# KRAFT'S CLASSIFICATION OF COMMUTATIVE GROUP SCHEMES KILLED BY $p$ OVER PERFECT FIELDS OF CHARACTERISTIC $p$

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## Contents



## 1. Introduction

This is an expository article on Kraft's classification of indecomposable commutative finite group schemes over perfect fields of characteristic $p$. Here $p$ is a prime number, fixed throughout this article. The special case when the base field is algebraically closed was used extensively in the literature, including [13], [11], [15], [4], [12], [14]. See 5.2.2 for the statement in this special case.

We fixed a *perfect* field $\kappa$ of characteristic $p$. Let $\kappa[\mathtt{F},\mathtt{V}]_\sigma$ be the Dieudonné ring modulo $p$; see 2.1.2. Explicitly, $\kappa[\mathtt{F},\mathtt{V}]_\sigma$ is a ring (with unity element 1) generated by the subring $\kappa$ and two elements $\mathtt{F}$ and $\mathtt{V}$, such that $\mathtt{F}\cdot\mathtt{V} = 0 = \mathtt{V}\cdot\mathtt{F}$, and

$$\mathtt{F}\cdot a = a^p\cdot\mathtt{F},\quad a\cdot\mathtt{V} = \mathtt{V}\cdot a^p\quad \forall\, a\in\kappa.$$

In particular $\kappa[\mathtt{F},\mathtt{V}]_\sigma$ is a (either left or right) vector space over $\kappa$, with $\{1,\mathtt{F}^n,\mathtt{V}^n : n\in\mathbb{N}_{\geq 1}\}$ as a basis. When $\kappa$ is the prime field, $\mathbb{F}_p[\mathtt{F},\mathtt{V}]_\sigma$ is a commutative $\mathbb{F}_p$-algebra, isomorphic to $\mathbb{F}_p[x,y]/(xy)$. The *covariant* Dieudonné theory establishes an equivalence of abelian categories between the category of commutative finite group schemes $G$ over $\kappa$ such that $[p]_G = 0$, and the category of left $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-modules of finite dimension over $\kappa$; c.f 2.1.2. So classification of indecomposable commutative finite group schemes killed by $p$ over $\kappa$ is the same as problem of classifying indecomposable finite dimensional left $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-modules.

We will define certain left $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-modules of finite dimension over $\kappa$, provisionally called *standard* $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-modules here. The finite group commutative schemes killed by $[p]$ attached to standard $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-modules are also said to be *standard*. Technically the input data for

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these standard $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-modules are $p$-linear representations of Kraft quivers; see 4.2 and 4.3.1 for the definitions of Kraft quivers and their $p$-linear representations.

A Kraft quiver is a finite direct graph whose arrows are labeled by F or V, such that if one reverses the all V-arrows are reverse, then each connected component of the resulting directed graph is an oriented line segment or an oriented circle. A circular Kraft quiver is *indecomposable* if it is not a nontrivial etale cover of another circular Kraft quiver. Connected Kraft quivers can be coded by words in the binary alphabet $\{\mathtt{F},\mathtt{V}^\sharp\}$. Under this encoding, finite words are in bijection with linear Kraft quivers, while periodic infinite words modulo cyclic permutations are in bijection with indecomposable circular Kraft quivers; see §3. In the case when the base field $\kappa$ is algebraically closed, every indecomposable left $\kappa[\mathtt{F},\mathtt{V}]$-module $M$ is isomorphic to the standard module attached to (the trivial representation of) either a linear Kraft quiver or an indecomposable circular Kraft quiver, and this Kraft quiver is determined by $M$ up to isomorphism.

For a general perfect base field $\kappa$ of characteristic $p$, Kraft's theorem asserts that every finite dimensional left $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-module $M$ is isomorphic to a standard $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-module attached to a $p$-linear representation $\rho_M$ of a Kraft quiver $\Gamma_M$, such that the connected components of $\Gamma$ are indecomposable and mutually non-isomorphic; see 5.2. Moreover the pair $(\Gamma_M,\rho_M)$ is determined by $M$ up to isomorphisms. However one should not be lured into thinking that there exist an equivalence between the category of finite dimensional left $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-modules and the category of $p$-linear representations of a suitable class of Kraft quivers.

It is worth noting that the only property of the Frobenius automorphism $\sigma$ in the proof of 5.2 is that it is a field automorphism of $\kappa$. Hence the statement 5.2 of Kraft's classification of finite dimensional left $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-modules holds also for rings of the form $K[X,Y]_\tau/(XY)$ as in 2.1.1, with the alphabet $\{\mathtt{F},\mathtt{V}^\sharp\}$ replaced by by $\{X,Y^\sharp\}$ of course. We will focus on $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-modules in this article.

The succinct proof in [8, Anhang] of the classification of finite dimensional left modules over the ring $\kappa[X,Y]_\tau/(XY)$-modules followed Gabriel's functorial interpretation of the Gelfand–Ponomarev classification of finite dimensional modules over the commutative $K$-algebra $K[X,Y]/(XY)$ in [5, Ch. 2]. The functorial framework of Gabriel's approach, which underlies Kraft's proof, can be found in [17, §3, p. 22]. See 6.1 for some historical discussion.

This article does not contain a complete proof of the main classification theorem 5.2. The reasons are twofolds: Firstly, extending results in [5, Ch. 2] to those in §4 is an exercise at the level of an undergraduate project. Secondly, we are unable to substantially shorten the argument in [5, Ch. 2], while keep the insights in [5, Ch. 2] as clearly visible at the same time. Instead we offer a roadmap to a proof of 5.2 in §6.

This article is structured as follows: Some background materials are collected in §2, followed by the definition of Kraft quivers in §3. The notion of $p$-linear representations of Kraft quivers and their associated "standard" $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-modules are explained in §4. The main classification results are formulated in §5. The last §6 contains a sketch of intermediate steps toward a proof of theorem 5.2, following [5, Ch. 2] closely. We hope it may help ambitious readers with the project mentioned in the previous paragraph.

## 2. Preliminaries: Krull–Remak–Schmidt theorem, semi-linear operators

### 2.1. Dieudonne rings modulo $p$ and their analogs.

**Definition 2.1.1.** Let $K$ be a field, and let $\tau$ be an automorphism of $K$.

(i) Denote by $K[X,Y]_\tau/(XY)$ the ring with 1 which contains $K$ as a subring with 1, and two elements $X, Y$, such that the following properties hold.
  - $XY = 0$.
  - $Xa = \tau(a)X$ and $aY = Y\tau(a)$ for all $a \in K$.
  - The elements $1, X, \ldots, X^n, \ldots, Y, \ldots, Y^n$, $n \geq 1$ form a $K$ basis for the (left) $K$-vector space $K[X,Y]_\tau/(XY)$.

(ii) Denote by $K[[X,Y]]_\tau/(XY)$ the formal completion of $K[X,Y]_\tau/(XY)$ with respect to the decreasing filtration formed by powers of the ideal $(X,Y)$.

Recall that for a ring endomorphism $\tau : K \to K$ of a field $K$, a map $\varphi : U \to V$ between vector spaces over $K$ is said to be *$\tau$-semilinear* if $\varphi$ is additive and $\varphi(au) = \tau(a)\varphi(u)$ for all $a \in K$ and all $u \in U$. When $K$ is a field of characteristic $p$ and $\tau = \sigma^r$ is the endomorphism $x \mapsto x^{p^r}$ for some natural number $r$, $p^r$-semilinear is synonymous to $\sigma^r$-semilinear.

Clearly a $K$-linear ring homomorphism $K[X,Y]_\tau \to \mathrm{End}_K(U)$ for a vector space $U$ over $K$ corresponds to a pair $(A, B)$, consisting of a $\tau$-linear operator $A$ and a $\tau^{-1}$-linear operator $B$ such that $A \circ B = 0 = B \circ A$. Similarly for a finite dimensional $K$-vector space $U$, a $K$-linear ring homomorphism $K[[X,Y]]_\tau \to \mathrm{End}_K(U)$ is given by a pair $(A, B)$ of operators as above, such that there exists a natural number $n \geq 1$ with $A^n = 0 = B^n$.

2.1.2. Recall that $\kappa$ is a perfect field of characteristic $p$, characteristic $p$, and $\sigma = \sigma_\kappa$ is the Frobenius automorphism $x \mapsto x^p$ of the perfect field $\kappa$. By definition the Dieudonné ring for $\kappa$ modulo $p$ is $\kappa[X,Y]_\sigma/(XY)$. We write $\mathtt{F}$ (respectively $\mathtt{V}$) for the image of $X$ (respectively $Y$) in $\kappa[X,Y]_\sigma/(XY)$, and write $\kappa[\mathtt{F},\mathtt{V}]_\sigma$ for $\kappa[X,Y]_\sigma/(XY)$.

2.2. The covariant Dieudonné functor establishes is a covariant functor $\mathbb{D}$ from the abelian category of commutative finite group schemes $G$ killed by $p$ over $\kappa$ to the category of finite dimensional left $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-modules. See [2, Appendix B.3] for a summary of versions of Dieudonné theories and references therein.

The following hold for a commutative finite group scheme $G$ with $[p]_G = 0$.

- The *order* of $G$ is equal to $\dim_\kappa(\mathbb{D}(G))$.
- $G$ is of multiplicative type (i.e. a twist of $(\mu_p)^h$ for some $h \in \mathbb{N}$) if and only if $\mathtt{F}$ is invertible on $\mathbb{D}(G)$.
- $G$ is etale if and only if $\mathtt{V}$ is invertible on $\mathbb{D}(G)$.
- $G$ is isomorphic to $(\alpha_p)^r$ for some $r \in \mathbb{N}$ if and only if $\mathtt{F}_{\mathbb{D}(G)} = 0 = \mathtt{V}_{\mathbb{D}(G)}$.
- Both $G$ and its Cartier dual $G^D$ are local if and only if $\mathtt{F}$ and $\mathtt{V}$ are both nilpotent on $\mathbb{D}(G)$.
- $G$ is a $\mathrm{BT}_1$-group if and only if $\mathrm{Ker}(\mathtt{F}_{\mathbb{D}(G)}) = \mathtt{V}(\mathbb{D}(G))$, or equivalently if and only if $\mathrm{Ker}(\mathtt{V}_{\mathbb{D}(G)}) = \mathtt{F}(\mathbb{D}(G))$.

We formulate the Krull–Remak–Schmidt theorem for objects in abelian categories which are both artinian and noetherian. It will be applied to the category of commutative finite group schemes killed by the operator $[p]$ over a base field of characteristic $p$.

**Proposition 2.3** (Krull–Remak–Schmidt)**.** *Let $\mathscr{C}$ be an abelian category, and let $M$ be an object in $\mathscr{C}$ such that the family of sub-objects of $M$ satisfies both the ascending and the descending chain condition.*

(a) *There exists a family of indecomposable objects $(M_i)_{i\in I}$ indexed by a finite set $I$, such that $M \cong \oplus_{i\in I} M_i$.*

(b) *If $(N_j)_{j\in J}$ is a collection of indecomposable objects in $\mathscr{C}$ such that $J$ is a finite set and $M \cong \oplus_{j\in J} N_j$, then there exists a bijection $f : I \to J$ such that $M_i \cong N_{f(i)}$ for each $i \in I$.*

**Remark 2.3.1.** (i) A non-zero object $N$ in an abelian category $\mathscr{C}$ is said to be *indecomposable* if $N$ is not isomorphic to a direct sum of two non-zero objects in $\mathscr{C}$

(ii) Proofs of 2.3 in the case when $\mathscr{C}$ is the category of left modules over a ring $R$ can be found in [1, §5 Th. 2] , [16, Ch. 5] and [9, Prop. X.7.4, Thm. X.7.5]. The same argument works for the general abelian categories.

(iii) For any indecomposable object $N$ in an abelian category $\mathscr{C}$, the ring $\mathrm{End}_{\mathscr{C}}(N)$ is a possibly non-commutative local ring; cf. [9, X.7.4]. Recall that a ring $R$ (with 1, and $1 \neq 0$) is said to be a *local ring* if the sum of any two non-units in $R$ is again a non-unit. If so then the Jacobson radical $\mathfrak{r}$ of $R$ is a maximal left ideal of $R$ and also a maximal right ideal of $R$, and the complement of $\mathfrak{r}$ is equal to the set the set consisting of all non-units in $R$.

We explain a classical result on $q$-semilinear operators on a finite dimensional vector space over an algebraically closed field of characteristic $p$, where $q$ is a positive power of $p$.

Proposition 2.4 below was first proved by Hasse and Witt (1936); it is also known as the Lang–Steinberg theorem. References for 2.4 are [6, §3 Satz 10], [3, §10, Prop. 5], and [18, §10]. We offer three proofs below for the convenience of the readers. The first two proofs use the fact that the Frobenius morphism of (open subschemes of) $\mathbb{A}^n$ in characteristic $p$ is purely inseparable. The third proof is more "ring-theoretic" in spirit.

**Proposition 2.4** (Hasse–Witt, Lang–Steinberg)**.** *Let $k$ be an algebraically closed field of characteristic $p$. The following two equivalent statements hold.*

(a) *Let $M$ be a finite dimensional vector space over $k$. Let $q = p^r$, $r \in \mathbb{N}_{\geq 1}$, and let $T : W \to W$ be a $q$-linear automorphism of $W$. Then there exists a basis $w_1, \dots, w_n$ of $W$ such that $T(w_i) = w_i$ for $i = 1, \dots, n$.*

(b) *Let $n \in \mathbb{N}_{\geq 1}$ be a positive integer. For any $A \in \mathrm{GL}_n(k)$, there exists an element $C \in \mathrm{GL}_n(k)$ such that*
$$C^{-1} A C^{(q)} = I_n,$$
*where $C^{(q)} := {}^{\sigma^r}C$ is the result of applying $\sigma^r$ to all entries of $C$, and $I_n$ is the unity element of $\mathrm{GL}_n(k)$.*

The equivalence of statements (a) and (b) in 2.4 is clear: (b) is the matricial form of (a).

Proof of proposition 2.4 (a). First we show that there exists a non-zero vector $w_1 \in W$ such that $T(w_1) = w_1$. We pick a basis $u_1, \dots, u_n$, and write $T(u_j) = \sum_{i=1}^n a_{ij} u_i$, $a_{ij} \in k$. We must show that there exists a non-zero element $(x_1, \dots, x_n) \in k^n$ such that $\sum_{j=1}^n a_{ij} x_j^{p^r} - x_i = 0$ for $i = 1, \dots, n$. The left hand side of this system equation defines a finite etale additive morphism from ${\mathbb{G}_\mathrm{a}}^n$ to ${\mathbb{G}_\mathrm{a}}^n$ whose kernel is a commutative finite etale group scheme of rank $p^{rn}$. In particular this group scheme has a non-zero $k$-point, which corresponds to a non-zero vector $w_1 \in W$ fixed under $T$.

It follow from induction on $\dim(W)$ that there exists a basis $w_1, v_2, \dots, v_n$ of $W$ and elements $b_2, \dots, b_n \in k$ such that $T(v_i) = v_i + b_i w_1$ for $i = 2, \dots, n$. Choose a root $c_i \in k$ of $Y^{p^r} - Y + b_i$ for $i = 2, \dots, n$. Then $w_i := v_i + c_i w_1$ satisfies $T(w_i) = w_i$ for $i = 2, \dots, n$ and $w_1, \dots, w_n$ is a basis of $W$. □

Proof of proposition 2.4 (b). For any given element $X \in \mathrm{GL}(d, k)$, let

$$f_X : \mathrm{GL}_n \to \mathrm{GL}_n$$

be the $k$-morphism from $\mathrm{GL}_n$ to itself which sends any functorial point $Y$ to $Y^{-1} X Y^{(q)}$. For any $B \in \mathrm{GL}_n(k)$, consider the derivative $df_X$

$$(df_X)_B : \mathfrak{t}_{G,B} \to \mathfrak{t}_{G, f_X(B)}.$$

of $f_X$, from the tangent space of $\mathrm{GL}_n$ at $B$ to the tangent space of $\mathrm{GL}_n$ at $f_X(B)$. Notice that the kernel of this tangential map is the same as the kernel of the derivative at $U$ of the morphism $g_X$, given by $Y \mapsto Y^{-1} X$ for all functorial points $Y$. The morphism $g_X$ is an isomorphism, hence $(df_X)_B$ is1 also an isomorphism. It follows that the morphism $f_X$ is etale, therefore there exists a Zariski dense open subset $U_X$ of $\mathrm{GL}_n$ such that $f_X(\mathrm{GL}_n(k)) \supseteq U(k)$. Applying this result to $X = A$ and to $X = I_n$, we obtain dense open subschemes $U_X$ and $U_{I_n}$ of $\mathrm{GL}_n$ such that

$$f_A(\mathrm{GL}_n(k)) \cap f_{I_d}(\mathrm{GL}_n(k)) \supseteq U_X(k) \cap U_{I_n}(k) = (U_X \cap U_{I_n})(k) \neq \emptyset,$$

because $U_X \cap U_{I_n}$ is again a dense open subscheme of $\mathrm{GL}_n$. In other words there exists elements $B, D \in \mathrm{GL}_n(k)$ such that

$$B^{-1} A B^{(q)} = D^{-1} I_n D^{(q)}.$$

Hence

$$(DB^{-1})\, A\, (BD^{-1})^{(q)} = I_n,$$

and the desired conclusion holds with $C = BD^{-1}$. □

2.4.1. Yet another proof of proposition 2.4 (a).

Let $r$ be a natural number. Consider the twisted polynomial algebra $k[T, \sigma^r]$, consisting of all $k$-linear combinations of monomials $X^n$, $n \in \mathbb{N}$, and the multiplication is determined by

$$a\, X^m \cdot b\, X^n = (a\, b^{p^m})\, X^{m+n} \quad \forall\, a, b \in k,\ \forall\, m, n \in \mathbb{N}.$$

The ring $k[X, \sigma^r]$ is a non-commutative principal ideal domain in the sense that the product of any two non-zero elements of $k[X, \sigma^r]$ is non-zero, every left ideal of $k[X, \sigma^r]$ is generated by one element, and every right ideal is also generated by one element; see [7, p. 29]. Moreover we have division algorithms in this ring $k[X, \sigma^r]$.

There is a structure theorem for finitely generated modules over not-necessarily commutative principal ideal domains which generalizes the usual structure theorem for finitely generated modules over principle ideal domains: Every finitely generated left module over a principal ideal domain is isomorphic to a direct sum of cyclic modules; see [7, Ch. 3, §§8–12].

We give $M$ the structure of a left $k[X, \sigma^r]$-module so that $X$ operates on $M$ as the operator $T$. The structure theorem above tells us that left module $M$ over the principal ideal domain$R := k[X, \sigma^r]$ is isomorphic to a direct sum of a finite number of cyclic $R$-modules of the form $R/(R\, f_i(X))$, where each $f_i(X)$ is a monic non-constant polynomial. Moreover the constant term of each polynomial $f_i(X)$ is non-zero, because $T$ is an automorphism of $M$.

Since $k$ is algebraically closed, for every non-constant polynomial $f(X) \in k[X, \sigma^r]$ whose constant term is non-zero, there exists a surjective $R$-module homomorphism $R/(R\cdot f(X)) \to R/(R\,(X-a))$ with $a \in k^\times$. To see this, write $f(X)$ as $f(X) = b_d X^d + \cdots + b_1 X + b_0$, with coefficients $b_j \in k$, $b_d \neq 0$. It suffices to pick an element $a \in k^\times$ such that $\sum_{0\le j\le d} b_j\, a^{p^{jd}} = 0$, i.e. a non-zero root of the separable polynomial $\sum_{0\le j\le d} b_j t^{p^{jd}}$. Note also that for every element $a \in k^\times$, there exists an element $c \in k^\times$ such that $a = c^{p^r-1}$, so $c^{p^r}(X-1) = (X-a)c$ in $k[X, \sigma^r]$, and the cyclic $R$-module $R/(R\,(X-a))$ is isomorphic to $R/(R\,(X-1)$

We claim that for every polynomial $f(X) \in R$ of degree $d \ge 1$ whose constant term is non-zero, there exists an $R$-module isomorphism $R/(R\,f(X)) \cong \big(R/(R\,(X-1))\big)^{\oplus d}$. An easy induction on $d$ plus the structure theorem for finitely generated $R$-modules shows that it suffices to prove the case when $f_2(X) = X^2 + bX + c$, $b, c \in k$, $c \neq 0$. There are $p^{2r}-1$ distinct non-zero roots in $k$ of the polynomial $t^{p^{2r}} + bt^{p^r} + ct$. It follows easily that there exist two $R$-module surjections from $h_1, h_2 : R/(R\,f_2(X)) \twoheadrightarrow R/(X-1)$ with $\mathrm{Ker}(h_1) \neq \mathrm{Ker}(h_2)$. So $R/(R\,f_2(X)) \cong (R/R\,(X-1))^{\oplus 2}$. $\square$

We record another consequence of the structure theory of finitely generated modules over non-commutative principal ideal domains.

**Lemma 2.4.2.** *Let $K$ be a field of characteristic $p$, and let $q = p^r$ for some positive integer $r \ge 1$. Let $M$ be a finite dimensional vector space over $K$, and let $T$ be a $q$-linear endomorphism of $M$.*

(i) *There exists a $K$-vector subspaces $N$ and $U$ of $M$ such that $M = N \oplus U$, $T(N) \subseteq N$, $T(U) \subseteq U$, such that $T$ acts on $N$ as a nilpotent operator, and $T$ induces an injective $q$-linear endomorphism of $M$. Moreover $N$ and $U$ are uniquely determined by $T$.*

(ii) *There exists $K$-vector subspaces $U_1, \ldots, U_r$ of $U$ satisfying the following properties.*
   - *$T(U_i) \subseteq U_i$ for each $i$, and $U = U_1 \oplus \cdots \oplus U_r$.*
   - *Each $U_i$ has a basis $v_{i,1}, v_{i,2}, \ldots, v_{i,s_i}$ such that*

$$T(v_{i,1}) = 0,\ T(v_{i,2}) = v_{i,1}, \ldots,\ T(v_{i,s_i}) = v_{i,s_i-1}\,.$$

(iii) *Suppose that the field $K$ is perfect. Then $T$ induces an isomorphism on $U$. Moreover there exist a natural number $n_0$ such that $N = \mathrm{Ker}(T^n)$ and $U = T^n(M)$ for all $n \ge n_0$.*

**Remark.** Combining 2.4.2 with 2.4, we obtain an analog of Jordan canonical form for a $q$-linear operators on a finite dimensional vector space over an algebraically closed field of characteristic $p$, but substantially simplified: We still need the same Jordan blocks with generalized eigenvalue 0, as in 2.4.2 (ii). Other than these nilpotent Jordan blocks, we only need the $1 \times 1$ identity matrix (as a $1 \times 1$ Jordan block).

## 3. Kraft quivers

**Definition 3.1.** A *finite directed graph with arrows (or directed edges) labeled by* F *or* V is a a quintuple $\Gamma = (\mathscr{V}, \mathscr{E}, \mathrm{label}, \mathrm{source}, \mathrm{target})$, where

- $\mathscr{V}$ is a finite set (called *vertices* of $\Gamma$),
- $\mathscr{E}$ is also a finite set (called *arrows* or*directed edges* of $\Gamma$),
- $\mathrm{label} : \mathscr{E} \to \{\mathtt{F}, \mathtt{V}\}$ is a map from $\mathscr{E}$ to the set consisting of two symbols F and V, and
- $\mathrm{source}, \mathrm{target} : \mathscr{E} \to \mathscr{V}$ are maps from $\mathscr{E}$ to $\mathscr{V}$.

For any arrow $E$, $\mathrm{source}(E)$ and $\mathrm{target}(E)$ are the *source* and the *target* of $E$ respectively; they are the vertices of $E$. An arrow $E$ is a *loop* if $\mathrm{source}(E) = \mathrm{target}(E)$. We say that $E$ is an F-arrow or a V-arrow according to whether $\mathrm{label}(E)$ is equal to F or V.

**Definition 3.2.** A finite directed graph $\Gamma = (\mathscr{V}, \mathscr{E}, \mathrm{label}, \mathrm{source}, \mathrm{target})$ with arrows labeled by F or V is a *Kraft quiver* if the following properties hold.

(i) For each vertex $v$ of $\Gamma$, there are at most two edges connected to $v$.
(ii) Two distinct F-arrows cannot share the same source nor the same target. Similarly two distinct V-arrows cannot share the same source nor the same target.
(iii) No vertex can be the target of an F-arrow and also the source of a V-arrow. Similarly no vertex can be the target of a V-arrow and also the source of an F-arrow.

3.3. (a) Recall that a finite (undirected) graph is a triple $(\mathscr{V}, \mathscr{E}, \mathrm{vertices})$, where $\mathscr{V}$ and $\mathscr{E}$ are finite sets, called the set of vertices and edges respectively, and vertices is a map from $\mathscr{E}$ to the family of subsets of $V$ with cardinality 1 or 2. Every finite graph has a *geometric realization*, which is a one-dimensional finite CW complex with $\mathscr{V}$ and $\mathscr{E}$ as its 0-dimensional (respectively 1-dimensional) cells.

For any Kraft quiver $\Gamma$, denote by $|\Gamma|$ the geometric realization of the finite graph underlying $\Gamma$.

(b) We have an obvious notion of morphisms between directed graphs (respectively Kraft quivers). We spell out the latter. Let

$$\Gamma_i = (\mathscr{V}_{\Gamma_i}, \mathscr{E}_{\Gamma_i}, \mathrm{label}_{\Gamma_i}, \mathrm{source}_{\Gamma_i}, \mathrm{target}_{\Gamma_i}) \quad i = 1, 2$$

be two Kraft quivers. A morphism $\xi$ from $\Gamma_1$ to $\Gamma_2$ is a pair of maps

$$\big(\, \xi^{(0)} : \mathscr{V}_{\Gamma_1} \to \mathscr{V}_{\Gamma_2},\ \xi^{(1)} : \mathscr{E}_{\Gamma_1} \to \mathscr{E}_{\Gamma_2} \,\big)$$

such that

- $\xi^{(0)}(\mathrm{source}_{\Gamma_1}(E)) = \mathrm{source}_{\Gamma_2}(\xi^{(1)}(E))$,
- $\xi^{(0)}(\mathrm{target}_{\Gamma_1}(E)) = \mathrm{target}_{\Gamma_2}(\xi^{(1)}(E))$, and
- $\mathrm{label}_{\Gamma_1}(E) = \mathrm{label}_{\Gamma_2}(\xi^{(1)}(E))$

for every arrow $E \in \mathscr{E}_{\Gamma_1}$.

(c) For any Kraft quiver $\Gamma = (\mathscr{V}, \mathscr{E}, \mathrm{label}, \mathrm{source}, \mathrm{target})$ be a Kraft quiver, denote by $\Gamma_{\mathtt{V}^\sharp}$ the directed graph with arrows labeled by $\mathtt{F}$ or $\mathtt{V}^\sharp$, obtained from $\Gamma$ by reversing all $\mathtt{V}$-arrows and changing their labels to $\mathtt{V}^\sharp$. More formally,

$$\Gamma_{\mathtt{V}^\sharp} = (\mathscr{V}, \mathscr{E}^\sharp, \mathrm{label}^\sharp, \mathrm{source}, \mathrm{target})$$

has the same set of vertices as $\Gamma$, and there is a bijection $\tau : \mathscr{E} \to \mathscr{E}^\sharp$ such that

- $\mathrm{label}^\sharp(\tau(E)) = \begin{cases} \mathtt{F} & \text{if } \ \mathrm{label}(E) = \mathtt{F} \\ \mathtt{V}^\sharp & \text{if } \ \mathrm{label}(E) = \mathtt{V} \end{cases}$
- $\mathrm{source}(\tau(E)) = \begin{cases} \mathrm{source}(E) & \text{if } \ \mathrm{label}(E) = \mathtt{F} \\ \mathrm{target}(E) & \text{if } \ \mathrm{label}(E) = \mathtt{V} \end{cases}$
- $\mathrm{target}(\tau(E)) = \begin{cases} \mathrm{target}(E) & \text{if } \ \mathrm{label}(E) = \mathtt{F} \\ \mathrm{source}(E) & \text{if } \ \mathrm{label}(E) = \mathtt{V} \end{cases}$

for all $E \in \mathscr{E}$. Clearly $\Gamma$ determines and is determined by $\Gamma_{\mathtt{V}^\sharp}$. Moreover $\tau$ determines a canonical isomorphism from $|\Gamma|$ to $|\Gamma_{\mathtt{V}^\sharp}|$. Here $\mathtt{V}^\sharp$ is treated as a symbol different from $\mathtt{V}$ and $\mathtt{F}$. It is tempting to use $\mathtt{V}^{-1}$ instead of $\mathtt{V}^\sharp$. We resisted that because $\mathtt{V}$ does not have an inverse at this point.

Examples of Kraft quivers $\Gamma$ and their associated quivers $\Gamma_{\mathtt{V}^\sharp}$ with $\mathtt{V}$-arrows inverted on pages 11 and 13.

As stated in 3.4 below, every connected component of the directed graph underlying $\Gamma_{\mathtt{V}^\sharp}$ is either a *directed* linear segment or a *directed* cycle.

**Lemma 3.4.** *Let* $\Gamma = (\mathscr{V}, \mathscr{E}, \mathrm{label}, \mathrm{source}, \mathrm{target})$ *be a Kraft quiver, and let* $\Gamma_{\mathtt{V}^\sharp}$ *be the directed graph with arrows labeled by* $\mathtt{F}$ *or* $\mathtt{V}^\sharp$ *attached to* $\Gamma$ *as in* 3.3 (c) *above.*

(i) *If an arrow* $E$ *in* $\Gamma$ *is a loop, then the vertex of* $E$ *is different from the vertices of all other arrows* $E' \neq E$. *In other words* $E$ *together with its vertex form a connected component of the geometric realization* $|\Gamma|$ *of* $\Gamma$.

(ii) *A connected component of* $|\Gamma|$ *which is neither a loop nor an isolated point is isomorphic to a line segment or a circle as a simplicial complex. More precisely, each connected component* $C$ *of the geometric realization* $|\Gamma_{\mathtt{V}^\sharp}|$ *of* $\Gamma_{\mathtt{V}^\sharp}$ *which does not come from a loop nor is an isolated point is of one of the following two types.*

(a) ($C$ is a line segment) *There exist arrows* $E_1^\sharp, \ldots, E_m^\sharp$ *in* $\Gamma_{\mathtt{V}^\sharp}$ *such that*
 - $\mathrm{target}(E_i^\sharp) = \mathrm{source}(E_{i+1}^\sharp)$ *for* $i = 1, \ldots, m-1$, *and*
 - $C$ *is the* disjoint *union of the* 0*-cells corresponding to the* $m{+}1$ *vertices* $\mathrm{source}(E_1)$, so *and the interiors of the* 1*-cells attached to the* $m$ *arrows* $E_1^\sharp, \ldots, E_m^\sharp$.

 *In other words, the arrows* $E_1^\sharp, \ldots, E_m^\sharp$ *determines an orientation of the line segment* $|C|$, *and trace* $C$ *from its origin to its end.*

(b) ($C$ is a simplicial circle) *There exist arrows* $E_1^\sharp, \ldots, E_m^\sharp$ *in* $\Gamma_{\mathtt{V}^\sharp}$ *such that*
 - $\mathrm{target}(E_i^\sharp) = \mathrm{source}(E_{i+1}^\sharp)$ *for* $i = 1, \ldots, m-1$, $\mathrm{target}(E_m^\sharp) = \mathrm{source}(E_1^\sharp)$,

– *$C$ is the* disjoint *union of the 0-cells corresponding to the $m$ vertices*

$$\mathrm{source}(E_1^\sharp), \ldots, \mathrm{source}(E_m^\sharp)$$

*and the interiors of the $m$ 1-cells attached to the arrows $E_1^\sharp, \ldots, E_m^\sharp$.*

*In other words the arrows $E_1^\sharp, \ldots, E_m^\sharp$ of $\Gamma_{\mathtt{V}^\sharp}$ successively go through the above $m$ points and define an orientation of the circle $C$.*

*Here is a more natural formulation. Both the set of vertices $\mathscr{V}_{\Gamma_{\mathtt{V}^\sharp}}$ and the set of arrows $\mathscr{E}_{\Gamma_{\mathtt{V}^\sharp}}$ of $\Gamma_{\mathtt{V}^\sharp}$ have natural structures as torsors under $\mathbb{Z}/m\mathbb{Z}$. Moreover*

– $\mathrm{source}(E^\sharp + 1 \bmod m) = \mathrm{source}(E^\sharp) + 1 \bmod m = \mathrm{target}(E^\sharp)$

– $\mathrm{target}(E^\sharp - 1 \bmod m) = \mathrm{target}(E^\sharp) - 1 \bmod m = \mathrm{source}(E^\sharp)$

*for every arrow $E^\sharp \in \mathscr{E}_{\Gamma_{\mathtt{V}^\sharp}}$.*

The proof of 3.4 is left as an exercise.

**Definition 3.4.1.** (a) A connected Kraft quiver $\Gamma$ is said to be *linear* if $|\Gamma|$ consists of a single point, or if $|\Gamma|$ is a line segment as in 3.4 (ii)(a).

(b) A connected Kraft quiver $\Gamma$ is said to be *circular* if $|\Gamma|$ is homeomorphic to a circle. In other words, either $\Gamma$ consists of a single vertex and a single loop, or if $|\Gamma|$ is a simplicial circle as in 3.4 (ii)(b).

(c) A morphism $\alpha : \Gamma_1 \to \Gamma_2$ between Kraft quivers is *etale* if the map $|\alpha| : |\Gamma_1| \to |\Gamma_2|$ induced $\alpha$ is a local homeomorphism at every point of $\Gamma_1$. Such an etale morphism $\alpha$ is said to be *surjective* if $\alpha$ induces surjections $\mathscr{V}_{\Gamma_1} \to \mathscr{V}_{\Gamma_2}$ and $\mathscr{E}_{\Gamma_1} \to \mathscr{E}_{\Gamma_2}$; equivalently, if $|\alpha| : |\Gamma_1| \to |\Gamma_2|$ is a surjection.

It is easy to see that if $\Gamma_1$ and $\Gamma_2$ are both connected and the etale morphism $\alpha$ is not an isomorphism, then both $\Gamma_1$ and $\Gamma_2$ are circular.

**Exercise 3.4.2.** Let $\Gamma$ be a Kraft quiver.

(i) Show that there exists a Kraft quiver $\bar{\Gamma}$ and an etale surjective morphism $\pi : \Gamma \to \bar{\Gamma}$ satisfying the following property: for any etale surjective morphism $\alpha : \Gamma \to \Gamma'$ of Kraft quivers, there exists a unique etale surjective morphism $\beta : \Gamma' \to \bar{\Gamma}$ such that $\pi = \beta \circ \alpha$. We call $\bar{\Gamma}$ the *indecomposable etale quotient* of $\Gamma$

(ii) Show that no two connected components of the indecomposable etale quotient $\bar{\Gamma}$ of $\Gamma$ are isomorphic.

**Definition 3.5.** Let $\Gamma = \big(\mathscr{V}_\Gamma, \mathscr{E}_\Gamma, \mathrm{label}_\Gamma, \mathrm{source}_\Gamma, \mathrm{target}_\Gamma\big)$ be a Kraft quiver. The Kraft quiver $\Gamma^\vee = \big(\mathscr{V}_{\Gamma^\vee}, \mathscr{E}_{\Gamma^\vee}, \mathrm{label}_{\Gamma^\vee}, \mathrm{source}_{\Gamma^\vee}, \mathrm{target}_{\Gamma^\vee}\big)$ *dual* to $\Gamma$ is defined as follows.

- The vertices of $\Gamma^\vee$ consists of elements written in the form $v^\vee$, $v \in \mathscr{V}_\Gamma$, so that $v \mapsto v^\vee$ is a bijection from $\mathscr{V}_\Gamma$ to $\mathscr{V}_{\Gamma^\vee}$.
- The arrows of $\Gamma^\vee$ consists of elements written in the form $E^\vee$, $E \in \mathscr{E}_\Gamma$, so that $E \mapsto E^\vee$ is a bijection from $\mathscr{E}_\Gamma$ to $\mathscr{E}_{\Gamma^\vee}$.
- The maps $\mathrm{label}_{\Gamma^\vee} : \mathscr{V}_{\Gamma^\vee} \to \{\mathtt{F}, \mathtt{V}\}$, $\mathrm{source}_{\Gamma^\vee} : \mathscr{E}_{\Gamma^\vee} \to \mathscr{V}_{\Gamma^\vee}$, $\mathrm{target}_{\Gamma^\vee} : \mathscr{E}_{\Gamma^\vee} \to \mathscr{V}_{\Gamma^\vee}$ are given by

  – $\mathrm{label}_{\Gamma^\vee}(v^\vee) = \begin{cases} \mathtt{F} & \text{if } \mathrm{label}_\Gamma(v) = \mathtt{V} \\ \mathtt{V} & \text{if } \mathrm{label}_\Gamma(v) = \mathtt{F} \end{cases}$ for every vertex $v^\vee$ of $\Gamma^\vee$, and

– $\mathrm{source}_{\Gamma^\vee}(E^\vee) = \mathrm{target}_\Gamma(E)$ for every arrow $E^\vee$ of $\Gamma^\vee$,
– $\mathrm{target}_{\Gamma^\vee}(E^\vee) = \mathrm{source}_\Gamma(E)\ \ \forall\, E^\vee \in \mathscr{E}_{\Gamma^\vee}$.

In other words, the Kraft quiver $\Gamma^\vee$ is obtained from $\Gamma$ by reversing the arrows of $\Gamma$ and flipping their labels at the same time.

Clearly we have a canonical isomorphism $\Gamma \xrightarrow{\sim} (\Gamma^\vee)^\vee$ of Kraft quivers, given by the pair of bijections $\big(\mathscr{V} \to (\mathscr{V}^\vee)^\vee, \mathscr{E} \to (\mathscr{E}^\vee)^\vee\big)$ defined by $v \mapsto ((v^\vee)^\vee\ \ \forall\, v \in \mathscr{V}$, and $E \mapsto ((E^\vee)^\vee\ \ \forall\, E \in \mathscr{E}$ respectively.

### 3.6. Kraft quivers attached to finite words in the alphabet $\{\mathtt{F}, \mathtt{V}^\sharp\}$

Lemma 3.4 provides an alternative description of *connected* Kraft quivers in terms of words in an alphabet consisting of two letters $\mathtt{F}$ and $\mathtt{V}^\sharp$. This alternative for linear Kraft quivers is explained here. Circular Kraft quivers will be discussed in 3.7.

Recall that a word in the alphabet $\{\mathtt{F}, \mathtt{V}^\sharp\}$ is a map

$$w : \{1, \dots, m\} \longrightarrow \{\mathtt{F}, \mathtt{V}\}$$

from a finite interval $\{1, \dots, m\}$ of $\mathbb{N}$ (or a linearly ordered finite set) to the set $\{\mathtt{F}, \mathtt{V}^\sharp\}$ consisting two symbols $\mathtt{F}$ and $\mathtt{V}^\sharp$. The *length* of such a word is $m$, the cardinality of the domain of $w$. The word with length 0 is the *empty word*.

**Convention.** When expressing a finite word $w$ as above, we will put $w(2)$ to the *left* of $w(1)$, $w(3)$ to the left of $w(2)$, etc., similar to when expressing natural numbers in base 10. For instance $\mathtt{F}\mathtt{V}^\sharp\mathtt{F}\mathtt{F}\mathtt{V}^\sharp$ denotes the word of length 5 such that $w(1) = w(4) = \mathtt{V}^\sharp$, and $w(2) = w(3) = w(5) = \mathtt{F}$. The reason for adopting this convention is the custom of writing an operator at the *left* of the argument. People often write $w_i$ for $w(i)$, so we will write a typical finite word $w$ as $(w_m, \dots, w_1)$.

**Definition 3.6.1.** Let $w = w_h \dots w_1$ be a finite word in the alphabet $\{\mathtt{F}, \mathtt{V}^\sharp\}$. Define the Kraft quiver $\Gamma(w)$ attached to $w$ as follows.

- The set $\mathscr{V}(w)$ of vertices of $\Gamma(w)$ is a set $\{z_0, \dots, z_h\}$ with $h + 1$ elements.
- The set $\mathscr{E}(w)$ of edges of $\Gamma(w)$ is a set $\{E_1, \dots, E_h\}$ with $h$ elements.
- The map

  $$\mathrm{label} : \mathscr{E}(w) = \{E_1, \dots, E_h\} \longrightarrow \{\mathtt{F}, \mathtt{V}^\sharp\}$$

  is given by

  $$\mathrm{label}(E_i) = \begin{cases} \mathtt{F} & \text{if } \ w_i = \mathtt{F} \\ \mathtt{V} & \text{if } \ w_i = \mathtt{V}^\sharp \end{cases}$$

- The map

  $$\mathrm{source} : \mathscr{E}(w) = \{E_1, \dots, E_h\} \longrightarrow \{z_0, \dots, z_h\} = \mathscr{V}(w)$$

  is given by

  $$\mathrm{source}(E_i) = \begin{cases} z_{i-1} & \text{if } \ w_i = \mathtt{F} \\ z_i & \text{if } \ w_i = \mathtt{V}^\sharp \end{cases}$$

- The map
$$\mathrm{target} : \mathscr{E}(w) = \{E_1, \ldots, E_h\} \longrightarrow \{z_0, \ldots, z_h\} = \mathscr{V}(w)$$
is given by
$$\mathrm{target}(E_i) = \begin{cases} z_i & \text{if } \ w_i = \mathtt{F} \\ z_{i-1} & \text{if } \ w_i = \mathtt{V}^\sharp \end{cases}$$

In particular each entry $w_i$ of the word $w$ corresponds to an arrow, whose label is $\mathtt{F}$ if and only if $w_i = \mathtt{F}$. The directed graph underlying $\Gamma(w)_{\mathtt{V}^\sharp}$ is a directed line segment with $h+1$ vertices $z_0, \ldots, z_h$, and $h$ arrows going from $z_{i-1}$ to $z_i$ for $i = 1, \ldots, h$.

**Example 3.6.2.** The Kraft squiver $\Gamma(\mathtt{FV^\sharp FFV^\sharp})_{\mathtt{V}^\sharp}$ attached to the word $\mathtt{FV^\sharp FFV^\sharp}$ of length 5 is
$$z_0 \xrightarrow{\mathtt{V}^\sharp} z_1 \xrightarrow{\mathtt{F}} z_2 \xrightarrow{\mathtt{F}} z_3 \xrightarrow{\mathtt{V}^\sharp} z_4 \xrightarrow{\mathtt{F}} z_5 \,.$$
The Kraft quiver $\Gamma(\mathtt{FV^\sharp FFV^\sharp})$
$$z_0 \xleftarrow{\mathtt{V}} z_1 \xrightarrow{\mathtt{F}} z_2 \xrightarrow{\mathtt{F}} z_3 \xleftarrow{\mathtt{V}} z_4 \xrightarrow{\mathtt{F}} z_5$$
is obtained from $\Gamma(\mathtt{FV^\sharp FFV^\sharp})_{\mathtt{V}^\sharp}$ by reversing all $\mathtt{V}^\sharp$-arrows and changing their labels to $\mathtt{V}$.

**Example 3.6.3.** The linear Kraft quiver attached to the finite word
$$w = w_{17} \ldots w_1 = (\mathtt{V}^\sharp)^2 \mathtt{F}^4 (\mathtt{V}^\sharp)^3 \mathtt{F}^2 \mathtt{V}^\sharp \mathtt{F}^5$$
is
$$\begin{tikzcd}
z_0 \arrow[r, "\mathtt{F}"] & z_1 \arrow[r, "\mathtt{F}"] & z_2 \arrow[r, "\mathtt{F}"] & z_3 \arrow[r, "\mathtt{F}"] & z_4 \arrow[r, "\mathtt{F}"] & z_5 \\
& & & & & z_6 \arrow[u, "\mathtt{V}"] \arrow[r, "\mathtt{F}"] & z_7 \arrow[r, "\mathtt{F}"] & z_8 \\
& & & & & & & z_9 \arrow[u, "\mathtt{V}"] \\
& & & & & & & z_{10} \arrow[u, "\mathtt{V}"] \\
& & & & & & & z_{11} \arrow[u, "\mathtt{V}"] \arrow[r, "\mathtt{F}"] & z_{12} \arrow[r, "\mathtt{F}"] & z_{13} \arrow[r, "\mathtt{F}"] & z_{14} \arrow[r, "\mathtt{F}"] & z_{15} \\
& & & & & & & & & & & z_{16} \arrow[u, "\mathtt{V}"] \\
& & & & & & & & & & & z_{17} \arrow[u, "\mathtt{V}"]
\end{tikzcd}$$

**Exercise 3.6.4.** Let $w = (w_m, \ldots, w_1)$ be a word in the binary alphabet $\{\mathtt{F}, \mathtt{V}^\sharp\}$, and let $\Gamma(w)$ be the linear Kraft quiver attached to $w$. Define the *dual word* of $w$ to be the word $w^\vee = (w_m^\vee, \ldots, w_2^\vee)$ of the same length as $w$, such that $w_i^\vee \neq w_i$ for all $i$. In other words
$$w_i^\vee = \begin{cases} \mathtt{V} & \text{if } w_i = \mathtt{F} \\ \mathtt{F} & \text{if } w_i = \mathtt{V} \end{cases}$$

for all $i = 1, \ldots, m$. Show that the the Kraft quiver $\Gamma(w^\vee)$ attached to the dual word $w^\vee$ of $w$ is isomorphic to the dual Kraft quiver $\Gamma(w)^\vee$ of $\Gamma(w)$.

### 3.7. Kraft quivers attached to cyclic words in the alphabet $\{\mathtt{F}, \mathtt{V}^\sharp\}$

**Definition 3.7.1.** (a) An infinite word in the alphabet $\{\mathtt{F}, \mathtt{V}^\sharp\}$ is an infinite sequence

$$\tilde{w} : \mathbb{N}_{\geq 1} \to \{\mathtt{F}, \mathtt{V}^\sharp\},$$

where $w(i)$ is either $\mathtt{F}$ or $\mathtt{V}^\sharp$ for every $i \in \mathbb{N}_{\geq 1}$.
(b) An infinite word $\tilde{w} = (w_i)_{i\geq 1}$ is *cyclic* if there is an integer $r \geq 1$ such that $w_{i+r} = w_i$ for every $i$. Such an integer $r$ is said to be a *period* of $\tilde{w}$. Often one denotes $w(i)$ by $w_i$, and write $\tilde{w}$ as $(w_i)_{i\geq 1}$.
(c) Let $w_h, \ldots, w_1$ be elements of $\{\mathtt{F}, \mathtt{V}^\sharp\}$, where $h \geq 1$ is a positive integer. Denote by $\lfloor w_h \ldots w_1 \rceil$ the cyclic word with $h$ as a period such that $w(j) = w_i$ for any $j \geq 1$ and any $i \in \{h, \ldots, 1\}$ such that $j \cong i \pmod{h}$.

**Definition 3.7.2.** Let $\tilde{w} = (w_i)_{i\geq 1}$ be a cyclic word in the alphabet $\{\mathtt{F}, \mathtt{V}^\sharp\}$, and let $h$ be a positive integer which is a period of $\tilde{w}$. Define a Kraft quiver $\Gamma(\tilde{w}, h)$ as follows.

- The set $\mathscr{V}(\tilde{w}, h)$ of vertices of $\Gamma(\tilde{w}, h)$ is in bijection with the set of all half-integers modulo translation by $h\mathbb{Z}$. We will denote by $v_{(2i-1)/2}$ the element of $\mathscr{V}(\tilde{w}, h)$ corresponding to the class $(2i-1)/2 + h\mathbb{Z}$ for any $i \in \mathbb{Z}$, so that $\mathscr{V}(\tilde{w}, h) = \{v_{1/2}, \ldots, v_{(2h-1)/2}\}$. Note that $v_{(2i-1)/2}$ depends only on the congruence class of $i$ modulo $h$.
- The set $\mathscr{E}(\tilde{w}, h)$ of arrows of $\Gamma(\tilde{w}, h)$ is in bijection of $\tilde{w}$ modulo translation by $h$. Denote by $E_i$ the arrow of $\Gamma(\tilde{w}, h)$ corresponding to $w_i$ for any $i \geq 1$. Thus $E_i = E_{i+h}$ for all $i$, and $\mathscr{E}(\tilde{w}, h) = \{E_h, \ldots, E_1\}$.
- The map
$$\text{label} : \mathscr{E}(\tilde{w}, h) = \{E_1, \ldots, E_h\} \to \{\mathtt{F}, \mathtt{V}\}$$
is given by
$$\text{label}(E_i) = \begin{cases} \mathtt{F} & \text{if } \ w_i = \mathtt{F} \\ \mathtt{V} & \text{if } \ w_i = \mathtt{V}^\sharp \end{cases}$$
- The map
$$\text{source} : \mathscr{E}(\tilde{w}, h) = \{E_1, \ldots, E_h\} \longrightarrow \{v_{1/2}, \ldots, v_{(2h-1)/2}\} = \mathscr{V}(\tilde{w}, h)$$
is given by
$$\text{source}(E_i) = \begin{cases} v_{(2i-1)/2} & \text{if } \ w_i = \mathtt{F} \\ v_{(2i+1)/2} & \text{if } \ w_i = \mathtt{V}^\sharp \end{cases}$$
- The map
$$\text{target} : \mathscr{E}(\tilde{w}, h) = \{E_1, \ldots, E_h\} \longrightarrow \{v_{1/2}, \ldots, v_{(2h-1)/2}\} = \mathscr{V}(\tilde{w}, h)$$
is given by
$$\text{target}(E_i) = \begin{cases} v_{(2i+1)/2} & \text{if } \ w_i = \mathtt{F} \\ v_{(2i-1)/2} & \text{if } \ w_i = \mathtt{V}^\sharp \end{cases}$$

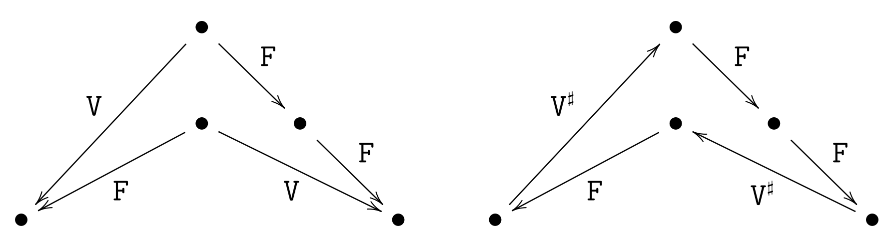
We say that $\Gamma(\tilde{w}, h)$ is *indecomposable* if $h$ divides all periods of the cyclic word $\tilde{w}$, i.e. if $h$ is the smallest positive period of $\tilde{w}$. This is equivalent to the condition that every etale surjection morphism $\alpha : \Gamma(\tilde{w}, h) \to \Gamma'$ from $\Gamma$ to another Kraft quiver $\Gamma'$ is an isomorphism. It is also equivalent to the condition that for every positive period $n$ of $\tilde{w}$, the natural map $\Gamma(\tilde{w}, n) \to \Gamma(\tilde{w}, h)$ identifies $\Gamma(\tilde{w}, h)$ as the indecomposable etale quotient of $\Gamma(\tilde{w}, n)$ in the sense of 3.4.2.

**Remark.** Let $\tilde{w} = (w_i)_{i\geq 1}$ and $\tilde{u} = (u_i)_{i\geq 1}$ are cyclic words in the alphabet $\{\mathtt{F}, \mathtt{V}^\sharp\}$ with a common period $h$. Suppose that $\tilde{u}$ is a *tail* of $\tilde{w}$, in the sense that there exists $n_0 \in \mathbb{N}$ such that $u_i = w_{n_0+i}$ for all $i \geq 1$. Then the Kraft quivers $\Gamma(\tilde{u}, h)$ and $\Gamma(\tilde{w}, h)$ are isomorphic. This statement is immediate from the definition.

**Example.** (a) The Kraft quiver $\Gamma(\lfloor \mathtt{V}^\sharp\mathtt{FF}\rceil, 3)$ attached to the cyclic word $\lfloor \mathtt{V}^\sharp\mathtt{FF}\rceil$ and the quiver $\Gamma(\lfloor \mathtt{V}^\sharp\mathtt{FF}\rceil, 3)_{\mathtt{V}^\sharp}$ labeled by $\mathtt{F}$ or $\mathtt{V}^\sharp$ are depicted below.


Arrows of the Kraft quiver $\Gamma(\lfloor \mathtt{V}^\sharp\mathtt{FF}\rceil)$ are labeled by $\mathtt{F}$ or $\mathtt{V}$. Reversing the $\mathtt{V}$-arrows in $\Gamma(\lfloor \mathtt{V}^\sharp\mathtt{FF}\rceil)$ and relabeling them by $\mathtt{V}^\sharp$ gives $\Gamma(\lfloor \mathtt{V}^\sharp\mathtt{FF}\rceil)_{V^\sharp}$. Notice that the arrows in $\Gamma(\lfloor \mathtt{V}^\sharp\mathtt{FF}\rceil)_{V^\sharp}$ determines an orientation of its geometric realization, which is a circle.

(b) The Kraft quiver $\Gamma(\lfloor \mathtt{V}^\sharp\mathtt{F}\mathtt{V}^\sharp\mathtt{FF}\rceil, 5)$ attached to the cyclic word $\lfloor \mathtt{V}^\sharp\mathtt{F}\mathtt{V}^\sharp\mathtt{FF}\rceil$ and its associated quiver $\Gamma(\lfloor \mathtt{V}^\sharp\mathtt{F}\mathtt{V}^\sharp\mathtt{FF}\rceil, 5)_{\mathtt{V}^\sharp}$ are given below.




(c) Consider the cyclic words $\lfloor \mathtt{F}\rceil$ and $\lfloor \mathtt{V}^\sharp\rceil$ with period 1. The Kraft quivers $\Gamma(\lfloor \mathtt{F}\rceil, 1)$ (respectively $\Gamma(\lfloor \mathtt{V}^\sharp\rceil, 1)$) consists of a single vertex and single loop, labeled by $\mathtt{F}$ (respectively $\mathtt{V}$).

**Exercise 3.7.3.** Exhibit an cyclic word $\lfloor w_h \ldots w_1\rceil$ with $h$ as small as possible, such that the Kraft quivers $\Gamma(\lfloor w_h \ldots w_1\rceil)$ and $\Gamma(\lfloor w_1 \ldots w_h\rceil)$ are not isomorphic.

**Exercise 3.7.4.** Let $\tilde{w} = (w_i)_{i\geq 1}$ be a cyclic word in the binary alphabet $\{\mathtt{F}, \mathtt{V}^\sharp\}$, let $h \geq 1$ be a period of $\tilde{w}$ and let $\Gamma(\tilde{w}, h)$ be the circular Kraft quiver attached to $(\tilde{w}, h)$. Define the *dual word* of $\tilde{w}$ to be the infinite word $\tilde{w}^\vee = (w_i^\vee)_{i\geq 1}$ such that $w_i^\vee \neq w_i$ for all $i$. In other words

$$w_i^\vee = \begin{cases} \mathtt{V} & \text{if } w_i = \mathtt{F} \\ \mathtt{F} & \text{if } w_i = \mathtt{V} \end{cases}$$

for all $i \geq 1$. Show that the dual Kraft quiver $\Gamma(\tilde{w}, h)^\vee$ of $\Gamma(\tilde{w}, h)$ is isomorphic to the Kraft quiver $\Gamma(\tilde{w}^\vee, h)$ attached to the dual cyclic word $w^\vee$ and period $h$.

## 4. Commutative groups schemes killed by $p$ attached to Kraft quivers

**Definition 4.1.** Let $\Gamma = (\mathscr{V}, \mathscr{E}, \text{label}, \text{source}, \text{target})$ be a Kraft quiver. Let $\kappa$ be a perfect field of characteristic $p$, and let $\sigma : \kappa \to \kappa$ be the Frobenius automorphism $x \mapsto x^p$ of $\kappa$.

(a) Define a left $\kappa[\mathtt{F}, \mathtt{V}]_\sigma$-module $M(\Gamma, \kappa)$ as follows.

(i) $M(\Gamma, \kappa)$ is a vector space over $\kappa$ with the set $\mathscr{V}$ of vertices as a basis.

(ii) $\mathtt{F}$ operates on $M(\Gamma)$ as the $\sigma$-linear (or $p$-linear) operator such that for every basis element $v \in \mathscr{V}$ of $M(\Gamma, \kappa)$,

$$\mathtt{F}(v) = \begin{cases} \text{target}(E) & \text{if } \exists\, E \in \mathscr{E} \text{ s.t. label}(E) = \mathtt{F} \text{ and source}(E) = v \\ 0 & \text{otherwise} \end{cases}$$

(iii) $\mathtt{V}$ operates on $M(\Gamma)$ as the $\sigma^{-1}$-linear (or $p^{-1}$-linear) operator such that for every basis element $v \in \mathscr{V}$ of $M(\Gamma, \kappa)$,

$$\mathtt{V}(v) = \begin{cases} \text{target}(E) & \text{if } \exists\, E \in \mathscr{E} \text{ s.t. label}(E) = \mathtt{V} \text{ and source}(E) = v \\ 0 & \text{otherwise} \end{cases}$$

In (ii) above, the vertex target$(E)$ is regarded as a standard basis element of $M(\Gamma, \kappa)$. Since there can be at most one $\mathtt{F}$-arrow $E$ with source$(E) = v$ according to 3.2 (ii), $\mathtt{F}(v)$ is well-defined. Similarly $\mathtt{V}(v)$ in (iii) is well-defined. Condition 3.2 (iii) ensures that the compositions of the operators $\mathtt{F} \circ \mathtt{V}$ and $\mathtt{V} \circ \mathtt{F}$ are both 0, so $M(\Gamma, \kappa)$ is indeed a left $\kappa[\mathtt{F}, \mathtt{V}]_\sigma$-module.

(b) Denote by $G(\Gamma)_\kappa$ the commutative finite group scheme over $\kappa$ attached to $M(\Gamma, \kappa)$ via the covariant Dieudonné theory.

Clearly $G(\Gamma)_\kappa$ is killed by $[p]$. Moreover $G(\Gamma)_\kappa$ is naturally isomorphic to

$$G(\Gamma)_{\mathbb{F}_p} \times_{\text{Spec}(\mathbb{F}_p)} \text{Spec}(\kappa),$$

so for an arbitrary (and not necessarily perfect) field $K$ of characteristic $p$, we have a commutative finite group scheme

$$G(\Gamma)_K := G(\Gamma)_{\mathbb{F}_p} \times_{\text{Spec}(\mathbb{F}_p)} \text{Spec}(K)$$

killed by $[p]$ attached to any given Kraft quiver $\Gamma$.

**Lemma 4.1.1.** *Let $K$ be a field of characteristic $p$.*

(a) *Let $\Gamma_1, \Gamma_2$ be Kraft quivers, and let $\Gamma$ be the disjoint union of $\Gamma_1$ and $\Gamma_2$. Then $G(\Gamma)_K$ is isomorphic to $G(\Gamma_1)_K \times_{\text{Spec}(K)} G(\Gamma_2)_K$.*

(b) *Suppose that $K$ contains a finite field $\mathbb{F}_{p^d}$ with $p^d$ elements. Then the group scheme $G(\lfloor \mathtt{F} \rceil, d)_K$ is isomorphic to the constant group scheme $\underline{(\mathbb{Z}/p\mathbb{Z})}^{\oplus d}$ over $K$. Similarly the group scheme $G(\lfloor \mathtt{V} \rceil, d)_K$ is isomorphic to $\mu_p^{\oplus d}$ over $K$.*

The proof of 4.1.1 is left as an exercise.

**Lemma 4.1.2.** *Let $K$ be a field of characteristic $p$, let $\Gamma$ be a Kraft quiver. Let $\Gamma^\vee$ be the dual Kraft quiver of $\Gamma$. The commutative finite group scheme $G(\Gamma^\vee)_K$ over $K$ attached to $\Gamma^\vee$ is isomorphic to the Cartier dual $G(\Gamma)_K^D$ of $G(\Gamma)_K$.*

**Lemma 4.1.3.** *Let $\Gamma = (\mathscr{V}, \mathscr{E}, \mathrm{label}, \mathrm{source}, \mathrm{target})$ be a Kraft quiver, and let $K$ be a field of characteristic $p$. The commutative group scheme $G(\Gamma)_K$ over $K$ is a $BT_1$ group if and only if every connected component of $\Gamma$ is circular. In other word, every connected component of the geometric realization $|\Gamma|$ of $\Gamma$ is homeomorphic to a circle.*

PROOF. Clearly we may assume that the base field $K$ is perfect. By 4.1.1, we may also assume that $\Gamma$ is connected. We need to show that

$$\dim_K \mathrm{Ker}(\mathtt{F}|_{M(\Gamma,K)}) = \dim_K \mathtt{V}(M(\Gamma, K)) \ \text{ if and only if } \ \Gamma \text{ is circular.}$$

The definition of the operators $\mathtt{F}, \mathtt{V}$ on $M(\Gamma, K)$ tells us the following.

(i) $\dim_K \mathtt{V}(M(\Gamma, K))$ is equal to the number of $\mathtt{V}$-arrows, i.e.

$$\dim_K \mathtt{V}(M(\Gamma, K)) = \mathrm{card}\,\{E \in \mathscr{E} \mid \mathrm{source}(E) = \mathtt{V}\}$$

(ii) $\dim_K \mathrm{Ker}(\mathtt{F}|_{M(\Gamma,K)})$ is equal to the number of vertices which are not the source of any $\mathtt{F}$-arrow. Hence

$$\begin{aligned}\dim_K \mathrm{Ker}(\mathtt{F}|_{M(\Gamma,K)}) &= \mathrm{card}\,\mathscr{V} - \mathrm{card}\,\{E \in \mathscr{E} \mid \mathrm{source}(E) = \mathtt{F}\} \\ &= \mathrm{card}\,\mathscr{V} - \mathrm{card}\,\mathscr{E} + \mathrm{card}\,\{E \in \mathscr{E} \mid \mathrm{source}(E) = \mathtt{V}\}\end{aligned}$$

Since the Kraft quiver $\Gamma$ is connected,

$$\mathrm{card}\,\mathscr{V} = \begin{cases} \mathrm{card}\,\mathscr{E} + 1 & \text{if } \ \Gamma \text{ is linear,} \\ \mathrm{card}\,\mathscr{E} & \text{if } \ \Gamma \text{ is circular.} \end{cases}$$

The lemma follows. □

The $\kappa[\mathtt{F}, \mathtt{V}]_\sigma$-modules $M(\Gamma, \kappa)$ in 4.1 (i) attached to Kraft quivers is a $p$-linear analog of linear representations attached to quivers, where a one-dimensional vector space is attached to each vertex and an isomorphism between one-dimensional vector space is attached to each arrow. In definition 4.2 below, the restriction on the dimension of vector spaces attached to vertices is lifted, resulting in a more general construction of finite commutative group schemes killed by $[p]$ over perfect base fields of characteristic $p$

**Definition 4.2.** Let $\Gamma = (\mathscr{V}, \mathscr{E}, \mathrm{label}, \mathrm{source}, \mathrm{target})$ be a Kraft quiver. Let $\kappa$ be a perfect field of characteristic $p$, and let $\sigma$ be the automorphism $x \mapsto x^p$ of $\kappa$.

(a) A *$p$-linear representation of $\Gamma$ over $\kappa$* is a family $(\rho, U) = \big(\rho_E, U_v\big)_{v\in\mathscr{V}, E\in\mathscr{E}}$, where $U_v$ is a finite dimensional vector space over $\kappa$ for each $v \in \mathscr{V}$, and

$$\rho_E : U_{\mathrm{source}(E)} \longrightarrow U_{\mathrm{target}(E)}$$

is a $\sigma$-linear *isomorphism* if label$(E) = \mathtt{F}$, and a $\sigma^{-1}$-linear *isomorphism* if label$(E) = \mathtt{V}$. Here $\sigma$ denotes the Frobenius automorphism $x \mapsto x^p$ of $\kappa$. Recall that a map $T : W_1 \to W_2$ between $\kappa$-vactor spaces is said to be $\sigma$-linear, or $p$-linear, if $T$ respects addition and $T(cx) = \sigma(c)T(x)$ for all $c \in \kappa$ and all $x \in W_1$.

The $p$-linear representation of $\Gamma$ over $\kappa$ such that $U_v = \kappa$ for each vertex $v$ and $\rho_E(1) = 1$ for each arrow $E$ is called the *trivial* $p$-linear representation of $\Gamma$ over $\kappa$, and denoted by $(\Gamma, \mathbf{1}_\kappa)$.

(b) Let $(\rho, U)$ and $(\phi, W)$ be $p$-linear representations over $\kappa$ of the Kraft quiver $\Gamma$. A homomorphism from $(\rho, U)$ to $(\phi, W)$ is a family $h = \big(h_v : U_v \to W_v\big)_{v \in \mathscr{V}}$, where $h_v$ is a $\kappa$-linear maps for each $v \in \mathscr{V}$, such that

$$h_{\mathrm{target}(E)} \circ \rho_E = \phi_E \circ h_{\mathrm{source}(E)}$$

for each arrow $E$ in $\Gamma$.

Clearly such a homomorphism $h$ is an isomorphism from $(\rho, U)$ to $(\phi, W)$, if and only if $h_v$ is an isomorphism from $U_v$ to $W_v$ for every vertex $v$.

**Definition 4.2.1.** (a) Let $\kappa'$ be a perfect field extension of $\kappa$. Let $(\rho, U)$ be a $p$-linear representation over $\kappa$ of a Kraft quiver $\Gamma$. Denote by $(\rho, U) \otimes_\kappa \kappa'$ the $p$-linear representation over $\kappa'$ of $\Gamma$, which associate to every vertex $v \in \mathscr{V}$ the $\kappa'$-vector space $U'_v = U_v \otimes_\kappa \kappa'$, and to every arrow $E \in \mathscr{E}$ the semi-linear map

$$\rho'_E = \rho_E \otimes_\kappa \kappa' : U_{\mathrm{source}} \otimes_\kappa \kappa' \longrightarrow U_{\mathrm{target}} \otimes_\kappa \kappa'$$

(b) Let $\Gamma', \Gamma$ be Kraft quivers, and let $\xi = \big(\xi^{(0)} : \mathscr{V}' \to \mathscr{V},\ \xi^{(1)} : \mathscr{E}' \to \mathscr{E}\big)$ be a morphism from $\Gamma'$ to $\Gamma$. Let $(\rho, U)$ be a $p$-linear representation of $\Gamma$ over a perfect field $\kappa$ of characteristic $p$. The pull-back $\xi^*(\rho, U) =: (\rho', U')$ of $(\rho, U)$ by $f$ is defined in the obvious way:

$$U_{v'} = U_{\xi^{(0)}(v')} \quad \forall\, v' \in \mathscr{V}',$$

and

$$\rho'_{E'} = \rho_{\xi^{(1)}(E')} : U_{\mathrm{source}(\xi^{(1)}(E'))} \longrightarrow U_{\mathrm{target}(\xi^{(1)}(E'))} \quad \forall\, E' \in \mathscr{E}'\,.$$

**Remark.** (i) The category of all $p$-linear representations of a Kraft quiver $\Gamma$ over a perfect field $\kappa$ of characteristic $p$ has a natural structure as an abelian category, with morphisms between $p$-linear representations of $\Gamma$ given by 4.2 (a).

More generally, the definition of pull-backs in 4.2.1 (b) makes the category of $p$-linear representations of Kraft quivers a *fibered category* over the category of Kraft quivers.

(ii) Suppose that $(\rho, U)$ and $(\phi, W)$ are two $p$-linear representations of $\Gamma$ over $\kappa$, then we can define a third $p$-linear representation $(\rho \otimes \phi, U \otimes W)$ of $\Gamma$ as follows:

- $(U \otimes W)_v := U_v \otimes_\kappa W_v$ for every vertex $v$ of $\Gamma$,
- $(\rho \otimes \phi)_E := \rho_E \otimes_\kappa \phi_E$ for every arrow $E$ of $\Gamma$.

We will not use this tensor product in the rest of this book.

(iii) There is no good "push-forward" construction for $p$-linear representations Kraft quivers under arbitrary morphisms of Kraft quivers. However 4.3 below tells us how to push $p$-linear representations forward under etale surjective morphisms of Kraft quivers.

**Definition 4.3.** Let $\Gamma_i = (\mathscr{V}_i, \mathscr{E}_i, \mathrm{label}_i, \mathrm{source}_i, \mathrm{target}_i)$ be a Kraft quiver for $i = 1, 2$, and let $\xi = (\xi^{(0)}, \xi^{(1)}) : \Gamma_1 \to \Gamma_2$ be an etale surjective morphism of Kraft quivers. Let $(\rho, U)$ be a $p$-linear representation of $\Gamma_1$ over a perfect field $\kappa$ of characteristic $p$. Define $\xi_*(\rho, U)$ to be the $p$-linear representation $(\phi, W)$ of $\Gamma_2$ such that

$$W_w = \bigoplus_{v \in (\xi^{(0)})^{-1}(w)} U_v \quad \forall\, w \in \mathscr{V}_2$$

and the semi-linear map $\phi_{E'} : W_{\mathrm{source}(E')} \longrightarrow W_{\mathrm{target}(E')}$ is given by

$$\phi_{E'} = \bigoplus_{E \in (\xi^{(0)})^{-1}(E')} \rho_E$$

for every arrow $E' \in \mathscr{E}_2$

It is easy to see that this push-forward functor $\xi_*$ from the category of $p$-linear representations of $\Gamma_1$ over $\kappa$ to the category of $p$-linear representations of $\Gamma_2$ over $\kappa$ is the right adjoint of the pull-back functor $\xi^*$ defined in 4.2.1 (b).

**Definition 4.3.1.** Let $(\rho, U)$ be a $p$-linear representation of $\Gamma$ as in 4.2 (a) above.
(i) The Dieudonné module $M(\Gamma, \rho, U)$ attached to $(\rho, U)$ is the left $\kappa[\mathtt{F}, \mathtt{V}]_\sigma$-module whose underlying $\kappa$-vector space is the direct sum

$$M(\Gamma, \rho, U) = \bigoplus_{v \in \mathscr{V}} U_v,$$

such that for each $v \in \mathscr{V}$, we have

$$\mathtt{F}|_{U_v} = \begin{cases} \rho_E & \text{if } \exists \text{ an } \mathtt{F}\text{-arrow } E \text{ with } \mathrm{source}(E) = v, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathtt{V}|_{U_v} = \begin{cases} \rho_E & \text{if } \exists \text{ an} \mathtt{V}\text{-arrow } E \text{ with } \mathrm{source}(E) = v, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Denote by $G(\Gamma, \rho, U)$ the commutative finite group scheme over $\kappa$ with $M(\Gamma, \rho, U)$ as its covariant Dieudonné module.

Clearly if $\rho$ is the trivial $p$-linear representation $\mathbf{1}_\kappa$ over $\kappa$ of $\Gamma$, then the left $\kappa[\mathtt{F}, \mathtt{V}]_\sigma$-module defined $M(\Gamma, \mathbf{1}_\kappa)$ defined in (a) is the module $M(\Gamma, \kappa)$ in 4.1, and the group scheme $G(\Gamma, \mathbf{1}_\kappa)$ defined in (b) becomes $G(\Gamma)_\kappa$.

**Remark 4.3.2.** (i) Let $\kappa'$ be a perfect extension field of $\kappa$. Let $G(\Gamma, U, \rho)_\kappa$ commutative finite group scheme over $\kappa$ attached to a $p$-linear representation $(\rho, U)$ of a Kraft quiver $\Gamma$, and let $(\rho', U')$ be the base extension to $\kappa'$ of $(\rho, U)$. Then $G(\Gamma, \rho', U')_\kappa$ is naturally isomorphic to $G(\Gamma, \rho, U) \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(\kappa')$.
(ii) Let $\left(\Gamma_i\right)_{i \in \pi_0(\Gamma)}$ be the family of connected components of $\Gamma$. Denote by $(\rho_i, U_i)$ the restriction to $\Gamma_i$ of the $p$-linear representation $(\rho, U)$ of $\Gamma$ for each connected component $\Gamma_i$ of the Kraft quiver $\Gamma$. Then the Dieudonné module $M(\Gamma, \rho, U)$ is canonically isomorphic to the direct sum $\bigoplus_{i \in \pi_0(\Gamma)} M(\Gamma_i, \rho_i, U_i)$, and the commutative group scheme $G(\Gamma, \rho, U)$ is canonically isomorphic to the product $\prod_{i \in \pi_0(\Gamma)} G(\Gamma_i, \rho_i, U_i)$.

**Lemma 4.3.3.** *Let $\xi : \Gamma_1 \to \Gamma_2$ be an etale surjective morphism of Kraft quivers, and let $(\rho, U)$ be a $p$-linear representation over a field $\kappa$ of characteristic $p$ as in* 4.3*. The commutative finite group scheme $G(\Gamma_2, \xi_*(\rho, U))_\kappa$ attached to the push-forward $\xi_*(\rho, U)$ of $(\rho, U)$ is canonically isomorphic to the group scheme $M(\rho, U)_\kappa$ attached to $(\rho, U)$.*

The proof of 4.3.3 is immediate from the definitions.

**Definition 4.3.4.** Let $\Gamma = \left(\mathscr{V}_\Gamma, \mathscr{E}_\Gamma, \mathrm{label}_\Gamma, \mathrm{source}_\Gamma, \mathrm{target}_\Gamma\right)$ be a Kraft quiver, and let $\kappa$ be a perfect field of characteristic $p$. Let $(\rho, U)$ be a $p$-linear representation of $\Gamma$ over $\kappa$. The $p$-linear representation $(\rho^\vee, U^\vee)$ of $\Gamma^\vee$ *dual* (or *contragradient*) to $(\rho, U)$ is defined as follows.

- $U^\vee_{v^\vee} = \mathrm{Hom}_\kappa(U_v, \kappa)$ for every vertex $v^\vee$ of $\Gamma^\vee$,
- $\rho^\vee_{E^\vee}(u^\vee)(u) = \begin{cases} \left(u^\vee(\rho_E(u))\right)^p & \text{if } \mathrm{label}_\Gamma(E) = \mathtt{V} \\ \left(u^\vee(\rho_E(u))\right)^{p^{-1}} & \text{if } \mathrm{label}_\Gamma(E) = \mathtt{F} \end{cases}$
  for each arrow $E^\vee$ in $\Gamma^\vee$, every $u^\vee \in U^\vee_{\mathrm{source}_{\Gamma^\vee}(E^\vee)} = \mathrm{Hom}_\kappa(U_{\mathrm{target}_\Gamma(E)}, \kappa)$ and every $u \in U_{\mathrm{source}_\Gamma(E)}$.

**Remark.** In the context of 4.3.4, we have a natural isomorphisms

$$(\rho, U) \xrightarrow{\sim} \left((\rho^\vee)^\vee, (U^\vee)^\vee\right).$$

of $p$-linear representations, compatible with the natural isomorphism

$$\Gamma \xrightarrow{\sim} (\Gamma^\vee)^\vee$$

of Kraft quivers.

**Lemma 4.3.5.** *Let $\Gamma$, $\Gamma^\vee$ be mutually dual Kraft quivers as in* 3.5*, and let $(\rho, U)$ and $(\rho^\vee, U^\vee)$ be mutually dual $p$-linear representations over $\kappa$ of $\Gamma$ and $\Gamma^\vee$ respectively. The Cartier dual of the commutative finite group scheme $G(\Gamma, \rho, U)$ is canonically isomorphic to $G(\Gamma^\vee, \rho^\vee, U^\vee)$, the commutative finite group scheme attached to the $p$-linear representation $(\rho^\vee, U^\vee)$ of $\Gamma^\vee$.*

Proof. The Dieudonné module $M_{\Gamma^\vee, \rho^\vee, U^\vee}$ of is naturally isomorphic to the $\kappa$-linear dual of $M(\Gamma, \rho, U)$. The canonical $\kappa$-bilinear pairing

$$M(\Gamma, \rho, U) \times M(\Gamma^\vee, \rho^\vee, U^\vee) \longrightarrow \kappa$$

respects the operators $\mathtt{F}$ and $\mathtt{V}$, according to the definition of the semi-linear operators $\mathtt{F}$ and $\mathtt{V}$ on $M(\Gamma^\vee, \rho^\vee, U^\vee)$, so it induces an isomorphism between the Cartier dual of $G(\Gamma, \rho, U)$ and the commutative finite group scheme $G(\Gamma^\vee, \rho^\vee, U^\vee)$ over $\kappa$. □

**Lemma 4.4.** *Let $\Gamma$ be a connected linear Kraft quiver. Let $\kappa$ be a perfect field of characteristic $p$. Let $(\rho, U)$ be a $p$-linear representation of $\Gamma$ over $\kappa$. Let $d := \dim_\kappa(U_v)$ for any vertex $v$ of $\Gamma(w)$. Then $(\rho, U)$ is isomorphic to the direct sum of $d$ copies of the trivial representation of $\Gamma$ over $\kappa$. Consequently $M(\Gamma, \rho, U)$ is isomorphic to $M(\Gamma, \kappa)^{\oplus d}$.*

Proof. Consider the directed graph $\Gamma_{V^\sharp}$, with arrows $E_1^\sharp, \ldots, E_m^\sharp$ such that the target of $E_i^\sharp$ is equal to the source of $E_{i+1}^\sharp$ for $i = 1, \ldots, m-1$ as in 3.4 (a). The vertices $v_0, \ldots, v_m$ in $\Gamma_{\mathtt{V}^\sharp}$ are identified with vertices in $\Gamma$, and the arrows $E_1^\sharp, \ldots, E_m^\sharp$ in $\Gamma_{\mathtt{V}^\sharp}$ correspond to arrows $E_1, \ldots, E_m$ of $\Gamma$. Pick a $\kappa$-basis $u_{v_0,1}, \ldots, u_{v_0,d}$ of $U_{v_0}$. Define inductively a basis $u_{v_i,1}, \ldots, u_{v_i,d}$ of $U_{v_i}$ for $i = 1, \ldots, m$ as follows:

$$u_{v_i,j} := \begin{cases} \rho_{E_i}(u_{v_{i-1,j}}) & \text{if } E_i = \mathtt{F} \\ \rho_{E_i}^{-1}(u_{v_{i-1,j}}) & \text{if } E_i = \mathtt{V} \end{cases} \qquad \forall\, j = 1, \ldots, d,$$

where $E_i$ is the unique arrow in $\Gamma_u$ connecting $v_{i-1}$ with $v_i$. The we have $p$-linear subrepresentations $(\rho_j, U_j)$ of $(\rho, U)$, $j = 1, \ldots, d$, with $U_{j,v_i} = \kappa \cdot v_{i,j}$ for all $i = 0, \ldots, m$, and

$$(\rho, U) = (\rho_1, U_1) \oplus \cdots \oplus (\rho_d, U_d).$$

Moreover the $p$-linear representation $(\rho_i, U_i)$ is isomorphic to the trivial representation, for each $i = 1, \ldots, d$. □

**Lemma 4.5.** *Let $\Gamma$ be a connected circular Kraft quiver, let $k$ be an* algebraically closed *field of characteristic $p$, and let $(\rho, U)$ be a $p$-linear representation of $\Gamma$ over $k$. Let $d := \dim_k(U_v)$ for any vertex $v$ of $\Gamma$. Then $(\rho, U)$ is isomorphic to the direct sum of $d$ copies of the trivial $p$-linear representations of $\Gamma$. Consequently the Dieudonné module $M(\Gamma, \rho, U)$ is isomorphic to $M(\Gamma)_k^{\oplus d}$.*

PROOF. Let $m = \mathrm{card}(\mathscr{V}_\Gamma) = \mathrm{card}(\mathscr{E}_\Gamma)$. We use the notation in 3.4 (b), so that the sets of vertices $\mathscr{V}^\sharp$ and edges $\mathscr{E}^\sharp$ of the directed graph $\Gamma_{\mathtt{V}^\sharp}$ are endowed with compatible $\mathbb{Z}/m\mathbb{Z}$-torsor structures. Recall that the set $\mathscr{V}^\sharp$ of vertices of $\Gamma_{\mathtt{V}^\sharp}$ is identified with the set $\mathscr{V}$ of vertices of $\Gamma$, and we have a bijection $\tau : \mathscr{E} \to \mathscr{E}^\sharp$ between the sets of vertices of $\Gamma$ and $\Gamma_{\mathtt{V}^\sharp}$.

For each arrow $E^\sharp \in \mathscr{E}^\sharp$, define a define a $p$-linear isomorphism

$$\phi_{E^\sharp} : U_{\mathrm{source}(E^\sharp)} \xrightarrow{\sim} U_{\mathrm{target}(E^\sharp)}$$

by

$$\phi_{E^\sharp} = \begin{cases} \rho_{\tau^{-1}(E^\sharp)} & \text{if } \ \mathrm{label}^\sharp(E^\sharp) = \mathtt{F}, \\ \rho_{\tau^{-1}(E^\sharp)}^{-1} & \text{if } \ \mathrm{label}^\sharp(E^\sharp) = \mathtt{V}^\sharp, \text{ or equivalently, if } \ \mathrm{label}(\tau^{-1}(E^\sharp)) = \mathtt{V} \end{cases}$$

Pick a vertex $v_0 \in \mathscr{V}^\sharp = \mathscr{V}$, Let $E_0^\sharp$ be the arrow in $\Gamma_{\mathtt{V}^\sharp}$ with $\mathrm{source}(E_0^\sharp) = v_0$. Consider the $\sigma^n$-linear automorphism

$$\Phi_{v_0} := \phi_{E_0^\sharp + n-1 \bmod n} \circ \cdots \circ \phi_{E_0^\sharp + 1 \bmod n} \circ \phi_{E_0^\sharp}$$

of $U_{v_0}$. By the Hasse–Witt theorem 2.4, there exists a $k$-basis $u_{v_0,1}, \ldots, u_{v_0,d}$ of $U_{v_0}$ such that $\Phi_{v_0}(u_{v_0,i}) = u_i$ for $i = 1, \ldots, d$. Propagate this basis inductively to the $k$-vector spaces $U_{v_0+1 \bmod n}, \ldots, U_{v_0+n-1 \bmod n}$ by the $p^n$-linear isomorphisms $\phi_v$, namely

$$u_{v_0+i+1 \bmod n,\, j} := \phi_{v_0+i}(u_{v_0+i \bmod n,\, j}) \quad \forall\, j = 1, \ldots, d$$

for $i = 0, 1, \ldots, n-2$. Then the subset $\{u_{v,j} \mid j = 1, \ldots, n\}$ of $U_v$ is a $k$-basis of $U_v$ for every vertex $v$. Let $(\rho_j, U_j)$ be the $p$-linear subrepresentation of $(\rho, U)$ such that $U_{j,v} = k \cdot u_{v,j}$ for each vertex $v$. Then $(\rho, U)$ is the direct sum of the $(\rho_j, U_j)$'s, $j = 1, \ldots, d$. Moreover the $p$-linear representation $(\rho_j, U_j)$ is isomorphic to the trivial representation of the Kraft quiver $\Gamma$, for $j = 1, \ldots, d$. □

**Definition 4.5.1.** For any connected circular Kraft quiver $\Gamma$, any perfect field $\kappa$ of characteristic $p$, and any $p$-linear representation $(\rho, U)$ of $\Gamma$ over $\kappa$, the argument in the proof of 4.5 gives us $\sigma^m$-linear automorphisms

$$\Phi_v : U_v \to U_v, \quad v \in \mathscr{V}_\Gamma,$$

where $m = \mathrm{card}(\mathscr{V}_\Gamma) = \mathrm{card}(\mathscr{E}_\Gamma)$. We call $\Phi_v$ the (forward) *monodromy operator* of $(\rho, U)$ at the vertex $v$.

**Lemma 4.5.2.** *Let $\Gamma$ be a connected circular Kraft quiver, with $m$ vertices and arrows. and let $\kappa$ be a perfect field. Let $(\rho, U)$ and $(\rho', U')$ be $p$-linear representations of $\Gamma$ over $\kappa$. Let $v_0$ be a vertex of $\Gamma$. Then $(\rho, U)$ is isomorphic to $(\rho', U')$ if and only if the monodromy operators $\Phi_{v_0} : U_{v_0} \to U_{v_0}^{(p^m)}$ for $(\rho, U)$ at $v_0$ is $\sigma^m$-conjugate to the monodromy operators $\Phi'_{v_0} : U'_{v_0} \to {U'_{v_0}}^{(p^m)}$ for $(\rho', U')$, in the sense that there exists a $\kappa$-linear isomorphism*

$$T : U_{v_0} \xrightarrow{\sim} U'_{v_0}$$

*such that*

$$\Phi'_{v_0} \circ T = T^{(p^m)} \circ \Phi_{v_0}$$

*where*

$$T^{(p^m)} = T \otimes_{(\kappa,\sigma^m)} \kappa \, : \, U_{v_0}^{(p^m)} = U_{v_0} \otimes_{(\kappa,\sigma^m)} \kappa \;\longrightarrow\; U'_{v_0} \otimes_{(\kappa,\sigma^m)} \kappa = {U'_{v_0}}^{(p^m)} .$$

## 5. Classification of commutative finite group schemes killed by $p$

**Proposition 5.1.** *Let $\kappa$ be a perfect field of characteristic $p$, and let $\Gamma$ be a connected Kraft quiver. Let $(\rho, U)$ be an indecomposable $p$-linear representation of $\Gamma$ over $\kappa$, i.e. $(\rho, U)$ is not isomorphic to the direct sum of two non-zero $p$-linear representations of $\Gamma$ over $\kappa$. Assume that the canonical $\Gamma \to \bar{\Gamma}$ from $\Gamma$ to its indecomposable etale quotient $\bar{\Gamma}$ described in* 3.4.2 *is an isomorphism. Equivalently, assume that $\Gamma$ is indecomposable if $\Gamma$ is a circular Kraft cover. Then the left $\kappa[\mathtt{F}, \mathtt{V}]_\sigma$-module $M(\Gamma, \rho, U)$ is indecomposable. Equivalently, the commutative finite group scheme $G(\Gamma, \rho, U)_\kappa$ over $\kappa$ is indecomposable.*

**Corollary 5.1.1.** *Let $K$ be a field of characteristic $p$, and let $\Gamma$ be a* connected *Kraft quiver. which is isomorphic to its indecomposable etale quotient. Then the finite commutative group scheme $G(\Gamma)_K$ is indecomposable in the abelian category of finite commutative group schemes over $K$. In other words $G(\Gamma)_K$ is not isomorphic to the direct sum of two non-trivial commutative group schemes over $K$.*

**Remark.** Clearly 5.1.1 is equivalent to its special case in which $K$ is assumed to be algebraically closed.

**Theorem 5.2.** *Let $\kappa$ be a perfect field of characteristic $p$. Let $G$ be a commutative finite group scheme over $\kappa$ killed by $p$.*

(i) *There exist*

- *a Kraft quiver $\Gamma$ isomorphic to its own indecomposable etale quotient such thatg no two connected components are isomorphic,*

- *a $p$-linear representation $(\rho, U)$ of $\Gamma$ over $\kappa$, and*
- *an isomorphism $G(\Gamma, \rho, U) \xrightarrow{\sim} G$ of group schemes over $\kappa$.*

(ii) *The pair $\big(\Gamma, (\rho, U)\big)$ in* (i) *is determined by $G$ up to isomorphism.*

(iii) *The commutative group scheme $G$ over $\kappa$ is indecomposable if and only $\Gamma$ is connected and the $p$-linear representation $(\rho, U)$ over $\kappa$ is indecomposable.*

**Corollary 5.2.1.** *Let $G$ be a commutative finite group scheme over a perfect field $\kappa$ of characteristic $p$, and let $(\Gamma, (\rho, U))$ be the corresponding Kraft quiver and $p$-linear representation as in* 5.2

(i) *$G$ is a* $\mathrm{BT}_1$ *group scheme over $\kappa$ if and only if every irreducible component of $\Gamma$ is circular.*
(ii) *$G$ is etale (respectively multiplicative) if and only if $\Gamma$ consists of a single $\mathtt{V}$-loop. (respectively a single $\mathtt{F}$-loop).*
(iii) *Both $G$ and its Cartier dual $G^D$ are local if and only if every circular connected component of $\Gamma$ contains both $\mathtt{F}$-arrows and $\mathtt{V}$-arrows.*

**Corollary 5.2.2.** *Let $k$ be an* algebraically closed *field of characteristic $p$. Let $G$ a commutative finite group scheme over $k$ killed by $[p]$.*

(i) *There exist*

- *a Kraft quiver $\Gamma$ such that every circular connected component of $\Gamma$ is indecomposable and no two connected components of $\Gamma$ are isomorphic, and*
- *an isomorphism $G(\Gamma)_k \xrightarrow{\sim} G$ over $k$.*

*Such a Kraft quiver $\Gamma$ is determined by $G$ up to isomorphism.*

(ii) *$G$ is indecomposable if and only if the Kraft quiver $\Gamma$ corresponding to $G$ is connected and indecomposable.*

(iii) *The set of isomorphism classes of indecomposable commutative finite group schemes killed by $p$ over $\kappa$ is in natural bijection with the set of isomorphism classes of connected indecomposable Kraft quivers.*

Corollary 5.2.2 follows from 5.2, 4.4, 4.3.3, and 4.5.

Corollary 5.2.3 below is an equivalent form of 5.2.2.

**Corollary 5.2.3.** *Let $k$ be an algebraically closed field of characteristic $p$.*

(i) *For any finite word $w = w_h \ldots w_1$ in the alphabet $\{\mathtt{F}, \mathtt{V}^\sharp\}$, the commutative group scheme $G(w)_k$ over $i$ attached to $w$ is indecomposable. Moreover $G(w)_k$ is* not *a* $\mathrm{BT}_1$ *group.*
(ii) *For any two finite words $w \neq w'$, the group schemes $G(w)_k$ and $G(w')_k$ are not isomorphic.*
(iii) *Let $\tilde{w}$ be an infinite periodic word in the alphabet $\{\mathtt{F}, \mathtt{V}^\sharp\}$, and let $h$ be the (smallest positive) period of $\tilde{w}$. The commutative group scheme $G(\tilde{w}, h)_k$ over $i$ attached to $\tilde{w}$ is an indecomposable* $\mathrm{BT}_1$ *group.*
(iv) *Let $\tilde{w}, \tilde{w}'$ be two infinite words with the same period $h$. Then $G(\tilde{w}, h)_k$ is isomorphic to $G(\tilde{w}', h)_k$ if and only if $\tilde{w}$ is a tail of $\tilde{w}'$.*

(v) *Every indecomposable commutative group scheme killed by* $p$ *is either isomorphic to* $G(w)$ *for an finite word* $w$ *or isomorphic to* $G(\tilde{w})$ *for an infinite periodic word* $\tilde{w}$.

**Corollary 5.2.4.** *Let* $G$ *be an indecomposable commutative finite group scheme over an algebraically closed field* $k$ *of characteristic* $p$, *and let* $\Gamma$ *be the connected Kraft quiver corresponding to* $G$ *as in* 5.2.2 (iii).

(i) $G$ *is a* $\mathrm{BT}_1$ *group scheme if and only if* $\Gamma$ *is circular.*
(ii) $G$ *is etale if and only if* $\Gamma$ *is circular and has only* $\mathtt{V}$*-arrows.*
(iii) $G$ *is multiplicative if and only if* $\Gamma$ *is circular and has only* $\mathtt{F}$*-arrows.*
(iv) *Both* $G$ *and its Cartier dual* $G^D$ *are local if and only if* $\Gamma$ *is either linear or has both* $\mathtt{F}$*-arrows and* $\mathtt{V}$*-arrows.*

**Exercise 5.2.5.** Let $k$ be an algebraically closed field of characteristic $p$. For every natural number $n$, denote by $C(n)$ the number of isomorphism classes of commutative group schemes over $k$ killed by $p$.

(a) Find explicit constants $a, B > 0$ such that

$$2^{an} \leq C(n) \leq 2^{Bn} \quad \forall n \geq 1.$$

(b) Find a sharp lower bound for $\liminf_{n\to\infty} \log C(n)$ and a sharp upper bound for $\limsup_{n\to\infty} \log C(n)$.

## 6. On the classification results in 5 and their proofs

6.1. In chapter 2 of their famous paper [5], Gelfand and Ponomarev determined explicitly up to isomorphisms, all indecomposable finite dimensional representations of the commutative algebra $K[x,y]/(xy)$ over an algebraically closed base field $K$. For an arbitrary base field $K$, they gave a similar description in terms of isomorphism classes of finite dimensional $K[T,T^{-1}]$-modules.

In a seminar talk in Bonn, P. Gabriel presented a functorial interpretation of the method in [5, Ch. 2]. Gabriel's seminar talk was never published, but the gist can be found in [17], where Ringel used the Gelfand–Ponomarev method, as streamlined by Gabriel, to classify all indecomposable modules over $K\langle x,y\rangle/(x^2,y^2)$, where $K$ is a field and $K\langle x,y\rangle$ is the free associative $K$-algebra in two variables $x,y$, and $x^2,y^2$ is the ideal of $K\langle x,y\rangle/(x^2,y^2)$ generated by $x^2$ and $y^2$.

The results stated in §5 classify finite dimensional representations of the ring $\kappa[\mathtt{F},\mathtt{V}]_\sigma$, where $\kappa$ is a perfect field of characteristic $p$. The rings $\kappa[\mathtt{F},\mathtt{V}]_\sigma$ and $\kappa[x,y]/(xy)$ look similar, and they are isomorphic when $\kappa = \mathbb{F}_p$. So it is reasonable to expect that the Gelfand–Ponomarev method can be generalized, and give a classification of pairs of semi-linear operators $\mathtt{F},\mathtt{V}$ on finite dimensional $\kappa$-vector spaces such that $\mathtt{F}\circ\mathtt{V} = 0 = \mathtt{V}\circ\mathtt{F}$. When $\kappa$ is algebraically closed, the classification of finite dimensional left $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-modules slightly simpler to state than the case of finite dimensional modules over $k[x,y]/(xy)$, because fo any integer $r \geq 1$, there is only one $\sigma^r$-conjugacy class of indecomposable $p^r$-linear automorphisms on finite dimensional $\kappa$-vector spaces, while conjugacy classes of indecomposable linear automorphisms are parametrized by Jordan blocks with non-zero eigenvalues.

In the appendix of [8] Kraft called his classification of finite dimensional (left) $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-modules "a *mild generalization* of results of Gelfand–Ponomarev", and said that he used Gabriel's functorial interpretation of [5, Ch. 2], and followed largely some unpublished notes of Gabriel's seminar.[1] Unfortunately notes on Gabriel's seminar talk on Bonn are unavailable, and the exposition in the appendix of [8] assumes familiarity with Gabriel's approach to [5, Ch. 2]. Readers who want to flesh out further details of the proofs in the appendix of [8] may want to consult [17, §3, p. 22] for the general functorial framework of Gabriel's approach, and also [17, pp. 23–30] for how Gabriel's idea works in the context of [17]. Those who wish to gain a thorough understanding of the classification results in §5 are urged to study chapter 2 of [5]. Some insights in [5] may be obscured in the streamlined approach in [8, Anhang].

We asserted in §1 that it is to write down detailed proofs of the assertions in §5, by following [5, Ch. 2] closely and changing linear maps and correspondences to semi-linear ones. In the rest of this subsection we outline the steps and main ingredients of such an exercise.

6.2. **Semilinear relations.** Gelfand–Ponomarev used the notion of linear relations first introduced in [10]. We will use a slightly more general notion of semi-linear relations.

**Definition 6.2.1.** Let $M$ be a vector space over be a perfect field $\kappa$ of characteristic $p$.
(i) For any integer $r$, a $\sigma^r$-linear (binary) relation on $M$ is an additive subgroup $B$ of $M\oplus M$ such that $(c\cdot x, \sigma^r(c)\cdot y)\in B$ for every element $(x,y)\in B$ and every scalar $c\in\kappa$.
(ii) Given a $\sigma^{r_1}$-linear relation $B_1$ on $M$ and a $\sigma^{r_2}$-linear relation on $M$, their composition $B_2\circ B_1$ is the $\sigma^{r_1+r_2}$-linear relation

$$B_2\circ B_1=\{(x,z)\in M\oplus M : \exists y\in M \text{ s.t. } (x,y)\in B_1 \text{ and } (y,z)\in B_2\}$$

on $M$. Clearly this composition operation is associative.
(iii) For any $\sigma^r$-linear relation $B$ on $M$, denote by $B^\sharp$ the $\sigma^{-r}$-linear relation on $M$ given by

$$B^\sharp := \{(x,y)\in M : (y,x)\in B\}.$$

Clearly $(B_2\circ B_1)^\sharp = B_1^\sharp\circ B_2^\sharp$ for any two semi-linear operators $B_1, B_2$.
(iv) For every $\kappa$-vector subspace $N\subseteq M$, the vector subspace $\theta_N := N\oplus(0)$ is a $\sigma^r$-linear relation for every $r\in\mathbb{Z}$. We often identify such a binary relation $\theta_N$ with the vector subspace $N$ of $M$, if confusion is unlikely. We often suppress the subscript in $\theta_M$, so that $\theta$ means $\theta_M$. The relation given by trivial vector subspace of $M\oplus M$ is denoted by $\mathbf{0}$.
(iv) For any $\sigma^r$-linear operator $T: M\to M$, the graph

$$\gamma_T := \{(x, T(x)) : x\in M\}$$

of $T$ is a $\sigma^r$-linear relation on $M$. We often write $T$ instead of $\gamma_T$, if confusion is unlikely. Of course $(\gamma_T)^\sharp$ corresponds to $T^{-1}$, in the sense that
(v) Let $B$ be a $\sigma^r$-linear binary relation on $M$.

[1] From the first paragraph of the appendix of [8]: Die folgende Klassifikation der Moduln enlicher Länge über dem Ring $\mathcal{A}=\kappa_\sigma[[a,b]]/(a\cdot b)$ is eine leichte Verallgemeinerung der Resultate von Gelfand–Ponomarev [5]. Dabei Benutzen wir eine fon P. Gabriel gegebene *funktiorielle Interpretation* dieser Resultate und folgend auch weitgehend seinen unveröffentlichten Aufzeichnungen. Ihm und auch C. Ringel danken wir für die Bemerkungen zum Text. Man vergleiche hierzu auch die Arbeit [17].

- The composition $\theta B$ corresponds to a vector subspace of $M$, called the *domain* of $B$ and denoted by $\mathrm{Dom}(B)$.
- The vector subspace corresponding to $\mathbf{0}B$ is called the *kernel* of $B$, and denoted by Ker(B).
- The vector subspace corresponding to $(B\mathbf{0})^\sharp = \mathbf{0}B^\sharp$ is *indeterminacy* of $B$, and denoted by $\mathrm{Indet}(B)$.
- The vector subspace corresponding to $(B\theta^\sharp)^\sharp = \theta B^\sharp$ is the *image* (or *value*) of $B$, denoted by $\mathrm{Im}(B)$.

Clearly
$$\mathbf{0}B \subseteq \theta B, \quad \mathbf{0}B^\sharp \subseteq \theta B^\sharp,$$
and the $\sigma^r$-linear relation $B$ induces a $\sigma^r$-linear isomorphism
$$\theta B/\mathbf{0}B \xrightarrow{\sim} \theta B^\sharp/\mathbf{0}B^\sharp \,.$$
(vi) For any vector subspace $N$ of $M$, let
$$B(N) := \{y \in M \mid \exists x \in N \text{ s.t. } (x,y) \in B\},$$
$$B^{-1}(N) := \{x \in M \mid \exists y \in N \text{s.t. } (x,y) \in B\}.$$

**Definition 6.2.2.** Let $B$ be a $\sigma^r$-linear binary relation on a finite dimensional vector space $M$ over a perfect field of characteristic $p$. Since
$$\mathbf{0}B^n \subseteq \mathbf{0}B^{n+1} \subseteq \theta B^{n+1} \subseteq \theta B^n \quad \forall\, n \in \mathbb{N},$$
there exists an integer $N \in \mathbb{N}$ such that
$$\mathbf{0}B^n = \mathbf{0}B^N \text{ and } \theta B^n = \theta B^N \qquad \forall\, n \geq N,$$
and
$$\mathbf{0}\cdot(B^\sharp)^n = \mathbf{0}\cdot(B^\sharp)^N \text{ and } \theta\cdot(B^\sharp)^n = \theta\cdot(B^\sharp)^N \qquad \forall\, n \geq N.$$
(a) We call the vector subspace of $M$ corresponding to $\mathbf{0}B^n$ for $n \geq N$ the *stable kernel* of $B$, and denote it by $\mathrm{Ker}(B^\infty)$. Similarly the subspace of $M$ corresponding to $\theta B^\infty$ stands for $\theta B^n$ for any $n \geq N$ is called the *stable domain* of and denoted by $\mathrm{Dom}(B^\infty)$.

(b) The $\kappa$-vector subspace of $M$ corresponding to $\theta(B^\sharp)^n$ for $n \geq N$ is called the *stable image* of $B$ and denoted by $\mathrm{Im}(B^\infty)$. The $\kappa$-subspace of $M$ corresponding to $\mathbf{0}(B^\sharp)^\infty$ is called the *stable indeterminacy* of $B$, and denoted by $\mathrm{Indet}(B^\infty)$.

**Exercise 6.2.3.** Let $B$ be a $\sigma^r$-linear relation on $M$ as in 6.2.2.
(a) Prove the following statements.
1. (1) $B\big(\mathrm{Dom}(B^\infty)\big) = \mathrm{Dom}(B^\infty) + \mathrm{Indet}(B)$.
2. (2) $B\big(\mathrm{Im}(B^\infty)\big) = \mathrm{Im}(B^\infty)$.
3. (3) $B^{-1}\big(\mathrm{Dom}(B^\infty)\big) = \mathrm{Dom}(B^\infty)$.
4. (4) $B^{-1}\big(\mathrm{Im}(B^\infty)\big) = \mathrm{Im}(B^\infty) + \mathrm{Ker}(B)$.
5. (5) $B^{-1}(\mathrm{Ker}(B^\infty)) = \mathrm{Ker}(B^\infty)$.
6. (6) $B(\mathrm{Ker}(B^\infty)) = \mathrm{Ker}(B^\infty) + \mathrm{Indet}(B)$.
7. (7) $B(\mathrm{Indet}(B^\infty)) = \mathrm{Indet}(B^\infty)$.
8. (8) $B^{-1}(\mathrm{Indet}(B^\infty)) = \mathrm{Indet}(B^\infty) + \mathrm{Ker}(B^\infty)$.

(b) Show that

$$\mathrm{Dom}(B^\infty)\cap\mathrm{Indet}(B)\ \subseteq\ \mathrm{Dom}(B^\infty)\cap\mathrm{Indet}(B^\infty)\ \subseteq\ \mathrm{Ker}(B^\infty).$$

Proposition 6.2.4 below says that every $\sigma^r$-linear binary relation on a finite dimensional $\kappa$-vector space is a direct sum of a "nilpotent $\sigma^r$-linear relation" and the graph of a $\sigma^r$-linear automorphism.

**Proposition 6.2.4.** *Let $B$ be a $\sigma^r$-linear binary relation on a finite dimensional vector space $M$ over $\kappa$, where $\kappa$ is a perfect field of characteristic $p$.*

(a) *The relation $B$ induces a $\sigma^r$-linear a $\sigma^r$-linear automorphism $T$ of $\mathrm{Dom}(B^\infty)/\mathrm{Ker}(B^\infty)$, in the sense that the image of $B\cap\big(\mathrm{Dom}(B^\infty)\oplus\mathrm{Dom}(B^\infty)\big)$ in $\mathrm{Dom}(B^\infty)\oplus\mathrm{Dom}(B^\infty)$ is equal to the graph of $T$.*

(b) *There exists a $\kappa$-vector subspace $S\subseteq\mathrm{Im}(B^\infty)\cap\mathrm{Dom}(B^\infty)$ and a $\kappa$-vector subspace $N$ of $M$ which contains $\mathrm{Ker}(B^\infty)+\mathrm{Indet}(B^\infty)$, such that the following properties hold.*

(i) $M=S\oplus N$.

(ii) $B=\big(B\cap(S\oplus S)\big)\oplus\big(B\cap(\mathrm{Ker}(B^\infty)+\mathrm{Indet}(B^\infty))\big)=\big(B\cap(S\oplus S)\big)\oplus\big(B\cap(N\oplus N)\big)$.

(iii) *$B\cap(S\oplus S)$ is the graph of a $\sigma^r$-linear automorphism of $S$. This automorphism of $S$ induces the $\sigma$-linear automorphism $T$ of* $\mathrm{Dom}(B^\infty)/\mathrm{Ker}(B^\infty)$ *in* (a) *above.*

(iv) $\mathrm{Dom}(B^\infty)=S\oplus\mathrm{Ker}(B^\infty)$.

(v) $\mathrm{Im}(B^\infty)=S\oplus\mathrm{Indet}(B^\infty)$.

Cf. [5, Thm. 3.1, p. 41] and also [17, Lemma, p. 21].

PROOF. Part (a) follows quickly from 6.2.3. We will prove part (b). Pick a $\kappa$-linear section

$$\iota:\mathrm{Dom}(B^\infty)/\mathrm{Ker}(B^\infty)\longrightarrow\mathrm{Dom}(B^\infty)$$

of the natural surjection $\mathrm{Dom}(B^\infty)\twoheadrightarrow\mathrm{Dom}(B^\infty)/\mathrm{Ker}(B^\infty)$. The gist of the statement (b) is the existence of a $\kappa$-linear map $f:\mathrm{Dom}(B^\infty)/\mathrm{Ker}(B^\infty)\longrightarrow\mathrm{Ker}(B^\infty)$ such that

$$\big(\iota(\bar x)+f(\bar x),\ \iota(T(\bar x))+f(T(\bar x)\big)\in B\qquad\forall\,\bar x\in\mathrm{Dom}(B^\infty)/\mathrm{Ker}(B^\infty).$$

It is clear that there exists a $\sigma^r$-linear map $\delta:\mathrm{Dom}(B^\infty)/\mathrm{Ker}(B^\infty)\longrightarrow\mathrm{Ker}(B^\infty)$ such that

$$\big(\iota(\bar x),\ \iota(T(\bar x))+\delta(\bar x)\big)\in B\qquad\forall\,\bar x\in\mathrm{Dom}(B^\infty)/\mathrm{Ker}(B^\infty).$$

One only needs to define $\delta$ on a $\kappa$-basis of $\mathrm{Dom}(B^\infty)/\mathrm{Ker}(B^\infty)$. Fix such a $\sigma^r$-linear map $\delta$. It suffices to show the existence of a $\kappa$-linear map $f:\mathrm{Dom}(B^\infty)/\mathrm{Ker}(B^\infty)\longrightarrow\mathrm{Ker}(B^\infty)$ such that

$$\big(f(\bar x),-\delta(\bar x)+f(T(\bar x))\big)\in B\qquad\forall\,\bar x\in\mathrm{Dom}(B^\infty)/\mathrm{Ker}(B^\infty).$$

We will construct such a map $f$ by a telescoping sum argument.

Similar to the construction of the map $\delta$, there exist a positive integer $N$ and $\sigma^{ri}$-linear maps $\delta_i:\mathrm{Dom}(B^\infty)/\mathrm{Ker}(B^\infty)\longrightarrow\mathrm{Ker}(B^\infty)$ for $i=2,\dots,m$ such that

$$\big(\delta_i(\bar x),\ \delta_{i+1}(\bar x)\big)\in B\quad\text{for } i=1,\dots,N,\quad\forall\,\bar x\in\mathrm{Dom}(B^\infty)/\mathrm{Ker}(B^\infty),$$

where $\delta_1:=\delta$ and $\delta_{N+1}:=0$ by convention. Define a map

$$f:\mathrm{Dom}(B^\infty)/\mathrm{Ker}(B^\infty)\longrightarrow\mathrm{Ker}(B^\infty)$$

by

$$f(\bar{x}) := \sum_{i=1}^{N} \delta_i(T^{-i}(\bar{x})) = \delta_1(T^{-1}(\bar{x})) + \delta_2(T^{-2}(\bar{x})) + \cdots + \delta_{N-1}(T^{-N+1}(\bar{x})) + \delta_N(T^{-N}(\bar{x}))$$

Then

$$-\delta(\bar{x}) + f(T(\bar{x})) = \delta_2(T^{-1}(\bar{x})) + \cdots + \delta_N(T^{-N+1}(\bar{x})),$$

and

$$\big(f(\bar{x}),\ -\delta(\bar{x}) + f(T(\bar{x}))\big) = \sum_{i=1}^{N-1} \big(\delta_i(T^{-i}(\bar{x})),\ \delta_{i+1}(T^{-i}(\bar{x}))\big) + \big(\delta_N(T^{-N}(\bar{x})), 0\big) \in B$$

for all $\bar{x} \in B$.

Let $S := (\iota + f)(\mathrm{Dom}(B^\infty)/\mathrm{Ker}(B^\infty))$, and let $N$ be a $\kappa$-vector subspace of $M$ which contains both $\mathrm{Ker}(B^\infty)$ and $\mathrm{Indet}(B^\infty)$ such that $M = S \oplus N$. Then conditions (i)–(v) hold. □

**Remark 6.2.5.** (1) In 6.2.4 (b), the $\kappa$-vector subspace $\mathrm{Ker}(B^\infty) + \mathrm{Indet}(B^\infty))$ of $M$ is of course uniquely determined by $B$. The $\sigma^r$-linear relation $B \cap (\mathrm{Ker}(B^\infty) + \mathrm{Indet}(B^\infty))$ on $\mathrm{Ker}(B^\infty) + \mathrm{Indet}(B^\infty))$ is called the *nilpotent component* of $B$.

(b) Proposition 6.2.4 is an analog of the theory of canonical forms for linear operators on finite dimensional vector spaces. However the pair $(S, N)$ of vector subspaces of $M$ in 6.2.4 (b) is *not* uniquely determined by $B$. Instead it is determined up to $\kappa$-linear conjugation: Suppose that $(S', N')$ is another pair of $\kappa$-linear subspaces of $M$ which satisfies conditions (i)–(v), then there exists an $\kappa$-linear automorphism $U$ of $M$) such that $U(S) = S'$, $U(N) = N'$,

$$U|_{\mathrm{Ker}(B^\infty)+\mathrm{Indet}(B^\infty)} = \mathrm{Id}_{\mathrm{Ker}(B^\infty)+\mathrm{Indet}(B^\infty)}.$$

In particular $U$ induces an automorphism of the binary relation $(M, B)$ on $B$.

6.3. Let $M$ be a finite dimensional left $\kappa[\mathtt{F}, \mathtt{V}]_\sigma$-module. We identify the operators $\mathtt{F}, \mathtt{V}^\sharp$ on $M$ with the relations given by their graphs.

6.3.1. Every (finite) word $w = (w_m \ldots w_1)$ in the binary alphabet $\{\mathtt{F}, \mathtt{V}\}$ defines a $\sigma^m$-linear relation $w_m \circ \cdots \circ w_1$. Semi-linear relations attached to words $w$ as above are also called *monomials* (in $\mathtt{F}$ and $cV^\sharp$); they will be denoted by $w_M$, or simply $w$ if no confusion is likely. Note that $\theta\mathtt{V}^\sharp \subseteq \mathbf{0}$ because $\mathtt{F} \circ \mathtt{V} = 0$. It follows that

$$\theta\mathtt{V}^\sharp w \subseteq \mathbf{0}\mathtt{F} w$$

for every (monomial attached to a) word $w$.

Consider the family

$$\mathfrak{F} = \mathfrak{F}_M := \big\{ N \subseteq M \mid \theta_N = \mathbf{0}w \text{ or } \theta w \ \text{ for some word } w \big\}$$

of $\kappa$-vector subspaces of $M$. Note that each member of the above family of vector subspaces is stable under both operators $\mathtt{F}$ and $\mathtt{V}$ on $M$. As an example, for the word $w = \mathtt{F}(\mathtt{V}^\sharp)^2\mathtt{F}^2\mathtt{V} = \mathtt{F}\mathtt{V}^\sharp\mathtt{V}^\sharp\mathtt{F}\mathtt{F}\mathtt{V}$, we have

$$\mathbf{0}w = \mathtt{V}\big(\mathtt{F}^{-2}\big(\mathtt{V}^2(\mathrm{Ker}(\mathtt{F}_M))\big)\big), \quad \theta w = \mathtt{V}\big(\mathtt{F}^{-2}(\mathtt{V}^2(M))\big),$$

where we have followed the general convention of identifying a vector subspace $N$ of $M$ with $\theta_N = N \oplus 0$.

An important fact is that the subspaces in the family $\mathfrak{F}$ form a flag

$$\mathbf{0} \subsetneqq \beta_1 \subsetneqq \cdots \subsetneqq \beta_s = \theta.$$

This is a consequence of the following lemma.

**Lemma 6.3.2.** *Let $w_1, w_2$ be distinct words. Then the intervals $(\mathbf{0}w_1, \theta w_1)$ and $(\mathbf{0}w_2, \theta w_2)$ are not interlaced: either one interval follows the other, or one interval is contained in the other. More precisely, after interchanging the indices for $w_1$ and $w_2$ if necessary, one of the following two cases occurs.*

- *If $w_1 = D_1 \mathtt{V}^\sharp D_0$ and $w_2 = D_2 \mathtt{F} D_0$ for some (possibly empty) words $D_0, D_1, D_2$, then the interval $(\mathbf{0}w_2, \theta w_2)$ follows $(\mathbf{0}w_1, \theta w_1)$, i.e.*

$$\mathbf{0}w_1 \subseteq \theta w_1 \subseteq \mathbf{0}w_2 \subseteq \theta w_2.$$

- *If $w_2 = D_3 w_1$ for some word $D_3$, then the interval $(\mathbf{0}w_2, \theta w_2)$ is contained in the interval $(\mathbf{0}w_1, \theta w_1)$, i.e.*

$$\mathbf{0}w_1 \subseteq \mathbf{0}w_2 \subseteq \theta w_2 \subseteq \theta w_1.$$

**Definition 6.3.3.** The intervals $(\beta_{i-1}, \beta_i)$, $i = 1, \ldots, s$ of the flag $\mathfrak{F}$ are called the *elementary intervals* of the $\kappa[\mathtt{F}, \mathtt{V}]_\sigma$-module $M$.

It turns out that each elementary interval corresponds to either a finite word or a weakly periodic infinite word, to be described next. These (finite or infinite) words in the alphabet $\{\mathtt{F}, \mathtt{V}\}$ provide the necessary combinatorial structure for decomposing the given $\kappa[\mathtt{F}, \mathtt{V}]_\sigma$-module $M$ as a direct sum of $\kappa[\mathtt{F}, \mathtt{V}]_\sigma$-modules attached to indecomposable $p$-linear representations of indecomposable Kraft quivers as in theorem 5.2.

**Exercise 6.3.4.** Consider the linear Kraft quiver $\Gamma = \Gamma\big((\mathtt{V}^\sharp)^2\mathtt{F}^4(\mathtt{V}^\sharp)^3\mathtt{F}^2\mathtt{V}^\sharp\mathtt{F}^5\big)$ attached to the word $w = w_{17}\dots w_1 = (\mathtt{V}^\sharp)^2\mathtt{F}^4(\mathtt{V}^\sharp)^3\mathtt{F}^2\mathtt{V}^\sharp\mathtt{F}^5$ of length 17 in 3.6.3.

$$\begin{tikzcd}
\bullet \arrow[r,"\mathtt{F}"] & \bullet \arrow[r,"\mathtt{F}"] & \bullet \arrow[r,"\mathtt{F}"] & \bullet \arrow[r,"\mathtt{F}"] & \bullet \arrow[r,"\mathtt{F}"] & \bullet \\
&&&&& \bullet \arrow[u,"\mathtt{V}"] \arrow[r,"\mathtt{F}"] & \bullet \arrow[r,"\mathtt{F}"] & \bullet \\
&&&&&&& \bullet \arrow[u,"\mathtt{V}"] \\
&&&&&&& \bullet \arrow[u,"\mathtt{V}"] \\
&&&&&&& \bullet \arrow[u,"\mathtt{V}"] \arrow[r,"\mathtt{F}"] & \bullet \arrow[r,"\mathtt{F}"] & \bullet \arrow[r,"\mathtt{F}"] & \bullet \arrow[r,"\mathtt{F}"] & \bullet \\
&&&&&&&&&&& \bullet \arrow[u,"\mathtt{V}"] \\
&&&&&&&&&&& \bullet \arrow[u,"\mathtt{V}"]
\end{tikzcd}$$

Thus $w_1 = \cdots = w_5 = \mathtt{F}$, $w_6 = \mathtt{V}^\sharp$, $w_7 = w_8 = \mathtt{F}$, $w_9 = w_{10} = w_{11} = \mathtt{V}^\sharp$, $w_{12} = w_{13} = w_{14} = w_{15} = \mathtt{F}$, $w_{16} = w_{17} = \mathtt{V}^\sharp$. Denote by $E_i$ the arrow in $\Gamma$ corresponding to the $i$-th letter $w_i$, $i = 1,\dots,17$, so that the directions of the arrows $E_i^\sharp$ in $\Gamma_{\mathtt{V}^\sharp}$ corresponding to $E_i$ are aligned. (Recall that the direction of $E_i$ is reversed if $E_i$ is a $\mathtt{V}$-arrow.) Denote by $z_0,\dots,z_{17}$ the vertices of $\Gamma$, so that $\mathrm{target}(E_i^\sharp) = z_i = \mathrm{source}(E_{i+1}^\sharp)$ for $i = 1,\dots,16$, $\mathrm{source}(E_1^\sharp) = z_0$, $\mathrm{target}(E_{17}^\sharp) = z_{17}$ as in 3.6. Determine explicitly the flag $\mathfrak{F}$ of the $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-module $M(\Gamma)_\kappa$. (Note: The length of flag $\mathfrak{F}$ is 18, and each $\kappa$-vector subspace $\beta_i$ is generated by a subset of $\{z_0,\dots,z_{17}\}$ with $i$ elements. However the naive guess that $\beta_i$ is generated by $\{z_0,\dots,z_i\}$ or $\{z_{17-i},\dots,z_{17}\}$ is far from the mark. Cf. [5, Example 4.1, p. 43].)

## 6.4. Summary of the classification of elementary intervals.

(1) Elementary intervals of the first kind

**Proposition 6.4.1.** *Suppose that $w$ is a finite (possibly empty) word such that $\theta\mathtt{V}^\sharp w \subsetneqq \mathtt{0F}w$. Then the interval $(\theta\mathtt{V}^\sharp w, \mathtt{0F}w)$ is elementary. Moreover if $w'$ is another finite word such that $\theta\mathtt{V}^\sharp w' \subsetneqq \mathtt{0F}w'$, then $(\theta\mathtt{V}^\sharp w', \mathtt{0F}w') = (\theta\mathtt{V}^\sharp w, \mathtt{0F}w)$*

Cf. [5, pp. 32–33].

By definition, an elementary interval $(\beta_{i-1},\beta_i)$ is *of the first kind* if there exists a (possibly empty) finite word $w$ such that

$$\beta_{i-1} = \theta\mathtt{V}^\sharp w \subsetneqq \mathtt{0F}w = \beta_i\,.$$

Elementary intervals not of the first kind are said to be *of the second kind.* We know from the above proposition that the set of elementary intervals of $M$ of the first kind are in bijection with the set of words in the alphabet $\{\mathtt{F}, \mathtt{V}^\sharp\}$ such that $\theta\mathtt{V}^\sharp w \subsetneqq \mathbf{0}\mathtt{F}w$. Such words will be called *words of the first kind* for $M$.

(2) Elementary intervals of the second kind

**Lemma 6.4.2.** *Let $(\beta_{i-1}, \beta_i)$ be an elementary interval of $M$ of the second kind.*
(i) *There exists an infinite word $\tilde{w} = (w_i)_{i\geq 1}$ in the alphabet $\{\mathtt{F}, \mathtt{V}\}$, uniquely determined by $(\beta_{i-1}, \beta_i)$, and an integer $N \geq 1$, such that*

$$(\beta_{i-1}, \beta_i) = (\mathbf{0}D_m, \theta D_m) \quad \forall\, m \geq N,$$

*where $D_m = w_m \ldots w_1$ is finite word consisting of the first $m$ letters of $\tilde{w}$.*
(ii) *The infinite word $\tilde{w} = (w_i)_{i\geq 1}$ in* (i) *is weakly periodic, in the sense that there exists a positive integers $d, h$ such that $w_{i+h} = w_i$ for all $i \geq d$.*

Cf. [5, pp. 33–34] and [5, p. 53]) for (i) and (ii) respectively.

Infinite words attached to elementary intervals of the second kind are called *infinite words of the second kind* attached to $M$. For each infinite word $\tilde{w}$ of the second kind, let

$$\mathbf{0}\tilde{w} := \mathbf{0}D_m, \quad \theta\tilde{w} := \theta D_m, \quad \forall\, m \gg 0,$$

so that $(\mathbf{0}\tilde{w}, \theta\tilde{w})$ is the elementary interval of the second kind corresponding which corresponds to $\tilde{w}$.

6.5. The sets $\Sigma_1 = \Sigma_1(M)$ and $\Sigma_2 = \Sigma_2(M)$) of words of the first and second kind in the binary alphabet $\{\mathtt{F}, \mathtt{V}\}$ associated to the given left $\kappa[\mathtt{F}, \mathtt{V}]_\sigma$-module $M$ have distinct combinatorial structures. Both $\Sigma_1$ and $\Sigma_2$ are finite sets. Elements of $\Sigma_1(M)$ are finite words in the alphabet $\{\mathtt{F}, \mathtt{V}\}$, while elements of $\Sigma_2(M)$ are infinite periodic words in $\{\mathtt{F}, \mathtt{V}\}$. We describe them separately.

(1) Words of the first kind

**Definition 6.5.1.** Define a partial order on the set $\Sigma_1$ of words of the first kind for $M$ as follows: For any two elements $D_1, D_2 \in \Sigma_1$, $D_1 \preceq D_2$ if and only $D_1$ is a left divisor of $D_2$, i.e.

$$D_1 \preceq D_2 \iff \exists \text{ a word } D \text{ s.t. } D_2 = D_1 D$$

**Lemma 6.5.2.** *Suppose that $\Sigma_1 = \Sigma_1(M) \neq \emptyset$. This finite poset $(\Sigma_1, \preceq)$ has the following property.*

(i) *$\Sigma_1$ has a unique minimal element, namely the empty word.*
(ii) *Each element of $\Sigma_1$ has at most one immediate predecessor and at most two immediate successor.*
(iii) *The directed graph $\gamma(\Sigma_1, \preceq)$ attached to the poset $(\Sigma_1, \preceq)$ is a connected tree, with a unique root corresponding to the empty word. Each vertex of $\gamma(\Sigma_1, \preceq)$ is the source of at most two arrows and the target of at most one arrow.*

Cf. [5, pp. 43–44].

**Remark.** By definition, an arrow in the directed graph attached to the poset $(\Sigma_1, \preceq)$ is a pair $(D_1, D_2)$ in $\Sigma_1$ such that $D_1 \precneqq D_2$, and there is no element $D' \in \Sigma_1$ such that $D_1 \precneqq D' \precneqq D_2$; such an arrow goes from $D_1$ to $D_2$. Clearly for any arrow from $D_1$ to $D_2$ as above, $D_2$ is either $D_1\mathtt{F}$ or $D_1\mathtt{V}^\sharp$. The statements 6.5.2 (ii)–(iii) follow from this observation.

**Definition 6.5.3** (the directed graph $\Gamma(M, 1\text{st})$ labeled by $\{\mathtt{F}, \mathtt{V}\}$)**.**
(a) We label the directed graph attached to the poset $(\Sigma_1, \preceq)$ with letters in $\{\mathtt{F}^\sharp, \mathtt{V}\}$ as follows. For any edge $E$ given by a pair $(D_1, D_2)$ of words of the first kind such that $D_2$ is an immediate successor of $D_1$ as in the previous paragraph, $D_1$ is the source of $E$ and $D_2$ is the target of $E$. Define the label of $E$ by

$$\text{label}(E) = \begin{cases} \mathtt{F}^\sharp & \text{if } \ D_2 = D_1\mathtt{F} \\ \mathtt{V} & \text{if } \ D_2 = D_1\mathtt{V}^\sharp \end{cases}$$

Denote by $\Gamma(M, 1\text{st})_{\mathtt{F}^\sharp}$ the resulting directed graph with arrows labeled by $\{\mathtt{F}^\sharp, \mathtt{V}\}$.

The last sentence of (iii) above can be strengthened: two distinct arrows sharing the same target have different labels.
(b) Let $\Gamma(M, 1\text{st})$ be the directed graph with arrows labeled by $\{\mathtt{F}, \mathtt{V}\}$ obtained from $\Gamma(M, 1\text{st})_{\mathtt{F}^\sharp}$ by reversing all $\mathtt{F}^\sharp$-arrows and changing their labels to $\mathtt{F}$.

Note that $\Gamma(M, 1\text{st})$ may not be a Kraft quiver: There be vertices $v$ of $\Gamma(M, 1\text{st})$ such that $v$ is the target an $\mathtt{F}$-arrow and also the source of a $\mathtt{V}$-arrow. This reflects the fact that there may exist vertices in $\Gamma(M, 1\text{st})_{\mathtt{F}^\sharp}$ which are connected to three other different vertices. However two distinct arrows in $\Gamma(M, 1\text{st})$ with the same source must have different labels.

(2) Words of the second kind
Let $\Sigma_2 = \Sigma_2(M)$ be the set of infinite words of the second kind for $M$. We already know that every element $\tilde{w} \in \Sigma_2$ is weakly periodic, but more is true.

**Lemma 6.5.4.** *Let $\tilde{w} = (w_i)_{i \geq 1} \in \Sigma_2$ be an infinite word of the second kind.*

(i) *The infinite word $\tilde{w}$ is cyclic, i.e. there exists a positive integer $h$ such that $w_i = w_{i+h}$ for every $i \geq 1$.*
(ii) *For each $j \in \mathbb{N}$, let $\tilde{w}(j)$ be the infinite word such that*

$$\tilde{w} = \tilde{w}(j) w_j \ldots w_1.$$

*In other words $\tilde{w}(j)$ is obtained from $\tilde{w}$ by lopping off the first $j$ letters of the word $\tilde{w}$. Then $\tilde{w}(j) \in \Sigma_2$ for every $j \in \mathbb{N}$.*
(iii) *The elementary intervals corresponding to the periodic words $\tilde{w}(j)$'s are have the same length, i.e.*

$$\dim_\kappa\bigl(\theta\tilde{w}(j)/\mathbf{0}\tilde{w}(j)\bigr) = \dim_\kappa\bigl(\theta\tilde{w}/\mathbf{0}\tilde{w}\bigr) \quad \forall\, j \in \mathbb{N}.$$

Cf. [5, Thm. 5.1, p. 53].

For an element $\tilde{w} \in \Sigma_2$, the subset $\{\tilde{w}(j) \mid j \in \mathbb{N}\}$ of $\Sigma_2$ is called the *cycle* containing $\tilde{w}$. Its cardinality is the (smallest positive) period of $\tilde{w}$. Thus the set $\Sigma_2$ of words of the second kind is the disjoint union of cycles in $\Sigma_2$.

**Definition 6.5.5** (the Kraft quiver $\Gamma(M, \text{2nd})$ with circular connected components)**.** Suppose that $\Sigma_2 \neq \emptyset$.

(a) Define a directed graph $\Gamma(M, \text{2nd})_{\mathtt{F}^\sharp}$ with arrows labeled by $\{\mathtt{F}^\sharp, \mathtt{V}\}$, which has $\Sigma_2$ as its set of vertices, as follows. Given any vertex $\tilde{w} \in \Sigma_2$, there is unique arrow $E$ with $\text{target}(E) = \tilde{w}$, and the source of this target is $\tilde{w}(1)$, the word obtained from $\tilde{w}$ by removing its first letter $w_1$. We have $\tilde{w} = \tilde{w}(1)w_1$, and $w_1$ is either $\mathtt{F}$ or $\mathtt{V}$. Define the label of $E$ by

$$\text{label}(E) = \begin{cases} \mathtt{F}^\sharp & \text{if } \ w_1 = \mathtt{F} \\ \mathtt{V} & \text{if } \ w_1 = \mathtt{V}^\sharp \end{cases}$$

Clearly the vertices of every connected component of $\Gamma(M, \text{2nd})_{\mathtt{F}^\sharp}$ is a cycle in $\Sigma_2$.

(b) Let $\Gamma(M, \text{2nd})$ be the directed graph with arrows labeled by $\{\mathtt{F}, \mathtt{V}\}$, obtained from $\Gamma(M, \text{2nd})_{\mathtt{F}^\sharp}$ by reversing all $\mathtt{F}^\sharp$-arrows and changing their labels to $\mathtt{F}$.

Clearly $\Gamma(M, \text{2nd})$ is a Kraft quiver, and the set of connected components of $\Gamma(M, \text{2nd})$ are in natural bijections with the set of cycles in $\Sigma_2$. For an infinite word $\tilde{w} = (w_i)_{i\geq 1} \in \Sigma_2$, the connected component of $\Gamma(M, \text{2nd})$ corresponding to the cycle in $\Sigma_2$ containing $\tilde{w}$ is isomorphic to the circular Kraft quiver $\Gamma(\lfloor w_h \dots w_1 \rceil)$, where $h$ is the smallest positive period of $\tilde{w}$.

6.6. Given a finite dimensional left $\kappa[\mathtt{F}, \mathtt{V}]_\sigma$ module $M$, we will define a $p$-linear representations $(\xi^{\text{2nd}}, U^{\text{2nd}})$ of the Kraft quiver $\Gamma(M, \text{2nd})$ and a $p$-linear representation of the directed graph $\Gamma(M, \text{1st})$ whose arrows are labeled by $\mathtt{F}$ or $\mathtt{V}$.

**(1)** The $p$-linear representation $(\xi^{\text{1st}}, U^{\text{1st}})$ of the directed graph $\Gamma(M, \text{1st})$ labeled by $\{\mathtt{F}, \mathtt{V}\}$.

**(1a)** To every word $D \in \Sigma_1$, the associated $\kappa$-vector space $U^{\text{1st}}_D$ is

$$U^{\text{1st}}_w = \mathtt{0F}D/\theta\mathtt{V}^\sharp D\,.$$

**(1b)** An $\mathtt{F}$-arrow $E$ corresponds to a pair of words $D_1$ and $D_2 = D_1\mathtt{F}$ in $\Sigma_1$, and

$$\text{source}(E) = D_2 = D_1\mathtt{F}, \quad \text{target}(E) = D_1.$$

We have

$$\mathtt{F}^{-1}(\mathtt{0F}D_1) = \mathbf{0}D_2, \quad \mathtt{F}^{-1}(\theta\mathtt{V}^\sharp D_1) = \theta\mathtt{V}^\sharp D_2,$$

and the map

$$\xi^{\text{1st}}_E : \mathtt{0F}D_2/\theta\mathtt{V}^\sharp D_2 \longrightarrow \mathtt{0F}D_1/\theta\mathtt{V}^\sharp D_1$$

attached to $E$ is induced by the $p$-linear operator

$$\mathtt{F}|_{\mathtt{0F}D_2} : \mathtt{0F}D_2 \longrightarrow \mathtt{0F}D_1$$

**(1c)** A $\mathtt{V}$-arrow $E'$ corresponds to a pair of words $D_1$ and $D_2 = D_1\mathtt{F}$ in $\Sigma_1$, and

$$\text{target}(E') = D_2 = D_1\mathtt{F}, \quad \text{source}(E') = D_1.$$

We have

$$\mathtt{V}(\mathtt{0F}D_1) = \mathtt{0F}D_2, \quad \mathtt{V}(\theta\mathtt{V}^\sharp D_1) = \theta\mathtt{V}^\sharp D_2,$$

and the map

$$\xi^{\text{2nd}}_{E'} : \mathtt{0F}D_1/\theta\mathtt{V}^\sharp D_1 \longrightarrow \mathtt{0F}D_2/\theta\mathtt{V}^\sharp D_2$$

is induced by the $\sigma^{-1}$-linear operator

$$\mathtt{V}|_{\mathbf{0}\mathtt{F}D_1} : \mathbf{0}\mathtt{F}D_1 \longrightarrow \mathbf{0}\mathtt{F}D_2.$$

**(1d)** Because two different arrows in $\Gamma(M, 1\mathrm{st})$ with the same source must have different labels, we obtain from the $p$-linear representation $(\xi^{1\mathrm{st}}, U^{1\mathrm{st}})$ a left module over $\kappa[\mathtt{V},\mathtt{V}]_\sigma$ just as in definition 4.3.1. The $\kappa$-vector space underlying this module is

$$\mathrm{gr}^{1\mathrm{st}}(M) := \sum_{D\in\Sigma_1} \mathbf{0}\mathtt{F}D/\theta\mathtt{V}^\sharp D,$$

and we will abuse the notation and denote this left $\kappa[\mathtt{V},\mathtt{V}]_\sigma$-module again by $\mathrm{gr}^{1\mathrm{st}}(M)$. Obviously the graded $\kappa$-vector space $\mathrm{gr}^{1\mathrm{st}}(M)$ is a direct summand of the graded $\kappa$-vector space $\oplus_{i=1}^s \beta_i/\beta_{i-1}$ attached to the flag $\mathfrak{F}$, in the sense that $\mathrm{gr}^{1\mathrm{st}}(M)$ is the sum of a subset of the summands $\oplus_{i=1}^s \beta_i/\beta_{i-1}$. The usual ordering of the index set $\{1,\dots,s\}$ does not play any role in the proof of theorem 5.2, and are best ignored.

**(2)** The $p$-linear representation $(\xi^{2\mathrm{nd}}, U^{2\mathrm{nd}})$ of the Kraft quiver $\Gamma(M, 2\mathrm{nd})$

**(2a)** To every infinite word $\tilde{w} \in \Sigma_2$, we associate the $\kappa$-vector space

$$U^{2\mathrm{nd}}_{\tilde{w}} = \theta\tilde{w}/\mathbf{0}\tilde{w}.$$

**(2b)** For every $\mathtt{F}$-arrow $E$, write the source and the target of $E$ as

$$\mathrm{source}(E) = \tilde{w}, \quad \mathrm{target}(E) = \tilde{w}(1), \quad \text{where } \tilde{w} = \tilde{w}(1)\mathtt{F},$$

i.e. $\tilde{w}(1)$ is obtained from $\tilde{w}$ by remove its first letter, which is $\mathtt{F}$. Consider the $p$-linear operator $\mathtt{F}$ and the $p^{-1}$-linear operator on $M$. From the definition of composition of semi-linear relations, we have

$$\mathtt{F}^{-1}(\theta\tilde{w}(1)) = \theta\tilde{w}, \quad \mathtt{F}^{-1}(\mathbf{0}\tilde{w}(1)) = \mathbf{0}\tilde{w},$$

Define the $p$-linear map $\xi^{2\mathrm{nd}}_E$ attached to $\mathtt{F}$-arrow $E$ to be the natural map

$$\theta\tilde{w}/\mathbf{0}\tilde{w} \longrightarrow \theta\tilde{w}(1)/\mathbf{0}\tilde{w}(1)$$

induced by the $p$-linear operator

$$\mathtt{F}|_{\theta\tilde{w}} : \theta\tilde{w} \to \theta\tilde{w}(1).$$

**(2c)** For every $\mathtt{V}$-arrow $E'$, write the target and the source of $E'$ as

$$\mathrm{tartet}(E') = \tilde{w}, \quad \mathrm{source}(E') = \tilde{w}(1), \quad \text{where } \tilde{w} = \tilde{w}(1)\mathtt{V}.$$

We have

$$\mathtt{V}(\theta\tilde{w}(1)) = \theta\tilde{w}, \quad \mathtt{V}(\mathbf{0}\tilde{w}(1)) = \mathbf{0}\tilde{w}.$$

Define the $\sigma^{-1}$-linear operator $\xi^{2\mathrm{nd}}_{E'}$ attached to the $\mathtt{V}$-arrow $E'$ to be the natural map

$$\theta\tilde{w}(1)/\mathbf{0}\tilde{w}(1) \longrightarrow \theta\tilde{w}/\mathbf{0}\tilde{w}$$

induced by the $\sigma^{-1}$-linear map

$$\mathtt{V}|_{\theta\tilde{w}(1)} : \theta\tilde{w}(1) \to \theta\tilde{w}.$$

We have defined the $p$-linear representation $(\xi^{\rm 2nd}, U^{\rm 2nd})$ attached to a left $\kappa[\mathtt{F},\mathtt{V}]$ module $M$. This $p$-linear representation defines a left $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-module whose underlying $\kappa$-vector space is the direct summand

$$\mathrm{gr}^{\rm 2nd}(M) := \sum_{\tilde{w}\in\Sigma_2} \theta\tilde{w}/\mathbf{0}\tilde{w}$$

of the graded $\kappa$-vector space

$$\mathrm{gr}_{\mathfrak{F}}(M) := \oplus_1^s \beta_i/\beta_{i-1}$$

attached to the flag $\mathfrak{F} = \big((0) = \beta_0 \subsetneqq \beta_1 \subsetneqq \cdots \subsetneqq \beta_s = M\big)$. We will abuse the notation and denote this left $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-module by $\mathrm{gr}^{\rm 2nd}(M)$.

Summary of 6.6. We have endowed the graded $\kappa$-vector space $\oplus_{i=1}^s \beta_i/\beta_{i-1}$ of the flag $\mathfrak{F}_M$ the structure of a left $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-module, namely the direct sum of $\mathrm{gr}^{\rm 1st}(M)$ and $\mathrm{gr}^{\rm 2nd}(M)$. Note that each member $\beta_i$ of the flag $\mathfrak{F}$ is a left $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-module, but induced left $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-module structure on $\oplus_{i=1}^s \beta_i/\beta_{i-1}$ is different from the left $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-module $\mathrm{gr}^{\rm 1st}(M) \oplus \mathrm{gr}^{\rm 2nd}(M)$.

From a more functorial perspective, we have defined two functors, from the category of finite dimensional left $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-modules, to the categories of $p$-linear representations of directed graphs labeled by $\{\mathtt{F},\mathtt{V}\}$ such that any two distinct arrows with the same label have distinct sources. It is not difficult to check what these two functors produce for modules attached to a $p$-linear representations of Kraft quivers.

(a) Let $D = w_m \dots w_1$ be a finite word, let $\Gamma(w_m \dots w_1)$ be the associated linear Kraft quiver, let $(\rho, U)$ be a $p$-linear representation of $\Gamma(w_m \dots w_1)$, and let

$$M = M(\Gamma(w_m \dots w_1), (\rho, U))$$

be the $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-module attached to $(\rho, U)$. Then the directed graph $\Gamma(M, \mathrm{1st})$ labeled by $\{\mathtt{F},\mathtt{V}\}$ is naturally isomorphic to $\Gamma(w_m \dots w_1)$, and we have a naturally isomorphism from the $p$-linear representation $(\rho, U)$ to $(\xi^{\rm 1st}, U^{\rm 1st})$. The latter induces an isomorphism of $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-modules from $M$ to $\mathrm{gr}^{\rm 1st}(M)$.

(b) Let $\lfloor w_h \dots w_1 \rceil$ be an indecomposable cyclic word, let $\Gamma(w_m \dots w_1)$ be the associated cyclic Kraft quiver, let $(\rho, U)$ be a $p$-linear representation of $\Gamma(w_m \dots w_1)$, and let

$$M = M(\Gamma(w_m \dots w_1), (\rho, U))$$

be the left $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-module attached to $(\rho, U)$. Then the Kraft quiver $\Gamma(M, \mathrm{2nd})$ is naturally isomorphic to $\Gamma(\lfloor w_m \dots w_1 \rceil)$, and we have a natural isomorphism of $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-modules from $M$ to $\mathrm{gr}^{\rm 2nd}(M)$, which comes from a natural isomorphism from $(\rho, U)$ to the $p$-linear representation $(\xi^{\rm 2nd}, U^{\rm 2nd})$ of $\mathrm{gr}^{\rm 2nd}(M)$.

6.7. Theorem 5.2 asserts that every finite dimensional left $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-module $M$ is isomorphic to a direct sum of non-trivial $\kappa[\mathtt{F},\mathtt{V}]_\sigma$-modules $(\Gamma_j, (\rho_j, U_j))$ attached to $p$-linear representations $(\rho_j, U_j)$ of connected Kraft quivers $\Gamma_j$, $j = 1, \dots, a+b$, where $a, b \in \mathbb{N}$, $\Gamma_1, \dots, \Gamma_a$ are linear Kraft quivers, and $\Gamma_{a+1}, \dots, \Gamma_{a+b}$ are indecomposable Kraft quivers. Moreover $\Gamma_i$ and $\Gamma_j$ are not isomorphic if $i \neq j$, and the $p$-linear representations $(\Gamma_j, (\rho_j, U_j))$ are determined by $M$ up to (non-canonical) isomorphism. Since the Kraft quivers $\Gamma_j$'s are all connected, the vector spaces attached to vertices of $\Gamma_j$ under $(\rho_j, U_j)$ have the same dimension, denoted by $\dim_\kappa(\rho_j, U_j)$.

It is natural to expect that the circular Kraft quivers $\Gamma_{a+1}, \ldots, \Gamma_{a+b}$ which appears are exactly the Kraft quivers attached to the infinite words of the second kind for $M$. Indeed that's what happens. Moreover the dimensions of the $p$-linear representations

$$(\rho_{a+1}, U_{a+1}), \ldots, (\rho_{a+b}, U_{a+b})$$

match the lengths of the elementary intervals in the corresponding cycle in $\Sigma_2$, i.e.

$$\dim(\rho_j, U_j) = \dim_\kappa(\theta\tilde{w}/\mathbf{0}\tilde{w})$$

if $\Gamma_j$ correspond to the cycle which contains a cyclic word $\tilde{w}$ of the second kind. This expectation turns out to be true. However the linear Kraft quivers $\Gamma_1, \ldots, \Gamma_a$ are not determined by the poset $(\Sigma_1, \preceq)$ alone. Instead the family

$$\{(\Gamma_j, \dim_\kappa(\rho_j, U_j)) \mid j = 1, \ldots a\}$$

is determined by the poset $(\Sigma_2, \preceq)$ together with the function

$$\ln : \Sigma_2 \to \mathbb{N}, \quad \ln(w) = \dim_\kappa(\mathtt{0}\mathtt{F}w/\theta\mathtt{V}^\sharp w) \quad \forall\, w \in \Sigma_2$$

given by the lengths of elementary intervals of the first kind.

The following proposition is the key step in the proof of theorem 5.2.

**Proposition 6.8.** *Let $M$ be a finite dimensional left $\kappa[\mathtt{F}, \mathtt{V}]_\sigma$-module. There exist $\kappa$ vectors subspaces $\gamma(D) \subseteq \mathtt{0}\mathtt{F}D$, $D \in \Sigma_1$ and $\gamma(\tilde{w}) \subseteq \theta\tilde{w})$, $\tilde{w} \in \Sigma_2$ indexed by $\Sigma_1$ and $\Sigma_2$, each of which is a complement in the corresponding elementary interval, such that these $\kappa$-vector subspaces are compatible with the operators $\mathtt{F}$ and $\mathtt{V}$ on $\mathrm{gr}^{1\mathrm{nd}}(M)$ and $\mathrm{gr}^{2\mathrm{nd}}(M)$. More explicitly:*

(1a) *$\gamma(D) \subseteq \mathtt{0}\mathtt{F}D$ and $\mathtt{0}\mathtt{F}D = \gamma(D) \oplus \theta\mathtt{V}^\sharp D$ for each $D \in \Sigma_1$.*

(1b) *For each $\mathtt{F}$-arrow in $\Gamma(M, 1\mathrm{st})$ corresponding to a pair of words $D_1$ and $D_2 = D_1\mathtt{F}$ in $\Sigma_1$, we have*

$$\mathtt{F}(\gamma(D_2)) \subseteq \gamma(D_1).$$

*under the operator $\mathtt{F}$ on $M$.*

(1c) *For each $\mathtt{V}$-arrow in $\Gamma(M, 1\mathrm{st})$ corresponding to a pair of words $D_1$ and $D_2 = D_1\mathtt{F}$, the operator $\mathtt{V}$ on $M$ satisfies $\mathtt{V}(\gamma(D_1)) = \gamma(D_2)$, i.e.*

$$\mathtt{V}|_{\gamma(D_1)} : \gamma(D_1) \twoheadrightarrow \gamma(D_2)$$

*is a $\sigma^{-1}$-linear surjection.*

(2a) *$\gamma(\tilde{w}) \subseteq \theta\tilde{w}$ and $\theta\tilde{w} = \gamma(\tilde{w}) \oplus \mathbf{0}\tilde{w}$ for each $\tilde{w} \in \Sigma_2$.*

(2b) *For each $\mathtt{F}$-arrow in $\Gamma(M, 2\mathrm{nd})$ given by cyclic words $\tilde{w}$ and $\tilde{w}(1)$ with $\tilde{w} = \tilde{w}(1)\mathtt{F}$, the operator $\mathtt{F}$ on $M$ induces a $\sigma$-linear bijection*

$$\mathtt{F}|_{\gamma(\tilde{w})} : \gamma(\tilde{w}) \xrightarrow{\sim} \gamma(\tilde{w}(1)).$$

(2c) *For each $\mathtt{V}$-arrow in $\Gamma(M, 2\mathrm{nd})$ given by cyclic words $\tilde{w}$ and $\tilde{w}(1)$ with $\tilde{w} = \tilde{w}(1)\mathtt{V}$, the operator $\mathtt{V}$ on $M$ induces a $\sigma^{-1}$-linear bijection*

$$\mathtt{V}|_{\gamma(\tilde{w}(1))} : \gamma(\tilde{w}(1)) \xrightarrow{\sim} \gamma(\tilde{w}).$$

**Remark 6.8.1.** (i) Proposition 6.8 is the semi-linear analogs of the combination of [5, Thm. 4.2, Ch. 2, p. 44] and [5, Thm. 5.3, Ch. 2, p. 54]. The proofs in [5] work without change, except that the maps and relations are semilinear in the context of 6.8.

(ii) The lemma at the end 6.2 on the structure of semi-linear binary relations is needed for the existence of good splittings $\gamma(\tilde{w})$ of elementary intervals of the second kind which satisfy conditions (2a)–(2c) in 6.8.

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