Néron models for semiabelian varieties: congruence and change of base field

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§1. Introduction

Let $\mathcal{O}$ be a henselian discrete valuation ring with perfect residue field. Denote by K the fraction field of $\mathcal{O} = \mathcal{O}_K$, and by $\mathfrak{p} = \mathfrak{p}_K$ the maximal ideal of $\mathcal{O}$. Then every abelian variety A over K has a Néron model $\underline{A}^{\mathrm{NR}}$ over $\mathcal{O}$. The Néron model $\underline{A}^{\mathrm{NR}}$ of A is a smooth group scheme of finite type over $\mathcal{O}$, characterized by the property that for every finite unramified extension L of K, every L-valued point of A_K extends uniquely to an $\mathcal{O}_L$-valued point of $\underline{A}^{\mathrm{NR}}$. We refer to the book [BLR] for a thorough exposition of the construction and basic properties of Néron models.

In general, the formation of Néron models does not commute with base change. Rather, for every finite extension field M of K, we have a canonical homomorphism

$$\mathrm{can}_{A,M/K} : \underline{A}^{\mathrm{NR}} \times_{\mathrm{Spec}\ \mathcal{O}_K} \mathrm{Spec}\ \mathcal{O}_M \to \underline{A_M}^{\mathrm{NR}}$$

from the base change of the Néron model to the Néron model of the base change, which extends the natural isomorphism between the generic fibers. If A has semistable reduction over $\mathcal{O}_K$, i.e. if the neutral component of $\underline{A}^{\mathrm{NR}}$ is a semiabelian scheme, then $\mathrm{can}_{A,M/K}$ is an open immersion. We define a numerical invariant $c(A,K)$ of A as follows. Let L/K be a finite separable extension such that the abelian variety A has semistable reduction over $\mathcal{O}_L$. Let $e(L/K)$ be the ramification index of L/K. Define

$$c(A,K) := \frac{1}{e(L/K)}\, \mathrm{length}_{\mathcal{O}_L}\left(\frac{\mathrm{Lie}\,\underline{A_L}^{\mathrm{NR}}}{\mathrm{can}_{A,L/K_*}(\mathrm{Lie}\,\underline{A}^{\mathrm{NR}} \otimes \mathcal{O}_L)}\right)$$

Notice that $c(A,K)$ does not depend on the choice of L/K. The invariant $c(A,K)$ measures the failure of A to have semistable reduction over $\mathcal{O}_K$; it is equal to zero if and only if A has semistable reduction over $\mathcal{O}_K$. Thus $c(A,K)$ may be regarded as a sort of "conductor" of A. We will call it the *base change conductor* of A. One motivation of this paper is to study the properties of the invariant $c(A,K)$ and determine whether it can be expressed in terms of more familiar ones, for instance the Artin conductor or the Swan conductor of the ℓ-adic Tate module attached to A.

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In a similar fashion one can attach to each semiabelian variety G over K a non-negative rational number $c(G, K)$; see 2.4 for the precise definition. For a torus T over K it has been shown, by E. de Shalit and independently by J.-K. Yu and the author, that

$$c(T, K) = \frac{1}{2}\, a\left(X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}\right)\,, \qquad (*)$$

one-half of the Artin conductor of the linear representation of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on the character group of T; see [CYdS]. This answers, in the affirmative, a question posed by B. Gross and G. Prasad.

The main geometric result of [CYdS] says that the congruence class of the Néron model $\underline{T}^{\mathrm{NR}}$ of T is determined by the congruence class, with perhaps a higher congruence level, of the Galois twisting data of T. More precisely, for a given torus T there exists an integer m such that for any $N \geq 1$ the congruence class of the Néron model $\underline{T}^{\mathrm{NR}}$ of T modulo $\mathfrak{p}_K^N$ is determined up to unique isomorphism, by the congruence class modulo $\mathfrak{p}_K^{N+m}$ in the sense of [D], of the quadruple $(\mathcal{O}_K, \mathcal{O}_L, \mathrm{Gal}(L/K), X^*(T))$. Here L is a finite Galois extension of K which splits T, and $X^*(T)$ is the character group of T with natural action by the Galois group $\mathrm{Gal}(L/K)$. With this congruence result at our disposal, the calculation of the base change conductor $c(T, K)$ in [CYdS] proceeds in several steps.

- For an induced torus of the form, $T = \mathrm{Res}_{M/K}\mathbb{G}_m$, where M is a finite separable extension of K, an easy computation shows that $c(T, K)$ is equal to half of the exponent of the discriminant $\mathrm{disc}(M/K)$. Therefore the formula (*) holds for products of induced tori.

- Suppose that the $\mathrm{char}(K) = 0$. One shows, as a consequence of Tate's Euler-Poincaré characteristic formula, that $c(T, K)$ depends only on the K-isogeny class of T. Therefore the base change conductor is an additive function on the Grothendieck group of finite dimensional $\mathbb{Q}$-rational representations of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. According to Artin's theorem on the characters of a finite group we know that the K-isogeny class of T is a $\mathbb{Q}$-linear combination of the isogeny classes of induced tori of the form $\mathrm{Res}_{M/K}\mathbb{G}_m$, where M runs through subextensions of L/K. So the formula (*) holds in general.

- When $\mathrm{char}(K) = p > 0$, one approximates $(\mathcal{O}_K, \mathcal{O}_L, \mathrm{Gal}(L/K), X^*(T))$ by a quadruple $(\mathcal{O}_{K_0}, \mathcal{O}_{L_0}, \mathrm{Gal}(L_0/K_0), X^*(T_0))$ with $\mathrm{char}(K_0) = 0$ as in [D]. One concludes by the geometric result on congruence of Néron models that the formula (*) and the isogeny invariance of $c(T, K)$ still hold.

Here what makes it possible to approximate tori over local fields of characteristic p by tori over local fields of characteristic 0 is the following, often under-appreciated, fact: The group of automorphisms of a split torus T of dimension d is isomorphic to $\mathrm{GL}_d(\mathbb{Z})$, and two split tori T_1, T_2 of the same dimension over local fields K_1, K_2 are congruent if $\mathcal{O}_{K_1}, \mathcal{O}_{K_2}$ are congruent.

In this article we examine how far the method used in [CYdS] can be generalized to the case of abelian varieties, or more generally semiabelian varieties. Happily the geometric result on congruence generalizes to Néron models for abelian varieties as expected: Given an abelian variety A over K, there exists a constant $m > 0$ such that for each $N > 0$, $\underline{A}^{\mathrm{NR}} \otimes_{\mathrm{Spec}\,\mathcal{O}} \mathrm{Spec}\,(\mathcal{O}/\mathfrak{p}^N)$ depends only on the $\pmod{\mathfrak{p}^{N+m}}$-congruence class of the Galois action on the degeneration data for A; see Theorem 7.6 for the precise statement and 7.1, 7.3 for the relevant definitions.

We make a digression in this paragraph to explain the above congruence statement. Suppose that L/K is a finite Galois extension of K such that A_L has semistable reduction over $\mathcal{O}_L$. Then the neutral component of the Néron model of A_L can be constructed, by a uniformization procedure, as a quotient of a semiabelian scheme $\widetilde{G}$ over $\mathcal{O}_L$ by a subgroup Y of periods; see [FC, chap. 2, 3]. The Galois L/K-descent data for A_K induces a semi-linear action of $\mathrm{Gal}(L/K)$ on $(\widetilde{G}, Y)$. There is a natural notion of congruence between two such degeneration data $(\widetilde{G_1}, Y_1), (\widetilde{G_2}, Y_2)$ with actions by $\mathrm{Gal}(L_1/K_1), \mathrm{Gal}(L_2/K_2)$ respectively, provided that the two sextuples

$$(\mathcal{O}_{K_1}, \mathfrak{p}_{K_1}, \mathcal{O}_{L_1}, \mathfrak{p}_{L_1}, \mathrm{Gal}(L_1/K_1), Y_1) \qquad \text{and} \qquad (\mathcal{O}_{K_2}, \mathfrak{p}_{K_2}, \mathcal{O}_{L_2}, \mathfrak{p}_{L_2}, \mathrm{Gal}(L_2/K_2), Y_2)$$

are congruent modulo a given level, say $m + N$. Our congruence result says that congruence between two degeneration data as above implies congruence between the Néron models $\underline{A_{K_1}}^{\mathrm{NR}}$ and $\underline{A_{K_2}}^{\mathrm{NR}}$. We illustrate the notion of congruence of degeneration data in the case of Tate curves. Suppose we are given an isomorphism α between

$$(\mathcal{O}_{K_1}, \mathfrak{p}_{K_1}, \mathcal{O}_{L_1}, \mathfrak{p}_{L_1}, \mathrm{Gal}(L_1/K_1), Y_1) \mod \mathfrak{p}_{K_1}^{m+N}$$

and

$$(\mathcal{O}_{K_2}, \mathfrak{p}_{K_2}, \mathcal{O}_{L_2}, \mathfrak{p}_{L_2}, \mathrm{Gal}(L_2/K_2), Y_2) \mod \mathfrak{p}_{K_2}^{m+N}.$$

After picking a generator for Y_1 and a corresponding generator for Y_2 under α, we obtain periods $q_i \in \mathfrak{p}_{L_i}$, $i = 1, 2$. Then we say that two degeneration data $(\mathbb{G}_m, q_1^{\mathbb{Z}}), (\mathbb{G}_m, q_2^{\mathbb{Z}})$ are "*congruent at level* $m + N$" if q_1 and q_2 have the same order a and their respective classes in $\mathfrak{p}_{L_1}^a \otimes (\mathcal{O}_{L_1}/\mathfrak{p}_{K_1}^{m+N}) \cong \mathfrak{p}_{L_2}^a \otimes (\mathcal{O}_{L_2}/\mathfrak{p}_{K_2}^{m+N})$ correspond under α.

However unlike the case of tori, one cannot deduce the validity of a general statement about the base change conductor for abelian varieties over local fields by "reducing to characteristic 0" using the congruence result explained above. The difficulty is that an abelian scheme $A_{\mathcal{O}_K}$ over $\mathcal{O}_K$ in positive characteristic may have "too many" automorphisms, such that no matter how one approximates the moduli point of $A_{\mathcal{O}_K}$ by $\mathcal{O}_{K'}$-valued points with $\mathrm{char}(K') = 0$, the resulting abelian scheme A' over $\mathcal{O}_{K'}$ will have "too small" a group of automorphisms, causing it impossible to approximate some Galois twist of A by abelian varieties over local fields of characteristic 0.

In some situations the base change conductor $c(A, K)$ has a simple expression. For instance if the abelian variety A has potentially totally multiplicative degeneration, i.e. there exists a finite extension L/K such that the neutral component of the closed fiber of the Néron model $\underline{A_L}^{\mathrm{NR}}$ of A_L is a torus, then $c(A, K)$ is equal to a quarter of the Artin conductor of the ℓ-adic Tate module $V_\ell(A)$ of A for a prime number ℓ which is invertible in the residue field κ. See Cor. 5.2, and also see Prop. 7.8 for a more general result.

Suppose either that the residue field κ of K is finite, or that $\mathrm{char}(K) = 0$, then the calculation of the base change conductor $c(A, K)$ for a general abelian variety A over K can be reduced to that of an abelian variety B over the completion $\widehat{K}$ of K with potentially good reduction, in the following sense. Recall that $c(A, K) = c(A, \widehat{K})$ since $\underline{A_{\widehat{K}}}^{\mathrm{NR}} = \underline{A}^{\mathrm{NR}} \times_{\mathrm{Spec}\, \mathcal{O}} \mathrm{Spec}\, \mathcal{O}_{\widehat{K}}$. According to the general theory of degeneration of abelian varieties, the abelian variety $A_{\widehat{K}}$ over $\widehat{K}$ can be "uniformized" as the quotient of a semi-abelian variety $\widetilde{G}$ over $\widehat{K}$ by a discrete lattice Y, where $\widetilde{G}$ is an extension of an abelian variety B over $\widehat{K}$ by a torus T, and B has potentially good reduction. In the above situation, Theorem 5.3 asserts that $c(A, K) = c(T, \widehat{K}) + c(B, \widehat{K})$.

Despite what the tori case may suggest, in general the base change conductor for abelian varieties over a local field K does change under K-rational isogenies: There exist abelian varieties A_1, A_2 which are isogenous over K, yet $c(A_1, K) \neq c(A_2, K)$. In 6.10 we give two such examples, one with $K = \mathbb{Q}_p$ and another with $K = \kappa[[t]]$, where κ is a perfect field of characteristic $p > 0$. In view of these examples, we see that $c(A, K)$ cannot be expressed in terms of invariants attached to the ℓ-adic Tate modules $V_\ell(A)$ of A. There is one positive result in this direction: If A_1, A_2 are abelian varieties over a local field K of characteristic 0 which are K-isogenous, and if there exists a finite separable extension L of K such that the neutral component of the closed fiber of the Néron models of each A_i over $\mathcal{O}_L$ is an extension of an *ordinary* abelian variety by a torus, then $c(A_1, K) = c(A_2, K)$; see Theorem 6.8. The examples mentioned earlier show that the ordinariness assumption cannot be dropped.

It has long been known since the creation of the Néron models that the formation of Néron models does not preserve exactness, nor does it commute with change of base fields. Thus the phenomenon that the base change conductor $c(-, K)$ has some nice properties, to the effect that many "defects" on the level of Néron models themselves often "cancel out" when measured by $c(-, K)$, may be unexpected. This may explain why the invariant $c(-, K)$ has attracted little attention before. This article and its predecessor [CYdS] are among the first to study the base change conductor; many basic questions concerning this invariant remain unsettled. A list of open problems, together with some comments, can be found in §8.

It is a pleasure to thank J.-K. Yu; this paper could not have existed without him. Indeed he was the first to observe that the Néron models for tori with congruent Galois representations should be congruent, and that this can be brought to bear on the problem of Gross and Prasad on the base change conductor $c(T, K)$ for tori. The author would also like to thank S. Bosch, E. de Shalit, X. Xavier and especially to S. Shatz for discussion and encouragement. Thanks are due to the referee for a very thorough reading of the manuscript. The seed of this work was sowed in the summer of 1999 during a visit to the National Center for Theoretical Science in Hsinchu, Taiwan; its hospitality is gratefully acknowledged.

§2. Notations

(2.1) Let $\mathcal{O} = \mathcal{O}_K$ be a discrete valuation ring with fraction field K and residue field κ. Let $\mathfrak{p} = \mathfrak{p}_K$ be the maximal ideal of $\mathcal{O}$ and let $\pi = \pi_K$ be a generator of $\mathfrak{p}$. The strict henselization (resp. the π-adic completion) of $\mathcal{O}$ will be denoted by $\mathcal{O}^{\mathrm{sh}}$ (resp. $\widehat{\mathcal{O}}$). Their fields of fractions will be denoted by K^{sh} and $\widehat{K}$ respectively. The residue field of $\mathcal{O}^{\mathrm{sh}}$ is κ^{sep}, the separable closure of κ.

(2.2) In this paper T (resp. G or $\widetilde{G}$, resp. A or B), sometimes decorated with a subscript, will be the symbol for an algebraic torus (resp. a semiabelian variety, resp. an abelian variety). Often $\widetilde{G}$ fits into a short exact sequence $0 \to T \to \widetilde{G} \to B$, so that the semi-abelian variety $\widetilde{G}$ is an extension of the abelian variety B by a torus T, called a *Raynaud extension*. Such usage conforms with the notation scheme used in [FC, Ch. 2, 3], which we will generally follow.

(2.3) It is well-known that semiabelian varieties over K have Néron models. The Néron models come in several flavors. For a be semiabelian variety G over K, we have

- the lft Néron model $\underline{G}^{\mathrm{lft\,NR}}$ as defined in [BLR],
- the open subgroup scheme $\underline{G}^{\mathrm{ft\,NR}}$ of $\underline{G}^{\mathrm{lft\,NR}}$ such that $\underline{G}^{\mathrm{ft\,NR}}(\mathcal{O}_K)$ is the maximal bounded subgroup of $G(K)$,
- the open subgroup scheme $\underline{G}^{\mathrm{conn\,NR}}$ of $\underline{G}^{\mathrm{ft\,NR}}$ such that the closed fiber of $\underline{G}^{\mathrm{conn\,NR}}$ is the neutral component of $\underline{G}^{\mathrm{ft\,NR}}$.

The lft Néron model $\underline{G}^{\mathrm{lft\,NR}}$ is a group scheme which is smooth, therefore locally of finite type, over $\mathcal{O}_K$; it satisfies $\underline{G}^{\mathrm{lft\,NR}}(\mathcal{O}_K^{\mathrm{sh}}) = G(K^{\mathrm{sh}})$. The other two models, $\underline{G}^{\mathrm{ft\,NR}}$ and $\underline{G}^{\mathrm{conn\,NR}}$, are smooth and of finite type over $\mathcal{O}_K$. We will abbreviate $\underline{G}^{\mathrm{ft\,NR}}$ to $\underline{G}^{\mathrm{NR}}$. Clearly $\underline{A}^{\mathrm{lft\,NR}} = \underline{A}^{\mathrm{NR}}$ for every abelian variety A over K.

(2.4) In this subsection we define numerical invariants $c(G,K)$ and

$$\underline{c}(G,K) = (c_1(G,K), \ldots, c_g(G,K))\,, \quad \text{where} \;\; g = \dim(G)\,,$$

for a semiabelian varietie G over K. For each semiabelian variety G over K there exists a finite extension field L/K such that $\underline{G_L}^{\mathrm{conn\,NR}}$ is a semiabelian scheme over $\mathcal{O}_L$. Actually there exists a finite extension L/K which has the additional property that it is separable, or even Galois, but this is not necessary for the definition. Let $\mathrm{can}_{G,L/K} : \underline{G}^{\mathrm{NR}} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\,\mathcal{O}_L \to \underline{G_L}^{\mathrm{NR}}$ be the canonical homomorphism which extends the natural isomorphism between the generic fibers. Define non-negative rational numbers $0 \le c_1(G,K) \le \cdots \le c_g(G,K)$, where $g = \dim(G)$, by

$$\frac{\mathrm{Lie}\,\underline{G_L}^{\mathrm{NR}}}{\mathrm{can}_{G,L/K\,*}(\mathrm{Lie}\,\underline{G}^{\mathrm{NR}} \otimes \mathcal{O}_L)} \cong \bigoplus_{i=1} \frac{\mathcal{O}_L}{\mathfrak{p}_L^{e(L/K)\,c_i(G,K)}}$$

Let $\underline{c}(G,K) = (c_1(G,K), \ldots, c_g(G,K))$ and $c(G,K) = c_1(G,K) + \cdots + c_g(G,K)$. Notice that $c(G,K)$ and the $c_i(G,K)$'s do not depend on the choice of the finite extension L/K such that G has semistable reduction over L. It is easy to see that $c(G,K)$ is equal to zero if and only if G has semistable reduction over K. We call $c(G,K)$ the *base change conductor* of G, and the $c_i(G,K)$'s the *elementary divisors of the base change conductor* of G.

(2.5) Let $0 \to T \to \widetilde{G} \to B \to 0$ be a Raynaud extension as above. We denote by $X = X^*(T)$ the character group of T; it is an étale sheaf of free abelian groups of finite rank over $\operatorname{Spec} K$. Therefore one can also think of X as a module for the Galois group $\operatorname{Gal}(K^{\mathrm{sep}}/K)$. Every Raynaud extension $0 \to T \to \widetilde{G} \to B \to 0$ corresponds to a K-rational homomorphism $c : X \to B^t$, where B^t is the dual abelian variety of B.

(2.6) Let Y be an étale sheaf of free abelian groups of finite rank over Spec K. A homomorphism $\iota : Y \to \widetilde{G}$ from Y to a semi-abelian variety $\widetilde{G}$ corresponds to a K-rational homomorphism $c^t : Y \to B$, together with a trivialization $\tau : 1_{Y \times X} \xrightarrow{\sim} (c^t \times c)^* \mathcal{P}^{-1}$ of the biextension $(c^t \times c)^* \mathcal{P}^{-1}$ over $Y \times X$.

§3. Uniformization of abelian varieties

Throughout this section $\mathcal{O}$ is assumed to be a complete discrete valuation ring. Our purpose here is to review the basic facts about uniformizing an abelian variety A over K as the quotient of a semi-abelian variety $\widetilde{G}$ over K by a discrete subgroup $\iota : Y \hookrightarrow \widetilde{G}$. Here Y is an étale sheaf of free abelian groups of finite rank over Spec K. The semi-abelian variety $\widetilde{G}$ fits into a Raynaud extension $0 \to T \to \widetilde{G} \to B \to 0$, such that the abelian variety B has potentially good reduction, and $\dim(T) = \operatorname{rank}(Y)$. Moreover the above data satisfies the positivity condition on page 59 of [FC], in the definition of the category DD, after a finite separable base field extension L of K such that T is split over L and B has good reduction over $\mathcal{O}_L$. This positivity condition is independent of the finite separable extension L one chooses. For future reference, an embedding $\iota : Y \hookrightarrow \widetilde{G}$ which satisfies the above positivity condition will be called a K-rational *degeneration data*, or a *degeneration data rational over* K. A K-rational degeneration data as above is said to be *split* over an extension L of K if Y is constant over K (equivalently, the torus T is split over L) and B has good reduction over L. Every degeneration data rational over K splits over some finite Galois extension of K.

(3.1) Proposition (i) *Let $\mathcal{O} = \mathcal{O}_K$ be a complete discrete valuation ring. Then every K-rational degeneration data $\iota : Y \hookrightarrow \widetilde{G}$ gives rise to an abelian variety A over K such that A is the quotient of $\widetilde{G}$ by Y in the rigid analytic category. Conversely every abelian variety A over K arises from a K-rational degeneration data $\iota : Y \hookrightarrow \widetilde{G}$.*

(ii) *Suppose that $\iota : Y \hookrightarrow \widetilde{G}$ is the K-rational degeneration data for an abelian variety A over K. Then for every Galois extension L/K of K such that the degeneration data* *splits over L (equivalently, A has semistable reduction over $\mathcal{O}_L$), there is a natural isomorphism $A(K) = (\widetilde{G}(L)/Y(L))^{\mathrm{Gal}(L/K)}$. This isomorphism is functorial in L/K.*

PROOF. The references for the quotient construction and the uniformization theorem are [M], [R1], [FC, Ch. 2, 3] and [BL]. They are written in the case when the abelian variety A has semistable reduction over $\mathcal{O}$, or equivalently when the degeneration data $\iota : Y \to \widetilde{G}$ is split over K. For instance [FC, Prop. 8.1, p. 78] is a reference for (ii) in the case when A has semistable reduction over K. The slightly more generally statement in the proposition follows from the semistable reduction case by descent. ∎

§4. The invariant $c(\widetilde{G}, K)$ for semiabelian varieties

The main result of this section is

(4.1) Theorem *Assume either that* $\mathrm{char}(K) = 0$ *and the residue field κ of $\mathcal{O} = \mathcal{O}_K$ is perfect, or that the residue field κ is finite. Then for every Raynaud extension $0 \to T \to \widetilde{G} \to B \to 0$ over K, we have $c(\widetilde{G}, K) = c(T, K) + c(B, K)$.*

(4.2) Remark (i) Our proof of Theorem 4.1 in the two cases are quite different technically. When the residue field κ is finite we use the Haar measure on the group of rational points on finite separable extensions of K. This proof is valid when K has characteristic p but we have difficulty translating it to the more general situation when the residue field is perfect but not finite.

(ii) The proof for the case when the $\mathrm{char}(K) = 0$ is somewhat indirect. First we establish it in Cor. 4.7 in the case when T is an induced torus; this part is valid for every discrete valuation ring $\mathcal{O}$. Then we show in Lemma 4.9 that the base change conductor $c(\widetilde{G}, K)$ stays the same under any K-isogeny whose kernel is contained in the torus part T of $\widetilde{G}$, if $\mathrm{char}(K) = 0$. The proof of Lemma 4.9 is valid only when $\mathrm{char}(K) = 0$.

The following lemma may indicate that the statement of Theorem 4.1 is plausible.

(4.3) Lemma *Assume that the residue field κ of $\mathcal{O}$ is algebraically closed and K is complete.*

(i) For every torus T over K, we have $\mathrm{H}^j(K, T) = (0)$ for all $j \geq 1$.

(ii) Let M be a free abelian group of finite rank with a continuous action by $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. Then $\mathrm{H}^1(\mathrm{Gal}(K^{\mathrm{sep}}/K), M)$ is finite, while $\mathrm{H}^j(\mathrm{Gal}(K^{\mathrm{sep}}/K), M) = (0)$ for all $j \geq 3$.

PROOF. This lemma is certainly known. We provide a proof for the readers' convenience. According to [S2, Chap. XII], the Brauer group of every finite extension of K is trivial, hence by [S3, Chap. II, §3, Propl 5] the cohomological dimension of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ is at most 1. So the strict cohomological dimension of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ is at most 2. The statement for $j \geq 3$ in both (i) and (ii) follows.

(i) Let L/K be a finite separable extension which splits T. Let $X_*(T)$ be the cocharacter group of T. The induced module $\mathrm{Ind}_{\mathrm{Gal}(K^{\mathrm{sep}}/L)}^{\mathrm{Gal}(K^{\mathrm{sep}}/K)} X_*(T)$ is the cocharacter group of an induced torus T' over K. The natural surjection

$$\mathrm{Ind}_{\mathrm{Gal}(K^{\mathrm{sep}}/L)}^{\mathrm{Gal}(K^{\mathrm{sep}}/K)} X_*(T) \twoheadrightarrow X_*(T)$$

gives an exact sequence

$$1 \to T'' \to T' \to T \to 1$$

of tori over K. By Shapiro's lemma,

$$\mathrm{H}^2(K, T') = X_*(T) \otimes_{\mathbb{Z}} \mathrm{H}^2(L, (L^{\mathrm{sep}})^\times) = (0)\,.$$

So $\mathrm{H}^2(K,T) = \mathrm{H}^3(K,T'') = (0)$, the second equality holds because $\mathrm{scd}(K) \le 2$. We have shown that $\mathrm{H}^2(K,T) = (0)$ for every torus T over K, especially $\mathrm{H}^2(K,T'') = (0)$. Again from the long exact sequence, we get $\mathrm{H}^1(K,T) = \mathrm{H}^2(K,T'') = (0)$. This proves (i).

Another proof of (i) is to observe that the $\mathrm{Gal}(L/K)$-module $L^\times$ is cohomological trivial by Tate-Nakayama, hence the $\mathrm{Gal}(L/K)$-module $T(L) := X_*(T) \otimes_{\mathbb{Z}} L^\times$ is also cohomologically trivial. See [S2, IX §5, Thm. 8, Thm. 9].

(ii) Let L/K be a finite separable extension such that $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ operates trivially on M. Let $M' := \mathrm{Ind}_{\mathrm{Gal}(K^{\mathrm{sep}}/L)}^{\mathrm{Gal}(K^{\mathrm{sep}}/K)}$. The natural embedding of M into M' gives a short exact sequence of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-modules $0 \to M \to M' \to M'' \to 0$. By Shapiro's lemma, $\mathrm{H}^1(\mathrm{Gal}(K^{\mathrm{sep}}/K), M') = M \otimes_{\mathbb{Z}} \mathrm{H}^1(\mathrm{Gal}(K^{\mathrm{sep}}/L), \mathbb{Z}) = (0)$. The asserted finiteness now follows from the easy fact that $(M' \otimes \mathbb{Q})^{\mathrm{Gal}(K^{\mathrm{sep}}/K)}$ surjects to $(M'' \otimes \mathbb{Q})^{\mathrm{Gal}(K^{\mathrm{sep}}/K)}$. ∎

(4.4) Remark The statements in Lemma 4.3 do not hold if κ is only assumed to be separably closed but may not be perfect. The reason is that the Brauer group of K may be non-trivial if κ is not perfect, see [S2, XIV, §5, exer. 2].

(4.5) Lemma *Suppose that $\mathcal{O}$ is strictly henselian. Let $T = \mathrm{Res}_{L/K} T_0$ be a torus over K induced from a split torus T_0 over a finite separable extension L of K. Let $f : X \to \mathrm{Spec}\ \mathcal{O}$ be a local scheme smooth over $\mathcal{O}$, and let $f_K^* T$ be the pull-back of T to $X_K := X \times_{\mathrm{Spec}\ \mathcal{O}} \mathrm{Spec}\ K$. Then $\mathrm{H}^1_{\text{ét}}(X_K, f_K^*T) = \mathrm{H}^1_{\mathrm{sm}}(X_K, f_K^*T) = \mathrm{H}^1_{\mathrm{fppf}}(X_K, f_K^*T) = (0)$.*

Proof. This result is known, see for instance [BX, 4.2]. We produce a proof for the convenience of the reader.

First we show that $\mathrm{H}^1_{\text{ét}}(X_K, f_K^*T) = 0$. We may and do assume that $T_0 = \mathbb{G}_m$. Let $S = \mathrm{Spec}\ \mathcal{O}$, $S' = \mathrm{Spec}\ \mathcal{O}_L$, $Y := X \times_S S'$, and $Y_K = Y \times_S \mathrm{Spec}\ K$. Let $g : Y \to X$ (resp. $g_K : Y_K \to X_K$) be the projection map from Y to X (resp. from Y_K to X_K). By the definition of Weil restriction of scalars, f_K^*T is canonically isomorphic to $(g_K)_*(\mathbb{G}_m)$. Since Y_K is finite étale over X_K, $(\mathrm{R}^i g_K)_{\text{ét}}(\mathbb{G}_m) = (0)$ for all $i > 0$. So $\mathrm{H}^1_{\text{ét}}(X_K, f_K^*T) = \mathrm{H}^1_{\text{ét}}(Y_K, \mathbb{G}_m)$. Now one observes that the scheme X_K is local, and that Y_K is finite over X_K, hence Y_K is semi-local. Therefore $\mathrm{H}^1_{\text{ét}}(X_K, f_K^*T) = (0)$.

Here is another proof of the vanishing of $\mathrm{H}^1_{\text{ét}}(X_K, f_K^* T)$. The scheme Y is smooth over S', hence is regular. Moreover Y is local since $\mathcal{O}_L$ is finite and totally ramified over $\mathcal{O}$. Each element of $\mathrm{H}^1_{\text{ét}}(Y_K, \mathbb{G}_m)$ corresponds to the isomorphism class of an invertible sheaf $\mathcal{L}_{\mathcal{K}}$ of $\mathcal{O}_{Y_K}$-modules. The scheme Y being regular, every invertible sheaf $\mathcal{L}_K$ on Y_K extends to an invertible sheaf $\mathcal{L}$ on Y. Such an extension $\mathcal{L}$ must be principal since Y is a local scheme, so $\mathcal{L}_K$ is also a principal invertible sheaf.

The same argument above also proves that $\mathrm{H}^1_{\text{sm}}(X_K, f_K^* T) = \mathrm{H}^1_{\text{fppf}}(X_K, f_K^* T) = (0)$ as well. Alternatively, one can invoke the general fact that $\mathrm{H}^i_{\text{ét}}(X_K, G) = \mathrm{H}^i_{\text{fppf}}(X_K, G) = \mathrm{H}^i_{\text{ét}}(X_K, G)$ for all $i \geq 0$ and every commutative smooth group scheme G over X_K; see Example 3.4 (c) and Theorem 3.9 of [Mi]. ∎

(4.6) Remark Let $S = \operatorname{Spec} \mathcal{O}, \eta = \operatorname{Spec} K$, and let $j : \eta \to S$ be the inclusion morphism. Let S_{sm} (resp. η_{sm}) the small smooth site attached to S (resp. η). Let $j_{\text{sm}} : \eta_{\text{sm}} \to S_{\text{sm}}$ be the morphism induced by j. Lemma 4.5 implies that $\mathrm{R}^1 j_{\text{sm}*} T = (0)$; this statement can also be regarded as a reformulation of Lemma 4.5.

(4.7) Corollary *Let $0 \to T \to \widetilde{G} \to B \to 0$ be a Raynaud extension over K such that T is isomorphic to the restriction of scalars $\operatorname{Res}_{L/K}(T_0)$ of a split torus T_0 over a finite extension L of K. Then the complex of smooth group schemes $0 \to \underline{T}^{\text{lft NR}} \to \underline{\widetilde{G}}^{\text{lft NR}} \to \underline{B}^{\text{NR}} \to 0$ over $\operatorname{Spec} \mathcal{O}$ is exact. Hence the complex $0 \to \operatorname{Lie}(\underline{T}^{\text{lft NR}}) \to \operatorname{Lie}(\underline{\widetilde{G}}^{\text{lft NR}}) \to \operatorname{Lie}(\underline{B}^{\text{NR}}) \to 0$ of free $\mathcal{O}$-modules is also exact.*

Proof. Lemma 4.5 implies that locally in the smooth topology, the morphism $\underline{\pi} : \underline{\widetilde{G}}^{\text{lft NR}} \to \underline{B}^{\text{NR}}$ admits a section; see also Remark 4.6. In particular the induced map $\operatorname{Lie}(\underline{\widetilde{G}}^{\text{lft NR}}) \to \operatorname{Lie}(\underline{B}^{\text{NR}})$ is surjective, and the morphism $\underline{\pi} : \underline{\widetilde{G}}^{\text{lft NR}} \to \underline{B}^{\text{NR}}$ is smooth. Denote by $\mathbb{T}$ the kernel of the homomorphism $\underline{\pi}$ between group schemes. By the Néron property, we get a homomorphism f_1 from $\mathbb{T}$ to $\underline{T}^{\text{lft NR}}$ which extends the natural isomorphism between the generic fibers. On the other hand, since $\underline{T}^{\text{lft NR}} \to \underline{\widetilde{G}}^{\text{lft NR}} \to \underline{B}^{\text{NR}}$ is a complex of group schemes, we get a homomorphism f_2 from $\underline{T}^{\text{lft NR}}$ to $\mathbb{T}$ extending the natural isomorphism between the generic fibers. Therefore f_1 and f_2 are inverse to each other, and $\underline{T}^{\text{lft NR}}$ is isomorphic to $\mathbb{T}$. ∎

(4.8) Remark (a) For *any* short exact sequence $0 \to T \to G_1 \to G_2 \to 0$ of semiabelian varieties over K with T as above, the same argument shows that the attached complex $0 \to \underline{T}^{\text{lft NR}} \to \underline{G_1}^{\text{lft NR}} \to \underline{G_2}^{\text{lft NR}} \to 0$ of Néron models is a short exact sequence of group schemes smooth over $\mathcal{O}$.

(b) The second part of the statement of [BX, 4.2], that $\mathrm{R}^1 j_{\text{sm}*} T_K = (0)$ for every torus T_K over K if the residue field of K is perfect, is incorrect. In fact, take T_K to be the norm-one torus attached to a totally ramified quadratic Galois extension L of K, G_1 be $\operatorname{Res}_{L/K}(\mathbb{G}_m)$, and G_2 be the quotient of G_1 by T_K. One verifies by explicit calculation of the Néron model of T_K that the canonical morphism from $\underline{T_K}^{\text{lft NR}}$ to $\underline{G_1}^{\text{lft NR}}$ is *not* a closed embedding. In view of (a) above, this implies that $\mathrm{R}^1 j_{\mathrm{sm}*} T_K \neq (0)$. Fortunately the rest of the results of [BX] are not affected [2]: The second part of [BX, 4.2] is used only in Thm. 4.11 (ii) of [BX] to show surjection $\phi_E \twoheadrightarrow \phi_B$ of the induced map on the component groups. For this purpose one only needs that $\mathrm{R}^1 j_{\acute{e}t*} T_K = (0)$, which is the statement of our Lemma 4.3 (i) which is also proved in [BX, 4.2].

(4.9) Lemma *Assume that* $\mathrm{char}(K) = 0$*. Let* $0 \to T_1 \to \widetilde{G}_1 \to B \to 0$ *be a Raynaud extension. Let*

$$0 \to T_2 \to \widetilde{G}_2 \to B \to 0$$

be the push-out of $0 \to T_1 \to \widetilde{G}_1 \to B \to 0$ *by a K-rational isogeny* $\alpha : T_1 \to T_2$*. Then* $c(\widetilde{G}_1, K) = c(\widetilde{G}_2, K)$*.*

Proof. The argument of Theorem 11.3 of [CYdS], which is an application of Tate's Euler-Poincaré characteristic formula, applies to the present situation, and shows that $c(\widetilde{G}_1, K) = c(\widetilde{G}_2, K)$. ∎

Proof of Theorem 4.1 when $\mathrm{char}(K) = 0$. Choose a torus T_3 over K such that the product torus $T_3 \times_{\mathrm{Spec}\,K} T$ is K-isogenous to $\mathrm{Res}_{L/K} T_0$ for some split torus T_0 over a finite extension L of K. Let $\underline{(T_3 \times \widetilde{G})}^{\text{lft NR}}$ be the Néron model for the product $T_3 \times_{\mathrm{Spec}\,K} \widetilde{G}$. Then we have $\underline{(T_3 \times \widetilde{G})}^{\text{lft NR}} \cong \underline{T_3}^{\text{lft NR}} \times_{\mathrm{Spec}\,\mathcal{O}} \underline{\widetilde{G}}^{\text{lft NR}}$, hence

$$c(T_3 \times \widetilde{G}, K) = c(T_3, K) + c(\widetilde{G}, K)\,.$$

On the other hand, the push-out of the Raynaud extension $0 \to T_3 \times T \to T_3 \times \widetilde{G} \to B \to 0$ via a K-isogeny $T_3 \times T \to \mathrm{Res}_{L/K} T_0$ is a Rayaud extension $0 \to \mathrm{Res}_{L/K} T_0 \to G_3 \to B \to 0$. From Lemma 4.9, Corollary 4.7 we get

$$c(T_3 \times \widetilde{G}, K) = c(G_3, K) = c(\mathrm{Res}_{L/K} T_0, K) + c(B, K) = c(T_3, K) + c(T, K) + c(B, K)\ .$$

The last equality uses the isogeny invariance of the base change conductor $c(T, K)$ for tori; see Theorem 11.3 of [CYdS], Comparing the two expressions for $c(T_3 \times \widetilde{G}, K)$, we get $c(\widetilde{G}, K) = c(T, K) + c(B, K)$. ∎

(4.10) Lemma *Let K be a local field with finite residue field κ. Let T be a torus over K which is split over a finite separable totally ramified extension L of K. Let $X^*(T)$ be the character group of T, consider as a module over* $\Gamma_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$*. For every finite separable extension K' of K, denote by $\Gamma_{K'}$ the Galois group* $\mathrm{Gal}(K^{\mathrm{sep}}/K')$*. Then for every finite unramified extension M of K, The restriction map*

$$\mathrm{res}^L_M : \mathrm{H}^1(\Gamma_K, X^*(T)) \to \mathrm{H}^1(\Gamma_M, X^*(T))$$

is an isomorphism.

[2] We thank S. Bosch for supplying the clarification.

(4.11) Corollary *Notations as in Lemma 4.10 above. Then the cardinality of the finite group* $\mathrm{H}^1(\Gamma_M, T(K^{\mathrm{sep}}))$ *is equal to that of* $\mathrm{H}^1(\Gamma_K, T(K^{\mathrm{sep}}))$ *for each finite unramified extension* M *of* K.

Proof of Corollary 4.11. According to Tate's local duality theorem, the cup-product pairing

$$\mathrm{H}^1(\Gamma_M, X^*(T)) \times \mathrm{H}^1(\Gamma_M, T(K^{\mathrm{sep}})) \to \mathrm{H}^2(\Gamma_M, K^{\mathrm{sep}}) \cong \mathbb{Q}/\mathbb{Z}$$

is a perfect pairing; see [S3, II §5.8, Thm. 6]. The Corollary follows. ∎

(4.12) Remark The restriction map from $\mathrm{H}^1(\Gamma_M, T(K^{\mathrm{sep}}))$ to $\mathrm{H}^1(\Gamma_K, T(K^{\mathrm{sep}}))$ is not an isomorphism, but rather $[M:K]$ times an isomorphism.

Proof of Lemma 4.10. We know that $H^1(\Gamma_L, X^*(T)) = (0)$ since T is split over L. Hence from the inflation-restriction sequence we get an isomorphism $\mathrm{H}^1(\mathrm{Gal}(L/K), X^*(T)) \xrightarrow{\sim} \mathrm{H}^1(\Gamma_K, X^*(T))$ via the inflation map. Similarly the inflation map gives an isomorphism $\mathrm{H}^1(\mathrm{Gal}(L\,M/M), X^*(T)) \xrightarrow{\sim} \mathrm{H}^1(\Gamma_M, X^*(T))$. Since L is totally ramified over K and M is unramified over K, the pairs $(\mathrm{Gal}(L/K), X^*(T))$ and $(\mathrm{Gal}(L\,M/M), X^*(T))$ are visibly isomorphic under the restriction map. Therefore the restriction map $\mathrm{H}^1(\Gamma_K, X^*(T)) \to \mathrm{H}^1(\Gamma_M, X^*(T))$ is an isomorphism. ∎

Proof of Theorem 4.1 when the residue field κ is finite.

Assume now that the residue field κ of the local field K is finite. The translation invariant differential forms of top degree on $\underline{G}^{\mathrm{lft\,NR}}$ form a free rank-one $\mathcal{O}$-module $\underline{\omega}_{\underline{G}^{\mathrm{NR}}}$. Let ω_G be a generator of $\underline{\omega}_{\underline{G}^{\mathrm{NR}}}$. For every finite separable extension F of K, let $\mu_{G,F}$ be the Haar measure on $G(F)$ attached to ω_G; it is independent of the choice of the generator ω_G of $\underline{\omega}_{\underline{G}^{\mathrm{NR}}}$. Similarly we get a Haar measure $\mu_{T_K,F}$ on $T(F)$ and a Haar measure $\mu_{B_K,F}$ on $T(F)$.

Let $\mu_{B_K,F} \times \mu_{T_K,F}$ be the "product Haar measure" on $G(F)$ such that the integral of a Schwartz function on $G(F)$ against $\mu_{B_K,F} \times \mu_{T_K,F}$ is given by first integrating over the fibers of $G(F) \to B(F)$ using the Haar measure $\mu_{T_K,F}$, and then integrate over the open subgroup $G(F)/T(F)$ of $B(F)$ using the Haar measure $\mu_{B_K,F}$ on $B(F)$. Here we have identified $G(F)/T(F)$ as an open subgroup of $B(F)$ using the exact sequence $0 \to T(F) \to G(F) \to B(F)$ attached to the short exact sequence $0 \to T \to G \to B \to 0$. The two Haar measures $\mu_{B_K,F} \times \mu_{T_K,F}$ and $\mu_{G_K,F}$ on $G(F)$ differ by a multiplicative constant, which can be described as follows. From the short exact sequence $0 \to T \to G \to B \to 0$ we obtain a canonical isomorphism

$$\beta : \underline{\omega}_B \otimes_K \underline{\omega}_T \xrightarrow{\sim} \underline{\omega}_G\,.$$

Write $\beta(\omega_B \otimes \omega_T) = a \cdot \omega(G)$ with $a \in K^\times$. Then for any finite separable extension F of K, the product measure $\mu_{B_K,F} \times \mu_{T_K,F}$ is equal to the Haar measure on $G(F)$ attached to $\beta(\omega_B \otimes \omega_T)$. Hence

$$\mu_{B_K,F} \cdot \mu_{T_K,F} = \|a\|_F \cdot \mu_{G_K,F}\,,$$

where $\|a\|_F = q_F^{\mathrm{ord}_F(a)}$, and q_F is the cardinality of the residue field κ_F of F. As we shall soon see, the element $a \in K$ turns out to be a unit of $\mathcal{O}$; this is the key point of the proof.

Let L be a finite separable extension of K such that T, G, B all have semistable reduction over $\mathcal{O}_L$. From the definition of the invariant $c(G,K)$ we see that

$$\mu_{G_L,L} = \|\pi_K\|_L^{c(G,K)}\, \mu_{G_K,L}\,.$$

Similar equalities hold for T and B. Therefore Theorem 4.1 will follow from the statement of the next proposition, for both fields K and L.

(4.13) Proposition *Notations as above. The Haar measure $\mu_{G_K,F}$ on $G(F)$ is equal to the product measure $\mu_{T_K,F} \times \mu_{B_K,F}$, for any finite separable extension F of K.*

PROOF. Since the formation of Néron models commutes with finite unramified extension of base fields, we may and do assume that T, G, B all have semistable reduction over a totally ramified finite separable extension L of K, and that the groups ϕ_T, ϕ_G, ϕ_B of connected components of the closed fiber of the Néron lft models of T, G, B are all constant over the residue field κ. By Lemma 4.5, we can divide out the maximal K-split subtorus of T, so we may and do assume that T is anisotropic over K, and hence anisotropic over all finite unramified extensions of K.

By what we have seen so far, it suffices to prove the proposition for one finite unramified extensions M of K. Since what we need to prove is that two Haar measures on the same group are equal, it suffices to integrate both against the constant function 1 on the group $G(M)$ and compare the two integrals; they differ by factor of the form $\|a\|_M$, where a is an element of $K^\times$ independent of M. Our strategy is to examine the growth behavior of the two integrals as M varies over the tower of finite unramified extensions of K using the Lang-Weil estimate to conclude that the factor in question is equal to 1 for all finite unramified extension M of K.

In the yet-unfinished proof of Thm. 4.1, We have written $\beta(\omega_B \otimes \omega_T)$ as $a \cdot \omega(G)$, where a is an element of $K^\times$. So $(\mu_{B_K,M} \times \mu_{T_K,M})(G(M)) = \|a\|_M \cdot \mu_{G_K,M}(G(M))$ for every finite unramified extension M of K. The projection $G(M) \to B(M)$ gives an embedding of $G(M)/T(F)$ into $B(M)$ as an open subgroup of finite index. By Fubini's theorem, we have

$$(\mu_{B_K,M} \times \mu_{T_K,M})(G(M)) = [B(M) : G(M)/T(M)]^{-1} \cdot \mu_{B_K,M}(B(M)) \cdot \mu_{T_K,M}(T(M))\,.$$

On the other hand, by the definition of the Haar measure $\mu_{G_K,M}$, we have

$$\mu_{G_K,M}(G(M)) = q_M^{-\dim(G)} \cdot |\underline{G_M}^{\mathrm{lft\,NR}}(\kappa_M)|\,.$$

Notice that $G(M)$ is compact since we assumed that T is anisotropic over M. Similarly, we have $\mu_{B_K,M}(B(M)) = q_M^{-\dim(B)} \cdot |\underline{B_M}^{\mathrm{NR}}(\kappa_M)|$ and $\mu_{T_K,M}(T(M)) = q_M^{-\dim(T)} \cdot |\underline{B_M}^{\mathrm{NR}}(\kappa_M)|$, where ϕ_T, ϕ_G, ϕ_B are the component groups of the closed fiber of the Néron models of T, G, B respectively. By the Lang-Weil estimate, we have

$$\lim_{q_M \to \infty} \frac{\mu_{G_K,M}(G(M))}{\mu_{B_K,M}(B(M)) \cdot \mu_{T_K,M}(T(M))} = \frac{|\phi_G|}{|\phi_B| \cdot |\phi_T|}$$

On the other hand by Cor. 4.11, we have the uniform bound $[B(M) : G(M)/T(M)] \leq |\mathrm{H}^1(\Gamma_K, X^*(T))|$, independent of M. As $q_M \to \infty$ we see that the element $a \in K^\times$ has to be a unit of $\mathcal{O}$, and moreover $[B(M) : G(M)/T(M)] \cdot |\phi_G| = |\phi_B| \cdot |\phi_T|$. This prove Prop. 4.13 and concludes the proof of the second part of Thm. 4.1. ∎

§5. Néron models and uniformization

(5.1) Proposition *Let $\mathcal{O}$ be a discrete valuation ring with fraction field K. Let A be an abelian variety over K, and let $\iota : Y \hookrightarrow \widetilde{G}$ be the corresponding degeneration data over $\widehat{K}$. Then $c(A, K) = c(\widetilde{G}, \widehat{K})$.*

Proof. We may and do assume that $\mathcal{O}$ is complete. Let $\alpha_K : \widetilde{G}^{\mathrm{rig}} \to A^{\mathrm{rig}}$ be the uniformization map over K in the category of rigid analytic varieties. By [BX, Thm. 2.3], the map α_K extends uniquely to a morphism of formal schemes

$$\alpha_{\mathcal{O}} : \left(\underline{\widetilde{G}}^{\mathrm{NR}}\right)^\wedge \to \left(\underline{A}^{\mathrm{NR}}\right)^\wedge ,$$

where $\left(\underline{\widetilde{G}}^{\mathrm{NR}}\right)^\wedge$ and $\left(\underline{A}^{\mathrm{NR}}\right)^\wedge$ denote the $\mathfrak{p}$-adic completion of the Néron models $\underline{\widetilde{G}}^{\mathrm{NR}}$ and $\underline{A}^{\mathrm{NR}}$ respectively. Moreover $\alpha_{\mathcal{O}}$ is étale, and gives an isomorphism from the neutral component of $\left(\underline{\widetilde{G}}^{\mathrm{NR}}\right)^\wedge$ to the neutral component of $\left(\underline{A}^{\mathrm{NR}}\right)^\wedge$. The same statement holds for every finite separable extension of K. We conclude the proof of Prop. 5.1 by the functoriality of $\alpha_{\mathcal{O}}$. ∎

(5.2) Corollary *Assume that the residue field κ of the discrete valuation ring $\mathcal{O}$ is perfect. Let A be an abelian variety over a K. Assume that over the ring of integers $\mathcal{O}_L$ of some finite separable extension L of K the neutral component of the closed fiber of the Néron model $\underline{A_L}^{\mathrm{NR}}$ of A_L is a torus, so that over the completion $\widehat{K}$ of K the abelian variety $A_{\widehat{K}}$ is uniformized as a quotient of a torus $\widetilde{G}$ over $\widehat{K}$ in the rigid category. Let ℓ be a prime number invertible in κ. Then $c(A, K) = \frac{1}{2}\, a\left(X^*(\widetilde{G})\right) = \frac{1}{4} a\left(\mathrm{V}_\ell(A)\right)$. Here $a\left(X^*(\widetilde{G})\right)$ is the Artin conductor of the representation of* $\mathrm{Gal}(\widehat{K}^{\mathrm{sep}}/\widehat{K})$ *on the character group of the torus $\widetilde{G}$, while $a\left(\mathrm{V}_\ell(A)\right)$ is Artin conductor of the ℓ-adic Tate module of A, considered as a $\mathbb{Q}_\ell$-representation of* $\mathrm{Gal}(\widehat{K}^{\mathrm{sep}}/\widehat{K})$.

Proof. The first equality is a special case of Prop. 5.1. The second equality is a consequence of the fact that $\mathrm{V}_\ell(A)/\mathrm{V}_\ell(\widetilde{G})$ is non-canonically isomorphic to the dual of $\mathrm{V}_\ell(\widetilde{G})$. ∎

(5.3) Theorem *Assume either that the residue field κ is finite, or that* $\mathrm{char}(K) = 0$ *and κ is perfect. Let A be an abelian variety over K. Let $Y \hookrightarrow \widetilde{G}$ be the corresponding degeneration data over $\widehat{K}$, and $\widetilde{G}$ fits into a Raynaud extension $0 \to T \to \widetilde{G} \to B \to 0$. Then*

$$c(A,K) = c(T,\widehat{K}) + c(B,\widehat{K}) = \frac{1}{2} a\left(X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}\right) + c(B,\widehat{K}) \ ,$$

where $a\left(X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}\right)$ is the Artin conductor of the linear representation of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ *on $X^*(T) \otimes \mathbb{Q}$.*

Proof. The first equality follows from Thm. 4.1 and Prop. 5.1. The second equality is proved in [CYdS, §11]. ∎

Although we can express the base change conductor $c(-,K)$ by a simple formula in the case of tori, in the case of abelian varieties a "simple formula" may be too optimistic to hope for. Since the statement that $c(T,K) = \frac{1}{2}\, a(X^*(T) \otimes \mathbb{Q})$ for all tori over K is equivalent to the statement that $c(T,K) = c(T',K)$ for all isogenous tori T, T', whether the invariant $c(A,K)$ for abelian varieties stays the same under isogenies may be a more intelligent question to ask. This question will be addressed in the next section.

§6. Study of isogeny invariance

In this section the $\mathcal{O} = \mathcal{O}_K$ denotes a henselian discrete valuation ring whose fraction field K has characteristic 0, unless otherwise stated. The residue field κ of $\mathcal{O}$ is assumed to be a perfect field of characteristic $p > 0$.

(6.1) Proposition *Let $u : B_1 \to B_2$ be an isogeny between abelian varieties over K such that for some finite Galois extension L of K, the map $\underline{u}_L : \underline{B}^{\mathrm{NR}}_{1,L} \to \underline{B}^{\mathrm{NR}}_{2,L}$ induced by u from the Néron models of $B_1 \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L$ to the Néron models of $B_2 \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L$ is étale. Then the map $\underline{u} : \underline{B_1}^{\mathrm{NR}} \to \underline{B_2}^{\mathrm{NR}}$ between the Néron models over $\mathcal{O}_K$ is étale.*

Proof. We may and do assume that $\mathcal{O}$ is complete and the residue field κ is algebraically closed. Let $M = \mathrm{Ker}(u)$, a finite group scheme over K. Consider the following diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & B_2(K)/u(B_1(K)) & \longrightarrow & \mathrm{H}^1(K,M) \\
 & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \mathrm{res}^L_K} \\
0 & \longrightarrow & B_2(L)/u(B_1(L)) & \longrightarrow & \mathrm{H}^1(L,M)
\end{array}$$

with exact rows. Since the homomorphism $\underline{u}_L$ is étale by assumption, it induces a surjection from the neutral component of $\underline{B}^{\mathrm{NR}}_{1,L}$ to the neutral component of $\underline{B}^{\mathrm{NR}}_{2,L}$, therefore $B_2(L)/u(B_1(L))$ is finite. On the other hand $\mathrm{Ker}(\mathrm{res}^L_K)$ is the image of $\mathrm{H}^1(\mathrm{Gal}(L/K), M(L))$ under the inflation map, hence is finite. We conclude that $B_2(K)/u(B_1(K))$ is finite from the above diagram.

For each $n > 0$ let $\underline{\underline{B}}_{i,n}$ be the Greenberg functor applied to the smooth group scheme $\underline{B_i}^{\mathrm{NR}} \times_{\mathrm{Spec}\, \mathcal{O}} \mathrm{Spec}(\mathcal{O}/\pi^n\mathcal{O})$ over $\mathrm{Spec}(\mathcal{O}/\pi^n\mathcal{O})$, $i = 1, 2$. Each $\underline{\underline{B}}_{i,n}$ is a smooth group scheme over $\mathrm{Spec}\,\kappa$, such that $\underline{\underline{B}}_{i,n}(\kappa) = \underline{B_i}^{\mathrm{NR}}(\mathcal{O}/\pi^n\mathcal{O})$. Consider the homomorphism $\underline{\underline{u}}_n : \underline{\underline{B}}_{1,n} \to \underline{\underline{B}}_{2,n}$ induced by $\underline{u}$. On the level of κ-points the cokernel

$$\mathrm{Coker}\big(\underline{\underline{u}}_n(\kappa) : \underline{\underline{B}}_{1,n}(\kappa) \to \underline{\underline{B}}_{2,n}(\kappa)\big)$$

is finite, because $\underline{\underline{B}}_{i,n}(\kappa)$ is a quotient of $\underline{B_i}^{\mathrm{NR}}(\mathcal{O}) = B_i(K)$ for $i = 1, 2$. Since $\dim\big(\underline{\underline{B}}_{1,n}\big) = \dim\big(\underline{\underline{B}}_{2,n}\big)$ and κ is algebraically closed, the kernel of $\underline{\underline{u}}_n$ is also a finite quasi-algebraic group over κ.

Choose an integer $N \gg 0$ such that the exponential map induces an isomorphism

$$\pi^N \,\mathrm{Lie}\, \underline{B_i}^{\mathrm{NR}} \xrightarrow{\sim} \mathrm{Ker}\big(\underline{B_i}^{\mathrm{NR}}(\mathcal{O}) \to \underline{B_i}^{\mathrm{NR}}(\mathcal{O}/\pi^N\mathcal{O})\big), \quad i = 1, 2.$$

Consider the following map between short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \pi^N \,\mathrm{Lie}\, \underline{B_1}^{\mathrm{NR}} \arrow[r] \arrow[d, "\mathrm{Lie}\, u"'] & \underline{B_1}^{\mathrm{NR}}(\mathcal{O}) \arrow[r] \arrow[d, "u_{\mathcal{O}}"] & \underline{B_1}^{\mathrm{NR}}(\mathcal{O}/\pi^N\mathcal{O}) \arrow[r] \arrow[d, "u_{\mathcal{O}/\pi^N\mathcal{O}}"] & 0 \\
0 \arrow[r] & \pi^N \,\mathrm{Lie}\, \underline{B_2}^{\mathrm{NR}} \arrow[r] & \underline{B_2}^{\mathrm{NR}}(\mathcal{O}) \arrow[r] & \underline{B_2}^{\mathrm{NR}}(\mathcal{O}/\pi^N\mathcal{O}) \arrow[r] & 0
\end{tikzcd}$$

induced by u. This gives us an exact sequence

$$\mathrm{Ker}\big(u_{\mathcal{O}/\pi^N\mathcal{O}}\big) \to \mathrm{Coker}\big(\mathrm{Lie}\, u\big) \to \mathrm{Coker}\big(u_{\mathcal{O}}\big)$$

Since $\mathrm{Ker}\big(u_{\mathcal{O}/\pi^N\mathcal{O}}\big)$ and $\mathrm{Coker}\big(u_{\mathcal{O}}\big)$ are finite, so is $\mathrm{Coker}\big(\mathrm{Lie}\, u\big)$. This implies that

$$\mathrm{Lie}\, \underline{u} : \mathrm{Lie}\, \underline{B_1}^{\mathrm{NR}} \xrightarrow{\sim} \mathrm{Lie}\, \underline{B_2}^{\mathrm{NR}}$$

since κ is infinite. ∎

(6.2) Remark A slight modification of the proof above works for the case $\mathrm{char}(K) = p > 0$ as well, but the hypothesis that the residue field κ is perfect is still needed.

(6.3) Lemma *(i) Let B_1, B_2 be abelian varieties over K which have good reduction over the ring of integers $\mathcal{O}_L$ of a finite Galois extension L of K. Let $u : B_1 \to B_2$ be a K-isogeny. Then there exists a unique factorization of u as a composition*

$$B_1 \xrightarrow{u_1} B_3 \xrightarrow{u_2} B_2$$

of isogenies between abelian varieties over K, such that over $\mathcal{O}_L$, $\mathrm{Ker}(u_1)$ extends to a finite flat subgroup scheme of $\underline{B}^{\mathrm{NR}}_{1,L}$ with geometrically connected closed fiber, and $\mathrm{Ker}(u_2)$ extends to a finite étale subgroup scheme of $\underline{B}^{\mathrm{NR}}_{3,L}$.

(ii) Notations as in (i) above. Assume moreover that the reduction of B_1 and B_2 over the residue field of $\mathcal{O}_L$ are ordinary abelian varieties. Then over $\mathcal{O}_L$, $\mathrm{Ker}(u_1^t)$ extends to a finite étale subgroup scheme of the $\mathcal{O}_L$-Néron model $\underline{B^t}_{3,L}^{\mathrm{NR}}$ of the dual abelian variety $B_3^t \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L$ of $B_3 \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L$.

Proof. Statement (i) is standard if B_1, B_2 have good reduction over $\mathcal{O}$. The statement (i) itself follows from descent. Statement (ii) follows from statement (i). ∎

(6.4) Definition Under the blanket assumption that $\mathrm{char}(K) = 0$ of this section, we can attach a non-zero ideal $\mathcal{D}_u$ to every isogeny u between abelian varieties over K, as follows. Let u be an isogeny from an abelian variety A_1 to an abelian variety A_2 over K. The homomorphism $\det(\mathrm{Lie}(\underline{u})) : \det(\mathrm{Lie}(\underline{A_1}^{\mathrm{NR}})) \to \det(\mathrm{Lie}(\underline{A_2}^{\mathrm{NR}}))$ between rank one free $\mathcal{O}$-modules is isomorphic to the inclusion of an ideal $\mathcal{D}_u$ of $\mathcal{O}$ in $\mathcal{O}$. In other words, $\mathcal{D}_u$ is defined by $\mathcal{O}/\mathcal{D}_u \cong \det(\mathrm{Lie}(\underline{A_2}^{\mathrm{NR}}))/\det(\mathrm{Lie}(\underline{u}))(\det(\mathrm{Lie}(\underline{A_1}^{\mathrm{NR}})))$

This definition applies to the dual isogeny $u^t : A_1^t \to A_2^t$ as well and gives us an ideal $\mathcal{D}_{u^t}$ in $\mathcal{O}$. The following is a result of Raynaud, see Thm. 2.1.1 and p. 208–210 of [R2].

(6.5) Proposition *Let $u : A_1 \to A_2$ be an isogeny between abelian varieties over K. Then the product ideal $\mathcal{D}_u \cdot \mathcal{D}_{u^t}$ is equal to the ideal of $\mathcal{O}$ generated by $\deg(u)$, the degree of the isogeny u.*

(6.6) Lemma *Let L be a finite separable extension of K such that A_1, A_2 have semistable reduction over $\mathcal{O}_L$. Then*

$$\mathrm{ord}_K(\mathcal{D}_u) + c(A_2, K) = \mathrm{ord}_K(\mathcal{D}_{u_L}) + c(A_1, K)\,.$$

Proof. We have a commutative diagram

$$\begin{array}{ccc} \underline{A_1}^{\mathrm{NR}} \times_{\mathrm{Spec}\,\mathcal{O}} \mathrm{Spec}\,\mathcal{O}_L & \xrightarrow{\underline{u} \times_{\mathrm{Spec}\,\mathcal{O}} \mathrm{Spec}\,\mathcal{O}_L} & \underline{A_2}^{\mathrm{NR}} \times_{\mathrm{Spec}\,\mathcal{O}} \mathrm{Spec}\,\mathcal{O}_L \\ {\scriptstyle \mathrm{can}_{A_1,L/K}} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{can}_{A_2,L/K}} \\ \underline{A_{1,L}}^{\mathrm{NR}} & \xrightarrow[\underline{u_L}]{} & \underline{A_{2,L}}^{\mathrm{NR}} \end{array}$$

of smooth group schemes over $\mathcal{O}_L$. Apply the functor Lie to the diagram above gives us a commutative diagram of free $\mathcal{O}_L$-modules of finite rank. We finish the proof by taking the determinants. ∎

(6.7) Theorem *For any abelian variety A over K, we have $c(A, K) = c(A^t, K)$.*

Proof. Let L be a finite separable extension of K such that A has semistable reduction over $\mathcal{O}_L$. Let $\lambda : A \to A^t$ be a polarization of A over K. Since polarizations are self-dual, Prop. 6.5 implies that

$$\mathrm{ord}_K(\mathcal{D}_\lambda) = \mathrm{ord}_K(\mathcal{D}_{\lambda_L}) = \frac{1}{2}\,\mathrm{ord}_K(\deg(\lambda))\,.$$

Apply Lemma 6.6 to λ, we get the desired equality $c(A, K) = c(A^t, K)$. ∎

(6.8) Theorem *Let A_1, A_2 be abelian varieties over K and let $u : A_1 \to A_2$ be a K-isogeny. Assume that A_i has semistable reduction over $\mathcal{O}_L$ for some finite Galois extension L of K, and the neutral component of the closed fiber of the Néron model of $A_{i,L}$ is an extension of an ordinary abelian variety by a torus, $i = 1, 2$. Then $c(A_1, K) = c(A_2, K)$.*

PROOF. Suppose first that A_1, A_2 have potentially good reduction. By Lemma 6.3, we can factor u as a composition of $u_1 : A_1 \to A_3$ and $u_2 : A_3 \to A_2$, where u_1^t is potentially étale and u_2 is potentially étale. By Prop. 6.1, we get $c(A_1^t, K) = c(A_3^t, K)$ and $c(A_3, K) = c(A_2, K)$. Thm. 6.7 tells us that $c(A_1^t, K) = c(A_1, K)$ and $c(A_3^t, K) = c(A_3, K)$, hence $c(A_1, K) = c(A_2, K)$ if A_1 and A_2 have potentially good reduction. The general case follows from Theorem 5.3 and the potentially good reduction case just proved. ∎

(6.9) Remark So far we have assumed that the discrete valuation ring $\mathcal{O}$ has mixed characteristics $(0, p)$. If both K and its residue field have characteristic 0, the proofs still work and actually become simpler. The statements of the results in sections 4, 5, 6 all hold.

(6.10) Counterexamples

We saw in Theorem 6.8 that for an abelian variety over a local field K with potentially good reduction, the base change conductor $c(A, K)$ stays the same under K-rational isogenies if K has characteristic zero and A has potentially ordinary reduction. We give two examples to show that the ordinariness hypothesis cannot be dropped.

(6.10.1) CM elliptic curves over $\mathbb{Q}$ with potentially supersingular reduction

Let E_1, E_2 be elliptic curves over $\mathbb{Q}_p$, such that E_1 has complex multiplication over $\mathbb{Q}_p$ by the ring of integers $\mathcal{O}_F$ of an imaginary quadratic field F. Assume the p is ramified in $\mathcal{O}_F$, and there exists a $\mathbb{Q}_p$-rational isogeny $u : E_1 \to E_2$ of degree p. There are plenty of such examples: A glance at the first three pages of [MF4, table 1] produces two such pairs $(27A, 27B)$ and$(36A, 36C)$, with $p = 3$ and $F = \mathbb{Q}(\sqrt{-3})$.

By [R2, p. 209], the elementary divisor of the map $\mathrm{Lie}\,\underline{u} : \mathrm{Lie}\,\underline{E_1}^{\mathrm{NR}} \to \mathrm{Lie}\,\underline{E_2}^{\mathrm{NR}}$ is either 1 or p, since its product with the elementary divisor of $\mathrm{Lie}\,\underline{u^t}$ is equal to p. On the other hand, let L be a finite extension of $\mathbb{Q}_p$ such that E_1, E_2 have good reduction over $\mathcal{O}_L$. Then $\mathcal{O}_F \otimes \mathbb{Z}_p \subseteq \mathcal{O}_L$, and the elementary divisor of the map $\mathrm{Lie}\,\underline{u}_L : \mathrm{Lie}\,\underline{E_{1,L}}^{\mathrm{NR}} \to \mathrm{Lie}\,\underline{E_{2,L}}^{\mathrm{NR}}$ is $\sqrt{-p}$. Therefore $c(E_1, \mathbb{Q}_p) \neq c(E_2, \mathbb{Q}_p)$.

(6.10.2) Isotrivial potentially supersingular elliptic curves

In this example we take $K = \kappa((t))$, $\mathcal{O} = \kappa[[t]]$, where κ is a perfect field of characteristic $p > 0$, $p \neq 3$. Let E_0 be a supersingular elliptic curve over κ, with endomorphisms by $\mathbb{Z}[\mu_3]$. (This forces p to be split in $\mathbb{Z}[\mu_3]$.) Let L be a tamely totally ramified extension of K of degree 3. Let E_1 be the twist of the constant elliptic curve $E_0 \times_{\mathrm{Spec}\,\kappa} \mathrm{Spec}\ K$ by a nontrivial element $\alpha \in \mathrm{Hom}(\mathrm{Gal}(L/K), \mu_3)$. Let E_2 be the Frobenius twist $E_1^{(p)}$ of E_1 and let $u : E_1 \to E_2$ be the relative Frobenius map. In other words E_2 is the quotient of E_1 by the connected subgroup of E_1 of order p. One can also obtain E_2 as the α-twist of $E_0^{(p)} \times_{\mathrm{Spec}\,\kappa} \mathrm{Spec}\, K$. Observe that μ_3 acts on the tangent space of E_0 and $E_0^{(p)}$ via two *different* characters of μ_3. Using the main result of [Ed], one can check that one of $c(E_1,K), c(E_2)$ is $\frac{1}{3}$, and the other is $\frac{2}{3}$. Especially $c(E_1,K) \neq c(E_2,K)$.

§7. Congruence of Néron models

For simplicity, we assume that K is complete in this section. Let L/K be a finite Galois extension with Galois group $\Gamma = \mathrm{Gal}(L/K)$. To fix the idea we adopt the convention that Γ operates on L (resp. Spec L) on the right (resp. on the left). In this section we will employ the notation system in chapters 2, 3 of [FC] for the degeneration data of a semiabelian variety over a local field, recalled at the end of section 2.

(7.1) Let A be an abelian variety over K. We have a natural isomorphism

$$\gamma(L/K) : \mathrm{Spec}\,(L \times_K L) \xrightarrow{\sim} \Gamma \times \mathrm{Spec}\; L\,,$$

The Galois descent data for $(L/K, \Gamma, A)$ is a semilinear left action of Γ on $A_L = A \times_{\mathrm{Spec}\,K}$ Spec L. It induces a semilinear left action

$$\mathrm{GDDintf}(L/K, \Gamma, A) : \Gamma \times \underline{A_L}^{\mathrm{NR}} \to \underline{A_L}^{\mathrm{NR}}$$

of Γ on $\underline{A_L}^{\mathrm{NR}}$ compatible with $\gamma(L/K)$, which we refer to as the *integral form of the Galois descent data for* $(L/K, \Gamma, A)$.

Suppose K_0 is another local field with a uniformizing element π_0, and L_0/K_0 is a finite Galois extension with Galois group Γ_0. Let A_0 be an abelian variety over K_0. Let $N > 0$ be a positive integer.

We say that the integral forms of the two Galois descent data $\mathrm{GDDintf}(L/K, \Gamma, A)$ and $\mathrm{GDDintf}(L_0/K_0, \Gamma_0, A_0)$ are congruent at level N, notation

$$\mathrm{GDDintf}(L/K, \Gamma, A) \equiv_{\alpha,\beta,\rho} \mathrm{GDDintf}(L_0/K_0, \Gamma_0, A_0) \quad (\mathrm{mod\ level}\ N)$$

if the following conditions hold:

- β is an isomorphism from Γ to Γ_0.
- α is an isomorphism

  $$\alpha : \mathrm{Spec}\left(\mathcal{O}_L/\pi^N \mathcal{O}_L\right) \xrightarrow{\sim} \mathrm{Spec}\left(\mathcal{O}_{L_0}/\pi_0^N \mathcal{O}_{L_0}\right)$$

  which is equivariant with respect to $(\Gamma \underset{\beta}{\xrightarrow{\sim}} \Gamma_0)$, and induces an isomorphism from $\mathcal{O}/\pi^N\mathcal{O}$ to $\mathcal{O}_0/\pi_0^N\mathcal{O}_0$.
- ρ is an isomorphism

  $$\rho : \underline{A_L}^{\mathrm{NR}} \times_{\mathrm{Spec}\ \mathcal{O}_L} \mathrm{Spec}(\mathcal{O}_L/\pi^N\mathcal{O}_L) \xrightarrow{\sim} \underline{A_{L_0}}^{\mathrm{NR}} \times_{\mathrm{Spec}\ \mathcal{O}_L} \mathrm{Spec}(\mathcal{O}_L/\pi^N\mathcal{O}_L)$$

  over α, and it is equivariant with respect to $\Gamma \underset{\beta}{\xrightarrow{\sim}} \Gamma_0$.

(7.2) Let $K_0, L_0, \pi_0, \Gamma_0$ be as in 7.1. For $j = \emptyset, 0$, let $\iota_j : Y_j \hookrightarrow \widetilde{G}_j$ be the K_j-rational degeneration data for an abelian variety A_j over K_j. Each $\widetilde{G}_j$ sits in a Raynaud extension $0 \to T_j \to \widetilde{G}_j \to B_j \to 0$ over K_j. We assume that B_j has good reduction $\underline{B}_j$ over $\mathcal{O}_{L_j}$, and that T_j is split over L_j. So the Raynaud extension above extends over $\mathcal{O}_{L_j}$ to

$$0 \to \underline{T}_j \to \underline{\widetilde{G}}_j \to \underline{B}_j \to 0\,,$$

a Raynaud extension over $\mathcal{O}_{L_j}$. The Galois group Γ_j operates naturally on $\underline{\widetilde{G}}_j$, Y_j and ι_j. We call this semi-linear action of Γ_j on $\iota_j : Y_j \hookrightarrow \widetilde{G}_j$ the (L_j/K_j)-*twisted degeneration data* for $(L_j/K_j, \Gamma_j, A_j)$, notation $\mathrm{DDtw}(L_j/K_j, \Gamma_j, A_j)$.

For $j = \emptyset, 0$, let $\mathcal{P}_j$ be the Poincaré sheaf on $\underline{B}_j \times_{\mathrm{Spec}\,\mathcal{O}_j} \underline{B}_j^t$. Let $c_j^t \times c_j : Y_j \times X_j \to \underline{B}_j \times \underline{B}_j^t$ be the homomorphism which corresponds to the Raynaud extension for $\underline{\widetilde{G}}_j$ over $\mathcal{O}_j$. Denote by τ_j the trivialization of $\left(c_j^t \times c_j\right)^* \mathcal{P}_j^{-1}$ over L_j which corresponds to ι_j, $j = \emptyset, 0$.

(7.3) Definition We say that the (L_j/K_j)-twisted degeneration data $\mathrm{DDtw}(L_j/K_j, \Gamma_j, A_j)$ for A_j, $j = \emptyset, 0$, are congruent at level N, notation

$$\mathrm{DDtw}(L/K, \Gamma, A) \equiv_{\alpha,\beta,\varrho,\upsilon,\xi} \mathrm{DDtw}(L_0/K_0, \Gamma_0, A_0) \quad (\text{mod level } N)$$

if the following conditions hold:

- β is an isomorphism from Γ to Γ_0.
- α is an $(\Gamma_0 \xrightarrow[\beta^{-1}]{\sim} \Gamma)$-equivariant isomorphism from the pair $\left(\mathrm{Tr}_{eN}(L_0), \mathrm{Tr}_N(K_0)\right)$ to the pair $\left(\mathrm{Tr}_{eN}(L), \mathrm{Tr}_N(K)\right)$. Here $e = e(L/K) = e(L_0/K_0)$, and we followed the notation in [D].
- ϱ is an isomorphism

  $$\underline{\widetilde{G}} \times_{\mathrm{Spec}(\mathcal{O}_K/\pi^N\mathcal{O}_K)} \mathrm{Spec}(\mathcal{O}_L/\pi^N\mathcal{O}_L) \xrightarrow{\sim} \underline{\widetilde{G}}_0 \times_{\mathrm{Spec}(\mathcal{O}_{K_0}/\pi_0^N\mathcal{O}_{K_0})} \mathrm{Spec}(\mathcal{O}_{L_0}/\pi_0^N\mathcal{O}_{L_0})$$

  which is compatible with β and α. It induces an isomorphism $\varrho_t : X^*(T) \xrightarrow{\sim} X^*(T_0)$ between the character groups.
- υ is an isomorphism from Y to Y_0. Together with ϱ, we get an isomorphism ξ from the biextension $(c^t \times c)^*\mathcal{P}^{-1} \otimes (\mathcal{O}_L/\pi^N\mathcal{O}_L)$ to $(c_0^t \times c_0)^*\mathcal{P}_0^{-1} \otimes (\mathcal{O}_{L_0}/\pi_0^N\mathcal{O}_{L_0})$.
- Choose a trivialization η of the biextension $(c^t \times c)^*\mathcal{P}^{-1}$ over $\mathcal{O}_L$ and a trivialization η_0 of the biextension $(c_0^t \times c_0)^*\mathcal{P}_0^{-1}$ over $\mathcal{O}_{L_0}$ which are compatible with respect to ξ and ρ. Let $b : Y \times X \to K^\times$ be the bilinear form such that $b \cdot \eta$ is equal to the trivialization τ of $(c^t \times c)^*\mathcal{P}^{-1}$ which corresponds to $\iota : Y \hookrightarrow \widetilde{G}$. Define b_0 similarly. Then for each $(y, \chi) \in Y \times X$, write $y_0 = \upsilon(y)$, $\chi_0 = \varrho_t(\chi)$, it is required that

  $$\mathrm{ord}_L(b(y,\chi)) = \mathrm{ord}_{L_0}(b_0(y_0,\chi_0))\,.$$

Moreover the element $\overline{b}(y,\chi) \in \mathfrak{p}_L^{\mathrm{ord}_L(b(y,\chi))} \otimes (\mathcal{O}_L/\pi^N\mathcal{O}_L)$ given by $b(y,\chi)$ corresponds under α^{-1} to the element $\overline{b_0}(y_0,\chi_0) \in \mathfrak{p}_{L_0}^{\mathrm{ord}_{L_0}(b_0(y_0,\chi_0))} \otimes (\mathcal{O}_{L_0}/\pi_0^N\mathcal{O}_{L_0})$ given by $\overline{b_0}(y_0,\chi_0)$.

It is easy to see that in the last item above, the condition does not depend on the choice of η and η_0.

(7.4) Proposition *Notation as above. If*

$$\mathrm{DDtw}(L/K,\Gamma,A) \equiv_{\alpha,\beta,\varrho,\upsilon,\xi} \mathrm{DDtw}(L_0/K_0,\Gamma_0,A_0) \quad (\text{mod level } N)\,,$$

then there is a naturally determined ϱ such that

$$\mathrm{GDDintf}(L/K,\Gamma,A) \equiv_{\alpha,\beta,\varrho} \mathrm{GDDintf}(L_0/K_0,\Gamma_0,A_0) \quad (\text{mod level } N)\,.$$

PROOF. This is a consequence of Mumford's construction as explained in [FC, Chap. 3]. Since the base $\mathcal{O}_L$ is a complete discrete valuation ring, one can construct a relatively complete model $\widetilde{P}$ which is regular. Then the quotient P contains the Néron model $\underline{A_L}^{\mathrm{NR}}$; see the proof of [FC, Prop. 8.1, p. 78]. One way to construct such relatively complete models $\widetilde{P}$ is to use the technique of torus embedding as explained in [KKMS, Chap. IV, §3]. With such construction, the congruence assumption implies congruence for the formal schemes P_{for} and $P_{0,\mathrm{for}}$. ∎

(7.5) As in [CYdS], for a scheme X constructed from the abelian variety A, "X is determined by the Galois descent data (mod level N)" means that if

$$\mathrm{GDDintf}(L/K,\Gamma,A) \equiv_{\alpha,\beta,\rho} \mathrm{GDDintf}(L_0/K_0,\Gamma_0,A_0) \quad (\text{mod level } N)\,,$$

then there is a canonical isomorphism determined by (α,β,ρ) from the scheme X to the scheme X_0 constructed from A_0 by the same procedure. Similarly, "X is determined by the degeneration data (mod level N)" means that if

$$\mathrm{DDtw}(L/K,\Gamma,A) \equiv_{\alpha,\beta,\varrho,\upsilon,\xi} \mathrm{DDtw}(L_0/K_0,\Gamma_0,A_0) \quad (\text{mod level } N)\,,$$

then there is a canonical isomorphism determined by $(\alpha,\beta,\varrho,\upsilon,\xi)$ from the scheme X to the scheme X_0.

(7.6) Theorem *Let A be an abelian variety over K. Then there exists an integer n such that for any integer $N \geq 1$, $\underline{A}^{\mathrm{NR}} \times_{\mathrm{Spec}\,\mathcal{O}} \mathrm{Spec}(\mathcal{O}/\pi^N\mathcal{O})$ is determined by the Galois descent data of A* (mod level $N+n$)*.*

PROOF. The proof of the main result Theorem 8.5 of [CYdS] works without change. Therefore we only sketch the argument here. Let $\underline{R}^{\mathrm{NR}} = \mathrm{Res}_{\mathcal{O}_L/\mathcal{O}}\big(\underline{A_L}^{\mathrm{NR}}\big)$. Let $\underline{A}^0$ be the schematic closure of A in $\underline{R}^{\mathrm{NR}}$. Let $\underline{R}' = \underline{R}^{\mathrm{NR}} \times_{\mathrm{Spec}\,\mathcal{O}} \mathrm{Spec}\ \mathcal{O}_L$. There is a natural map $\alpha : \underline{R}' \to \Gamma \times \underline{A_L}^{\mathrm{NR}}$, where the target $\Gamma \times \underline{A_L}^{\mathrm{NR}}$ is naturally identified with the Néron model of the generic fiber of $\mathrm{Res}_{\mathcal{O}_L/\mathcal{O}}(\underline{A_L}^{\mathrm{NR}})$. Let $\beta : \underline{A_L}^{\mathrm{NR}} \hookrightarrow \Gamma \times \underline{A_L}^{\mathrm{NR}}$ be the diagonal embedding. Taking the fiber product of α and β, we get a closed subscheme $\underline{A}'$ of $\underline{R}'$ whose generic fiber is the abelian variety A naturally embedded in $\mathrm{Res}_{L/K}(A_L)$. Clearly $\underline{A}'$ is locally defined $g(d-1)$ equations as a subscheme of $\underline{R}'$, where $g = \dim(A)$, $d = [L : K]$. Moreover $\underline{A}'$ contains $\underline{A}^0 \times_{\mathrm{Spec}\ \mathcal{O}} \mathrm{Spec}\ \mathcal{O}_L$.

Let $\pi^h\mathcal{O}$ be the intersection of the Jacobian ideal of $\underline{A}' \subset \underline{R}'$ with $\mathcal{O}$. The argument of [CYdS, §5] show that the defect of smoothness δ of $\underline{A}^0$ is at most h. Then the main argument in [CYdS, §8], which uses the approximation theorem of [El], shows that $\underline{A}^{\mathrm{NR}} \times_{\mathrm{Spec}\ \mathcal{O}} \mathrm{Spec}(\mathcal{O}/\pi^N\mathcal{O})$ is determined by the Galois descent data of A (mod level m) for any $m \geq \max(N + \delta + 2h, 3h + 1)$. Especially we can take n to be $3h$ in the statement of Theorem 7.6 ∎

(7.7) Remark Prop. 7.4 and Thm. 7.6 together says that in the situation of Thm. 7.6, a congruence

$$\mathrm{DDtw}(L/K, \Gamma, A) \equiv_{\alpha,\beta,\varrho,\upsilon,\xi} \mathrm{DDtw}(L_0/K_0, \Gamma_0, A_0) \quad (\text{mod level } N + n)$$

between the degeneration data of A and A_0 at level $N+n$ uniquely determines a congruence between the Néron models $\underline{A}^{\mathrm{NR}}$ and $\underline{A_0}^{\mathrm{NR}}$ at level N, for every $N \geq 1$.

(7.8) Proposition *Let $\mathcal{O}$ be a discrete valuation ring with perfect residue field κ. Let A be an abelian variety over the fraction field K of $\mathcal{O}$, and let $\iota : Y \hookrightarrow \widetilde{G}$ be the degeneration data for A over the completion $\widehat{K}$ of K. Suppose that in the Raynaud extension*

$$0 \to T \to \widetilde{G} \to B \to 0$$

the abelian variety B has good reduction over $\mathcal{O}_{\widehat{K}}$. Then

$$c(A, K) = c(T, K) = \frac{1}{2}\, a(X^*(T) \otimes \mathbb{Q}) = \frac{1}{4}\, a\left(\mathrm{V}_\ell(A)\right),$$

where ℓ is a prime number invertible in κ, and $a(\cdot)$ denotes the Artin conductor for linear representations of $\mathrm{Gal}(\widehat{K}^{\mathrm{sep}}/\widehat{K})$.

PROOF. From the definition one sees that the invariant $c(A, K)$ is determined by $\underline{A}^{\mathrm{NR}} \times_{\mathrm{Spec}\ \mathcal{O}} \mathrm{Spec}(\mathcal{O}/\pi^N\mathcal{O})$ for $N \gg 0$. So by Theorem 7.6 $c(A, K)$ is determined by the Galois descent data of A (mod level m) for $m \gg 0$. Hence by Proposition 7.4 $c(A, K)$ is determined by $\mathrm{DDtw}(L_j/K_j, \Gamma_j, A_j)$ (mod level m), the L/K-twisted degeneration data modulo level m for $m \gg 0$. Since B has good reduction, one can find a local field K_0 of characteristic 0 and a finite Galois extension L_0 of K_0 such that

$$\mathrm{DDtw}(L/K, \Gamma, A) \equiv_{\alpha,\beta,\varrho,\upsilon,\xi} \mathrm{DDtw}(L_0/K_0, \Gamma_0, A_0) \quad (\text{mod level } m)\,.$$

For $m \gg 0$ we get

$$c(A, K) = c(A_0, K) = c(T_0, K) = c(T, K) = \frac{1}{2}\, a(X^*(T) \otimes \mathbb{Q})$$

by Theorem 5.3. This proves the first two equalities of Prop. 7.8. The last equality in the statement of Prop. 7.8 is a consequence of the general fact that

$$a\,(\mathrm{V}_\ell(A)) = 2\,a(X^*(T)\otimes\mathbb{Q})$$

when A is uniformaized by $(\widetilde{G}, Y)$ and $\widetilde{G}$ sits in a short exact sequence $0\to T\to\widetilde{G}\to B\to 0$. ∎

§8. Some open questions

In this section we compile a list of unresolved questions.

(8.1) Question ADDITIVITY OF $c(-,K)$

Theorem 4.1 is prove under an awkward assumption: *either the local field K has characteristic 0 or the residue field κ is finite.* One would like to find a better proof so that this assumption can be replaced by: *the residue field κ is perfect.*

Here is a related question. Even under the more restrictive "awkward assumption" above, it is not yet known whether the base change conductor $c(G,K)$ is "*additive*" for short exact sequence of semiabelian varieties, under the "awkward assumption" above. Abelian varieties with potentially good reduction would be the first "test case" to be considered.

One can reformulate this additivity question for the base change conductor as an exactness question for the volume form of the Néron models, as follows. Let

$$0\to G_1\to G_2\to G_3\to 0$$

be a short exact sequence of semiabelian varieties over the local field K. As before let $\underline{\omega_{\underline{G_i}^{\mathrm{NR}}}}$ be the rank-one free $\mathcal{O}_K$-module of translation-invariant differential forms of top-degree on the Néron model $\underline{G_i}^{\mathrm{NR}}$ of G_i, and let $\underline{\omega}_{G_i} = \underline{\omega}_{\underline{G_i}^{\mathrm{NR}}}\otimes_{\mathcal{O}_K} K$, $i=1,2,3$. From the short exact sequence above we obtain a natural isomorphism $\beta : \underline{\omega}_{G_1}\otimes_K\underline{\omega}_{G_3}\xrightarrow{\sim}\underline{\omega}_{G_2}$. The statement that $c(G_2)=c(G_1)+c(G_3)$ is equivalent to the following exactness statement, that

$$\beta\left(\underline{\omega}_{\underline{G_1}^{\mathrm{NR}}}\otimes_{\mathcal{O}_K}\underline{\omega}_{\underline{G_3}^{\mathrm{NR}}}\right) = \underline{\omega}_{\underline{G_2}^{\mathrm{NR}}}\,.$$

This reformulation is implicit in the proof of Thm. 4.1 when the residue field κ is finite; in that case the exactness statement displayed above is equivalent to the statement of Prop. 4.13.

(8.2) Question DOES $c(A,K)=c(A^t,K)$ IN GENERAL?

We have seen in Thm. 6.7 that $c(A,K)=c(A^t,K)$ if $\mathrm{char}(K)=0$ and K is perfect, and one would like to know whether this is also true when $K\cong\kappa((t))$ and κ is a perfect field of characteristic $p>0$.

(8.3) Question NON-PERFECT RESIDUE FIELDS

The results in §7 on congruence of Néron models hold for all discrete valuation rings. But almost everything else we showed about the base change conductor $c(G,K)$ depends on the hypothesis that the residue field κ is perfect. Even for tori we do not know whether $c(T,K)$ is an isogeny invariant if the residue field is not perfect. It will be interesting to know to what extent the perfectness assumption on κ is really necessary.

(8.4) Question ESTIMATE $c(-,K)$

We have seen that in general the base change conductor $c(G,K)$ may change under K-isogenies. So it is impossible to have a "simple formula" for $c(G,K)$ in terms of the Artin or the Swan conductor of the ℓ-adic Tate module $\mathrm{V}_\ell(G)$ of G, where ℓ is a prime number invertible in the residue field κ. Even in the case when A is an abelian variety with potentially ordinary reduction over a local field K of characteristic 0, $c(A,K)$ is not necessarily determined by the Galois representation $\mathrm{V}_\ell(G)$; CM elliptic curves with the same CM field but different CM-types provide counter-examples. However it is desirable, if possible at all, to have *estimates* of $c(G,K)$, in terms of the Artin or the Swan conductor of $\mathrm{V}_\ell(G)$, and/or other "more familiar" numerical invariants

(8.5) Question THE REFINED INVARIANTS $c_i(-,K)$ FOR TORI

The elementary divisors $c_1,\dots,c_{\dim(G)}$ of the base change conductor are defined in 2.4. They are the exponents of the elementary divisors of the homomorphism ${\mathrm{can}_{G,L/K}}_*$: $\mathrm{Lie}\,\underline{G}^{\mathrm{NR}}\otimes\mathcal{O}_L \to \mathrm{Lie}\,\underline{G_L}^{\mathrm{NR}}$. One would like to study their behavior, and estimate them in terms of more familiar invariants.

In the case of tori the elementary divisors $c_i(T,K)$ of the base change conductor can be viewed as invariants attached to linear representations of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on free abelian groups of finite ranks. Their sum is equal to one half of the Artin conductor. They do not seem to have been studied in the literature. The following are a few facts about the $c_i(T,K)$'s. The proofs are omitted.

(a) EXAMPLE: INDUCED TORI. Let L be a finite separable totally ramified extension of K of degree d. We assume that the residue field of K is perfect. Let π_L be a uniformizing element of $\mathcal{O}_L$. let $T=\mathrm{Res}_{L/K}\mathbb{G}_m$. Let $\sigma_1,\dots,\sigma_d$ be the K-linear embeddings of L into K^{sep}. Denote by ord_K the valuation on K^{sep} such that the $\mathrm{ord}_K(\pi_K)=1$. Then the elementary divisors $c_1(T,K),\dots,c_d(T,K)$ of the base change conductor are just the valuations of the elementary divisors of the $d\times d$ matrix $\big(\sigma_i(\pi_L^j)\big)_{1\le i,j\le d}$. Using the Vandermonde determinant, one can compute $c_1(T,K),\dots,c_d(T,K)$ explicitly; the answer is as follows. For $n=1,\dots,d-1$ let

$$b_n = \max_{\substack{I\subseteq\{1,\dots,d\}\\ \mathrm{I}=n+1}} \left(\sum_{\substack{j<k\\ j,k\in I}} \mathrm{ord}_K\left(\sigma_j(\pi_L)-\sigma_k(\pi_L)\right)\right)$$

Then $c_1(T,K)=0, c_2(T,K)=b_1$ and

$$c_i(T,K)=b_{i-1}-b_{i-2}, \quad i=3,\dots,d.$$

By Cor. 4.7, the invariants for $T'':=\mathrm{Res}_{L/K}(\mathbb{G}_m)/\mathbb{G}_m$ are

$$c_1(T'',K)=c_2(T,K),\ c_2(T'',K)=c_3(T,K),\dots,c_{d-1}(T'',K)=c_d(T,K).$$

On the other hand these invariants for the norm-one torus

$$T':=\mathrm{Ker}\left(\mathrm{Nm}_{L/K}:\mathrm{Res}_{L/K}(\mathbb{G}_m)\to\mathbb{G}_m\right)$$

attached to L/K seem to be more difficult to compute.

(b) The elementary divisors of the base change conductor may change under K-isogenies in general. For instance let L be a separable biquadratic extension of K. Let M_1, M_2, M_3 be the three quadratic subextension of L/K, suitably indexed so that $\mathrm{ord}_K(\mathrm{disc}_{M_1/K})\le \mathrm{ord}_K(\mathrm{disc}_{M_2/K})\le \mathrm{ord}_K(\mathrm{disc}_{M_3/K})$. Let T_i be the norm-one torus attached to M_i/K, $i=1,2,3$. Then $T_4:=T_1\times T_2\times T_3$ is K-isogenous to the quotient torus $T'':=\mathrm{Res}_{L/K}(\mathbb{G}_m)/\mathbb{G}_m$. We have $c_i(T_4,K)=\frac{1}{2}\mathrm{ord}_K(\mathrm{disc}_{M_i/K})$, $i=1,2,3$. On the other hand it is not difficult to construct biquadratic extensions L/K such that the $c_i(T'',K)$'s are different from the $\frac{1}{2}\mathrm{ord}_K(\mathrm{disc}_{M_i/K})$'s.

(c) The multiplicity of (the occurence of) 0 in the sequence $(\, c_i(T,K)\,)_{1\le i\le \dim(T)}$ is invariant under K-isogenies; this multiplicity is equal to the dimension of the largest subtorus of T which is split over the maximal unramified extension of K.

(d) Let T be a torus over K, and let T' (resp. T'') be a subtorus (resp. a quotient torus) of T over K. Then fore all $1\le i\le \dim(T')$ we have

$$\sum_{1\le j\le i} c_j(T,K)\le \sum_{1\le j\le i} c_j(T',K).$$

Similarly we have

$$\sum_{1\le j\le i} c_j(T,K)\le \sum_{1\le j\le i} c_j(T'',K)$$

for all $1\le i\le \dim(T'')$.

(e) Using the inequalities in (d) and the explicit calculation in (a) above, one obtains estimates of $\sum_{1\le j\le i} c_j(T,K)$ in terms of the breaks of the Galois group of a finite Galois splitting field of T. These estimates do not appear to be sharp.

(8.6) Question Isogeny invariance of $c(A,K)$, A potentially ordinary

It is natural to ask whether the assumption that $\mathrm{char}(K)=0$ in Thm. 6.8 is superfluous. This question and related issues will be addressed in another paper.

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