Riemann's theta formula

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There is a myriad of identities satisfied by the Riemann theta function $\theta(z;\Omega)$ and its close relatives $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z;\Omega)$. The most famous among these theta relations is a quartic relation known to Riemann, associated to a 4×4 orthogonal matrix with all entries ± 1; see 1.3. It debuted as formula (12) on p. 20 of [10], and was named *Riemann's theta formula* by Prym. In the preface of [10] Prym said that he learned of this formula from Riemann in Pisa, where he was with Riemann for several weeks in early 1865, and that he wrote down a proof following Riemann's suggestions.

For any fixed abelian variety, these theta identities give a set of *quadratic* equations which defines this abelian variety. The coefficients of these quadratic equations are theta constants, or "thetanullwerte", which vary with the abelian variety. At the same time, the Riemann theta identities give a set of *quartic* equations satisfied by the theta constants, which gives a systems of defining equations of the moduli space of abelian varieties (endowed with suitable theta level structures).

§1. Riemann's theta formula

We will first formulate a generalized Riemann theta identity, for theta functions attached to a quadratic form on a lattice.

(1.1) DEFINITION. (THETA FUNCTIONS ATTACHED TO QUADRATIC FORMS) Let Q be a $\mathbb{Q}$-valued *positive definite* symmetric bilinear form on an h-dimensional $\mathbb{Q}$-vector space $\Gamma_{\mathbb{Q}}$, where h is a positive integer. Let $\Gamma\subset\Gamma_{\mathbb{Q}}$ be a $\mathbb{Z}$-lattice in $\Gamma_{\mathbb{Q}}$, i.e. a free abelian subgroup of $\Gamma_{\mathbb{Q}}$ of rank h. Denote by $\Gamma_{\mathbb{Q}}^{\vee}$ the $\mathbb{Q}$-linear dual of $\Gamma_{\mathbb{Q}}$, and let $\Gamma^{\vee} := \{\lambda\in\Gamma_{\mathbb{Q}}^{\vee}\,|\,\lambda(\Gamma)\subset\mathbb{Z}.\}$. We identify elements of $\mathbb{Q}^g\otimes_{\mathbb{Q}}\Gamma_{\mathbb{Q}}$ with g-tuples of elements of $\Gamma_{\mathbb{Q}}$ and similarly for $\mathbb{Q}^g\otimes_{\mathbb{Q}}\Gamma_{\mathbb{Q}}^{\vee}$.

(i) The pairing $\mathbb{Q}^g\times\mathbb{C}^g\ni(n,z)\mapsto {}^t n\cdot z\in\mathbb{C}$ on $\mathbb{Q}^g\times\mathbb{C}^g$ and the natural pairing $\Gamma_{\mathbb{Q}}\times\Gamma_{\mathbb{Q}}^{\vee}\to\mathbb{Q}$ induces a pairing $\langle\ ,\ \rangle : (\mathbb{Q}^g\otimes_{\mathbb{Q}}\Gamma_{\mathbb{Q}})\times(\mathbb{C}^g\otimes_{\mathbb{Q}}\Gamma_{\mathbb{Q}}^{\vee}\to\mathbb{C}$.

(ii) Let $\tilde{Q} : (\mathbb{Q}^g\otimes\Gamma_{\mathbb{Q}})\times(\mathbb{Q}^g\otimes\Gamma_{\mathbb{Q}})\to \mathrm{M}_g(\mathbb{Q})$ be the matrix-valued symmetric bilinear pairing

$$\tilde{Q} : (u,v) = ((u_1,\dots,u_g),(v_1,\dots,v_g)) \to \tilde{Q}(u,v) = \big(Q(u_i,v_j)\big)_{1\le i,j\le g} \quad \forall u,v\in\mathbb{Q}^g\otimes_{\mathbb{Q}}\Gamma.$$

(iii) For every $A\in\mathbb{Q}^g\otimes_{\mathbb{Q}}\Gamma_{\mathbb{Q}}$, every $B\in\mathbb{Q}^g\otimes_{\mathbb{Q}}\Gamma_{\mathbb{Q}}^{\vee}$ and every element $\Omega\in\mathfrak{H}_g$ of the Siegel upper-half space of genus g, define the theta function $\theta^{Q,\Gamma}\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right]$ on the gh-dimensional $\mathbb{C}$-vector space

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$\mathbb{C}\otimes_{\mathbb{Q}}\Gamma_{\mathbb{Q}}^{\vee}$ attached to (Q,Γ) by

$$\theta^{Q,\Gamma}\left[\begin{smallmatrix}A\\B\end{smallmatrix}\right](Z;\Omega) := \sum_{N\in\mathbb{Z}^g\otimes_{\mathbb{Z}}\Gamma} \mathbf{e}\big(\tfrac{1}{2}\operatorname{Tr}(\Omega\cdot\tilde{Q}(N+A,N+A)\big)\cdot\mathbf{e}(\langle N+A,Z+B\rangle),$$

where $\mathbf{e}(z) := \exp(2\pi\sqrt{-1}\,z)$ for all $z\in\mathbb{C}$.

Note that we have $\theta^{Q\oplus Q',\Gamma\oplus\Gamma'}\left[\begin{smallmatrix}(A,A')\\(B,B')\end{smallmatrix}\right]((Z,Z');\Omega) = \theta^{Q,\Gamma}\left[\begin{smallmatrix}A\\B\end{smallmatrix}\right](Z;\Omega)\cdot\theta^{Q',\Gamma'}\left[\begin{smallmatrix}A'\\B'\end{smallmatrix}\right](Z';\Omega)$ for the orthogonal direct sum $(Q\oplus Q',\Gamma\oplus\Gamma')$ of (Q,Γ) and (Q',Γ'). In particular if (Q,Γ) is the orthogonal direct sum of h one-dimensional quadratic forms, then $\theta^{Q,\Gamma}$ is a product of h "usual" theta functions with characteristics.

Let (Q,Γ) be a $\mathbb{Q}$-valued positive definite quadratic form. Let $T: L_{\mathbb{Q}}\to\Gamma_{\mathbb{Q}}$ be a $\mathbb{Q}$-linear isomorphism of vector spaces over $\mathbb{Q}$, and let L be a $\mathbb{Z}$-lattice in $L_{\mathbb{Q}}$. Let $T^{\vee}:\Gamma_{\mathbb{Q}}^{\vee}\to L_{\mathbb{Q}}$ be the $\mathbb{Q}$-linear dual of T. Let $Q': L_{\mathbb{Q}}\times L_{\mathbb{Q}}\to\mathbb{Q}$ be the positive definite quadratic form on $L_{\mathbb{Q}}$ induced by Q through the isomorphism T. Let $1\otimes T:\mathbb{Q}^g\otimes_{\mathbb{Q}}L_{\mathbb{Q}}\to\Gamma_{\mathbb{Q}}$ be the linear map induced by T; similarly for $1\otimes T^{\vee}:\mathbb{C}^g\otimes_{\mathbb{Q}}\Gamma_{\mathbb{Q}}^{\vee}\to\mathbb{C}^g\otimes_{\mathbb{Q}}L_{\mathbb{Q}}^{\vee}$. Let

$$\begin{aligned} K &= (1\otimes T)(\mathbb{Z}^g\otimes_{\mathbb{Z}}L)/\big((\mathbb{Z}^g\otimes_{\mathbb{Z}}\Gamma)\cap(1\otimes T)(\mathbb{Z}^g\otimes_{\mathbb{Z}}L)\big) \xrightarrow{\sim} \big[(1\otimes T)(\mathbb{Z}^g\otimes_{\mathbb{Z}}L)+(\mathbb{Z}^g\otimes_{\mathbb{Z}}L)\big]/\mathbb{Z}^g\otimes_{\mathbb{Z}}L \\ \Delta &= (1\otimes T^{\vee})^{-1}(\mathbb{Z}^g\otimes_{\mathbb{Z}}L^{\vee})/\big((\mathbb{Z}^g\otimes_{\mathbb{Z}}\Gamma^{\vee})\cap(1\otimes T^{\vee})^{-1}(\mathbb{Z}^g\otimes_{\mathbb{Z}}L^{\vee})\big) \xrightarrow{\sim} \big[(1\otimes T^{\vee})^{-1}(\mathbb{Z}^g\otimes_{\mathbb{Z}}L^{\vee})+(\mathbb{Z}^g\otimes_{\mathbb{Z}}L^{\vee})\big]/\mathbb{Z}^g\otimes_{\mathbb{Z}}L^{\vee} \end{aligned}$$

(1.2) THEOREM. (GENERALIZED RIEMANN THETA IDENTITY) *For every $A\in\mathbb{Q}^g\otimes\Gamma_{\mathbb{Q}}$ and every $B\in\mathbb{Q}^g\otimes\Gamma_{\mathbb{Q}}^{\vee}$, the equality*

$$(\mathrm{R}_{\mathrm{ch}}^{Q,T})\qquad \theta^{Q',L}\left[\begin{smallmatrix}(1\otimes T)^{-1}A\\(1\otimes T^{\vee})B\end{smallmatrix}\right]((1\otimes T^{\vee})Z;\Omega) = \#(\Delta)^{-g}\cdot\sum_{A'\in K,\,B'\in\Delta}\mathbf{e}(-\langle A,B'\rangle)\cdot\theta^{Q,\Gamma}\left[\begin{smallmatrix}A+A'\\B+B'\end{smallmatrix}\right](Z;\Omega)$$

holds for all $\Omega\in\mathcal{H}_g$ and all $Z\in\mathbb{C}^g\otimes\Gamma_{\mathbb{Q}}^{\vee}$.

Note that each term on the right hand side of $(\mathrm{R}_{\mathrm{ch}}^{Q,T})$ is independent of the choice of B' in its congruence class modulo $\mathbb{Z}^g\otimes_{\mathbb{Z}}L^{\vee}$.

(1.3) Theorem 1.2 is very easy to prove once stated in that form. In two examples below (Q',L) is the diagonal quadratic form $x_1^2+\cdots+x_h^2$ on $\mathbb{Z}^h$, Γ is also $\mathbb{Z}^n$, and T is (given by) a matrix such that $T\cdot{}^tT$ is a multiple of the identity matrix I_h.

(a) When $h=4$, Q and Q' are both the diagonal quadratic form $x_1^2+x_2^2+x_3^2+x_4^2$ on $\mathbb{Z}^4$, $A=B=0$ and T is given by the orthogonal matrix $\begin{pmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{pmatrix}$, the equation $(\mathrm{R}_{\mathrm{ch}}^{Q,T})$ is the classical **Riemann's theta formula** [10, p. 20].

(b) When $h=2$, $\Gamma=L=\mathbb{Z}^2$, $B=0$, Q' is $x_1^2+x_2^2$, Q is $2x_1^2+2x_2^2$, T is given by the matrix $\frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$, and $(\mathrm{R}_{\mathrm{ch}}^{Q,T})$ becomes

$$\theta\begin{bmatrix}a\\0\end{bmatrix}(z,2\Omega)\cdot\theta\begin{bmatrix}b\\0\end{bmatrix}(w,2\Omega) = 2^{-g}\cdot\sum_{c\in 2^{-1}\mathbb{Z}^g/\mathbb{Z}^g}\theta\begin{bmatrix}c+(a+b)/2\\0\end{bmatrix}(z+w,2\Omega)\cdot\theta\begin{bmatrix}c+(a-b)/2\\0\end{bmatrix}(z-w,2\Omega)$$

for all $z,w\in\mathbb{C}^g$ and all $a,b\in\mathbb{Q}^g$.

Much more about theta identities can be found in [8, Ch. II §6], [9, §6], [2, Ch. IV §1], and classical sources such as [4, Ch. VII §1], [5] and [10].

§2. Theta relations

(2.1) Notation. Let $d_1, d_2, \ldots, d_g$ be positive integers such that $4|d_1|d_2|\cdots|d_g$, fixed in this section.

- Let $\delta = (d_1, \ldots, d_g)$. For any positive integer n, let

$$K(n\delta) := \oplus_{j=1}^{g} n^{-1} d_i^{-1}\mathbb{Z}/\mathbb{Z}.$$

- For any positive integer m, let

$$K_m := \oplus_{j=1}^{g} m^{-1}\mathbb{Z}/\mathbb{Z}, \quad \text{and let} \quad K_m^* := \mathrm{Hom}(K_m, \mathbb{C}^*).$$

- For any non-negative integer n and any $a \in K(2^n\delta)$, we will use the following notation

$$q_n(a) = q_{n,\delta}(a) := \theta[\begin{smallmatrix}-a\\0\end{smallmatrix}](0, 2^n\Omega), \qquad Q_n(a) = Q_{n,\delta}(a) := \theta[\begin{smallmatrix}-a\\0\end{smallmatrix}](2^n z, 2^n\Omega)$$

  for theta constants and theta functions, where $\Omega \in \mathcal{H}_g$ has been suppressed.

Clearly the following symmetry condition holds:

$$(\Theta_{\mathrm{ev}}) \qquad q_n(a) = q_n(-a) \qquad \forall a \in K(2^n\delta).$$

(2.2) The generalized Riemann theta identities implies a whole family of relations between theta functions and theta constants. Among them are the quadratic relations $(\Theta_{\mathrm{quad}}^{n,\delta})$ between theta functions with theta constants as coefficients below, as well as the quartic relations $(\Theta_{\mathrm{quad}}^{n,\delta})$ between theta constants, for all $n \geq 0$. The theta constants satisfy strong non-degeneracy conditions, represented by $(\Theta_{\mathrm{nv}}^{n+1,\delta})$ below.

(2.2.1) For any $a, b, c \in K(2n\delta)$ satisfying $a \equiv b \equiv c \pmod{K(n\delta)}$ and any $l \in K_2^*$, we have

$$(\Theta_{\mathrm{quad}}^{n,\delta}) \qquad \begin{aligned} 0 = &\Big[\sum_{\eta \in K_2} l(\eta)\cdot q_{n+1}(c+r)\Big]\cdot\Big[\sum_{\eta\in K_2} l(\eta)\cdot Q_n(a+b+\eta)\cdot Q_n(a-b+r)\Big] \\ &-\Big[\sum_{\eta \in K_2} l(\eta)\cdot q_{n+1}(b+r)\Big]\cdot\Big[\sum_{\eta\in K_2} l(\eta)\cdot Q_n(a+c+\eta)\cdot Q_n(a-c+r)\Big]. \end{aligned}$$

(2.2.2) For any $a, b, c, d \in K(2n\delta)$ satisfying $a \equiv b \equiv c \equiv d \pmod{K(n\delta)}$ and any $l \in K_2^*$, we have

$$(\Theta_{\mathrm{quar}}^{n,\delta}) \qquad \begin{aligned} 0 = &\Big[\sum_{\eta \in K_2} l(\eta)\cdot q_n(a+b+\eta)\, q_n(a-b+\eta)\Big]\cdot\Big[\sum_{\eta\in K_2} l(\eta)\cdot q_n(c+d+\eta)\, q_n(c-d+\eta)\Big] \\ &-\Big[\sum_{\eta \in K_2} l(\eta)\cdot q_n(a+d+\eta)\, q_n(a-d+\eta)\Big]\cdot\Big[\sum_{\eta\in K_2} l(\eta)\cdot q_n(b+c+\eta)\, q_n(b-d+\eta)\Big]. \end{aligned}$$

(2.2.3) For any $n \geq 0$, any $a_1, a_2, a_3 \in K(2^{n+1}\delta)$ and any $l_1, l_2, l_3 \in K_4^*$, there exists an element $b \in K_8 \subset K(2\delta)$ and an element $\lambda \in 2K_4^*$ such that

$$(\Theta_{\text{nondeg}}^{n+1,\delta}) \qquad 0 \neq \prod_{i=1}^{3} \Big[\sum_{\eta \in K_4} (l_i + \lambda)(\eta) \cdot q_{n+1}(a_i + b + \eta) \Big].$$

(2.3) Equations defining abelian varieties. The geometric significance of these theta relations are two folds, for abelian varieties and also their moduli. Recall that $d_1, \ldots, d_n$ are positive integers with $d_1 | \cdots | d_g$. Let $N = (\prod_{i=1}^{g} d_i) - 1$.

1. Let $A_{\Omega,\delta} := \mathbb{C}^g/(\Omega \cdot \mathbb{Z}^g + D(\delta) \cdot \mathbb{Z}^g$ be the abelian variety whose period lattice is generated by the columns of $(\Omega\, D)$, where $D(\delta)$ is the diagonal matrix with $d_1, \ldots, d_g$ as its diagonal entries. It was proved by Lefschetz that the theta functions $\big\{\{Q_{0,\delta}(a) \,|\, a \in K(\delta)\big\}$ define a projective embedding $j : A_{\Omega,\delta} \hookrightarrow \mathbb{P}^N$ if $4|d_1|\cdots|d_g$. Mumford showed in [6, I §4] that

   > *the quadratic equations $(\Theta_{\text{quar}}^{0,\delta})$ in the projective coordinates of $\mathbb{P}^N$ cuts out the abelian variety $A_{\Omega,\delta}$ as a subvariety inside $\mathbb{P}^N$ if $4|d_1$.*

   In particular an abelian variety is determined by their theta constants $q_{0,\delta}(a)$'s if the *level* δ is divisible by 4. The group law on the abelian variety can also be recovered from these theta constants.

2. The next question is: do the Riemann quartic equations $(\Theta_{\text{quar}}^{n,\delta})$ on the theta constants cut out the moduli of abelian varieties? The answer given in [6, II §6] is basically "*yes if* $8|d_1$" with a suitable non-degeneracy condition:

   Suppose that $8|d_1|\cdots|d_g$, and $\{q(a) \,|\, a \in K(\delta)\}$ is a family of complex numbers indexed by $K(\delta)$. Assume that this given $\prod_{i=1}^{g} d_i)$-tuple of numbers has the following properties.

   - $q(a) = q(-a)$ for all $a \in K(\delta)$.
   - *All quartic equations in $(\Theta_{\text{quar}}^{0,\delta})$ hold.*
   - *All quartic equations in $(\Theta_{\text{quar}}^{0,\delta})$ hold.*
   - (The non-degeneracy condition) *There exists a family of complex numbers $\{q_1(u) \,|\, u \in K(\delta)\}$ indexed by $K(2\delta)$ which satisfies $q_1(u) = q(-u)$ for all $u \in K(2\delta)$, the quartic equations $(\Theta_{\text{quar}}^{1,\delta})$, the condition $(\Theta_{\text{nondeg}}^{1,\delta})$, and*

     $$q_0(a) \cdot q_0(b) = \sum_{u,v \in K(2\delta),\, u+v=a,\, u-v=b} q_1(u) \cdot q_1(v) \qquad \forall a, b \in K(\delta).$$

   Then there exists an element $\Omega \in \mathfrak{H}_g$ such that

   $$q_{0,\delta}(a) := \theta\big[\begin{smallmatrix} -a \\ 0 \end{smallmatrix}\big](0, \Omega) \quad \forall\, a \in K(\delta).$$

(2.4) Adelic Heisenberg groups and theta measures. The analysis in [6] of theta relations is based on a finite adelic version of the Heisenberg group, which is a central extension of $\mathbb{A}_f^{2g}$ by the multiplicative group scheme $\mathbb{G}_m$ over a base field k in which 2 is invertible. Here $\mathbb{A}_f = \prod_p' \mathbb{Q}_p$, the restricted product of p-adic numbers, over all primes p which are invertible in k. One gets such a group scheme $\widehat{\mathcal{G}}(\mathscr{L})$ whenever one is handed a symmetric ample line bundle of degree one over a g-dimensional *principally polarized* abelian variety over k, plus a compatible family of theta structure for torsion points of order invertible in k, which induces an isomorphism from $\mathbb{A}_f^{2g}$ to the set of all torsion points as above.

Such a Heisenberg group $\widehat{\mathcal{G}}(\mathscr{L})$ is isomorphic to "the standard finite adelic Heinsenberg group" $\mathrm{Heis}(2g,\mathbb{A}_f)$ over the algebraic closure k^{alg} of k. The definition of $\mathrm{Heis}(2g,\mathbb{A}_f)$ is similar to that of the real Heisenberg group $\mathrm{Heis}(2g,\mathbb{R})$ in [1, §2.1], with the following changes: (a) the field $\mathbb{R}$ is replaced by the ring $\mathbb{A}_f$, (b) the unit circle group $\mathbb{C}_1^\times$ is replaced by the multiplicative group scheme $\mathbb{G}_m$ over k^{alg}, and (c) the isomorphism $\mathbf{e}:\mathbb{R}/\mathbb{Z}\xrightarrow{\sim}\mathbb{C}_1^\times$ is replaced by an isomorphism from $\mathbb{A}_f/\hat{\mathbb{Z}}\cong\oplus_p\mathbb{Q}_p/\mathbb{Z}_p$ to the group $\mu_\infty(k^{\mathrm{alg}})$ of all roots of 1 in k^{alg}.

The group $\widehat{\mathcal{G}}(\mathscr{L})$ acts on the direct limit $\varinjlim_n \Gamma(A,\mathscr{L}^{\otimes n}) =: \widehat{\Gamma}(\mathscr{L})$, where n runs through all positive integers which are invertible in k. This action of $\widehat{\mathcal{G}}(\mathscr{L})$ on $\widehat{\Gamma}(\mathscr{L})$ is an $\mathbb{A}_f$-version of the dual Schrödinger representation discussed in [1, §2.3] At this point the representation-theoretic formalism for theta functions discssed in [1, §2] carry over to th present situation.

The insight gained from the systematic use of the adelic Schrödinger representation produces not only the two theorems in (2.3), but also a new way to look at theta constants: There exists a *measure* μ on $\mathbb{A}_f^g$ which satisfies the properties (i)–(iv) below. All theta relations are encoded in the simple equality in (i), and the non-degeneracy condition becomes (iv).

(i) There exists another measure ν on $\mathbb{A}_f^g$ such that $\xi_*(\mu\times\mu)=\nu\times\nu$ as measures on $\mathbb{A}_f^g\times\mathbb{A}_f^g$, where $\xi:\mathbb{A}_f^g\times\mathbb{A}_f^g\to\mathbb{A}_f^g\times\mathbb{A}_f^g$ is the map $(x,y)\mapsto(x+y,x-y)$.

(ii) The algebraic theta constants are integrals over suitable compact open subsets of $\mathbb{A}_f^g$.

(iii) The algebraic theta function attached to a non-zero global section s_0 of the one-dimensional vector space $\Gamma(A,\mathscr{L})$ as above, is the function

$$x\mapsto\theta^{\mathrm{alg}}(x)=\int(U_{(1,-x)}\cdot\delta_0)\,d\mu$$

on $\mathbb{A}_f^{2g}$, where δ_0 is the characteristic function for the compact open subset $\prod_p'\mathbb{Z}_p\subset\mathbb{A}_f^{2g}$, and $U_{(1,-x)}\cdot\delta_0$ is the result of the action on δ_0 by the the element $U_{(1,-x)}\in\mathrm{Heis}(2g,\mathbb{A}_f)$ under the dual Schrödinger representation.

(iv) For every $x\in\mathbb{A}_f^{2g}$, there exists an element $\eta\in\frac{1}{2}\prod_p'\mathbb{Z}_p$ such that $\theta^{\mathrm{alg}}(x+\eta)\neq0$.

When the base field is $\mathbb{C}$, the theta measure μ_Ω attached to $\Omega\in\mathfrak{H}_g$ is

$$\mu_\Omega(V)=\sum_{n\in V\cap\mathbb{Q}^g}\mathbf{e}\big(\tfrac{1}{2}\cdot{}^tn\cdot\Omega\cdot n\big)$$

for all compact open subset $V\subset\mathbb{A}_f^g$, and the companion measure ν_Ω is

$$\nu_\Omega(V)=\sum_{n\in V\cap\mathbb{Q}^g}\mathbf{e}\big(\tfrac{1}{2}\cdot{}^tn\cdot\Omega\cdot n\big).$$

(2.5) The best introduction to the circle of ideas in this section [9]; the readers may also consult [2, Ch. IV] and the original papers [6].

Anyone who had more than a casual look at the papers [6] would agree that the results are both fundamental and deep, opening up a completely new direction in the study of theta functions. These paper are "however, not easy to read", and the ideas in them "have not been developed very far".[1] It is indeed curious that there has been no "killer application" of the theory of algebraic theta functions so far. However it should be a safe bet that this anomaly won't last much longer.

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[1] The quotes Mumford's words in the preface of [9].