THE GROUP ACTION ON THE CLOSED FIBER OF THE LUBIN-TATE MODULI SPACE

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Abstract. The action of the automorphism group of a 1-dimensional formal A-module on the closed fiber of the Lubin-Tate deformation space is studied using the Cartier-Dieudonné theory. We obtain a formula for the leading term of this action and give some applications.

0. Introduction

Let G_0 be a 1-dimensional formal group of height $h < \infty$ over an algebraically closed field k of characteristic $p > 0$. It is known that the deformations of G_0 are unobstructed. The universal deformation space of G_0 is isomorphic to $\operatorname{Spf} W(k)[[u_1, \ldots, u_{h-1}]]$, and the equicharacteristic deformation space of G_0 is isomorphic to $\operatorname{Spf} k[[u_1, \ldots, u_{h-1}]]$. The group $Aut(G_0)$ of automorphisms of G_0 is isomorphic to the group of units $\mathcal{O}_B^\times$ of the maximal order $\mathcal{O}_B$ in the central division algebra $B = B_{\frac{1}{h}}$ over $\mathbb{Q}_p$ with invariant $\frac{1}{h}$. There is a natural action of $Aut(G_0)$ on $\operatorname{Spf} W(k)[[u_1, \ldots, u_{h-1}]]$ and $\operatorname{Spf} k[[u_1, \ldots, u_{h-1}]]$ "by changing the marking of the closed fiber". Even though the deformation space looks so simple, the action of the compact p-adic group $Aut(G_0)$ on it is extremely complicated. For instance, the equivariant cohomologies of some naturally defined vector bundles contain deep informations about the stable homotopy theory. Very little computation of these equivariant cohomology groups has been done. This action itself remains a mystery in most aspects.

Much insight about this action comes from the p-adic period map

$$\rho \ : \ \operatorname{Spf} W(k)[[u_1, \ldots, u_{h-1}]] \otimes \operatorname{frac}(W(k)) \to \mathbb{P}^{h-1}_{\operatorname{frac}(W(k))} ,$$

which was described explicitly in [GH]. Here $\operatorname{Spf} W(k)[[u_1, \ldots, u_{h-1}]] \otimes \operatorname{frac}(W(k))$ denotes the rigid open unit ball, thought of as the generic fiber of $\operatorname{Spf} W(k)[[u_1, \ldots, u_{h-1}]]$. The p-adic period map comes from comparing the position of the Hodge filtration relative to the space of horizontal sections of the covariant isocrystal attached to G_0. The target space is the projective space of 1-dimensional quotients of the covariant isocrystal attached to G_0, therefore $(End(G_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^\times$ operates on it via a natural linear representation on the isocrystal. It is a pleasant fact that the period map ρ is étale and surjective ([GH], [Laf]). The period map ρ is equivariant with respect to the inclusion $Aut(G_0) \subset (End(G_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^\times$, so the complicated action of $Aut(G_0)$ is 'linearized' by the period map ρ. However the period map is far from

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being defined over $W(k)$. This makes it extremely difficult to extract information about the action of $Aut(G_0)$ on the closed fiber Spf $k[[u_1, \dots, u_{h-1}]]$.

This article is a report on my attempt to study the orbit structure of this action, after simplifying the problem by taking the Zariski closure of the orbits. On the one hand this problem is similar to the problem on the deformation of superspecial abelian varieties studied in [Ch], which in turn was motivated by Milne's sharp formulation of the conjecture on the existence of canonical integral model. On the other hand I was particularly influence by the inspiring paper [GH] by Gross and Hopkins. It is a pleasure to thank Gross, Hopkins and Milne for many useful conversations and correspondences. I would also like to thank the referee for a very careful and thorough reading of the manuscript, and for useful suggestions. The bulk of this work was done during a visit to the Mathematical Sciences Research Institute at Berkeley in the spring of 1993, and I would like to thank the University of Pennsylvania and MSRI for their financial support, and MSRI for hospitality.

The main results of this paper are stated in Theorems 1 and 2 in section 2. They give the main term of the action of an automorphism of G_0 on $k[[u_1, \dots, u_{h-1}]]$, with respect to two different filtrations on $k[[u_1, \dots, u_{h-1}]]$. Two applications are given in section 3. We also formulate a naive guess on the possible structure of the Zariski closure of orbits of open subgroups of $Aut(G_0)$ (see the "Hope" in section 3). It is well-known that much of the theory and results for one-dimensional commutative formal groups generalize to one-dimensional formal A-modules, where A is a complete discrete valuation ring with finite residue field. We therefore formulate things in this more general context. As a result q takes the place of p. The reader may recover the formal group case by taking $A = \mathbb{Z}_p$ and $q = p$.

1. The universal group law and its Cartier-Dieudonné module

In this section we set up the notation and the coordinates of the Lubin-Tate space. Let K be a nonarchimedian local field with finite residue field $\mathbb{F}_q$, and let A be the ring of integers in K. (So K is either a finite extension of $\mathbb{Q}_p$ or isomorphic to $\mathbb{F}_q[[t]]$.) Denote by $\mathfrak{m}$ the maximal ideal of A and π a uniformizer of A. Let h be a positive integer. Let G_1 be the one-dimensional formal A-module with logarithm

$$g_1(x) = \sum_{m \geq 0} \frac{x^{q^{hm}}}{\pi^m} \ .$$

The power series $g_1(x)$ satisfies the functional equation

$$g_1(x) = x + \frac{1}{\pi} g_1(x^{q^h}) \ .$$

It follows from Hazewinkel's functional equation lemma ([Ha], Chap. 1, sec. 2, especially pp. 9–10) that $g_1(x)$ is the logarithm of a formal A-module G_1 over A. Explicitly, the group law is

$$G_1(x, y) = g_1^{-1}(g_1(x) + g_1(y)) \quad ;$$

the A-module structure is

$$[a](x) = g_1^{-1}(a \cdot g_1(x))$$

for $a \in A$. The A-module G_1 reduces modulo $\mathfrak{m}$ to a one-dimensional formal A-module G_0 over $\kappa = A/\mathfrak{m}$ of height h. It is well-known that any one-dimensional formal A-module of height h over an algebraically closed overfield k of κ is isomorphic to $G_0 \otimes_\kappa k$ over k. This A-module G_1 may be called a "canonical lifting" of G_0. (In the sense that it underlies a Lubin-Tate formal group for the unramified extension of K of degree h, and not to be confused with canonical liftings of ordinary abelian varieties. The usual convention is that a canonical lifting of G_0 is determined by an embedding of the maximal order of a finite separable extension of K of degree h into the ring of endomorphisms of G_0.)

The deformation functor $Def(G_0)$ of G_0 assigns to any artinian local A-algebra R the set

$$Def(G_0)(R_{/A}) = \{(G, \alpha)\}/\text{isomorphisms} \ ,$$

where G is a one-dimensional formal A-module over R and

$$\alpha : G \otimes_R R/\mathfrak{m}_R \xrightarrow{\sim} G_0 \otimes_k R/\mathfrak{m}_R$$

is an isomorphism of formal A-modules. It is well-known that the functor $Def(G_0)$ is formally smooth over A, and is represented by a formal scheme isomorphic to the formal spectrum of a power series ring Spf $A[[u_1, \dots, u_{h-1}]]$. For any perfect field k containing κ, we can restrict $Def(G_0)$ to the category of artinian local A-algebras whose residue fields are overfields of k, and then this deformation functor is represented by Spf $W_A(k)[[u_1, \dots, u_{h-1}]]$, where $W_A(k)$ is the ring of ramified Witt vectors soon to be explained in this section.

A beautiful way to describe the universal deformation G of G_0 is to give a recursive formula for its logarithm, when the coordinates $u_1, \dots, u_{h-1}$ and the coordinate x for G are carefully chosen. Let $g_{\underline{u}}(x)$ be the logarithm of G, so that the group law for G is

$$G(x, y) = g_{\underline{u}}^{-1}(g_{\underline{u}}(x) + g_{\underline{u}}(y)) \ .$$

We can choose good coordinates so that $g_{\underline{u}}(x)$ satisfies the following functional equation

$$g_{\underline{u}}(x) = x + \frac{u_1}{\pi} g_{\underline{u}}^{(q)}(x^q) + \frac{u_2}{\pi} g_{\underline{u}}^{(q^2)}(x^{q^2}) + \cdots + \frac{u_{h-1}}{\pi} g_{\underline{u}}^{(q^{h-1})}(x^{q^{h-1}}) + \frac{1}{\pi} g_{\underline{u}}^{(q^h)}(x^{q^h}) \ .$$

Here $g_{\underline{u}}^{(q^i)}$ is obtained from $g_{\underline{u}}$ by the change of rings $A[[\underline{u}]] \to A[[\underline{u}]]$, $u_j \mapsto u_j^{q^i}$. This functional equation is really a recursive definition of the coefficients $a_n(\underline{u})$ of

$$g_{\underline{u}} = \sum_{n \geq 0} a_n(\underline{u}) \ x^{q^n} \quad ,$$

namely

$$a_0(\underline{u}) = 1, \quad a_n(\underline{u}) = 0 \quad \forall n < 0,$$

$$a_n(\underline{u}) = \frac{u_1}{\pi}a_{n-1}(\underline{u}^{(q)}) + \frac{u_2}{\pi}a_{n-2}(\underline{u}^{(q^2)}) + \cdots + \frac{u_{h-1}}{\pi}a_{n-h+1}(\underline{u}^{(q^{h-1})}) + \frac{1}{\pi}a_{n-h}(\underline{u}^{(q^h)}) \,, \quad \forall n \geq 1 \,.$$

From this one can write down a closed formula for the coefficients $a_n(\underline{u})$'s due to Jiu-Kang Yu. The closed formula involves 'ordered partition' and will not be reproduced here.

Besides the power series method with functional equations, there is another method of describing formal A-modules using Cartier-Dieudonné theory due to Cartier and Drinfel'd. Here we review some aspects of the theory in order to introduce the relevent notation. For any A-algebra R, let $W_A(R)$ be the ring of ramified Witt vectors $(r_0, r_1, \dots, r_n, \dots)$, where each $r_n \in R$, and the ring structure is given by the family of Witt polynomials $(w_n^A)_{n\geq 0}$, where

$$w_n^A(\underline{r}) = r_0^{q^n} + \pi r_1^{q^{n-1}} + \cdots + \pi^{n-1}r_{n-1}^q + \pi^n r_n \,,$$

so that each w_n^A is a functorial homomorphism of commutative A-algebras. Although the Witt polynomials depend on the choice of the uniformizer π, the rings $W_A(R)$ do not. The Teichmüller representative $(r, 0, 0, \dots) \in W_A(R)$ will be denoted $\langle r \rangle$. The Witt ring $W_A(R)$ has the following properties:

- The Verschiebung map

  $$\upsilon = \upsilon_A : (r_0, r_1, \dots, r_n, \dots) \mapsto (0, r_0, r_1, \dots, r_n, \dots)$$

  is an additive homomorphism.
- There is a unique functorial homomorphism of rings $\tau = \tau_A : W_A \to W_A$ such that $w_n^A \circ \tau_A = w_{n+1}^A$.
- The natural A-algebra structure of $W_A(R)$ is given by the composite

  $$A \xrightarrow{\Delta(A)} W_A(A) \to W_A(R) \,,$$

  where $\Delta(A)$ is the unique homomorphism from A to $W_A(A)$ such that $w_n^A \circ \Delta(A) = id_A$, $\forall n \geq 0$.
- For every perfect field k over κ, $W_A(k)$ is a complete discrete valuation ring over A with residue field k and π is a uniformizer; moreover $W_A(k)$ is characterized by this property. Especially when k is an algebraic extension of κ, $W_A(k)$ is the π-adic completion of the unramified extension of A with residue field k.
- $(r_0, r_1, \dots, r_n, \dots) = \sum_{n\geq 0} \upsilon^n \langle r_n \rangle$. When R is a perfect overfield of κ, we have

  $$(r_0, r_1, \dots, r_n, \dots) = \sum_{n\geq 0} \pi^n \langle r_n^{q^{-n}} \rangle \,.$$

- $\langle a \rangle \langle b \rangle = \langle ab \rangle \,, \quad \forall a, b \in R$.
- $\tau \langle a \rangle = \langle a^q \rangle \,, \quad \forall a \in R$.
- $\tau \upsilon x = \pi x \,, \quad \forall x \in W_A(R)$.
- $(\upsilon x)y = \upsilon(x \cdot \tau y) \,, \quad \forall x, y \in W_A(R)$. Especially $\upsilon \tau x = (0, 1, 0, 0, \dots) \cdot x$.

- When $\pi = 0$ in R, we have $(0, 1, 0, 0, \dots) = \pi$ in $W_A(R)$, and

$$\tau(r_0, r_1, \dots, r_n, \dots) = (r_0^q, r_1^q, \dots, r_n^q, \dots) \ .$$

Moreover, $\langle a \rangle + \langle b \rangle = \langle a+b \rangle \ , \quad \forall a, b \in R\,.$

We defined the ring of ramified Witt vectors $W_A(R)$ in order to describe the ramified Cartier ring $Cart_A(R)$. For any A-algebra R, $Cart_A(R)$ is a noncommutative ring functorial in R. Set theoretically $Cart_A(R)$ consists of all elements of the form

$$\sum_{m,n \geq 0} V_q^m \langle r_{m,n} \rangle F_\pi^n$$

such that $r_{m,n} \in R$ for all m, n, and for any m, $r_{m,n} = 0$ for all but finitely many n's.

There is an embedding of rings $W_A(R) \hookrightarrow Cart_A(R)$

$$(r_0, r_1, \dots) \mapsto \sum_{n \geq 0} V_q^n \langle r_n \rangle F_\pi^n \ .$$

This embedding determines the additive structures of $Cart_A(R)$. It also gives an embedding of A in $Cart_A(R)$. Explicitly, for any $a \in A$, there exisits a uniquely determined element $(b_0, b_1, \dots) \in W_A(A)$ such that $w_n^A(b_0, b_1, \dots) = a \ \forall n \geq 0$. Then the image of a in $Cart_A(R)$ is the element $\sum_{n \geq 0} V_q^n \langle b_n \rangle F_\pi^n$. When there is no danger of confusion, we abbreviate V_q by V and F_π by F. The Cartier ring $Cart_A(R)$ satisfies the following commutation relations:

- For any $a \in A$, the image of a in $Cart_A(R)$ commutes with both V and F.
- $F_\pi \langle r \rangle = \langle r^q \rangle F_\pi$, $\forall r \in R$.
- $\langle r \rangle V_q = V_q \langle r^q \rangle$, $\forall r \in R$.
- $F_\pi V_q x = \pi x = x \pi$, $\forall x \in Cart_A(R)$.

The main result in the Cartier-Dieudonné theory, extended to formal A-modules (not necessarily one-dimensional) by Drinfel'd, states that the functor which assigns to each smooth formal A-module its group of q-typical curves, establishes an equivalence of categories between the category of smooth formal A-modules over any A-algebra R and the category of V_q-reduced left $Cart_A(R)$-modules, see [Dr2], [Ha]. This translates problems about formal A-modules into linear algebra problems for modules over the Cartier ring.

In order to use this theory, we need to find an explicit form of the Cartier-Dieudonné module of the universal deformation $G = G_{\underline{u}}$. It turns out that our previous choice of coordinates also gives a very simple presentation of the covariant Dieudonné module of $G_{\underline{u}}$.

Let $u_1, \dots, u_{h-1}$ be the coordinates for the deformation space of G_0 and x be the coordinate for the group G. We shall use t as the parameter for curves in $G_{\underline{u}}$ (i.e. elements of $G(A[[\underline{u}]][[t]])$. The curve $\gamma(t) = t$ is a q-typical curve in $G_{\underline{u}}$ since the group law for $G_{\underline{u}}$ is q-typical. (Its logarithm involves only powers of q.) The following lemma explictly describes the Dieudonné module of $G_{\underline{u}}$.

Lemma 1. *The Cartier-Dieudonné module for the universal formal A-module $G_{\underline{u}}$ over $A[[\underline{u}]]$ is isomorphic to*

$$Cart_A(A[[\underline{u}]])/Cart_A(A[[\underline{u}]]) \cdot (F - V^{h-1} - \sum_{1 \leq i \leq h-1} V^{i-1}\langle u_i \rangle) \ .$$

The generator here is the tautological curve $\gamma(t) = t$. The logarithm $g_{\underline{u}}(x)$ of the formal group law (in coordinate x) is determined by the functional equation

$$g_{\underline{u}}(x) = x + \frac{u_1}{\pi} g_{\underline{u}}^{(q)}(x^q) + \frac{u_2}{\pi} g_{\underline{u}}^{(q^2)}(x^{q^2}) + \cdots + \frac{u_{h-1}}{\pi} g_{\underline{u}}^{(q^{h-1})}(x^{q^{h-1}}) + \frac{1}{\pi} g_{\underline{u}}^{(q^h)}(x^{q^h}) \ .$$

The group law for $G_{\underline{u}}$ is

$$G_{\underline{u}}(x,y) \ = \ g_{\underline{u}}^{-1}\left(g_{\underline{u}}(x) + g_{\underline{u}}(y)\right) \ \in A[\underline{u}][[x,y]] \subset A[[\underline{u}]][[x,y]] \ .$$

The A-module structure for $G_{\underline{u}}$ is given by

$$[a](x) \ = \ g_{\underline{u}}^{-1}(a \cdot g_{\underline{u}}(x)) \ \in \ A[\underline{u}][[x]] \subset A[[\underline{u}]][[x]] \ , \qquad a \in A \quad .$$

Lemma 1 is a specialization of [Ha] (29.5.3); see also [Ha] (27.7.15) and (27.7.19 (ii)). It is also possible to prove it directly. □

Let ζ_h be a primitive $(q^h - 1)$st root of 1 in $\overline{K}$. The ring $A_h = A[\zeta_h]$ is the unramified extension of degree h of A with residue field $\kappa_h = A[\zeta_h]/\pi A[\zeta_h] = \kappa(\zeta_h)$, the cyclic extension of degree h over κ. Every element of $End(G_0 \otimes \overline{\kappa})$ is defined over κ_h, and $End(G_0 \otimes \overline{\kappa})$ is isomorphic to the maximal order in the central division algebra over K with invariant $\frac{1}{h}$. The ring $A[\zeta_h]$ is canonically identified with $End(G_{1/A_h})$, such that the action of ζ_h on G_1 sends x to $\zeta_h x$. The ring $End(G_0 \otimes \overline{\kappa})$ is generated by A_h and $\phi = Frob_q \in End(G_0)$:

$$End(G_0 \otimes \overline{\kappa}) \ = \ A_h \oplus A_h\phi \oplus \cdots \oplus A_h\phi^{h-1} \ .$$

The action of an element $\Phi_0 \in Aut(G_0 \otimes \kappa_h)$ on $A_h[[\underline{u}]]$ can be characterized as follows: There is a unique ring isomorphism

$$h(\Phi_0) : A_h[[\underline{u}]] \to A_h[[\underline{u}]]$$

such that there is a homomorphism

$$\Phi : h(\Phi_0)_* G_{A_h[[\underline{u}]]} \to G_{A_h[[\underline{u}]]}$$

which reduces to Φ_0 modulo π. It is clear that $A^\times$ operates trivially on $A_h[[\underline{u}]]$, as $A \subseteq End(G_{A_h[[\underline{u}]]})$. The action of an element $\zeta \in \mu_{q^h-1}(\overline{K})$ is easy to compute:

Lemma 2. *For $\zeta \in \mu_{q^h-1}(\overline{K}) \subseteq Aut(G_0 \otimes \kappa_h)$, the corresponding A_h-algebra homomorphism $h(\zeta) : A_h[[\underline{u}]] \to A_h[[\underline{u}]]$ sends u_i to $\zeta^{q^i-1} \cdot u_i$ for each $1 \le i \le h-1$.*

Proof. There is a homomorphism from

$$Cart_A(A_h[[\underline{u}]]) \Big/ Cart_A(A_h[[\underline{u}]]) \cdot \left(F_\pi - V_q^{h-1} - \sum_{1 \le i \le h-1} V_q^{i-1} \langle \zeta^{q^i-1} u_i \rangle \right)$$

to

$$Cart_A(A_h[[\underline{u}]]) \Big/ Cart_A(A_h[[\underline{u}]]) \cdot \left(F_\pi - V_q^{h-1} - \sum_{1 \le i \le h-1} V_q^{i-1} \langle u_i \rangle \right)$$

sending the class of $\langle 1 \rangle$ to the class of $\langle \zeta \rangle$. This finishes the proof of Lemma 2. It is also possible to prove it using the power series method and Hazewinkel's functional equation. $\square$

The Cartier-Dieudonné module of $G_0 \otimes \kappa(\zeta_h)$ is

$$M_0 \;=\; Cart_A(\kappa_h)/Cart_A(\kappa_h) \cdot \left(F_\pi - V_q^{h-1}\right) \;.$$

M_0 is a free module of rank h over $W_A(\kappa_h) = A_h$; the classes of $1, V, \dots, V^{h-1}$ is a free A_h-basis of M_0. An endomorphism Φ_0 of $G_0 \otimes \kappa(\zeta_h)$ corresponds to an endomorphism of M_0, which in turn is completely determined by the image of $e :=$ the class of 1 under Φ_0, say $\Phi_0 : e \mapsto \sum_{i=0}^{h-1} a_i\, V^i \cdot e$, $a_i \in A_h$. This is just the previous decomposition

$$End(G_0 \otimes \kappa(\zeta_h)) = \bigoplus_{i=0}^{h-1} A\phi^i \;.$$

The filtration of $End(G_0 \otimes \kappa(\zeta_h))$ by the powers of its maximal ideal is determined by congruence of the coefficients a_i's of the $V^i e$'s: if $m = h \cdot r + s$, $m, r, s \in \mathbb{N}$, $0 \le s \le h-1$, then $\Phi_0 \equiv 0$ modulo the mth power of the maximal ideal of $End(G_0 \otimes \kappa(\zeta_h))$ iff

$$a_0 \equiv \cdots \equiv a_{s-1} \equiv 0 \;(mod\ \pi^{r+1}) \quad \text{and} \quad a_s \equiv \cdots \equiv a_{h-1} \equiv 0 \;(mod\ \pi^r)\;.$$

In the next section we shall examine the action of an element Φ_0 of $Aut(G_0 \otimes \kappa(\zeta_h)) = End(G_0 \otimes \kappa(\zeta_h))^\times$ which is congruent to the identity modulo a high power of the maximal ideal of $End(G_0 \otimes \kappa(\zeta_h))$.

2. Computation of the group action

Let $\Phi_0 \in End(G_0 \otimes \kappa(\zeta_h))$, and we assume that

$$\Phi_0(e) = e + \xi \cdot V^\ell e,\ \xi \in \mu_{q^h-1}(\overline{K}),\ \ell \in \mathbb{N},\ \ell \geq 1\ .$$

Clearly the Φ_0's of this type form a system of representatives of

$$Fil_V^\ell(Aut(G_0 \otimes \kappa(\zeta_h))) \overset{def}{=} id\ +\ V^\ell \cdot End(G_0 \otimes \kappa(\zeta_h))$$

modulo $Fil_V^{\ell+1}(Aut(G_0 \otimes \kappa(\zeta_h)))$. The action of Φ_0 on the closed fiber $Spf\ \kappa_h[[\underline{u}]]$ corresponds to a homomorphism

$$h(\Phi_0) : \kappa_h[[\underline{u}]] \to \kappa_h[[\underline{u}]],\quad u_i \mapsto s_i\ .$$

We would like to compute the s_i's. Φ_0 extends to a homomorphism Φ from

$$\begin{aligned} &Cart_A(\kappa_h[[\underline{u}]]) \cdot e' \ \cong \\ &\qquad Cart_A(\kappa_h[[\underline{u}]]) \Big/ Cart_A(\kappa_h[[\underline{u}]]) \cdot \left(F - V^{h-1} - \sum_{i=1}^{h-1} V^{i-1}\langle s_i\rangle\right) \end{aligned}$$

to

$$\begin{aligned} &Cart_A(\kappa_h[[\underline{u}]]) \cdot e \ \cong \\ &\qquad Cart_A(\kappa_h[[\underline{u}]]) \Big/ Cart_A(\kappa_h[[\underline{u}]]) \cdot \left(F - V^{h-1} - \sum_{i=1}^{h-1} V^{i-1}\langle u_i\rangle\right) \end{aligned}$$

Write $\ell = n \cdot h + b$, $0 \leq b \leq h-1$, and

$$\Phi(e') = e\ +\ \pi^n \cdot \xi \cdot V^b\ e + \sum_{m\geq 0} V^m \langle a_m(\underline{u})\rangle \cdot e\ ,$$

where $a_m(\underline{u})$ are elements of $\kappa_h[[\underline{u}]]$, uniquely determined by Φ_0. Computing the action of Φ_0 hinges largely on calculating the $a_m(\underline{u})$'s. Clearly $\Phi(e')$ must satisfy the equation

$$\left(F - V^{h-1} - \sum_{i=1}^{h-1} V^{i-1}\langle s_i\rangle\right) \cdot \Phi(e')\ =\ 0\ .$$

Using the fact that every element of $Cart_A(\kappa_h[[\underline{u}]]) \cdot e$ can be uniquely written in the form

$$\sum_{n\geq 0} V^n\langle x_n\rangle \cdot e \qquad\qquad x_n\ \in\ \kappa_h[[\underline{u}]]\ ,$$

the above equation for $\Phi(e')$ can be translated into a system of equations for the s_i's and $a_m(\underline{u})$'s. Then we solve this system of equations modulo some suitable ideal I (which depends on Φ_0) to compute the leading term of s_i's. This would give us the leading term of the action of Φ_0 on $\kappa_h[[\underline{u}]]$. The computation itself is a bit long and quite messy. Roughly, we first see that for large m's $a_m(\underline{u})$ is congruent to 0 modulo I. From some m_0 down $a_m(\underline{u})$ becomes nonzero modulo I. The system of equations give a recursive formula which computes $a_m(\underline{u})$ in terms of the $a_n(\underline{u})$'s with $n > m$. Going from m_0 down to 0 , we compute all $a_m(\underline{u})$'s modulo I. The system of equations for the s_i's can then be solved easily modulo I. In fact only $a_0(\underline{u}), \ldots, a_{h-1}(\underline{u})$ are needed, but we had to compute the rest of the $a_m(\underline{u})$'s before we can get to them. Now the computation begins. We use the assumed form of $\Phi(e')$, the commutation relations in the Cartier ring, and the fact that

$$Fe = V^{h-1}e + \sum_{i=1}^{h-1} V^{i-1}\langle u_i\rangle \cdot e$$

to get

$$\begin{aligned}
&\left(F - V^{h-1}\right) \cdot \Phi(e') \\
&= \sum_{i=1}^{h-1} V^{i-1}\langle u_i\rangle \cdot e \;+\; V^{n+b}\, \xi^{\sigma^{n+b+1}} \sum_{i=1}^{h-1} V^{i-1}\langle u_i^{q^n}\rangle\ F^n \cdot e \\
&\quad + \sum_{m\geq 0} V^m\langle a_m(\underline{u})^q\rangle \left(\sum_{i=1}^{h-1} V^{i-1}\langle u_i\rangle \cdot e + V^{h-1}\cdot e\right) - \sum_{m\geq 0} V^{h-1+m}\langle a_m(\underline{u})\rangle \cdot e \\
&= \sum_{i=1}^{h-1} V^{i-1}\langle u_i\rangle \cdot e + \sum_{i=1}^{h-1} V^{n+b+i-1}\langle \xi^{\sigma^{n+b+i}}\ u_i^{q^n}\rangle\ F^n \cdot e \\
&\quad + \sum_{m\geq 0}\sum_{i=1}^{h-1} V^{m+i-1}\langle a_m(\underline{u})^{q^i}\ u_i\rangle \cdot e + \sum_{m\geq 0} V^{m+h-1}\langle a_m(\underline{u})^{q^h}\rangle \cdot e \\
&\quad - \sum_{m\geq 0} V^{m+h-1}\langle a_m(\underline{u})\rangle \cdot e
\end{aligned}$$

Here σ denotes Fr_q, and the ξ in the second equality denotes the image of ξ in κ_h. On the other hand

$$\begin{aligned}
&\sum_{i=1}^{h-1} V^{i-1}\langle s_i\rangle \cdot \Phi(e') \\
&= \sum_{i=1}^{h-1} V^{i-1}\langle s_i\rangle \cdot e + \sum_{i=1}^{h-1} V^{n+i-1+b}\langle \xi^{\sigma^{n+b}} \cdot s_i^{q^{n+b}}\rangle\ F^n \cdot e \\
&\quad + \sum_{i=1}^{h-1}\sum_{m\geq 0} V^{i-1+m}\langle s_i^{q^m} \cdot a_m(\underline{u})\rangle \cdot e
\end{aligned}$$

Assume now that $n \geq 1$. We make some simple estimate:

$$
\begin{aligned}
\langle u_i^{q^n} \rangle \; F^n \cdot e = & \langle u_i^{q^n} \rangle \left(V^{h-1} + \sum_{j=1}^{h-1} V^{j-1} \langle u_j^{q^{n-1}} \rangle \right) \cdots \left(V^{h-1} + \sum_{j=1}^{h-1} V^{j-1} \langle u_j^{q} \rangle \right) \\
& \cdot \left(V^{h-1} + \sum_{j=1}^{h-1} V^{j-1} \langle u_j \rangle \right) \cdot e \\
\equiv & \; 0 \qquad (\mathrm{mod}\ (\underline{u})^{\sigma^n + \cdots + \sigma + 1}) \ .
\end{aligned}
$$

Here we have abused notation and used σ to denote the κ_h-algebra homomorphism from $\kappa_h[[\underline{u}]]$ to itself sending each u_i to u_i^q; $(\underline{u})^{\sigma^i}$ denotes the ideal $\left(u_1^{q^i}, \ldots , u_{h-1}^{q^i} \right) \subseteq \kappa_h[[\underline{u}]]$, and

$$
(\underline{u})^{\sigma^n + \cdots + \sigma + 1} \;=\; (\underline{u})^{\sigma^n} \cdot (\underline{u})^{\sigma^{n-1}} \cdots (\underline{u})^{\sigma} \cdot (\underline{u}) \ .
$$

Similarly

$$
\langle s_i^{q^{n+b}} \rangle \; F^n \cdot e \;\equiv\; 0 \qquad (\mathrm{mod}\ (\underline{u})^{\sigma^n + \cdots + \sigma + 1}) \ .
$$

Therefore

$$
\begin{cases}
\quad s_i \equiv u_i \quad (\mathrm{mod}\ (\underline{u})^{\sigma^n + \cdots + \sigma + 1}) & \forall \ 1 \leq i \ \leq \ h-1 \\
a_m(\underline{u}) \equiv 0 \quad (\mathrm{mod}\ (\underline{u})^{\sigma^n + \cdots + \sigma + 1}) & \forall \ m \geq 0
\end{cases}
$$

is a solution modulo $(\underline{u})^{\sigma^n + \cdots + \sigma + 1}$, hence it is the unique solution modulo the ideal $(\underline{u})^{\sigma^n + \cdots + \sigma + 1}$. Now we 'peel down the layers' and try to dig out the first few nonvanishing terms. From the previous expression for $\langle u_i^{q^n} \rangle \; F^n \cdot e$ we get

$$
\langle u_i^{q^n} \rangle \; F^n \cdot e \equiv \langle u_i^{q^n} \cdot u_1^{q^{n-1} + \cdots + q + 1} \rangle \cdot e \qquad (\mathrm{mod}\ (\underline{u})^{\sigma^{n+1} + \sigma^{n-2} + \sigma^{n-3} + \cdots + \sigma + 1}) \ .
$$

Similarly, modulo $(\underline{u})^{\sigma^{n+1} + \sigma^{n-2} + \sigma^{n-3} + \cdots + \sigma + 1}$ we have for $0 \leq b \leq h-1$,

$$
\langle s_i^{q^{n+b}} \rangle \; F^n \cdot e \equiv
\begin{cases}
\langle u_i^{q^n} \cdot u_1^{q^{n-1} + \cdots + q + 1} \rangle \cdot e & \text{if } b = 0 \\
\ 0 & \text{if } b \neq 0 \ .
\end{cases}
$$

We shall work for a while modulo the ideal $(\underline{u})^{\sigma^{n+1} + \sigma^{n-2} + \sigma^{n-3} + \cdots + \sigma + 1}$. Observe that $s_i^{q^j} \equiv u_i^{q^j}$ if $j \geq 1$, and $a_m(\underline{u})^{q^j} \equiv 0$ if $j \geq 1$. Let us consider first the easier case $char(A) = p \geq 0$, i.e. $A \cong \mathbb{F}_q[[t]]$ for some $q = p^r$. Modulo

$(\underline{u})^{\sigma^{n+1}+\sigma^{n-2}+\sigma^{n-3}+\cdots+\sigma+1}$ we have

$$
\begin{aligned}
&\left(F - V^{h-1}\right) \cdot \Phi(e') \\
\equiv &\sum_{i=1}^{h-1} V^{i-1} \langle u_i \rangle \cdot e - \sum_{0 \le m \le n-h+b} V^{m+h-1} \langle a_m(\underline{u}) \rangle \cdot e \\
&+ \sum_{i=1}^{h-1} V^{n+b+i-1} \langle \xi^{\sigma^{n+b+i}} \ u_i^{q^n} \ u_1^{q^{n-1}+\cdots+q+1} \rangle \cdot e - \sum_{i=1}^{h-1} V^{n+b+i-1} \langle a_{n+b+i-h}(\underline{u}) \rangle \cdot e \\
&- \sum_{m \ge n+b+h-1} V^m \langle a_{m-h+1}(\underline{u}) \rangle \cdot e \\
\equiv &\sum_{i=1}^{h-1} V^{i-1} \langle u_i \rangle \cdot e - \sum_{h-1 \le m \le n+b-1} V^m \langle a_{m-h+1}(\underline{u}) \rangle \cdot e \\
&+ \sum_{i=1}^{h-1} V^{n+b+i-1} \langle \xi^{\sigma^{n+b+i}} \ u_i^{q^n} \ u_1^{q^{n-1}+\cdots+q+1} - a_{n+b+i-h}(\underline{u}) \rangle \cdot e \\
&- \sum_{m \ge n+b+h-1} V^m \langle a_{m-h+1}(\underline{u}) \rangle \cdot e
\end{aligned}
$$

We also have

$$
\begin{aligned}
&\sum_{i=1}^{h-1} V^{i-1} \langle s_i \rangle \cdot \Phi(e') \\
\equiv &\sum_{i=1}^{h-1} V^{i-1} \langle s_i \rangle \cdot e + \sum_{1 \le i \ \le h-1} V^{n+b+i-1} \langle \xi^{\sigma^{n+b}} \cdot u_i^{q^{n+b}} \cdot u_1^{q^{n-1}+\cdots+q+1} \rangle \cdot e \\
&+ \sum_{\substack{0 \le m \le n \\ 1 \le i \le h-1}} V^{i-1+m} \langle s_i^{q^m} \ a_m(\underline{u}) \rangle \cdot e \\
\equiv &\begin{cases}
\displaystyle\sum_{i=1}^{h-1} V^{i-1} \langle s_i + \sum_{1 \le j \le i} s_j^{q^{i-j}} \ a_{i-j}(\underline{u}) \rangle \cdot e \\
\displaystyle + \sum_{h-1 \le m \le n+h-2} V^m \langle \sum_{1 \le i \le h-1} s_i^{q^{m-i+1}} \ a_{m-i+1}(\underline{u}) \rangle \cdot e \\
\qquad \text{(if } b \ne 0\text{), or} \\
\displaystyle\sum_{i=1}^{h-1} V^{i-1} \langle s_i + \sum_{1 \le j \le i} s_j^{q^{i-j}} \ a_{i-j}(\underline{u}) \rangle \cdot e \\
\displaystyle + \sum_{h-1 \le m \le n-1} V^m \langle \sum_{1 \le i \le h-1} s_i^{q^{m-i+1}} \ a_{m-i+1}(\underline{u}) \rangle \cdot e \\
\displaystyle + \sum_{1 \le i \le h-1} V^{n+i-1} \langle \xi^{\sigma^n} \cdot u_i^{q^n} \cdot u_1^{q^{n-1}+\cdots+q+1} + \sum_{1 \le j \le h-1} s_j^{q^{n+i-j}} \ a_{n+i-j}(\underline{u}) \rangle \cdot e \\
\qquad \text{(if } b = 0\text{)}
\end{cases}
\end{aligned}
$$

Now we can compare "the coefficients of the V^j's" to get information. When $b \neq 0$, we get modulo $(\underline{u})^{\sigma^{n+1}+\sigma^{n-2}+\sigma^{n-3}+\cdots+\sigma+1}$:

$$a_m \equiv 0 \quad \forall\, m \geq n+b$$

$$u_i \equiv s_i \;+\; \sum_{1\leq j\leq i-1} u_j^{q^{i-j}}\; a_{i-j}(\underline{u}) \;+\; s_i\; a_0(\underline{u}) \qquad (1 \leq i \leq h-1)$$

Thus we need $a_0(\underline{u}), \dots, a_{h-1}(\underline{u})$ in order to get the functions $s_1, \dots, s_{h-1}$. They are given recursively as follows:

$$a_{n+b+i-h}(\underline{u}) \equiv \xi^{\sigma^{n+b+i}}\; u_i^{q^n} \cdot u_1^{q^{n-1}+\cdots+q+1} \qquad \forall\, 1 \leq i \leq h-1\ .$$

This gives $a_{n+b-1}(\underline{u}), \dots, a_{n+b+1-h}(\underline{u})$; and for $h-1 \leq m \leq n+b-1$ we have

$$a_{m-h+1}(\underline{u}) \equiv -\sum_{1\leq i\leq h-1} s_i^{q^{m-i+1}}\; a_{m-i+1}(\underline{u})\ .$$

In other words, for $0 \leq m \leq n+b-h$,

$$a_m(\underline{u}) \equiv -\sum_{1\leq i\leq h-1} s_i^{q^{m+h-i}}\; a_{m+h-i}(\underline{u})\ .$$

Hence for $b \neq 0$ (i.e. $1 \leq b \leq h-1$), we get

$$\begin{cases} a_m(\underline{u}) \equiv 0 \qquad \forall\, m \geq n+b \\ a_{n+b+i-h}(\underline{u}) \equiv \xi^{\sigma^{n+b+i}}\; u_i^{q^n} \cdot u_1^{q^{n-1}+\cdots+q+1} \qquad \forall\, 1 \leq i \leq h-1 \\ a_m(\underline{u}) \equiv -\displaystyle\sum_{1\leq i\leq h-1} u_i^{q^{m+h-i}}\; a_{m+h-i}(\underline{u}) \qquad \forall\, 0 \leq m \leq n+b-h \\ u_i \equiv s_i\;\; (1+a_0(\underline{u})) \;+\; \displaystyle\sum_{1\leq j\leq i-1} u_j^{q^{i-j}}\; a_{i-j}(\underline{u}) \\ \qquad (\mathrm{mod}\ (\underline{u})^{\sigma^{n+1}+\sigma^{n-2}+\sigma^{n-3}+\cdots+\sigma+1}) \end{cases}$$

When $b = 0$, again modulo $(\underline{u})^{\sigma^{n+1}+\sigma^{n-2}+\sigma^{n-3}+\cdots+\sigma+1}$, we have

$$a_m(\underline{u}) \equiv 0 \qquad \forall\, m \geq n$$

$$\begin{aligned} &\xi^{\sigma^{n+i}}\; u_i^{q^n}\; u_1^{q^{n-1}+\cdots+q+1} \;-\; a_{n+i-j}(\underline{u}) \\ &\equiv \xi^{\sigma^n}\; u_i^{q^n}\; u_1^{q^{n-1}+\cdots+q+1} \;+\; \sum_{i+1\leq j\leq h-1} s_j^{q^{n+i-j}}\; a_{n+i-j}(\underline{u}) \qquad (1 \leq i \leq h-1) \end{aligned}$$

This gives $a_{n-1}(\underline{u}), \dots, a_{n-h+1}(\underline{u})$. Also

$$a_{m-h+1}(\underline{u}) \;\equiv\; -\sum_{1\leq i\leq h-1} s_i^{q^{m-i+1}}\; a_{m-i+1}(\underline{u}) \qquad (h-1 \leq m \leq n-1)$$

This gives $a_{n-h}(\underline{u}), \ldots, a_0(\underline{u})$. Moreover

$$u_i \;\equiv\; s_i \;+\; \sum_{1\le j\le i-1} s_j^{q^{i-j}}\; a_{i-j}(\underline{u}) \;+\; s_i\; a_0(\underline{u}) \qquad (1\le i\le h-1)$$

In other words we have

$$\begin{cases} a_m(\underline{u}) \equiv 0 \qquad \forall\; m\ge n \\ a_{n+i-h}(\underline{u}) \;\equiv\; \left(\xi^{\sigma^{n+i}} - \xi^{\sigma^n}\right)\; u_i^{q^n}\; u_1^{q^{n-1}+\cdots+q+1} \\ \qquad\qquad - \sum_{i+1\le j\le h-1} s_j^{q^{n+i-j}}\; a_{n+i-j}(\underline{u}) \qquad (\forall\; 1\le i\le h-1) \\ a_m(\underline{u}) \;\equiv\; -\sum_{1\le i\le h-1} u_i^{q^{m+h-i}}\; a_{m+h-i}(\underline{u}) \qquad (\forall\; 0\le m\le n-h) \\ u_i \;\equiv\; s_i\;\; (1+a_0(\underline{u})) + \sum_{1\le j\le i-1} u_j^{q^{i-j}}\; a_{i-j}(\underline{u}) \qquad (\forall\; 1\le i\le h-1) \\ \qquad\qquad (\mathrm{mod}\; (\underline{u})^{\sigma^{n+1}+\sigma^{n-2}+\sigma^{n-3}+\cdots+\sigma+1}) \end{cases}$$

Consider now the case when $char(A)=0$. Let e be the absolute ramification index of A, $q=p^r$. We shall see need some simple calculation about ramified Witt vectors. We need information about the Witt components of the sum of two Teichmüller representatives. Suppose

$$\langle x\rangle \;+\; \langle y\rangle \;=\; (S_0, S_1, S_2, \ldots)\;,$$

so $S_0 = x+y$, $S_1 = \frac{1}{\pi}\left(x^q + y^q - (x+y)^q\right), \ldots$.
If $e=1$, $q=p$, i.e. $A=\mathbb{Z}_p$, then

$$S_1 \equiv -x^{p-1}y \quad \left(\mathrm{mod} \sum_{2\le i\le p-1} (x)^{p-i}(y)^i + (\pi)\right)$$

$$S_2 \equiv -x^{p^2-1}y \quad \left(\mathrm{mod} \sum_{2\le i\le p^2-1} (x)^{p^2-i}(y)^i + (\pi)\right)$$

$$\vdots$$

$$S_m \equiv -x^{p^m-1}y \quad \left(\mathrm{mod} \sum_{2\le i\le p^m-1} (x)^{p^m-i}(y)^i + (\pi)\right)$$

$$\vdots$$

If $e>1$, then no matter whether $q=p$ or not,

$$S_m \equiv 0 \quad (\mathrm{mod}\;\pi) \quad if\;\; e\nmid m$$

$$S_m \equiv -\,x^{q^m-1}y \left(\mathrm{mod} \sum_{2\le i\le q^m-1} (x)^{q^m-i}(y)^i + (\pi)\right) \quad if\;\; e\mid m$$

While if $q = p^r$, $r > 1$, then

$$S_m \equiv 0 \quad \left(\bmod \sum_{p^{r-1} \le i \le q^m - p^{r-1}} (x)^{q^m - i}(y)^i + (\pi)\right) \quad \forall\, m \ge 1 \, .$$

Moreover, we always have

$$S_m \equiv 0 \quad \left(\bmod\ (x,y)^{q^m}\right) \, .$$

With the above estimates, we can go back to our calculation. We have

$$\begin{aligned}
&\left(F - V^{h-1}\right) \Phi(e') \\
\equiv & \sum_{i=1}^{h-1} V^{i-1}\langle u_i\rangle\ e \ - \sum_{0 \le m \le n-h+b} V^{m+h-1}\langle a_m(\underline{u})\rangle\ e \\
& + \sum_{i=1}^{h-1} V^{n+b+i-1}\langle \xi^{\sigma^{n+b+i}}\ u_i^{q^n}\ u_1^{q^{n-1}+\cdots+q+1}\rangle\ e \\
& - \sum_{i=1}^{h-1} V^{n+b+i-1}\ \langle a_{n+b+i-h}(\underline{u})\rangle\ e \ - \sum_{m \ge n+b+h-1} V^m\langle a_{m-h+1}(\underline{u})\rangle\ e \\
& \qquad\qquad \left(\bmod\ (\underline{u})^{\sigma^{n+1}+\sigma^{n-2}+\cdots+\sigma+1}\right) \\
& \equiv \sum_{i=1}^{h-1} V^{i-1}\langle u_i\rangle\ e \ - \sum_{h-1 \le m \le n+b-1} V^m\langle a_{m-h+1}(\underline{u})\rangle\ e \\
& + \sum_{i=1}^{h-1} V^{n+b+i-1}\ \langle \xi^{\sigma^{n+b+i}}\ u_i^{q^n}\ u_1^{q^{n-1}+\cdots+q+1}\ - \ a_{n+b+i-h}(\underline{u})\rangle\ e \\
& - \sum_{m \ge n+b+h-1} V^m\ \langle a_{m-h+1}(\underline{u})\rangle\ e \\
& \qquad\qquad \left(\bmod\ \ (\underline{u})^{\sigma^{n+1}+\sigma^{n-2}+\cdots+\sigma+1}\ +\ (\underline{u})^{q(\sigma^n+\sigma^{n-1}+\cdots+\sigma+1)}\right)
\end{aligned}$$

Case 1. $q = p^r$, $r > 1$.

Then modulo $(\underline{u})^{p^{r-1}(p-1)+p^{r-1}(\sigma^n+\sigma^{n-1}+\cdots+\sigma+1)}$, we get

$$
\begin{aligned}
&\sum_{i=1}^{h-1} V^{i-1}\ \langle s_i\rangle\ \Phi(e') \\
\equiv &\begin{cases}
\text{if } b \neq 0: \\
\quad \displaystyle\sum_{i=1}^{h-1} V^{i-1} \left\langle s_i \ +\sum_{1\le j\le i} s_j^{q^{i-j}}\ a_{i-j}(\underline{u})\right\rangle\ e \\
\quad +\displaystyle\sum_{h-1\le m\le n+h-2} V^m \left\langle \sum_{i=1}^{h-1} s_i^{q^{m-i+1}}\ a_{m-i+1}(\underline{u})\right\rangle\ e \\
\\
\text{or, if } b = 0: \\
\quad \displaystyle\sum_{i=1}^{h-1} V^{i-1} \left\langle s_i \ +\sum_{1\le j\le i} s_j^{q^{i-j}}\ a_{i-j}(\underline{u})\right\rangle\ e \\
\quad +\displaystyle\sum_{h-1\le m\le n-1} V^m \left\langle \sum_{i=1}^{h-1} s_i^{q^{m-i+1}}\ a_{m-i+1}(\underline{u})\right\rangle\ e \\
\quad +\displaystyle\sum_{i=1}^{h-1} V^{n+i-1} \left\langle \xi^{\sigma^n}\cdot u_i^{q^n}\ u_1^{q^{n-1}+\cdots+q+1} + \sum_{1\le j\le h-1} s_j^{q^{n+i-j}}\ a_{n+i-j}(\underline{u})\right\rangle\ e
\end{cases} \qquad \blacksquare
\end{aligned}
$$

Comparing this with the formula for the case $char(A) = p$, we see that the formula is the same if read modulo the ideal $(\underline{u})^{p^{r-1}(p-1)+p^{r-1}(\sigma^n+\sigma^{n-1}+\cdots+\sigma+1)}$.

Case 2. $q = p$.

This time we get a slightly different formula:

$$
\begin{aligned}
&\sum_{i=1}^{h-1} V^{i-1}\ \langle s_i\rangle\ \Phi(e') \\
&\equiv \sum_{i=1}^{h-1} V^{i-1}\langle s_i\rangle\ e\ +\ \sum_{i=1}^{h-1} V^{i-1} \left\langle \sum_{1\le j\le i} s_j^{q^{i-j}}\ a_{i-j}(\underline{u})\right\rangle\ e \\
&+\begin{cases}
\text{if } b \neq 0: \\
\quad \displaystyle\sum_{h-1\le m\le n+h-2} V^m \left\langle \sum_{i=1}^{h-1} s_i^{q^{m-i+1}}\ a_{m-i+1}(\underline{u})\right\rangle\ e \\
\\
\text{or, if } b = 0: \\
\quad \displaystyle\sum_{h-1\le m\le n-1} V^m \left\langle \sum_{i=1}^{h-1} s_i^{q^{m-i+1}}\ a_{m-i+1}(\underline{u})\right\rangle\ e \\
\quad +\displaystyle\sum_{i=1}^{h-1} V^{n+i-1} \left\langle \xi^{\sigma^n}\cdot u_i^{q^n}\ u_1^{q^{n-1}+\cdots+q+1} + \sum_{1\le j\le h-1} s_j^{q^{n+i-j}}\ a_{n+i-j}(\underline{u})\right\rangle\ e
\end{cases}
\end{aligned}
$$

modulo the ideal $(\underline{u})^{q(\sigma^n+\cdots+\sigma+1)}$.

Modulo the ideal $(\underline{u})^{p^e+2(\sigma^n+\cdots+\sigma)} + (\underline{u})^{p^e-1} \sum_{i=0}^{h-2} (\underline{u})^{\sigma^i} (a_i(\underline{u}))$, we still have

$$\begin{aligned}
&\sum_{i=1}^{h-1} V^{i-1} \langle s_i \rangle \Phi(e') \\
\equiv &\sum_{i=1}^{h-1} V^{i-1}\langle s_i \rangle e + \sum_{i=1}^{h-1} V^{i-1} \left\langle \sum_{1\le j\le i} s_j^{q^{i-j}} a_{i-j}(\underline{u}) \right\rangle e \\
+ &\begin{cases}
\text{if } b \neq 0 : \\
\displaystyle\sum_{h-1\le m\le n+h-2} V^m \left\langle \sum_{i=1}^{h-1} s_i^{q^{m-i+1}} a_{m-i+1}(\underline{u}) \right\rangle e \\
\\
\text{or, if } b = 0 : \\
\displaystyle\sum_{h-1\le m\le n-1} V^m \left\langle \sum_{i=1}^{h-1} s_i^{q^{m-i+1}} a_{m-i+1}(\underline{u}) \right\rangle e \\
\displaystyle\quad + \sum_{i=1}^{h-1} V^{n+i-1} \left\langle \xi^{\sigma^n} \cdot u_i^{q^n} u_1^{q^{n-1}+\cdots+q+1} + \sum_{1\le j\le h-1} s_j^{q^{n+i-j}} a_{n+i-j}(\underline{u}) \right\rangle e
\end{cases}
\end{aligned}$$

In the case $q = p^r > p$, we conclude that the following formulae $(*)$ for the case $b \neq 0$ (resp. $(*)'$ for the case $b = 0$) hold modulo $(\underline{u})^{p^{r-1}(p-1)+p^{r-1}(\sigma^n+\cdots+\sigma+1)}$:

$$(*) \begin{cases}
a_m(\underline{u}) \equiv 0 & \forall\, m \ge n+b \\
a_{n+b+i-h}(\underline{u}) \equiv \xi^{\sigma^{n+b+i}} u_i^{q^n} u_1^{q^{n-1}+\cdots+q+1} & 1 \le i \le h-1 \\
a_m(\underline{u}) \equiv -\displaystyle\sum_{i=1}^{h-1} u_i^{q^{m+h-i}} a_{m+h-i}(\underline{u}) & 0 \le m \le n+b-h \\
u_i \equiv s_i\ (1 + a_0(\underline{u})) + \displaystyle\sum_{j=1}^{i-1} u_j^{q^{i-j}} a_{i-j}(\underline{u}) & 1 \le i \le h-1
\end{cases}$$

$$(*)' \begin{cases}
a_m(\underline{u}) \equiv 0 & \forall\, m \ge n \\
a_{n+i-h}(\underline{u}) \equiv \left(\xi^{\sigma^{n+i}} - \xi^{\sigma^n}\right) u_i^{q^n} u_1^{q^{n-1}+\cdots+q+1} & \\
\qquad\qquad - \displaystyle\sum_{i+1\le j\le h-1} s_j^{q^{n+i-j}} a_{n+i-j}(\underline{u}) & 1 \le i \le h-1 \\
a_m(\underline{u}) \equiv -\displaystyle\sum_{i=1}^{h-1} u_i^{q^{m+h-i}} a_{m+h-i}(\underline{u}) & 0 \le m \le n-h \\
u_i \equiv s_i\ (1 + a_0(\underline{u})) + \displaystyle\sum_{j=1}^{i-1} u_j^{q^{i-j}} a_{i-j}(\underline{u}) & 1 \le i \le h-1
\end{cases}$$

Notice that $(*)$ and $(*)'$ differ only in how $a_{n+b-1}, \ldots, a_{n+b-h+1}$ are defined.

When $q = p$, although the congruence becomes a little worse, we can still conclude that the formulae for the s_i's given by $(*)$ and $(*)'$ at least give the correct initial terms of $s_i - u_i$ in the $(\underline{u})$-adic filtration of $k[[\underline{u}]]$:

Theorem 1. *Let $\Phi_0 \in Aut(G_0)$ be an automorphism of G_0 such that $\Phi_0(e) = e + \xi\, V^\ell e$, where $\xi \in \mu_{q^h-1}\big(W_A(\mathbb{F}_{q^h})\big)$, $\ell = nh + b$, with $0 \le b \le h-1$, $n \ge 1$. Let $s_i = h(\Phi_0)(u_i)$ for $i = 1, \ldots, h-1$.*

(1) If $char(A) = p > 0$ (i.e. $A \cong \mathbb{F}_q[[t]]$), then modulo $(\underline{u})^{\sigma^{n+1}+\sigma^{n-2}+\cdots+1}$, the s_i's are given recursively by the formulae $(*)$ and $(*)'$.

(2) In the case $char(A) = 0$, when $q = p^r > p$, the s_i's are given recursively by the formulae $(*)$ and $(*)'$ modulo $(\underline{u})^{p^{r-1}(p-1)+p^{r-1}(\sigma^n+\cdots+\sigma+1)}$. When $q = p$, the same formulae give at least the correct initial term of $s_i - u_i$'s in the $(\underline{u})$-adic filtration of $k[[\underline{u}]]$.

Remarks. (1) When $n = 0$, some slight modification is necessary. If $1 \le b \le h-1$, the formula which gives the leading term of $s_i - u_i$'s is:

$$\begin{cases} a_0(\underline{u}) \equiv \xi\, u_{h-b} & \\ u_i \equiv s_i\,(1 + a_0(\underline{u})\,) & 1 \le i \le b \\ u_i + \xi^{\sigma^i}\, u_{i-b} \equiv s_i\,(1 + a_o(\underline{u})\,) & b+1 \le i \le h-1 \end{cases}$$

We leave the proof to the patient reader. When $n = 0$, $b = 0$, then Φ_0 is congruent to a $(q^h - 1)$st root of unity modulo V, and the main term of the action is given by lemma 2.

(2) These formulae relate the π-adic filtration of $Aut(G_0)$ with the filtration

$$Fil^n = \left\{ \phi \in Aut(G_0) \;\middle|\; \begin{array}{l} \phi \text{ extends to an automorphism of} \\ G \times_{\mathrm{Spec}\ \kappa_h[[\underline{u}]]} \mathrm{Spec}\ \kappa_h[[\underline{u}]]/(\underline{u})^{n+1} \end{array} \right\} .$$

(3) Similar results can be obtained for the action of $Aut(G_0)$ on

$$\mathrm{Spf}\ \kappa_h[[u_1, \ldots, u_{h-1}]]/(u_1, \ldots, u_{h_1-1})\ ,$$

the maximal subscheme of Spf $\kappa_h[[\underline{u}]]$ over which the height of the universal deformation is at least h_1. K. Keating has investigated the case $h_1 = h-1$ in [Ke].

We shall reformulate theorem 1 using matrix notation so that the role of the Hasse invariant u_1 becomes apparent. Let $\underline{u}$ and $\underline{s}$ denote the column vectors

$$\underline{u} = \begin{pmatrix} u_{h-1} \\ \vdots \\ u_1 \end{pmatrix}, \quad \underline{s} = \begin{pmatrix} s_{h-1} \\ \vdots \\ s_1 \end{pmatrix}$$

respectively. Denote by R the $(h-1) \times (h-1)$ matrix

$$R = \begin{pmatrix} u_1^{q^{h-2}} & u_2^{q^{h-3}} & \ldots & u_{h-2}^q & u_{h-1} \\ 0 & u_1^{q^{h-3}} & \ldots & u_{h-3}^q & u_{h-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & u_1^q & u_2 \\ 0 & 0 & \ldots & 0 & u_1 \end{pmatrix},$$

and let R' be the $(h-1)\times(h-1)$ matrix

$$R' \;=\; \begin{pmatrix} u_1^{q^{h-2}} & u_2^{q^{h-3}} & \dots & u_{h-2}^{q} & s_{h-1} \\ 0 & u_1^{q^{h-3}} & \dots & u_{h-3}^{q} & s_{h-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & u_1^{q} & s_2 \\ 0 & 0 & \dots & 0 & s_1 \end{pmatrix} .$$

Let C_m denote the $(h-1)\times(h-1)$ matrix

$$C_m \;=\; \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -u_1^{q^{m+h-1}} & -u_2^{q^{m+h-2}} & -u_3^{q^{m+h-3}} & \dots & -u_{h-2}^{q^{m+2}} & -u_{h-1}^{q^{m+1}} \end{pmatrix} .$$

Notice that $\det(R) \;=\; u_1^{q^{h-2}+\cdots+q+1}$, and $\det(C_m) \;=\; (-1)^{h-1}\; u_1^{q^{m+h-1}}$.

For $b = 1, \dots, h-1$, let $\underline{d}_{b,n,\xi}$ be the $(h-1)\times 1$ column vector

$$\begin{aligned} \underline{d}_{b,n,\xi} \;&=\; \begin{pmatrix} \xi^{\sigma^{n+b+h-1}} \cdot u_{h-1}^{q^n} \cdot u_1^{q^{n-1}+\cdots+q+1} \\ \xi^{\sigma^{n+b+h-2}} \cdot u_{h-2}^{q^n} \cdot u_1^{q^{n-1}+\cdots+q+1} \\ \vdots \\ \xi^{\sigma^{n+b+2}} \cdot u_2^{q^n} \cdot u_1^{q^{n-1}+\cdots+q+1} \\ \xi^{\sigma^{n+b+1}} \cdot u_1^{q^n} \cdot u_1^{q^{n-1}+\cdots+q+1} \end{pmatrix} \\ &=\; \begin{pmatrix} u_{h-1}^{q^n} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & u_1^{q^n} \end{pmatrix} \cdot \begin{pmatrix} \xi^{\sigma^{n+b+h-1}} \\ \vdots \\ \xi^{\sigma^{n+b+2}} \\ \xi^{\sigma^{n+b+1}} \end{pmatrix} \cdot u_1^{q^{n-1}+\cdots+q+1} . \end{aligned}$$

Let $\underline{d}'_{0,n,\xi}$ be the $(h-1)\times 1$ column vector

$$\begin{aligned} \underline{d}'_{0,n,\xi} \;&=\; \begin{pmatrix} (\xi^{\sigma^{n+h-1}} - \xi^{\sigma^n}) \cdot u_{h-1}^{q^n} \cdot u_1^{q^{n-1}+\cdots+q+1} \\ (\xi^{\sigma^{n+h-2}} - \xi^{\sigma^n}) \cdot u_{h-2}^{q^n} \cdot u_1^{q^{n-1}+\cdots+q+1} \\ \vdots \\ (\xi^{\sigma^{n+2}} - \xi^{\sigma^n}) \cdot u_2^{q^n} \cdot u_1^{q^{n-1}+\cdots+q+1} \\ (\xi^{\sigma^{n+1}} - \xi^{\sigma^n}) \cdot u_1^{q^n} \cdot u_1^{q^{n-1}+\cdots+q+1} \end{pmatrix} \\ &=\; \begin{pmatrix} u_{h-1}^{q^n} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & u_1^{q^n} \end{pmatrix} \cdot \begin{pmatrix} \xi^{\sigma^{n+h-1}} - \xi^{\sigma^n} \\ \vdots \\ \xi^{\sigma^{n+2}} - \xi^{\sigma^n} \\ \xi^{\sigma^{n+1}} - \xi^{\sigma^n} \end{pmatrix} \cdot u_1^{q^{n-1}+\cdots+q+1} . \end{aligned}$$

For $i = 1, \ldots, h-1$, let D_i be the $(h-1) \times (h-1)$ matrix

$$D_i \;=\; \begin{array}{cc} & \begin{array}{ccccccccc} \phantom{-u_{h-i}^{n-1}} & \phantom{-u_{h-i+1}^{n-2}} & & \phantom{-u_{h-2}^{q^{n-i+1}}} & i^{th} & & & & \end{array} \\ i^{th} & \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 1 & 0 & 0 & \ldots & 0 \\ -u_{h-i}^{n-1} & -u_{h-i+1}^{n-2} & \ldots & -u_{h-2}^{q^{n-i+1}} & -u_{h-1}^{q^{n-i}} & 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix} \end{array},$$

and let $\underline{d}_{0,n,\xi}$ be the $(h-1) \times 1$ column vector

$$\underline{d}_{0,n,\xi} \;=\; D_{h-1} \cdots D_2 \cdot D_1 \cdot \underline{d}'_{0,n,\xi} \;.$$

With the above notation, Theorem 1 says that

$$-R' \cdot C_0 \cdot C_1 \cdots C_{n+b-h} \cdot \underline{d}_{b,n,\xi} \;\equiv\; \underline{s} - \underline{u} \qquad \left(\mathrm{mod}\ (\underline{u})^{\sigma^{n+1}+\sigma^{n-2}+\cdots+\sigma+1}\right)$$

if $char(A) = p > 0$. If $char(A) = 0$ and $q = p^r > p$, this congruence holds modulo $(\underline{u})^{p^{r-1}(p-1)+p^{r-1}(\sigma^n+\cdots+\sigma+1)}$. Replacing σ by q in the above congruence, we get a cruder congruence, and one sees that the column vector $-R{\cdot}C_0{\cdot}C_1 \cdots C_{n+b-h}{\cdot}\underline{d}_{b,n,\xi}$ gives the initial form of $\underline{s} - \underline{u}$ with respect to the $(\underline{u})$-adic filtration on $k[[\underline{u}]]$.

The matricial reformulation above makes it clear that theorem 1 is not well-suited for many applications: for each row of the matrix R, the entries are monomials of different degrees, and similarly for the C_m's. However a natural remedy suggests itself. Instead of giving $u_1, \ldots, u_{h-1}$ weight 1, we decree that u_i has weight q^{i-1}, and let

$$w(\underline{u}^I) \;=\; i_1 + i_2\, q + \cdots + i_{h-1}\, q^{h-2}$$

for a multi-index $I = (i_1, \ldots, i_{h-1}) \in \mathbb{N}^{h-1}$. Define

$$Fil^w \;=\; \left\{ \sum_{w(I) \geq w} a_I \cdot \underline{u}^I \;\in \kappa_h[[\underline{u}]] \;\middle|\; a_I \in \kappa_h \right\} \;\subseteq\; \kappa_h[[\underline{u}]] \;.$$

Clearly each Fil^w is an ideal of $\kappa_h[[\underline{u}]]$. Now we shall re-examine the calculation using this filtration. Computing as before:

$$\begin{aligned} \langle u_i^{q^n} \rangle\, F^n\, e \;&=\; \langle u_i^{q^n} \rangle \left(V^{h-1} + \sum_{i=1}^{h-1} V^{i-1} \langle u_i^{q^{n-1}} \rangle \right) \cdots \\ &\quad \cdots \left(V^{h-1} + \sum_{i=1}^{h-1} V^{i-1} \langle u_i^{q} \rangle \right) \left(V^{h-1} + \sum_{i=1}^{h-1} V^{i-1} \langle u_i \rangle \right) \\ &\equiv 0 \quad \left(\mathrm{mod}\ \ Fil^{q^{n+i-1}+q^{n-1}+\cdots+q+1}\right) . \end{aligned}$$

Similarly,

$$\langle s_i^{q^{n+b}} \rangle \; F^n \; e \;\equiv\; 0 \quad (\bmod \;\; Fil^{q^{n+i+b-1}+q^{n-1}+\cdots+q+1}) \; .$$

Therefore

$$\begin{cases} \qquad s_i \;\equiv u_i \\ a_m(\underline{u}) \;\equiv 0 \;\; \forall \; m \geq 0 \end{cases} \quad \left(\bmod \;\; Fil^{q^n+\cdots+q+1} \right)$$

Moreover,

$$\langle u_i^{q^n} \rangle \; F^n \; e \;\equiv\; \langle u_i^{q^n} \cdot u_1^{q^{n-1}+\cdots+q+1} \rangle \; e \quad \left(\bmod \;\; Fil^{q^{n+i}+q^{n-2}+\cdots+q+1} \right) \; ,$$

$$\langle s_i^{q^{n+b}} \rangle \; F^n \; e \;\equiv\; \langle u_i^{q^{n+b}} \cdot u_1^{q^{n-1}+\cdots+q+1} \rangle \; e \quad \left(\bmod \;\; Fil^{q^{n+b+i}+q^{n-2}+\cdots+q+1} \right) \; .$$

We get

$$\begin{aligned} & \left(F - V^{h-1} \right) \; \Phi(e') \\ \equiv & \sum_{i=1}^{h-1} \; V^{i-1} \; \langle u_i \rangle \; e \; + \; V^{n+b} \; \langle \xi^{\sigma^{n+b+1}} \; u_1^{q^n+q^{n-1}+\cdots+1} \rangle \; e \\ & \quad - \sum_{m \geq 0} \; V^{m+h-1} \; \langle a_m(\underline{u}) \rangle \; e \qquad \left(\; \bmod \;\; Fil^{q^{n+1}+q^{n-2}+\cdots+q+1} \; \right) \; , \end{aligned}$$

$$\begin{aligned} & \sum_{i=1}^{h-1} \; V^{i-1} \; \langle s_i \rangle \; \Phi(e') \\ \equiv & \sum_{i=1}^{h-1} V^{i-1} \; \langle s_i \rangle \; e \; + \; V^{n+b} \; \langle \xi^{\sigma^{n+b}} \; s_1^{q^n} \; u_1^{q^{n-1}+\cdots+q+1} \rangle \; e \\ & \quad + \sum_{\substack{1 \leq i \leq h-1 \\ m \geq 0}} \; V^{m+i-1} \; \langle s_i^{q^m} \; a_m(\underline{u}) \rangle \; e \qquad \left(\; \bmod \;\; Fil^{q^{n+1}+q^{n-2}+\cdots+q+1} \; \right) \; . \end{aligned}$$

If $b \neq 0$, then

$$\begin{aligned} & \sum_{i=1}^{h-1} \; V^{i-1} \; \langle s_i \rangle \; \Phi(e') \\ & \equiv \; \sum_{i=1}^{h-1} V^{i-1} \; \langle s_i \rangle \; e \; + \; \sum_{m \geq o} \; V^m \; \left\langle \; \sum_{i=1}^{h-1} \; s_i^{q^{m-i+1}} \; a_{m-i+1}(\underline{u}) \; \right\rangle \; e \\ & \equiv \; \sum_{i=1}^{h-1} \; V^{i-1} \; \left\langle \; s_i \; + \sum_{1 \leq j \leq i} \; s_j^{q^{i-j}} \; a_{i-j}(\underline{u}) \; \right\rangle \; e \\ & \quad + \sum_{h-1 \leq m \leq n} \; V^m \; \left\langle \; \sum_{i=1}^{h-1} \; s_i^{q^{m-i+1}} \; a_{m-i+1}(\underline{u}) \; \right\rangle \; e \\ & \qquad\qquad \left(\; \bmod \;\; Fil^{q^{n+1}+q^{n-2}+\cdots+q+1} \; \right) \; . \end{aligned}$$

While if $b = 0$, then

$$\begin{aligned}
&\sum_{i=1}^{h-1} V^{i-1} \langle s_i \rangle \ \Phi(e') \\
&\equiv \sum_{i=1}^{h-1} V^{i-1} \left\langle s_i + \sum_{1 \leq j \leq i} s_j^{q^{i-j}} \ a_{i-j}(\underline{u}) \right\rangle e \\
&\quad + \sum_{h-1 \leq m \leq n-1} V^m \left\langle \sum_{i=1}^{h-1} s_i^{q^{m-i+1}} \ a_{m-i+1}(\underline{u}) \right\rangle e \\
&\quad + V^n \left\langle \xi^{\sigma^n} \ u_1^{q^n + \cdots + q + 1} + \sum_{j=1}^{h-1} s_j^{q^{n+1-j}} \ a_{n+1-j}(\underline{u}) \right\rangle e \\
&\qquad \left(\mathrm{mod} \ \ Fil^{q^{n+1} + q^{n-2} + \cdots + q + 1} \right) .
\end{aligned}$$

When $b \neq 0$,

$$\begin{aligned}
\left(F - V^{h-1}\right) \ \Phi(e') \equiv &\sum_{i=1}^{h-1} V^{i-1} \langle u_i \rangle \ e - \sum_{h-1 \leq m \leq n+b-1} V^m \ \langle a_{m-h+1}(\underline{u}) \ \rangle \ e \\
&+ V^{n+b} \left\langle \xi^{\sigma^{n+b+1}} \ u_1^{q^n + q^{n-1} + \cdots + q + 1} - a_{n+b-h+1}(\underline{u}) \right\rangle e \\
&- \sum_{m \geq n+b+1} V^m \ \langle a_{m-h+1}(\underline{u}) \rangle \ e \\
&\qquad \left(\mathrm{mod} \ \ Fil^{q^{n+1} + q^{n-2} + \cdots + q + 1} \right) .
\end{aligned}$$

Comparing this with the expression for $\sum_{i=1}^{h-1} V^{i-1} \langle s_i \rangle \ \Phi(e')$, we get

$$\begin{cases}
u_i \equiv s_i + \displaystyle\sum_{j=1}^{i} s_j^{q^{i-j}} \ a_{i-j}(\underline{u}) \qquad 1 \leq i \leq h-1 \\
a_m(\underline{u}) \equiv 0 \qquad \forall m \geq n+b-h+2 \\
a_{n+b-h+1}(\underline{u}) \equiv \xi^{\sigma^{n+b+1}} \ u_1^{q^n + q^{n-1} + \cdots + q + 1} \\
a_{m-h+1}(\underline{u}) \equiv 0 \qquad m = n+b-1, \ldots , n+1 \\
a_{m-h+1}(\underline{u}) \equiv -\displaystyle\sum_{i=1}^{h-1} s_i^{q^{m-i+1}} \ a_{m-i+1}(\underline{u}) \qquad h-1 \leq m \leq n \\
\left(\Leftrightarrow \ a_m(\underline{u}) \equiv -\displaystyle\sum_{i=1}^{h-1} s_i^{q^{m+h-i}} \ a_{m+h-i}(\underline{u}) \qquad 0 \leq m \leq n-h+1 \right) \\
\qquad \left(\mathrm{mod} \ \ Fil^{q^{n+1} + q^{n-2} + \cdots + q + 1} \right)
\end{cases}$$

From the above recursive formulae we can extract the leading terms of the $a_m(\underline{u})$'s:

(1) $a_{n+b-h+1}(\underline{u}) \in Fil^{q^n+q^{n-1}+\cdots+q+1}$, and its leading term is

$$\xi^{\sigma^{n+b+1}} \; u_1^{q^n+q^{n-1}+\cdots+q+1} \; .$$

(2) $a_{n-h+b}(\underline{u}) \equiv \cdots \equiv a_{n-h+2}(\underline{u}) \equiv 0 \qquad \left(\bmod \; Fil^{q^{n+1}+q^{n-2}+\cdots+q+1} \right)$

(3) $a_{n-h+1}(\underline{u}) \in Fil^{2q^n+q^{n-1}+\cdots+q+1}$, with leading term

$$-\xi^{\sigma^{n+b+1}} \; u_{h-b}^{q^{n+b-h+1}} \; u_1^{q^n+q^{n-1}+\cdots+q+1} \; ;$$

$a_{n-h}(\underline{u})$ has leading term

$$-\xi^{\sigma^{n+b+1}} \; u_{h-b-1}^{q^{n+b-h+1}} \; u_1^{q^n+q^{n-1}+\cdots+q+1} \; ;$$

and similarly for $a_{n-h-1}(\underline{u}), \ldots, a_{n-2h+b+2}(\underline{u})$. For instance, $a_{n-2h+b+2}(\underline{u})$ has leading term

$$-\xi^{\sigma^{n+b+1}} \; u_1^{q^{n+b-h+1}} \; u_1^{q^n+q^{n-1}+\cdots+q+1} \; .$$

(4) $a_{n-2h+b+1}(\underline{u})$ has leading term

$$\xi^{\sigma^{n+b+1}} \; u_{h-1}^{q^{n-2h+b+2}} \; u_1^{q^{n+b-h+1}+q^n+q^{n-1}+\cdots+q+1} \; ;$$

$a_{n-2h+b}(\underline{u})$ has leading term

$$\xi^{\sigma^{n+b+1}} \; u_{h-2}^{q^{n-2h+b+2}} \; u_1^{q^{n+b-h+1}+q^n+q^{n-1}+\cdots+q+1} \; ;$$

similarly for $a_{n-2h+b-1}(\underline{u}), \ldots, a_{n-3h+b+3}(\underline{u})$. For instance, $a_{n-3h+b+3}(\underline{u})$ has leading term

$$\xi^{\sigma^{n+b+1}} \; u_1^{q^{n-2h+b+2}} \; u_1^{q^{n+b-h+1}+q^n+q^{n-1}+\cdots+q+1} \; .$$

The pattern for the leading terms of the $a_m(\underline{u})$'s should be very clear by now. We shall give the leading term of the first few $a_m(\underline{u})$'s; they appear in the expression for the $s_i - u_i$'s.

(5) Write $n+b = (h-1)\cdot d+c$, with $c, d \in \mathbb{Z}$, $0 \le c \le h-2$. Then $n+b-(h-1)(d-1) = c+h-1$, and $a_{c+h-1}(\underline{u})$ has leading term

$$(-1)^d \; \xi^{\sigma^{n+b+1}} \; u_1^{q^{c+2(h-1)}+\cdots+q^{n+b-2(h-1)}+q^{n+b-(h-1)}+q^n+q^{n-1}+\cdots+q+1} \; .$$

$a_{c+h-2}(\underline{u})$ has leading term

$$(-1)^{d-1} \; \xi^{\sigma^{n+b+1}} \; u_{h-1}^{q^{c+h-1}} \; u_1^{q^{c+2(h-1)}+\cdots+q^{n+b-2(h-1)}+q^{n+b-(h-1)}+q^n+q^{n-1}+\cdots+q+1} \; ;$$

similarly for $a_{c+h-3}(\underline{u}), \ldots, a_c(\underline{u})$. For instance, $a_c(\underline{u})$ has leading term

$$(-1)^{d-1} \; \xi^{\sigma^{n+b+1}} \; u_1^{q^{c+(h-1)}+q^{c+2(h-1)}+\cdots+q^{n+b-2(h-1)}+q^{n+b-(h-1)}+q^n+q^{n-1}+\cdots+q+1} \; .$$

For the rest of the $a_m(\underline{u})$'s: $a_{c-1}(\underline{u})$ has leading term

$$(-1)^d \, \xi^{\sigma^{n+b+1}} \; u_{h-1}^{q^c} \, u_1^{q^{c+(h-1)}+q^{c+2(h-1)}+\cdots+q^{n+b-2(h-1)}+q^{n+b-(h-1)}+q^n+q^{n-1}+\cdots+q+1} \, ,$$

$a_{c-2}(\underline{u})$ has leading term

$$(-1)^d \, \xi^{\sigma^{n+b+1}} \; u_{h-2}^{q^c} \, u_1^{q^{c+(h-1)}+q^{c+2(h-1)}+\cdots+q^{n+b-2(h-1)}+q^{n+b-(h-1)}+q^n+q^{n-1}+\cdots+q+1} \, ,$$

similarly for $a_{c-3}(\underline{u}), \ldots , a_0(\underline{u})$. Especially, $a_0(\underline{u})$ has leading term

$$(-1)^d \, \xi^{\sigma^{n+b+1}} \; u_{h-c}^{q^c} \, u_1^{q^{c+(h-1)}+q^{c+2(h-1)}+\cdots+q^{n+b-2(h-1)}+q^{n+b-(h-1)}+q^n+q^{n-1}+\cdots+q+1} \, .$$

For $1 \le i \le h-1$,

$$u_i \;\equiv\; s_i \;+ \sum_{1\le j\le i} s_j^{q^{i-j}} \; a_{i-j}(\underline{u}) \qquad \left(\bmod \;\; Fil^{q^{n+1}+q^{n-2}+\cdots+q+1} \right) \, .$$

Hence $s_{h-1} - u_{h-1}$ has leading term

$$(-1)^d \, \xi^{\sigma^{n+b+1}} \; u_{h-c-1}^{q^c} \, u_1^{q^{c+(h-1)}+q^{c+2(h-1)}+\cdots+q^{n+b-2(h-1)}+q^{n+b-(h-1)}+q^n+q^{n-1}+\cdots+q+1} \, ,$$

similarly for $s_{h-2} - u_{h-2}, \ldots , s_{c+1} - u_{c+1}$, with the leading term of $s_{c+1} - u_{c+1}$ being

$$(-1)^d \, \xi^{\sigma^{n+b+1}} \; u_1^{q^c} \; u_1^{q^{c+(h-1)}+q^{c+2(h-1)}+\cdots+q^{n+b-2(h-1)}+q^{n+b-(h-1)}+q^n+q^{n-1}+\cdots+q+1} \, .$$

Next, $s_c - u_c$ has leading term

$$(-1)^{d-1} \, \xi^{\sigma^{n+b+1}} \; u_c \, u_{h-c}^{q^c} \, u_1^{q^{c+(h-1)}+q^{c+2(h-1)}+\cdots+q^{n+b-2(h-1)}+q^{n+b-(h-1)}+q^n+q^{n-1}+\cdots+q+1} \, ,$$

$s_{c-1} - u_{c-1}$ has leading term

$$(-1)^{d-1} \, \xi^{\sigma^{n+b+1}} \; u_{c-1} \, u_{h-c}^{q^c} \, u_1^{q^{c+(h-1)}+q^{c+2(h-1)}+\cdots+q^{n+b-2(h-1)}+q^{n+b-(h-1)}+q^n+q^{n-1}+\cdots+q+1} \, ,$$

similarly for $s_{c-2} - u_{c-2}, \ldots , s_1 - u_1$. The leading term of $s_1 - u_1$ is

$$(-1)^{d-1} \, \xi^{\sigma^{n+b+1}} \; u_1 \, u_{h-c}^{q^c} \, u_1^{q^{c+(h-1)}+q^{c+2(h-1)}+\cdots+q^{n+b-2(h-1)}+q^{n+b-(h-1)}+q^n+q^{n-1}+\cdots+q+1} \, .$$

Notice that the leading terms of $s_{h-1} - u_{h-1}, \ldots , s_1 - u_1$ have different weights; the weight of $s_{c+1} - u_{c+1}$ is the lowest. So the leading terms of the matrix

$$\begin{pmatrix} s_{h-1} - u_{h-1} \\ \vdots \\ s_{c+1} - u_{c+1} \\ \vdots \\ s_1 - u_1 \end{pmatrix}$$

is

$$
{}_{(c+1)^{st}} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (-1)^d \ \xi^{\sigma^{n+b+1}} \ u_1^{q^c + q^{c+(h-1)} + q^{c+2(h-1)} + \cdots + q^{n+b-(h-1)} + q^n + q^{n-1} + \cdots + q + 1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}
$$

This finishes our discussion of the case $b \neq 0$.

When $b = 0$,

$$
\begin{aligned}
& \left(F - V^{h-1}\right) \ \Phi(e') \\
\equiv & \sum_{i=1}^{h-1} V^{i-1} \ \langle u_i \rangle \ e \ - \sum_{h-1 \leq m \leq n-1} V^m \ \langle a_{m-h+1}(\underline{u}) \rangle \ e \\
& + \ V^n \ \left\langle \ \xi^{\sigma^{n+1}} \ u_1^{q^n + q^{n-1} + \cdots + q + 1} \ - \ a_{n-h+1}(\underline{u}) \ \right\rangle \ e \\
& - \sum_{m \geq n+1} V^m \ \langle a_{m-h+1}(\underline{u}) \rangle \ e \qquad \left(\bmod \ Fil^{q^{n+1} + q^{n-2} + \cdots + q + 1} \right)
\end{aligned}
$$

Again by comparing with the expression for $\sum_{i=1}^{h-1} \ V^{i-1} \ \langle s_i \rangle \ \Phi(e')$, we get

$$
\begin{cases}
a_m(\underline{u}) \ \equiv 0 \qquad \forall \ m \geq n - h + 2 \\
a_{n-h+1}(\underline{u}) \ \equiv \left(\xi^{\sigma^{n+1}} - \xi^{\sigma^n}\right) \ u_1^{q^n + q^{n-1} + \cdots + q + 1} \\
a_m(\underline{u}) \ \equiv \ - \sum_{i=1}^{h-1} \ s_i^{q^{m+h-i}} \ a_{m+h-i}(\underline{u}) \qquad 0 \leq m \leq n - h \\
u_i \ \equiv s_i \ + \sum_{1 \leq j \leq i} \ s_j^{q^{i-j}} a_{i-j}(\underline{u}) \qquad 1 \leq i \leq h - 1
\end{cases}
$$

It follows from the above inductive formulae that

$$
a_{n-h}(\underline{u}) \ \equiv \ -\left(\xi^{\sigma^{n+1}} - \xi^{\sigma^n}\right) \ u_{h-1}^{q^{n-h+1}} \ u_1^{q^n + q^{n-1} + \cdots + q + 1} \ ;
$$

$a_{n-h-1}(\underline{u})$ has leading term

$$
-\left(\xi^{\sigma^{n+1}} - \xi^{\sigma^n}\right) \ u_{h-2}^{q^{n-h+1}} \ u_1^{q^n + q^{n-1} + \cdots + q + 1} \ ;
$$

similarly for $a_{n-h-2}(\underline{u}), \ldots, a_{n-2h+2}(\underline{u})$; $a_{n-2h+2}(\underline{u})$ has leading term

$$
-\left(\xi^{\sigma^{n+1}} - \xi^{\sigma^n}\right) \ u_1^{q^{n-h+1}} \ u_1^{q^n + q^{n-1} + \cdots + q + 1} \ .
$$

The leading terms for the rest of the $a_m(\underline{u})$'s follow the same pattern, descending in blocks of h-1 steps. When going down from one block to another, an additional factor has to be added. Within the same block, most factors stay the same. The changing factor is the extra factor which was inserted when stepping one block down and has the form $-u_i^{q^{n-(h-1)k}}$. Here k is a natural number fixed for the block and increases by one when going down from one block to the next, and i varies within the block downward from $h-1$ to 1. We write n as $n=(h-1)\cdot d+c$, with $c,d\in\mathbb{N}$, $0\le c\le h-2$. Then $n-(h-1)\cdot(d-1)=c+h-1$, and the leading term of $a_{c+h-1}(\underline{u})$ is

$$(-1)^d\ \left(\xi^{\sigma^{n+1}}-\xi^{\sigma^n}\right)\ u_1^{q^{c+2(h-1)}+q^{c+3(h-1)}+\cdots+q^{n-(h-1)}+q^n+q^{n-1}+\cdots+q+1}\ .$$

$a_{c+h-2}(\underline{u})$ has leading term

$$(-1)^{d-1}\ \left(\xi^{\sigma^{n+1}}-\xi^{\sigma^n}\right)\ u_{h-1}^{q^{c+(h-1)}}\ \ u_1^{q^{c+2(h-1)}+q^{c+3(h-1)}+\cdots+q^{n-(h-1)}+q^n+q^{n-1}+\cdots+q+1}\ ;$$

similarly for $a_{c+h-3}(\underline{u}),\dots,a_c(\underline{u})$; $a_c(\underline{u})$ has leading term

$$(-1)^{d-1}\ \left(\xi^{\sigma^{n+1}}-\xi^{\sigma^n}\right)\ u_1^{q^{c+(h-1)}}\ \ u_1^{q^{c+2(h-1)}+q^{c+3(h-1)}+\cdots+q^{n-(h-1)}+q^n+q^{n-1}+\cdots+q+1}\ .$$

$a_{c-1}(\underline{u})$ has leading term

$$(-1)^d\ \left(\xi^{\sigma^{n+1}}-\xi^{\sigma^n}\right)\ u_{h-1}^{q^c}\ u_1^{q^{c+(h-1)}+q^{c+2(h-1)}+\cdots+q^{n-(h-1)}+q^n+q^{n-1}+\cdots+q+1}\ ;$$

similarly for $a_{c-2}(\underline{u}),\dots,a_0(\underline{u})$; the leading term of $a_0(\underline{u})$ is

$$(-1)^d\ \left(\xi^{\sigma^{n+1}}-\xi^{\sigma^n}\right)\ u_{h-c}^{q^c}\ u_1^{q^{c+(h-1)}+q^{c+2(h-1)}+\cdots+q^{n-(h-1)}+q^n+q^{n-1}+\cdots+q+1}\ .$$

$s_{h-1}-u_{h-1}$ has leading term

$$(-1)^d\ \left(\xi^{\sigma^{n+1}}-\xi^{\sigma^n}\right)\ u_{h-c-1}^{q^c}\ u_1^{q^{c+(h-1)}+q^{c+2(h-1)}+\cdots+q^{n-(h-1)}+q^n+q^{n-1}+\cdots+q+1}\ ,$$

similarly for $s_{h-2},\dots,s_{c+1}-u_{c+1}$; the leading term of $s_{c+1}-u_{c+1}$ is

$$(-1)^d\ \left(\xi^{\sigma^{n+1}}-\xi^{\sigma^n}\right)\ u_1^{q^c}\ u_1^{q^{c+(h-1)}+q^{c+2(h-1)}+\cdots+q^{n-(h-1)}+q^n+q^{n-1}+\cdots+q+1}\ .$$

The leading term of s_c-u_c is

$$(-1)^{d-1}\ \left(\xi^{\sigma^{n+1}}-\xi^{\sigma^n}\right)\ u_c\ u_{h-c}^{q^c}\ u_1^{q^{c+(h-1)}+q^{c+2(h-1)}+\cdots+q^{n-(h-1)}+q^n+q^{n-1}+\cdots+q+1}\ ;$$

the leading term of $s_{c-1}-u_{c-1}$ is

$$(-1)^{d-1}\ \left(\xi^{\sigma^{n+1}}-\xi^{\sigma^n}\right)\ u_{c-1}\ u_{h-c}^{q^c}\ u_1^{q^{c+(h-1)}+q^{c+2(h-1)}+\cdots+q^{n-(h-1)}+q^n+q^{n-1}+\cdots+q+1}\ ;$$

similarly for $s_{c-2}-u_{c-2},\dots,s_1-u_1$. For instance, the leading term of s_1-u_1 is

$$(-1)^{d-1}\ \left(\xi^{\sigma^{n+1}}-\xi^{\sigma^n}\right)\ u_1\ u_{h-c}^{q^c}\ u_1^{q^{c+(h-1)}+q^{c+2(h-1)}+\cdots+q^{n-(h-1)}+q^n+q^{n-1}+\cdots+q+1}\ .$$

Notice that the leading terms of the matrix

$$\begin{pmatrix} s_{h-1} - u_{h-1} \\ \vdots \\ s_{c+1} - u_{c+1} \\ \vdots \\ s_1 - u_1 \end{pmatrix}$$

is

$$\begin{array}{r c} & \\ & \\ & \\ (c+1)^{st} & \\ & \\ & \\ & \end{array} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (-1)^d \, \left(\xi^{\sigma^{n+1}} - \xi^{\sigma^n}\right) \, u_1^{q^c+q^{c+(h-1)}+q^{c+2(h-1)}+\cdots+q^{n-(h-1)}+q^n+q^{n-1}+\cdots+q+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

In the case when $char(A) = 0$, the calculation is more complicated, but the conclusion about the leading term still holds. We leave this assertion to the patient reader to verify. We summarize our discussion above as

Theorem 2. *Let $(\ Fil^w\ )_{w\in\mathbb{N}}$ be the decreasing filtration of $\kappa_h[[\underline{u}]]$ by*

$$Fil^w = \left\{ \sum_{w(I)\geq w} a_I \ \underline{u}^I \in \kappa_h[[\underline{u}]] \ \middle| \ a_I \in \kappa_h \right\} ,$$

where

$$w(I) = \sum_{j=1}^{h-1} i_j \cdot q^{j-1} \quad for \quad I = (i_1, \ldots, i_{h-1}) \in \mathbb{N} .$$

For $\Phi_0 \in Aut(G_0)$ such that $\Phi_0(e) = e + \xi \ V^\ell \cdot e$, where $\xi \in \mu_{q^h-1}\big(W_A(\mathbb{F}_{q^h})\big)$, $\ell = nh + b$, $0 \leq b \leq h-1$, $n \geq 1$, $n, b \in \mathbb{N}$. Write $n + b = (h-1)d + c$, with $c, d \in \mathbb{N}$, $0 \leq c \leq h-2$. Then for $1 \leq i \leq h-c-1$, the leading term of $s_{c+i} - u_{c+i}$ with respect to $(\ Fil^w\ )_{w\in\mathbb{N}}$ is

$$(-1)^d \cdot \ \eta \cdot u_i^{q^c} \cdot u_1^{\sum_{j=1}^{d-1} \ q^{c+j(h-1)} + \sum_{a=0}^{n} \ q^a} ,$$

where

$$\eta = \begin{cases} \xi^{\sigma^{n+b+1}} & if\ 1 \leq b \leq h-1 \\ \left(\xi^{\sigma^{n+1}} - \xi^{\sigma^n}\right) & if\ b = 0 \end{cases} .$$

For $1 \leq i \leq c$, the leading term of $s_i - u_i$ with respect to $(\ Fil^w\ )_{w\in\mathbb{N}}$ is

$$(-1)^d \cdot \ \eta \cdot u_i \cdot u_{h-c}^{q^c} \cdot u_1^{\sum_{j=1}^{d-1} \ q^{c+j(h-1)} + \sum_{a=0}^{n} \ q^a} ,$$

and η is the same as before. Especially, the leading term of the vector

$$(s_1 - u_1, \ldots, s_{h-1} - u_{h-1})$$

is

$$\left(0, \ldots, 0, (-1)^d \cdot \ \eta \cdot u_1^{\sum_{j=0}^{d-1} \ q^{c+j(h-1)} + \sum_{a=0}^{n} \ q^a}, 0, \ldots, 0\right) \quad ,$$

where only the $(c+1)$st entry is nonzero.

3. Consequences of the calculation

The first application of Theorem 2 is

Proposition 3. *Let k be an overfield of κ_h. If a function $f \in k[[\underline{u}]]$ is invariant under a compact open subgroup of $Aut(G_0)$, then f is a constant, i.e. $f \in k$.*

Corollary. *Let k be an overfield of κ_h. If $W \subseteq k[[\underline{u}]]$ is a finite dimensional k-vector subspace stable under an open subgroup of $Aut(G_0)$, then $W \subseteq k$.*

Remarks.

(1) Heuristically, the quotient of $\mathrm{Spec}\,(k[[\underline{u}]])$ by an open subgroup H of $Aut(G_0)$ in the category of affine schemes should be the spectrum of the H-invariants in $k[[\underline{u}]]$. From this point of view Proposition 3 indicates that "the generic orbits" of any compact open subgroup of $Aut(G_0)$ operating naturally on $\mathrm{Spec}\,(k[[\underline{u}]])$ (or $\mathrm{Spf}\,(k[[\underline{u}]])$) are likely to be big in the sense that they are Zariski dense in $\mathrm{Spec}\,(k[[\underline{u}]])$.

(2) The infinite dimensional continuous representation $(\underline{u}) \cdot k[[\underline{u}]]$ of $Aut(G_0)$ is a very poorly understood representation. One can view it as being "topologically nilpotent". It is highly reducible yet indecomposable.

Proof of Corollary. Suppose that W is a finite dimensional k-vector subspace of $k[[\underline{u}]]$ stable under a compact open subgroup $H \subseteq Aut(G_0)$. Since the decreasing sequence of vector spaces

$$W \cap (\underline{u})^n$$

stablizes as $n \to \infty$, and

$$\bigcap_n (\underline{u})^n \ = \ 0 \quad ,$$

there exists $n_0 \in \mathbb{N}$ such that $W \cap (\underline{u})^{n_0} \ = 0$. There exists an open subgroup $H' \subseteq H$ such that

$$\phi(f) - f \in (\underline{u})^{n_0} \quad \forall \ \phi \in H', \ \forall \ f \in k[[\underline{u}]] \ .$$

Hence all elements of W are invariant under H', therefore $W \subseteq k$ by Proposition 3. $\square$

Proof of Proposition 3. Suppose that $f \in (\underline{u})k[[\underline{u}]]$ is fixed by an open subgroup $H \subseteq Aut(G_0)$. There exists a positive integer n_0 such that $id + \pi^{n_0} End(G_0) \subseteq H$. There exists a unique $s \in \mathbb{N}$ and a unique $f_0 \in (\underline{u})k[[\underline{u}]]$ such that $f = f_0^{p^s}$ and

$$\nabla f_0 := \left(\frac{\partial f_0}{\partial u_1}, \ldots, \frac{\partial f_0}{\partial u_{h-1}} \right) \neq 0 \ .$$

f_0 is fixed by H since f is. Therefore we may and do assume that ∇f is non-zero, hence for a suitable choice of a positive integer ℓ (subject to some congruence condition) and $\Phi_0 \in Aut(G_0)$ with $\Phi_0(e) = e + \xi V^\ell e$, $\xi \in \mu_{q^h-1} \setminus \mu_{q-1}$, the pairing of the initial form of ∇f with the initial form of

$$(\ h(\Phi_0)(u_1) - u_1, \ldots, h(\Phi_0)(u_{h-1}) - u_{h-1}\ )$$

is not 0. Note that the condition $\xi \notin \mu_{q-1}$ ensures that the element η defined as in Theorem 2 is not equal to zero. The Taylor expansion of f gives a contradiction if ℓ is sufficiently large to force $\Phi_0 \in H$. $\square$

The same argument using the Taylor expansion also proves

Proposition 4. *Suppose that k is an overfield of κ_h. Let $I \subseteq k[[\underline{u}]]$ be an ideal stable under an open subgroup of $Aut(G_0)$. Suppose that the image of u_1 in $gr^{\bullet}_{Fil^w}(k[[\underline{u}]]/I)$ is not a zero-divisor of $gr^{\bullet}_{Fil^w}(k[[\underline{u}]]/I)$. Then $I = (0)$.*

The two propositions above are only pretty weak statements concerning the orbit structure of Spf $k[[u]]$ with respect to the action of $Aut(G_0)$ or an open subgroup of $Aut(G_0)$. These orbits under the compact group $Aut(G_0)$ are very complicated, but we can take the Zariski closure of the orbits to simplify the problem. Then an optimist may harbor the following

Hope. *Any reduced irreducible formal subscheme of Spf$(k[[\underline{u}]])$ stable under an open subgroup of $Aut(G_0)$ is of the form*

$$\mathit{Spf}\,(k[[\underline{u}]]/(u_1, u_2, \ldots, u_{i-1})) \qquad \text{for some } i\,, \quad 1 \leq i \leq h\ .$$

(Notice that Spf $(k[[\underline{u}]]/(u_1, u_2, \ldots, u_{i-1}))$ is the locus over which the universal deformation has height at least i.)

Proposition 4 can be viewed as a weak evidence for the hope. To prove the hope by computation, one would need much better control of the error term than what is provided by this paper.

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