p^2)}(y_2) & = & y_1 \\ \quad\vdots & & \\ V^{(p^{N-1})}(y_{N-1}) & = & y_{N-2} \end{cases}$$

Moreover if one requires that $y_0 \neq 0$, then y_{N-1} has order p^N.

(3.2) We would like to estimate the valuation of elements $y_0, y_1, \ldots, y_i, \ldots$ as above, with $y_0 \neq 0$ and $V^{(p^i)}(y_i) = y_{i-1}$ for all $i \geq 1$, using the method of Newton polygons. Let $v = v_K$ denote the normalized valuation for K, extended to the algebraic closure of K. We want to show that $v(y_i) = p^{-(h-1)}\, v(y_{i-1})$ for all sufficiently large i. Here is the simple strategy: once one has a lower bound for $v(y_i)$, the Newton polygon of the equation $V^{(p^i)}(x) = y_{i-1}$ will have to be a straight line segment for i large, since the valuation of the coefficients of all but the leading term of the monic polynomial $V^{(p^i)}(x)$ grow with i. This estimate itself is a simple consequence of the convexity of the Newton polygon.

(3.3) Theorem *Let G_R be a one-dimensional formal group over $R = k[[t]]$, where k is an algebraically closed field of characteristic $p > 0$. Suppose that the closed fiber G_0 of G has finite height h and the generic fiber G_K of G over $K = k((t))$ has height one. Then there exists a natural number n_1 such that for any sequence $0 \neq y_0, y_1, y_2, \ldots$ in K^{sep} with $V(y_0) = 0$, $V^{(p^i)}(y_i) = y_{i-1}$ for all $i \geq 1$, we have $v_K(y_n) = p^{-(n-n_1)(h-1)} \cdot v_K(y_{n_1})$ for all $n \geq n_1$. Furthermore, for any sequence $0 \neq y_0, y_1, y_2, \ldots$ as above, there exists a natural number $N_0 \geq n_1$, such that the extension $K(y_n)/K(y_{N_0})$ is totally ramified of degree $p^{(n-N_0)(h-1)}$ for all $n \geq N_0$.*

Proof. First we make a trivial estimate. Let $c_1 = v(A_1(t))$. Looking at the Newton polygon of $V(x)$, clearly we have $v(y_0) \leq c_1$. For the same reason, we have $v(y_i) \leq c_1$ for all $i \geq 0$.

Now we look at the equation $V^{(p^i)}(x) = y_{i-1}$ for y_i when $i \gg 0$. Since the $v(y_i)$'s are bounded, and $v(A_j(t)^{(p^i)}) = p^i v(A_j(t))$ goes to infinity as $i \to \infty$, the Newton polygon for the equation

$$-y_{i-1} + A_1(t)^{p^i} \cdot x + A_2(t)^{p^i} \cdot x^p + \cdots + A_{h-1}^{p^i}(t) \cdot x^{p^{h-2}} + x^{p^{h-1}} = 0$$

is the line segment connecting $(0, v(y_{i-1}))$ and $(p^{h-1}, 0)$ for $i \gg 0$. Hence all solutions of the equation $V^{(p^i)}(x) = y_{i-1}$ have the same valuation, equal to $p^{-(h-1)} v(y_{i-1})$, again for $i \gg 0$. This argument shows that there exists a constant n_1, independent of the sequence $y_0, y_1, \ldots$ such that $v(y_n) = p^{-(h-1)} v(y_{n-1})$ for all $n \geq n_1$. This will be the case for instance if $p^{n_1} \cdot v(A_j(t)) \geq c_1$ for all $j = 1, \ldots, h-1$. This proves the first part of the theorem.

The growth behavior of the denominator of the valuation of the y_i's as i increases clearly implies that the ramification index $e(K(y_n)/K)$ is at least $p^{(n-n_1)(h-1)}$ for all $n \geq n_1$. On the other hand we have the trivial bound $[K(y_i) : K(y_{i-1}] \leq p^{h-1}$ for all $i \geq 1$. Therefore $[K(y_i) : K(y_{i-1}] = e(K(y_i)/K(y_{i-1})) = p^{h-1}$ for all sufficiently large i's. ∎

(3.4) Remark As pointed out by the referee, the same argument actually proves more: Let $\epsilon = \min\{v_K(A_1(t), \ldots, v_K(A_{h-1}(t)\}$. Then the convexity of the Newton polygon implies that

$$0 < v(y_i) \leq \max\left\{\frac{v(y_{i-1})}{p^{h-1}}, v(y_{i-1}) - p^i \epsilon\right\},$$

and $v(y_i) = \frac{v(y_{i-1})}{p^{h-1}}$ if $v(y_{i-1}) \leq p^i \epsilon$. In particular, the number N_0 in the statement of Theorem 3.3 is *independent* of the sequence $0 \neq y_0, y_1, y_2, \ldots$ one considers. This also follows from a compactness argument below; see Corollary 3.6 (ii).

The image H of ρ_G operates linearly on $\mathrm{T_p(G)}$, and its orbits are p-adic analytic subvarieties of $\mathrm{T_p(G)}$. Denote $\mathrm{T_p(G)} - \mathrm{p} \cdot \mathrm{T_p(G)}$ by U; clearly U is stable under the action of H.

(3.5) Theorem *Every non-zero orbit of H on $\mathrm{T_p(G)}$ is open. Especially the linear representation ρ of H on the $\mathbb{Q}_p$-vector space $\mathrm{T_p(G)} \otimes_{\mathbb{Z}_\mathrm{p}} \mathbb{Q}_\mathrm{p}$ is irreducible.*

Proof. Let $\{y_i\}$ be a family of elements in K^{sep} as in Theorem 3.3. From Theorem 3.3 we know that for any $i \geq N_0$, the orbit of y_n under $\mathrm{Gal}(K^{\mathrm{sep}}/K(y_{n-1}))$ has p^{h-1} elements. The family $\{y_i\}$ above corresponds to an element $z \in U$, and the $\mathrm{Gal}(K^{\mathrm{sep}}/K(y_{n-1}))$-orbit of y_n corresponds to the coset

$$\mathrm{Gal}(K^{\mathrm{sep}}/K(y_{n-1})) \cdot z + p^{n+1}\mathrm{T_p(G)}/\mathrm{p^{n+1}T_p(G)}$$

So for each orbit Z of H contained in U and for each point z of Z, there exists a natural number N_1 such that

$$((z + p^n \mathrm{T_p(G)}) \cap \mathrm{Z}) + \mathrm{p^{n+1}T_p(G)} = \mathrm{z} + \mathrm{p^n T_p(G)}$$

for all $n \geq N_1$. Since Z is a p-adic analytic subvariety of $\mathrm{T_p(G)}$, this is possible only when $\dim(Z) = \dim(\mathrm{T_p(G)})$, that is Z is an open subvariety. So every H-orbit in U is open. Clearly this implies that every H-orbit in $\mathrm{T_p(G)} - \{0\}$ is open. The last statement of the theorem follows immediately. ∎

(3.6) Corollary *(i) There are only a finite number of H-orbits in U.*

(ii) The constant N_0 in the last part of Theorem 3.3 can be chosen uniformly for all sequences $0 \neq y_0, y_1, y_2, \ldots$. In other words there exists a natural number N_0 such that any sequence $0 \neq y_0, y_1, y_2, \ldots$ as in Theorem 3.3, the extension $K(y_n)/K(y_{N_0})$ is totally ramified of degree $p^{(n-N_0)(h-1)}$ for all $n \geq N_0$.

Proof. Statement (i) follows from the compactness of U. As for (ii), the families $\{y_i\}$ considered in Theorem 3.3 correspond to points of U. Since the one can use the same number N_0 on each Galois orbit in U, statement (ii) follows from statement (i). ∎

§4. The determinant of ρ_G and its breaks

(4.1) Theorem *The image of the one-dimensional Galois representation $\det \circ \rho_G$ is an open subgroup of $\mathbb{Z}_p^\times$. In other words the image is infinite.*

The First Proof. According to [G, Thm. 2.7], $\det \circ \rho_G$ is equal to the inverse of the action of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on the character group of G_K. So it suffices to show that the formal group G_L is not isomorphic to $\widehat{\mathbb{G}_m}$ over any finite separable extension L of K.

As before, choose a parameter x for G_R such that

$$[p]_G(x) = A_1(t) \cdot x^p + \cdots + A_{h-1}(t) \cdot x^{p^{h-1}} + x^p \ .$$

Suppose that there exists an isomorphism ϕ from G to $\widehat{\mathbb{G}_m}$ over a finite extension L of K. This isomorphism ϕ is given by a power series

$$\phi(x) = a_1 \cdot x + a_2 \cdot x^2 + \cdots \ , \quad a_i \in L, \ a_1 \neq 0 \, ,$$

such that $\phi(x)^p = \phi([p]_G(x))$. From this one quickly deduces that $a_i = 0$ unless i is equal to a power of p, and that all the a_i's are integral over L. To be more explicit, let $\phi_i = a_{p^i}$, then the condition $\phi(x)^p = \phi([p]_G(x))$ becomes the following system of equations.

$$\begin{array}{lcl}
\phi_0^{p-1} - A_1 & = & 0 \\
\phi_1^p - \phi_1 A_1^p - \phi_0 A_2 & = & 0 \\
\phi_2^p - \phi_2 A_1^{p^2} - \phi_1 A_2^p - \phi_0 A_3 & = & 0 \\
\quad \vdots & & \\
\phi_{h-2}^p - \phi_{h-2} A_1^{p^{h-2}} - \phi_{h-3} A_2^{p^{h-3}} - \cdots - \phi_1 A_{h-2}^p - \phi_0 A_{h-1} & = & 0 \\
\phi_{h-1}^p - \phi_{h-1} A_1^{p^{h-1}} - \phi_{h-2} A_2^{p^{h-2}} - \cdots - \phi_1 A_{h-1}^p - \phi_0 & = & 0 \\
\phi_h^p - \phi_h A_1^{p^h} - \phi_{h-1} A_2^{p^{h-1}} - \cdots - \phi_2 A_{h-1}^{p^2} - \phi_1 & = & 0 \\
\phi_{h+1}^p - \phi_{h+1} A_1^{p^{h+1}} - \phi_h A_2^{p^h} - \cdots - \phi_3 A_{h-1}^{p^3} - \phi_2 & = & 0 \\
\vdots & & \\
\phi_m^p - \phi_m A_1^{p^m} - \phi_{m-1} A_2^{p^{m-1}} - \cdots - \phi_{m-h+2} A_{h-1}^{p^{m-h+2}} - \phi_{m-h+1} & = & 0 \\
\qquad \forall m \geq h-1 & &
\end{array}$$

Let $F(x_1, x_2) \in R[[x_1, x_2]]$ be the group law for G_R with respect to the chosen parameter x. We have $\phi(F(x_1, x_2)) = \phi(x_1) \cdot \phi(x_2)$ in $L[[x_1, x_2]]$. Since $\phi(x) \in \mathcal{O}_L[[x]]$, ϕ extends to a homomorphism $\phi_{R'}$ from G to $\widehat{\mathbb{G}_m}$ over $R' = \mathcal{O}_L$. Over the residue field $\kappa' \cong k$ of R', the reduction of $\phi_{R'}$ gives a homomorphism from G_k to $\widehat{\mathbb{G}_m}$, which must be trivial since G_k has height $h \geq 2$. (This also follows from the system of equations in last paragraph.) On the other hand, $\phi_{R'}$ defines a homomorphism $\phi[p^\infty]_{R'}$ from the Barsotti-Tate group $G[p^\infty]_R \times_{\operatorname{Spec} R} \operatorname{Spec} \mathcal{O}_L$ to the Barsotti-Tate group $\mathbb{G}_m[p^\infty]_{\mathcal{O}_L}$, which induces the trivial homomorphism between the closed fibers. This implies that $\phi[p^\infty]_{R'}$ induces the trivial homomorphism over $\mathcal{O}_L/\pi_L^n\mathcal{O}_L$ for every $n \geq 1$. Therefore $\phi_{R'}$ is trivial, a contradiction. ∎

THE SECOND PROOF. The previous proof uses a rigidity property of homomorphisms between Barsotti-Tate groups. Here is a more elementary alternative to the last paragraph of the first proof. Take any sequence $0 \neq y_0, y_1, y_2, \ldots$ of elements in K^{sep} such that $V(y_0) = 0$ and $V^{(p^i)}(y_i) = y_{i-1}$ for all $i \geq 1$. Let $z_i = y_i^{i+1}$, $i = 0, 1, \ldots$. Then $[p]_G(z_i) = z_{i-1}$ for all $i \geq 1$. Hence $\phi(z_i) = \phi(z_{i-1})^p$ for all $i \geq 1$. We want to show that $\phi(z_0) = 0$, i.e. $\phi(z_i) = 0$ for all i. This will force ϕ to be trivial and we would have proved the theorem.

Suppose that $\phi(z_0) \neq 0$. Then $v_K(\phi(z_i)) = p^{-i}\delta_1$, where $\delta_1 = v_K(\phi(z_0))$. On the other hand, Theorem 3.3 tells us that there exists $\delta_2 \in \mathbb{Q}_{>0}$ such that $v_K(z_i) = p^{-ih}\,\delta_2$ for $i \gg 0$. Write

$$\phi(x) = \phi_0\, x + \phi_1\, x^p + \phi_2\, x^{p^2} + \cdots \qquad \phi_i \in \mathcal{O}_L\,.$$

Suppose that ϕ_m is such that $v_K(\phi_m) < v_K(\phi_0) \ldots, v_K(\phi_{m-1})$, and $v_K(\phi_m) \leq v_K(\phi_i)$ for all $i > m$. Then we have

$$v_K(\phi(z_i)) = v(\phi_m\, z_i^{p^m}) = v_K(\phi_m) + \delta_2\, p^{m-ih}$$

for all $i \gg 0$. This is a contradiction. ∎

(4.2) Remark The first proof can be interpreted as a direct verification of a special case of the theorem of de Jong in [dJ].

(4.3) Once one knows that the image of $\det \circ \rho_G : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathbb{Z}_p^\times$ is open, one can determine the breaks of the corresponding abelian extension of K asymptotically. We shall need a modified version of Tate's lemma in [G, Lemma 1.5]. The proof is essentially the same as in [G], hence omitted. A systematic exposition of the upper-number filtration in the generality we need can be found in the appendix of [De], see also [G].

(4.4) Lemma *Let K be the fraction field of a complete discrete valuation ring $\mathcal{O}_K$ with perfect residue field. Let*

$$f(Y) = Y^e + a_{e-1} \cdot Y^{e-1} + \cdots + a_1 \cdot Y + a_0$$

be an irreducible separable polynomial in $K[Y]$ such that for any root β of $f(Y)$, the field $K(\beta)$ is totally ramified. Pick a root β, let $E = K(\beta)$, and let π_E be a uniformizer of E. Let $\Gamma_{E/K} = \mathrm{Hom}(E, K^{sep})$ be the set of embeddings of E into K^{sep}, endowed with the upper-numbering filtration. Consider the polynomial

$$h(Y) = \left(\frac{1}{\pi_E^e}\right) \cdot f(\pi_E\, Y + \beta) \;\in E[Y]$$

and the Newton polygon $N(h)$ of $h(Y)$ on the (x, y)-plane constructed using the normalized valuation v_K of K. We have

(a) *The upper breaks of* $\Gamma_{E/K}$ *occur at the y-intercepts of the non-trivial sides of* $N(h)$.

(b) *For each non-trivial side of* $N(h)$ *with slope* λ, *let* y_λ *be the corresponding upper break. Then the horizontal length of this non-trivial side is equal to*

$$\mathrm{Card}\left(\Gamma_{\mathrm{E/K}}^{\mathrm{y}_\lambda} - \Gamma_{\mathrm{E/K}}^{\mathrm{y}_\lambda^+}\right) .$$

(4.5) Let

$$\phi(x) = \sum_{i=0}^{\infty} \phi_i(t) \cdot x^{p^i} \ , \quad \phi_i(t) \in K^{\mathrm{sep}} \ \forall i, \ \phi_0(t) \neq 0$$

be an isomorphism from the toric part of $G[p^\infty]_K$ to $\widehat{\mathbb{G}_m}[p^\infty]$ over K^{sep}, regarded as a generator of the character group of the toric part of $G[p^\infty]_K$. The openness of $\det(H)$ in $\mathbb{Z}_p^\times$ implies that there exists a natural number $n_2 \geq h-1$ such that for all $n \geq n_2$, the extension $K_n = K(\phi_0, \dots, \phi_{n-1}, \phi_n)$ over $K_{n_2} = K(\phi_0, \dots, \phi_{n_2})$ is totally ramified and cyclic of degree p^{n-n_2}. Especially the polynomial

$$f_n(Y) = Y^p - A_1^{p^n} Y - (\phi_{n-1} A_2^{p^{n-1}} + \phi_{n-2} A_3^{p^{n-2}} + \cdots + \phi_{n-h+2} A_{h-1}^{p^{n-h+2}} + \phi_{n-h+1})$$

is the minimal polynomial of ϕ_n over $K_{n-1} = K(\phi_0, \dots, \phi_{n-1})$. Let π_n be a uniformizer for K_n, and let v_{n-1} be the normalized valuation for K_{n-1}. The Newton polygon for

$$h_n(Y) = \frac{1}{\pi_n^p} \cdot f_n(\pi_n Y + \phi_n) = \frac{1}{\pi_n^p}\left(\pi_n^p Y^p - A_1^{p^n} \pi_n Y\right)$$

using v_{n-1} is the line segment connecting $(1, p^n v_{n-1}(A) - \frac{p-1}{p})$ to $(p, 0)$. Hence the only upper break for K_n/K_{n-1} is $\frac{p^{n+1}}{p-1} v_{n-1}(A) - 1$ according to the modified Tate's lemma.

It is convenient to write the positive integer $\frac{v_{n-1}(A)}{p-1}$ as $c \cdot p^{n-1}$, i.e. we let $c = \frac{v_{n-1}(A)}{(p-1)p^{n-1}}$. From the above information, the upper breaks for K_n/K_{n-2} works out to be $c \cdot p^{2n-2} - 1$ and $c \cdot p^{2n-3}(p^2-1) + c \cdot p^{2n-2} - 1$ when $n - 2 \geq n_2$. Continuing this way, one finds that whenever $n > m \geq n_3$, the upper breaks of the totally ramified cyclic extension K_n/K_m are

$$\begin{array}{l} c \cdot (p+1)(p^{n-m-1} - 1)p^{2m+1} + c \cdot p^{2m+2} - 1 \\ c \cdot (p+1)(p^{n-m-2} - 1)p^{2m+1} + c \cdot p^{2m+2} - 1 \\ \quad \vdots \\ c \cdot (p+1)(p-1)p^{2m+1} + c \cdot p^{2m+2} - 1 \\ c \cdot p^{2m+2} - 1 \end{array}$$

In other words, the $n-m$ upper breaks for K_n/K_m occur at

$$c \cdot p^{2m+1}(p^{i+1} + p^i - 1) \ , \qquad i = 0, 1, \dots, n-m-1 \ .$$

We summarize our calculations as

(4.6) Theorem (a) *The image of the determinant* $\det \circ \rho_G$ *of the p-adic Galois representation* ρ_G *attached to the maximal étale quotient of* $G[p^\infty]_K$ *is an open subgroup of* $\mathbb{Z}_p^\times$.

(b) *Pick an integer $n_2 \geq h-1$ such that* $\mathrm{Im}(\det \circ \rho_G) \supseteq 1 + p^{n_2+1}\mathbb{Z}_p$. *For each $n \geq n_2$, let K_n be the fixed field corresponding to the subgroup $1 + p^{n+1}\mathbb{Z}_p$ of* $\mathrm{Im}(\det \circ \rho_G)$. *Then for every $n > n_2$, K_n is a totally ramified cyclic extension of K_{n_2} with Galois group*

$$\mathrm{Gal}(K_n/K_{n_2}) \cong 1 + p^{n_2+1}\mathbb{Z}_p/1 + p^{n+1}\mathbb{Z}_p \cong \mathbb{Z}/p^{n-n_2}\mathbb{Z} \ ,$$

and the upper breaks occur at

$$c \cdot p^{2n_2+1}(p^{i+1} + p^i - 1) \ , \qquad i = 0, 1, \dots , n - n_2 - 1 \ ,$$

where $c := \frac{v_{m-1}(A)}{(p-1)p^{m-1}}$ *for* $m \gg 0$.

§5. A weak classification

(5.1) In this section we use Theorem 3.5 and Theorem 4.1 to classify the possible p-adic representations ρ_G's when $h = 3, 4$. Before doing so, it may be helpful to consider the general case first. As usual one first passes to the Zariski closure M of the image of ρ in $\mathrm{GL}(\mathrm{T_p(G)} \otimes_{\mathbb{Z}_\mathrm{p}} \mathbb{Q}_\mathrm{p})$. Let M° be the neutral component of M. The group M° is a connected reductive algebraic group over $\mathbb{Q}_p$ since it has a faithful irreducible representation on $\mathrm{T_p(G)} \otimes_{\mathbb{Z}_\mathrm{p}} \mathbb{Q}_\mathrm{p}$. Let $\mathfrak{m}$ be the Lie algebra of M in this section. By [B, §7, Cor. 7. 9], we know that

$$[\mathfrak{m}, \mathfrak{m}] \subseteq [\mathfrak{h}, \mathfrak{h}] \subsetneq \mathfrak{h} \ .$$

Since $\mathfrak{h} \subseteq \mathfrak{m}$, $[\mathfrak{h}, \mathfrak{h}]$ is equal to $[\mathfrak{m}, \mathfrak{m}]$. In particular, H contains an open subgroup of the $\mathbb{Q}_p$-points of the derived group $M^{\circ,\mathrm{der}}$ of M°.

(5.2) Prehomogeneous vector spaces

Recall that a prehomogeneous vector space is a algebraic linear action of an algebraic group on a vector space with a Zariski dense open orbit. Clearly the algebraic linear representation $\rho : M^\circ \to \mathrm{T_p(G)} \otimes_{\mathbb{Z}_\mathrm{p}} \mathbb{Q}_\mathrm{p}$ contains a Zariski dense open orbit. Therefore $(M^\circ, \rho, \mathrm{T_p(G)} \otimes_{\mathbb{Z}_\mathrm{p}} \mathbb{Q}_\mathrm{p})$ is a prehomogeneous vector space over $\mathbb{Q}_p$. Moreover it is defined over $\mathbb{Q}_p$, and has the additional property that every non-zero orbit which contains a $\mathbb{Q}_p$-rational point is open in the Zariski topology.

There is an extensive literature on the classification of prehomogeneous vector spaces. The fundamental results are due to Sato and Kimura, and there are important contributions by Chen, Igusa, Kac, Rubenthaler and Vinberg. The standard reference for prehomogenous vector spaces seems to be [SK]; the survey article [Ki] is a convenient place to look up the definitions and results. This theory provides a classification, over algebraically closed field of characteristic 0, of prehomogenous vector spaces whose underlying representation is absolutely irreducible, under the so called *castling equivalence*. Combining this classification with the results of [Ti], in principle one can work out a weak classification of possible triples $(M^\circ, \rho, \mathrm{T_p(G)} \otimes_{\mathbb{Z}_\mathrm{p}} \mathbb{Q}_\mathrm{p})$ which satisfy the restrictions imposed by Theorem 3.5 and Theorem 4.6. The interested readers are invited to consult [Ki, §3] and [Ti, §7].

For the convenience of the reader, we reproduce the definition of castling equivalence here; see [Ki, p. 227] for details. For any representation (H, ρ) of a linear algebraic group H and any two integers $m, n \geq 1$, the pair $(H \times \mathrm{GL}_n, \rho \otimes \mathrm{std})$ is a prehomogeneous vector space if and only if the pair $(H \times \mathrm{GL}_m, \rho^* \otimes \mathrm{std}_n)$ is a prehomogeneous vector space, where ρ^* denotes the contragradient representation of ρ and std_n denotes the standard representation for GL_n. Going from $(H\times \mathrm{GL}_n, \rho\otimes\mathrm{std})$ to $(H\times\mathrm{GL}_m, \rho^*\otimes\mathrm{std}_m)$ is called a *castling transform*. Two prehomogeneous vector spaces are *castling equivalent* if one can be obtained from the other by a finite number of castling transforms.

Here is a list of simple factors (allowing a central $\mathbb{G}_m$) that may occur in absolutely irreducible prehomogeneous vector spaces. See [Ki, p. 228–229]

- the classical groups and their standard representations;
- the second symmetric product of the standard representation of GL_n;
- the second exterior product of the standard representation of GL_n with n even;
- the second exterior product of the standard representation of GL_5;
- the 14-dimensional fundamental representation for GSp_6 which corresponds to the long root;
- the third exterior product of GL_6, GL_7, GL_8;
- the spin representation for Spin_7, Spin_9, Spin_{11};
- a half-spin representation for Spin_{10}, Spin_{12}, Spin_{14};
- the 7-dimensional irreducible representation of G_2;
- the 27-dimensional irreducible representation of E_6;
- the 56-dimensional irreducible representation of E_7.

(5.3) It is instructive to look first at the case when M° is commutative. In other words M° is an algebraic torus. The $\mathbb{Q}_p$-subalgebra of $\mathrm{End}(\mathrm{T_p(G)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)$ generated by the image of its Lie algebra $\mathfrak{m}$ is a commutative semisimple algebra contained in $\mathrm{End}(\mathrm{T_p(G)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)$, therefore its dimension is at most $d = h-1$. On the other hand, $\mathfrak{h}\subseteq\mathfrak{m}$ has dimension at least d, since H has at least one open orbit in $\mathrm{T_p(G)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. So $\mathfrak{h}=\mathfrak{m}$, and they are equal to a commutative semi-simple subalgebra F of $\mathrm{End}(\mathrm{T_p(G)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)$ of dimension d over $\mathbb{Q}_p$. The fact that every non-zero orbit of H in $\mathrm{T_p(G)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is open implies that this commutative semi-simple subalgebra F is a field extension of $\mathbb{Q}_p$ with $[F:\mathbb{Q}_p]=d$. Clearly this is the smallest size $\mathfrak{h}$ can possibly be under the restrictions we have imposed on it.

(5.4) Next we consider the first non-trivial case $h=3$. If $\mathfrak{m}$ is not commutative, $[\mathfrak{m},\mathfrak{m}]$ is a nontrivial semi-simple Lie algebra over $\mathbb{Q}_p$ contained in the traceless subalgebra of $\mathrm{End}(\mathrm{T_p(G)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)\cong \mathrm{M}_2(\mathbb{Q}_p)$. Hence $[\mathfrak{m},\mathfrak{m}]=\mathfrak{sl}(\mathrm{T_p(G)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)$. Therefore $\mathfrak{h}$ is equal to $\mathrm{End}(\mathrm{T_p(G)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)\cong \mathrm{M}_2(\mathbb{Q}_p)$ since it is not traceless. On the other hand if $\mathfrak{m}$ is commutative, then $\mathfrak{h}$ is equal to some quadratic extension field of $\mathbb{Q}_p$ embedded in $\mathrm{End}(\mathrm{T_p(G)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)\cong\mathrm{M}_2(\mathbb{Q}_p)$.

(5.5) Suppose now that $h=4$. Of course we have two possibilities at the two extremes, that is $\mathfrak{h}$ can be either $\mathrm{End}(\mathrm{T_p(G)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)\cong\mathrm{M}_3(\mathbb{Q}_p)$, or a cubic extension field of $\mathbb{Q}_p$ embedded in $\mathrm{End}(\mathrm{T_p(G)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)\cong\mathrm{M}_3(\mathbb{Q}_p)$. But there is now a third possibility, namely that $[\mathfrak{m},\mathfrak{m}]$ is a nontrivial semisimple proper subalgebra of $\mathfrak{sl}(\mathrm{T_p(G)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)\cong\mathfrak{sl}_3(\mathbb{Q}_p)$. The dimension of the irreducible representation involved is so low that the only possibility left is that there exists a quadratic form Q on $\mathrm{T_p(G)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ such that $[\mathfrak{m},\mathfrak{m}]=\mathfrak{so}(Q)$, consequently $\mathfrak{h}=\mathfrak{go}(Q)$. Since all non-zero orbits of H are open, the quadratic form Q on $\mathrm{T_p(G)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ has to be anisotropic.

We summarize the above discussions in the following

(5.6) Theorem *Let ρ_G be the Galois representation on the p-adic Tate module* $\mathrm{T_p(G)}$ *of the maximal étale quotient of $G[p^\infty]_K$, let H be the image of* $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ *in* $\mathrm{GL}(\mathrm{T_p(G)})$*, and let M° be the neutral component of the Zariski closure of the image of H in* $\mathrm{GL}(\mathrm{T_p(G)})$*. Then*

(1) The connected algebraic group M° over $\mathbb{Q}_p$ is reductive, and its natural action ρ on $\mathrm{T_p(G)}\otimes_{\mathbb{Z}_\mathrm{p}}\mathbb{Q}_\mathrm{p}$ is an irreducible representation of M° over $\mathbb{Q}_p$ with nontrivial determinant.

(2) The p-adic analytic group H contains an open subgroup of the derived group $M^{\circ,\mathrm{der}}$ of M^{der}.

(3) The triple $(M^\circ, \rho, \mathrm{T_p(G)}\otimes_{\mathbb{Z}_\mathrm{p}}\mathbb{Q}_\mathrm{p})$ is a prehomogeneous vectors space over $\mathbb{Q}_p$, such that the every non-zero M°-orbit which contains a $\mathbb{Q}_p$-rational point is open in the Zariski topology.

(4) The Lie algebra $\mathfrak{m}$ of M° is commutative if and only if the Lie algebra $\mathfrak{h}$ of H is commutative. If this is the case, then $\mathfrak{m}=\mathfrak{h}$, and they are equal to a field extension F of $\mathbb{Q}_p$ of degree $d=h-1$ embedded in $\mathrm{End}_{\mathbb{Q}_p}(\mathrm{T_p(G)}\otimes_{\mathbb{Z}_\mathrm{p}}\mathbb{Q}_\mathrm{p})\cong M_\mathrm{d}(\mathbb{Q}_\mathrm{p})$.

(5) If $h=3$, then there are two possibilities for $\mathfrak{h}$. It can be either

- $\mathrm{End}_{\mathbb{Q}_p}(\mathrm{T_p(G)}\otimes_{\mathbb{Z}_\mathrm{p}}\mathbb{Q}_\mathrm{p})\cong M_2(\mathbb{Q}_\mathrm{p})$*, or*
- *a quadratic extension field F of $\mathbb{Q}_p$, embedded in the matrix algebra* $\mathrm{End}_{\mathbb{Q}_p}(\mathrm{T_p(G)}\otimes_{\mathbb{Z}_\mathrm{p}}\mathbb{Q}_\mathrm{p})$*.*

(6) If $h=4$, there are three possibilities for $\mathfrak{h}$. It can be equal to

- $\mathrm{End}_{\mathbb{Q}_p}(\mathrm{T_p(G)}\otimes_{\mathbb{Z}_\mathrm{p}}\mathbb{Q}_\mathrm{p})\cong M_3(\mathbb{Q}_\mathrm{p})$*, or*
- *a cubic extension F of $\mathbb{Q}_p$ embedded in* $\mathrm{End}_{\mathbb{Q}_p}(\mathrm{T_p(G)}\otimes_{\mathbb{Z}_\mathrm{p}}\mathbb{Q}_\mathrm{p})$*, or*
- *the Lie algebra of the group of orthogonal similitudes $GO(Q)$ attached to an anisotropic quadratic form Q on the three-dimension vector space $\mathrm{T_p(G)}\otimes_{\mathbb{Z}_\mathrm{p}}\mathbb{Q}_\mathrm{p}$ over $\mathbb{Q}_p$.*

(5.7) Remark (a) If we knew that M° is the generic fiber of a reductive group scheme $\mathfrak{M}^\circ$ over $\mathbb{Z}_p$, and that the action of M° on $\mathrm{T_p(G)}\otimes_{\mathbb{Z}_\mathrm{p}}\mathbb{Q}_\mathrm{p}$ is the generic fiber of a linear action of $\mathfrak{M}^\circ$ on a free $\mathbb{Z}_p$-module such that every non-zero orbit of the closed fiber of $\mathfrak{M}^\circ$ is Zariski open, then the classification would be a lot cleaner and shorter. Namely we would be able to conclude that $(M^\circ, \rho, \mathrm{T_p(G)}\otimes_{\mathbb{Z}_\mathrm{p}}\mathbb{Q}_\mathrm{p})$ is the restriction of scalar from an unramified extension of $\mathbb{Q}_p$ to $\mathbb{Q}_p$ of an unramified twist of the standard representation of a classical group, just by looking up the classification list of [SK] and use Lang's theorem. Unfortunately such information does not seem to be available.

(b) The author does not know any example where $h\geq 3$ and $\mathfrak{h}$ is commutative. As mentioned in the introduction, a "wishful thinking" is that under the general assumptions of this paper, we have $\mathfrak{h}=\mathrm{End}_{\mathbb{Q}_p}(\mathrm{T_p(G)}\otimes_{\mathbb{Z}_\mathrm{p}}\mathbb{Q}_\mathrm{p})$. In other words, H is an open subgroup of $\mathrm{GL}(\mathrm{T_p(G)})$.

References

[B] A. Borel. *Linear Algebraic Groups.* Benjamin, 1969. Notes taken by Hyman Bass.

[C1] C.-L. Chai. Every ordinary symplectic isogeny class in positive characteristic is dense in the moduli. *Invent. Math.*, 121:439–479, 1995.

[C2] C.-L. Chai. The group action on the closed fiber of the Lubin-Tate moduli space. *Duke Math. J.*, 82:725–754, 1996.

[1] C.-L. Chai. Tate-linear subvarieties and Hecke orbits in the moduli of ordinary abelian varieties. In preparation.

[De] P. Deligne. Les corps locaux de caractéristique p, limites de corps locaux de caractéristique 0. In *Représentations des groupes réductifs sur un corps local*, pages 119–157. Hermann, 1984.

[dJ] A. J. de Jong. Homomorphisms of Barsotti-Tate groups and crystals in positive characteristic. *Invent. Math.*, 134:301–333, 1998.

[Dr] V. G. Drinfel'd. Elliptic modules. *Math. USSR Sb.*, 23:561–591, 1974.

[E] T. Ededahl. The action of monodromy on torsion points of jacobians. In G. van der Geer, F. Oort, and J. Steenbrink, editors, *Arithmetic Algebraic Geometry*, volume 89 of *Progress in Math.*, pages 41–49. Birkhäuser, 1991.

[EO] T. Ekedahl and F. Oort. Connected subsets of a moduli space of abelian varieties. Preprint, 1993.

[GH] B. H. Gross and M. J. Hopkins. Equivariant vector bundles on the Lubin-Tate moduli space. In *Topology and Representation Theory*, volume 158 of *Contemp. Math.*, pages 23–88. Amer. Math. Soc., 1994.

[G] B. Gross. Ramification in p-adic Lie extensions. In *Journées de Géométrie Algébrique de Rennes III*, number 65 in Astérisque, pages 81–102, 1979.

[Ka] N. Katz. P-adic properties of modular schemes and modular forms. In *Modular Functions of One Variable III*, volume 350 of *Lecture Notes in Math.*, pages 69–190. Springer-Verlag, 1973.

[Ki] T. Kimura. A classification theory of prehomogeneous vector spaces. In *Representations of Lie Groups, Kyoto, Hiroshima, 1986*, volume 14 of *Advanced Studies in Pure Math.*, pages 223–256, 1988.

[O] F. Oort. Newton polygons and formal groups: conjectures by Manin and Grothendieck. Utrecht preprint 995, January 1997.

[Se] S. Sen. Ramification in p-adic Lie extensions. *Invent. Math.*, 17:44–50, 1972.

[SK] M. Sato and T. Kimura. A classification of irreducible prehomogeneous vector spaces and their relative invariants. *Nagoya Math. J.*, 65:1–155, 1977.

[Ta] J. Tate. p-divisible groups. In *Proc. of a Conference on Local Fields (at Driebergen)*, pages 158–183. Springer-Verlag, 1967.

[Ti] J. Tits. Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque. *J. Reine Angew. Math.*, 247:196–220, 1071.

Ching-Li Chai
Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104
U.S.A.

Email: `chai@math.upenn.edu`