The period matrices and theta functions of Riemann

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This article deals with four notions due to Riemann: (A) Riemann bilinear relations, (B) Riemann forms, (C) Riemann theta functions and (D) Riemann's theta formula, in four parts. Following the original instruction from the editors, a short explanation is given for each concept, with complete definitions and key theorems. The goal was to provide a short and quick exposition of these concepts to students and non-experts, plus some historical information to help the readers appreciate a small portion of Riemann's monumental contributions.

That instruction with clear perimeters persuaded me to accept the assignment, despite the apprehensions due to my limited perspectives in these topics, and my ignorance in the history of mathematics in and before the nineteenth centuray. As a result the present article is more like four entries in [9] instead of an essay in the style found in Book reviews of the Bulletin of the American Mathematical Society, which may be more attractive in several aspects.

Although the editorial policy of this volume has changed somewhat, I have not attempted anything more ambitious, except that the four short articles have been consolidated into one, as suggestion by the referee. The careful reading of the referee saved me from many embarrassing errors, for which I am truly thankful.

Part A. Riemann bilinear relations

The **Riemann bilinear relations**, also called the **Riemann period relations**, are quadratic relations for **period matrices**. The ones considered by Riemann are of two sorts: (a) periods of holomorphic one-forms on a compact Riemann surface, and (b) periods of holomorphic one-forms on an abelian variety.

§1. Period relations for abelian integrals of the first kind

The statements (1.2 a) and (1.2 b) in Theorem 1.2 are the *Riemann bilinear relations* for the period integrals of differentials of the first kind on a a compact Riemann surface.

(1.1) Notation and terminology

- Let S be a compact connected Riemann surface of genus $g \geq 1$.
- Let $\omega_1, \ldots, \omega_g$ be a $\mathbb{C}$-basis of the space $\Gamma(S, K_S)$ of holomorphic differential one-forms on S.

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- Let $\gamma_1, \dots, \gamma_{2g}$ be a $\mathbb{Z}$-basis of the first Betti homology group $H_1(S,\mathbb{Z})$.
- Let $J = J_{2g}$ be the $2g \times 2g$ matrix $\begin{pmatrix} 0_g & I_g \\ -I_g & 0_g \end{pmatrix}$, where I_g is the $g \times g$ identity matrix, and $0_g = 0 \cdot I_g$.
- For any i, j with $1 \le i, j \le 2g$, let $\Delta_{ij} = \gamma_i \frown \gamma_j \in \mathbb{Z}$ be the intersection product of γ_i with γ_j. Let $\Delta = \Delta(\gamma_1, \dots, \gamma_{2g})$ be the $2g \times 2g$ skew-symmetric matrix with entries Δ_{ij}.
- $\gamma_1, \dots, \gamma_{2g}$ is said to be a *canonical basis* of $H_1(S,Z)$ if $\Delta(\gamma_1, \dots, \gamma_{2g}) = J_{2g}$.
- It is well-known that $H_1(S,Z)$ admits a canonical basis. In other words there exists an element $C \in \mathrm{GL}_{2g}(\mathbb{Z})$ such that ${}^tC \cdot \Delta \cdot C = J_{2g}$.
- The $g \times 2g$ matrix $P = P(\omega_1, \dots, \omega_g; \gamma_1, \dots, \gamma_{2g})$ whose (r,i)-th entry is $\int_{\gamma_i} \omega_r$ for every $r = 1, \dots, g$ and every $i = 1, \dots, 2g$ is called the **period matrix** defined by the one-cycles $\gamma_1, \dots, \gamma_{2g}$ and the holomorphic one-forms $\omega_1, \dots, \omega_g$.

(1.2) Theorem. *Let $P = P(\omega_1, \dots, \omega_g; \gamma_1, \dots, \gamma_{2g})$ be the period matrix for a $\mathbb{C}$-basis of the space $\Gamma(S, K_S)$ of holomorphic one-forms on S and a $\mathbb{Z}$-basis $\gamma_1, \dots, \gamma_{2g}$ of $H_1(S,\mathbb{Z})$. We have*

$$P \cdot \Delta(\gamma_1, \dots, \gamma_{2g})^{-1} \cdot {}^tP = 0_g \tag{1.2 a}$$

and

$$-\sqrt{-1} \cdot P \cdot \Delta(\gamma_1, \dots, \gamma_{2g})^{-1} \cdot {}^t\overline{P} > 0_g \tag{1.2 b}$$

in the sense that $-\sqrt{-1} \cdot P \cdot \Delta(\gamma_1, \dots, \gamma_{2g})^{-1} \cdot {}^t\overline{P}$ is a $g \times g$ hermitian positive definite *matrix.*

Note that the validity of the statements (1.2 a) and (1.2 b) is independent of the choice of the $\mathbb{Z}$-basis $\gamma_1, \dots, \gamma_{2g}$ of $H_1(S,\mathbb{R})$: $\Delta(\gamma_1, \dots, \gamma_{2g})^{-1}$ is the skew-symmetric real matrix $\left(\gamma_i^\vee \frown \gamma_j^\vee\right)_{1 \le i,j \le 2g}$, where the intersection pairing $\frown$ on $H_1(S,\mathbb{Z})$ has been $\mathbb{R}$-linearly extended to $H_1(S,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$, and $\gamma_1^\vee, \dots, \gamma_{2g}^\vee \in H_1(S,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ are characterized by $\gamma_k^\vee \frown \gamma_j = \delta_{kj}$ for all $j, k = 1, \dots, 2g$.

(1.3) Corollary. *Suppose that $\Delta(\gamma_1, \dots, \gamma_{2g}) = J_{2g}$. Write the period matrix P in block form as $P = (P_1\ P_2)$, where P_1 (respectively P_2) is the $g \times g$ matrix whose entries are period integrals of $\omega_1, \dots, \omega_g$ with respect to $\gamma_1, \dots, \gamma_g$ (respectively $\gamma_{g+1}, \dots, \gamma_{2g}$).*

(i) *The $g \times g$ matrix P_1 is* non-singular *i.e.* $\det(P_1) \neq 0$.

(ii) *Let $\Omega := P_1^{-1} \cdot P_2$. Then Ω is a* symmetric *$g \times g$ matrix and its imaginary part* $\mathrm{Im}(\Omega)$ *is* positive definite.

(1.4) The basic idea for the proof of Theorem 1.2 is as follows. First one "cuts open" the Riemann surface S along $2g$ oriented simple closed paths $C_1,\dots,C_{2g}$ in S with a common base point so that the properties (a)–(d) below hold.

(a) For any pair $i \neq j$, C_i meets C_j only at the base point.

(b) The image of $C_1,\dots,C_{2g}$ in $H_1(S,\mathbb{Z})$ is a canonical basis $\gamma_1,\dots,\gamma_{2g}$ of $H_1(S,\mathbb{Z})$.

(c) The "remaining part" $S \smallsetminus (C_1 \cup \cdots \cup C_{2g})$ is 2-cell S_0.

(d) The boundary ∂S_0 of S_0 (in the sense of homotopy theory) consists of

$$C_1, C_{g+1}, C_1^{-1}, C_{g+1}^{-1}, C_2, C_{g+2}, C_2^{-1}, C_{g+2}^{-1}, \dots, C_g, C_{2g}, C_g^{-1}, C_{2g}^{-1}$$

oriented cyclically.

For any non-zero holomorphic one-form ω on S, there exists a holomorphic function f on the simply connected domain S_0 such that $df = \omega$. Then for every holomorphic one-form η on S, we have

$$\int_{\partial S_0} f \cdot \eta = \int_{S_0} d(f \cdot \eta) = 0$$

and also

$$-\sqrt{-1} \cdot \int_{\partial S_0} \bar{f} \cdot \omega = -\sqrt{-1} \cdot \int_{S_0} d\bar{f} \wedge df > 0,$$

by Green's theorem. The bilinear relations (1.2) and (1.2) follow. Details of this are carried out in [26, Ch. 3 §3], [6, pp. 231–232] and [19, pp. 139–141].

As remarked by Siegel on page 113 of [26], these two bilinear relations were discovered and proved by Riemann, using the argument sketched in the previous paragraph. It is remarkable that Riemann's original proof is still the optimal one 150 years later. The readers are encouraged to consult Riemann's famous memoir [22], especially §§20–21.

§2. Riemann bilinear relations for abelian functions

The Riemann bilinear relations provide a necessary and sufficient condition for a set of $2g$ $\mathbb{R}$-linearly independent vectors in $\mathbb{C}^g$ to be the periods of g holomorphic differentials on a g-dimensional abelian variety.

(2.1) Definition. (a) An **abelian function** on a complex vector space V is a meromorphic function f on V such that there exists a lattice $\Lambda \subset V$ with the property that $f(z+\xi) = f(z)$ for all $z \in V$ and all $\xi \in \Lambda$.[1]

(b) An abelian function f on a g-dimensional vector space V over $\mathbb{C}$ is **degenerate** if its period group

$$\mathrm{Periods}(f) := \{\, \eta \in V \mid f(z+\eta) = f(z) \ \ \forall z \in V \,\}$$

is *not* a lattice in V. (Then $\mathrm{Periods}(f)$ contains a positive dimensional $\mathbb{R}$-vector subspace of V, and in fact also a positive dimensional $\mathbb{C}$-vector subspace of V.)

[1] Recall that a *lattice* in a finite dimensional vector space V over $\mathbb{C}$ is a discrete free abelian subgroup of V rank $2\dim_{\mathbb{C}}(V)$, equivalently V/Λ is a compact torus.

(2.2) DEFINITION. Let $g \geq 1$ be a positive integer.

(a) A $g \times 2g$ matrix Q with entries in $\mathbb{C}$ is a **Riemann matrix** if there exists a skew symmetric integral $2g\times 2g$ matrix E with $\det(E) \neq 0$ satisfying the two conditions (2.2 a), (2.2 b) below.

$$Q \cdot E^{-1} \cdot {}^tQ = 0_g \tag{2.2 a}$$

and

$$\sqrt{-1} \cdot Q \cdot E^{-1} \cdot {}^t\overline{Q} > 0_g. \tag{2.2 b}$$

Such an integral matrix E is called a *principal part* of Q.

(b) The **Siegel upper-half space** $\mathfrak{H}_g$ of genus g is the set of all symmetric $g\times g$ complex matrix Ω such that $(\Omega\ I_g)$ is a Riemann matrix with principal part $\begin{pmatrix} 0_g & I_g \\ -I_g & 0_g \end{pmatrix}$, or equivalently Ω is symmetric and the imaginary part $\mathrm{Im}(\Omega)$ of Ω is positive definite.[2]

(2.3) THEOREM. *Let Q be a $g\times 2g$ matrix with entries in $\mathbb{C}^g$ such that the subgroup Λ of $\mathbb{C}^g$ generated by the $2g$ columns of Q is a lattice in $\mathbb{C}^g$. There exists a non-degenerate abelian function f on $\mathbb{C}^g$ whose period group is equal to Λ if and only if Q is a Riemann period matrix.*

A proof of theorem 2.3 can be found in [17, Ch. 1] and also in [6, Ch. 2 §6]. For a classical treatment of theorem 2.3, chapter 5 §§ 9–11 of [27] is highly recommended.

Recall that an *abelian variety* over $\mathbb{C}$ is a complex projective variety with a an algebraic group law, or equivalently a compact complex torus which admits an holomorphic embedding to a complex projective space. It is a basic fact that the existence of a non-degenerate abelian function on $\mathbb{C}^g$ with respect to the lattice $Q\cdot\mathbb{Z}^{2g} \subset \mathbb{C}^g$ is equivalent to the statement that the quotient $\mathbb{C}^g/(Q\cdot\mathbb{Z}^{2g})$ of $\mathbb{C}^g$ by the lattice Q is an abelian variety. So an equivalent statement of Theorem 2.3 is:

(2.3.1) THEOREM. *A compact complex torus of the form $\mathbb{C}^g/(Q\cdot\mathbb{Z}^{2g})$ for a $g\times(2g)$ complex matrix Q is an abelian variety if and only if Q is a Riemann period matrix.*

It is easy to see that g-dimensional compact complex tori vary in a g^2-dimensional analytic family. Theorem 2.3.1 says that the elements of the set of all g-dimensional abelian varieties vary in a countable union of $g(g+1)/2$-dimensional analytic families. More precisely the set of all g-dimensional abelian varieties with a fixed principal part E is parametrized by the quotient of the Siegel upper-half space $\mathfrak{H}_g$ under the action of a discrete group of $\mathrm{Sp}_{2g}(\mathbb{R})$ with finite stabilizer subgroups.

(2.4) Historical Remarks. The statement of Theorem 2.3 did not appear in Riemann's published papers, but Riemann was aware of it. On page 75 of [27] Siegel wrote:

> *Riemann was the first to recognize that the period relations are necessary and sufficient for the existence of non-degenerate abelian functions. However, his formulation was incomplete and he did not supply a proof. Later, Weierstrass also failed to establish a complete proof despite his many efforts in this direction. Complete proofs were finally attained by Appell for the case $g = 2$ and by Poincaré for arbitrary g.*

[2]Elements of $\mathfrak{H}_g$ are also called "Riemann matrices" by some authors. We do not do so here.

Krazer's comments on page 120 of [13] are similar but more polite. He also said that Riemann communicated his discovery to Hermite in 1860, citing (the German translation of) [7]. One can feel the excitement brought by Riemann's letter[3] in Hermite's exposition of Riemann's "extremely remarkable discovery" of the symmetry conditions on a period lattice $\Lambda \subset \mathbb{C}^n$, necessary for the existence of (non-degenerate) abelian functions, see pages 148–150 of [7]. On page 120 of [13] one finds references to subsequent works on abelian integrals and abelian functions during the last two decades of the nineteenth century, by Weierstrass, Hurwitz, Poincaré, Picard, Appell and Frobenius. Riemann's ideas were developed in these papers, and rigorous proofs were given to his assertions for the first time in some cases, c.f. the first paragraph of Remark 5.6. It took forty years for these original ideas of Riemann to be assimilated.

(2.4.1) Let S be a Riemann surface, let $\omega_1,\ldots,\omega_g$ be a $\mathbb{C}$-basis of holomorphic differentials on S, and let $\gamma_1,\ldots,\gamma_{2g}$ be a $\mathbb{Z}$-basis of the first homology group $H_1(S,\mathbb{Z})$. Theorem 1.2 says that $P(\omega_1,\ldots,\omega_g;\gamma_1,\ldots,\gamma_{2g})$ is a Riemann matrix with principle part $-\Delta(\gamma_1,\ldots,\gamma_{2g})$, and theorem 2.3 tells us that the quotient of $\mathbb{C}^g$ by the lattice $P(\omega_1,\ldots,\omega_g;\gamma_1,\ldots,\gamma_{2g})\cdot\mathbb{Z}^{2g}$ is an abelian variety. This abelian variety $\mathrm{Jac}(S)$ is called the *Jacobian variety* of the Riemann surface S. Two lines of investigation open up immediately.

A. Choose and fix a base point x_0 on S. Considering abelian integrals from x_0 to a variable point $x \in S$, one get the a map

$$X \in x \mapsto \begin{pmatrix} \int_{x_0}^{x} \omega_1 \\ \vdots \\ \int_{x_0}^{x} \omega_1 \end{pmatrix} \bmod P(\omega_1,\ldots,\omega_g;\gamma_1,\ldots,\gamma_{2g})\cdot\mathbb{Z}^{2g} \in \mathrm{Jac}(S).$$

from S to $\mathrm{Jac}(S)$. Through this Abel-Jacobi map one can analyze further geometric properties of the Riemann surface S.

B. As the Riemann surface S varies in its moduli space, so does the corresponding Jacobian variety $\mathrm{Jac}(S)$. A natural question is: which abelian varieties arise this way? Can we characterize the Jacobian locus either analytically or algebraically, as a subvariety of the moduli space of abelian varieties?

The best introduction to this circle of ideas is [18], which also contains a nice "guide to the literature of references". See also [5], [6, Ch. 2 §7] and [19, Ch. 2 §§2–3] for Jacobian varieties and the Abel-Jacobi map. To get an idea on the immense landscape of modern-day research which grew out of ideas of Riemann on the moduli of curves and abelian varieties, we recommend the three-volume collection of survey articles in [4].

Part B. Riemann forms

The notion of *Riemann forms* began as a coordinate-free reformulation of the *Riemann period relations* for abelian functions discovered by Riemann. This concept has evolved with the progress in mathematics in the 150 years after Riemann. Nowadays it is often viewed from the perspective of abstract Hodge theory: a Riemann form gives an alternating pairing on a (pure) Hodge $\mathbb{Z}$-structure of Hodge type $\{(0,-1),(-1,0)\}$ and weight -1, with values in the "Tate twisted" version $\mathbb{Z}(1)$ of $\mathbb{Z}$.

[3] This letter wasn't mentioned in [7].

Along more algebraic lines, the concept of Riemann forms was extended to abelian varieties over arbitrary base fields, after Weil [28] developed the theory of abelian varieties over an arbitrary base fields. For any (co)homology theory, for instance de Rham, Hodge, étale or crystalline cohomology theory, a Riemann form for an abelian variety is an alternating pairing induced by a *polarization* of A, on the first homology group $H_1(A)$ of an abelian variety A, with values in $H_1(\mathbb{G}_m)$, the first Tate twist of the homology of a point. A synopsis for the case of étale cohomology is given in §4.

§3. From Riemann matrices to Riemann forms

A version of theorem 3.1 appeared in [7, pp. 148–150], preceded by the high praise

> "il est extrèmement remarquable est c'est à M. le Dr. Riemann, de Göttingen, qu'on doit cette découverte analytique ..."

from Hermite. This theorem is stated in terms of period matrices for abelian functions in the article on *Riemann bilinear relations*; see [27, Ch. 5,§§9–11] for a classical treatment. The definition 3.2 of Riemann forms delivers the same conditions with a better perspective. The reader may consult [17, Ch. I §§2-3] and [8, II §3, III §6] for more information.

(3.1) THEOREM. *Let Λ be a lattice in a finite dimension complex vector space V. In other words Λ is a discrete free abelian subgroup of V whose rank is equal to $2\dim_{\mathbb{C}}(V)$. The compact complex torus X of the form V/Λ is isomorphic to (the $\mathbb{C}$-points of) a complex abelian variety if and only if the pair (V,Λ) admits a* **Riemann form**.

Recall that a complex abelian variety is a *projective* irreducible algebraic variety over $\mathbb{C}$ with an algebraic group law.

(3.2) DEFINITION. Let V be a finite dimensional vector space over $\mathbb{C}$ of dimension $g \geq 1$, and let Λ be a lattice V, i.e. a discrete free abelian subgroup of V rank $2g$. A **Riemann form** for (V,Λ) is a skew symmetric $\mathbb{Z}$-bilinear map

$$\mu : \Lambda \times \Lambda \to \mathbb{Z}$$

such that the map

$$(v_1, v_2) \ \mapsto \ \mu_{\mathbb{R}}(\sqrt{-1} \cdot v_1, v_2) \quad \forall v_1, v_2 \in V$$

is a symmetric positive definite $\mathbb{R}$-bilinear form on V, where $\mu_{\mathbb{R}} : V \times V \to \mathbb{R}$ is the unique skew-symmetric $\mathbb{R}$-bilinear which extends μ. Note that the last condition implies that

$$\mu_{\mathbb{R}}(\sqrt{-1} \cdot v_1, \sqrt{-1} \cdot v_2) = \mu_{\mathbb{R}}(v_1, v_2) \quad \forall v_1, v_2 \in V.$$

REMARK. The map $H : V \times V \to \mathbb{C}$ defined by

$$H(v_1, v_1) = \mu(\sqrt{-1} \cdot v_1, v_2) + \sqrt{-1} \cdot \mu(v_1, v_2) \quad \forall v_1, v_2 \in V,$$

is a *positive definite* Hermitian form whose imaginary part is equal to $\mu_{\mathbb{R}}$.[4]

[4] Clearly μ and H determines each other. Some authors call H a Riemann form, focusing more on the Hermitian form H instead of $\mathrm{Im}(H)$.

(3.2.1) Example. Let S be a Riemann surface of genus $g \geq 1$. Regard $H_1(S,\mathbb{Z})$ as a lattice in the $\mathbb{C}$-linear dual $\Gamma(S,K_S)^\vee := \mathrm{Hom}_{\mathbb{C}}(\Gamma(S,K_S),\mathbb{C})$ of the space $\Gamma(S,K_S)$ of all holomorphic one-forms on S via the injection

$$j(\gamma)(\omega) = \int_\gamma \omega \qquad \forall \gamma \in H_1(S,K_S),\ \forall \omega \in \Gamma(S,K_S)$$

Let $\frown : H_1(S,\mathbb{Z}) \times H_1(S,\mathbb{Z}) \longrightarrow \mathbb{Z}$ be the intersection product on S. Then $(-1)\cdot \frown$, the additive inverse of the intersection product, is a Riemann form for the lattice $H_1(S,\mathbb{Z})$ in $\Gamma(S,K_S)^\vee$.[5]

(3.2.2) DEFINITION. Let $\mu : V \times V \to \mathbb{Z}$ be a Riemann form of (V,Λ). A **canonical basis** for (Λ,μ) is a $\mathbb{Z}$-basis $v_1,\ldots,v_g,v_{g+1},\ldots,v_{2g}$ of Λ such that there exist positive integers $d_1,\ldots,d_g > 0$ with $d_i | d_{i+1}$ for $i = 1,\ldots,g-1$ and

$$\mu(v_i,v_j) = \begin{cases} d_i & \text{if } j = i+g \\ -d_i & \text{if } j = i-g \\ 0 & \text{if } j-i \neq \pm g \end{cases} \qquad \forall 1 \leq i,j \leq 2g.$$

The positive integers $d_1,\ldots,d_g$ are uniquely determined by μ, called the *elementary divisors* of μ. A Riemann form of a pair (V,Λ) is **principal** if all of its elementary divisors are equal to 1.

Lemma 3.3 below gives a dictionary between the more traditional notion of Riemann matrices and the coordinate-free notion of Riemann forms. Lemma 3.3.1 shows that a Riemann form plus a choice of a *canonical basis* of Λ leads to a "normal form" for the Riemann matrix in terms of the Siegel upper-half space $\mathfrak{H}_g$. Thus the family of all Riemann forms with elementary divisors $d_1,\ldots,d_g$ on lattices in g-dimensional complex vector spaces are holomorphically parametrized by the Siegel upper-half space $\mathfrak{H}_g$ up to integral symplectic transformations in $\mathrm{Sp}_{2g}(\mathbb{Z}^{2g},J)$, where $J = J(d_1,\ldots,d_g)$ is the skew-symmetric pairing on $\mathbb{Z}^{2g}$ given by the matrix $\begin{pmatrix} 0_g & D \\ -D & 0_g \end{pmatrix}$ and D is the diagonal matrix with $d_1,\ldots,d_g$ along its diagonal.

(3.3) LEMMA. *Suppose that μ is a Riemann form for (V,Λ). Let H be the positive definite Hermitian form on V with μ as its imaginary part. Let $v_1,\ldots,v_{2g}$ be a $\mathbb{Z}$-basis of Λ. Let E be the skew symmetric $2g\times 2g$ integer matrix whose (i,j)-th entry is $\mu(v_i,v_j)$ for all $i,j = 1,\ldots,2g$. For every $\mathbb{C}$-basis $z_1,\ldots,z_g$ of V, the $g\times 2g$ matrix*

$$P = \big(z_r(v_j)\big)_{1\leq r\leq g,\ 1\leq j\leq 2g}$$

is a Riemann matrix with principal part E, i.e.

$$P \cdot E^{-1} \cdot {}^tP = 0_g, \quad \sqrt{-1}\cdot P \cdot E^{-1} \cdot {}^t\overline{P} > 0_g$$

Conversely every Riemann matrix P with a principle part E arises this way from a Riemann form on a pair (V,Λ).

[5] This annoying sign is an unfortunate consequence of the choice of sign in the definition of Riemann forms as given in 3.2. In many ways it would be more natural to require that $(v_1,v_2) \mapsto \mu(v_1,\sqrt{-1}\cdot v_2)$ be positive definite in 3.2, but a number of changes will be required if one adopts that.

(3.3.1) Lemma. *Suppose that μ is a Riemann form for (V,Λ) and $v_1,\dots,v_{2g}$ is a canonical basis for (Λ,μ) with elementary divisors $d_1|\cdots|d_g$ as in 3.2.2. Let $z_1,\dots,z_g$ be the $\mathbb{C}$-linear function on V determined by $z_r(v_{g+j})=\delta_{rj}\cdot d_j$ for all $r,j=1,\dots,g$. Then the Riemann matrix $P=\big(z_r(v_j)\big)_{1\le r\le g,\ 1\le j\le 2g}$ has the form*

$$P=(\Omega\, D)$$

for some element Ω in the Siegel upper-half space $\mathfrak{H}_g$, where D is the $g\times g$ diagonal matrix with entries $d_1,\dots,d_g$ and the matrix $\Omega=\big(\Omega_{ij}\big)_{1\le i,j\le g}$ is determined by

$$v_j=\sum_{i=1}^{g}\Omega_{ij}\cdot\frac{v_{g+i}}{d_i}\quad \text{for } \ j=1,\dots,g.$$

(3.3.2) Remark. In the context of 3.3.1, with the $\mathbb{Z}$-basis $v_1,\dots,v_{2g}$ for Λ and the $\mathbb{C}$-coordinates $z_1,\dots,z_g$ for V, the Hermitian form H on V becomes

$$(\vec{z},\vec{w})\ \mapsto\ {}^t\vec{z}\cdot\mathrm{Im}(\Omega)^{-1}\cdot\overline{\vec{w}},\qquad \text{for } \ \vec{z},\vec{w}\in\mathbb{C}^g.$$

and the matrix representation of the $\mathbb{R}$-linear endomorphism $J: v\mapsto\sqrt{-1}\cdot v$ of V for the $\mathbb{R}$-basis $v_1,\dots,v_{2g}$ is

$$\begin{pmatrix}\mathrm{Im}(\Omega)^{-1}\cdot\mathrm{Re}(\Omega) & \mathrm{Im}(\Omega)^{-1}\cdot D\\ -D^{-1}\cdot\mathrm{Im}(\Omega)-D^{-1}\cdot\mathrm{Re}(\Omega)\cdot\mathrm{Im}(\Omega)^{-1}\cdot\mathrm{Re}(\Omega) & -D^{-1}\cdot\mathrm{Re}(\Omega)\cdot\mathrm{Im}(\Omega)^{-1}\cdot D\end{pmatrix}$$

(3.3.3) Remark. (1) The bilinear conditions for the Riemann matrix $(\Omega\, D)$ has an alternative equivalent form:

(3.3.3 a)
$$\begin{pmatrix}I_g & -{}^t\Omega\cdot D^{-1}\end{pmatrix}\cdot\begin{pmatrix}0_g & D\\ -D & 0\end{pmatrix}\cdot\begin{pmatrix}I_g\\ -D^{-1}\cdot\Omega\end{pmatrix}=0_g$$

(3.3.3 a)
$$-\sqrt{-1}\cdot\begin{pmatrix}I_g & -{}^t\Omega\cdot D^{-1}\end{pmatrix}\cdot\begin{pmatrix}0_g & D\\ -D & 0\end{pmatrix}\cdot\begin{pmatrix}I_g\\ -D^{-1}\cdot\overline{\Omega}\end{pmatrix}>0_g$$

(2) The columns of the matrix $\begin{pmatrix}I_g\\ -D^{-1}\cdot\overline{\Omega}\end{pmatrix}$ correspond to vectors

$$v_j\otimes 1-\sum_{j=1}^{g}\Omega_{ij}\cdot(v_{i+g}\otimes 1)\ \in(\Lambda\otimes_{\mathbb{Z}}\mathbb{C}),\quad j=1,\dots,g,$$

which form a basis of the kernel of the $\mathbb{C}$-linear surjection

$$\mathrm{pr}: V\otimes_{\mathbb{R}}\mathbb{C}\twoheadrightarrow V,\qquad v_k\otimes 1\mapsto v_k\ \text{ for } k=1,\dots,2g$$

So (3.3.3 a) says that $\mathrm{Ker}(\mathrm{pr})$ is a Lagrangian subspace for the alternating form $\mu\otimes\mathbb{C}$ on $\Lambda\otimes_{\mathbb{Z}}\mathbb{C}$.

(3) There is a natural Hodge $\mathbb{Z}$-structure of type $\{(0,-1),(-1,0)\}$ on Λ such that the kernel $\mathrm{Ker}(\mathrm{pr})$ of $\mathrm{pr}: V\otimes_{\mathbb{R}}\mathbb{C}\twoheadrightarrow V$ is the $(0,-1)$-part of the Hodge filtration on $\Lambda\otimes_{\mathbb{Z}}\mathbb{C}$. Moreover $2\pi\sqrt{-1}\cdot\mathrm{Im}(H)$ defines a morphism $\Lambda\times\Lambda\to\mathbb{Z}(1)$ between Hodge $\mathbb{Z}$-structures, where $\mathbb{Z}(1)$ is the pure Hodge $\mathbb{Z}$-structure of type $\{(-1,-1)\}$ on the abelian group $2\pi\sqrt{-1}\subset\mathbb{C}$. The reader may consult [2] for a survey of modern Hodge theory.

§4. Algebraic incarnation of Riemann forms

(4.1) The étale version. We explain the definition of the Riemann form attached to an ample divisor D of an abelian variety A over a field k, as an alternating pairing for the first étale homology group of A,[6] due to Weil [28]. Details can be found in [17, IV §20]. We assume for simplicity that the base field k is algebraically closed.

Let $H_1^{\mathrm{et}}(A) := \varprojlim_n A[n](k)$ and let $H_1^{\mathrm{et}}(\mathbb{G}_{\mathrm{m}}) := \varprojlim_n \mathbb{G}_{\mathrm{m}}[n](k)$, where n runs through all positive integers which are invertible in k. Here $A[n]$ denotes the group of all n-torsion points of A, and $\mathbb{G}_{\mathrm{m}}[n](k)$ denotes the group of all n-th roots of 1 in k. The groups $H_1(A)$ and $H_1(\mathbb{G}_{\mathrm{m}})$ are naturally identified with the first étale homology groups of A and $\mathbb{G}_{\mathrm{m}}$ respectively.

The Riemann form attached to the ample divisor D will be a bilinear pairing

$$E^D : H_1^{\mathrm{et}}(A) \times H_1^{\mathrm{et}}(A) \to H_1^{\mathrm{et}}(\mathbb{G}_{\mathrm{m}}).$$

Given two elements $\underline{a} = (a_n)$ and $\underline{b} = (b_n)$ in $H_1^{\mathrm{et}}(A)$, represented as compatible systems of torsion points $a_n, b_n \in A[n]$, we need to produce a compatible system $\underline{c} = (c_n)$ of roots of unity in k.

(1) Since b_n is an n-torsion point of A, the divisors $[n]_A^{-1}(T_{-b_n}D) - [n]_A^{-1}(D)$ and $n \cdot (T_{-b_n}D) - n \cdot D$ are both principal by the theorem of the cube, where $T_{-b_n} : A \to A$ is the map "translation by $-b_n$". So there exist rational functions f_n, g_n on A such that the principal divisor (f_n), (g_n) are equal to $[n]_A^{-1}(T_{-b_n}D) - [n]_A^{-1}(D)$ and $n \cdot (T_{-b_n}D) - n \cdot D$ respectively.

(2) Because $[n]_A^*(g_n)$ and f_n^n have the same divisor, their ratio is a non-zero constant in k. Hence

$$T_{a_n}^*(f_n^n)/f_n^n = T_{a_n}^*[n]_A^*(g_n)/[n]_A^*(g_n) = 1.$$

Let

$$c_n := \frac{f_n}{T_{a_n}^*(f_n)} \in \mathbb{G}_{\mathrm{m}}[n](k).$$

(3) One verifies that $c_{mn}^m = c_n$ for all $m, n \in \mathbb{N}$ which are invertible in k, so the roots of unity c_n's produced in (2) form a compatible system $\underline{c} \in H_1^{\mathrm{et}}(\mathbb{G}_{\mathrm{m}})$. Define $E_{\mathrm{et}}^D(\underline{a}, \underline{b})$ to be this element $\underline{c} \in H_1^{\mathrm{et}}(\mathbb{G}_{\mathrm{m}})$.

(4.2) The general formalism for constructing Riemann forms. The following formal procedure produces an alternating pairing on the first homology group of an abelian variety A and a *polarization*[7] $\lambda : A \to A^t$ of A, for any "good" cohomology theory H^* on a suitable category of algebraic varieties.

[6] Any divisor on A algebraically equivalent to D will give the same alternating pairing.

[7] Every ample divisor D on an abelian variety A gives rise to a homomorphism $\phi_D : A \to A^t$ defined as follows. It is the homomorphism from A to its dual abelian variety A^t, which sends any point $x \in A(k)$ to the isomorphism class of the $\mathscr{O}_A(T_{-x}D) \otimes_{\mathscr{O}_A} \mathscr{O}_A(D)^{\otimes -1}$. Note that the last invertible $\mathscr{O}_A$-module is algebraically equivalent to 0, hence corresponds to a point of A^t. A *polarization* of an abelian scheme A over base scheme S is by definition an S-homomorphism ϕ from A to its dual abelian scheme A^t such that for every geometric point $\bar{s} \to S$ of S, the fiber $\phi_{\bar{s}}$ is equal to $\phi_{D_{\bar{s}}}$ for some an ample divisorn $D_{\bar{s}}$ on $A_{\bar{s}}$.

(1) Consider the Poincaré line bundle $\mathscr{P}$ on $A \times A^t$. The first Chern class $c_1(\mathscr{P})$ of $\mathscr{P}$ is an element of $H^2(A \times A^t)(1) \cong \mathrm{Hom}(H_1(A) \otimes H_1(A^t), H_1(\mathbb{G}_{\mathrm{m}}))$.

(2) The polarization λ of A induces a map $H_*(\lambda) : H_1(A) \to H_1(A^t)$.

(3) Combining (1) and (2), one gets a bilinear map

$$(u,v) \mapsto c_1(\mathscr{P})(u, H_*(\lambda)(v))$$

from $H_1(A) \times H_1(A)$ to $H_1(\mathbb{G}_{\mathrm{m}})$, which turns out to be an alternating pairing. This is the Riemann form on $H_1(A)$ attached to λ.

Examples of such good cohomology theories include Hodge, étale and crystalline cohomologies. The cases of Hodge and étale cohomologies have been explained in previous paragraphs. The crystalline case is documented in [1, §5.1], where the commutativity of various diagrams are carefully verified.

Part C. Riemann's theta function

Riemann's theta function $\theta(z,\Omega)$ was born in the famous memoir [22] on abelian functions. Its cousins, theta functions with characteristics $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z,\Omega)$, are essentially translates of $\theta(z;\Omega)$. These theta functions can be viewed in several ways:

(a) They were first introduced and studied as holomorphic functions in the z and/or the Ω variable.

(b) Geometrically these theta functions can be identified with sections of ample line bundles on abelian varieties and can be thought of as projective coordinates of abelian varieties. Their values at the zero, known as *thetanullwerte*, give projective coordinates of moduli of abelian varieties.

(c) They are matrix coefficients, for the Schr̆odinger representation of the Heisenberg group and the oscillator (or Segal-Shale-Weil) representation of the metaplectic group.

The geometric point of view will be summarized in §5 and the group theoretic viewpoint in §6.

§5. Theta functions as sections of line bundles

For any positive integer g, denote by $\mathfrak{H}_g$ the Siegel upper-half space of genus g, consisting of all $g \times g$ symmetric complex matrices with positive definite imaginary part. We will use the following version of exponential function: $\mathbf{e}(z) := \exp(2\pi\sqrt{-1} \cdot z)$ for any $z \in \mathbb{C}$.

(5.1) DEFINITION. (i) The **Riemann theta function** $\theta(z;\Omega)$ of genus g is the holomorphic function in two (blocks of) variables $(z,\Omega) \in \mathbb{C}^g \times \mathfrak{H}_g$, defined by the theta series

$$\theta(z;\Omega) := \sum_{m \in \mathbb{Z}^g} \mathbf{e}(\tfrac{1}{2}\,{}^t n \cdot \Omega \cdot n/2) \cdot \mathbf{e}({}^t n \cdot z).$$

(ii) Let $a,b\in\mathbb{R}^g$. The theta function $\theta\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right](z;\Omega)$ with **characteristics** a,b is the is the holomorphic function on $\mathbb{C}^g\times\mathfrak{H}_g$ defined by

$$\begin{aligned}\theta\begin{bmatrix}a\\ b\end{bmatrix}(z;\Omega) &= \sum_{m\in\mathbb{Z}^g}\mathbf{e}(\tfrac{1}{2}{}^t(n+a)\cdot\Omega\cdot(n+a))\cdot\mathbf{e}({}^t(n+a)\cdot(z+b))\\ &= \mathbf{e}(\tfrac{1}{2}{}^ta\cdot\Omega\cdot a+{}^ta\cdot(z+b))\cdot\theta(z+\Omega\cdot a+b;\Omega)\end{aligned}$$

Note that $\theta\left[\begin{smallmatrix}a+m\\ b+n\end{smallmatrix}\right](z;\Omega)=\mathbf{e}({}^ta\cdot n)\cdot\theta\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right](z;\Omega)$ for all $m,n\in\mathbb{Z}^g$, so that $\mathbf{e}({}^ta\cdot b)\cdot\theta\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right](z;\Omega)$ depends only on $a,b \bmod \mathbb{Z}^g$.

(5.2) The theta series in the definition (5.1) is classical but its significance is not immediately clear, and the many transformation formulas they satisfy may look daunting when one encounters them for the first time. We will first look at the effect of translation by elements of the lattice $\Omega\mathbb{Z}^g+\mathbb{Z}^g$ in the variable z.

The theta function $\theta\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right](z;\Omega)$ satisfies the following functional equation

$$\theta\begin{bmatrix}a\\ b\end{bmatrix}(z+\Omega\cdot m+n;\Omega)=\mathbf{e}({}^ta\cdot n-{}^tb\cdot m)\cdot\mathbf{e}(-\tfrac{1}{2}{}^tm\cdot\Omega\cdot m-{}^tm\cdot z)\cdot\theta\begin{bmatrix}a\\ b\end{bmatrix}(z;\Omega)\quad\forall m,n\in\mathbb{Z}^g.$$

For any $a,b\in\mathbb{R}^g$, the family of holomorphic functions

$$u^{a,b}_{\Omega\cdot m+n}(z):=\mathbf{e}({}^ta\cdot n-{}^tb\cdot m)\cdot\mathbf{e}(-\tfrac{1}{2}{}^tm\cdot\Omega\cdot m-{}^tm\cdot z)\qquad m,n\in\mathbb{Z}^g$$

on V with values in $\mathbb{C}^\times$ forms a 1-cocycle $\mu^{a,b}$ for the lattice $\Omega\cdot\mathbb{Z}^g+\mathbb{Z}^g$ in $\mathbb{C}^g$, in the sense that

$$\mu^{a,b}_{\xi_1}(z)\cdot\mu^{a,b}_{\xi_2}(z+\xi_1)=\mu^{a,b}_{\xi_1+\xi_2}(z)\qquad\forall\,\xi_1,\xi_2\in\Omega\cdot\mathbb{Z}^g+\mathbb{Z}^g.$$

This 1-cocycle defines a line bundle $\mathscr{L}^{a,b}_\Omega$ on the compact complex torus $\mathbb{C}^g/(\Omega\cdot\mathbb{Z}^g+\mathbb{Z}^g)$, so that $\theta\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right](z;\Omega)$ can be interpreted as a section of $\mathscr{L}^{a,b}_\Omega$. The line bundles $\mathscr{L}^{a,b}_\Omega$ are all algebraically equivalent; each one defines a principal polarization for the abelian variety $\mathbb{C}^g/(\Omega\cdot\mathbb{Z}^g+\mathbb{Z}^g)$

It turns out that the same idea goes further and applies to all ample line bundles on complex abelian varieties. We will see in the Appell–Humbert Theorem 5.3 how to classify all line bundles on a complex abelian variety. Then we will see in Proposition 5.4.2 (3) how to use the theta functions $\theta\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right](z;\Omega)$ to produce all sections of all ample line bundles on any complex abelian variety, which do not necessarily admit a principal polarization. In Theorem 5.5, a version of the Lefschetz embedding theorem, we will see how to use the theta functions to write down an explicit projective embedding for a given abelian variety, as well as a set of generators of the field of abelian functions on it. The last was, of course, Riemann's motivation for introducing his theta functions.

(5.3) THEOREM. (APPELL–HUMBERT) *Let Λ be a lattice in a finite dimensional complex vector space V.*

(1) *Every holomorphic line bundle on V/Λ is isomorphic to the quotient of the trivial line bundle on V via a 1-cocycle $(\xi,z)\mapsto u_\xi(z)$ for Λ, where $u_\xi(z)$ is an entire function on V with values in $\mathbb{C}^\times$ for every $\xi\in\Lambda$ and $u_{\xi_1+\xi_2}(z)=u_{\xi_1}(z)\cdot u_{\xi_2}(z+\xi_1)$ for all $\xi_1,\xi_2\in\Lambda$.*

(2) *For any* 1*-cocycle* $u_\xi(z)$ *as in* (1)*, there exists a quadruple* (H,S,ℓ,ψ)*, where*

(2a) $H: V\times V\to\mathbb{C}$ *is a Hermitian form, conjugate linear in the second argument, such that* $\mathrm{Im}(H)$ *is* $\mathbb{Z}$*-valued on* Λ*, and* $\psi:\Lambda\to\mathbb{C}_1^\times$ *is a complex function with absolute values* 1 *such that* $\psi(\xi_1+\xi_2)\cdot\psi(\xi_1)^{-1}\cdot\psi(\xi_2)^{-1}=(-1)^{\mathrm{Im}(H)(\xi_1,\xi_2)}$ *for all* $\xi_1,\xi_2\in\Lambda$*,*

(2b) $S: V\times V\to\mathbb{C}$ *is a symmetric* $\mathbb{C}$*-bilinear form,*

(2c) $\ell: V\to\mathbb{C}$ *is a* $\mathbb{C}$*-linear function, and*

(2d) $\psi:\Lambda\to\mathbb{C}_1^\times$ *is a complex function with absolute values* 1 *such that*

$$\psi(\xi_1+\xi_2)\cdot\psi(\xi_1)^{-1}\cdot\psi(\xi_2)^{-1}=(-1)^{\mathrm{Im}(H)(\xi_1,\xi_2)}\qquad\forall\,\xi_1,\xi_2\in\Lambda,$$

such that

$$u_\xi(z)=\mathbf{e}\big(\tfrac{1}{2\sqrt{-1}}\cdot(H(z,\xi)+S(z,\xi))\big)\cdot\mathbf{e}\big(\tfrac{1}{4\sqrt{-1}}\cdot(H(\xi,\xi)+S(\xi,\xi))\big)\cdot\mathbf{e}(\ell(\xi))\cdot\psi(\xi)\quad\forall\xi\in\Lambda.$$

The quadruple (H,S,ℓ,ψ) *is uniquely determined by the* 1*-cocycle* $u_\xi(z)$*. Conversely every quadruple* (H,S,ℓ,ψ) *satisfying conditions* (2a)–(2d) *determines a* 1*-cocyle for* (V,Λ)*.*

(3) *Let* $\mathscr{L}$ *and* $\mathscr{L}'$ *be two line bundles attached to two quadruples* (H,S,ℓ,ψ) *and* (H',S',ℓ',ψ') *as in* (2)*.*

(3a) $\mathscr{L}$ *is isomorphic to* $\mathscr{L}'$ *if and only if* $H=H'$ *and* $\psi=\psi'$*.*

(3b) $\mathscr{L}$ *is algebraically equivalent to* $\mathscr{L}'$ *if and only if* $H=H'$*.*

(3c) $\mathscr{L}$ *is ample if and only if* $\mathrm{Im}(H)$ *is a Riemann form of* (V,Λ)*, i.e.* H *is positive definite.*

(5.4) It is explained in the article on Riemann forms that for any Riemann form μ on a compact complex torus V/Λ and any choice of canonical $\mathbb{Z}$-basis $v_1,\dots,v_{2g}$ of Λ with elementary divisors $d_1|\cdots|d_g$, there exists a unique element $\Omega\in\mathfrak{H}_g$ such that the $\mathbb{C}$-linear map from V to $\mathbb{C}^g$ which sends each v_i to d_i times the i-th standard basis of $\mathbb{C}^g$, induces a biholomorphic isomorphism from V/Λ to $\mathbb{C}^g/(\Omega\cdot\mathbb{Z}^g+D\cdot\mathbb{Z}^g)$, where D is the $g\times g$ diagonal matrix with $d_1,\dots,d_g$ as diagonal entries. Under this isomorphism, the Riemann form μ becomes the alternating pairing

$$\mu_D:(\Omega\cdot m_1+n_1,\Omega\cdot m_2+n_2)\mapsto{}^t m_1\cdot D\cdot n_2-{}^t n_1\cdot D\cdot m_2\qquad\forall\,m_1,n_2,m_2,n_2\in\mathbb{Z}^g$$

on the lattice $\Lambda_{\Omega,D}:=\Omega\cdot\mathbb{Z}^g+D\cdot\mathbb{Z}^g$, and the Hermitian form on $\mathbb{C}^g$ whose imaginary part is the $\mathbb{R}$-bilinear extension of μ_D is given by the formula

$$H_\Omega:(z,w)\mapsto{}^t z\cdot\mathrm{Im}(\Omega)^{-1}\cdot\overline{w}'\qquad\forall\,z,w\in\mathbb{C}^g.$$

(5.4.1) An easy calculation shows that for any given $a,b\in\mathbb{R}^g$, the restriction to $\Lambda_{\Omega,D}$ of the 1-cocycle $u^{a,b}$ corresponds to the quadruple $(H_\Omega,S_\Omega,0,\psi_0\cdot\chi_D^{a,b})$, where

- S_Ω is the symmetric $\mathbb{C}$-bilinear form

$$S_\Omega\colon (z,w)\mapsto -{}^t z\cdot\mathrm{Im}(\Omega)^{-1}\cdot w\qquad\forall\,z,w\in\mathbb{C}^g,$$

  or equivalently S_Ω is the unique $\mathbb{C}$-bilinear form on $\mathbb{C}^g$ which coincides with $-H_{\Omega,D}$ on the subset $\mathbb{C}^g\times(D\cdot\mathbb{Z}^g)\subset\mathbb{C}^g\times\mathbb{C}^g$,

- ψ_0 is the quadratic unitary character on $\Lambda_{\Omega,D}$ defined by

$$\psi_0(\Omega\cdot m+D\cdot n)=(-1)^{{}^t m\cdot n}\qquad \forall m,n\in\mathbb{Z}^g,$$

- and $\chi_D^{a,b}$ is the unitary character on $\Lambda_{\Omega,D}$ defined by

$$\chi_D^{a,b}(\Omega\cdot m+D\cdot n)=\mathbf{e}({}^t a\cdot D\cdot n-{}^t b\cdot m)\qquad \forall m,n\in\mathbb{Z}^g.$$

Note that $\chi_D^{a,b}=\chi_D^{a',b'}$ if and only if $a-a'\in D^{-1}\cdot\mathbb{Z}^g$ and $b-b'\in\mathbb{Z}^g$. Let $\mathscr{L}_{\Omega,D}^{a,b}$ be the line bundle on $\mathbb{C}^g/\Lambda_{\Omega,D}$ given by the restriction to $\Lambda_{\Omega,D}$ of the 1-cocycle $u^{a,b}$. Clearly for every $a'\in a+D^{-1}\cdot\mathbb{Z}^g$, $\theta\begin{bmatrix}a'\\ b\end{bmatrix}(z,\Omega)$ defines a section of the line bundle $\mathscr{L}_{\Omega,D}^{a,b}$.

More generally, for every positive integer r, every $a'\in a+D^{-1}\cdot\mathbb{Z}^g$ and every $b'\in r^{-1}b+r^{-1}\mathbb{Z}^g$, $\theta\begin{bmatrix}a'\\ b'\end{bmatrix}(z,r^{-1}\Omega)$ defines a section of the line bundle $(\mathscr{L}_{\Omega,D}^{a,b})^{\otimes r}$. Underlying this statement is the fact that the pull-back of $\mathscr{L}_{r^{-1}\Omega,D}^{a',b'}$ via the isogeny $\mathbb{C}^g/\Lambda_{\Omega,D}\to\mathbb{C}^g/\Lambda_{r^{-1}\Omega,D}$ of degree r^g is isomorphic to $(\mathscr{L}_{\Omega,D}^{a,b})^{\otimes r}$, for every a',b' as above.

(5.4.2) Proposition. (1) *For every line bundle $\mathscr{L}$ on the compact complex torus $\mathbb{C}^g/\Lambda_{\Omega,D}$ with Riemann form μ_D, there exist elements $a,b\in\mathbb{R}^g$ such that $\mathscr{L}$ is isomorphic to $\mathscr{L}_{\Omega,D}^{a,b}$. Moreover for any $a',b'\in\mathbb{R}^g$, $\mathscr{L}_{\Omega,D}^{a,b}$ is isomorphic to $\mathscr{L}_{\Omega,D}^{a',b'}$ if and only if $a'\in a+D^{-1}\cdot\mathbb{Z}^g$ and $b'\in b+\mathbb{Z}^g$.*

(2) *We have* $\dim_{\mathbb{C}}\Gamma\big(\mathbb{C}^g/\Lambda_{\Omega,D},\mathscr{L}_{\Omega,D}^{a,b}\big)=\det(D)=\prod_{i=1}^g d_i$. *Moreover as a' runs through a set of representatives of $(a+D^{-1}\cdot\mathbb{Z}^g)/\mathbb{Z}^g$, the theta functions $\theta\begin{bmatrix}a'\\ b\end{bmatrix}(z,\Omega)$ give rise to a $\mathbb{C}$-basis of $\Gamma(\mathscr{L}_{\Omega,D}^{a,b})$.*

(3) *For every positive integer r, we have* $\dim_{\mathbb{C}}\Gamma\big(\mathbb{C}^g/\Lambda_{\Omega,D},(\mathscr{L}_{\Omega,D}^{a,b})^{\otimes r}\big)=r^g\cdot\prod_{i=1}^g d_i$. *Moreover as a' runs through a set of representatives of $(a+D^{-1}\cdot\mathbb{Z}^g)/\mathbb{Z}^g$ and b' runs through a set of representatives of $(r^{-1}b+r^{-1}\mathbb{Z}^g)/\mathbb{Z}^g$, the global sections corresponding to $\theta\begin{bmatrix}a'\\ b'\end{bmatrix}(z,r^{-1}\Omega)$ form a $\mathbb{C}$-basis of $\Gamma\big((\mathscr{L}_{\Omega,D}^{a,b})^{\otimes r}\big)$.*

(5.5) Theorem. *Let r be a positive integer, $r\geq 3$. For given $a,b\in\mathbb{R}^g$, let $\{a_k|1\leq k\leq r^g\}$ be a system of representatives of $(a+D^{-1}\cdot\mathbb{Z}^g)/\mathbb{Z}^g$ and let $\{b_l|1\leq l\leq\prod_{i=1}^g d_i\}$ be a system of representatives of $(r^{-1}b+r^{-1}\mathbb{Z}^g)/\mathbb{Z}^g$. The family of functions*

$$\Big\{\theta\begin{bmatrix}a_k\\ b_l\end{bmatrix}(z,r^{-1}\Omega)\colon\ 1\leq k\leq r^g,\ 1\leq l\leq\textstyle\prod_{i=1}^g d_i\Big\}$$

defines a projective embedding $\mathbb{C}^g/\Lambda_{\Omega,D}\hookrightarrow\mathbb{P}^{(\prod_{i=1}^g d_i)\cdot r^g-1}$ of the compact complex torus $\mathbb{C}^g/\Lambda_{\Omega,D}$. In particular the field consisting of all abelian functions for the pair $(\mathbb{C}^g,\Lambda_{\Omega,D})$ is generated over $\mathbb{C}$ by the following family of quotients of theta functions

$$\theta\begin{bmatrix}a_{k_1}\\ b_{l_1}\end{bmatrix}(z,r^{-1}\Omega)\Big/\theta\begin{bmatrix}a_{k_2}\\ b_{l_2}\end{bmatrix}(z,r^{-1}\Omega)\qquad 1\leq k_1,k_2\leq r^g,\ 1\leq l_1,l_2\leq\textstyle\prod_{i=1}^g d_i.$$

(5.6) Remark. The Riemann theta function was defined in §17 of Riemann's famous memoir [22] on abelian functions. The notion of theta functions $\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](z;\Omega)$ with characteristics is due to Prym [21, p. 25]. The second part of theorem 5.5 is a version of the "*theta theorem*" stated by Riemann in §26 of [22] and proved by Poincaré; see Siegel's formulation and comments in [27, p. 91]. Theorem 5.3 was proved by Appell and Humbert for abelian surfaces, called "hyperelliptic surfaces" at their time. The higher dimensional cases of theorem 5.3 and theorem 5.5 are due to Lefschetz. For more information related to this section, we recommend [17, Ch. I & Ch. III §17], [19, Ch. II, §§1–1], [8, Ch. II–III] and [12, Ch. I–II].

Theta functions in dimension $g=1$ go back to Jacobi, who obtained their properties by algebraic methods through his theory of elliptic functions; see [10] and [11].[8] We refer to [31, Ch. XXI] for a classical treatment and [31, pp. 462–463] for historical comments on theta functions, which appeared first in Euler's investigation on the *partition function* $\prod_{n=1}^{n}(1-x^n z)^{-1}$.

§6. Theta functions as matrix coefficients

There are several infinite dimensional representations, closely related to theta functions, which are inspired by quantum mechanics. First one has the Schrödinger representation, an irreducible unitary projective representation of $\mathbb{R}^{2g}$. There is also a projective representation of the symplectic group $\mathrm{Sp}(2g,\mathbb{R})$, on the space underlying the Schrödinger representation, called the oscillator representation. When viewed in the z-variable, the Riemann theta function $\theta(z;\Omega)$, is essentially a matrix representation of the Schrödinger representation. When viewed as a function in the Ω-variable, the theta functions become matrix coefficients of the oscillator representation. This section provides a synopsis of this group-theoretic approach. More systematic treatments can be found in [20, §§1–4 &§8] [8, Ch. 1], as well as the papers [29] and [3].

(6.1) The Heisenberg group. Heisenberg groups are central extensions of abelian groups. We will use the following version of the real Heisenberg groups $\mathrm{Heis}(2g,\mathbb{R})$, whose underlying set is $\mathbb{C}_1^\times \times \mathbb{R}^{2g}$, and a typical element will be written as a pair (λ,x), where $\lambda\in\mathbb{C}_1^\times$, $x=\begin{pmatrix}x_1\\x_2\end{pmatrix}$, $x_1,x_2\in\mathbb{R}^g$. Often we write (λ,x_1,x_2) for (λ,x). The group law on $\mathrm{Heis}(2g,\mathbb{R})$ is

$$(\lambda,x)\cdot(\mu,y)=\lambda\cdot\mu\cdot\mathbf{e}(\tfrac{1}{2}({}^t x_1\cdot y_2-{}^t x_2\cdot y_1))$$

so that the $(\lambda,0)$'s form the center of $\mathrm{Heis}(2g,\mathbb{R})$, identified with $\mathbb{C}_1^\times$. This group law induces a $\mathbb{C}_1^\times$-valued commutator pairing

$$\mathbb{R}^{2g}\times\mathbb{R}^{2g}\longrightarrow\mathbb{C}_1^\times,\quad (x,y)\ \mapsto\ (\lambda,x)\cdot(\mu,y)\cdot(\lambda,x)^{-1}\cdot(\mu,y)^{-1}=\mathbf{e}({}^t x_1\cdot y_2-{}^t x_2\cdot y_1)$$

on $\mathbb{R}^{2g}$. Underlying the notation for $\mathrm{Heis}(2g,\mathbb{R})$ is a continuous section $\mathbf{s}$ of the projection $\mathrm{pr}:\mathrm{Heis}(2g,\mathbb{R})\twoheadrightarrow\mathbb{R}^{2g}$, $\mathbf{s}:x\mapsto(1,x)$ for all $x\in\mathbb{R}^{2g}$, which is a group homomorphism when restricted to any Lagrangian subspace of $\mathbb{R}^{2g}$ for the $\mathbb{R}$-bilinear alternating pairing $E:(x,y)\mapsto {}^t x_1\cdot y_2-{}^t x_2\cdot y_1$.

[8] The theta functions $\theta_1(z,q)$, $\theta_2(z,q)$ $\theta_3(z,q)$, $\theta_4(z,q)$ in Jacobi's notion [11] [31, Ch. XXI], where $z\in\mathbb{C}$ and $q=\mathbf{e}(\tau/2)$, $\tau\in\mathfrak{H}_1$, are equal to: $-\theta\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right](z/\pi;\tau)$, $\theta\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right](z/\pi;\tau)$, $\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](z/\pi;\tau)$ and $\theta\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right](z/\pi;\tau)$ respectively. In Jacobi's earlier notation in [10] are four theta functions $\Theta(u,q),\Theta_1(u,q),\mathrm{H}(u,q)$ and $\mathrm{H}_1(u,q)$, where the symbol H denotes capital eta. They are related to his later notation by: $\Theta(u,q)=\theta_4\big(u\cdot\theta_3(0,q)^{-2},q\big)$, $\Theta_1(u,q)=\theta_3\big(u\cdot\theta_3(0,q)^{-2},q\big)$, $\mathrm{H}(u,q)=\theta_1\big(u\cdot\theta_3(0,q)^{-2},q\big)$ and $\mathrm{H}_1(u,q)=\theta_2\big(u\cdot\theta_3(0,q)^{-2},q\big)$.

(6.2) The Schrödinger representation. The main theorem about the representations of the Heisenberg group $\mathrm{Heis}(2g,\mathbb{R})$, due to Stone, Von Neumann and Mackey is this:

> *There is a unique irreducible unitary representation* $U : \mathrm{Heis}(2g,\mathbb{R}^{2g}) \to \mathrm{Aut}(\mathcal{H})$ *whose restriction to the center* $\mathbb{C}_1^\times$ *of* $\mathrm{Heis}(2g,\mathbb{R}^{2g})$ *is* $\lambda \mapsto \lambda \cdot \mathrm{Id}_{\mathcal{H}}$.

This infinite dimensional unitary representation of the Heisenberg group will be call the Schrödinger representation. It can be produced, or realized, in many ways. We will explain how to obtain a model of the Schrödinger representation for each pair (K,σ), consisting of a closed maximal isotropic subgroup $K \subset \mathbb{R}^{2g}$ and a splitting $\sigma : K \to \mathrm{Heis}(2g,\mathbb{R})$ of the projection $\mathrm{pr} : \mathrm{Heis}(2g,\mathbb{R}) \to \mathbb{R}^{2g}$ over K.

Given such a pair (K,σ), we have the induced representation on the space $L^2(\mathbb{R}^{2g}//K)$ of all essentially L^2-functions on $\sigma(K)\backslash\mathrm{Heis}(2g,\mathbb{R})$ whose restriction to $\mathbb{C}_1^\times$ is $\lambda \mapsto \lambda \cdot \mathrm{Id}$. In the case that $K = 0 \oplus \mathbb{R}^g \subset \mathbb{R}^g \oplus \mathbb{R}^g$ and $\sigma = \mathbf{s}|_K$, we have the usual position-momentum realization on the Hilbert space $\mathcal{H}_{\mathrm{PM}} = L^2(\mathbb{R}^{2g})$, where the action of a typical element $(\lambda, y_1, y_2) \in \mathrm{Heis}(2g,\mathbb{R})$ is given by

$$(U_{(\lambda,y_1,y_2)}\phi)(x) = \lambda \cdot \mathbf{e}({}^t x \cdot y_2) \cdot \mathbf{e}(\tfrac{1}{2}\,{}^t y_1 \cdot y_2) \cdot \phi(x+y_1) \quad \forall \phi(x) \in L^2(\mathbb{R}^g).$$

Let $C, A_1, \ldots, A_g, B_1, \ldots, B_g$ be the basis of the Lie algebra $\mathfrak{heis}(2g,\mathbb{R})$ of $\mathrm{Heis}(2g,\mathbb{R})$ such that $\exp(t\cdot C) = (\mathbf{e}(t), \vec{0}, \vec{0})$, $\exp(t\cdot A_i) = (1, t\vec{e}_i, \vec{0})$ and $\exp(t\cdot B_i) = (1, \vec{0}, t\vec{e}_i)$ for all $i = 1, \ldots, g$, where $\vec{e}_1, \ldots, \vec{e}_g$ are the standard basis elements of $\mathbb{R}^g$. Their Lie brackets are

$$[A_i,A_j] = [B_i,B_j] = [C,A_i] = [C,B_j] = 0 \text{ and } [A_i,B_j] = \delta_{ij}\cdot C \quad \forall\, i,j = 1,\ldots,g.$$

The action of $\mathfrak{heis}(2g,\mathbb{R})$ on $\mathcal{H}_{\mathrm{PM}}$ is given by

$$U_{A_i}(f) = \frac{\partial f}{\partial x_i}, \quad U_{B_i}(f) = 2\pi\sqrt{-1}\,x_i \cdot f, \quad U_C(f) = 2\pi\sqrt{-1}\cdot f.$$

(6.3) $\mathcal{H}_\infty$ **and** $\mathcal{H}_{-\infty}$. Let $\mathcal{H}_\infty$ be the set of all smooth vectors in the Schrödinger representation $\mathcal{H}$, consisting of all elements $f \in \mathcal{H}$ such that all higher derivatives

$$\Big(\big(\prod_{i=1}^{g} U_{B_i}^{n_i}\big)\circ\big(\prod_{i=1}^{g} U_{A_i}^{m_i}\big)\Big)(f)$$

of f exists in $\mathcal{H}$, for all $m_1,\ldots,m_g,n_1,\ldots,n_g \in \mathbb{N}$. By Sobolev's lemma, the space $\mathcal{H}_\infty^{\mathrm{PM}}$ of all smooth vectors in the position-momentum realization $\mathcal{H}_{\mathrm{PM}}$ is equal to the Schwartz space $\mathscr{S}(\mathbb{R}^g)$ of $\mathbb{R}^g$, consisting of all rapidly decreasing smooth functions on $\mathbb{R}^g$.

Let $\mathcal{H}_{-\infty}$ be the space of all $\mathbb{C}$-linear functionals on $\mathcal{H}_\infty$ which are continuous for the topology defined by the family of seminorms

$$\big\{ f \ \mapsto \ \big\|\big(\big(\textstyle\prod_{i=1}^{g} U_{A_i}^{m_i}\big)\circ\big(\prod_{i=1}^{g} U_{B_i}^{n_i}\big)f\big\|\,\big\}_{m_i,n_i\in\mathbb{N}\ \forall i=1,\ldots,g}$$

on $\mathcal{H}_\infty$. For the position-momentum realization, this "distribution completion" of $\mathcal{H}$ is the space of all tempered distributions on $\mathbb{R}^g$.

(6.4) The smooth vectors f_Ω. Every element $\Omega \in \mathfrak{H}_g$ determines a Lie subalgebra W_Ω of the complexification $\mathfrak{heis}(2g,\mathbb{C}) := \mathfrak{heis}(2g,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathfrak{heis}(2g,\mathbb{R})$, given by

$$W_\Omega := \textstyle\sum_{i=1}^{g} \mathbb{C} \cdot (A_i - \sum_{j=1}^{g} \Omega_{ij} B_j).$$

A direct computation shows that for every $\Omega \in \mathfrak{H}_g$, the subspace of all elements in the Schrödinger representation killed by W_Ω is a one-dimensional subspace of $\mathcal{H}_\infty$; let f_Ω be a generator of this subspace. In the position-momentum realization, f_Ω can be taken to be the function

$$f_\Omega(x) = \mathbf{e}(\tfrac{1}{2}\, {}^t x \cdot \Omega \cdot x).$$

(6.5) The theta distribution $e_{\mathbb{Z}} \in \mathcal{H}_{-\infty}$. Inside $\mathrm{Heis}(2g,\mathbb{R})$ we have a discrete closed subgroup

$$\sigma(\mathbb{Z}^{2g}) := \left\{ \left((-1)^{{}^t m \cdot n}, m, n\right) \in \mathrm{Heis}(2g,\mathbb{R}) \;\middle|\; m, n \in \mathbb{Z}^g \right\}.$$

One sees by a direct computation that the set of all elements in $\mathcal{H}_{-\infty}$ fixed by $\sigma(\mathbb{Z}^{2g})$ is a one-dimensional vector subspace over $\mathbb{C}$. A generator $e_{\mathbb{Z}}$ of this subspace in $\mathcal{H}_{\mathrm{PM}}$ is

$$e_{\mathbb{Z}} = \sum_{n \in \mathbb{Z}^g} \delta_n, \quad \text{where } \delta_n = \text{the delta function at } n.$$

This element $e_{\mathbb{Z}} \in \mathcal{H}_{-\infty}$ might be thought of as a "universal theta distribution".

(6.6) Proposition. *For any $\Omega \in \mathfrak{H}_g$, any generator f_Ω of $\mathcal{H}^{W_\Omega}$ and any generator $e_{\mathbb{Z}}$ of $\mathcal{H}^{\sigma(\mathbb{Z}^{2g})}$, there exists a constant $c \in \mathbb{C}^\times$ such that*

$$\begin{aligned} \langle U_{(1,x_1,x_2)} f_\Omega,\, e_{\mathbb{Z}} \rangle &= c \cdot \mathbf{e}\big(\tfrac{1}{2}\,({}^t x_1 \Omega x_1 + {}^t x_1 \cdot x_2)\big) \cdot \theta(\Omega x_1 + x_2; \Omega) \\ &= c \cdot \mathbf{e}(-\tfrac{1}{2}\, {}^t x_1 \cdot x_2) \cdot \theta \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} (0; \Omega) \end{aligned}$$

for all elements $(1,x_1,x_2) \in \mathrm{Heis}(2g,\mathbb{R})$. This constant c is 1 if f_Ω and $e_{\mathbb{Z}}$ are those specified in 6.4 and 6.5 for the position-momentum realization.

(6.7) The big metaplectic group. Let $\widetilde{\mathrm{Mp}}(2g,\mathbb{R})$ be the subgroup of of the group $\mathrm{U}(\mathcal{H})$ of all unitary automorphisms of the Hilbert space $\mathcal{H}$, defined by

$$\widetilde{\mathrm{Mp}}(2g,\mathbb{R}) := \left\{ T \in \mathrm{U}(\mathcal{H}) \,|\, \exists\, \gamma \in \mathrm{Sp}(2g,\mathbb{R}) \text{ s.t. } T \cdot U_{(\lambda,x)} \cdot T^{-1} = U_{(\lambda,\gamma x)} \;\; \forall\, (\lambda,x) \in \mathrm{Heis}(2g,\mathbb{R}) \right\}$$

The uniqueness of the Schrödinger representation implies that the natural homomorphism

$$\rho : \widetilde{\mathrm{Mp}}(2g,\mathbb{R}) \longrightarrow \mathrm{Sp}(2g,\mathbb{R})$$

is surjective and $\mathrm{Ker}\rho = \mathbb{C}_1^\times \cdot \mathrm{Id}$, so that $\widetilde{\mathrm{Mp}}$ is a central extension of $\mathrm{Sp}(2g,\mathbb{R})$ by $\mathbb{C}_1^\times$. Being a subgroup of $\mathrm{U}(\mathcal{H})$, we have a tautological unitary representation $\widetilde{\mathrm{Mp}}(2g,\mathbb{R})$ on $\mathcal{H}$.

The natural action of the symplectic group $\mathrm{Sp}(2g,\mathbb{R})$ on the set of all Lagrangian subspaces of $\mathbb{R}^{2g}$ induces an action on the set of all Lie subalgebras $W_\Omega \subset \mathfrak{heis}(2g,\mathbb{C})$ associted to elements of $\mathfrak{H}_g$ as defined in 6.4. Explicitly, we have $\gamma \cdot W_\Omega = W_{\gamma\cdot\Omega}$, where

$$\gamma \cdot \Omega = (D \cdot \Omega - C) \cdot (-B \cdot \Omega + A)^{-1} \qquad \forall\, \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g,\mathbb{R}),$$

which is the composition of the usual action

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \Omega \mapsto (A\cdot\Omega+B)\cdot(C\cdot\Omega+D)^{-1}$$

of $\mathrm{Sp}(2g,\mathbb{R})$ on $\mathfrak{H}_g$ with the involution

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;\mapsto\; {}^t\gamma^{-1} = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}\cdot\gamma\cdot\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}^{-1} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}$$

of $\mathrm{Sp}(2g,\mathbb{R})$. Let $j(\gamma,\Omega) := \det(-B\cdot\Omega+A)$ for γ as above and $\Omega\in\mathfrak{H}_g$; it is a 1-cocycle for $(\mathrm{Sp}(2g,\mathbb{R}),\mathfrak{H}_g)$.

(6.8) Proposition. *For every $\Omega\in\mathfrak{H}_g$, let f_Ω be the element $f_\Omega(x)=\mathbf{e}(\frac{1}{2}\,{}^tx\Omega x)$ in $\mathcal{H}_{\mathrm{PM}}^{W_\Omega}$.*

(1) *For every $\tilde{\gamma}\in\widetilde{\mathrm{Mp}}(2g,\mathbb{R})$ with image $\gamma=\rho(\tilde{\gamma})$ in $\mathrm{Sp}(2g,\mathbb{R})$ and every $\Omega\in\mathfrak{H}_g$, there exists a unique element $C(\tilde{\gamma},\Omega)\in\mathbb{C}^\times$ such that*

$$\tilde{\gamma}\cdot f_\Omega = C(\tilde{\gamma},\Omega)\cdot f_{\gamma\cdot\Omega}.$$

(2) *There is a unitary character $\chi:\widetilde{\mathrm{Mp}}(2g,\mathbb{R})\to\mathbb{C}_1^\times$ of $\mathrm{Mp}(2g,\mathbb{R})$ such that*

$$C(\tilde{\gamma},\Omega)^2 = \chi(\tilde{\gamma})\cdot j(\gamma,\Omega) \quad \forall\,\Omega\in\mathfrak{H}_g.$$

(6.9) The metaplectic group and the theta level subgroup. The metaplectic group, defined by $\mathrm{Mp}(2g,\mathbb{R}) := \mathrm{Ker}(\chi)$, is a double cover of the symplectic group $\mathrm{Sp}(2g,\mathbb{R})$. The tautological action of $\mathrm{Mp}(2g,\mathbb{R})$ on $\mathcal{H}$ is called the *oscillator representation*.

Let $\Gamma_{1,2}$ be the subgroup of $\mathrm{Sp}(2g,\mathbb{Z})$ consisting of all $\gamma\in\mathrm{Sp}(2g,\mathbb{Z})$ leaving fixed the function $(m,n)\mapsto(-1)^{{}^tm\cdot n}$ on $\mathbb{Z}^{2g}$. Its inverse image $\widetilde{\Gamma_{1,2}}$ in $\mathrm{Mp}(2g,\mathbb{R})$ is the subgroup consisting of all elements $\tilde{\gamma}\in\mathrm{Mp}(2g,\mathbb{R})$ such that $\tilde{\gamma}\cdot\sigma(\mathbb{Z}^{2g})\cdot\tilde{\gamma}^{-1}=\sigma(\mathbb{Z}^{2g})$.

(6.10) Proposition. *Let μ_8 be the group of all 8th roots of unity. There exists a surjective group homomorphism $\eta:\widetilde{\Gamma_{1,2}}\to\mu_8$ such that $\tilde{\gamma}\cdot e_{\mathbb{Z}}=\eta(\tilde{\gamma})\cdot e_{\mathbb{Z}}$ for all $\tilde{\gamma}\in\widetilde{\Gamma_{1,2}}$.*

Because $\mathrm{Mp}(2g,\mathbb{R})$ is contained in the normalizer of the image of the Heisenberg group in $\mathrm{U}(\mathcal{H})$ by construction, we have a unitary representation of their semi-direct product $\mathrm{Heis}(2g,\mathbb{R})\rtimes\mathrm{Mp}(2g,\mathbb{R})$ on $\mathcal{H}$ which combines the Schrödinger and the oscillator representation. Proposition 6.11 below, which is a reformulation of 6.8 (1), says that the Riemann theta function $\theta(z;\Omega)$ is essentially a matrix coefficient for the representation of $\mathrm{Heis}(2g,\mathbb{R})\rtimes\mathrm{Mp}(2g,\mathbb{R})$ up to some elementary exponential factor.

(6.11) Proposition. *Let $e_{\mathbb{Z}}$ be the $\sigma(\mathbb{Z}^{2g})$-invariant distribution $\sum_{n\in\mathbb{Z}^{2g}}\delta_n$ in $\mathcal{H}_{-\infty}^{\mathrm{PM}}$ and let $f_{\sqrt{-1}\mathrm{I}_g}$ be the smooth vector $f_{\sqrt{-1}\mathrm{I}_g}(x)=\exp(-\pi\,{}^txx)$ in $\mathcal{H}_\infty^{\mathrm{PM}}$ fixed by $W_{\sqrt{-1}\mathrm{I}_g}$. We have*

$$\left\langle U_{(1,x_1,x_2)}\cdot\tilde{\gamma}\cdot f_{\sqrt{-1}\mathrm{I}_g},\; e_{\mathbb{Z}}\right\rangle = C(\tilde{\gamma},\sqrt{-1}\mathrm{I}_g)\cdot\mathbf{e}(-\tfrac{1}{2}\,{}^tx_1x_2)\cdot\theta\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(0,\gamma\cdot\sqrt{-1}\,\mathrm{I}_g)$$

for all $\tilde{\gamma}\in\mathrm{Mp}(2g,\mathbb{R})$ and all $(1,x_1,x_2)\in\mathrm{Heis}(2g,\mathbb{R})$, where γ is the image of $\tilde{\gamma}$ in $\mathrm{Sp}(2g,\mathbb{R})$.

In the above formula $\gamma\cdot\sqrt{-1}\,\mathrm{I}_g = (\sqrt{-1}D-C)\cdot(A-\sqrt{-1}B)^{-1} =: \Omega_\gamma$ if $\gamma = \begin{pmatrix} A & B \\ C & D\end{pmatrix}$, and we have $C(\tilde{\gamma},\sqrt{-1}\mathrm{I}_g)^2 = \det(A-\sqrt{-1}\,B)$, which determines $C(\tilde{\gamma},\sqrt{-1}\mathrm{I}_g)$ up to ± 1. Note that

$$
\begin{aligned}
&C(\tilde{\gamma},\sqrt{-1}\mathrm{I}_g)\cdot\mathbf{e}(-\tfrac{1}{2}\,{}^t x_1\,x_2)\cdot\theta\Big[\begin{smallmatrix} x_1 \\ x_2\end{smallmatrix}\Big](0,\gamma\cdot\sqrt{-1}\,\mathrm{I}_g)\\
&= C(\tilde{\gamma},\sqrt{-1}\mathrm{I}_g)\cdot\mathbf{e}(-\tfrac{1}{2}\,({}^t x_1\cdot\Omega_\gamma\cdot x_2)+{}^t x_1\cdot x_2)\cdot\theta(\Omega_\gamma x_1+x_2;\Omega_\gamma),
\end{aligned}
$$

so up the the elementary factors $C(\tilde{\gamma},\sqrt{-1}\mathrm{I}_g)\cdot\mathbf{e}(-\frac{1}{2}\,({}^t x_1\cdot\Omega_\gamma\cdot x_2)+{}^t x_1\cdot x_2)$, the Riemann theta function is indeed a matrix coefficient of the group $\mathrm{Heis}(2g,\mathbb{R})\rtimes\mathrm{Mp}(2g,\mathbb{R})$ on the Hilbert space $\mathcal{H}$ for the unitary representation which combines both the Schrödinger and the oscillator representation.

(6.12) Propositions 6.8 and 6.10 provide a modern version of the transformation theory of theta functions. They reflect the fact that theta constants $\theta\big[\begin{smallmatrix} a \\ b\end{smallmatrix}\big](0;\Omega)$ with rational characteristics are Siegel modular forms of weight $1/2$. The readers may consult Part 2 of [14] and [13, Ch. V] for what this theory looked like at the end of the nineteenth century.

(6.13) So far we have only looked at matrix coefficients attached to special elements of the form $f_\Omega\otimes e_{\mathbb{Z}}$ in the tensor product $\mathcal{H}_\infty\otimes\mathcal{H}_{-\infty}$. The group $\mathrm{Heis}(2g,\mathbb{R})\times\mathrm{Heis}(2g,\mathbb{R})$ acts on the tensor product $\mathcal{H}_\infty\otimes\mathcal{H}_{-\infty}$, such that the $\mathbb{C}_1^\times$ in the first copy of $\mathrm{Heis}(2g,\mathbb{R})$ acts on through $z\mapsto z\cdot\mathrm{Id}$ and the $\mathbb{C}_1^\times$ in the second copy of $\mathrm{Heis}(2g,\mathbb{R})$ acts through via $z\mapsto z^{-1}\cdot\mathrm{Id}$. Each element of $\mathbf{v}\in\mathcal{H}_\infty\otimes\mathcal{H}_{-\infty}$ gives rise to a function $f_{\mathbf{v}}$ on $\mathbb{R}^{2g}$, defined by $f_{\mathbf{v}}(x) = \varepsilon\big((U_{(1,x)},1)\cdot\mathbf{v}\big)$, where $\varepsilon:\mathcal{H}_\infty\otimes\mathcal{H}_{-\infty}\to\mathbb{C}$ is the linear functional corresponding to the pairing between $\mathcal{H}_\infty$ and $\mathcal{H}_{-\infty}$. The action of the first and second copy of $\mathrm{Hein}(2g,\mathbb{R})$ on $\mathbf{v}\in\mathcal{H}_\infty\otimes\mathcal{H}_{-\infty}$ becomes two commuting actions of $\mathrm{Hein}(2g,\mathbb{R})$ on the space of all "good functions" on $\mathbb{R}^{2g}$,

$$U^{\mathrm{right}}_{(\lambda,y_1,y_2)}f(x_1,x_2) = \lambda\cdot\mathbf{e}\big(\tfrac{1}{2}\,({}^t x_1\cdot y_2-{}^t x_2\cdot y_1)\big)\cdot f(x_1+y_1,x_2+y_2)$$

and

$$U^{\mathrm{left}}_{(\lambda,y_1,y_2)}f(x_1,x_2) = \lambda^{-1}\cdot\mathbf{e}\big(\tfrac{1}{2}\,({}^t x_1\cdot y_2-{}^t x_2\cdot y_1)\big)\cdot f(x_1-y_1,x_2-y_2)$$

respectively. This provides a very nice way to organize all theta functions through symmetries arising from the action of the group $\mathrm{Hein}(2g,\mathbb{R})\times\mathrm{Hein}(2g,\mathbb{R})/\{(\lambda,\lambda)|\lambda\in\mathbb{C}_1^\times\}$. Of course the last group is isomorphic to the Heisenberg group $\mathrm{Hein}(4g,\mathbb{R})$, and its action on $\mathcal{H}_\infty\otimes\mathcal{H}_{-\infty}$ is an incarnation of the Schrödinger representation of this bigger Heisenberg group! This compelling story is elegantly presented in [20, §§1-5]; see also [8, Ch. I] and [3].

(6.14) The Heisenberg groups play an important role in the study of the mathematical foundations of quantum mechanics by Weyl in Ch. II §11 and Ch. IV §15 of [30]. Weil recognized that theta series should be interpreted as automorphic forms for the metaplectic group and developed the adelic theory of oscillator representations in [29] for this purpose. This representation-theoretic approach has since become a core part of the theory of automorphic forms and L-functions.

Part D. Riemann's theta formula

There is a myriad of identities satisfied by the Riemann theta function $\theta(z;\Omega)$ and its close relatives $\theta\big[\begin{smallmatrix} a \\ b\end{smallmatrix}\big](z;\Omega)$. The most famous among these theta relations is a quartic relation known to Riemann, associated to a 4×4 matrix all of whose entries are ± 1 and is 2 times an orthogonal matrix; see 7.3. It debuted as formula (12) on p. 20 of [21], and was named *Riemann's theta formula* by Prym. In the preface of [21] Prym said that he learned of this formula from Riemann in Pisa, where he was with Riemann for several weeks in early 1865, and that he wrote down a proof following Riemann's suggestions.

For any fixed abelian variety, these theta identities give a set of *quadratic* equations which defines this abelian variety. The coefficients of these quadratic equations are theta constants, or "thetanullwerte", which vary with the abelian variety. At the same time, the Riemann theta identities give a set of *quartic* equations satisfied by the theta constants, which forms a system of defining equations of the moduli space of abelian varieties (endowed with suitable theta level structures).

§7. Riemann's theta formula

We will first formulate a generalized Riemann theta identity, for theta functions attached to a quadratic form on a lattice.

(7.1) DEFINITION. (THETA FUNCTIONS ATTACHED TO QUADRATIC FORMS) Let Q be a $\mathbb{Q}$-valued *positive definite* symmetric bilinear form on an h-dimensional $\mathbb{Q}$-vector space $\Gamma_{\mathbb{Q}}$, where h is a positive integer. Let $\Gamma\subset\Gamma_{\mathbb{Q}}$ be a $\mathbb{Z}$-lattice in $\Gamma_{\mathbb{Q}}$, i.e. Γ is a free abelian subgroup of $\Gamma_{\mathbb{Q}}$ of rank h. Denote by $\Gamma_{\mathbb{Q}}^{\vee}$ the $\mathbb{Q}$-linear dual of $\Gamma_{\mathbb{Q}}$, and let $\Gamma^{\vee} := \{\lambda\in\Gamma_{\mathbb{Q}}^{\vee}\,|\,\lambda(\Gamma)\subset\mathbb{Z}.\}$. We identify elements of $\mathbb{Q}^g\otimes_{\mathbb{Q}}\Gamma_{\mathbb{Q}}$ with g-tuples of elements of $\Gamma_{\mathbb{Q}}$ and similarly for $\mathbb{Q}^g\otimes_{\mathbb{Q}}\Gamma_{\mathbb{Q}}^{\vee}$.

(i) The pairing $\mathbb{Q}^g\times\mathbb{C}^g\ni(n,z)\mapsto {}^t n\cdot z\in\mathbb{C}$ on $\mathbb{Q}^g\times\mathbb{C}^g$ and the natural pairing $\Gamma_{\mathbb{Q}}\times\Gamma_{\mathbb{Q}}^{\vee}\to\mathbb{Q}$ induce a pairing $\langle\ ,\ \rangle : (\mathbb{Q}^g\otimes_{\mathbb{Q}}\Gamma_{\mathbb{Q}})\times(\mathbb{C}^g\otimes_{\mathbb{Q}}\Gamma_{\mathbb{Q}}^{\vee})\to\mathbb{C}$.

(ii) Let $\tilde{Q} : (\mathbb{Q}^g\otimes\Gamma_{\mathbb{Q}})\times(\mathbb{Q}^g\otimes\Gamma_{\mathbb{Q}})\to \mathrm{M}_g(\mathbb{Q})$ be the matrix-valued symmetric bilinear pairing

$$\tilde{Q} : (u,v) = ((u_1,\dots,u_g),(v_1,\dots,v_g)) \to \tilde{Q}(u,v) = \big(Q(u_i,v_j)\big)_{1\le i,j\le g} \qquad \forall u,v\in\mathbb{Q}^g\otimes_{\mathbb{Q}}\Gamma.$$

(iii) For every $A\in\mathbb{Q}^g\otimes_{\mathbb{Q}}\Gamma_{\mathbb{Q}}$, every $B\in\mathbb{Q}^g\otimes_{\mathbb{Q}}\Gamma_{\mathbb{Q}}^{\vee}$ and every element $\Omega\in\mathfrak{H}_g$ of the Siegel upper-half space of genus g, define the theta function $\theta^{Q,\Gamma}\left[\begin{smallmatrix}A\\B\end{smallmatrix}\right]$ on the (gh)-dimensional $\mathbb{C}$-vector space $\mathbb{C}\otimes_{\mathbb{Q}}\Gamma_{\mathbb{Q}}^{\vee}$ attached to (Q,Γ) by

$$\theta^{Q,\Gamma}\left[\begin{smallmatrix}A\\B\end{smallmatrix}\right](Z;\Omega) := \sum_{N\in\mathbb{Z}^g\otimes_{\mathbb{Z}}\Gamma} \mathbf{e}\big(\tfrac{1}{2}\,\mathrm{Tr}(\Omega\cdot\tilde{Q}(N+A,\,N+A)\big)\cdot\mathbf{e}(\langle N+A,\,Z+B\rangle),$$

where $\mathbf{e}(z) := \exp(2\pi\sqrt{-1}\,z)$ for all $z\in\mathbb{C}$.

Note that we have $\theta^{Q\oplus Q',\Gamma\oplus\Gamma'}\left[\begin{smallmatrix}(A,A')\\(B,B')\end{smallmatrix}\right]((Z,Z');\Omega) = \theta^{Q,\Gamma}\left[\begin{smallmatrix}A\\B\end{smallmatrix}\right](Z;\Omega)\cdot\theta^{Q',\Gamma'}\left[\begin{smallmatrix}A'\\B'\end{smallmatrix}\right](Z';\Omega)$ for the orthogonal direct sum $(Q\oplus Q',\Gamma\oplus\Gamma')$ of (Q,Γ) and (Q',Γ'). In particular if (Q,Γ) is the orthogonal direct sum of h one-dimensional quadratic forms, then $\theta^{Q,\Gamma}$ is a product of h "usual" theta functions with characteristics.

Let (Q,Γ) be a $\mathbb{Q}$-valued positive definite quadratic form. Let $T : L_{\mathbb{Q}} \to \Gamma_{\mathbb{Q}}$ be a $\mathbb{Q}$-linear isomorphism of vector spaces over $\mathbb{Q}$, and let L be a $\mathbb{Z}$-lattice in $L_{\mathbb{Q}}$. Let $T^\vee : \Gamma_{\mathbb{Q}}^\vee \to L_{\mathbb{Q}}$ be the $\mathbb{Q}$-linear dual of T. Let $Q' : L_{\mathbb{Q}} \times L_{\mathbb{Q}} \to \mathbb{Q}$ be the positive definite quadratic form on $L_{\mathbb{Q}}$ induced by Q through the isomorphism T. Let $1 \otimes T : \mathbb{Q}^g \otimes_{\mathbb{Q}} L_{\mathbb{Q}} \to \Gamma_{\mathbb{Q}}$ be the linear map induced by T; similarly for $1 \otimes T^\vee : \mathbb{C}^g \otimes_{\mathbb{Q}} \Gamma_{\mathbb{Q}}^\vee \to \mathbb{C}^g \otimes_{\mathbb{Q}} L_{\mathbb{Q}}^\vee$. Define two finite abelian groups K and Δ by

$$\begin{aligned} K &:= (1\otimes T)(\mathbb{Z}^g \otimes_{\mathbb{Z}} L)/\big((\mathbb{Z}^g\otimes_{\mathbb{Z}}\Gamma)\cap(1\otimes T)(\mathbb{Z}^g\otimes_{\mathbb{Z}} L)\big) \xrightarrow{\sim} \big[(1\otimes T)(\mathbb{Z}^g\otimes_{\mathbb{Z}} L)+(\mathbb{Z}^g\otimes_{\mathbb{Z}} L)\big]/\mathbb{Z}^g\otimes_{\mathbb{Z}} L \\ \Delta &:= (1\otimes T^\vee)^{-1}(\mathbb{Z}^g\otimes_{\mathbb{Z}} L^\vee)/\big((\mathbb{Z}^g\otimes_{\mathbb{Z}}\Gamma^\vee)\cap(1\otimes T^\vee)^{-1}(\mathbb{Z}^g\otimes_{\mathbb{Z}} L^\vee)\big) \xrightarrow{\sim} \big[(1\otimes T^\vee)^{-1}(\mathbb{Z}^g\otimes_{\mathbb{Z}} L^\vee)+(\mathbb{Z}^g\otimes_{\mathbb{Z}} L^\vee)\big]/\mathbb{Z}^g\otimes_{\mathbb{Z}} L^\vee \end{aligned}$$

(7.2) THEOREM. (GENERALIZED RIEMANN THETA IDENTITY) *For every $A \in \mathbb{Q}^g \otimes \Gamma_{\mathbb{Q}}$ and every $B \in \mathbb{Q}^g \otimes \Gamma_{\mathbb{Q}}^\vee$, the equality*

$$(\mathrm{R}_{\mathrm{ch}}^{Q,T}) \qquad \theta^{Q',L\left[\begin{smallmatrix}(1\otimes T)^{-1}A\\ (1\otimes T^\vee)B\end{smallmatrix}\right]}((1\otimes T^\vee)Z;\Omega) = \#(\Delta)^{-g}\cdot \sum_{A'\in K,\,B'\in\Delta} \mathbf{e}(-\langle A,B'\rangle)\cdot \theta^{Q,\Gamma\left[\begin{smallmatrix}A+A'\\ B+B'\end{smallmatrix}\right]}(Z;\Omega)$$

holds for all $\Omega \in \mathcal{H}_g$ and all $Z \in \mathbb{C}^g \otimes \Gamma_{\mathbb{Q}}^\vee$.

Note that each term on the right hand side of $(\mathrm{R}_{\mathrm{ch}}^{Q,T})$ is independent of the choice of B' in its congruence class modulo $\mathbb{Z}^g\otimes_{\mathbb{Z}} L^\vee$.

(7.3) Theorem 7.2 is very easy to prove once stated in that form. In two examples below (Q',L) is the diagonal quadratic form $x_1^2+\cdots+x_h^2$ on $\mathbb{Z}^h$, Γ is equal to $\mathbb{Z}^n$, and T is (given by) a matrix such that $T\cdot{}^tT$ is a multiple of the identity matrix I_h.

(a) When $h=4$, Q and Q' are both the diagonal quadratic form $x_1^2+x_2^2+x_3^2+x_4^2$ on $\mathbb{Z}^4$, $A=B=0$ and T is given by the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

the equation $(\mathrm{R}_{\mathrm{ch}}^{Q,T})$ is the classical **Riemann's theta formula** [21, p. 20].

(b) When $h=2$, $\Gamma=L=\mathbb{Z}^2$, $B=0$, Q' is $x_1^2+x_2^2$, Q is $2x_1^2+2x_2^2$, T is given by the matrix $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and $(\mathrm{R}_{\mathrm{ch}}^{Q,T})$ becomes

$$\theta\begin{bmatrix} a \\ 0 \end{bmatrix}(z,2\Omega)\cdot\theta\begin{bmatrix} b \\ 0 \end{bmatrix}(w,2\Omega) = 2^{-g}\cdot\sum_{c\in 2^{-1}\mathbb{Z}^g/\mathbb{Z}^g}\theta\begin{bmatrix} c+(a+b)/2 \\ 0 \end{bmatrix}(z+w,2\Omega)\cdot\theta\begin{bmatrix} c+(a-b)/2 \\ 0 \end{bmatrix}(z-w,2\Omega)$$

for all $z,w\in\mathbb{C}^g$ and all $a,b\in\mathbb{Q}^g$.

Much more about theta identities can be found in [19, Ch. II §6], [20, §6], [8, Ch. IV §1], and classical sources such as [13, Ch. VII §1], [14] and [21].

§8. Theta relations

(8.1) Notation. Let $d_1, d_2, \ldots, d_g$ be positive integers such that $4|d_1|d_2|\cdots|d_g$, fixed in this section.

- Let $\delta = (d_1, \ldots, d_g)$. For any positive integer n, let

$$K(n\delta) := \bigoplus_{j=1}^{g} n^{-1} d_i^{-1} \mathbb{Z}/\mathbb{Z}.$$

- For any positive integer m, let

$$K_m := \bigoplus_{j=1}^{g} m^{-1}\mathbb{Z}/\mathbb{Z}, \quad \text{and let} \quad K_m^* := \mathrm{Hom}(K_m, \mathbb{C}^*).$$

- For any non-negative integer n and any $a \in K(2^n\delta)$, we will use the following notation

$$q_n(a) = q_{n,\delta}(a) := \theta[\begin{smallmatrix}-a\\0\end{smallmatrix}](0, 2^n\Omega), \qquad Q_n(a) = Q_{n,\delta}(a) := \theta[\begin{smallmatrix}-a\\0\end{smallmatrix}](2^n z, 2^n\Omega)$$

for theta constants and theta functions, where $\Omega \in \mathcal{H}_g$ has been suppressed.

Clearly the following symmetry condition holds:

$$(\Theta_{\mathrm{ev}}^{n,\delta}) \qquad q_n(a) = q_n(-a) \qquad \forall a \in K(2^n\delta).$$

(8.2) The generalized Riemann theta identities implies a whole family of relations between theta functions and theta constants. Among them are the quadratic relations ($\Theta_{\mathrm{quad}}^{n,\delta}$) between theta functions with theta constants as coefficients below, as well as the quartic relations ($\Theta_{\mathrm{quad}}^{n,\delta}$) between theta constants, for all $n \geq 0$. The theta constants satisfy strong non-degeneracy conditions, represented by ($\Theta_{\mathrm{nv}}^{n+1,\delta}$) below.

(8.2.1) For any $a, b, c \in K(2n\delta)$ satisfying $a \equiv b \equiv c \pmod{K(n\delta)}$ and any $l \in K_2^*$, we have

$$(\Theta_{\mathrm{quad}}^{n,\delta}) \qquad \begin{aligned} 0 = {} & \Big[\sum_{\eta \in K_2} l(\eta) \cdot q_{n+1}(c+r)\Big] \cdot \Big[\sum_{\eta \in K_2} l(\eta) \cdot Q_n(a+b+\eta) \cdot Q_n(a-b+r)\Big] \\ & - \Big[\sum_{\eta \in K_2} l(\eta) \cdot q_{n+1}(b+r)\Big] \cdot \Big[\sum_{\eta \in K_2} l(\eta) \cdot Q_n(a+c+\eta) \cdot Q_n(a-c+r)\Big]. \end{aligned}$$

(8.2.2) For any $a, b, c, d \in K(2n\delta)$ satisfying $a \equiv b \equiv c \equiv d \pmod{K(n\delta)}$ and any $l \in K_2^*$, we have

$$(\Theta_{\mathrm{quar}}^{n,\delta}) \qquad \begin{aligned} 0 = {} & \Big[\sum_{\eta \in K_2} l(\eta) \cdot q_n(a+b+\eta)\, q_n(a-b+\eta)\Big] \cdot \Big[\sum_{\eta \in K_2} l(\eta) \cdot q_n(c+d+\eta)\, q_n(c-d+\eta)\Big] \\ & - \Big[\sum_{\eta \in K_2} l(\eta) \cdot q_n(a+d+\eta)\, q_n(a-d+\eta)\Big] \cdot \Big[\sum_{\eta \in K_2} l(\eta) \cdot q_n(b+c+\eta)\, q_n(b-d+\eta)\Big]. \end{aligned}$$

(8.2.3) For any $n \geq 0$, any $a_1, a_2, a_3 \in K(2^{n+1}\delta)$ and any $l_1, l_2, l_3 \in K_4^*$, there exists an element $b \in K_8 \subset K(2\delta)$ and an element $\lambda \in 2K_4^*$ such that

$$(\Theta_{\mathrm{nondeg}}^{n+1,\delta}) \qquad 0 \neq \prod_{i=1}^{3} \Big[\sum_{\eta \in K_4} (l_i + \lambda)(\eta) \cdot q_{n+1}(a_i + b + \eta)\Big].$$

(8.3) Equations defining abelian varieties. The geometric significance of these theta relations are twofold, for abelian varieties and also their moduli. Recall that $d_1,\ldots,d_n$ are positive integers with $d_1|\cdots|d_g$. Let $N=(\prod_{i=1}^g d_i)-1$.

1. Let $A_{\Omega,\delta}:=\mathbb{C}^g/(\Omega\cdot\mathbb{Z}^g+D(\delta)\cdot\mathbb{Z}^g)$ be the abelian variety whose period lattice is generated by the columns of $(\Omega\ D(\delta))$, where $D(\delta)$ is the diagonal matrix with $d_1,\ldots,d_g$ as its diagonal entries. It was proved by Lefschetz that the theta functions $\{\{Q_{0,\delta}(a)\,|\,a\in K(\delta)\}$ define a projective embedding $j: A_{\Omega,\delta}\hookrightarrow\mathbb{P}^N$ if $4|d_1|\cdots|d_g$. Mumford showed in [16, I §4] that

   > *the quadratic equations* $(\Theta^{0,\delta}_{\text{quar}})$ *in the projective coordinates of* $\mathbb{P}^N$ *cut out the abelian variety* $A_{\Omega,\delta}$ *as a subvariety inside* $\mathbb{P}^N$ *if* $4|d_1$.

   In particular an abelian variety is determined by its theta constants $q_{0,\delta}(a)$'s if the *level* δ is divisible by 4. The group law on the abelian variety can also be recovered from these theta constants.

2. The next question is: do the Riemann quartic equations $(\Theta^{n,\delta}_{\text{quar}})$ on the theta constants cut out the moduli of abelian varieties? The answer given in [16, II §6] is basically "*yes if* $8|d_1$" with a suitable non-degeneracy condition:

   Suppose that $8|d_1|\cdots|d_g$, *and* $\{q(a)\,|\,a\in K(\delta)\}$ *is a family of complex numbers indexed by* $K(\delta)$. *Assume that this given* $(\prod_{i=1}^g d_i)$*-tuple of numbers has the following properties.*

   - $q(a)=q(-a)$ for all $a\in K(\delta)$.
   - *All quartic equations in* $(\Theta^{0,\delta}_{\text{quar}})$ *hold.*
   - (The non-degeneracy condition) *There exists a family of complex numbers* $\{q_1(u)\,|\,u\in K(\delta)\}$ *indexed by* $K(2\delta)$ *which satisfies conditions* $(\Theta^{1,\delta}_{\text{ev}})$, $(\Theta^{1,\delta}_{\text{quar}})$ *and* $(\Theta^{1,\delta}_{\text{nondeg}})$, *and*

   $$q_0(a)\cdot q_0(b)=\sum_{u,v\in K(2\delta),\,u+v=a,\,u-v=b} q_1(u)\cdot q_1(v)\qquad\forall a,b\in K(\delta).$$

   Then there exists an element $\Omega\in\mathfrak{H}_g$ *such that*

   $$q_{0,\delta}(a):=\theta[\begin{smallmatrix}-a\\0\end{smallmatrix}](0,\Omega)\quad\forall a\in K(\delta).$$

(8.4) Adelic Heisenberg groups and theta measures.

(8.4.1) The analysis in [16] of theta relations is based on a finite adelic version of the Heisenberg group, which is a central extension of $\mathbb{A}_f^{2g}$ by the multiplicative group scheme $\mathbb{G}_m$ over a base field k in which 2 is invertible. Here $\mathbb{A}_f=\prod'_p\mathbb{Q}_p$, the restricted product of p-adic numbers, over all primes p which are invertible in k. One gets such a group scheme $\widehat{\mathcal{G}}(\mathscr{L})$ whenever one is handed a symmetric ample line bundle of degree one over a g-dimensional *principally polarized* abelian variety over k, plus a compatible family of theta structure for torsion points of order invertible in k, which induces an isomorphism from $\mathbb{A}_f^{2g}$ to the set of all torsion points as above.

(8.4.2) Such a Heisenberg group $\widehat{\mathcal{G}}(\mathscr{L})$ is isomorphic to "the standard finite adelic Heisenberg group" $\mathrm{Heis}(2g,\mathbb{A}_f)$ over the algebraic closure k^{alg} of k. The definition of $\mathrm{Heis}(2g,\mathbb{A}_f)$ is similar to that of the real Heisenberg group $\mathrm{Heis}(2g,\mathbb{R})$ in (6.1), with the following changes: (a) the field $\mathbb{R}$ is replaced by the ring $\mathbb{A}_f$, (b) the unit circle group $\mathbb{C}_1^\times$ is replaced by the multiplicative group scheme $\mathbb{G}_m$ over k^{alg}, and (c) the isomorphism $\mathbf{e}:\mathbb{R}/\mathbb{Z}\xrightarrow{\sim}\mathbb{C}_1^\times$ is replaced by an isomorphism from $\mathbb{A}_f/\hat{\mathbb{Z}}\cong\oplus_p\mathbb{Q}_p/\mathbb{Z}_p$ to the group $\mu_\infty(k^{\mathrm{alg}})$ of all roots of 1 in k^{alg}.

The group $\widehat{\mathcal{G}}(\mathscr{L})$ acts on the direct limit $\varinjlim_n\Gamma(A,\mathscr{L}^{\otimes n})=:\widehat{\Gamma}(\mathscr{L})$, where n runs through all positive integers which are invertible in k. This action of $\widehat{\mathcal{G}}(\mathscr{L})$ on $\widehat{\Gamma}(\mathscr{L})$ is an $\mathbb{A}_f$-version of the dual Schrödinger representation discussed in (6.2). At this point the representation-theoretic formalism for theta functions discussed in §6 carries over to the present situation.

(8.4.3) The insight gained from the systematic use of the adelic Schrödinger representation produces not only the two theorems in (8.3), but also a new way to look at theta constants: There exists a *measure* μ on $\mathbb{A}_f^g$ which satisfies the properties (i)–(iv) below. All theta relations are encoded in the simple equality in (i), and the non-degeneracy condition becomes (iv).

(i) There exists another measure ν on $\mathbb{A}_f^g$ such that $\xi_*(\mu\times\mu)=\nu\times\nu$ as measures on $\mathbb{A}_f^g\times\mathbb{A}_f^g$, where $\xi:\mathbb{A}_f^g\times\mathbb{A}_f^g\to\mathbb{A}_f^g\times\mathbb{A}_f^g$ is the map $(x,y)\mapsto(x+y,x-y)$.

(ii) The algebraic theta constants are integrals over suitable compact open subsets of $\mathbb{A}_f^g$.

(iii) The algebraic theta function attached to a non-zero global section s_0 of the one-dimensional vector space $\Gamma(A,\mathscr{L})$ as above, is the function

$$x\mapsto\theta^{\mathrm{alg}}(x)=\int(U_{(1,-x)}\cdot\delta_0)\,d\mu$$

on $\mathbb{A}_f^{2g}$, where δ_0 is the characteristic function for the compact open subset $\prod_p'\mathbb{Z}_p\subset\mathbb{A}_f^{2g}$, and $U_{(1,-x)}\cdot\delta_0$ is the result of the action on δ_0 by the the element $U_{(1,-x)}\in\mathrm{Heis}(2g,\mathbb{A}_f)$ under the dual Schrödinger representation.

(iv) For every $x\in\mathbb{A}_f^{2g}$, there exists an element $\eta\in\frac{1}{2}\prod_p'\mathbb{Z}_p$ such that $\theta^{\mathrm{alg}}(x+\eta)\neq0$.

Remark. When the base field is $\mathbb{C}$, the theta measure μ_Ω attached to $\Omega\in\mathfrak{H}_g$ is

$$\mu_\Omega(V)=\sum_{n\in V\cap\mathbb{Q}^g}\mathbf{e}\big(\tfrac{1}{2}\cdot{}^tn\cdot\Omega\cdot n\big)$$

for all compact open subset $V\subset\mathbb{A}_f^g$, and the companion measure ν_Ω is

$$\nu_\Omega(V)=\sum_{n\in V\cap\mathbb{Q}^g}\mathbf{e}\big(\tfrac{1}{2}\cdot{}^tn\cdot\Omega\cdot n\big).$$

(8.5) The best introduction to the circle of ideas in this section is [20], which also motivates you to reach for the real candies in [16]. The readers may also consult [8, Ch. IV].

Anyone who had more than a casual look at the papers [16] would agree that the results are both fundamental and deep, opening up a completely new direction in the study of theta functions.

These papers are “however, not easy to read”, and the ideas in them “have not been developed very far”.[9] It is indeed curious that there has been no “killer application” of the theory of algebraic theta functions so far. However it should be a safe bet that this anomaly won’t last too much longer.

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