10

Orbital rigidity for biextensions

In this chapter we prove that orbital rigidity holds for biextensions of p-divisible formal groups over a perfect field κ of characteristic p. The main theorem 10.6.7 is a precise generalization of theorem 7.1.1 on the orbital rigidity of p-divisible formal groups. For a sustained deformation space $\mathcal{D}ef\,(X_0)_{\mathrm{sus}}$, theorem 7.1.1 implies that orbital rigidity holds for $\mathcal{D}ef\,(X_0)_{\mathrm{sus}}$ when the p-divisible group X_0 over κ has exactly two slopes, and theorem 10.6.7 implies that $\mathcal{D}ef\,(X_0)_{\mathrm{sus}}$ is orbitally rigid when X_0 has three slopes.

The orbital rigidity of $\mathcal{D}ef\,(X_0)_{\mathrm{sus}}$ when X_0 has three slopes is the first nontrivial case after the two-slope case. Its proof requires a new notion, called *tempered perfections* of augmented Noetherian local domains over a perfect base field κ of characteristic p. This new tool can be applied to prove the orbital rigidity of $\mathcal{D}ef\,(X_0)_{\mathrm{sus}}$ for all p-divisible groups X_0 with no restriction on the number of its slopes, and also the orbital rigidity for *Tate-linear formal varieties*, a class of smooth formal varieties which include sustained deformation spaces $\mathcal{D}ef\,(X_0)_{\mathrm{sus}}$ and $\mathcal{D}ef\,(X_0,\mu_0)_{\mathrm{sus}}$. In the introductory section 10.1 we will explain the general idea of orbital rigidity, and the notion of Tate-linear formal varieties. The orbital rigidity of Tate-linear formal varieties follows from the method of tempered perfections and induction on the nilpotency class of the nilpotent group governing the Tate-linear formal variety.

10.1. What is an orbitally rigid equivariant formal variety?

In 10.1.1–10.1.2 we describe the general idea of "orbitally rigid equivariant formal varieties with extra structures" in a categorical setting. The title of this chapter acquires a precise meaning when one specialize to the structure of biextensions of p-divisible formal groups. In 10.1.3–10.1.6.3 we explain the motivation of the orbital rigidity question for biextensions of p-divisible formal groups, and outline how a new class of complete augmented rings in characteristic p enters the proof of orbital rigidity for biextensions of p-divisible formal groups.

10.1.1. Orbitally rigid equivariant formal varieties.

First we illustrate the idea of orbital rigidity in the category of equivariant local formal varieties over an algebraically closed base field k of characteristic p. The objects in this category are triples $(\mathscr{D}, \tilde{G}, \mu)$, where $\mathscr{D} = \mathrm{Spf}(R)$ for a complete augmented Noetherian local domain R over k, $\tilde{G}$ is a topological group and $\mu : \tilde{G} \times \mathscr{D} \to \mathscr{D}$ is a left action of $\tilde{G}$ on $\mathscr{D}$.

(a) For any subgroups G' of $\tilde{G}$, a *G'-equivariant subquotient* of $\mathscr{D}$ is a triple

$$(\mathscr{D}_1, \mathscr{D}_2,\ \mathscr{D}_1 \xrightarrow{f} \mathscr{D}_2\ ),$$

where $\mathscr{D}_1$ is a closed formal subvariety of $\mathscr{D}$ over k, $\mathscr{D}_2$ is a G'-equivariant formal subvariety over k, and f is a G'-equivariant formal morphism.

(b) We say that a subgroup G of $\tilde{G}$ operates *strongly nontrivial* on $\mathscr{D}$ if for every *open* subgroup G' of G and every G'-equivariant subquotient $(\mathscr{D}_1, \mathscr{D}_2,\ \mathscr{D}_1 \xrightarrow{f} \mathscr{D}_2\ )$ of $\mathscr{D}$ with $\dim(\mathscr{D}_2) > 0$, the G'-action on $\mathscr{D}_2$ is nontrivial.

(c) We say that $(\mathscr{D}, \tilde{G})$ is *orbitally rigid* if for every subgroup G of $\tilde{G}$ operating strongly nontrivially on $\mathscr{D}$, every irreducible closed formal subvariety of $\mathscr{D}$ over k stable under the action of G is of a certain special form, with a nice structure.

An assertion that an equivariant formal variety $(\mathscr{D}, \tilde{G})$ is strongly rigid must be accompanied by a family of *special formal subvarieties* of $\mathscr{D}$, defined directly in a *structural* way, such that

- every formal subvariety W of $\mathscr{D}$ which is stable under some unspecified subgroup G of $\tilde{G}$ such that $(\mathscr{D}, G)$ is strongly nontrivial, is a special formal subvariety, and
- "most", if not all, special subvarieties are stable under the action of some subgroup G of $\tilde{G}$ acting strongly nontrivially on $\mathscr{D}$.

We emphasize that the definition of special formal subvarieties must not make the orbital rigidity of $(\mathscr{D}, \tilde{G})$ an obvious tautology.

10.1.2. Orbitally rigid equivariant formal varieties with extra structures.

An $\tilde{G}$-equivariant formal variety $\mathscr{D}$ over k considered for possible strong rigidity phenomenon usually has a nice structure $\mathscr{S}$ which is respected by $\tilde{G}$-action. Suppose that this is the case.

(a) It is natural to use a more restricted class of *G'-equivariant $\mathscr{S}$-subquotients*, by requiring in addition that the G'-equivariant maps $\mathscr{D}_1 \hookrightarrow \mathscr{D}$ and $f : \mathscr{D}_1 \to \mathscr{D}_2$ respect the structure $\mathscr{S}$.

(b) We say that a subgroup G of $\tilde{G}$ operates *strongly $\mathscr{S}$-nontrivially* on $\mathscr{D}$ if for every open subgroup G' of G, G' operates nontrivially on every positive-dimensional $\mathscr{S}$-subquotient of $(\mathscr{D}, G')$.

(c) Replacing "strongly nontrivially" by "strongly $\mathscr{S}$-nontrivially" in 10.1.1 (c) results in a corresponding notion of *orbitally $\mathscr{S}$-rigid equivariant formal varieties with $\mathscr{S}$-structure*.

Clearly $(\mathscr{D}, G')$ is strongly $\mathscr{S}$-nontrivial if it is strongly nontrivial. So orbital $\mathscr{S}$-rigidity implies orbital rigidity in the sense of 10.1.1 (c). For certain structures $\mathscr{S}$, for instance when $\mathscr{S} = p$-$\mathcal{D}iv$, the converse, with "if" replaced by "only if" in the preceding sentence, is also true. If this is the case, then orbital $\mathscr{S}$-rigidity is equivalent to orbital rigidity in the sense of 10.1.1 (c).

When $\mathscr{S}$ is p-$\mathcal{D}iv$ (respectively when $\mathscr{S}$ is the structure $\mathcal{B}iext$-rigid of biextensions of p-divisible formal groups), the explicit definition of strongly $\mathscr{S}$-nontrivial equivariant formal varieties is given in 7.3.1 (respectively 10.2.7.4). In the main body 7.2–10.6 of this chapter, strong nontriviality refers to either 7.3.1 or 10.2.7.4.

The statement "$(\mathscr{D}, \mathrm{Aut}(\mathscr{D}, \mathscr{S}))$ is orbitally $\mathscr{S}$-rigid" is often shortened to "$\mathscr{D}$ is orbitally $\mathscr{S}$-rigid", or "the $\mathscr{S}$-structure $\mathscr{D}$ is orbitally rigid". For instance every p-divisible formal group over a perfect field is strongly rigid, meaning that it is orbitally p-$\mathcal{D}iv$-rigid. The main result of this chapter is that every biextension of p-divisible formal groups is orbitally $\mathcal{B}iext$-rigid.

Remark. (i) In algebraic geometry, "rigid" usually means "does not deform", i.e. all deformations are trivial. For instance tori and formal tori are rigid. As another example, an abelian subvariety A of an abelian variety B has no nontrivial deformation inside B, hence is rigid as abelian subvarieties; similarly p-divisible subgroups are rigid.

The notion of orbital rigidity discussed here is not based on deformation. It spirit is closer to the rigidity theorems of Margulis, Mostow, Prasad and Ratner's theorems on unipotent flows.

(ii) In known examples, a special formal subvariety W in a orbitally rigid equivariant formal variety $\mathscr{D}$ does not deform algebraically. More precisely, if $\mathcal{W} \subseteq \mathscr{D} \times \mathrm{Spf}(k[[t]])$ is a reduced irreducible closed formal subscheme of $\mathscr{D} \times \mathrm{Spf}(k[[t]])$ flat over $\mathrm{Spf}(k[[t]])$, the closed fiber of $\mathcal{W}$ is W, and the generic fiber of $\mathcal{W}$ is a special formal subvariety of $\mathscr{D} \times_{\mathrm{Spec}(k)} \mathrm{Spec}\big(k((t))\big)$, then $\mathcal{W} = W \times \mathrm{Spf}(k[[t]]) \subseteq \mathscr{D} \times \mathrm{Spf}(k[[t]])$.

On the other hand, special formal subvarieties are often organized into families parameterized by suitable "p-adic varieties". As an examples special subvariety of a formal torus T over $\overline{\mathbb{F}_p}$ are formal subtori of T. All d-dimensional formal subtori of a formal torus T are parametrized by the set of all d-dimensional $\mathbb{Q}_p$-vector subspaces of $\mathrm{X}_*(T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, where $\mathrm{X}_*(T)$ is the co-character group of T.

10.1.3. First examples of orbitally rigid formal varieties.

The simplest example of the orbital rigidity phenomenon is the case when $\mathscr{D}$ is a formal torus over an algebraically closed field k of characteristic p and $\tilde{G} = {\mathbb{Z}_p}^{\times}$.

> *Let T be a formal torus over k. If W is a reduced irreducible closed formal subscheme of T which is stable under $[1+p^n]_T$ for some integer $n \geq 2$, then W is a formal subtorus of T.*

It turns out that orbital rigidity also holds for p-divisible formal groups:

> *Let X be a p-divisible formal group over k. If W is a reduced irreducible closed formal subscheme of X which is stable under a strongly non-trivial action of a subgroup G of $\mathrm{Aut}(X)$, where $\mathrm{Aut}(X)$ consists of all group automorphisms of X. Then W is a p-divisible subgroup of X.*

See theorem 7.1.1. The assumption that G operates *strongly nontrivially* means that for every non-trivial p-divisible subquotient Y of X stable under the action of an open subgroup

G' of G, the action of G' on Y is nontrivial. Equivalently, no Jordan–Hölder component of the $\mathrm{Lie}(G)$-module $\mathbb{D}(X)_{\mathbb{Q}}$ is the trivial $\mathrm{Lie}(G)$-module.

The discovery of the orbital rigidity phenomenon for p-divisible formal groups was motivated by the Hecke orbit problem, for central leaves in modular varieties of PEL type such that the corresponding families of abelian varieties have only exactly two slopes. For an $\overline{\mathbb{F}_p}$-point z_0 of such a central leaf $\mathcal{C}$, the formal completion $\mathcal{C}^{/z_0}$ of $\mathcal{C}$ at z_0 has a natural structure as (a trivial torsor for) an isoclinic p-divisible formal group over $\overline{\mathbb{F}_p}$. Moreover there is a compact p-adic Lie group T_{z_0}, an open subgroup of the group of $\mathbb{Q}_p$-points of the "Frobenius torus" attached to z_0, which operates strongly nontrivially on $\mathcal{C}^{/z_0}$.

Suppose we are given an irreducible closed subvariety Z of $\mathcal{C}$ stable under all prime-to-p Hecke correspondences of the ambient modular variety, and we want to prove that Z is equal to $\mathcal{C}$ as predicted by the Hecke orbit conjecture. The assumption that Z is stable under all prime-to-p Hecke correspondences implies that the formal completion $Z^{/z_0} \subseteq \mathcal{C}^{/z_0}$ of $Z^{/z_0}$ is stable under the action of T_{z_0}. So we obtain from orbital rigidity for p-divisible formal groups that for every $\overline{\mathbb{F}_p}$-point z_0 of the normal locus of Z, the formal completion $Z^{/z_0}$ corresponds to a p-divisible subgroup of the p-divisible formal group $\mathcal{C}^{/z_0}$. This does not prove the prediction, but it's a good start. Here is a catchphrase of this initial result.

> *Every Hecke-invariant subvariety inside a central leaf with two slopes is Tate-liner at every point.*

10.1.4. In search of a good definition of Tate-linear formal varieties.
Naturally one tries to extend the orbital rigidity result for leaves with two slopes to a general leaf $\mathcal{C}$ in a modular variety of PEL type, so that the above catchphrase holds. The question below sums up the challenges.

Question Q1. How to define a *good* notion of "Tate-linear formal varieties", and the related notion of "special formal subvarieties" of a Tate-linear formal variety", so that every Tate-linear formal variety is orbitally rigid?

(In other words, the definitions of Tate-linear formal varieties and special formal subvarieties should ensure that every formal subvariety which is stable under a strongly nontrivial action of a p-adic Lie group G on the ambient Tate-linear formal variety is a special formal subvariety. We stress again that definitions which make the orbital rigidity assertion an obvious tautology, being worse than useless, do not qualify.)

Given an $\overline{\mathbb{F}_p}$-point z_0 of $\mathcal{C}$, the formal completion $\mathcal{C}^{/z_0}$ is "assembled from" a family of fibrations $\pi_i : \mathcal{T}_i \to \mathcal{T}_{i+1}$, $i = 0, \ldots, a-1$, such that π_i is the projection map of a torsor for a p-divisible group Z_i over $\overline{\mathbb{F}_p}$, $\mathcal{T}_0 = \mathcal{C}^{/z_0}$, and $\mathcal{T}_a = \mathrm{Spec}(\overline{\mathbb{F}_p})$. We regard such a family of fibrations as a weak form of a "Tate-linear structure on $\mathcal{C}^{/z_0}$", take a leap of faith, and set forth the working hypothesis T1 and its variant T2 below.

Working hypothesis T1. *The formal completion $\mathcal{C}^{/z_0}$ at a closed point z_0 of a central leaf $\mathcal{C}$ in a Siegel modular variety $\mathscr{A}_{g,1,n,\overline{\mathbb{F}_p}}$, $n \geq 3$, $\gcd(n,p) = 1$, is a Tate-linear formal variety.*

Working hypothesis T2. *The sustained deformation space $\mathcal{D}ef\,(X_0)_{\rm sus}$ of a p-divisible group X_0 over $\overline{\mathbb{F}}_p$ is a Tate-linear formal variety.*

10.1.5. Testing the first nontrivial case: biextensions of formal groups. With a "Tate-linear formal variety" $\mathcal{T}$ being of the form $\mathcal{C}^{/z_0}$ or $\mathcal{D}ef\,(X_0)_{\rm sus}$ as in T1 or T2, the general question Q1 on orbital rigidity becomes more specific. In order to make progress, it is a good idea to focus on the first nontrivial case, spelled out below.

Let X_1, X_2, X_3 be isoclinic p-divisible groups over $\overline{\mathbb{F}}_p$ with slopes $\mu_1 < \mu_2 < \mu_3$. The sustained deformation space $\mathcal{D}ef\,(X_1 \times X_2 \times X_3)_{\rm sus}$ has a natural structure as a *biextension* of $\mathscr{H}om'_{\rm div}(X_1, X_2) \times \mathscr{H}om'_{\rm div}(X_2, X_3)$ by $\mathscr{H}om'_{\rm div}(X_1, X_3)$.

Recall that given three commutative group schemes X, Y, Z over a base field k, a *biextension* of $X \times Y$ by Z is a morphism $E \to X \times Y$ plus two compatible relative group laws. The first group law, for relative to Y, makes $E \to Y$ an extension of $X_Y := X \times Y$ by $Z_Y := Z \times Y$ over Y, while the second group law, relative to X, makes $E \to X$ an extension of Y_X by Z_X over X. The best-known example is the Poincare bundle for an abelian variety A; it is a biextension of $A \times A^t$ by $\mathbb{G}_m$, where A^t is the dual abelian variety of A. Mumford invented the concept of bi-extension in [76] to treat deformation and lifting problems for polarized abelian varieties. See §10.2 for a review of the notion of biextensions.

Buoyed by optimism, we make a further working hypothesis that all biextensions of p-divisible formal groups are Tate-linear, and arrive at the orbital rigidity question for biextensions Q2 below.

Working hypothesis T3. *Every biextension of p-divisible formal groups over $\overline{\mathbb{F}}_p$ is Tate-linear.*

We emphasize that the "working hypotheses" T1–T3 will acquire mathematical meaning only after precise definitions of "special formal subvarieties" are given.

Challenge Q2 (Orbital rigidity question for biextensions in loose form). Let X, Y, Z be p-divisible formal groups over $\overline{\mathbb{F}}_p$, and let $E \to X \times Y$ be a biextension of $X \times Y$ by Z. Let G be a closed subgroup of $\mathrm{Aut}_{\rm biext}(E)$ acting strongly nontrivially on E.

(a) Define a good notion of "special formal subvarieties" of a biextension E as above.
(b) Show that every reduced irreducible formal subscheme W of E closed under the action of G is a special formal subvariety of E.

By definition, the group of automorphisms of a biextension of $X \times Y$ by Z as above is a subgroup of $\mathrm{Aut}(X) \times \mathrm{Aut}(Y) \times \mathrm{Aut}(Z)$. The assumption that G operates strongly nontrivially on E means that the induced actions of G on X, Y, Z are all strongly nontrivial as explained in 10.1.3. See 10.2.3.1 and 10.2.7.3 for more information about automorphisms of a biextension.

For the question Q2, a reasonable expectation is that a special formal subvariety T of a biextension $\pi : E \to X \times Y$ should be "almost" a torsor for a p-divisible subgroup $Z' \subseteq Z$, over a p-divisible subgroup U of $X \times Y$. To be more specific, we expect that every special formal subvariety T of a biextension E as in Q2 satisfies the following expectations.

(E1) The intersection of T with the closed fiber $\pi^{-1}(0,0) \cong Z$, with reduced structure, is a p-divisible subgroup Z' of Z.

(E2) The formal subvariety $T \subseteq E$ is stable under the translation action by Z' for the Z-torsor structure attached to the biextension $E \to X \times Y$.

(E3) The map $T/Z' \longrightarrow X \times Y$ induced by π, from the reduced irreducible formal subscheme T/Z' of E/Z' to $X \times Y$, factors as a composition

$$T/Z' \xrightarrow{q_T} U \hookrightarrow X \times Y \ ,$$

where U is a p-divisible subgroup of $X \times Y$, and the formal morphism q_T is finite, dominant and purely inseparable.

The expectations (E1)–(E3) enables us to formulate a more precise version Q2$'$ of the question Q2.

Challenge Q2$'$ (Orbital rigidity question for biextensions). Let X, Y, Z be p-divisible formal groups over $\overline{\mathbb{F}_p}$, let $\pi : E \to X \times Y$ be a biextension of $X \times Y$ by Z, and let G be a closed subgroup of $\mathrm{Aut}_{\mathrm{biext}}(E)$ operating strongly nontrivially on E.

(a$'$) Let U be a p-divisible subgroup of $X \times Y$ stable under the action of G. Find a necessary and sufficient condition on U, for the existence of a G-invariant reduced irreducible formal subscheme W of E above U such that the morphism $W \to U$ is finite, dominant and purely inseparable.

(b$'$) Suppose that W is a reduced irreducible closed formal subscheme of E stable under the action of G. Prove the following statements.

(i) The formal scheme $(W \cap Z)_{\mathrm{red}}$, the intersection $W \cap Z$ with reduced structure, is a p-divisible subgroup of Z.

(ii) The formal morphism $q_{W,(W\cap Z)_{\mathrm{red}}} : W/(W \cap Z)_{\mathrm{red}} \longrightarrow X \times Y$ is finite and purely inseparable.

Note that in the situation of (ii), orbital rigidity for p-divisible formal groups tells us that the schematic image of $q_{W,(W\cap Z)_{\mathrm{red}}}$ is a p-divisible subgroup of $X \times Y$.

It turns out that the question (a$'$) can be answered using orbital rigidity for p-divisible formal groups. The necessary and sufficient condition asked in (a$'$) is

$$\theta_n^E(\mathrm{pr}_X(u_1), \mathrm{pr}_Y(u_2)) = \theta_n^E(\mathrm{pr}_X(u_2), \mathrm{pr}_Y(u_1))$$

for all $n \geq 1$ and all functorial points (u_1, u_2) of $U \times U$; see 10.3.2 and 10.3.4.1. Here $\mathrm{pr}_X : E \to X$ and $\mathrm{pr}_Y : E \to Y$ are the composition of $\pi : E \to X \times Y$ with the projections $\mathrm{pr}_1 : X \times Y \to X$ and $\mathrm{pr}_2 : X \times Y \to Y$ respectively, and

$$\left(\theta_n^E : X[p^n] \times Y[p^n] \to Z[p^n]\right)_{n\geq 1}$$

is the family of Weil pairings attached to the biextension E, whose construction is reviewed in 10.2.5.1.

10.1.6. The method of hypocotyle elongation through tempered perfections.

10.1.6.1. The question (b′) is harder and requires a new idea. We will first describe the main tool in proving orbital rigidity for p-divisible formal groups, theorem 7.2.1 and its important special case 7.2.2. It works like this. Suppose you have a reduced irreducible Noetherian local formal scheme $\mathcal{Y}$ over a perfect field κ of characteristic p and a sequence of congruence relations on $\mathcal{Y} = \mathrm{Spf}(R, \mathfrak{m})$, which can be interpolated by "a single formula" consisting of

- a formal power series $f(u_1, \ldots, u_a, v_1, \ldots, v_b)$, and
- formal functions $g_1, \ldots, g_a, h_1, \ldots h_b \in \mathfrak{m}$ on $\mathcal{Y}$,

so that the infinite sequence of congruence relations can be written in the form

$$(*) \qquad f(g_1, \ldots, g_a, h_1^{p^{rn}}, \ldots, h_b^{p^{rn}}) \equiv 0 \pmod{\mathfrak{m}^{d_n}}, \quad n \geq n_0,$$

where r is a fixed positive integer and d_n is a sequence of positive integers such that $\lim_{n\to\infty} \frac{d_n}{p^{rn}} = 0$. This method, which we call *hypocotyl elongation*, says that the congruences $(*)$ imply the equality

$$f(g_1 \otimes 1, \ldots, g_a \otimes 1, 1 \otimes h_1, \ldots, 1 \otimes h_b) = 0$$

on $\mathcal{Y} \times \mathcal{Y}$. Thus the congruences $(*)$ modulo $\mathfrak{m}^{p^{d_n}}$, of formal functions on $\mathcal{Y}$ belonging to $\mathfrak{m}^{p^{rn}}$, "grows" to an equality of formal functions on $\mathcal{Y} \times \mathcal{Y}$. See 7.1.2 for a more detailed introduction to hypocotyl elongation.

The key step in proving orbital rigidity for p-divisible formal groups is as follows. We are given a p-divisible group $X = X_1 \times X_2$ over an algebraically closed base field κ of characteristic p, such that X_1 is isoclinic of slope μ_1, $X_1[F^r] = X_1[p^c]$ for suitable positive integers r, c with $\frac{c}{r} = \mu_1$, while all slopes of X_2 are strictly smaller than μ_1. There is a p-adic Lie group G acting strongly non-trivially on X, and we are given a reduced irreducible formal subscheme W of X stable under G. For any element $C \in \mathrm{Lie}(X) \cap \mathrm{End}(X)$, let $\Phi_C : X \times X \longrightarrow X$ be the morphism which sends every functorial point (x, x') of $X \times X$ to $x +_X C(\mathrm{pr}_{X_1}(x'))$. We need to show that $\Phi_C(W \times W) \subseteq W$, or equivalently, the formal function $\Phi_C^*(f)|_{W\times W}$ on $W \times W$ is equal to 0 for every formal function $f \in I_W$, where I_W consists of all formal functions on X which vanishes on W.

Consider the one-parameter subgroup $\exp(p^2\mathbb{Z}_p \cdot C)$ in G. The G-invariance of W tells us that $\big(\exp(p^nC)^*f\big)\big|_W = 0$ for every $f \in I_W$. For $n \gg 0$, we have a "first order approximation" $\exp(p^{nc}C) \equiv \mathrm{id}_X + p^{nc}C \pmod{p^{2nc}}$ from the Taylor series expansion of the automorphism $\exp(p^{nc}C))$ of X. Using such first order approximations, the equalities $\big(\exp(p^{nc}C)^*f\big)\big|_W = 0$ gives us an infinite sequence of congruence relations. With a suitable choices of regular parameters for the coordinate rings of X_1 and X, one sees that these congruence relations has precisely the form $(*)$, for the formal function Φ_C^*f on $X \times X$. Applying the method of hypocotyl elongation to the function $\Phi_C^*(W)$, we obtain the desired conclusion that $\Phi_C(W \times W) \subseteq W$ for every element $C \in \mathrm{Lie}(X) \cap \mathrm{End}(X)$.

10.1.6.2. It is natural to try to generalize the above method to prove (b′) in question Q2′, but one encounters several difficulties.

- Unless the biextension E of $X \times Y$ by Z is split, there is no natural "projection map" from E to Z with good properties.
- Any "explicit formula" of the action of $G \subseteq \mathrm{Aut}_{\mathrm{biext}}(E)$ through an exponential map from an subgroup of the Lie algebra $\mathrm{Lie}(G)$ must involve the Weil pairings for E, which complicates things.
- But the most serious obstacle has to do with the method of hypocotyl elongation. For a given one parameter subgroup $\exp(p^2\mathbb{Z}_p \cdot v)$ in G, the infinitely many congruence relations resulting from the assumption that W is stable under G cannot be interpolated by a "single formula" as in 10.1.6 $(*)$, which consists of a suitable formal power series in several variables and a finite number of elements of the affine coordinate ring of W.

Because of these difficulties, especially the last one, for a very long time it was completely unclear whether orbital rigidity actually holds for biextensions, or it is a pipe dream stemming from excessive optimism.

10.1.6.3. The way to solve this conundrum is to introduce a suitable class of rings of "generalized formal functions". They provide extension rings of any given complete augmented Noetherian local domain $(R, \mathfrak{m})$ over a perfect base field κ of characteristic p, and lie between R and the completion of the perfection of R. They are not Noetherian, unless $R = \kappa$, but they satisfy certain weak version of finiteness properties enjoyed by Noetherian local domains. We call them *completed tempered perfections* of the input complete Noetherian local domain $(R, \mathfrak{m})$, or *tempered perfections* for short. Elements of tempered perfections of R are called *tempered virtual functions* on the formal scheme $\mathrm{Spf}(R)$.

The usefulness of tempered virtual functions for the orbital rigidity question Q2$'$ is threefold. First, there are many "tempered virtual morphism" from a biextension E to the fiber group Z with good properties. Secondly for each one parameter subgroup $\exp(p^2\mathbb{Z}_p \cdot v)$ in G, the infinite sequence of congruence relation resulting from the first order approximation of the Taylor expansion of $\exp(p^{na} \cdot v)$, for a suitable positive integer a, can be interpolated by single formula which involves only a finite number of tempered virtual functions. Thirdly the method of hypocotyl prolongation extends to tempered virtual functions; see 10.5.6 and 10.5.3. Armed with the tempered virtual functions, the previous strategy for proving orbital rigidity for p-divisible groups also works for biextensions. The final result is stated in theorem 10.6.7.

Readers are advised to go to 10.7.1 for an introduction to tempered perfections. The definitions of several families of tempered perfections and their basic properties are collected in the appendix 10.7 of this chapter. Most of the basic algebraic properties of this new class of rings are still unexplored. We have resisted the temptation of developing a theory of tempered perfections of formal schemes and formal group. Instead proofs are given directly in terms of these rings and related co-algebras and co-actions.

10.1.7. The notion of Tate-linear formal varieties revealed. The desired properties of Tate-linear formal varieties have been discussed in 10.1.4. It is time to reveal what we believe is a good notion of Tate-linear formal varieties.

10.1.7.1. Definition. Let κ be a field of characteristic p. A *Tate-linear unipotent group* N over κ is a projective system $(N_i)_{i\geq 1}$ of finite group schemes N_i over κ with epimorphic transition homomorphisms $\pi_{i,i+1} : N_{i+1} \twoheadrightarrow N_i$ together with a compatible system of decreasing filtrations $\mathrm{Fil}_{\mathrm{sl}}^{\bullet} N_i$ indexed by $(0,1]$, satisfying the following properties.

(i) $\pi_{i,i+1}(\mathrm{Fil}_{\mathrm{sl}}^{s} N_{i+1}) = \mathrm{Fil}_{\mathrm{sl}}^{s} N_i$ for all $i \geq 1$ and all $s \in (0,1]$.

(ii) There exists a finite subset $\mathrm{slope}(N)$ of $(0,1] \cap \mathbb{Q}$, such that for every $s \in (0,1]$ and every $i \geq 1$, the quotient group scheme $\mathrm{gr}^s N_i := \mathrm{Fil}_{\mathrm{sl}}^{s} N_i / \mathrm{Fil}_{\mathrm{sl}}^{>s} N_i$ is trivial if and only if $s \in \mathrm{slope}(N)$.

(iii) For every $i \geq 1$, the distinct elements in the filtration $\mathrm{Fil}_{\mathrm{sl}}^{\bullet} N_i$ form a finite *central* series of subgroup schemes of N_i.

(iv) For each $s \in (0,1]$, there exists an p-divisible group Y_s over κ which is either 0 or isoclinic with slope s, such that the projective system

$$\mathrm{gr}^s N := \big(\mathrm{gr}^s N_i := \mathrm{Fil}_{\mathrm{sl}}^{s} N_i / \mathrm{Fil}_{\mathrm{sl}}^{>s} N_i\big)_{i\geq 1}$$

is isomorphic to the projective system $\big(Y_s[p^i], Y_s[p^{i+1}] \xrightarrow{[p]} Y_s[p^i]\big)_{i\geq 1}$ attached to Y_s.

Elements of the finite subset $\mathrm{slope}(N) \subseteq (0,1] \cap \mathbb{Q}$ are said to be the slopes of N. Note that N is a projective system of nilpotent groups of class at most $\mathrm{card}(\mathrm{slope}(N))$. Moreover N is uniquely ℓ-divisible for every prime number $\ell \neq p$.

10.1.7.2. Definition. Let $N = \big(N_i\big)_{i\geq 1}$ be a Tate-linear unipotent group over a field κ of characteristic p.

(a) The Tate module $\mathbf{T}_p(N)$ of N is the limit

$$\mathbf{T}_p(N) := \varprojlim_i N_i$$

of the projective system N, as a sheaf of groups on the category of κ-schemes with the fpqc topology. Clearly $\mathbf{T}_p(N)$ is a sheaf of torsion free nilpotent group of class at most $\mathrm{card}(\mathrm{slope}(N))$. Moreover it is uniquely ℓ-divisible for every prime number $\ell \neq p$.

(b) Define $\mathbf{V}_p(N) = \mathbf{T}_p(N)_{\mathbb{Q}}$ to be the Mal'cev completion of $\mathbf{T}_p(N)$. It is a sheaf of torsion free divisible nilpotent group of class at most $\mathrm{card}(\mathrm{slope}(N))$ for the fpqc topolgy on the category of κ-schemes.

Just as the stabilized Aut groups discussed in 5.4, we have a Lie theory for Tate-linear unipotent groups.

- Let $N = \big(N_i\big)_{i\geq 1}$ be a Tate-linear unipotent group over a field κ of characteristic p. Define $\mathrm{Lie}(\mathbf{V}_p(N))$ to be the sheaf of Lie $\mathbb{Q}_p$-algebras on the big fpqc site of $\mathrm{Spec}(\kappa)$, with the addition law and the Lie bracket on $\mathrm{Lie}(\mathbf{V}_p(N))$ given by the inverse Baker-Campbell-Hausdorff formula.
- The construction of $\mathrm{Lie}(\mathbf{V}_p(N))$ comes with isomorphisms of sheaves

$$\exp : \mathrm{Lie}(\mathbf{V}_p(N)) \xrightarrow{\sim} \mathbf{V}_p(N) \quad \text{and} \quad \log : \mathbf{V}_p(N) \xrightarrow{\sim} \mathrm{Lie}(\mathbf{V}_p(N)),$$

inverse to each other, such that $\mathbb{Q}_p$-Lie subalgebras of $\mathrm{Lie}(\mathbf{V}_p(N))$ correspond to torsion free divisible subgroups of $\mathbf{V}_p(N)$. This is the *Mal'cev correspondence* in the theory of nilpotent groups.

- There exists a p-divisible group L over κ, such that the additive group underlying $\mathrm{Lie}(\mathbf{V}_p(N))$ is isomorphic to $\big(\varprojlim_i L[p^i]\big)\otimes_{\mathbb{Z}}\mathbb{Q}$, where the projective limit is taken in the category of fpqc sheaves on the category of κ-schemes.
- Let $c \leq \mathrm{card}(\mathrm{slope}(N))$ be the smallest positive integer such that N is nilpotent of class at most c. If $p > c$, there is an integral version of the Lie theory of N, known as the *Lazard correspondence* in the theory of nilpotent groups. In this case we have a sheaf $\mathrm{Lie}(\mathbf{T}_p(N))$ of Lie $\mathbb{Z}_p$-algebras, and mutually inverse isomorphisms

$$\exp : \mathrm{Lie}(\mathbf{T}_p(N)) \xrightarrow{\sim} \mathbf{T}_p(N) \quad \text{and} \quad \log : \mathbf{T}_p(N) \xrightarrow{\sim} \mathrm{Lie}(\mathbf{T}_p(N)).$$

  Moreover there is a p-divisible group L over κ, such that the additive group underlying $\mathrm{Lie}(\mathbf{T}_p(N))$ is isomorphic to $\varprojlim_i L[p^i]$.

10.1.7.3. Definition. Let $N = \big(N_i\big)_{i\geq 1}$ be a Tate-linear unipotent group over a field κ of characteristic p. Define the Tate-linear formal variety $\mathrm{TL}(N)$ attached to N to be the smooth formal scheme over κ which represents the sheaf $\mathbf{V}_p(N)/\mathbf{T}_p(N)$, i.e.

$$\mathrm{TL}(N) := \mathbf{V}_p(N)/\mathbf{T}_p(N).$$

Alternatively, $\mathrm{TL}(N)$ can be defined as the deformation space of compatible systems of right N-torsors. In other words there exists a canonical isomorphism between $\mathbf{V}_p(N)/\mathbf{T}_p(N)$ and the deformation space of right N-torsors. The argument in the case when N is either $\mathscr{A}ut^{\mathrm{st}}(X)$ for some p-divisible group X over κ or $\mathscr{A}ut^{\mathrm{st}}(Y,\lambda)$ for some polarized p-divisible group (Y,λ) over κ shows that these two definitions are equivalent.

Clearly every homomorphism $h : N_1 \to N_2$ between Tate-linear unipotent groups induces a morphism $h_* : \mathrm{TL}(N_1) \to \mathrm{TL}(N_2)$ between Tate-linear formal varieties.

This group theoretic definition of Tate-linear formal varieties also allows us to define isogenies and Hecke correspondences in the context of Tate-linear formal varieties.

10.1.7.4. Definition. Let N be a Tate-linear unipotent group over a field κ of characteristic p. A closed formal subscheme Z of the Tate-linear formal variety $\mathrm{TL}(N)$ over κ is a *Tate-linear formal subvariety* if there exist a Tate-linear unipotent subgroup N' of N such that $\mathbf{T}_p(N')$ is co-torsion free in $\mathbf{T}_p(N)$ and the morphism $\mathrm{TL}(N') \rightarrowtail \mathrm{TL}(N)$ associated to $N' \hookrightarrow N$ is the closed embedding $Z \hookrightarrow \mathrm{TL}(N)$.

Note that the map $\mathrm{TL}(N') \to \mathrm{TL}(N)$ attached to the embedding $N' \hookrightarrow N$ is a mono because $\mathbf{T}_p(N')$ is co-torsion free in $\mathbf{T}_p(N)$.

10.1.7.5. With the notion of Tate-linear formal varieties specified, the general orbital rigidity property below acquires a precise meaning. We set up the notations first.

Let N be a Tate-linear unipotent group over a perfect field κ of characteristic p, and let $\mathrm{TL}(N)$ be the Tate-linear formal variety attached to N. For each slope s of N, let Y_s

be the p-divisible group over κ such that $\mathrm{gr}^s(N)$ is isomorphic to the projective system $\big(Y_s[p^i], Y_s[p^{i+1}] \xrightarrow{[p]_{Y_s}} Y_s[p^i]\big)_{i\geq 1}$ attached to Y_s.

(Orbital rigidity for Tate-linear formal varieties) *We use the notation in the preceding paragraph. Let G be a p-adic Lie group acting on N, such that the induced action of G on Y_s is strongly nontrivial for every slope s of N. If W is a reduced irreducible subvariety of* $\mathrm{TL}(N)$ *stable under the action of G, then W is a Tate-linear formal variety.*

As mentioned already, the orbital rigidity for Tate-linear formal varieties as stated above can be proved by the argument used for the case of biextension. The key is the method of hypocotyle elongation via tempered perfections, combined with induction on the number of slopes of the Tate-linear unipotent group N governing the Tate-linear formal variety $\mathrm{TL}(N)$. A detailed proof, together with general properties of Tate-linear formal varieties, will be published elsewhere.

We decided to present the proof of the special case of biextension, instead of general Tate-linear formal varieties, for several reasons. The biextension case corresponds to Tate-linear formal varieties $\mathrm{TL}(N)$ where N is nilpotent of class at most 2. All essential difficulties after the case of p-divisible formal groups show up in the biextension case. Its proof has the advantage of simplicity, with the main ideas clearly exhibited, and not shrouded by induction or the theory of Tate-linear formal varieties. In addition, the easier part of the proof of orbital rigidity for all Tate-linear formal varieties, which generalizes 10.3.4.1 and does not involve tempered perfections, can be reduced to the biextension case. This book is too long already, and we hope the readers would not mind not seeing a proof of the most general here.

10.2. Biextension basics

The notion of biextensions of commutative groups was first introduced by Mumford in [**76**] and further developed by Grothendieck in expositions VI, VII of [**18**].

10.2.1. Definition. Let R be a Noetherian complete local ring whose residue $R/\mathfrak{m}$ is a field of characteristic p, and $S := \mathrm{Spf}(R)$. Let X, Y, Z be p-divisible groups over R (resp. commutative formal groups) over R. A *biextension* of $X \times_S Y$ by Z is a 5-tuple

$$(\pi : E \to X \times_S Y,\ +_1 : E \times_Y E \to E,\ +_2 : E \times_X E \to E,\ \epsilon_1 : Y \to E,\ \epsilon_2 : X \to E)$$

where E is the formal spectrum of a Noetherian complete local ring formally smooth over R, π is an S-morphism, $+_1$ and ϵ_1 are Y-morphisms, $+_2$ and ϵ_2 are X-morphisms. In addition the following properties are satisfied.

(0) The morphism π is formally smooth and faithfully flat.

(1a) The pair $(+_1, \epsilon_1)$ makes E a p-divisible group (resp. commutative smooth formal group) over Y with 0-section ϵ_1.

(1b) The projection map $\pi : E \to X \times_S Y$ is a group homomorphism for the group law $+_1$ and the base change to Y of the group law $+_X : X \times_S X \to X$ of the p-divisible group X.

(2a) The pair $(+_2, \epsilon_2)$ makes E a p-divisible group (resp. commutative smooth formal group) over X with 0-section ϵ_2.

(2b) The projection map $\pi : E \to X \times_S Y$ is a group homomorphism for the group law $+_2$ and the base change to X of the group law $+_Y : Y \times_S Y \to Y$ of the p-divisible group Y.

(3a) The S-morphism

$$Z \times_S Y \to E, \quad (z, y) \mapsto z +_2 \epsilon_1(y)$$

defines an S-isomorphism from $Z \times_S Y$ to $E \times_{(X \times_S Y)} (0_X \times_S Y)$.

(3b) The S-morphism

$$Z \times_S X \to E, \quad (z, x) \mapsto z +_1 \epsilon_2(x)$$

defines an S-isomorphism from $Z \times_S X$ to $E \times_{(X \times_S Y)} (X \times_S 0_Y)$.

(4) (compatibility of the two relative group laws) For any formal scheme T over S and any four T-valued points $w_{11}, w_{12}, w_{21}, w_{22}$ of E such that

$$\pi_1(w_{11}) = \pi_1(w_{12}), \ \pi_1(w_{21}) = \pi_1(w_{22}), \ \pi_2(w_{11}) = \pi_2(w_{21}), \ \pi_2(w_{12}) = \pi_1(w_{22})$$

where $\pi_1 := \mathrm{pr}_1 \circ \pi$ and $\pi_2 := \mathrm{pr}_2 \circ \pi$ are the two projections from E to X and Y respectively, the equality

$$(w_{11} +_2 w_{12}) +_1 (w_{21} +_2 w_{22}) = (w_{11} +_1 w_{21}) +_2 (w_{12} +_1 w_{22})$$

holds.

Remark. (i) Conditions (1a) and (1b) assert that the relative group law $+_1$ on E over Y is an extension of (the base change to Y of) X by (the base change to Y of) Z. Similarly (2a) and (2b) say that the relative group law $+_2$ on E over X is an extension of (the base change to X of) Y by (the base change to X of) Z.

(ii) In 10.2.1 the group law $+_1$ (respectively $+_2$) denotes "addition along the first (respectively the second) of the two variables (X, Y)". This is consistent with the usage in [**76**, p. 320] but different from the convention in [**76**, p. 310].

(iii) Of course the definition 10.2.1 of biextension works in other contexts, for instance sheaves of commutative groups for the fppf site for a general scheme S. For our purpose the case when X, Y and Z are all p-divisible groups will be sufficient. For the main result on orbital rigidity for p-divisible groups, S will be the spectrum of a field k of characteristic $p > 0$ and X, Y, Z are p-divisible formal groups over k.

10.2.1.1. The following properties are easily verified.

(i) For any formal scheme T over S, any T-valued points y_1, y_2 of Y and any T-valued points x_1, x_2 of X, we have

$$\epsilon_1(y_1) +_2 \epsilon_1(y_2) = \epsilon_1(y_1 + y_2), \quad \epsilon_2(x_1) +_1 \epsilon_2(x_2) = \epsilon_2(x_1 + x_2).$$

(ii) For any formal scheme T over S, any T-valued points z of Z and any T-valued point w of E, we have

$$(z +_1 \epsilon_2(\pi_1(w))) +_2 w = (z +_2 \epsilon_1(\pi_2(w))) +_1 w.$$

This equality means that the Z-actions on E induced by the relative group laws $+_1$ and $+_2$ are equal, given $\pi : E \to X \times_S Y$ a natural structure as a Z-torsor. Let

$$* : Z \times_S E = (Z \times_S (X \times_S Y)) \times_{(X\times Y)} E \to E$$

be the morphism defining this Z-torsor structure on E.

(iii) The restriction of $+_1$ to $Z\times_S Z \subseteq E\times_Y E$ is equal to the group law of Z. Similarly for the restriction of $+_2$ to $Z \times_S Z \subseteq E \times_X E$.

(iv) The S-isomorphism $(z, y) \mapsto z +_2 \epsilon_1(y)$ in (3a) is a group isomorphism from the product group $Z \times_S Y$ to the group law on $E \times_{(X\times Y)} (0_X \times Y)$ induced by $+_2$. In other words the restriction to $\{0_X\} \subseteq X$ of the extension of Y by Z over X, given by the relative group law $+_2$, splits canonically. Similarly for the S-isomorphism $(z, x) \mapsto z +_1 \epsilon_2(x)$ in (3b) is a group isomorphism from the product group $Z \times_S X$ to the group law on $E \times_{(X\times Y)} (X \times 0_Y)$ induced by $+_1$.

(v) The restriction of ϵ_1 to 0_Y is equal to the restriction of ϵ_2 to 0_X. Over the scheme-theoretic union Δ of the images of $X \times_S 0_Y$ and $0_X \times_S Y$, i.e. the smallest closed subscheme of $X \times_S Y$ containing both, we have an S-morphism $\epsilon : \Delta \to E$ such that $\pi \circ \Delta = \mathrm{id}_\Delta$ which is equal to ϵ_2 on $X \times_S 0_Y$ and equal to ϵ_1 on $0_X \times_S Y$. Because $\pi : E \to X\times_S Y$ is formally smooth, there exists a section $s : X\times_S Y \to E$ of π which extends ϵ.

10.2.1.2. The *trivial biextension* of $X \times_S Y$ by Z is by definition the natural biextension structure on $X \times_S Y \times Z$, where the two relative group laws are given by

$$(x_1, y, z_1) +_1 (x_2, y, z_2) = (x_1 + x_2, y, z_1 + z_2), \quad (x, y_1, z_1) +_2 (x, y_2, z_2) = (x, y_1 + y_2, z_1 + z_2).$$

A biextension $E \to X \times_S Y$ by Z is *trivial* if there is an biextension isomorphism ψ from the trivial biextension to E which induces $\mathrm{id}_X, \mathrm{id}_Y, \mathrm{id}_Z$ on X, Y, Z respectively; ψ to $X \times_S Y \times_S 0_Z$ is called the a *splitting* of a trivial biextension of $X \times_S Y$ by Z. We will see in 10.2.3.6 that when X and Y are both p-divisible, such an isomorphism ψ is unique if one exists.

10.2.2. Cocycle description of biextensions.

10.2.2.1. Definition. Let X, Y, Z be smooth formal groups over S, let $\pi : E \to X \times_S Y$ be a biextension of $X \times_S Y$ by Z as in 10.2.1, and let $s : X \times_S Y \to E$ be a section of π which extends both ϵ_1 and ϵ_2 as in 10.2.1.1 (v). Define S-morphisms

$$\tau : (X \times_S X) \times_S Y \to Z \quad \text{and} \quad \sigma : X \times_S (Y \times_S Y) \to Z$$

associated to the section s by the following formulas expressed in terms of T-valued points x, x_1, x_2, y, y_1, y_2 in X and Y for formal schemes T over S:

(a) $$s(x_1, y) +_1 s(x_2, y) = \tau(x_1, x_2; y) * s(x_1 + x_2, y)$$

(b) $$s(x, y_1) +_2 s(x, y_2) = \sigma(x; y_1, y_2) * s(x, y_1 + y_2)$$

10.2.2.2. Cocycle identities. The S-morphisms τ and σ satisfy properties (1)–(5) below, for all formal schemes T over S, all T-valued points x, x_1, x_2, x_3 of X and all points y, y_1, y_2, y_3 of Y. Identities (1) and (2) are consequences of the fact that the section s of π extends ϵ_1 and ϵ_2. Identities (3) and (4) hold because the two relative group laws $+_1$ and $+_2$ are commutative and associative. The identity (5) follows from the compatibility of the two relative group laws.

(1) $\sigma(x; 0, y_2) = 0 = \sigma(x; y_1, 0)$, $\tau(0, x_2; y) = 0 = \tau(x_1, 0; y)$.
(2) $\sigma(0; y_1, y_2) = 0$, $\tau(x_1, x_2; 0) = 0$.
(3) (symmetry)
$$\sigma(x; y_1, y_2) = \sigma(x; y_2, y_1), \quad \tau(x_1, x_2; y) = \tau(x_2, x_1; y)$$
(4) (associativity)
$$\begin{aligned} \sigma(x; y_1, y_2) + \sigma(x; y_1 + y_2, y_3) &= \sigma(x; y_1, y_2 + y_3) + \sigma(x; y_2, y_3) \\ \tau(x_1, x_2; y) + \tau(x_1 + x_2, x_3; y) &= \tau(x_1, x_2 + x_3; y) + \tau(x_2, x_3; y) \end{aligned}$$
(5) (compatibility)
$$\begin{aligned} &\sigma(x_1 + x_2; y_1, y_2) - \sigma(x_1; y_1, y_2) - \sigma(x_2; y_1, y_2) \\ &= \tau(x_1, x_2; y_1 + y_2) - \tau(x_1, x_2; y_1) - \tau(x_1, x_2; y_2) \end{aligned}$$

10.2.2.3. Coboundary. Suppose that we replace $s(x, y)$ by a another section

(10.2.2.3.1) $$s'(x, y) = f(x, y) * s(x, y),$$

where $f(x, y) : X \times_S Y \to Z$ is an S-morphism such that $f(x, 0) = 0 = f(0, y)$. Then the maps $\tau' : (X \times_S X) \times_Y \to Z$ and $\sigma' : X \times_S (Y \times_S Y) \to Z$ associated to the section s' are related to the maps σ and τ by

(10.2.2.3.2) $$\tau'(x_1, x_2; y) - \tau(x_1, x_2; y) = f(x_1, y) + f(x_2, y) - f(x_1 + x_2, y),$$

(10.2.2.3.3) $$\sigma'(x; y_1, y_2) - \sigma(x; y_1, y_2) = f(x, y_1) + f(x, y_2) - f(x, y_1 + y_2).$$

10.2.2.4. Conversely given a pair (α, β) of S-morphisms satisfying equations (1)–(5) in 10.2.2.2, there exists a biextension of $X \times_S Y$ by Z naturally attached to the cocycle (α, β). Moreover the biextensions attached to two cocycles (α, β), (α', β') are isomorphic as biextensions of $X \times_S Y$ by Z in the sense of 10.2.3.1 (c) below if and only if the two cocycles differ by a coboundary in the sense that there exists an S-morphism $f : X \times_S Y \to Z$ such that 10.2.2.3.2 and 10.2.2.3.3 hold.

10.2.3. Homomorphisms between biextensions.

10.2.3.1. Definition. Let X, Y, Z, X', Y', Z' be commutative smooth formal groups (respectively p-divisible groups) over $S = \mathrm{Spf}(R)$ as in 10.2.1. Let $\pi : E \to X \times_S Y$ be a biextension of $X \times_S Y$ by Z, and $\pi' : E' \to X' \times_S Y'$ be a biextension of $X' \times_S Y'$ by Z'.

(a) An *S-homomorphism of biextensions*, or an *S-bihomomorphism* for short, from the biextension E to the biextension E' is a quadruple of S-morphisms

$$(\psi : E \to E', \alpha : X \to X', \beta : Y \to Y', \gamma : Z \to Z')$$

where α, β, γ are S-homomorphisms of commutative formal groups (respectively p-divisible groups), and ψ is compatible with the biextension structure of E and E', in the sense that the following properties are satisfied.
(i) $\pi' \circ \psi = (\alpha \times \beta) \circ \pi$,
(ii) $\psi \circ +_1 = +_1' \circ (\psi \times_Y \psi)$, $\psi \circ +_2 = +_2' \circ (\psi \times_X \psi)$,
(iii) $\psi \circ \epsilon_1 = \epsilon_1' \circ \beta$, $\psi \circ \epsilon_1 = \epsilon_2' \circ \alpha$.
If $X = X'$, $Y = Y'$, $Z = Z'$, $E = E'$ and $\pi = \pi'$, such an S-bihomomorphism from E to E' is said to be an S-endomorphism of the biextension E.

(b) An S-bihomomorphism $(\psi, \alpha, \beta, \gamma)$ is an *isomorphism of biextensions*, if ψ, α, β and γ are all isomorphism of formal schemes (respectively p-divisible groups), in which case the quadruple $(\psi^{-1}, \alpha^{-1}, \beta^{-1}, \gamma^{-1})$ "is" an S-bihomomorphism from E' to E.

(c) Suppose that $X' = X$, $Y' = Y$ and $Z' = Z$. We say that the E and E' are isomorphic as biextensions of $X \times_S Y$ by Z if there exists an isomorphism $(\psi, \mathrm{id}_X, \mathrm{id}_Y, \mathrm{id}_Z)$ from E to E'.

(d) An S-bihomomorphism $(\psi, \alpha, \beta, \gamma)$ between biextensions of p-divisible groups (respectively commutative smooth formal groups) is an *isogeny* if the homomorphisms α, β and γ between formal groups (respectively p-divisible groups) are all isogenies.

Note that an isomorphism $(\psi, \alpha, \beta, \gamma)$ from E to E' as in 10.2.3.1 (b) above induces an isomorphism $(\psi', \mathrm{id}_X, \mathrm{id}_Y, \mathrm{id}_Z)$ from $\gamma_* E$ to $(\alpha \times \beta)^* E'$, so that the two biextensions $\gamma_* E$ and $(\alpha \times \beta)^* E'$ of $X \times Y$ by Z' are isomorphic in the sense of 10.2.3.1 (c).

When the biextensions E and E' are specified by cocycles (τ, σ) and (τ', σ') respectively, an S-homomorphism of biextensions from E to E' is given by a map $\mu : X \times_S Y \to Z'$ satifying (10.2.3.3.2) and (10.2.3.3.3) below.

10.2.3.2. Remark. (i) Let E, E' be biextensions as in 10.2.3.1. The set $\mathrm{Hom}_{\mathrm{biext}}(E, E')$ of all biextension homomorphisms from E to E' does not have a natural group structure. Instead there are two relative group laws

$$\mathrm{Hom}_{\mathrm{biext}}(E, E') \times_{\mathrm{Hom}(Y,Y')} \mathrm{Hom}_{\mathrm{biext}}(E, E') \longrightarrow \mathrm{Hom}_{\mathrm{biext}}(E, E')$$
$$\mathrm{Hom}_{\mathrm{biext}}(E, E') \times_{\mathrm{Hom}(X,X')} \mathrm{Hom}_{\mathrm{biext}}(E, E') \longrightarrow \mathrm{Hom}_{\mathrm{biext}}(E, E')$$

However the natural map

$$\mathrm{Hom}_{\mathrm{biext}}(E,E') \to \mathrm{Hom}(X,X') \times \mathrm{Hom}(Y,Y')$$

may not be surjective. So in general the set $\mathrm{Hom}_{\mathrm{biextn}}(E,E')$ does not have a natural structure as a biextension of $\mathrm{Hom}(X,X') \times \mathrm{Hom}(Y,Y')$ by $\mathrm{Hom}(Z,Z')$.

(ii) In 10.2.3.1 we did *not* consider quadruples

$$\tilde{\psi} : E \to E',\ \tilde{\alpha} : Y \to X',\ \tilde{\beta} : X \to Y',\ \gamma : Z \to Z'$$

of S-morphisms such that the diagram

$$\begin{tikzcd} E \arrow[rr, "\tilde{\psi}"] \arrow[d, "\pi"] & & E' \arrow[d, "\pi'"] \\ X \times Y \arrow[rr, "(\tilde{\alpha}\circ \mathrm{pr}_2)\times(\tilde{\beta}\circ \mathrm{pr}_1)"] & & X \times Y \end{tikzcd}$$

commutes,

$$\tilde{\psi} \circ +_1 = +'_2 \circ (\tilde{\psi} \times_X \tilde{\psi}), \quad \tilde{\psi} \circ +_2 = +'_1 \circ (\tilde{\psi} \times_Y \tilde{\psi}),$$
$$\tilde{\psi} \circ \epsilon_1 = \epsilon'_2 \circ \tilde{\alpha}, \quad \text{and} \quad \tilde{\psi} \circ \epsilon_2 = \epsilon'_1 \circ \tilde{\beta}.$$

Had we done so, we would have introduced a "parity" in the definition of homomorphisms, endomorphisms and automorphisms of biextensions, so that the composition of two homomorphisms with the same parity is even, while the composition of two homomorphisms with different parities is odd.

10.2.3.3. Cocycle description of homorphisms of biextensions.

Let X, Y, Z, X', Y', Z' be smooth formal groups over S. Let E, E' be biextensions over S as in 10.2.3.1. Let $\psi : E \to E'$ be a homomorphism of bi-extensions over S as in 10.2.3.1, which induces S-homomorphisms $\alpha : X \to X'$, $\beta : Y \to Y'$ and $\gamma : Z \to Z'$. Let $s(x,y)$ be a section of $\pi : E \to X \times_S Y$ extending ϵ_1 and ϵ_2, and let $\tau : (X \times_S X) \times_S Y \to Z$, $\sigma : X \times_S (Y \times_S Y) \to Z$ be the maps associated to the section $s(x,y)$ as defined in 10.2.2. Similarly let $s'(x',y')$ be a section of $\pi : E' \to X' \times_S Y'$ extending ϵ'_1 and ϵ'_2, and define $\tau' : (X' \times_S X') \times_S Y' \to Z'$ and $\sigma' : X' \times_S (Y' \times_S Y') \to Z'$ in the same way. Define an S-morphism

$$\mu = \mu_\psi : X \times_S Y \to Z'$$

by

$$\psi(s(x,y)) = \mu(x,y) * s'(\alpha(x), \beta(y)) \tag{10.2.3.3.1}$$

for all points x of X and all points y of Y with values in the same formal scheme over S. It is easy to verify that

$$\gamma(\tau(x_1,x_2;y)) - \tau'(\alpha(x_1),\alpha(x_2);\beta(y)) = \mu(x_1,y) + \mu(x_2,y) - \mu(x_1+x_2,y) \tag{10.2.3.3.2}$$

$$\gamma(\sigma(x;y_1,y_2)) - \sigma'(\alpha(x);\beta(y_1),\beta(y_2)) = \mu(x,y_1) + \mu(x,y_2) - \mu(x,y_1+y_2) \tag{10.2.3.3.3}$$

for all formal schemes T over S, all T-points x, x_1, x_2 of X and all T-points y, y_1, y_2 of Y.

Conversely it is easy to verify that every S-morphism $\mu : X \times_S Y \to Z'$ which the two displayed equations (10.2.3.3.2) (10.2.3.3.3) indeed defines a homomorphism of biextensions from E to E'.

10.2.3.4. Let $E \to X \times_S Y$ be a biextension of formal groups $X \times_S Y$ by Z.

(a) For any formal group Z' over S and any S-homomorphism $\xi : Z \to Z'$, the standard push-forward construction yields a biextension $\xi_*(E \to X \times_S Y)$ of $X \times_S Y$ by Z', plus a homomorphism ψ_1 from $E \to X \times_S Y$ to $\xi_*(E \to X \times_S Y)$, which induces $\mathrm{id}_X, \mathrm{id}_Y, \xi$ on X, Y, Z respectively. In addition $\xi_*(E \to X \times_S Y)$ satisfies the universal property that every biextension homomorphisms $(\psi, \alpha, \beta, \xi)$ from E to a biextension E' of $X' \times Y'$ by Z' factors through ψ_1.

(b) Similarly for any formal groups X_1, Y_1 over S and any homomorphisms $\zeta : X_1 \to X$, $\eta : Y_1 \to Y$, the standard pull-back construction yields a biextension $(\zeta, \eta)^*(E \to X \times_S Y)$ of $X_1 \times_S Y_1$ by Z, which satisfies an obvious universal property among biextension homomorphisms $(\psi_1, \alpha_1, \beta_1, \gamma_1)$ from biextensions $E_1 \to X_1 \times_S Y_1$ to E with $\alpha_1 = \alpha$ and $\beta_1 = \beta$.

10.2.3.5. Lemma. *Let X and Y be p-divisible groups over S. Every bi-additive morphism $g : X \times_S Y \to Z$ from $X \times_S Y$ to a sheaf of groups Z over S is identically zero.*

Proof. The proof is completely formal.

(a) The bi-additivity of g implies that $g([p^n]_X(x_1), [p^n]_Y(y_1)) = [p^{2n}]_Z(g(x_1, y_1)) = 0$ for all S-scheme T_1, all $x_1 \in X[p^{2n}](T_1)$ and all $y_1 \in Y[p^{2n}(T_1)$.

(b) Recall that the morphisms

$$[p^n]_{X[p^{2n}]\to X[p^n]} : X[p^{2n}] \to X[p^n] \quad \text{and} \quad [p^n]_{Y[p^{2n}]\to X[p^n]} : Y[p^{2n}] \to Y[p^n]$$

induced by "multiplication by p^n" are both faithfully flat. So for every S-scheme T, every $x \in X[p^n](T)$, and every $y \in Y[p^n](T)$, there exists a faithfully flat morphism $f : T_1 \to T$, an element $x_1 \in X[p^{2n}](T_1)$ and an element $y_1 \in Y[p^{2n}](T_1)$ such that

$$x \circ f = [p^n]_{X[p^{2n}]\to X[p^n]} \circ x_1 \quad \text{and} \quad y \circ f = [p^n]_{Y[p^{2n}]\to Y[p^n]} \circ y_1.$$

The desired conclusion that $g : X \times_S Y \to Z$ is equal to the zero map follows immediately from (a) and (b). $\square$

10.2.3.6. Corollary. *Let X, Y, Z, X', Y', Z' be smooth formal groups over S. Let $\pi : E \to X \times_S Y$ be a biextension of $X \times_S Y$ by Z and let $\pi' : E' \to X' \times_S Y'$ be a biextension of $X' \times_S Y'$ by Z'. Let $(\psi, \alpha, \beta, \gamma)$ be an S-homomorphism of biextensions.*

(a) *The maps α, β and γ are uniquely determined by the morphism ψ.*

(b) *Suppose that X and Y are both p-divisible. The morphism $\psi : E \to E'$ is uniquely determined by the homomorphisms $\alpha : X \to X'$, $\beta : Y \to Y'$ and $\gamma : Z \to Z'$.*

Proof. The statement (a) is obvious. It remains to prove (b). Suppose that $(\psi_1, \alpha, \beta, \gamma)$ and $(\psi_2, \alpha, \beta, \gamma)$ are two S-homomorphisms of biextensions from E to E'. We need to show that $\psi_1 = \psi_2$.

Let $g : X \times_S Y \to Z'$ be the S-morphsm such that

$$\psi_2 = (g \circ \pi') * \psi_1.$$

It is easy to see that the map $g : X \times_S Y \to Z'$ is a *bi-additive* in the sense that

$$g(x_1 + x_2, y) = g(x_1, y) + g(x_2, y),\ \ g(x, y_1 + y_2) = g(x, y_1) + g(x, y_2)$$

for all formal scheme T over S, all T-valued points x, x_1, x_2 of X and all T-valued points y, y_1, y_2 of Y. Such a bi-additive map $g : X \times_S Y \to Z'$ is necessarily equal to the zero map by lemma 10.2.3.5. Therefore the natural map

$$\begin{array}{rcl} \mathrm{Hom}_{\mathrm{biext}}(E, E') & \longrightarrow & \mathrm{Hom}(X, X') \times \mathrm{Hom}(Y, Y') \times \mathrm{Hom}(Z, Z') \\ (\psi, \alpha, \beta, \gamma) & \mapsto & (\alpha, \beta, \gamma) \end{array}$$

is injective when X and Y are both p-divisible groups over S. $\square$

10.2.3.7. Corollary. *Let X, Y, Z be p-divisible groups over a scheme S, and let $\pi : E \to X \times_S Y$ be a biextension. Let $T \to S$ be a faithfully flat morphism. If the base change $\pi_T : E_T \to X_T \times_T Y_T$ to T of the biextension E is split, then E is a split biextension.*

PROOF. Let $\zeta_T : X_T \times_T Y_T \to E_T$ be a splitting of E_T. By 10.2.3.5 and 10.2.3.6 the pull-backs of ζ_T via the two projections $\mathrm{pr}_1, \mathrm{pr}_2 : T \times_S T \to T$ are canonically isomorphic, and the canonical isomorphism satisfies the cocycle condition. So ζ_T descends to a splitting $\zeta : X \times_S Y \to E$ of E. $\square$

10.2.4. The Weil pairings of a biextension of p-divisible groups.

Let R be a Noetherian complete local ring whose residue field $R/\mathfrak{m}$ has characteristic p. Let X, Y, Z be p-divisible groups over $S = \mathrm{Spf}(R)$ as in 10.2.1.

10.2.4.1. For every biextension E of $X \times_S Y$ by Z, there is an associated family

$$\theta_E = \big(\theta_n^E\big)_{n \in \mathbb{N}_{\geq 1}}$$

of bilinear pairings

$$\theta_n = \theta_n^E : X[p^n] \times_S Y[p^n] \to Z[p^n], \quad n \in \mathbb{N}$$

called the *Weil pairing*, attached to this biextension $E \to X \times_S Y$. A definition of the Weil pairing and its basic properties will be reviewed in 10.2.5.

A biextension E of p-divisible groups is determined by its Weil pairing up to unique isomorphism; this is a consequence of 10.2.5.8. In particular a biextension E is trivial if and only if $\theta_n = 0$ for every $n \geq 1$.

Remark. As we will see in 10.2.5.2, there are actually two families of Weil pairings associated to a given biextensions of p-divisible groups. The first family $(\theta_n)_n$, denoted by $(\beta_n)_n$ in [76], is normalized by the relative group law $+_1$. The other family $(\omega_n)_n$ is normalized by the relative group law $+_2$. The two Weil parings differ by a sign: $\theta_n +_Z \omega_n = 0$ for all n.

10.2.4.2. These bilinear pairings θ_n are compatible in the sense that

(10.2.4.2.1) $$\theta_n([p]_X(x_{n+1}), [p]_Y(y_{n+1})) = [p]_Z\,(\theta_{n+1}(x_{n+1}, y_{n+1}))$$

for all $x_{n+1} \in X[p^{n+1}]$, all $y_{n+1} \in Y[p^{n+1}]$ and all $n \in \mathbb{N}$; or equivalently,

(10.2.4.2.2) $$\theta_{n+1}(x_n, y_{n+1}) = \theta_n(x_n, [p]_Y(y_{n+1}))$$

(10.2.4.2.3) $$\theta_{n+1}(x_{n+1}, y_n) = \theta_n([p]_X(x_{n+1}), y_n)$$

for all $x_n \in X[p^n]$, $x_{n+1} \in X[p^{n+1}]$, $y_n \in Y[p^n]$, $y_{n+1} \in Y[p^{n+1}]$ and all $n \in \mathbb{N}$. See [**76**, Prop. 4] and also Exp. VIII of [**18**] more information.

Exercise. Suppose that a, r are natural numbers with $a < r$ such that

$$p^{am} \cdot \theta_{rm}(x_{rm}, y_{rm}) = 0.$$

for all $m \geq 1$ and all functorial points (x_{rm}, y_{rm}) of $X[p^{rm}] \times Y[p^{rm}]$. Show that $\theta_n = 0$ for all $n \geq 1$.

10.2.4.3. Functoriality of Weil pairings. Let X, Y, Z, X', Y', Z' be p-divisible groups over S, let E be a biextension of $X \times_S Y$ by Z, and let E' be a biextension of $X' \times_S Y'$ by Z'. Let $(\theta_n^E)_{n\in\mathbb{N}}$ and $(\theta_n^{E'})_{n\in\mathbb{N}}$ be the Weil pairings attached to E and E' respectively. Suppose that $(\psi, \alpha, \beta, \gamma)$ is a homomorphism of biextensions from E to E'. Then

$$\gamma(\theta_n^E(x_n, y_n)) = \theta_n^{E'}(\alpha(x_n), \beta(y_n))$$

for all $x_n \in X[p^n]$ and all $y_n \in Y[p^n]$.

The following statements follow easily from the functoriality of Weil pairings.

(a) For any isogeny $\xi : Z \to Z'$, the push-forward biextension $\xi_*(E \to X \times_S Y)$ is trivial if and only if $E \to X \times_S Y$ is.

(b) For any pair of isogenies $\zeta : X_1 \to X$, $\eta : Y_1 \to Y$, the pull-back biextension $(\zeta \times \eta)^*(E \to X \times_S Y)$ is trivial if and only if $E \to X \times_S Y$ is.

10.2.4.4. Lemma. *Let X, Y, Z be p-divisible groups over a field κ of characteristic p. Let $\pi : E \to X \times Y$ be a biextension of $X \times Y$ by Z, and let $\big(\theta_n\big)_{n\geq 1}$ be the associated Weil pairings. Suppose that for every slope λ of X and every slope ν of Y, $\lambda + \nu$ is not a slope of Z. Then $\theta_n = 0$ for all n. In other words, the biextension $E \to X \times Y$ is trivial.*

PROOF. One can use the Dieudonné theory for biextensions stated in 10.2.7.2 to prove 10.2.4.4. Here is a direct proof.

It suffices to prove the statement after extending the base field to an algebraic closure of κ and modifying X, Y, Z by isogenies. Using the bilinearity of the Weil pairings, we are reduced to the following special case.

> *The p-divisible groups X, Y, Z are isoclinic, and there exist natural numbers a, b, c, r, $r > 0$, $a, b, c \leq r$, $a + b \neq c$, and isomorphisms $\alpha : X \xrightarrow{\sim} X^{(p^r)}$, $\beta : Y \xrightarrow{\sim} Y^{(p^r)}$, $\gamma : Z \xrightarrow{\sim} Z^{(p^r)}$, such that*
>
> $$\mathrm{Fr}^r_{X/\kappa} = \alpha \circ [p^a]_X, \quad \mathrm{Fr}^r_{Y/\kappa} = \alpha \circ [p^b]_Y, \quad \mathrm{Fr}^r_{Z/\kappa} = \alpha \circ [p^c]_Z.$$

Functoriality with respect to the κ-bihomomorphism $\big(\mathrm{Fr}^r_{E/\kappa}, \mathrm{Fr}^r_{X/\kappa}, \mathrm{Fr}^r_{Y/\kappa}, \mathrm{Fr}^r_{Z/\kappa}\big)$ from E to $E^{(p^r)}$ tells us that

$$\theta_n^{E^{(p^r)}}(\mathrm{Fr}^r_{X/\kappa}x_n, \mathrm{Fr}^r_{Y/\kappa}y_n) = \mathrm{Fr}^r_{Z/\kappa}\theta_n^E(x_n, y_n)$$

for all functorial points (x_n, y_n) of $X[p^n] \times Y[p^n]$ and all $n \geq 1$, i.e.

$$(*) \qquad [p^{a+b}]_Z\big(\theta_n^{E^{(p^r)}}(\alpha(x_n), \beta(y_n))\big) = [p^c]_Z\big(\gamma(\theta_n^E(x_n, y_n))\big) \quad \forall\, n \geq 1.$$

Suppose that $a + b > c$. We claim that

$$(**) \qquad [p^{a+b-c}]_{Z^{(p^r)}}\big(\theta_n^{E^{(p^r)}}(\alpha(x_n), \beta(y_n))\big) = \gamma(\theta_n^E(x_n, y_n))$$

for every commutative κ-algebra R, every R-valued points $(x_n, y_n) \in X[p^n](R) \times Y[p^n](R)$, and every $n \geq 1$. There exists a finite locally free commutative R-algebra R' and an R'-point $(x_{n+c}, y_{n+c}) \in (X[p^{n+c}] \times Y[p^{n+c}])(R')$, such that

$$[p^c]_X(x_{n+c}) = x_n \quad \text{and} \quad [p^c]_Y(y_{n+c}) = y_n.$$

Since

$$\begin{aligned}[p^{a+b-c}]_{Z^{(p^r)}}\big(\theta_n^{E^{(p^r)}}(\alpha(x_n), \beta(y_n))\big) &= [p^{a+b-c}]_{Z^{(p^r)}}\big(\theta_{n+c}^{E^{(p^r)}}(p^c\alpha(x_{n+c}), \beta(y_n))\big)\\ &= [p^{a+b}]_{Z^{(p^r)}}\big(\theta_{n+c}(\alpha(x_{n+c}), \beta(y_{n+c}))\big)\end{aligned}$$

and similarly

$$\theta_n^E(x_n, y_n) = [p^c]_Z\big(\theta_{n+c}^E(x_{n+c}, y_{n+c})\big),$$

the claim follows from $(*)$. Iterating $(**)$, we get

$$\gamma^N\big(\theta_n^E(x_n, y_n)\big) = [p^{N(a+b-c)}]_{Z^{(p^{Nr})}}\big(\theta_n^{E^{(p^{Nr})}}(\alpha^N(x_n), \beta^N(y_n))\big)$$

for all $N \in \mathbb{N}$, where $\alpha^N = \alpha^{(p^{N-1})} \circ \cdots \circ \alpha^{(p)} \circ \alpha$ is the N-th iterate of α, and similarly for β^N and γ^N. With $N > \frac{n}{a+b-c}$, we see that $\theta_n^E = 0$ for all $n \geq 1$ when $a + b > c$. The case when $a + b - c < 0$ is proved by a similar argument. $\square$

10.2.5. The Weil pairing as descent data over torsion subgroup schemes.

Let X, Y, Z be p-divisible groups over a base scheme S. Let $\pi : E \to X \times_S Y$ be a biextension of $X \times_S Y$ by Z over S. We review in 10.2.5.1

(a) the definition of the Weil pairing attached to a biextension $E \to X \times_S Y$, and

(b) how to construct a biextension E_n of $X[p^n] \times_S Y[p^n]$ by Z by descending the split biextension

$$Z \times X[p^n] \times_S Y[p^{2n}] \to X[p^n] \times_S Y[p^{2n}]$$

along the faithfully flat morphism

$$1_{X[p^n]} \times_S [p^n]_{Y[p^{2n}]} : X[p^n] \times_S Y[p^{2n}] \to X[p^n] \times_S Y[p^n]$$

using the descent datum given by a bihomomorphism $\theta_n : X[p^n] \times_S Y[p^n] \to Z[p^n]$.

The descent construction reviewed in 10.2.5.1 (iii), (iv) has many applications. For instance it implies that if the Weil pairings $\theta_{n_1,E}$, $\theta_{n_1,E'}$ attached biextensions E, E' of p-divisible groups $X \times_S Y$ by Z at a fixed level $[p^{n_1}]$ coincide, then there exists a canonical isomorphism between the restrictions of the biextensions E and E' to $X[p^{n_1}] \times_S Y[p^{n_1}]$; see 10.2.5.8 and its Dieudonné theory version 10.2.7.2, 10.2.7.3. More importantly it allows us to compute the leading term of the Taylor expansion of actions of p-adic Lie groups on biextensions; see 10.4.1.2.

10.2.5.1. We recall the explicit construction of Weil pairings

$$\theta_n = \theta_n^E : X[p^n] \times_S Y[p^n] \to Z[p^n], \quad n \geq 1$$

in [76, pp. 320–321].

(i) The first ingredient is a canonical trivialization ξ_n of the biextension

$$(1_{X[p^n]} \times [p^n]_{Y[p^{2n}]})^* E_n = E_n \times_{(X[p^n]\times_S Y[p^n], 1_{X[p^n]}\times_S [p^n]_{Y[p^{2n}]})} (X[p^n] \times_S Y[p^{2n}]),$$

the pull-back of $E_n = \pi^{-1}(X[p^n] \times_S Y[p^n])$ via the finite locally free bi-additive homomorphism $1_{X[p^n]} \times [p^n]_{Y[p^{2n}]} : X[p^n] \times_S Y[p^{2n}] \to X[p^n] \times_S Y[p^n]$. In other words we will construct a natural bi-additive map

$$\xi_n = \xi_n^E : X[p^n] \times_S Y[p^{2n}] \to E_n$$

such that the diagram

$$\begin{tikzcd} X[p^n] \times_S Y[p^{2n}] \arrow[r, "\xi_n"] \arrow[d, "="'] & E_n \arrow[d, "\pi_n"] \\ X[p^n] \times_S Y[p^{2n}] \arrow[r, "1\times[p^n]_Y"] & X[p^n] \times_S Y[p^n] \end{tikzcd}$$

commutes.

Given any S-scheme T, any $x_n \in X[p^n](T)$, any $y_{2n} \in Y[p^{2n}](T)$, there exist (a) a scheme T_1 faithfully flat and locally of finite presentation over T, and (b) an element $e_1 \in E(T_1)$ which lies above (x_n, y_{2n}), such that when one multiplies e_1 by p^n with respect to the first relative group law $+_1$, we have

$$[p^n]_{+_1}(e_1) = \epsilon_1(y_{2n}).$$

Such an element e_1 is not unique, but any two choices differ by an element of $Z[p^n]$. Define $\xi_n(x_n, y_{2n})$ as p^n times e_1 with respect to the second group law $+_2$:

$$\xi_n(x_n, y_{2n}) := [p^n]_{+_2}(e_1).$$

Clearly the right hand side of the above equality is independent of the choice of the element e_1, where we have used the first group law $+_1$ to produce a $Z[p^n]$-torsor lying above the T-point (x_n, y_{2n}) of $X[p^n] \times_S Y[p^{2n}]$. By descent we conclude that $\xi_n(x_n, y_{2n}) \in E_n(T)$.

We have defined a morphism $\xi_n : X[p^n] \times_S Y[p^{2n}] \to E_n$. This morphism corresponds to a section of the biextension

$$(1_{X[p^n]} \times [p^n]_{Y[p^{2n}]})^* E_n \to X[p^n] \times_S Y[p^{2n}],$$

denoted again by ξ_n, abusing the notation. It is easy to see that ξ_n is a bihomomorphism which splits the biextension $(1_{X[p^n]} \times [p^n]_{Y[p^{2n}]})^* E_n$.

(ii) Define a morphism $\alpha_n : Z \times_S X[p^n] \times_S Y[p^{2n}] \longrightarrow E_n = \pi^{-1}(X[p^n] \times_S Y[p^n])$ by

$$\alpha_n(z, x_n, y_{2n}) := z * \xi_n(x_n, y_{2n})$$

for all S-scheme T, all $z \in Z(T)$, all $x_n \in X[p^n](T)$ and all $y_{2n} \in Y[p^{2n}](T)$. It is easy to see that the following commutative diagram

$$\begin{tikzcd}
Z \times_S X[p^n] \times_S Y[p^{2n}] \arrow[r, "\alpha_n"] \arrow[d, "\mathrm{pr}_{23}"'] & E_n \arrow[d, "\pi|_{E_n}"] \\
X[p^n] \times_S Y[p^{2n}] \arrow[r, "1 \times p^n"] & X[p^n] \times_S Y[p^n]
\end{tikzcd}$$

is cartesian. So the biextension $\pi_n : E_n \to X[p^n] \times_S Y[p^n]$ is descended from the trivial biextension $\mathrm{pr}_{23} : Z \times_S X[p^n] \times_S Y[p^{2n}] \longrightarrow X[p^n] \times_S Y[p^{2n}]$ along the faithfully flat morphism $1_{X[p^n]} \times [p^n]_{Y[p^{2n}]} : X[p^n] \times_S Y[p^{2n}] \longrightarrow X[p^n] \times_S Y[p^n]$.

(iii) Construct a bihomomorphism

$$\theta_n = \theta_n^E : X[p^n] \times_S Y[p^n] \longrightarrow Z[p^n]$$

using the descent datum for α_n.

The effect of translation by elements of $Y[p^n]$ on the isomorphism α_n is recorded by a map $\theta'_n : X[p^n] \times_S Y[p^{2n}] \times_S Y[p^n] \to Z$, defined by

$$\alpha_n(\lambda, x_n, y_{2n}) = \alpha_n(\lambda + \theta'_n(x_n, y_{2n}, b_n), x, y_{2n} + b_n)$$

for all S-scheme T, all $\lambda \in Z(T)$, all $x_n \in X[p^n](T)$, all $y_{2n} \in Y[p^{2n}](T)$ and all $b_n \in Y[p^n](T)$. An easy calculation shows that $\theta'_n(x_n, y_{2n}, b_n)$ is independent of y_{2n}. In other words there exists an S-morphism $\theta_n : X[p^n] \times_S Y[p^n] \to Z$ such that the last displayed equation simplifies to

$$\alpha_n(\lambda, x_n, y_{2n}) = \alpha_n(\lambda + \theta_n(x_n, b_n), x_n, y_{2n} + b_n).$$

An easy calculation shows that θ_n is a bihomomorphism, hence it factors through the closed subgroup scheme $Z[p^n] \hookrightarrow Z$.

Reversing the construction, it is easy to see that θ_n encodes the descent datum from the trivial biextension $Z \times_S X[p^n] \times_S Y[p^{2n}]$ down to E_n: the bihomomorphism θ_n gives an $X[p^n]$-action of the base change to $X[p^n]$ of the group scheme $Y[p^n]$, on the $X[p^n]$-scheme $Z \times_S X[p^n] \times_S Y[p^{2n}]$. So $E_n \to X[p^n] \times_S Y[p^n]$ can be reconstructed from θ_n, and the biextension $\pi : E \to X \times_S Y$ can be reconstructed from the family $(\theta_n)_{n \geq 1}$ of Weil pairings.

10.2.5.2. The two relative group laws play different roles in the definition the morphisms ξ_n and θ_n. We will say that ξ_n and θ_n are normalized by the first group law $+_1$ (or by the first factor X in the product $X \times_S Y$), referring to the condition $[p^n]_{+_1}(z_1) = \epsilon_1(y)$ above on the element z_1 above (x, y).

If the roles played by the two relative laws are interchanged, then we get a canonical splitting

$$\psi_n = \psi_n^E : X[p^{2n}] \times_S Y[p^n] \to E_n$$

of $([p^n]_{X[p^{2n}]} \times 1_{Y[p^n]})^* E_n$ normalized by the relative group law $+_2$, and a bi-additive map

$$\omega_n = \omega_n^E : X[p^n] \times_S Y[p^n] \to Z[p^n]$$

such that

$$\omega_n(a_n, y_n) * \psi_n(x_{2n} + a_n, y_n) = \psi_n(x_{2n}, y_n)$$

$\forall\, a_n \in X[p^n]$, $\forall\, x_{2n} \in X[p^{2n}]$, $\forall\, y_n \in Y[p^n]$. Note that

$$\psi_n^E = \xi_n^{\iota^* E} \circ (\iota|_{X[p^{2n}] \times Y[p^n]}) \quad \text{and} \quad \omega_n^E = \theta_n^{\iota^* E} \circ (\iota|_{X[p^n] \times Y[p^n]})$$

for each $n \geq 1$, where $\iota : X \times_S Y \to Y \times_S X$ is isomorphism $(x, y) \mapsto (y, x)$ on functorial points of $X \times_S Y$, and $\iota^* E \to Y \times_S X$ is the pull-back by ι of the biextension $E \to X \times_S Y$.

Claim. The bi-additive map $\omega_n : X[p^n] \times_S Y[p^n] \to Z[p^n]$ is equal to $-\theta_n$.

Before proving the claim, it is convenient to rephrase the definition of θ_n as follows.

(a) The fiber product

$$\mathfrak{T}_n := \pi^{-1}(X[p^n] \times_S Y[p^n]) \times_{([p^n]_{+_1}, E, \epsilon_1)} Y$$

has a natural structure as a biextension of $X[p^n] \times_S Y[p^n]$ by $Z[p^n]$, contained in the biextension $\pi^{-1}(X[p^n] \times_S Y[p^n])$, of $(X[p^n] \times_S Y[p^n])$ by Z.

(b) The bi-additive map $\theta_n : X[p^n] \times_S Y[p^n] \to Z[p^n]$ is characterised by the property that

$$[p^n]_{+_2}|_{\mathfrak{T}_n} = (\theta_n \circ \pi|_{\mathfrak{T}_n}) * (\epsilon_2 \circ \mathrm{pr}_1)|_{\mathfrak{T}_n}$$

Interchanging the two relative group laws $+_1$ and $+_2$ gives us a rephrased definition of ω_n. We will prove the above claim using this rephrased definition of ω_n and descent.

Proof of claim. Suppose that $\mathrm{Spec}(R) \to S$ is an affine scheme over the base scheme S, and we are given elements $x_n \in X[p^n](R)$, $y_n \in Y[p^n](R)$, and an element $e \in E(R)$ with $\pi(e) = (x_n, y_n)$ which satisfy the normalization condition $[p^n]_{+_1}(e) = \epsilon_1(y_n)$ with respect to the group law $+_1$. By definition $\theta_n(x_n, y_n)$ is the unique element in $Z[p^n](R)$ such that $[p^n]_{+_2}(e) = \theta_n(x_n, y_n) * \epsilon_2(x_n)$.

Pick a finite faithfully flat R-algebra S such that there exists an element $z \in Z[p^{2n}](S)$ with $[p^n]_Z(z) = -\theta_n(x_n, y_n)$. Then we have $[p^n]_{+_2}(z * e) = \epsilon_2(x_n)$, so the element $\xi * e$ in $E(S)$, which lies above (x_n, y_n), satisfies the normalization condition with respect to the group law $+_2$. Moreover we have

$$[p^n]_{+_1}(z * e) = [p^n]_Z(z) * \epsilon_1(x_n).$$

So $\omega_n(x_n, y_n) = [p^n]_Z(z) = -\theta_n(x_n, y_n)$ according to the rephrased definition of ω_n. $\square$

Remark. See also 10.2.6.1, in the same setup as the above argument.

10.2.5.3. It is instructive to compare the splitting ξ_n of $(1_{X[p^n]} \times [p^n]_{Y[p^{2n}]})^* E_n$ and the splitting ψ_n of $([p^n]_{X[p^{2n}]} \times 1_{Y[p^n]})^* E_n$, by pulling back both splittings to $X[p^{2n}] \times_S Y[p^{2n}]$. Define a splitting Ξ_n of $(1_{X[p^n]} \times [p^n]_{Y[p^{2n}]})^* E_n$ by

$$\Xi_n := \xi_n \circ ([p^n]_{X[p^{2n}]} \times 1_{Y[p^{2n}]}) : X[p^{2n}] \times_S Y[p^{2n}] \to E_n.$$

Similarly let $\psi_n = \psi_n^E : X[p^{2n}] \times_S Y[p^n] \to Z$ be the canonical splitting of the biextension $([p^n]_{X[p^{2n}]} \times 1_{Y[p^n]})^* E_n$ of $X[p^{2n}] \times_S Y[p^n]$ by Z, by switching the role of the two relative group laws: For any S-scheme T and T-points $x_{2n} \in X[p^{2n}](T)$ and $y_n \times_S Y[p^n](T)$, pick a finite locally free cover $T_1 : T_1 \to T$ and an element $e_1 \in E(T_1)$ lying above (x_{2n}, y_n) such that $[p^n]_{+_2}(e_1) = \epsilon_2(x_{2n})$, and $\psi_n(x_{2n}, y_n)$ is defined to be the element $[p^n]_{+_1}(e_1)$ of $E(T)$ which lies above $(p^n x_{2n}, y_n)$. Define

$$\Psi_n := \psi_n \circ (1_{X[p^{2n}]} \times [p^n]_{Y[p^{2n}]}) : X[p^{2n}] \times_S Y[p^{2n}] \to E_n.$$

Both Ξ_n and Ψ_n are splittings of the biextension

$$([p^n]_{X[p^{2n}]} \times [p^n]_{Y[p^{2n}]})^* E_n \to X[p^{2n}] \times_S Y[p^{2n}].$$

Define a bi-additive map $\Gamma_{2n} : X[p^{2n}] \times_S Y[p^{2n}] \to Z$ by the requirement that

$$\Gamma_{2n}(x_{2n}, y_{2n}) * \Xi_n(x_{2n}, y_{2n}) = \Psi_n(x_{2n}, y_{2n}) \quad \forall\, x_{2n} \in X[p^{2n}], \forall\, y \in Y[p^{2n}].$$

We see from the defining properties of Ξ_n, ξ_n, Ψ_n and ψ_n that

$$\Gamma_{2n}(a_n, y_{2n}) * \epsilon_1(p^n y_{2n}) = \Psi_n(a_n, y_{2n}) = -\omega_n(a_n, p^n y_{2n}) * \epsilon_1(p^n y_{2n})$$

and

$$(\Gamma_{2n}(x_{2n}, b_n) -_Z \theta_n(p^n x_{2n}, b_n)) * \epsilon_2(p^n x_{2n}) = \Gamma_{2n}(x_{2n}, b_n) * \Xi_n(x_{2n}, b_n) = \epsilon_2(p^n x_{2n})$$

for all $a_n \in X[p^n]$, all $b_n \in Y[p^n]$, all $x_{2n} \in X[p^{2n}]$, and all $y_{2n} \in Y[p^{2n}]$. It follows that

$$\theta_n(p^n x_{2n}, p^n y_{2n}) = p^n \Gamma_{2n}(x_{2n}, y_{2n}) = -\omega_n(p^n x_{2n}, p^n y_{2n})$$

for all $x_{2n} \in X[p^{2n}]$ and all $y_{2n} \in Y[p^{2n}]$. Note that we have shown again that $\theta_n +_Z \omega_n = 0$.

10.2.5.4. Exercise. Prove the following the compatibility properties of the trivializations $(\xi_n)_{n \geq 1}$ and $(\psi_n)_{n \geq 1}$:

$$\xi_n(p x_{n+1}, p^2 y_{2n+2}) = [p]_{+_1} [p]_{+_2} \xi_{n+1}(x_{n+1}, y_{2n+2}) \ \ \forall x_{n+1} \in X[p^{n+1}], \forall y_{2n+2} \in Y[p^{2n+2}]$$
$$\psi_n(p^2 x_{2n+2}, p y_{n+1}) = [p]_{+_1} [p]_{+_2} \psi_{n+1}(x_{2n+1}, y_{n+1}) \ \ \forall x_{2n+2} \in X[p^{2n+2}], \forall y_{n+1} \in Y[p^{n+1}].$$

Equivalently,

$$\Xi_n(p^2 x_{2n+2}, p^2 y_{2n+2}) = [p]_{+_1}\, [p]_{+_2} \Xi_{n+1}(x_{2n+2}, y_{2n+2})$$
$$\Psi_n(p^2 x_{2n+2}, p^2 y_{2n+2}) = [p]_{+_1}\, [p]_{+_2} \Psi_{n+1}(x_{2n+2}, y_{2n+2})$$

for all $x_{2n+2} \in X[p^{2n+2}]$ and all $y_{2n+2} \in Y[p^{2n+2}]$.

10.2.5.5. Exercise. Prove the following properties of the bi-additive map Γ_{2n}.

(a) The bi-additive map $\Gamma_{2n} : X[p^{2n}] \times_S Y[p^{2n}] \to Z$ factors through $Z[p^{2n}] \hookrightarrow Z$.
(b) For all $x_{2n+2} \in X[p^{2n+2}]$ and all $y_{2n+2} \in Y[p^{2n+2}]$, we have

$$p^2\Gamma_{2n+2}(x_{2n+2}, y_{2n+2}) = \Gamma_{2n}(p^2 x_{2n+2}, p^2 y_{2n+2}).$$

(c) $\Gamma_{2n} = \theta_{2n} = -\omega_{2n}$ for all $n \geq 1$.

10.2.5.6. Remark. We saw in 10.2.5.2 that the two Weil pairings normalized by the two group laws differ by a sign. In the case of the Poincaré biextension

$$\mathcal{P}[p^\infty] \to A[p^\infty] \times_S A^t[p^\infty]$$

associated to an abelian scheme A over a base scheme S, it is natural to ask which one of the two Weil pairings is equal to the standard Weil pairing $e_n^A : A[p^n] \times_S A^t[p^n] \to \mu_{p^n}$.

A careful comparison with the definition of e_n in the first three pages of §20 of Mumford's book [77] reveals that e_n^A is equal to the pairing ω_n normalized by the "second" relative group law $+_2$ of the biextension $\mathcal{P}[p^\infty] \to A[p^\infty] \times_S A^t[p^\infty]$. Recall that $+_2$ makes $\mathcal{P}[p^\infty]$ a p-divisible group over the base scheme $A[p^\infty]$, which is an extension of $A^t[p^\infty]$ (base changed to $A[p^\infty]$) by μ_{p^∞} (also base changed to $A[p^\infty]$).

10.2.5.7. Lemma. *Let $\pi : E \to X \times Y$ be a biextension of p-divisible groups $X \times_S Y$ by a p-divisible group Z over a base scheme Y. For each positive integer n, let $\theta_n : X[p^n] \times_S Y[p^n] \to Z[p^n]$ be the Weil pairing as described in 10.2.5.1.*

(1) *Suppose that n_1 is a positive integer and θ_{n_1} is equal to the trivial bi-additive map from $X[p^{n_1}] \times_S Y[p^{n_1}]$ to $Z[p^{n_1}]$. Then the biextension $\pi^{-1}(X[p^{n_1}] \times_S Y[p^{n_1}])$ of $X[p^{n_1}] \times_S Y[p^{n_1}]$ by Z splits canonically. In other words there exists a canonical isomorphism*

$$\zeta_{n_1} : \pi^{-1}(X[p^{n_1}] \times_S Y[p^{n_1}]) \xrightarrow{\sim} Z \times_S X[p^{n_1}] \times_S Y[p^{n_1}].$$

(2) *Suppose that n_2 is a positive integer, $n_2 > n_1$ and θ_{n_2} is equal to the trivial bi-additive map. Then θ_{n_1} is also equal to the trivial bi-additive map. Moreover the canonical trivializations $\zeta_{n_1}^{\mathrm{can}}$ and $\zeta_{n_2}^{\mathrm{can}}$ are compatible, i.e. $\zeta_{n_1}^{\mathrm{can}}$ is equal to the restriction to $\pi^{-1}(X[p^{n_1}] \times_S Y[p^{n_1}])$ of $\zeta_{n_2}^{\mathrm{can}}$.*

Proof. We saw in 10.2.5.1 that the pull-back of $\pi^{-1}(X[p^{n_1}] \times_S Y[p^{n_1}])$ to $X[p^{n_1}] \times_S Y[p^{2n_1}]$ by the faithfully flat morphism $1_{X[p^{n_1}} \times [p^{n_1}]_{Y[p^{2n_1}]} : X[p^{n_1}] \times_S Y[p^{2n_1}] \to X[p^{n_1}] \times_S Y[p^{n_1}]$ is canonically trivial, and the bihomomorphism θ_{n_1} corresponds to the descent data from the trivial biextension $Z \times X[p^{n_1}] \times_S Y[p^{2n_1}]$ down to π_{n_1} along the morphism $1 \times p^{n_1}$: $X[p^{n_1}] \times_S Y[p^{2n_1}] \to X[p^{n_1}] \times_S Y[p^{n_1}]$. So if θ_{n_1} is the trivial homomorphism, then this descent datum defines a canonical isomorphism between the $\pi^{-1}(X[p^{n_1}] \times_S Y[p^{n_1}])$ and the trivial biextension $Z \times X[p^{n_1}] \times Y[p^{n_1}])$. We have proved statement (1).

The first part of (2) follows from the compatibility of Weil pairings (10.2.4.1.2) and (10.2.4.1.3). The compatibility statement (2) follows from the same descent argument used in the proof of (1). □

Proposition 10.2.5.8 and Corollary 10.2.5.9 below are applications of 10.2.5.1. It enables us to determine the restriction of a homomorphism between two biextensions to torsion subgroups schemes $X[p^n] \times Y[p^n]$.

10.2.5.8. Lemma. *Let $\pi : E \to X \times_S Y$ and $\pi' : E' \to X \times_S Y$ be two biextensions of p-divisible groups $X \times_S Y$ by a p-divisible group Z over S. Let*

$$\left(\theta_n, \theta'_n : X[p^n] \times_S Y[p^n] \to Z[p^n]\right)_{n\in\mathbb{N}}$$

be the Weil parings attached normalized by the first relative group laws $+_{1,E}, +_{1,E'}$ associated to the biextensions E, E' respectively.

(1) *If n_1 is a positive integer and $\theta_{n_1} = \theta'_{n_1}$, then there exists a canonical isomorphism*

$$\zeta_n : \pi^{-1}(X[p^{n_1}] \times_S Y[p^{n_1}]) \xrightarrow{\sim} (\pi')^{-1}(X[p^{n_1}] \times_S Y[p^{n_1}])$$

determined by θ_n and θ'_n.

(2) *Suppose that $n_2 > n_1$ and $\theta_{n_2} = \theta'_{n_2}$. Then $\theta_{n_1} = \theta'_{n_1}$ and the canonical isomorphism*

$$\zeta_{n_1} : \pi^{-1}(X[p^{n_1}] \times_S Y[p^{n_1}]) \xrightarrow{\sim} (\pi')^{-1}(X[p^{n_1}] \times_S Y[p^{n_1}])$$

is compatible with the canonical isomorphism

$$\zeta_{n_2} : \pi^{-1}(X[p^{n_2}] \times_S Y[p^{n_2}]) \xrightarrow{\sim} (\pi')^{-1}(X[p^{n_2}] \times_S Y[p^{n_2}]).$$

(3) *Suppose that $\theta_n = \theta'_n$ for all $n \in \mathbb{N}$. Then the collection of canonical isomorphisms*

$$\zeta_n : \pi^{-n}(X[p^n] \times_S Y[p^n]) \xrightarrow{\sim} (\pi')^{-n}(X[p^n] \times_S Y[p^n]), \qquad n \in \mathbb{N}$$

defines an isomorphism from the biextension E to the biextension E' which induces $\mathrm{id}_X, \mathrm{id}_Y$ and id_Z on the p-divisible groups X, Y and Z.

(4) *Suppose that $\zeta : E \to E'$ is an isomorphism of biextensions which induces $\mathrm{id}_X, \mathrm{id}_Y$ and id_Z on the p-divisible groups X, Y and Z. Then $\theta_n = \theta'_n$ for all $n \in \mathbb{N}$, and the restriction of ζ to $\pi^{-1}(X[p^n] \times_S Y[p^n])$ is equal to the canonical isomorphism*

$$\zeta_n : \pi^{-1}(X[p^{n_1}] \times_S Y[p^{n_1}]) \xrightarrow{\sim} (\pi')^{-1}(X[p^{n_1}] \times_S Y[p^{n_1}])$$

attached to θ_n and θ'_n, for all $n \in \mathbb{N}$.

Proof. The biextension structures on E and E' endow the Z-torsor $E \wedge^Z ([-1]_Z)_* E'$ over $X \times Y$ a structure of a biextension of $X \times_S Y$ by Z. The statements (1), (2) follow from 10.2.5.7 applied to $E \wedge^Z ([-1]_Z)_* E'$. The statement (3) follows from (2).

To prove the statement (4), we observe first that the functoriality of the Weil pairings tell us that $\theta_n = \theta'_n$ for all n. By (3), the canonical isomorphisms ζ_n are compatible and defines an isomorphism of biextensions $\zeta' : E \to E'$ over $X \times_S Y$. There exists a unique morphism

$$b : X \times_S Y \to Z$$

such that

$$\zeta'(e) = b(\pi(e)) * \zeta(e)$$

for all S-scheme T and all $e \in E(T)$. Clearly $b : X \times_S Y \to Z$ is bi-additive in the sense that

$$b(x_1 + x_2, y) = b(x_1, y) \quad \text{and} \quad b(x, y_1 + y_2) = b(x, y_1) + b(x, y_2)$$

for all S-schemes T, all $x, x_1, x_2 \in X(T)$ and all $y, y_1, y_2 \in Y(T)$. We know from 10.2.3.5 that such a bi-additive map is necessarily zero. We have shown that $\zeta' = \zeta$. □

10.2.5.9. Corollary. *Let X, Y, Z, X', Y', Z' be p-divisible groups over S. Let E be a biextension of $X \times_S Y$ by Z, and let E' be a biextension of $X' \times_S Y'$ by Z'. There is a natural bijection from the set $\mathrm{Hom}_{\mathrm{biext}}(E, E')$ of all S-bihomomorphisms from E to E', to the set of all triples $(\alpha, \beta, \gamma) \in \mathrm{Hom}_S(X, X') \times \mathrm{Hom}_S(Y, Y') \times \mathrm{Hom}_S(Z, Z')$ such that*

$$\gamma(\theta_{n,E}(x_n, y_n)) = \theta_{n,E'}(\alpha(x_n), \beta(y_n))$$

for all $n \in \mathbb{N}$, all schemes T over S, all $x_n \in X[p^n](T)$, and all $y_n \in Y[p^n](T)$.

Remark. (i) Denote by $\mathrm{Biext}^1(X, Y; Z)$ the set of all biextensions of $X \times_S Y$ by Z up to isomorphisms which induce $\mathrm{id}_X, \mathrm{id}_Y, \mathrm{id}_Z$ on X, Y and Z; c.f. 10.2.3.1 (c). By 10.2.5.8 and 10.2.5.9, the map $E \mapsto \theta_E$ establishes a functorial bijection from $\mathrm{Biext}^1(X, Y; Z)$ to the set of all compatible families of bilinear pairings $(\theta_n : X[p^n] \times Y[p^n] \to Z[p^n])_{n \in \mathbb{N}}$. See also [76, Prop. 4, p. 319], Exp. VIII of [18] and 10.2.7.2.

(ii) One knows from [18, VII 3.6.5] that for sheaves of abelian groups P, Q, G over a topos, the set $\mathrm{Biext}^1(P, Q; G)$ of isomorphism classes of biextensions of $P \times Q$ by G is naturally isomorphic to $\mathrm{Ext}^1(P \otimes^{\mathbb{L}} Q, G)$. On the other hand, for p-divisible groups X, Y we have isomorphisms $\mathscr{T}\!or^1_{\mathbb{Z}_p}(X[p^n], Y[p^n]) \cong X[p^n] \otimes_{\mathbb{Z}_p} Y[p^n]$ of fppf-sheaves. The construction of the Weil pairing attached to a biextension reflects these two facts.

10.2.5.10. Lemma. *Let X, Y, Z be p-divisible formal groups over a field κ of characteristic p, let $E \to X \times Y$ be a biextension of $X \times Y$ by Z. Suppose that X, Y factor as products $X = X_1 \times X_2$, $Y = Y_1 \times Y_2$, where X_1, X_2, Y_1, Y_2 are p-divisible groups over κ, such that all slopes of $X_1 \times Y_1$ are $\geq \mu_1$. Suppose moreover that the Weil pairings*

$$\left(\theta_n^{E,\pi} : X[p^n] \times Y[p^n] \to Z[p^n]\right)_{n \geq 1}$$

attached to the biextension (E, π) vanish on $X_1[p^n] \times Y[p^n]$ and also on $X[p^n] \times Y[p^n]$, for every $n \geq 1$. Then E has a natural structure

$$(\pi' : E \to X_2 \times Y_2, +'_1 : E \times_{Y_2} E \to E, +'_1 : E \times_{X_2} E \to E, \epsilon'_1 : Y_2 \to E, \epsilon'_2 : X_2 \to E)$$

as a biextension of $X_2 \times Y_2$ by $Z' = X_1 \times Y_1 \times Z$, such that the following properties hold.

(1) $e +_1 e' = (-\mathrm{pr}_{Y_1}(e)) *' (e +'_1 e') = (-\mathrm{pr}_{Y_1}(e')) *' (e +'_1 e')$
for all functorial points e, e' of E such that $\mathrm{pr}_Y(e) = \mathrm{pr}_Y(e')$,

(2) $e +_2 e' = (-\mathrm{pr}_{X_1}(e)) *' (e +'_2 e') = (-\mathrm{pr}_{X_1}(e')) *' (e +'_2 e')$
for all functorial points e, e' of E such that $\mathrm{pr}_X(e) = \mathrm{pr}_X(e')$,

(3) $e +'_1 e' = \left(\mathrm{pr}_{Y_1}(e) +_1 \mathrm{pr}_{Y_1}(e')\right) *' \left((-\mathrm{pr}_{Y_1}(e) *' e) +_1 (-\mathrm{pr}_{Y_1}(e') *' e')\right)$
for all functorial points e, e' of E such that $\mathrm{pr}_{Y_2}(e) = \mathrm{pr}_{Y_2}(e')$,

(4) $e +'_2 e' = \big(\mathrm{pr}_{X_1}(e) +_1 \mathrm{pr}_{X_1}(e')\big) *' \big((-\mathrm{pr}_{X_1}(e) *' e) +_1 (-\mathrm{pr}_{X_1}(e') *' e')\big)$ *for all functorial points e, e' of E such that $\mathrm{pr}_{X_2}(e) = \mathrm{pr}_{X_2}(e')$,*
(5) $\epsilon_1(y_1, y_2) = y_1 *' \epsilon'_1(y_2)$ *for all functorial points (y_1, y_2) of $Y = Y_1 \times Y_2$, and*
(6) $\epsilon_2(x_1, x_2) = x_1 *' \epsilon'_2(x_2)$ *for all functorial points (x_1, x_2) of $X = X_1 \times X_2$.*
(7) $z * e{=}z *' e$ *for all functorial points $(z, e) \in Z \times E$.*
(8) *Every automorphism of the biextension (E, π) is an automorphism of the biextension (E, π'), and vice versa.*

Here

- $*' : Z' \times E \to E$ *is the Z'-torsor structure associated with the structure on E as a biextension of $X_2 \times Y_2$ by Z'.*
- $\pi' : E \to X_2 \times Y_2$ *is the composition of $\pi : E \to X \times Y$ with the projection $X \times Y = (X_1 \times Y_1) \times (X_2 \times Y_2) \longrightarrow X_2 \times Y_2$.*
- $\mathrm{pr}_X : E \to X$ *is the composition of $\pi : E \to X \times Y$ with the projection $X \times Y \to X$.*
- $\mathrm{pr}_Y : E \to Y$ *is the composition of $\pi : E \to X \times Y$ with the projection $X \times Y \to Y$.*
- $\mathrm{pr}_{X_i} : E \to X_i$ *is the composition of pr_X with the projection $X = X_1 \times X_2 \to X_i$, $i = 1, 2$.*
- $\mathrm{pr}_{Y_i} : E \to Y_i$ *is the composition of pr_Y with the projection $Y = Y_1 \times Y_2 \to Y_i$, $i = 1, 2$.*

PROOF. The assumption on the Weil pairings means that $\theta_n^{E,\pi}$ factors through the projection $X[p^n] \times Y[p^n] \longrightarrow X_2[p^n] \times Y_2[p^n]$ and induces a compatible family of bilinear pairings $\big(\theta'_n : X_2[p^n] \times Y_2[p^n] \to Z[p^n]\big)_{n\geq 1}$. Define

$$\theta_n^{E',\pi'} : X_2[p^n] \times Y_2[p^n] \to Z'[p^n] = (X_1 \times Y_1 \times Z)[p^n]$$

to be the composition

$$X_2[p^n] \times Y_2[p^n] \xrightarrow{\theta'_n} Z[p^n] \hookrightarrow X_1[p^n] \times Y_1[p^n] \times Z[p^n] = Z'[p^n]\ .$$

The bilinear pairings $\big(\theta^{E',\pi'}\big)_{n\geq 1}$ define a biextension $\pi' : E' \to X_2 \times Y_2$ of $X_2 \times Y_2$ by Z'. Moreover we have a natural isomorphism of formal schemes $E \xrightarrow{\sim} E'$, which sends

$$z_n * \xi_n^E\big((x_{1,n}, x_{2,n}), (y_{1,2n}, y_{2,2n})\big)$$

to

$$\big(x_{1,n}, [p^n](y_{1,2n}), z_n\big) *' \xi^{E'}(x_{2,n}, y_{2,2n})$$

for all functorial points $x_{1,n} \in X_1[p^n]$, $x_{2,n} \in X_2[p^n]$, $y_{1,2n} \in Y_1[p^{2n}]$, $y_{2,2n} \in Y_2[p^{2n}]$, $z_n \in Z$ with values in the same κ-scheme S, in the notation of 10.2.5.1. We identify E' with E via this isomorphism. Properties (1)–(8) follow immediately. □

10.2.6. Let $\pi : E \to X \times_S Y$ be a biextension of p-divisible groups X, Y, Z. We will construct a family $(\eta_n)_{n\geq 1}$ of morphisms $\eta_n : E_n \to Z$ and a similar family of morphisms

$\rho_n : E_n \to Z$, such that

$$\eta_n\big|_Z = [p^n]_Z = \rho_n\big|_Z$$
$$[p]_Z \circ \eta_n = \eta_{n+1} \circ (E_n \hookrightarrow E_{n+1})$$
$$[p]_Z \circ \rho_n = \rho_{n+1} \circ (E_n \hookrightarrow E_{n+1})$$

for all $n \geq 1$, where $E_n := \pi^{-1}(X[p^n] \times_S Y[p^n])$. See 10.2.6.1 for their definitions, and 10.2.6.3 for a basic congruence estimate of η_n and ρ_n.

10.2.6.1. Definition. Let $\pi : E \to X \times_S Y$ be a biextension of p-divisible groups X, Y, Z over a scheme S.

For any positive integer n, we have a canonical map $\xi_n : X[p^n] \times_S Y[p^{2n}] \to E_n$ such that $\pi \circ \xi_n = 1_{X[p^n]} \times_S [p^n]_{Y^{p^{2n}}}$ and $\xi_n(x_n, y_{2n}) = \theta_n(x_n, b_n) * \xi_n(x_n, y_{2n} + b_n)$ for all $x_n \in X[p^n]$, all $y_{2n} \in Y[p^{2n}]$ and all $b_n \in Y[p^n]$. The map $\alpha_n : X[p^n] \times_S Y[p^{2n}] \times_S Z \to E_n$ which sends functorial points (x_n, y_{2n}, z) to $z * \xi_n(x_n, y_{2n})$ is a faithfully flat homomorphisms of biextensions.

(1) Definition of η_n. Let $\tilde{\eta}_n : X[p^n] \times_S Y[p^{2n}] \times_S Z \to Z$ be the map given by

$$\tilde{\eta}_n(x_n, y_{2n}, z) = p^n z$$

for all functorial points (x_n, y_{2n}, z) of $\tilde{\eta}_n : X[p^n] \times_S Y[p^{2n}] \times_S Z$.

Define the map $\eta_n : E_n \to Z$ by descending $\tilde{\eta}_n$ along α_n, i.e. η_n is the unique morphism from E_n to Z such that

$$\eta_n \circ \alpha_n = \tilde{\eta}_n.$$

The map η_n is induced by the relative group law $+_1$ on E_n in the sense that

$$[p^n]_{+_1}(e_n) = \eta(e_n) * \epsilon_1(y_n)$$

for every $e_n \in E_n$ above $(x_n, y_n) \in X[p^n] \times_S Y[p^n]$. This relation can be regarded as an alternative definition of η_n.

(2) Definition of ρ_n. Consider the map

$$\alpha_n \circ [p^n]_{+_2} := \alpha_n \circ (1_{X[p^n]} \times [p^n]_{Y[p^{2n}]} \times [p^n]_Z) : X[p^n] \times_S Y[p^{2n}] \times_S Z \longrightarrow Z,$$

which sends every functorial point (x_n, y_{2n}, z) of $X[p^n] \times_S Y[p^{2n}] \times_S Z$ to

$$\begin{aligned}(\alpha_n \circ [p^n]_{+_2})(x_n, y_{2n}, z) &= \alpha_n(x_n, p^n y_{2n}, p^n z) = -\theta_n(x_n, p^n y_{2n}) * \alpha_n(x_n, 0, p^n z) \\ &= (-\theta_n(x_n, p^n y_{2n}) +_Z p^n z) * \epsilon_2(x_n).\end{aligned}$$

Let $\tilde{\rho}_n : X[p^n] \times_S Y[p^{2n}] \times_S Z \to Z$ be the map which sends functorial points (x_n, y_{2n}, z) of $X[p^n] \times_S Y[p^{2n}] \times_S Z$ to

$$\tilde{\rho}_n(x_n, y_{2n}, z) = -\theta_n(x_n, p^n y_{2n}) +_Z p^n z.$$

Clearly $\tilde{\rho}_n(x_n, y_{2n} + b_n, z) = \tilde{\rho}_n(x_n, y_{2n} + b_n, z)$ for every functorial point b_n of $Y[p^n]$.

Define $\rho_n : E_n \to Z$ to be the morphism such that

$$\tilde{\rho}_n = \rho_n \circ \alpha_n.$$

In other words, $\tilde{\rho}_n$ descends from the finite locally free cover $\alpha_n : X[p^n] \times_S Y[p^{2n}] \times_S Z \to E_n$ to the morphism $\rho_n : E_n \to Z$.

The map ρ_n is related to the relative group law $+_2$ through the equality

$$[p^n]_{+_2}(e_n) = \rho_n(e_n) * \epsilon_2(x_n)$$

for every functorial point $e_n \in E_n$ above $(x_n, y_n) \in X[p^n] \times_S Y[p^n]$, which provides an alternative definition of ρ_n.

10.2.6.2. Exercise. (i) Show that

$$\begin{aligned} [p]_Z \circ \eta_n &= \eta_{n+1} \circ (E_n \hookrightarrow E_{n+1}) \\ [p]_Z \circ \rho_n &= \rho_{n+1} \circ (E_n \hookrightarrow E_{n+1}) \\ \eta_n &= (\theta_n \circ \pi|_{E_n}) +_Z \rho_n \end{aligned}$$

for all $n \geq 1$.

(ii) Show that both η_n and ρ_n are bi-additive maps from E_n to Z.

10.2.6.3. Proposition. *Let $\pi : E \to X \times Y$ be a biextension of p-divisible formal groups over a field κ of characteristic p. Let*

$$\big(\eta_n : \pi^{-1}(X[p^n] \times Y[p^n]) \to Z\big)_{n \in \mathbb{N}}, \quad \big(\rho_n : \pi^{-1}(X[p^n] \times Y[p^n]) \to Z\big)_{n \in \mathbb{N}}$$

be the two compatible families of morphisms defined in 10.2.6.1. Let $\mu = \mu_{Z,\max}$ be the maximum among the slopes of Z. There exist positive integers n_2, c_2 such that

$$\eta_n|_{E_n \cap E[F^{\lfloor n/\mu \rfloor - c_2}]} = 0 \quad \text{and} \quad \rho_n|_{E_n \cap E[F^{\lfloor n/\mu \rfloor - c_2}]} = 0$$

for all $n \geq n_2$. Here $E[F^{\lfloor n/\mu \rfloor - c_2}]$ denotes the inverse image of the base point of $E^{(p^{\lfloor n/\mu \rfloor - c_2})}$ under the iterated relative Frobenius morphism $\mathrm{Fr}_{E/\kappa}^{\lfloor n/\mu \rfloor - c_2} : E \to E^{(p^{\lfloor n/\mu \rfloor - c_2})}$.

Proof. We only need to show the existence of n_2, c_2 such that $\eta_n|_{E_n \cap E[F^{\lfloor n/\mu \rfloor - c_2}]} = 0$. The other half, i.e. $\rho_n|_{E_n \cap E[F^{\lfloor n/\mu \rfloor - c_2}]} = 0$, follows by symmetry.

For every positive integer n, we have a finite locally free cover

$$\alpha_n : X[p^n] \times Y[p^{2n}] \times Z \longrightarrow E_n, \quad \alpha_n(x_n, y_{2n}, z) = z * \xi_n^E(x_n, y_{2n})$$

defined in 10.2.5.1. Write $\mu = \frac{a}{r}$, where a, r are positive integers.

Suppose we are given an element $e \in E_n(R) \cap E[F^i](R)$, where R is a commutative κ-algebra and $i \in \mathbb{N}$. Let T be a finite locally free R-algebra such that there exist

$x_n \in X[p^n](T)$, $y_{2n} \in Y[p^{2n}](T)$, and $z \in Z[T]$ such that $z * \xi_n(x_n, y_{2n}) = e$. The assumption that $\mathrm{Fr}^i_{E/\kappa}(e)$ is the base point of $E^{(p^i)}$ implies that $\mathrm{Fr}^i_{X/\kappa}(x_n) = 0$, therefore $\xi_n^{E^{(p^i)}}\big(\mathrm{Fr}^i_{X/\kappa}(x_n), \mathrm{Fr}^i_{Y/\kappa}(y_{2n})\big)$ is equal to the base point of $E^{(p^i)}$. From

$$\mathrm{Fr}^i_{Z/\kappa}(e) = \mathrm{Fr}^i_{Z/\kappa}(z) * \xi_n^{E^{(p^i)}}\big(\mathrm{Fr}^i_{X/\kappa}(x_n), \mathrm{Fr}^i_{Y/\kappa}(y_{2n})\big),$$

we see that $\mathrm{Fr}^i_{E/\kappa}(e) \in E[F^i]$ if and only if $z \in Z[F^i]$. Recall also that $\eta_n(z * \xi_n^E(x_n, y_{2n}) = [p^n]_Z(z)$.

The assumption that $\frac{a}{r}$ is the largest slope of Z implies the existence of positive integers n_3, c_3 such that $Z[F^i] \subseteq Z[p^{\lceil ia/r \rceil + c_3}]$ for all $i \geq n_3$. For a point

$$e = z * \xi_n(x_n, y_{2n}) \in E_n \cap E[F^i]$$

as in the previous paragraph, if $i \geq n_3$ and $\lceil ia/r \rceil + c_3 \leq n$, then $z \in Z[F^i] \subseteq Z[p^n]$ and $\eta_n(z) = p^n z = 0$. Let $c_2 := \lceil c_3 r/a \rceil$ and $n_2 := \lceil \frac{a}{r}(n_3 + c_2) \rceil$. A simple calculation shows that the restriction of η_n to $E_n \cap E[F^{\lfloor n/\mu \rfloor}]$ is identically 0. ∎

The following proposition 10.2.6.4 is a more precise version of 10.2.6.3 in two special situations.

10.2.6.4. Proposition. *Let X, Y, Z be isoclinic p-divisible groups over a field κ of characteristic p, and let $\pi : E \to X \times Y$ be a biextension of $X \times Y$ by Z.*

(1) *If for every slope λ of X and ever slope ν of Y, $\lambda + \nu$ is* not *a slope of Z, then*

$$\eta_n = 0 \quad \text{and} \quad \rho_n = 0$$

for every $n \in \mathbb{N}$.

(2) *Let $a, a_1, a_2, r > 0$ be positive integers such that $a_1 + a_2 = a$, $a \leq r$. Suppose that*

$$X[p^{a_1}] = X[F^r], \quad Y[p^{a_2}] = Y[F^r], \quad Z[p^r] = Z[F^r].$$

Then $E[F^{mr}] \subseteq E_{ma}$, $\eta^E_{ma}|_{E[F^{mr}]} = 0$, $\rho^E_{ma}|_{E[F^{mr}]} = 0$, and $\theta^E_n|_{E[F^{mr}]} = 0$ for every positive integer m.

PROOF. The assumption in (1) implies that the Weil pairings θ^E_n attached to E vanish identically, and the biextension E splits canonically; see 10.2.4.4. So the maps η_n and ρ_n are equal to 0 for all n.

Under the assumptions of (2), we have

$$X[F^{mr}] = X[p^{ma_1}] = \mathrm{Im}\Big(X[p^{ma}] \xrightarrow{[p^{ma_2}]_{X[p^{ma}]}} X[p^{ma}] \Big),$$

$$Y[F^{mr}] = Y[p^{ma_2}] = \mathrm{Im}\Big(Y[p^{ma}] \xrightarrow{[p^{ma_1}]_{Y[p^{ma}]}} Y[p^{ma}] \Big).$$

The assertion that $\theta^E_n|_{E[F^{mr}]} = 0$ follows, because θ^E_n is bi-additive. The assertion that $\eta^E_{ma}|_{E[F^{mr}]} = 0 = \rho^E_{ma}|_{E[F^{mr}]}$ follows from the relation of the two relative group laws with η_n and ρ_n, and the above displayed formulas. Alternatively, the argument in the proof of

10.2.6.3 using the descent data of E_n also shows that $\eta^E_{ma}|_{E[F^{mr}]}$ and $\rho^E_{ma}|_{E[F^{mr}]}$ are both equal to 0. □

Remark. One can also deduce 10.2.6.3 from 10.2.6.4, using lemma 10.2.6.5 below to reduce to the case when the base field κ is algebraically closed and X, Y, Z are all product of isoclinic p-divisible groups, then to the case when X, Y, Z are all isoclinic. Another application of 10.2.6.5 allows us to modify X, Y, Z by isogenies so that 10.2.6.4 is applicable. Details are left as an exercise.

10.2.6.5. Lemma. *Let R_1, R_2, S_1, S_2 be Noetherian local rings, and let $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{n}_1, \mathfrak{n}_2$ be their maximal ideals. Let $h_1 : R_1 \to S_1$ and $h_2 : R_2 \to S_2$ be injective local homomorphisms such that S_i is a finitely generated R_i-module via h_i for $i = 1, 2$. There exist positive integers C, d with the following property:*

> *Let $f, g : R_1 \to R_2$ and $f', g' : S_1 \to S_2$ be local homomorphisms such that $h_2 \circ f = f' \circ h_1$ and $h_2 \circ g = g' \circ h_1$. If $n \in \mathbb{N}$ and $f'(y) - g'(y) \equiv 0 \pmod{\mathfrak{n}_2^{Cn+d}}$ for all $y \in S_1$, then $f(x) - g(x) \equiv 0 \pmod{\mathfrak{n}_1^n}$ for all $x \in R_1$.*

PROOF. There exists a positive integer $a > 0$ such that $\mathfrak{n}_2^C \subseteq \mathfrak{n}_1 S_2$. By the Artin–Rees lemma, there exists a natural number e such that

$$S_1 \cap \mathfrak{n}_1^{m+e} S_2 \subseteq n_1^m \qquad \forall\, n \in \mathbb{N}.$$

Lemma 10.2.6.5 holds for $C = a$ and $d = ae$. □

10.2.7. Dieudonné theory for biextensions. Suppose that κ is a perfect field of characteristic $p > 0$. We recall the covariant Dieudonné theory for biextensions of p-divisible groups over κ the associated Weil pairings.

We use the same notation scheme for covariant Dieudonné theory as in previous chapters.

- Let $\Lambda = \Lambda(\kappa)$ be the ring of all p-adic Witt vectors with entries in κ.
- Let $\sigma = \sigma|_{\Lambda(\kappa)} : \Lambda(\kappa) \to \Lambda(\kappa)$ be the ring endomorphism
$$x = (x_0, x_1, x_2, \ldots) \mapsto {}^{\sigma}x = (x_0^p, x_1^p, x^p, \ldots),$$
and let $\mathtt{V} = \mathtt{V}_{\Lambda(\kappa)} : \Lambda(\kappa) \to \Lambda(\kappa)$ be the additive endomorphism
$$x = (x_0, x_1, x_2, \ldots) \mapsto {}^{\mathtt{V}}x = (0, x_0, x_1, x_2, \ldots)$$
of $\Lambda(\kappa)$. Recall that $\mathtt{V}_{\Lambda(\kappa)} \circ \sigma_{\Lambda(\kappa)} = \sigma_{\Lambda(\kappa)} \circ \mathtt{V}_{\Lambda(\kappa)} = [p]_{\Lambda(\kappa)}$.
- The classical covariant Dieudonné theory attaches to every p-divisible formal group X over κ a free $\Lambda(\kappa)$-module $\mathbb{D}_*(X)$ whose rank is equal to height(X), together with additive endomorphisms
$$\mathtt{F},\ \mathtt{V} : \mathbb{D}_*(X) \longrightarrow \mathbb{D}_*(X)$$
of $\mathbb{D}_*(X)$ such that
$$\mathtt{F}(a\,x) = {}^{\sigma}a\,\mathtt{F}(x), \qquad \mathtt{V}({}^{\sigma}a\,x) = a\,\mathtt{V}(x) \quad \text{and} \quad \mathtt{F}(\mathtt{V}(x)) = p\,x = \mathtt{V}(\mathtt{F}(x))$$

for all $a \in \Lambda(\kappa)$ and all $x \in \mathbb{D}_*(X)$.

- The main theorem of the classical covariant Dieudonné theory asserts that the assignment
$$X \mapsto \mathbb{D}_*(X)$$
establishes an equivalence of categories from the additive category of p-divisible groups over κ to the additive category of Dieudonné modules for the perfect base field κ.

10.2.7.1. Let X, Y, Z, X', Y', Z' be p-divisible groups over κ. We have seen in 10.2.5.8 and 10.2.5.9 that the map which to every biextension E of $X \times Y$ associates the compatible family of Weil pairing $(\theta_{n,E})_{n\in\mathbb{N}}$ establishes an equivalence of categories, from the category of biextensions of $X \times Y$ by Z, to the category of compatible families of bilinear pairings
$$\big(\, b_n : X[p^n] \times Y[p^n] \to Z[p^n] \,\big)_{n\in\mathbb{N}}.$$
Moreover the set of all bihomomorphisms $\psi : E \to E'$ from a biextension E of $X \times Y$ by Z to a biextension E' of $X' \times Y'$ by Z' is in natural bijection with the set of all triples
$$(\alpha, \beta, \gamma) \in \mathrm{Hom}_k(X, X') \times \mathrm{Hom}_k(Y, Y') \times \mathrm{Hom}_k(Z, Z')$$
such that
$$\gamma(\theta_{n,E}(x_n, y_n)) = \theta_{n,E'}(\alpha(x_n), \beta(y_n))$$
for all k-schemes T, all $x_n \in X[p^n](T)$ and all $y_n \in Y[p^n](T)$. We will explain how to express these statements in terms of Dieudonné modules.

Proposition 10.2.7.2 below is a longer version of 5.3.5.4.

10.2.7.2. Proposition. *Notation as above.*

(i) *To every biextension E of $X \times Y$ by Z, there is an associated $\Lambda(k)$-bilinear map*
$$\Theta_E : \mathbb{D}_*(X) \times \mathbb{D}_*(Y) \longrightarrow \mathbb{D}_*(Z)$$
such that
$$\begin{aligned}\Theta_E(\mathtt{F}_{\mathbb{D}_*(X)}(x), y) &= \mathtt{F}_{\mathbb{D}_*(Z)}\big(\Theta_E(x, \mathtt{V}_{\mathbb{D}_*(Y)}y)\big)\\ \Theta_E(x, \mathtt{F}_{\mathbb{D}_*(Y)}(y)) &= \mathtt{F}_{\mathbb{D}_*(Z)}\big(\Theta_E(\mathtt{V}_{\mathbb{D}_*(X)}x, y)\big)\\ \Theta_E\big(\mathtt{V}_{\mathbb{D}_*(X)}x, \mathtt{V}_{\mathbb{D}_*(Y)}y\big) &= \mathtt{V}_{\mathbb{D}_*(Z)}\big(\Theta_E(x, y)\big)\end{aligned}$$
for all $x \in \mathbb{D}_*(X)$ and all $y \in \mathbb{D}_*(Y)$.

(ii) *For every $\Lambda(k)$-bilinear map*
$$\Theta : \mathbb{D}_*(X) \times \mathbb{D}_*(Y) \longrightarrow \mathbb{D}_*(Z)$$
which satisfies the conditions that
$$\begin{aligned}\Theta(\mathtt{F}_{\mathbb{D}_*(X)}(x), y) &= \mathtt{F}_{\mathbb{D}_*}(Z)\big(\Theta(x, \mathtt{V}_{\mathbb{D}_*(Y)}(y))\big),\\ \Theta(x, \mathtt{F}_{\mathbb{D}_*(Y)}(y)) &= \mathtt{F}_{\mathbb{D}_*(Z)}\big(\Theta(\mathtt{V}_{\mathbb{D}_*(X)}x, y)\big)\\ \Theta\big(\mathtt{V}_{\mathbb{D}_*(X)}x, \mathtt{V}_{\mathbb{D}_*(Y)}y\big) &= \mathtt{V}_{\mathbb{D}_*(Z)}\big(\Theta(x, y)\big)\end{aligned}$$

for all $x \in \mathbb{D}_*(X)$ *and all* $y \in \mathbb{D}_*(Y)$*, there exists a biextension* E *of* $X \times Y$ *by* Z*, unique up to unique isomorphism, such that* $\Theta = \Theta_E$*. In particular the biextension* E *is split if and only if* $\Theta_E = 0$*.*

(iii) *Given a biextension* E *of* $X \times Y$ *by* Z *and a biextension* E' *of* $X' \times Y'$ *by* Z'*, the natural map from the set of all homomorphisms of biextensions*

$$(\psi : E \to E', \alpha : X \to X', \beta : Y \to Y', \gamma : Z \to Z') \in \mathrm{Hom}_{\mathrm{biext}}(E, E')$$

to the set of all triples (f, g, h) *satisfying the conditions*

- $f \in \mathrm{Hom}_{\Lambda(\kappa),\mathtt{F},\mathtt{V}}(\mathbb{D}_*(X), \mathbb{D}_*(X'))$,
- $g \in \mathrm{Hom}_{\Lambda(\kappa),\mathtt{F},\mathtt{V}}(\mathbb{D}_*(Y), \mathbb{D}_*(Y'))$,
- $h \in \mathrm{Hom}_{\Lambda(\kappa),\mathtt{F},\mathtt{V}}(\mathbb{D}_*(Z), \mathbb{D}_*(Z'))$,
- $h(\Theta_E(x, y)) = \Theta_{E'}(f(x), g(y)) \qquad \forall\, x \in \mathbb{D}_*(X),\ \forall\, y \in \mathbb{D}_*(y)$

is a bijection.

Remark. A bilinear pairing $\Theta : \mathbb{D}_*(X) \times \mathbb{D}_*(Y) \to \mathbb{D}_*(Z)$ satisfying the properties in 10.2.7.2 (i) corresponds to a family of bi-additive maps

$$\theta_n : X[p^n] \times Y[p^n] \to Z[p^n], \quad n \geq 1$$

according to general Dieudonné theory. Our choice of the *sign* of the correspondence E between Θ_E in 10.2.7.2 is that Θ_E corresponds to the Weil pairing $\theta_E = (\theta_{E,n})_{n\geq 1}$.

10.2.7.3. Corollary. *Notation as in 10.2.7.2. In particular* $E \to X \times Y$ *is a biextension of* $X \times Y$ *by* Z *and* Θ_E *is the* $\Lambda(\kappa)$*-bilinear map from* $\mathbb{D}_*(X) \times \mathbb{D}_*(Y)$ *to* $\mathbb{D}_*(Z)$ *attached to the biextension* Z*.*

(1) *The group* $\mathrm{Aut}_{\mathrm{biext}}(E)$ *of all automorphisms of the biextension* E *has a natural structure as a compact* p*-adic Lie group. It is naturally isomorphic to the closed subgroup of*

$$\mathrm{Aut}_{\Lambda,\mathtt{F},\mathtt{V}}(\mathbb{D}_*(X)) \times \mathrm{Aut}_{\Lambda,\mathtt{F},\mathtt{V}}(\mathbb{D}_*(Y)) \times \mathrm{Aut}_{\Lambda}(\mathbb{D}_*(Z))$$

consisting of all triples

$$(\alpha, \beta, \gamma) \in \mathrm{Aut}_{\Lambda,\mathtt{F},\mathtt{V}}(\mathbb{D}_*(X)) \times \mathrm{Aut}_{\Lambda,\mathtt{F},\mathtt{V}}(\mathbb{D}_*(Y)) \times \mathrm{Aut}_{\Lambda,\mathtt{F},\mathtt{V}}(\mathbb{D}_*(Z))$$

such that

$$\gamma(\Theta_E(x, y)) = \Theta_E(\alpha(x), \beta(y)) \qquad \forall\, x \in \mathbb{D}_*(X),\ \forall\, y \in \mathbb{D}_*(Y).$$

(2) *The Lie algebra of the compact* p*-adic Lie group* $\mathrm{Aut}_{\mathrm{biext}}(E)$ *is naturally isomorphic to the Lie subalgebra of*

$$\mathrm{End}_{\Lambda_{\mathbb{Q}},\mathtt{F},\mathtt{V}}(\mathbb{D}_*(X)_{\mathbb{Q}}) \oplus \mathrm{End}_{\Lambda_{\mathbb{Q}}\,\mathtt{F},\mathtt{V}}(\mathbb{D}_*(Y)_{\mathbb{Q}}) \oplus \mathrm{End}_{\Lambda_{\mathbb{Q}},\mathtt{F},\mathtt{V}}(\mathbb{D}_*(Z)_{\mathbb{Q}})$$

consisting of all triples (A, B, C) *in the above direct sum such that*

$$C(\Theta_E(x, y)) = \Theta_E(Ax, y) + \Theta_E(x, By)$$

for all $x \in \mathbb{D}_*(X)$ *and all* $y \in \mathbb{D}_*(Y)$*. Here*

- $\Lambda_{\mathbb{Q}} = \Lambda(\kappa)_{\mathbb{Q}} = \Lambda(\kappa) \otimes_{\mathbb{Z}} \mathbb{Q}$,
- $\mathbb{D}_*(X)_{\mathbb{Q}} := \mathbb{D}_*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$*, and similarly for* $\mathbb{D}_*(Y)_{\mathbb{Q}}$ *and* $\mathbb{D}_*(Z)_{\mathbb{Q}}$*,*

– $\mathrm{End}_{\Lambda_{\mathbb{Q}},\mathtt{F},\mathtt{V}}(\mathbb{D}_*(X)_{\mathbb{Q}})$ *denotes the set of all endomorphisms of the* $\Lambda_{\mathbb{Q}}$*-module* $\mathbb{D}_*(X)_{\mathbb{Q}}$ *which commute with* $\mathtt{F}$ *and* $\mathtt{V}$*; it is naturally isomorphic to the Lie algebra of the compact* p*-adic Lie group* $\mathrm{Aut}_{\Lambda,\mathtt{F},\mathtt{V}}(\mathbb{D}_*(X)) \cong \mathrm{Aut}(X)$*. Similarly for* $\mathrm{End}_{\Lambda_{\mathbb{Q}},\mathtt{F},\mathtt{V}}(\mathbb{D}_*(Y)_{\mathbb{Q}})$ *and* $\mathrm{End}_{\Lambda_{\mathbb{Q}},\mathtt{F},\mathtt{V}}(\mathbb{D}_*(Z)_{\mathbb{Q}})$.

10.2.7.4. Definition. Let G be a compact p-adic Lie group, which is a closed subgroup of the group of all $\mathbb{Q}_p$-points of a linear algebraic group over $\mathbb{Q}_p$. Let X, Y, Z be p-divisible groups over a field κ of characteristic p. Let $E \to X \times_{\mathrm{Spec}(\kappa)} Y$ be a biextension of $X \times_{\mathrm{Spec}(\kappa)} Y$ by Z. Let $\rho : G \to \mathrm{Aut}_{\mathrm{biext}}(E)$ be a continuous action of G on E which respects the biextension structure of E.

We say that the action of G on E is *strongly non-trivial* if the actions of G on X, Y, Z induced by the action of G on E are all strongly nontrivial in the sense of 7.3.1.

10.3. Equivariant sections and special formal subvarieties

Given a biextension of p-divisible formal groups $\pi : E \to X \times Y$ over a field κ of characteristic p and an strongly nontrivial action of a p-adic Lie group G on E, we will first show that the existence of a G-equivariant section of π implies that the biextension E splits; see 10.3.1. Then we will consider the case $X = Y$ and show that the existence of a G-equivariant section of the restriction of π the diagonal $\Delta_X \subseteq X \times X$ implies that the Weil pairings $\theta_{E,n}$ of E are symmetric; see 10.3.2. This train of thought lead to the notion of *special* formal subvarieties in a biextension; see 10.3.4.3.

10.3.1. Proposition. *Let* κ *be a field of characteristic* p*. Let* X, Y, Z *be* p*-divisible formal groups over* κ*. Let* $\pi : E \to X \times_{\mathrm{Spec}(\kappa)} Y$ *be a biextension of* $X \times_{\mathrm{Spec}(\kappa)} Y$ *by* Z*. Let* G *be a compact* p*-adic Lie group, and let* $\rho : G \to \mathrm{Aut}_{\mathrm{biext}}(E)$ *be an action of* G *on the biextension* $E \to X \times Y$*. Let* $\rho_X : G \to \mathrm{Aut}(X)$*,* $\rho_Y : G \to \mathrm{Aut}(Y)$*,* $\rho_Z : G \to \mathrm{Aut}(Z)$ *be the induced actions of* G *on* X, Y, Z *respectively. Let* $s : X \times_{\mathrm{Spec}(k)} Y \to E$ *be a* G*-equivariant section of* π*, i.e.* $\pi \circ s = \mathrm{id}_{X \times_{\mathrm{Spec}(k)} Y}$ *and* $\rho(g) \cdot s = s \circ (\rho_X(g), \rho_Y(g))$*. Suppose that the action of* G *on* E *is strongly nontrivial. Then the biextension* $\pi : E \to X \times_{\mathrm{Spec}(k)} Y$ *is trivial, and the section* s *is its canonical splitting.*

Proof. Following the notation in 10.2.2.1, let

$$\tau : X \times X \times Y \to Z \quad \text{and} \quad \sigma : X \times Y \times Y \to Z$$

be the maps associated to the section s defined by the formulas

$$s(x_1, y) +_1 s(x_2, y) = \tau(x_1, x_2; y) * s(x_1 + x_2, y)$$
$$s(x, y_1) +_2 s(x, y_2) = \sigma(x; y_1, y_2) * s(x, y_1 + y_2)$$

for functorial points x, x_1, x_2, y, y_1, y_2 of X and Y respectively. We will show that orbital rigidity for p-divisible formal groups 7.1.1 implies that the maps τ and σ are both 0.

By theorem 7.1.1, the graph of τ is a p-divisible subgroup of $X \times X \times Y \times Z$. In other words τ is a group homomorphism from the product group $X \times X \times Y$ to Z. So

$$\tau(x_1, x_2; y) = \tau(x_1, x_2; 0) + \tau(0, 0; y)$$

for all $x_1, x_2 \in X$ and all $y \in Y$.

Clearly $\tau(0,0;y) = 0$ for all $y \in Y$. Since $\big(\pi^{-1}(X \times \{0\}), +_1|_{\pi^{-1}(X\times\{0\})}\big)$ is a p-divisible group, theorem 7.1.1 implies that the graph of $s|_{X\times\{0\}}$ is a p-divisible subgroup of $\big(\pi^{-1}(X \times \{0\}), +_1|_{\pi^{-1}(X\times\{0\})}\big)$. So

$$\tau(x_1, x_2; 0) = 0 \quad \text{for all } x_1, x_2 \in X.$$

Therefore $\tau(x_1, x_2; y) = 0$ for all $x_1, x_2 \in X$ and all $y \in Y$. Similarly $\sigma(x; y_1, y_2) = 0$ for all $x \in X$ and all $y_1, y_2 \in Y$. We have shown that s is a splitting of the biextension E. □

10.3.1.1. Lemma. *Let X, Z be p-divisible groups over a scheme S. Let $\pi : E \to X \times X$ be a biextension of $X \times_S X$ by Z. Let $(\theta_n^E)_{n\geq 1}$ be the family of Weil pairings of E.*

(i) *For all $n \geq 1$ and all $x_n, x'_n \in X[p^n]$, we have*

$$\theta_n^{\iota^* E}(x_n, x'_n) = -\theta_n^E(x_n, x'_n).$$

(ii) *The biextensions $\iota^* E$ and E are isomorphic if and only if*

$$\theta_n^E(x_n, x'_n) = -\theta_n^E(x'_n, x_n)$$

for all $n \geq 1$ and all $x_n, x'_n \in X[p^n]$.

(iii) *The biextension $\iota^* E$ is isomorphic to $([-1]_Z)_* E$ if and only if*

$$\theta_n^E(x_n, x'_n) = \theta_n^E(x'_n, x_n)$$

for all $n \geq 1$ and all $x_n, x'_n \in X[p^n]$.

PROOF. We know from 10.2.5.2 that $\theta_n^{\iota^* E} \circ (\iota|_{X[p^n]\times X[p^n]}) = \omega_n^E = -\theta_n^E$ for all $n \geq 1$, where $\iota : X \times_S X \to X \times_S X$ be the isomorphism $(x, x') \mapsto (x', x)$ on functorial points of $X \times_S X$. The statement (i) follows. The statements (ii), (iii) are corollaries of (i). □

10.3.1.2. Definition. Let X, Z be p-divisible formal groups over a field κ of characteristic p. Let $\pi : E \to X \times X$ be a biextension. Let $\iota : X \times X \to X\times$ be isomorphism $(x, x') \mapsto (x', x)$ on functorial points of $X \times X$.

Suppose that the Weil pairings θ_n^E are symmetric in the sense that

$$\theta_n^E(x_n, x'_n) = \theta_n^E(x'_n, x_n)$$

for all $n \geq 1$ and all $x_n, x'_n \in X[p^n]$. Let δ be the unique isomorphism of biextensions from $([-1]_Z)_* E$ to $\iota^* E$, whose existence is guaranteed by 10.3.1.1 (iii).

Define the *involution* τ of such a biextension E with symmetric Weil pairings to be the composition

$$\tau := c \circ \delta \circ \zeta$$

of the top horizontal arrows in the following commutative diagram

$$\begin{tikzcd} E \arrow[r, "\zeta"] \arrow[d, "\pi"'] & ([-1]_Z)_* E \arrow[r, "\delta"] \arrow[d, "\pi''"'] & \iota^* E \arrow[r, "c"] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\ X \times X \arrow[r, "="] & X \times X \arrow[r, "="] & X \times X \arrow[r, "\iota"] & X \times X, \end{tikzcd}$$

where the commutative square at the left is the push-out diagram for the biextension $([-1]_Z)_*E$, and the commutative square at the right is the pull-back diagram for the biextension ι^*E. Clearly $\tau\circ\tau = \mathrm{id}_E$ and $\tau_Z = [-1]_Z$.

10.3.1.3. Remark. Suppose that the Weil pairings θ_n^E are skew symmetric in the sense that
$$\theta_n^E(x_n, x_n') = -\theta_n^E(x_n, x_n')$$
for all $n \geq 1$. There is an involution ς on E in this situation as well, defined below. We won't use it in the rest of this chapter.

Let δ' be the unique isomorphism of biextensions from E to ι^*E, whose existence is guaranteed by 10.3.1.1 (ii). Define the involution ς of such a biextension E with skew symmetric Weil pairings to be the composition
$$\varsigma := c\circ\delta'$$
of the top horizontal maps of the following commutative diagram
$$\begin{tikzcd} E \arrow[r, "\delta'"] \arrow[d, "\pi"'] & \iota^*E \arrow[r, "c"] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\ X\times X \arrow[r, "="] & X\times X \arrow[r, "\iota"] & X\times X, \end{tikzcd}$$
where the commutative square at the right is the pull-back diagram for ι^*E as before.

10.3.1.4. Corollary. *Let $\pi : E \to X\times X$ be a biextension of p-divisible formal groups $X\times X$ by Z with symmetric Weil pairings as in 10.3.1.2. Let $E^{\tau=1}$ be the fixed-point subscheme of the involution τ of E with $\tau|_Z = [-1]_Z$. Let $\Delta_X \subseteq X\times X$ be the diagonal subscheme of $X\times X$.*

(i) *If $p\neq 2$, then π induces an isomorphism from $E^{\tau=1}$ to Δ_X.*
(ii) *Suppose that $p = 2$. Then $E^{\tau=1}$ has a natural structure as a $Z[2]$-torsor over Δ_X, and π induces an isomorphism from the reduced formal scheme $(E^{\tau=1})_{\mathrm{red}}$ to Δ_X.*

10.3.2. Proposition. *Let X, Z be p-divisible formal groups over a field κ of characteristic p. Let $\pi : E \to X\times X$ be a biextension of $X\times X$ by Z. Let G be a p-adic Lie group, and let $\rho : G \to \mathrm{Aut}_{\mathrm{biext}}(E)$ be a strongly non-trivial action of G on E such that the actions of G on the two factors of $X\times X$ are the same. Suppose that there exists a G-equivariant section ζ of $\pi^{-1}\Delta_X \to \Delta_X$ over the diagonal $\Delta_X \subseteq X\times X$. Then the Weil pairings θ_n^E of E are symmetric for all $n\geq 1$.*

10.3.3. Corollary. *We keep the notation and assumptions in 10.3.2. In particular $\pi : E \to X\times X$ is a biextension of p-divisible formal group $X\times X$ by Z over $\kappa \supseteq \mathbb{F}_p$, G is a p-adic Lie group acting strongly nontrivially on E and induces the same action on both factors of $X\times X$, and ζ is a G-equivariant section ζ of π over the diagonal formal subscheme Δ_X of $X\times X$. If the Weil pairings θ_n^E of E are skew-symmetric, then $\theta_n^E = 0$ for all $n\geq 1$. In other words the biextension E splits.*

10.3.3.1. Remark. The statements 10.3.2 and 10.3.3 are equivalent. Clearly 10.3.2 $\Longrightarrow$ 10.3.3. Assume that 10.3.3 holds. By 10.2.5.2, we know that $\theta_n^{\iota^* E}(x_n, x'_n) = -\theta_n^E(x_n, x'_n)$ for all functorial points x_n, x'_n of $X[p^n]$.

Consider the biextension $E_d := E \times_{(X\times X)} \iota^* E$ of $X \times X$ by the product group $Z \times Z$, and the biextension $E' = h_* E_d$ of $X \times X$ by Z otained from E_d by the push-forward construction via the homomorphism $h = +_Z$ from $Z \times Z$ to Z, $h(z_1, z_2) = z_1 + z_2$ for all $z_1, z_2 \in Z$. The construction of E' tells us that

$$\theta_n^{E'}(x_n, x'_n) = \theta_n^E(x_n, x'_n) - \theta_n^E(x'_n, x_n),$$

for all $x_n, x'_n \in X[p^n]$, so $\theta_n^{E'}$ is skew-symmetric. The section ζ of E over Δ_X induces a section $\iota^*\zeta$ of $\iota^* E$ over Δ_X, and the section $\zeta_d = (\zeta, \iota^*\zeta)$ of $E_d \times_{(X\times X)} \Delta_X$ gives section ζ' of $E' \times_{(X\times X)} \Delta_X$. It is clear that the sections $\iota^*\zeta$, ζ_d and ζ' are all G-equivariant. Corollary 10.3.3 applied to the biextension E' tells us that $\theta_n^{E'} = 0$ for all $n \geq 1$. Therefore θ_n is symmetric. We have shown that 10.3.3 $\Longrightarrow$ 10.3.2. □

10.3.3.2. PROOF OF 10.3.3.

Step 1. Reduction to the case when the slopes of X and Z are disjoint.

Clearly we may assume that κ is algebraically closed. Let $\alpha : Z \to Z'$ be an isogeny such that there exist isoclinic p-divisible groups $Z_1, \ldots, Z_m$ over κ and an isomorphism $Z' \cong Z_1 \times \cdots \times Z_m$. It suffices to prove the assertion of 10.3.3 for each of the biextension $(\mathrm{pr}_i \circ \alpha)_* E$ of $X \times X$ by Z_i, $i = 1, \ldots, m$. So we may and do assume that Z is an isoclinic p-divisible formal group over κ.

Suppose that the slope of Z appears in X. Choose a isogeny $\beta : X_1 \times X_2 \to X$ such that X_2 is isoclinic with the same slope as Z, and all slopes of X_1 are different from the slope of Z. Clearly it suffice to prove the assertion of 10.3.3 for the biextension $\beta^* E \to X_1 \times X_2$. But $\theta_n^{\beta^* E}(x_{2,n}, y_n) = 0 = \theta_n^{\beta^* E}(y_n, x_{2,n})$ for all functorial points $y_n \in (X_1 \times X_2)[p^n]$ and all $x_{2,n} \in X_2[p^n]$. So it suffices to prove the assertion of 10.3.3 for the biextension $\beta^* E \times_{(X\times X)} (X_1 \times X_1)$ of $X_1 \times X_1$ by Z.

In the rest of 10.3.3.2 we assume that the p-divisible formal groups X and Z have no slope in common.

Step 2. Represent the G-equivariant section ζ of $E \times_{(X\times X)} \Delta_X$ in terms a family of Z-valued functions $(f_n)_{n\geq 1}$ on $X[p^{2n}] \times X[p^{2n}]$, using the canonical splitting Ξ_n of the biextension

$$([p^n]_X \times [p^n]_X)^* E\big|_{X[p^{2n}]\times X[p^{2n}]} \longrightarrow X[p^{2n}] \times X[p^{2n}]$$

defined in 10.2.5.3.

Recall that the bi-additive map Φ_n satisfies the functional equation

$$\theta_n^E(p^n x_{2n}, b_n) * \Xi_n(x_{2n} + a_n, y_{2n} + b_n) = \Xi_n(x_{2n}, y_{2n}) \tag{10.3.3.2.1}$$

for all $x_{2n}, y_{2n} \in X[p^{2n}]$ and all $a_n, b_n \in X[p^n]$, and the compatibility relations in 10.2.5.4:
(10.3.3.2.2)

$$\Xi_n(p^2 x_{2n+2}, p^2 y_{2n+2}) = [p]_{+_1}[p]_{+_2}\Xi_{n+1}(x_{2n+2}, y_{2n+2}) = \Phi_{n+1}(p x_{2n+2}, p y_{2n+2})$$

for all $x_{2n+2}, y_{2n+2} \in X[p^{2n+2}]$. Define morphisms

$$f_n : X[p^{2n}] \to Z, \quad n \geq 1$$

by

$$\big([p^n]_X^* \zeta\big)\big|_{X[p^{2n}]} = f_n * \big(\Xi_n|_{\Delta_{X[p^{2n}]}}\big)$$

i.e.

(10.3.3.2.3) $$\zeta(p^n x_{2n}) = f_n(x_{2n}) * \Phi_n(x_{2n}, x_{2n})$$

for all $x_{2n} \in X[p^{2n}]$, and of course $f_n(0) = 0$. From (10.3.3.2.3) we get

$$\begin{aligned} f_n(p^2 x_{2n+2}) * \Phi_n(p^2 x_{2n+2}, p^2 x_{2n+2}) &= \zeta(p^{n+2} x_{2n+2}) \\ &= f_{n+1}(p x_{2n+2}) * \Phi_{n+1}(p x_{2n+2}, p x_{2n+2}) \end{aligned}$$

So we deduce from (10.3.3.2) that

(10.3.3.2.4) $$f_n(p y_{2n+1}) = f_{n+1}(y_{2n+1})$$

for all $y_{2n+1} \in Y[p^{2n+1}]$. The functional equation (10.3.3.1) implies that

(10.3.3.2.5) $$f_n(x_{2n} + b_n) = \theta_n^E(p^n x_{2n}, b_n) + f_n(x_{2n})$$

for all $x_{2n} \in X[p^{2n}]$ and all $b_n \in X[p^n]$.

Step 3. Show that the functions f_n satisfy the theorem of the cube, using orbital rigidity for p-divisible formal groups.

Define maps $\gamma_n : X[p^{2n}] \times X[p^{2n}] \times X[p^{2n}] \to Z$, $n \geqslant 1$, by

(10.3.3.2.6)
$$\begin{aligned} \gamma_n(x_{2n}, y_{2n}, z_{2n}) := f_n(x_{2n} + y_{2n} + z_{2n}) - f_n(x_n + y_n) - f_n(y_{2n} + z_{2n}) - f_n(x_{2n} + z_{2n}) \\ + f_n(x_{2n}) + f_n(y_{2n}) + f_n(z_{2n}) \end{aligned}$$

for all $x_{2n}, y_{2n}, z_{2n} \in X[p^{2n}]$. An easy calculation using the functional equations (10.3.3.2.5) shows that

(10.3.3.2.7) $$\gamma_n(x_{2n} + a_n, y_{2n} + b_n, z_{2n} + c_n) = \gamma_n(x_{2n}, y_{2n}, z_{2n})$$

for all $x_{2n}, y_{2n}, z_{2n} \in X[p^{2n}]$ and all $a_n, b_n, c_n \in X[p^n]$. Therefore there exist uniquely defined morphisms

$$\bar{\gamma}_n : X[p^n] \times X[p^n] \times X[p^n] \to Z, \quad n \geq 1$$

such that

$$\gamma_n(x_{2n}, y_{2n}, z_{2n}) = \bar{\gamma}_n(p^n x_{2n}, p^n y_{2n}, p^n z_{2n}) \quad \forall\, x_{2n}, y_{2n}, z_{2n} \in X[p^{2n}].$$

It is easy to deduce from the compatibility relation (10.3.3.2.4) between the f_n's that

$$\bar{\gamma}_{n+1}\big|_{X[p^n] \times X[p^n] \times X[p^n]} = \bar{\gamma}_n \quad \forall\, n \geq 1.$$

Thus $\bar{\gamma} := \varinjlim_n \bar{\gamma}_n$ is a G-equivariant morphism from the $X \times X \times X$ to Z. By orbital rigidity for p-divisible formal groups, the graph of $\bar{\gamma}$ is a p-divisible formal subgroup of $X \times X \times X \times Z$, i.e. $\bar{\gamma}$ is a homomorphism from $X \times X \times X$ to Z. Since the slopes of X and Z are disjoint, $\bar{\gamma}$ is 0. Therefore $\gamma_n = 0$ for all $n \geq 1$.

Step 4. Define morphisms $g_n : X[p^{2n}] \times X[p^{2n}] \to Z$ by

$$g_n(x_{2n}, y_{2n}) := f_n(x_{2n} + y_{2n}) - f_n(x_{2n}) - f_n(y_{2n})$$

for all $x_{2n}, y_{2n} \in X[p^{2n}]$. The fact that $f_n = 0$ for all n implies that the map $g_n : X[p^{2n}] \times X[p^{2n}] \to Z$ is bi-additive for every $n \geq 1$.

From (10.3.3.2.5) we get

$$g_n(x_{2n}, a_n) = \theta_n^E(p^n x_{2n}, a_n) \tag{10.3.3.2.8}$$

for all $x_{2n} \in X[p^{2n}]$ and all $a_n \in X[p^n]$. Since θ_n^E is assumed to be skew symmetric for all n, $\theta_n^E(p^n x_{2n}, p^n x_{2n}) = 0$ for all $x_{2n} \in X[p^{2n}]$ and all $n \geq 1$. So

$$0 = g_n(x_{2n}, p^n x_{2n}) = p^n g_n(x_{2n}, x_{2n}) \tag{10.3.3.2.9}$$

for all $x_{2n} \in X[p^{2n}]$ and all $n \geq 1$.

From (10.3.3.2.4) we get

$$g_{n+1}(y_{2n+1}, z_{2n+1}) = g_n(py_{2n+1}, pz_{2n+1})$$

for all $y_{2n+1}, z_{2n+1} \in X[p^{2n+1}]$. Iterating, we get

$$g_{n+m}(y_{2n+m}, z_{2n+m}) = g_n(p^m y_{2n+m}, p^m z_{2n+m}) \tag{10.3.3.2.10}$$

for all $y_{2n+m}, z_{2n+m} \in X[p^{2n+m}]$ and all $m, n \geq 1$.

Given any $n \geq 1$, any $m \geq n$, a commutative κ-algebra R and any element $y_{2n} \in X[p^n](R)$, there exists a finite locally free commutative R-algebra R' and elements $y_{2n+2m} \in X[p^{2n+2m}]$ such that $p^{2m} y_{2n+2m} = y_{2n}$ and $p^{2m} z_{2n+2m} = z_{2n}$. Apply (10.3.3.2.10) with $y_{2n+m} = p^m y_{2n+2m}$, $z_{2n+m} = p^m y_{2n+2m}$, we get

$$g_n(y_{2n}, y_{2n}) = g_{n+m}(p^m y_{2n+2m}, p^m y_{2n+2m}) = p^{2m} g_{n+m}(y_{2n+2m}, y_{2n+2m}).$$

the last equality follows from (10.3.3.2.9) because $2m \geq n + m$. We have shown that $g_n(y_{2n}, y_{2n}) = 0$ for all $y_n \in X[p^n]$ and all $n \geq 1$.

The map $g_n : X[p^{2n}] \times X[p^{2n}] \to Z$ is obviously symmetric by definition, therefore

$$2g_n(x_{2n}, y_{2n}) = g_n(x_{2n} + y_{2n}, x_{2n} + y_{2n}) - g_n(x_{2n}, x_{2n}) - g_n(y_{2n}, y_{2n}) = 0$$

for all $x_{2n}, y_{2n} \in X[p^{2n}]$. This immediately implies that $g_n = 0$ if $p \neq 2$. In the case $p = 2$, an argular similar to but simpler than the argument used in the last two paragraphs again shows that $g_n = 0$. We conclude from (10.3.3.2.8) that $\theta_n^E = 0$ for all $n \geq 1$. We have proved corollary 10.3.3 and proposition 10.3.2. ∎

10.3.4. Special formal subvarieties of a biextension.

Let X, Y, Z be p-divisible formal group over a field κ of characteristic p, and let $\pi : E \to X \times Y$ be a biextension of $X \times Y$ by Z. We will define a class of reduced irreducible closed formal subschemes of E, called *special formal subvarieties*, guided by the proposition 10.3.4.1 below, which is a reformulation of 10.3.2.

10.3.4.1. Proposition. *Let X, Y, Z be p-divisible groups over a field κ of characteristic p. Let $\pi : E \to X \times Y$ be a biextension of $X \times Y$ by Z with Weil pairings $\big(\theta_n^E\big)_{n\geq 1}$. Let G be a p-adic Lie group, and let $\rho : G \to \mathrm{Aut}_{\mathrm{biext}}(E)$ be a strongly nontrivial action of G on E. Let $U \subseteq X \times Y$ be a p-divisible subgroup of $X \times Y$ stable under the action of G. Let $q_X : U \to X$ be the composition $U \hookrightarrow X \times Y \xrightarrow{\mathrm{pr}_1} X$ and let $q_Y : U \to Y$ be the composition $U \hookrightarrow X \times Y \xrightarrow{\mathrm{pr}_2} Y$. Suppose that there exists a reduced irreducible closed formal subscheme $\tilde{U}$ of E which is stable under the action of G such that π induces a purely inseparable dominant morphism $\tilde{U} \to U$. Then*

$$\theta_n^E(q_X(u_n), q_Y(u_n)) = \theta_n^E(q_Y(u_n), q_X(u_n))$$

for all functorial points u_n of $U[p^n]$ and all $n \geq 1$.

PROOF. There exists a positive integer n_0 and a morphism $\zeta : U \to \tilde{U}$, necessarily G-equivariant, such that

$$\big(\pi|_{\pi^{-1}U}\big) \circ \zeta = [p^{n_0}]_U.$$

Consider the biextension

$$E' := \big([p^{n_0}]_X \circ q_X) \times ([p^{n_0}]_Y \circ q_Y)\big)^* E \xrightarrow{\pi'} U \times U$$

of $U \times U$ by Z, with the induced strongly nontrivial action by G. The G-equivariant map $\zeta : U \to E$ defines a G-equivariant section ζ' of E' over the diagonal subgroup $\Delta_U \subseteq U \times U$. Proposition 10.3.2 tells us that

$$p^{n_0} \cdot \big(\theta_n^E(q_X(u_n), q_Y(u_n)) - \theta_n^E(q_Y(u_n), q_X(u_n))\big) = 0$$

for all $n \geq 1$ and all functorial points u_n of $U[p^n]$. Therefore

$$\theta_n^E(q_X(u_n), q_Y(u_n)) - \theta_n^E(q_Y(u_n), q_X(u_n)) = 0$$

for all $u_n \in U[p^n]$ and all $n \geq 1$. □

10.3.4.2. Proposition. *Let X, Y, Z be p-divisible formal groups over a field κ of characteristic p, and let $\pi : E \to X \times Y$ be a biextension of $X \times Y$ by Z. Let U be a p-divisible subgroup of $X \times Y$. Assume that the Weil pairings θ_n^E of E satisfy the symmetry condition*

$$\theta_n^E(q_X(u_n), q_Y(u_n)) = \theta_n^E(q_Y(u_n), q_X(u_n)) \quad \forall\, n \geq 1,\ \forall\, u_n \in U[p^n]$$

with respect to U. Then we have a natural commutative diagram

$$\begin{tikzcd}
 & E'' := ([p^\epsilon n]_U \times [p^\epsilon]_U)^* E' \arrow[r, "\delta_\epsilon"] \arrow[d, "\pi''"] & E' := (q_X \times q_Y)^* E \arrow[r, "\gamma"] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\
U \arrow[r, "(1_U, 1_U)"'] \arrow[ur, "\zeta_{E''}"] & U \times U \arrow[r, "{[p^\epsilon]_U \times [p^\epsilon]_U}"'] & U \times U \arrow[r, "q_X \times q_Y"'] & X \times Y
\end{tikzcd}$$

with the following the following properties.

(i) *Both squares are Cartesian.*

(ii) $\begin{cases} \epsilon = 0 \text{ and } \delta_\epsilon = \mathrm{id}_{E'} & \text{if } p \neq 2, \\ \epsilon = 1 & \text{if } p = 2. \end{cases}$

(iii) *If $p \neq 2$, then the schematic image $\zeta_{E''}(U)$ of the map $U \xrightarrow{\zeta_{E''}} E''$ is*

$$\zeta_{E''}(U) = (E'')^{\tau''=1} = (E')^{\tau'=1},$$

where $(E'')^{\tau''=1}$ is the fixed-point subscheme of the involution τ'' of the biextension E'' with symmetric Weil pairings as in 10.3.1.4.

(iv) *If $p = 2$, then the schematic image $\zeta_{E''}(U)$ of the map $U \xrightarrow{\zeta_{E''}} E''$ is*

$$\zeta_{E''}(U) = \left((E'')^{\tau''=1}\right)_{\mathrm{red}},$$

and the schematic image $(\delta_\epsilon \circ \zeta_{E''})(U)$ of $U \xrightarrow{\delta_\epsilon \circ \zeta_{E''}} E'$ is

$$(\delta_\epsilon \circ \zeta)(U) = \left((E')^{\tau'=1}\right)_{\mathrm{red}},$$

where $\left((E')^{\tau'=1}\right)_{\mathrm{red}}$ is the fixed-point subscheme of E' under the involution τ', with the reduced structure.

PROOF. The assumption that $\theta_n^E(q_X(u_n), q_Y(u_n)) = \theta_n^E(q_Y(u_n), q_X(u_n))$ for all $u_n \in U[p^n]$ and all $n \geq 1$ means that the Weil pairings $\theta_n^{E''}$ of E'' are symmetric. The statement (i) and (ii) are part of the definiton of the commutative diagram. The existence of the map $\zeta_{E''}$ with the asserted properties (iii) and (iv) follows from 10.3.1.4. □

10.3.4.3. Definition. We keep the notation and assumptions in 10.3.4.2.

(a) For every natural number $m \geq 0$, define a morphism

$$U \xrightarrow{[p^m]_U^* \zeta_{E''}} ([p^m]_U \times [p^m]_U)^* E''$$

to be the unique map which makes the following diagram with a Cartesian square at the lower right corner

$$\begin{tikzcd}
U \arrow[dr, "{[p^m]_U^* \zeta_{E''}}"] \arrow[drr, bend left, "{\zeta_{E''} \circ [p^m]_U}"] \arrow[ddr, bend right, "{(1_U, 1_U)}"'] & & \\
& ([p^m]_U \times [p^m]_U)^* E'' \arrow[r, "\delta_m"] \arrow[d, "\pi''_m"] & E'' \arrow[d, "\pi''"] \\
& U \times U \arrow[r, "{[p^m]_U \times [p^m]_U}"] & U \times U
\end{tikzcd}$$

commutative.

(b) For every homomorphism $h : U \to Z$ of p-divisible groups, denote by

$$U \xrightarrow{h * [p^m]_U^* \zeta_{E''}} ([p^m]_U \times [p^m]_U)^* E''$$

the map given by

$$(h * [p^m]_U^* \zeta_{E''})(u) := h(u) * [p^m]_U^* \zeta_{E''}(u) \quad \forall\, u \in U.$$

10.3.4.4. Proposition. *We keep the notation and assumptions in 10.3.4.2. In particular $\pi : E \to X \times Y$ is a biextension of $X \times Y$ by Z, and U is a p-divisible subgroup of $X \times Y$ satisfying the symmetry condition with respect to U. Suppose that G is a p-adic. Lie group acting strongly nontrivially on the biextension E, and U is stable under G.*

(1) *The map $\zeta_{E''} : U \to E''$ is G-equivariant. The schematic image $(\gamma \circ \delta_\epsilon \circ \zeta_{E''})(U)$ of the composition $U \xrightarrow{\gamma\circ\delta_\epsilon\circ\zeta_{E''}} E$ is a reduced irreducible closed formal subscheme of E stable under G, and the morphism $(\gamma \circ \delta_\epsilon \circ \zeta_{E''})(U) \longrightarrow U$ induced by π is dominant and purely inseparable.*

(2) *Suppose that T is a reduced irreducible closed formal subscheme of E which is stable under the action of G and the map π induces a purely inseparable dominant map $T \to U$. Then there exist a G-quivariant homomorphism $h : U \to Z$ of p-divisible groups and a natural number m, such that the schematic image*

$$\big(\gamma \circ \delta_\epsilon \circ \delta_m \circ (h * [p^m]_U^* \zeta_{E''})\big)(U)$$

of the compositon of the following maps

$$U \xrightarrow{h*[p^m]_U^*\zeta_{E''}} ([p^m]_U \times [p^m]_U)^* E'' \xrightarrow{\delta_m} E'' \xrightarrow{\delta_\epsilon} E' \xrightarrow{\gamma} E$$

is equal to T, where the map $U \xrightarrow{h*[p^m]_U^*\zeta_{E''}} ([p^m]_U \times [p^m]_U)^* E''$ is defined in 10.3.4.3 and the maps $\delta_m, \delta_\epsilon$ and γ are as in 10.3.4.2.

In particular if U and Z do not have any slope in common, then $(\gamma\circ\delta_\epsilon\circ\zeta_{E''})(U)$ is the only G-invariant reduced irreducibel closed formal subscheme $T \subseteq E$ lying above U such that the morphism $T \to U$ induced by π is dominant and purely inseparable.

Proof. The assertions in (1) and the first paragraph of (2) are consequences of 10.3.1.4 and 10.3.4.2. The uniqueness of T in the last paragraph of (2) is a corollary of the main assertion of (2) and the orbital rigidity of p-divisible formal groups. □

Remark. Definition 10.3.4.5 is formulated so that the schematic image $\gamma \circ \delta_\epsilon \circ \zeta_{E''}(U)$ in 10.3.4.4 (1) and reduced irreducible closed formal subschemes T of E satisfying the assumption in 10.3.4.4 (2) are special formal subvarietis of the biextension E.

10.3.4.5. Definition. Let X, Y, Z be p-divisible groups over a field κ of characteristic p. Let $\pi : E \to X \times Y$ be a biextension of $X \times Y$ by Z. Let $T \subseteq E$ be a reduced irreducible closed formal subscheme of E. We say that T is a *special* formal subvariety of E if the following there exist a p-divisible formal subgroup Z_1 of Z and a p-divisible formal subgroup U of $X \times Y$ such that the following statements hold

(i) The formal subscheme T of E is stable under the translation action of the subgroup Z_1 of Z.

Let $\bar{E} := E/Z_1$, so that the induced morphism $\bar{\pi} : \bar{E} \to X \times Y$ is a biextension of $X \times Y$ by $\bar{Z} := Z/Z_1$, and $\bar{T} := T/Z_1$ is a reduced irreducible closed formal subscheme of $\bar{E}$.

(ii) The p-divisible subgroup $U \subseteq X \times Y$ is the schematic image of T under π. Equivalently U is the schematic image of $\bar{T}$ under $\bar{\pi}$.

(iii) The morphism $\bar{T} \to U$ induced by $\bar{\pi}$ is dominant and purely inseparable.

(iv) For every $n \geq 1$ and every functorial point u_n of $U[p^n]$, we have

$$\theta_n^{\bar{E}}(q_X(u_n), q_Y(u_n)) = \theta_n^{\bar{E}}(q_Y(u_n), q_X(u_n)),$$

where $q_X : U \to X$ and $q_Y : U \to Y$ are the projections of U to X and Y respectively, and $\big(\theta_n^{\bar{E}}\big)_{n\geq 1}$ is the Weil pairing of the biextension $\bar{E}$.

According to 10.3.4.2 condition (iv) implies that we have the following commutative diagram

$$\begin{tikzcd}
 & \bar{E}'' := ([p^\epsilon]_U \times [p^\epsilon]_U)^*\bar{E}' \arrow[r, "\delta_\epsilon"] \arrow[d, "\bar{\pi}''"] & \bar{E}' := (q_X \times q_Y)^*\bar{E} \arrow[r, "\gamma"] \arrow[d, "\bar{\pi}'"] & \bar{E} \arrow[d, "\bar{\pi}"] \\
U \arrow[ur, "\zeta_{\bar{E}''}"] \arrow[r, "(1_U, 1_U)"'] & U \times U \arrow[r, "{[p^\epsilon]_U \times [p^\epsilon]_U}"'] & U \times U \arrow[r, "q_X \times q_Y"'] & X \times Y,
\end{tikzcd}$$

where

- both squares are Cartesian,
- $\begin{cases} \epsilon = 0, \bar{E}'' = \bar{E}' \text{ and } \delta_\epsilon = \mathrm{id}_{\bar{E}'} & \text{if } p \neq 2, \\ \epsilon = 1 & \text{if } p = 2, \end{cases}$
- the schematic image $(\delta_\epsilon \circ \zeta_{\bar{E}''})(U)$ of $\delta_\epsilon \circ \zeta_{\bar{E}''}$ is

$$(\delta_\epsilon \circ \zeta_{\bar{E}''})(U) = \begin{cases} (\bar{E}')^{\bar{\tau}'=1} & \text{if } p \neq 2, \\ \big((\bar{E}')^{\bar{\tau}'=1}\big)_{\mathrm{red}} & \text{if } p = 2. \end{cases}$$

Here $(\bar{E}')^{\bar{\tau}'=1}$ is the fixed-point subscheme of the involution $\bar{\tau}'$ of the biextension $\bar{E}'$ with symmetric Weil pairings as in 10.3.1.4.

(v) There exist a homomorphism $h : U \to Z$ of p-divisible groups and a natural number $m \in \mathbb{N}$ such that the schematic image of the map

$$\gamma \circ \delta_\epsilon \circ \delta_m \circ (h * [p^m]_U^* \zeta_{\bar{E}''})$$

from the lower-left corner to the upper-right corner of the commutative diagram

$$\begin{tikzcd}
 & ([p^m]_U \times [p^m]_U)^*\bar{E}'' \arrow[r, "\delta_m"] \arrow[d, "\bar{\pi}''_m"] & ([p^m]_U \times [p^m]_U)^*\bar{E}'' \arrow[r, "\delta_\epsilon"] \arrow[d, "\bar{\pi}''"] & \bar{E}' \arrow[r, "\gamma"] \arrow[d, "\bar{\pi}'"] & \bar{E} \arrow[d, "\bar{\pi}"] \\
U \arrow[ur, "h*{[p^m]_U^*}\zeta_{\bar{E}''}"] \arrow[r, "(1_U, 1_U)"'] & U \times U \arrow[ur, "\zeta_{\bar{E}''}"] \arrow[r, "{[p^m]_U \times [p^m]_U}"'] & U \times U \arrow[r, "1_U \times 1_U"'] & U \times U \arrow[r, "q_X \times q_Y"'] & X \times Y
\end{tikzcd}$$

is

$$\big(\gamma \circ \delta_\epsilon \circ \delta_m \circ (h * [p^m]_U^* \zeta_{\bar{E}''})\big)(U) = \bar{T},$$

where $h * [p^m]_U^* \zeta_{\bar{E}''} : U \longrightarrow \bar{E}''$ is the map in definition 10.3.4.3 applied to the biextension $\bar{E}'' \to U \times U$.

The statements 10.3.4.6 and 10.3.4.7 below follow quickly from definition 10.3.4.5.

10.3.4.6. Corollary. *Let X, Y, Z be p-divisible groups over a field κ of characteristic p. Let $\pi : E \to X \times Y$ be a biextension of $X \times Y$ by Z. Let $T \subseteq E$ be a reduced irreducible formal subscheme of E. Suppose that there exists a p-divisible subgroup $Z_2 \subseteq Z$ of Z which satisfies the following properties.*

(i) *T is stable under the translation action by Z_2.*

(ii) *The quotient T/Z_2 is a special formal subvariety of the biextension*

$$E/Z_2 = (Z \twoheadrightarrow Z/Z_2)_* E \longrightarrow X \times Y$$

of $X \times Y$ by Z/Z_2.

Then T is a special formal subvariety of E.

10.3.4.7. Corollary. *Let X, Y, Z be p-divisible formal groups over a field κ of characteristic p. Let $\pi : E \to X \times Y$ be a biextension of $X \times Y$ by Z. Let T be a special formal subvariety of E. If the sets of slopes of X, Y, Z are pairwise disjoint,then T is a sub-biextension of E. In other words T has a structure as a biextension of $X' \times Y'$ by Z', where $X' \subseteq X$, $Y' \subseteq Y$ and $Z' \subseteq Z$ are p-divisible subgroups, such that $(T \hookrightarrow E, X' \hookrightarrow X, Y' \hookrightarrow Y, Z' \hookrightarrow Z)$ is a homomorphism of biextensions.*

10.4. Action of a one-parameter subgroup on a biextension

In this section k is a *perfect* field of characteristic p , X, Y, Z are p-divisible formal groups over k, and $\pi : E \to X \times Y$ is a biextension of $X \times Y$ by Z.

10.4.1. Suppose we have a one-dimensional p-adic Lie group Γ acting on a biextension E of $X \times Y$ by Z. We will extract from such an action a collection of congruence relations; see proposition 10.7.3.3. This collection of congruence relations comes from the "leading term" of the action of a sequence (γ_m) in Γ with $\lim_{m\to\infty} \gamma_m = 1$, and can be regarded as a substitute for the "derivative" of the action of Γ on E.[1]

Recall from 10.2.7.3 that the Lie algebra of the p-adic Lie group $\mathrm{Aut}_{\mathrm{biext}}(E)$ consists of all triples (A, B, C) which kill the bilinear form Θ_E, i.e.

$$C(\Theta_E(x, y)) - \Theta_E(Ax, y) - \Theta_E(x, By) = 0 \quad \forall\, x \in \mathbb{D}_*(X),\ \forall\, y \in \mathbb{D}_*(Y).$$

10.4.1.1. Lemma. *Let $v = (A, B, C)$ be an element of the Lie algebra of $\mathrm{Aut}_{\mathrm{biext}}(E)$. Suppose that $A \in \mathrm{End}(X)$, $B \in \mathrm{End}(Y)$ and $C \in \mathrm{End}(Z)$. Then $\exp(p^2 t\, A) \in \mathrm{Aut}(X)$, $\exp(p^2 t\, B) \in \mathrm{Aut}(Y)$, $\exp(p^2 t\, C) \in \mathrm{Aut}(Z)$ and $\exp(p^2 t\, v) \in \mathrm{Aut}_{\mathrm{biext}}(E)$ for all $t \in \mathbb{Z}_p$.*

[1] The challenge of finding a good notion of "derivative" can be seen in a simple example: the standard action of ${\mathbb{Z}_p}^\times$ on the formal completion $\widehat{\mathbb{G}_m} = \mathrm{Spf}(\overline{\mathbb{F}_p}[[t]])$ of $\mathbb{G}_m$ over $\overline{\mathbb{F}_p}$. The action of an element $a \in {\mathbb{Z}_p}^\times$ on $\widehat{\mathbb{G}_m}$ sends the coordinate t to $(1 + t)^a - 1$.

Proof. The Taylor series for $\exp(p^2t\,A) \in \mathrm{Aut}(X)$ converges p-adically and defines an element of $\mathrm{Aut}(X)$. Similarly $\exp(p^2t\,B) \in \mathrm{Aut}(Y)$ and $\exp(p^2t\,C) \in \mathrm{Aut}(Z)$. That $\exp(p^2t\,v) \in \mathrm{Aut}_{\mathrm{biext}}(E)$ follows from 10.2.7.3. □

10.4.1.2. Proposition. *Let $v = (A, B, C)$ be an element of the Lie algebra of $\mathrm{Aut}_{\mathrm{biext}}(E)$ such that $A \in \mathrm{End}(X)$, $B \in \mathrm{End}(Y)$ and $C \in \mathrm{End}(Z)$. Let n be a positive intger.*

(i) *For every integer $n \geq 2$, The infinite series*

$$\sum_{j\geq 2} \frac{p^{n(j-1)}}{j!} C^j$$

converges to an element of $\mathrm{End}(Z)$ if $n \geq 2$.

(ii) *Suppose that $n \geq 3$. The restriction to $E_n = \pi^{-1}(X[p^n] \times Y[p^n])$ of the automorphism $\exp(p^n v)$ of E is equal to*

$$\Big(-\theta_n\circ(1_X\times B)\circ(\pi|_{E_n}) + C\circ\eta_n + \sum_{j\geq 2}\frac{p^{n(j-1)}}{j!}C^j\circ\eta_n\Big) * \mathrm{id}_{E_n}$$

$$= \Big(-\theta_n\circ(1_X\times B)\circ(\pi|_{E_n}) + C\circ\theta_n\circ(\pi|_{E_n}) + C\circ\rho_n + \sum_{j\geq 2}\frac{p^{n(j-1)}}{j!}C^j\circ\rho_n\Big) * \mathrm{id}_{E_n}$$

$$= \Big(\theta_n\circ(A\times 1_Y)\circ(\pi_{E_n}) + C\circ\rho_n + \sum_{j\geq 2}\frac{p^{n(j-1)}}{j!}C^j\circ\rho_n\Big) * \mathrm{id}_{E_n},$$

where the maps $\eta_n, \rho_n : E_n \to Z$ are defined in 10.2.6.1.

Proof. The statement (i) follows from the easy estimate

$$\mathrm{ord}_p(j!) < \frac{j}{p-1} \leq j,$$

which implies that

$$\mathrm{ord}_p\Big(\frac{p^{2(j-1)}}{j!}\Big) \geq (n-1)(j-1) - 1.$$

Clearly $(n-1)(j-1) - 1 \geq 0$ for all $j \geq 2$ and $(n-1)(j-1) - 1 \to 0$ as $j \to \infty$. The statement (i) follows.

For (ii), note first that $\mathrm{ord}_p\Big(\frac{p^{2(j-1)}}{j!}\Big) \geq (n-1)(j-1)-1 > 0$ for all $j \geq 2$ because $n \geq 3$. The automorphism $\exp(p^nA) \times \exp(p^nB) \times \exp(p^nC)$ of $X[p^n] \times Y[p^{2n}] \times Z$ descents, via the finite locally free morphism $\alpha_n : X[p^n]\times Y[p^{2n}]\times Z \to E_n$ in 10.2.5.1, to the restriction to E_n of the automorphism $\exp(p^nv)$ of E_n. The statement (ii) follows from an easy calculation, the definition of η_n, ρ_n in 10.2.6.1 and the Taylor expansion of $\exp(p^nC)$, as follows.

For each functorial point (x_n, y_{2n}, z) of $X[p^n] \times Y[p^{2n}] \times Z$, we have

$$\begin{aligned}\exp(p^n v)\big(\alpha_n(x_n, y_{2n}, z)\big) &= \alpha_n\big(\exp(p^n A)(x_n), \exp(p^n B)(y_{2n}), \exp(p^n C)(z)\big)\\ &= \alpha_n\big(x_n, y_{2n} + p^n B y_{2n}, z + \textstyle\sum_{j\ge 1} \frac{p^{nj}}{j!} C^j z\big)\\ &= \alpha_n\big(x_n, y_{2n}, z - \theta_n(x_n, Bp^n y_{2n}) + \textstyle\sum_{j\ge 1} \frac{p^{nj}}{j!} C^j z\big).\end{aligned}$$

Since $\eta_n(\alpha_n(x_n, y_{2n}, z)) = p^n z$ and $(x_n, Bp^n y_{2n}) = (1_X \times B)(\pi(\alpha_n(x_n, y_{2n}, z)))$, we have proved that $\exp(p^n v)|_{E_n}$ is given by the first line of the formula in (ii). The first two expressions in the displayed formula are equal because $\eta_n = (\theta_n \circ \pi|_{E_n}) + \rho_n$ and $\frac{p^{n(j-1)}}{j!}$ kills θ_n for all $j \ge 2$. The second and third expression are equal because

$$\theta_n(Ax_n, y_n) + \theta_n(x_n, By_n) = C\theta_n(x_n, y_n)$$

for all functorial points (x_n, y_n) of $X[p^n] \times Y[p^n]$. $\square$

Remark. The equality of the first and the third expression in 10.4.1.2 (ii) exhibits a clear symmetry if we take into account the fact that $\theta_n = -\omega_n$, and the relation of η_n to $+_1$ (respectively ρ_n to $+_2$) in 10.2.6.1.

10.4.1.3. Definition. Let $v = (A, B, C) \in \mathrm{Lie}(\mathrm{Aut}_{\mathrm{biext}}(E)) \cap \big(\mathrm{End}(X) \oplus \mathrm{End}(Y) \oplus \mathrm{End}(Z)\big)$ be an element of the Lie algebra of $\mathrm{Aut}_{\mathrm{biext}}(E)$ as in 10.4.1.2. Define a map

$$\mathfrak{z}_n[v] : E_n \to Z$$

by

$$\begin{aligned}\mathfrak{z}_n[v] &:= \theta_n \circ (1_X \times (-B)) \circ (\pi|_{E_n}) + C \circ \eta_n\\ &= -\theta_n \circ (1_X \times B) \circ (\pi|_{E_n}) + C \circ \theta_n \circ (\pi|_{E_n}) + C \circ \rho_n\\ &= \theta_n \circ (A \times 1_Y) \circ (\pi_{E_n}) + C \circ \rho_n\,.\end{aligned}$$

It is the "linear part", in v, of the terms before "$*\mathrm{id}_{E_n}$" in the formula in 10.4.1.2 (ii).

10.4.1.4. Lemma. *For every element $v = (A, B, C) \in \mathrm{End}(X) \oplus \mathrm{End}(Y) \oplus \mathrm{End}(Z)$ of the Lie algebra of $\mathrm{Aut}_{\mathrm{biext}}(E)$, the maps $\big(\mathfrak{z}_n[v]\big)_n$ satisfy the compatibility relations*

$$[p]_Z \circ \mathfrak{z}_n[v] = \mathfrak{z}_{n+1} \circ (E_n \hookrightarrow E_{n+1})$$

for all $n \ge 1$.

Proof. This assertion is immediate from the definition 10.4.1.3 of $\mathfrak{z}_n[v]$ and the similar compatibility relations

$$[p]_Z \circ \eta_n = \eta_{n+1} \circ (E_n \hookrightarrow E_{n+1}), \quad [p]_Z \circ \rho_n = \rho_{n+1} \circ (E_n \hookrightarrow E_{n+1})$$

and

$$[p]_Z \circ \theta_n = \theta_{n+1} \circ (X[p^n] \times Y[p^n] \hookrightarrow X[p^{n+1}] \times Y[p^{n+1}]). \quad \square$$

10.4.2. The following assumptions and notation for a biextension $\pi : E \to X \times Y$ of $X \times Y$ by Z will be used in a number of situations below, where X, Y, Z are p-divisible formal group over a perfect field κ of characteristic p.

(i) Let $v = (A, B, C) \in \mathrm{Lie}(\mathrm{Aut}_{\mathrm{biext}}(E)) \cap \big(\mathrm{End}(X) \oplus \mathrm{End}(Y) \oplus \mathrm{End}(Z)\big)$ be an element of the Lie algebra of $\mathrm{Aut}_{\mathrm{biext}}(E)$ with components $A \in \mathrm{End}(X)$, $B \in \mathrm{End}(Y)$ and $C \in \mathrm{End}(Z)$.

(ii) Assume that a, s, r are three positive integers such that
- $0 < r < s$, and $\frac{a}{r}$ is the largest slope of Z
- $\frac{a}{s}$ is strictly bigger than every slope of X and every slope of Y.

From general properties of slopes of p-divisible groups we know that there exist natural numbers $n_0, c_0 \in \mathbb{N}$ with $n_0 \geq \min(2, c_0/r)$ such that

$$X[p^{na}] \supset \mathrm{Ker}(\mathrm{Fr}_X^{ns}) \quad \text{and} \quad Y[p^{na}] \supset \mathrm{Ker}(\mathrm{Fr}_Y^{ns})$$

and

$$Z[p^{na}] \supset \mathrm{Ker}(\mathrm{Fr}_Z^{nr-c_0})$$

for all $n \geq n_0$, where $\mathrm{Fr}_X^{ns} : X \to X^{(p^{ns})}$ (respectively Fr_Y^{ns}) is the (ns)-th iterate of the relative Frobenius for X (respectively Y). Similarly for $\mathrm{Fr}_Z^{nr-c_0}$.

(iii) Let $R = R_E$ be the affine coordinate ring of the smooth formal scheme E, so that $E = \mathrm{Spf}(R)$ and R is non-canonically isomorphic to a formal power series ring in d variables, where $d = \dim(E)$. Let $\mathfrak{m} = \mathfrak{m}_E$ be the maximal ideal of R. Let $\phi = \phi_R$ be the absolute Frobenius endomorphism of R, which sends every element $x \in R$ to x^p.

For every natural number j, define an ideal of R by

$$\mathfrak{m}^{(p^j)} := \phi^j(\mathfrak{m})R.$$

Note that

$$\mathfrak{m}^{d \cdot p^j} \subseteq \mathfrak{m}^{(p^j)} \subseteq \mathfrak{m}^{p^j}.$$

Denote by $E \bmod \mathfrak{m}^{(j)}$ the Artinian scheme

$$E \bmod \mathfrak{m}^{(p^j)} := \mathrm{Spec}(R/\mathfrak{m}^{(p^j)})$$

10.4.3. Proposition. *We use the notation and assumption in 10.4.2 and 10.4.1.2. In particular $v = (A, B, C)$ is an element of the Lie algebra of $\mathrm{Aut}_{\mathrm{biext}}(E)$, $A \in \mathrm{End}(X)$, $B \in \mathrm{End}(Y)$, and $C \in \mathrm{End}(Z)$. There exist positive integers n_3, c_3 such that the congruence*

$$\begin{aligned}\psi(\exp(p^{na}v)) &\equiv \mathfrak{z}_{na}[v] \pmod{\mathfrak{m}^{(p^{\min(ns, 2nr-c_3)})}} \\ &\equiv \big(-\theta_{na} \circ (1_X \times B) \circ (\pi|_{E_{na}}) + C \circ \eta_{na}\big) * \mathrm{id}_{E_{na}} \pmod{\mathfrak{m}^{(p^{\min(ns, 2nr-c_3)})}}\end{aligned}$$

for the action $\psi(\exp(p^{na}v))$ of the element $\exp(p^{na}v) \in G$ on E holds for all integers $n \geq n_3$. In other words, the restrictions to the Artinian scheme $\mathrm{Spec}(R/\mathfrak{m}^{(p^{\min(ns,2nr-c_3)})})$ of the two automorphisms $\psi(\exp(p^{na}v))$ and $\mathfrak{z}_{na}[v] * \mathrm{id}_{E_{na}}$ of the formal scheme E coincide. Here $E_{na} = \pi^{-1}(X[p^{na}] \times Y[p^{na}])$ as before, and $\mathfrak{z}_{na}$ is the map from E_{na} to Z defined in 10.4.1.3.

PROOF. This proposition is a straight-forward consequence of 10.2.6.3 and 10.4.1.2.

1. The assumption 10.4.2 (ii) tells us that $E_{na} \supset \mathrm{Spec}\big(R/\mathfrak{m}^{(p^{ns})}\big)$ for all $n \geq n_0$.
2. We know from 10.4.1.2 that the restriction of $\psi(\exp(p^{na}\,v))$ to E_{na} is equal to
$$\Big(-\theta_{na}\circ(1_X\times B)\circ(\pi|_{E_{na}}) + C\circ\eta_{na} + \sum_{j\geq 2}\frac{p^{na(j-1)}}{j!}C^j\circ\eta_{na}\Big) * \mathrm{id}_{E_{na}}.$$
3. We know from 10.2.6.3 that there exist positive integers n_2, c_2 such that
$$\eta_{na} \equiv 0 \left(\bmod\ \mathfrak{m}^{(p^{nr-c_2})}\right)$$
for all $n \geq \frac{n_2}{a}$.
4. An elementary calculation shows that
$$\mathrm{ord}_p\tfrac{p^{na(j-1)}}{j!} > na(j-1) - \tfrac{j}{p-1} \geq na-2 \qquad \forall j \geq 2$$

Let $n_3 := \mathrm{Min}(n_0, \lceil n_2/a\rceil)$. Combining 3 and 4 above we get an estimate of the typical "error term" $\frac{p^{na(j-1)}}{j!}C^j\circ\eta_{na}$:
$$\frac{p^{na(j-1)}}{j!}C^j\circ\eta_{na} \equiv 0 \left(\bmod\ \mathfrak{m}^{(p^{2nr-c_3})}\right)$$
where $c_3 := 2c_0 + c_2$, for all $n \geq n_3$ and all $j \geq 2$. ∎

10.4.4. Corollary 10.4.5 below is a variant of 10.4.3 and will be convenient for our purpose.

The setup for 10.4.5 is as follows. We will use the general notation scheme in 10.4.2 and 10.4.3: X, Y, Z are p-divisible groups over a perfect field $k \supset \mathbb{F}_p$, $\pi : E \to X \times Y$ is a biextension of $X \times Y$ by Z. Let $(R, \mathfrak{m}) = (R_E, \mathfrak{m}_E)$ be the coordinate ring of E.

(i) Assume that X, Y, Z are p-divisible formal groups, i.e. every slope of X, Y, Z is strictly positive.
(ii) Let $\nu = (A, B, C)$ be an element of the Lie algebra of $\mathrm{Aut}_{\mathrm{biext}}(E \to X \times Y)$, $A \in \mathrm{End}(X)$, $B \in \mathrm{End}(Y)$ and $C \in \mathrm{End}(Z)$.
(iii) Assume that Z is a product of isoclinic p-divisible groups; write Z as a product of isoclinic p-divisible subgroups with distinct slopes: $Z = \prod Z_l$, where each Z_l is isoclinic, the slopes of the Z_l are mutually distinct, and the slope of Z_l is the biggest among slopes of Z.
(iv) Assume that the slope of Z_1 is strictly bigger than every slope of $X \times Y$.
(v) Choose positive integers a, r, s, n_3 with $r < s$ such that the following conditions hold.
   - slope$(Z_1) = \frac{a}{r}$
   - $X[p^{na}] \supset \mathrm{Ker}(\mathrm{Fr}_X^{ns})$ and $Y[p^{na}] \supset \mathrm{Ker}(\mathrm{Fr}_Y^{ns})$ for all $n \geq n_3$.
   - $Z_l[p^{na}] \supset \mathrm{Ker}(\mathrm{Fr}_{Z_l}^{ns})$ for all $l \neq 1$ and all $n \geq n_3$.

(vi) For every $n \geq n_3$, let

$$\bar{\mathfrak{z}}_{na}[v] := \mathrm{pr}_{Z_1} \circ \mathfrak{z}_{na}[v] : E_{na} \longrightarrow Z_1$$

be the composition of $E_{na} \xrightarrow{\mathfrak{z}_{na}[v]} Z$ with the projection $Z \xrightarrow{\mathrm{pr}_{Z_1}} Z_1$ from Z to its first factor.

10.4.5. Corollary. *Notation and assumptions as in 10.4.4. In particular a, r, s are positive integers, $0 < a < r < s$, $\frac{a}{r}$ is the largest slope of Z, Z_1 is the maximal p-divisible subgroup of Z with slope $\frac{a}{r}$, $\frac{a}{s}$ is strictly bigger than any slope of $X \times Y \times (Z/Z_1)$, and $Z_1[p^a] = Z_1[F^r] = \mathrm{Ker}(\mathrm{Fr}^r_{Z_1/k})$. There exist positive integers n_4, c_4 such that*

$$\exp(p^{na}v) \equiv \bar{\mathfrak{z}}_{na}[v] * \mathrm{id}_{E_{na}} \pmod{\mathfrak{m}^{(p^{\min(ns, 2nr - c_4)})}}$$

for all $n \geq n_4$.

Corollary 10.4.5 is an easy consequence of 10.4.3.

10.5. Hypocotyl elongation in tempered perfections

The main results in section 10.5 are proposition 10.5.3 and theorem 10.5.6. The are generalizations of 10.5.3 and theorem 7.2.1, to tempered perfections of formal power series rings and augmented complete Noetherian local domains respectively. See 10.7 for the definitions and basic properties of tempered perfections.

The base field κ in this section is a perfect field of characteristic p, unless stated otherwise.

10.5.1. We reproduce some notations related to tempered perfections for the convenience of the readers.

1. Let $E, C > 0$, $d \geq 0$ be real numbers. The support subset

$$\mathrm{supp}(m : \flat : E; C, d) \subseteq \mathbb{N}[\tfrac{1}{p}]^m$$

with parameters $(E; C, d)$ defined in 10.7.3.6, and abbreviated to $\mathrm{supp}(m : E; C, d)$ in this subsection, is

$$\mathrm{supp}(m : E; C, d) = \mathrm{supp}(m : \flat : E; C, d) = \left\{ I \in \mathbb{N}[\tfrac{1}{p}]^m : |I|_p \leq C \cdot (|I|_\sigma + d)^E \right\}$$

2. Let $\underline{x} = (x_1, \ldots, x_m)$ be a tuple of variables,

(i) The total degree of monomials in $\underline{x}$ gives rise to a decreasing filtration

$$\mathrm{Fil}^{\geq \bullet}_{\mathrm{t.deg}}$$

on $\kappa\langle\langle x_1, \ldots, x_m \rangle\rangle^{E,\flat}_{C;\,d}$, indexed by real numbers:

$$\mathrm{Fil}^{\geq u}_{\mathrm{t.deg}}\big(\kappa\langle\langle x_1, \ldots, x_m \rangle\rangle^{E,\flat}_{C;\,d}\big) := \left\{ \sum_{I \in \mathrm{supp}(m:E;C,d)} a_I \cdot \underline{x}^I \;\middle|\; a_J \in \kappa \;\; \forall I, \;\; a_I = 0 \;\text{ if }\; |I|_\sigma < u \right\}$$

for every $u \in \mathbb{R}$.

(ii) For every real number u, define $\mathrm{Fil}^{>u}_{\mathrm{t.deg}}$ by

$$\mathrm{Fil}^{>u}_{\mathrm{t.deg}}\big(k\langle\langle x_1,\ldots,x_m\rangle\rangle^{E,\flat}_{C;\,d}\big) := \Big\{ \textstyle\sum_{I\in\mathrm{supp}(m:E;C,d)} a_I\cdot \underline{x}^I \;\Big|\; a_J\in\kappa \;\; \forall I, \;\; a_I = 0 \;\text{ if }\; |I|_\sigma \le u \Big\}.$$

The following lemma deals with the perfection

$$\kappa[x_1^{p^{-\infty}},\ldots,x_m^{p^{-\infty}}] = \cup_{n\in\mathbb{N}}\, \kappa[x_1^{p^{-n}},\ldots,x_m^{p^{-n}}]$$

of the polynomial ring $\kappa[x_1,\ldots,x_m]$ over the perfect base field κ. Notice that one can evaluate any element of $\kappa[x_1^{p^{-\infty}},\ldots,x_m^{p^{-\infty}}]$ at any m-tuple $(c_1,\ldots,c_m)\in\kappa^m$. Lemma 10.5.2 provides a dichotomy when an element $F(x_1,\ldots,x_m)\in\kappa[x_1^{p^{-\infty}},\ldots,x_m^{p^{-\infty}}]$ is evaluated at all Fr_q-powers

$$\{\,(c_1^{q^n},\ldots,c_m^{q^n})\;:\; n\in\mathbb{N}$$

of a gien m-tuple $(c_1,\ldots,c_m)$, where $q=p^r$ is a power of p, $r\in\mathbb{N}_{>0}$:

- either $F(c_1^{q^n},\ldots,c_m^{q^n})=0$ for infinitely many natural numbers,
- or $F(c_1^{q^n},\ldots,c_m^{q^n})=0$ for all $n\in\mathbb{Z}$.

10.5.2. Lemma. *Let r be a positive integer, and let $q=p^r$. Let $F(x_1,\ldots,x_m)$ be an element of $\kappa[x_1^{p^{-\infty}},\ldots,x_m^{p^{-\infty}}]$. Suppose that $(c_1,\ldots,c_m)\in\kappa^m$ is an element of κ^m and n_0 is a natural number such that*

$$F(c_1^{q^n},\ldots,c_n^{q^n})=0$$

for all integers $n\ge n_0$. Then $F(c_1^{q^n},\ldots,c_n^{q^n})=0$ for all $n\in\mathbb{Z}$. In particular

$$F(c_1,\ldots,c_n)=0.$$

Proof. When $F(x_1,\ldots,x_n)\in\kappa[x_1,\ldots,x_n]$, this statement was proved in 7.2.3.1; see also [9, 2.2]. The general case follows because there exists a positive integer i such that $F(x_1,\ldots,x_m)^{p^i}\in\kappa[x_1,\ldots,x_m]$. □

10.5.3. Proposition. *Let $\underline{x}=(x_1,\ldots,x_m)$, $\underline{y}=(y_1,\ldots,y_m)$, $\underline{u}=(u_1,\ldots,a)$ and $\underline{v}=(v_1,\ldots,v_b)$ be four tuples of variables. Let $(E_1;C_1,d_1)$ and $(E_2;C_2,d_2)$ be two triples of real parameters with $E_1,E_2>0$ and $C_1,C_2,d_1,d_2\ge 1$. Let*

$$f(\underline{u},\underline{v})\in\kappa\langle\langle u_1^{p^{-\infty}},\ldots,u_a^{p^{-\infty}},v_1^{p^{-\infty}},\ldots,v_b^{p^{-\infty}}\rangle\rangle^{E_1,\flat}_{C_1;\,d_1}$$

be an element of $\kappa\langle\langle u_1^{p^{-\infty}},\ldots,u_a^{p^{-\infty}},v_1^{p^{-\infty}},\ldots,v_b^{p^{-\infty}}\rangle\rangle^{E_1,\flat}_{C_1;\,d_1}$ such that the support $\mathrm{supp}(f)$ *of f is contained in the product* $\mathrm{supp}(a:E_1;C_1,d_1)\times\mathrm{supp}(b:E_1;C_1,d_1)$*:*

$$\mathrm{supp}(f)\subseteq\mathrm{supp}(a:E_1;C_1,d_1)\times\mathrm{supp}(b:E_1;C_1,d_1). \tag{10.5.3.1}$$

In other words f lies in the closure in $\kappa\langle\langle u_1^{p^{-\infty}},\ldots,u_a^{p^{-\infty}},v_1^{p^{-\infty}},\ldots,v_b^{p^{-\infty}}\rangle\rangle^{E_1,\flat}_{C_1;\,d_1}$ of the subring

$$\kappa\langle\langle u_1^{p^{-\infty}},\ldots,u_a^{p^{-\infty}}\rangle\rangle^{E_1,\flat}_{C_1;\,d_1}\otimes_\kappa\kappa\langle\langle v_1^{p^{-\infty}},\ldots,v_b^{p^{-\infty}}\rangle\rangle^{E_1,\flat}_{C_1;\,d_1}.$$

Let

$$(g_1(\underline{x}),\dots,g_a(\underline{x}))\in\big(\mathrm{Fil}^{>0}_{\mathrm{t.deg}}\kappa\langle\langle x_1^{p^{-\infty}},\dots,x_m^{p^{-\infty}}\rangle\rangle^{E_2,\flat}_{C_2;\,d_2}\big)^a$$

be an a-tuple of elements in $\kappa\langle\langle x_1^{p^{-\infty}},\dots,x_m^{p^{-\infty}}\rangle\rangle^{E_2,\flat}_{C_2;\,d_2}$ whose constant terms are 0. Let

$$(h_1(\underline{y}),\dots,h_b(\underline{y}))\in\big(\mathrm{Fil}^{>0}_{\mathrm{t.deg}}k\langle\langle y_1^{p^{-\infty}},\dots,y_m^{p^{-\infty}}\rangle\rangle^{E_2,\flat}_{C_2;\,d_2}\big)^b$$

be a b-tuple of elements in $k\langle\langle y_1^{p^{-\infty}},\dots,y_m^{p^{-\infty}}\rangle\rangle^{E_2,\flat}_{C_2;\,d_2}$ whose constant terms are 0. Let $q=p^r$ be a power of p, where $r>0$ is a positive integer. Let n_0 be a natural number. Suppose that there exists a sequence $(d_n)_{n\geq n_0}$ of natural numbers such that

$$\lim_{n\to\infty}\frac{q^n}{d_n}=0 \tag{10.5.3.2}$$

and

$$f(g_1(\underline{x}),\dots,g_a(\underline{x}),h_1(\underline{x})^{q^n},\dots,h_b(\underline{x})^{q^n})\equiv 0\pmod{\mathrm{Fil}^{d_n}_{\mathrm{t.deg}}}\qquad\forall\,n\geq n_0. \tag{10.5.3.3}$$

Then

$$f(g_1(\underline{x}),\dots,g_a(\underline{x}),h_1(\underline{y}),\dots,h_b(\underline{y}))=0. \tag{10.5.3.4}$$

In the above the congruence relation 10.5.3.3 takes place in $\kappa\langle\langle x_1^{p^{-\infty}},\dots,x_m^{p^{-\infty}}\rangle\rangle^{E_3,\flat}_{C_3;\,d_3}$, and the equation 10.5.3.4 holds in the ring $\kappa\langle\langle x_1^{p^{-\infty}},\dots,x_m^{p^{-\infty}},y_1^{p^{-\infty}},\dots,y_m^{p^{-\infty}}\rangle\rangle^{E_3,\flat}_{C_3;\,d_3}$, where

- *$E_3=E_1+E_2+E_1E_2$,*
- *$C_3=C_1^{1+E_2}\cdot C_2^{1+E_1+E_1E_2}\cdot(1+d)^{E_1E_2(1+E_2)}$, and*
- *d_3 is a sufficiently large constant depending on $(E_1;C_1,d_1)$ and $(E_2;C_2,d_2)$.*

See 10.7.6.5 and the trivial lower bound for e_2 there.

10.5.4. Proof of proposition 10.5.3. Let

$$\underline{t}=(t_{i,j})_{(i,j)\in\{1,\dots,b\}\times(\mathrm{supp}(m:E_2;C_2,d_2)\smallsetminus\underline{0})}$$

be an infinite array of variables indexed by $\{1,\dots,b\}\times\big(\mathrm{supp}(m:E_2;C_2,d_2)\smallsetminus\{\underline{0}\}\big)$, where $\underline{0}$ is the zero element of the support subset $\mathrm{supp}(m:E_2;C_2,d_2)\subseteq\mathbb{N}[1/p]^m$ defined in 10.5.1. For each $i=1,\dots,b$,

$$h_i(\underline{y})=\sum_{\underline{0}\neq K\in\mathrm{supp}(m:E_2;C_2,d_2)}c_{i,K}\ \underline{y}^J$$

with $c_{i,K}\in k$ for all $J\in S(m:E_2;C_2,d_2)\smallsetminus\{\underline{0}\}$. Let

$$H_i(\underline{t};\underline{y}):=\sum_{\underline{0}\neq K\,\in\,\mathrm{supp}(m:E_2;C_2,d_2)}\underline{t}_{i,K}\ \underline{y}^K$$

The assumption 10.5.3.1 implies that the composition

$$f\big(g_1(\underline{x}),\dots,g_a(\underline{x}),H_1(\underline{t};\underline{y}),\dots,H_1(\underline{t};\underline{y})\big)$$

is a well-defined formal series $\kappa\langle\langle x_1^{p^{-\infty}},\dots,x_m^{p^{-\infty}},y_1^{p^{-\infty}},\dots,y_m^{p^{-\infty}}\rangle\rangle_{C_3;\,d_3}^{E_3,\flat}$ whose support is contained in the product $\mathrm{supp}(m:E_3;C_3,d_3)\times\mathrm{supp}(m:E_3;C_3,d_3)$:

$$(10.5.4.1)\qquad f\big(\underline{g}(\underline{x}),\underline{H}(\underline{t};\underline{y})\big)=\sum_{(I,J)\in\mathrm{supp}(m:E_3;C_3,d_3)\times\mathrm{supp}(m:E_3;C_3,d_3)}A_{I,J}(\underline{t})\ \underline{x}^I\underline{y}^J$$

Moreover each coefficient $A_{I,J}(\underline{t})$ is an element in the perfection

$$\kappa[\underline{t}^{p^{\infty}}]=\kappa\big[t_{i,K}^{p^{-\infty}}\big]_{i\in\{1,\dots,b\},\,K\in\mathrm{supp}(m:E_2;C_2,d_2)\smallsetminus\{\underline{0}\}}$$

of the polynomial ring

$$\kappa[\underline{t}^{p^{-\infty}}]=\kappa[t_{i,K}]_{i\in\{1,\dots,b\},\,K\in\mathrm{supp}(m:E_2;C_2,d_2)\smallsetminus\{\underline{0}\}}$$

in infinitely many variables $t_{i,K}$. Clearly For every $n\in\mathbb{N}$, we have

$$(10.5.4.2)\qquad f\big(\underline{g}_1(\underline{x}),\dots,\underline{g}_a(\underline{x}),\underline{h}_1(\underline{x})^{q^n},\dots,\underline{h}_b(\underline{x})^{q^n}\big)=\sum_{I,J}A_{I,J}(\underline{c}^{q^n})\ \underline{x}^{I+q^nJ}\,.$$

In particular

$$(10.5.4.3)\qquad f\big(\underline{g}_1(\underline{x}),\dots,\underline{g}_a(\underline{x}),\underline{h}_1(\underline{x}),\dots,\underline{h}_b(\underline{x})\big)=\sum_{I,J}A_{I,J}(\underline{c})\ \underline{x}^{I+J}\,.$$

By assumption 10.5.3.2, we get

$$(10.5.4.4)\qquad \sum_{(I,J)\ \text{s.t.}\ |I+q^nJ|_\sigma<d_n}A_{I,J}(\underline{c}^{q^n})\ \underline{x}^{I+q^nJ}=0\qquad\forall\,n\geq n_0\,.$$

We want to show that $A_{I,J}(\underline{c})=0$ for all $(I,J)\in\mathrm{supp}(m:E_3;C_3,d_3)\times\mathrm{supp}(m:E_3;C_3,d_3)$. Suppose to the contrary that $A_{I_0,J_0}(\underline{c})\neq 0$ for some $(I_0,J_0)\in\mathrm{supp}(m:E_3;C_3,d_3)\times\mathrm{supp}(m:E_3;C_3,d_3)$. By lemma 10.5.2, there exist infinitely many natural numbers n such that $A_{I_0,J_0}(\underline{c}^{q^n})\neq 0$. Define a subset

$$T\subseteq\mathrm{supp}(m:E_3;C_3,d_3)\times\mathrm{supp}(m:E_3;C_3,d_3)$$

by

$$T:=\big\{(I,J)\,:\ I,J\in\mathrm{supp}(m:E_3;C_3,d_3),\ A_{I,J}(\underline{c}^{q^n})\neq 0\ \text{ for infinitely many } n\in\mathbb{N}\big\}\,.$$

This set T is non-empty because it contains (I_0,J_0). Again by lemma 10.5.2 we know that

$$A_{I,J}(\underline{c}^{q^n})=0\qquad\forall\,n\in\mathbb{Z}\ \text{ if }\ (I,J)\notin T\,,$$

and equation 10.5.4.5 becomes

$$(10.5.4.5)\qquad \sum_{(I,J)\in T\ \text{s.t.}\ |I+q^nJ|_\sigma<d_n}A_{I,J}(\underline{c}^{q^n})\ \underline{x}^{I+q^nJ}=0\qquad\forall\,n\geq n_0\,.$$

Let

$$M_2:=\min\{\,|J|_\sigma:\ (I,J)\in T\}$$

and let

$$M_1:=\min\{\,|I|_\sigma:\ (I,J)\in T\ \text{ and }\ |J|_\sigma=M_2\}\,.$$

The minimum which definesM_2 (respectively M_1) exists because every subset $\mathrm{supp}(m : E_3; C_3, d_3)$ whose archimedean norm is bounded above is a finite set. This finiteness property for $\mathrm{supp}(m : E_3; C_3, d_3)$ also implies that there exists a positive number $\epsilon_2 > 0$ such that

$$J \in \mathrm{supp}(m : E_3; C_3, d_3) \ \text{ and } \ |J|_\sigma > M_2 \implies |J|_\sigma > M_2 + \epsilon_2. \tag{10.5.4.6}$$

The subset

$$T_1 := \{(I, J) \in T \ : \ |J|_\sigma = M_2,\ |I|_\sigma = M_1\}$$

is a non-empty finite set. There exists a natural number $n_1 \geq n_0$ such that properties 10.5.4.7–10.5.4.9 below hold.

$$M_1 + q^n M_2 < d_n - 2 \quad \forall\ n \geq n_1,\ n \in \mathbb{N} \tag{10.5.4.7}$$

$$q^n \cdot \epsilon_2 > M_1 \quad \forall n \geq n_1,\ n \in \mathbb{N} \tag{10.5.4.8}$$

(10.5.4.9)

$$(I_1, J_1), (I_2, J_2) \in T_1, \quad I_1 + q^n J_1 = I_2 + q^n J_2 \ \text{ and } \ n \geq n_1 \implies (I_1, J_1) = (I_2, J_2)$$

Consider the set

$$S_n := \left\{ (I, J) \in T \ : \ |I + q^n J|_\sigma = M_1 + q^n M_2 \right\}.$$

The property 10.5.4.8 and the inequality 10.5.4.6 imply that $S_n = T_1$ for all $n \geq n_1$. Because $S_n = T_1$, when we examine terms of total degree $M_1 + q^n M_2$ in equation 10.5.4.5, we find that

$$\sum_{(I,J)\in T_1} A_{I,J}(\underline{c}^{q^n})\ \underline{x}^{I+q^n J} = 0 \qquad \forall\, n \geq n_1\,. \tag{10.5.4.10}$$

By property 10.5.4.9 and equation 10.5.4.9, we see that

$$A_{I,J}(\underline{c}^{q^n}) = 0$$

for all $(I, J) \in T_1$ and all $n \geq n_1$, therefore T_1 is the empty set. This is a contradiction. We have proved proposition 10.5.3. ∎

Remark. (a) The assumption 10.5.3.1 on the support of $f(\underline{u}, \underline{v})$ implies the uniform bound 10.5.4.1 on the support of the composition $f\big(g_1(\underline{x}), \ldots, g_a(\underline{x}), H_1(\underline{t}; \underline{y}), \ldots, H_b(\underline{t}; \underline{y})\big)$. This observation allows us to take advantage of the finiteness property of the support set $\mathrm{supp}(m; E_3; C_3, d_3)$. The rest of the argument in the proof of 10.5.3 is identical with the proof of [9, 3.1].

(b) For application to orbital rigidity of biextensions of p-divisible formal groups, we will need only the special case of 10.5.3 when $f(\underline{u}, \underline{v}) \in \kappa[[u_1, \ldots, u_a, v_1, \ldots, v_b]]$, i.e. $f(\underline{u}, \underline{v})$ is a usual power series.

(c) Our proof is not strong enough to show that 10.5.3 holds for every element f in $k\langle\langle \underline{u}^{p^{-\infty}}, \underline{u}^{p^{-\infty}} \rangle\rangle^{E_1, \flat}_{C_1; d_1}$. But we don't have a counter-example either. It will be interesting if one can find a larger class of formal series $f(\underline{u}, \underline{v})$ in $k\langle\langle \underline{u}^{p^{-\infty}}, \underline{u}^{p^{-\infty}} \rangle\rangle^{E_1, \flat}_{C_1; d_1}$ for which the statement 10.5.3 holds.

10.5.5. The setup of theorem 10.5.6.

1. Let $(R, \mathfrak{m})$ be an augmented complete Noetherian local domain over a perfect field κ of characteristic p. Let $(R, \mathfrak{m})^{\mathrm{perf}, \flat}_{A,b;d}$ be a tempered perfection of R, where A, b, d are real numbers, $A, b > 0$, and $d \geq b$. See 10.7.4.2 for the definition of $(R, \mathfrak{m})^{\mathrm{perf}, \flat}_{A,b;d}$.

2. The tempered perfection $(R, \mathfrak{m})^{\mathrm{perf}, \flat}_{A,b;d}$ of $(R, \mathfrak{m})$ carries a filtration

$$\left(\mathrm{Fil}^{\bullet}_{(R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}, \deg}\right)_{\bullet},$$

which is induced by the filtration $\mathrm{Fil}^{\bullet}_{R^{\mathrm{perf}}, \deg}$ on the perfection R^{perf} of R. See 10.7.4.2 for details.

3. Let $m, m' > 0$ be a positive integers, and let

$$\kappa\langle\langle \underline{u}^{p^{-\infty}}, \underline{v}^{p^{-\infty}} \rangle\rangle^{E, \flat}_{C;\, d} = \kappa\langle\langle u_1^{p^{-\infty}}, \dots, u_m^{p^{-\infty}}, v_1^{p^{-\infty}}, \dots, v_{m'}^{p^{-\infty}} \rangle\rangle^{E, \flat}_{C;\, d}$$

be a tempered perfection of $\kappa[[\underline{u}, \underline{v}]] = \kappa[[u_1, \dots, u_m, v_1, \dots, v_{m'}]]$, where E, C, d are real numbers, $E, C > 0$, and $d \geq 0$.

4. Let $g_1, \dots, g_m, h_1, \dots, h_{m'}$ be elements of the maximal ideal of $(R, \mathfrak{m})^{\mathrm{perf}, \flat}_{A,b;d}$.

5. Let $A' > 0, b' > 0, d' \geq b'$ be real numbers such that the following conditions hold.
   - The continuous ring homomorphism

   $$\mathrm{ev}_{\underline{g}\otimes 1, 1\otimes \underline{h}} : \kappa[[u_1, \dots, u_m, v_1, \dots, v_{m'}]] \longrightarrow \left(R\hat{\otimes}_{\kappa} R, \mathfrak{m}_{R\hat{\otimes}_{\kappa} R}\right)^{\mathrm{perf}, \flat}_{A,b;d}$$

   which sends a typical formal power series

   $$f(u_1, \dots, u_m, v_1, \dots, v_{m'}) \in \kappa[[u_1, \dots, u_m, v_1, \dots, v_{m'}]]$$

   to

   $$f(g_1 \otimes 1, \dots, g_m \otimes 1, 1 \otimes h_1, \dots, 1 \otimes h_{m'}) \in \left(R\hat{\otimes}_{\kappa} R, \mathfrak{m}_{R\hat{\otimes}_{\kappa} R}\right)^{\mathrm{perf}, \flat}_{A,b;d},$$

   extends to a continuous ring homomorphism

   $$\mathrm{ev}_{\underline{g}\otimes 1, 1\otimes \underline{h}} : \kappa\langle\langle \underline{u}^{p^{-\infty}}, \underline{v}^{p^{-\infty}} \rangle\rangle^{E, \flat}_{C;\, d} \longrightarrow \left(R\hat{\otimes}_{\kappa} R, \mathfrak{m}_{R\hat{\otimes}_{\kappa} R}\right)^{\mathrm{perf}, \flat}_{A',b';d'}.$$

   The existence of such a triple (A', b', d') is straight-forward from the definitions. See 10.7.6.5 for case when $(R, \mathfrak{m})$ is a formal power series ring.
   - The continuous ring homomorphism

   $$\mathrm{ev}_{\underline{g}, \underline{h}} : \kappa[[u_1, \dots, u_m, v_1, \dots, v_{m'}]] \longrightarrow (R, \mathfrak{m})^{\mathrm{perf}, \flat}_{A,b;d},$$

   which sends a typical formal power series

   $$f(u_1, \dots, u_m, v_1, \dots, v_{m'}) \in \kappa[[u_1, \dots, u_m, v_1, \dots, v_{m'}]]$$

   to

   $$f(g_1, \dots, g_m, h_1, \dots, h_{m'}) \in (R, \mathfrak{m})^{\mathrm{perf}, \flat}_{A,b;d},$$

extends to a continuous ring homomorphism

$$\mathrm{ev}_{\underline{g},1\underline{h}} : \kappa\langle\langle \underline{u}^{p^{-\infty}}, \underline{v}^{p^{-\infty}} \rangle\rangle_{C;\,d}^{E,\,\flat} \longrightarrow (R,\mathfrak{m})_{A',b';d'}^{\mathrm{perf},\,\flat}.$$

- The diagram

$$\begin{array}{ccc} \kappa\langle\langle \underline{u}^{p^{-\infty}}, \underline{v}^{p^{-\infty}} \rangle\rangle_{C;\,d}^{E,\,\flat} & \xrightarrow{\mathrm{ev}_{\underline{g}\otimes 1, 1\otimes \underline{h}}} & \big(R\hat{\otimes}_\kappa R, \mathfrak{m}_{R\hat{\otimes}_\kappa R}\big)_{A',b';d'}^{\mathrm{perf},\,\flat} \\ \Big\downarrow{=} & & \Big\downarrow{\Delta^*} \\ \kappa\langle\langle \underline{u}^{p^{-\infty}}, \underline{v}^{p^{-\infty}} \rangle\rangle_{C;\,d}^{E,\,\flat} & \xrightarrow{\mathrm{ev}_{\underline{g},\underline{h}}} & (R,\mathfrak{m})_{A',b';d'}^{\mathrm{perf},\,\flat} \end{array}$$

commutes, where the vertical arrow Δ^* is induced by the multiplication map $\Delta^* : R\otimes R \to R$ for the κ-algebra R.

6. For every element $f \in \kappa\langle\langle u_1^{p^{-\infty}}, \ldots, u_m^{p^{-\infty}}, v_1^{p^{-\infty}}, \ldots, v_{m'}^{p^{-\infty}} \rangle\rangle_{C;\,d}^{E,\,\flat}$, define elements

$$f(\underline{g},\underline{h}) \in (R,\mathfrak{m})_{A',b';d'}^{\mathrm{perf},\,\flat} \quad \text{and} \quad f(\underline{g}\otimes 1, 1\otimes \underline{h}) \in \big(R\hat{\otimes}_\kappa R, \mathfrak{m}_{R\hat{\otimes}_\kappa R}\big)_{A',b';d'}^{\mathrm{perf},\,\flat}$$

by

$$f(\underline{g},\underline{h}) = f(g_1,\ldots,g_m,h_1,\ldots,h_{m'}) := \mathrm{ev}_{\underline{g},\underline{h}}(f),$$
$$f(\underline{g}\otimes 1, 1\otimes \underline{h}) = f(g_1\otimes 1,\ldots,g_m\otimes 1, 1\otimes h_1,\ldots,1\otimes h_{m'}) := \mathrm{ev}_{\underline{g}\otimes 1,1\otimes\underline{h}}(f).$$

10.5.6. Theorem (Hypocoptyl elongation for tempered virtual functions). *We use the notation in 10.5.5. Let $(R,\mathfrak{m})$ be an augmented complete Noetherian local domain over a perfect field κ of characteristic p.*

- *Let $g_1,\ldots,g_m,h_1,\ldots,h_{m'}$ be elements of the maximal ideal of $(R,\mathfrak{m})_{A,b;d}^{\mathrm{perf},\,\flat}$.*
- *Let $f(u_1,\ldots,u_m,v_1,\ldots,v_{m'})$ be an element of*

$$\kappa\langle\langle u_1^{p^{-\infty}}, \ldots, u_m^{p^{-\infty}}, v_1^{p^{-\infty}}, \ldots, v_{m'}^{p^{-\infty}} \rangle\rangle_{C;\,d}^{E,\,\flat}$$

  which lies in the closure of the image of

$$\kappa\langle\langle \underline{u}^{p^{-\infty}} \rangle\rangle_{C;\,d}^{E,\,\flat} \otimes_\kappa \kappa\langle\langle \underline{v}^{p^{-\infty}} \rangle\rangle_{C;\,d}^{E,\,\flat} \longrightarrow \kappa\langle\langle \underline{u}^{p^{-\infty}}, \underline{v}^{p^{-\infty}} \rangle\rangle_{C;\,d}^{E,\,\flat}.$$

- *Let $q = p^r$ be a power of p for some positive integer r. Let $(d_n)_{n\in\mathbb{N},\, n\geq n_0}$ be a sequence of positive integers such that $\lim_{n\to\infty} \frac{q^n}{d_n} = 0$.*

Suppose that

$$(\dagger) \qquad f(g_1,\ldots,g_m,h_1^{q^n},\ldots,h_{m'}^{q^n}) \equiv 0 \quad \Big(\mathrm{mod}\ \mathrm{Fil}^{d_n}_{(R,\mathfrak{m})_{A',b';d'}^{\mathrm{perf},\,\flat},\mathrm{deg}}\Big)$$

in $(R,\mathfrak{m})_{A',b';d'}^{\mathrm{perf},\,\flat}$ for all $n \geq n_0$. Then

$$f(g_1\otimes 1,\ldots,g_m\otimes 1, 1\otimes h_1,\ldots,1\otimes h_{m'}) = 0$$

in the completed tempered perfection $\big(R\hat{\otimes}_\kappa R, \mathfrak{m}_{R\hat{\otimes}_\kappa R}\big)_{A',b';d'}^{\mathrm{perf},\,\flat}$ of $R\hat{\otimes}_\kappa R$.

PROOF. Extending the base field κ if necessary, we may and do assume that κ is algebraically closed. By 7.2.2.1, there exists a κ-linear injective local homomorphism $\iota : R \hookrightarrow \kappa[[t_1, \ldots, t_m]]$. By 10.7.6.4, the homomorphism

$$\iota^\flat : (R, \mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d} \longrightarrow (\kappa[[\underline{t}]], (\underline{t}))^{\mathrm{perf},\flat}_{A,b;d}$$

induced by ι is also an injection. So it suffices to show that

$$f(\iota^\flat(g_1) \otimes 1, \ldots, \iota^\flat(g_m) \otimes 1, 1 \otimes \iota^\flat(h_1), \ldots, 1 \otimes \iota^\flat(h_{m'})) = 0.$$

Moreover the congruence relations (†) implies that

$$(\ddagger) \qquad f(\iota^\flat(g_1), \ldots, \iota^\flat(g_m), \iota^\flat(h_1)^{q^n}, \ldots, \iota^\flat(h_{m'})^{q^n}) \equiv 0 \ \left(\mathrm{mod}\ \mathrm{Fil}^{d_n}_{(\kappa[[\underline{t}]],(\underline{t}))^{\mathrm{perf},\flat}_{A',b';d'},\deg}\right)$$

for all $n \geq n_0$. We know from 10.7.5 that there exist real numbers E_2, C_2, d_2 such that

$$(\kappa[[\underline{t}]], (\underline{t}))^{\mathrm{perf},\flat}_{A,b;d} \subseteq \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^{E_2,\flat}_{C_2;\, d_2}.$$

So we can apply proposition 10.5.3 and conclude that $f(\iota^\flat(\underline{g}) \otimes 1, 1 \otimes \iota^\flat(\underline{h})) = 0$. ∎

10.6. Orbital rigidity for bi-extensions of p-divisible formal groups

10.6.1. Notation and basic setup. In this section k is a perfect field of characteristic $p > 0$. The proofs of the main theorems 10.6.2 and other consequences of the argument are immediately reduced to the case when k is algebraically closed.

(i) Let X, Y, Z be p-divisible formal groups, and $\pi : E \to X \times Y$ is a biextension of $X \times Y$ by Z.

(ii) Let G be a compact p-adic Lie group. Let $(\rho, \alpha, \beta, \gamma)$ be an action of G on the biextension $E \to X \times Y$, where $\rho : G \to \mathrm{Aut}_{\mathrm{biext}}(E \to X \times Y)$ is a continuous injective homomorphism, and $\alpha : G \to \mathrm{Aut}(X)$ (respectively $\beta : G \to \mathrm{Aut}(Y)$, $\gamma : G \to \mathrm{Aut}(Z)$) is the action of G on X (respectively Y, Z) underlying ρ. We know from 10.2.7.3 that the group homomorphism

$$(\alpha, \beta, \gamma)\colon G \longrightarrow \mathrm{Aut}(X) \times \mathrm{Aut}(Y) \times \mathrm{Aut}(Z)$$

is a closed embedding of compact p-adic Lie groups, and the induced map

$$(d\alpha, d\beta, d\gamma) : \mathrm{Lie}(G) \longrightarrow \mathrm{End}(X)_{\mathbb{Q}} \oplus \mathrm{End}(Y)_{\mathbb{Q}} \oplus \mathrm{End}(Z)_{\mathbb{Q}}$$

is an injective homomorphism of finite dimensional Lie algebras over $\mathbb{Q}_p$. We often use the map (α, β, γ) to identify G with a subgroup of $\mathrm{Aut}(X) \times \mathrm{Aut}(Y) \times \mathrm{Aut}(Z)$, and regard $\mathrm{Lie}(G)$ as a $\mathbb{Q}_p$-vector subspace of $\mathrm{Lie}(G)\, \mathrm{End}(X)_{\mathbb{Q}} \oplus \mathrm{End}(Y)_{\mathbb{Q}} \oplus \mathrm{End}(Z)_{\mathbb{Q}}$.

(iii) Let $W \subseteq E$ be a formal subvariety of E, in the sense that there exists a prime ideal I_W of the coordinate ring R_E of E such that $W = \mathrm{Spf}(R_E/I_W)$. Assume that W is stable under the action of G.

(iv) The formal subscheme $V = \mathrm{Spf}\left(R_{X \times_{\mathrm{Spec}(k)} Y}/(I_W \cap R_{X\times Y})\right) \subseteq X \times_{\mathrm{Spec}(k)} Y$ will be called the *image* of W in $X \times_{\mathrm{Spec}(k)} Y$.

10.6.2. Theorem. *Let W be a formal subvariety of E stable under the action of G. Let μ_1 be the maximum of the slopes of Z. Assume that μ_1 is strictly bigger than every slope of X and every slope of Y. Let Z_1 be the maximal p-divisible subgroup of Z which is isoclinic of slope μ_1. Let Z_1' be a p-divisible subgroups of Z_1 which is contained in W and stable under the action of G. Let $\Upsilon_{Z_1'} : Z_1' \times E \to E$ be the morphism*

$$\Upsilon : Z_1' \times E \to E \qquad (z_1', e) \mapsto z_1' * e\,,$$

corresponding to the restriction to Z_1' of the action of Z on E. For every element $v = (A, B, C) \in \mathrm{End}(X) \oplus \mathrm{End}(Y) \oplus \mathrm{End}(Z)$ of the Lie algebra of G, we have

$$\big(\Upsilon \circ (C|_{Z_1'} \times \mathrm{id}_W)\big)(Z_1' \times W) \subseteq W.$$

In other words the formal subvariety $W \subseteq E$ is stable under translation by the p-divisible subgroup $C(Z_1')$ of Z.

Theorem 10.6.2 will be proved in 10.6.3.

10.6.2.1. Corollary. *In the situation of 10.6.2, assume in addition that the action of G on Z_1' is strongly nontrivial. Then*

$$\Upsilon(Z_1' \times W) \subseteq W\,.$$

Proof. The assumption that the action of G on Z_1' is strongly non-trivial implies that there exists elements $v_{ij} = (A_{ij}, B_{ij}, C_{ij}) \in \mathrm{Lie}(G)$, indexed by a finite subset

$$\big\{(i,j) \in \mathbb{N}^2 : \ i \in \{1, \ldots, m\},\ j \in \{1, \ldots, n_i\}\big\}\,,$$

where $n_i \in \mathbb{N}_{\geq 1}$ for each $i = 1, \ldots, m$, such that

$$\sum_{1 \leq i \leq m} C_{i,1}|_{Z_1'} \circ \cdots \circ C_{i,n_i}|_{Z_1'} \in \mathrm{End}(Z_1')_{\mathbb{Q}}^{\times}.$$

Here $C_{i,j} \in \mathrm{End}(Z_1')_{\mathbb{Q}}$ stands for the restriction to Z_1' of the element $C_{i,j} \in \mathrm{End}(Z)_{\mathbb{Q}} = \mathrm{End}(Z) \otimes_{\mathbb{Z}} \mathbb{Q}$. See [9, 4.1.1] for this lemma on representation theory. The statement (2) follows from statement (1) and the above linear algebra consequence of the assumption that G operates strongly non-trivially on Z_1'.

10.6.2.2. Corollary. *Let W be a formal subvariety of E stable under the action of G as in 10.6.2. Suppose that the largest slope μ_1 of Z is strictly bigger than every slope of $X \times Y$, and the action of G on Z is strongly nontrivial. Then the intersection $W \cap Z_1$ with reduced structure is a p-divisible subgroup of Z_1, and W is stable under the translation action by Z_1' via the Z-torsor structure of E. Here Z_1 is the largest isoclinic p-divisible subgroup of Z of slope μ_1 as in 10.6.2.*

Proof. We know from orbital rigidity of p-divisible groups 7.1.1 that $(W \cap Z_1)_{\mathrm{red}} = U_1 \cup \cdots U_m$, where each U_i is a p-divisible subgroup of Z_1. Corollary 10.6.2.1 implies that $W \cap Z_1$ is stable under the translation action of U_i for $i = 1, \ldots, m$. So $(W \cap Z_1)_{\mathrm{red}}$ is equal to the p-divisible subgroup $U_1 + \cdots + U_m$ of Z_1. ∎

10.6.3. PROOF OF THEOREM 10.6.2.

Step 1. Preliminary reduction steps.

(a) It suffices to verify the statement of 10.6.2 after extending the base field k to an algebraic closure of k. So we may and do assume that k is algebraically closed.

(b) If $E \to E'$ is an isogeny of biextensions, the statement of 10.6.2 holds for E if and only if it holds for E'.

Modifying E by suitable isogenies, we may and do assume that X, Y, Z are product of isoclinic p-divisible groups. Moreover we may assume that for each isotypic factor U of X, Y, or Z, there exist positive integers a', r' such that $U[p^{a'}] = U[F^{r'}] := \mathrm{Ker}(\mathrm{Fr}^{r'}_{U/k})$. In particular there exist positive integers a, r such that $\mu_1 = \frac{a}{r}$ and $Z_1[p^a] = Z_1[F^r] := \mathrm{Ker}(\mathrm{Fr}^r_{Z_1/k})$.

(c) Choose a suitable regular system of parameters $(u_1, \ldots, u_b)$ for the coordinate ring Z_1 such that $Z_1 = \mathrm{Spf}(k[[u_1, \ldots, u_b]])$ and

$$[p^a]^*_{Z_1}(u_i) = u_i^{p^r}$$

for $i = 1, \ldots, b$.

(d) The largest slope μ_1 of Z is assumed to be strictly bigger than every slope appearing in $X \times Y$. Multiplying a, r by a suitable positive integer, we may and do assume that there exists positive intergers s, n_0 such that $s > r$ and $\frac{a}{s}$ is strictly bigger than every slope of $X \times Y$, and

$$X[p^{na}] \supset \mathrm{Ker}(\mathrm{Fr}^{ns}_{X/k}) \quad \text{and} \quad Y[p^{na}] \supset \mathrm{Ker}(\mathrm{Fr}^{ns}_{Y/k})$$

for all $n \geq n_0$.

Step 2. By 10.4.1.4, after suitably adjusting the positive integers s, r, a with $s > r > a > 0$, $\mu_1 = \frac{a}{r}$, there exist positive integers $n_4 \geq n_0$ and c_4 such that

$$\exp(p^{na}v) \equiv \bar{\mathfrak{z}}_{na}[v] * \mathrm{id}_{E \bmod \mathfrak{m}} \pmod{\mathfrak{m}^{(p^{\min(ns, 2nr-c_4)})}} \tag{10.6.3.1}$$

for all $n \geq n_4$, where

$$\bar{\mathfrak{z}}_{na}[v] = \left(\mathrm{pr}_{Z_l} \circ \mathfrak{z}_{na}\right)\big|_{\pi^{-1}(\mathrm{Ker}(\mathrm{Fr}^{ns}_X) \times \mathrm{Ker}(\mathrm{Fr}^{ns}_Y))} : \pi^{-1}\big(\mathrm{Ker}(\mathrm{Fr}^{ns}_X) \times \mathrm{Ker}(\mathrm{Fr}^{ns}_Y)\big) \longrightarrow Z_1$$

is the restriction to $\pi^{-1}\big(\mathrm{Ker}(\mathrm{Fr}^{ns}_X) \times \mathrm{Ker}(\mathrm{Fr}^{ns}_Y)\big)$ of the composition of $\mathfrak{z}_{na}[v]$ with the projection

$$\mathrm{pr}_{Z_l} : Z \to Z_l.$$

For each $j = 1, \ldots, b$, defined a ϕ^r-compatible sequence $(a_{j,n})_{n \geq n_4}$ with respect to ϕ^s in the sense of 10.7.2.3, by

$$a_{j,n} := \bar{\mathfrak{z}}_{na}[v]^*(u_j) \bmod \mathfrak{m}_E^{(p^{ns})} \in R_E/\mathfrak{m}_E^{(p^{ns})}$$

for all $n \geq n_4$. Let $i_1 := \max\big(s - r, \lceil \frac{n_4}{r} \rceil\big)$. For each $j = 1, \ldots, b$, let

$$\tilde{a}_j \in (R_E, \mathfrak{m}_E)^{\mathrm{perf},\#}_{s:\phi^r;[i_1]}$$

be the formal series corresponding to the ϕ^r-compatible sequence $(a_{j,n})_{n \geq n_4}$.

Although $(R_E, \mathfrak{m}_E)^{\mathrm{perf},\#}_{s:\phi^r;[i_1]}$ is more tightly related to ϕ^r-compatible sequences through the construction in 10.7.3.2, we will pass to the larger ring $(R_E, \mathfrak{m}_E)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}$, and consider the $\tilde{a}_j$'s as elements of $(R_E, \mathfrak{m}_E)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}$ in the rest of the proof.

Step 3. The elements $\tilde{a}_1, \ldots, \tilde{a}_b \in (R_E, \mathfrak{m}_E)^{\mathrm{perf},\#}_{s:\phi^r;[i_1]}$ define a ring homomorphism

$$\tilde{\mathfrak{z}}[v]^* \colon R_{Z_1} = k[[u_1, \ldots, u_b]] \longrightarrow (R_E, \mathfrak{m}_E)^{\mathrm{perf},\#}_{s:\phi^r;[i_1]}\,.$$

Let

$$\omega_1 \colon\ (R_E, \mathfrak{m}_E)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} \longrightarrow (R_{Z_1}, \mathfrak{m}_{Z_1})^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}$$

be the ring homomorphism induced by the inclusion $Z_1 \hookrightarrow E$. Because the restriction to Z of the morphism $\mathfrak{z}_n[v] : \pi^{-1}(X[p^n] \times Y[p^n]) \to Z$ is equal to $[p^n]_Z \circ C|_Z$ for every $n \in \mathbb{N}$, We see that

$$\omega_1 \circ \tilde{\mathfrak{z}}[v] = j_{R_{Z_1}} \circ (C|_{Z_1})^* \tag{10.6.3.2}$$

where $j_{R_{Z_1}} : R_{Z_1} \hookrightarrow (R_{Z_1}, \mathfrak{m}_{Z_1})^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}$ is the natural injection from R_{Z_1} to its completed tempered perfection $(R_{Z_1}, \mathfrak{m}_{Z_1})^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}$.

Step 4. We also have the following ring homomorphisms.

(a) The canonical homomorphism $\tau : R_E \to R_E/I_W = R_W$ gives rise to a homomorphism

$$\tau^\flat : (R_E, \mathfrak{m}_E)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} \longrightarrow (R_W, \mathfrak{m}_W)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}\,.$$

(b) The injective local homomorphism $\iota : R_W \to k[[t_1, \ldots, t_m]]$ induces a injective continuous homomorphism

$$\tilde{\iota} : (R_W, \mathfrak{m}_W)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} \longrightarrow k\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^\flat_{s:\phi^r;[i_1]}\,.$$

(c) Continuous ring homomorphisms

$$\Delta_1 : R_E \to R_{Z_1} \widehat{\otimes} R_E \quad \text{and} \quad \Delta_1' : R_E \to R_{Z_1'} \widehat{\otimes} R_E$$

corresponding to the actions $Z_1 \times E \to E$ and $Z_1' \times E \to E$ of Z_1 and Z_1' on E.

(d) The ring homomorphism

$$\omega_1' : (R_W, \mathfrak{m}_W)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} \longrightarrow \left(R_{Z_1'}, \mathfrak{m}_{Z_1'}\right)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}\,.$$

induced by the surjective ring homomorphism $R_W \twoheadrightarrow R_{Z_1'}$ which corresponds to the inclusion $Z_1' \hookrightarrow W$.

(e) The ring endomorphisms $C|^*_{Z_1} : R_{Z_1} \to R_{Z_1}$ and $C|^*_{Z_1'} : R_{Z_1} \to R_{Z_1'}$ corresponding to the endomorphisms $C|_{Z_1}$ (respectively $C|_{Z_1'}$) of the p-divisible group Z_1 (respectively Z_1').

(f) The ring homomorphism

$$q^\flat : (R_{Z_1}, \mathfrak{m}_{Z_1})^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_1]} \longrightarrow \left(R_{Z_1'}, \mathfrak{m}_{Z_1'}\right)^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_1]}$$

induced by the canonical surjection $q : R_{Z_1} \twoheadrightarrow R_{Z_1'}$

Clearly we have

$$\omega_1' \circ \tau^\flat = q^\flat \circ \omega_1 \quad \text{and} \quad C|^*_{Z_1'} \circ q = q \circ C|^*_{Z_1} \tag{10.6.3.3}$$

The following diagram

$$\begin{tikzcd}
R_{Z_1} \arrow[r, "\tilde{\mathfrak{z}}{[v]}^*"] \arrow[d, "C|^*_{Z_1}"'] & (R_E, \mathfrak{m}_E)^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_1]} \arrow[r, "\tau"] \arrow[d, "\omega_1"'] & (R_W, \mathfrak{m}_W)^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_1]} \arrow[d, "\omega_1'"'] \\
R_{Z_1} \arrow[r, "j_{R_{Z_1}}"'] & (R_{Z_1}, \mathfrak{m}_{Z_1})^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_1]} \arrow[r, "q^\flat"'] & \left(R_{Z_1'}, \mathfrak{m}_{Z_1'}\right)^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_1]}
\end{tikzcd} \tag{10.6.3.4}$$

commutes by 10.6.3.2. It follows that the diagram
(10.6.3.5)

$$\begin{tikzcd}
 & R_{Z_1'}\hat{\otimes} R_E \arrow[r, "C|^*_{Z_1'}\otimes\tau"] \arrow[d, "1_{R_{Z_1'}}\otimes\tau"] & R_{Z_1'}\hat{\otimes} R_W \arrow[d, "="] \\
R_E \arrow[ur, "\Delta_1'"] \arrow[d, "\Delta_1"'] & R_{Z_1'}\hat{\otimes} R_W \arrow[r, "C|^*_{Z_1'}\otimes 1_{R_W}"] & R_{Z_1'}\hat{\otimes} R_W \arrow[d, "j_{R_{Z_1'}}\otimes j_{R_W}"] \\
R_{Z_1}\hat{\otimes} R_E \arrow[ur, "q\otimes 1"] \arrow[d, "\tilde{\mathfrak{z}}{[v]}^*\otimes j_{R_E}"'] \arrow[drr, "(j_{R_{Z_1}}\circ C|^*_{Z_1})\otimes(\tau^\flat\circ j_{R_E})"] & & \left(R_{Z_1'}, \mathfrak{m}_{Z_1'}\right)^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_1]} \hat{\otimes} (R_W, \mathfrak{m}_W)^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_1]} \\
(R_E, \mathfrak{m}_E)^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_1]} \hat{\otimes} (R_E, \mathfrak{m}_E)^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_1]} \arrow[rr, "\omega_1\otimes\tau^\flat"] \arrow[d, "\tau^\flat\otimes\tau^\flat"'] & & (R_{Z_1}, \mathfrak{m}_{Z_1})^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_1]} \hat{\otimes} (R_W, \mathfrak{m}_W)^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_1]} \arrow[u, "q^\flat\otimes 1"'] \arrow[d, "q^\flat\otimes 1"] \\
(R_W, \mathfrak{m}_W)^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_1]} \hat{\otimes} (R_W, \mathfrak{m}_W)^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_1]} \arrow[rr, "\omega_2\otimes 1"] & & \left(R_{Z_1'}, \mathfrak{m}_{Z_1'}\right)^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_1]} \hat{\otimes} (R_W, \mathfrak{m}_W)^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_1]}
\end{tikzcd}$$

also commutes.

Step 5. Recall that I_W is the prime ideal of the coordinate ring of E consisting of all functions on E which vanishes on the G-invariant formal subvariety $W \subseteq E$. We want to show that

$$(C|^*_{Z_1'} \otimes \tau) \circ \Delta_1'(f) = 0 \quad \forall\, \chi \in I_W \tag{A}$$

We know from diagram (10.6.3.5) that

$$\big((C|^*_{Z_1'} \otimes \tau) \circ \Delta_1'\big)(\chi) = \big((q^\flat \otimes 1) \circ ((j_{R_{Z_1}} \circ C|^*_{Z_1}) \otimes (\tau^\flat \circ j_{R_E})) \circ \Delta_1\big)(\chi).$$

Because $j_{R_{Z_1'}}$ and j_{R_W} are both injective, our goal (A) is to equivalent to

(B) $$\big((q^\flat \otimes 1) \circ ((j_{R_{Z_1}} \circ C|^*_{Z_1}) \otimes (\tau^\flat \circ j_{R_E})) \circ \Delta_1\big)(\chi) = 0 \quad \forall\, \chi \in I_W.$$

The commutative diagram (10.6.3.5) tells us that

$$\begin{aligned}(q^\flat \otimes 1) \circ \big((j_{R_{Z_1}} \circ C|^*_{Z_1}) \otimes (\tau^\flat \circ j_{R_E})\big) \circ \Delta_1 &= (\omega_1 \otimes \tau) \circ (\tilde{\mathfrak{z}}[v]^* \otimes j_{R_E}) \circ \Delta_1 \\ &= (\omega_2 \otimes 1) \circ (\tau \otimes \tau) \circ (\tilde{\mathfrak{z}}[v]^* \otimes j_{R_E}) \circ \Delta_1.\end{aligned}$$

We will show the stronger statement

(C) $$\big((\tau \otimes \tau) \circ (\tilde{\mathfrak{z}}[v]^* \otimes j_{R_E}) \circ \Delta_1\big)(\chi) = 0 \quad \forall\, \chi \in I_W.$$

In other words, the composition of the three vertical arrows at the left edge of the diagram (10.6.3.5) kills every element of the prime ideal I_W. Since

$$\text{(C)} \implies \text{(B)} \iff \text{(A)},$$

it suffices to prove (C).

Step 6. Suppose that χ is an element of I_W. Define an element

$$f_\chi \in (R_E, \mathfrak{m}_E)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} \,\hat{\otimes}\, (R_E, \mathfrak{m}_E)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}$$

by

$$f_\chi := \big((\tilde{\mathfrak{z}}[v]^* \otimes j_{R_E}) \circ \Delta_1\big)(\chi),$$

where $\tilde{\mathfrak{z}}[v]^* \otimes j_{R_E} \circ \Delta_1$ is the composition

$$R_{Z_1} \hat{\otimes} R_E \xrightarrow{\Delta_1} R_E \hat{\otimes} R_E \xrightarrow{\tilde{\mathfrak{z}}[v]^* \otimes j_{R_E}} (R_E, \mathfrak{m}_E)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} \,\hat{\otimes}\, (R_E, \mathfrak{m}_E)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}\ .$$

We want to show the image of f_χ under the map

$$(R_E, \mathfrak{m}_E)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} \,\hat{\otimes}\, (R_E, \mathfrak{m}_E)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} \xrightarrow{\tau^\flat \otimes \tau^\flat} (R_W, \mathfrak{m}_W)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} \,\hat{\otimes}\, (R_W, \mathfrak{m}_W)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}$$

is 0.

Step 7. Let ϕ be the Frobenius endomorphism $x \mapsto x^p$ on $(R_W, \mathfrak{m}_W)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}$, Let

$$\nu_W \colon\ (R_W, \mathfrak{m}_W)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} \,\hat{\otimes}\, (R_W, \mathfrak{m}_W)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} \longrightarrow (R_W, \mathfrak{m}_W)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}$$

be map which defines multiplication for the ring $(R_W, \mathfrak{m}_W)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}$. Geometrically ν_W corresponds to the diagonal morphism from $\mathrm{Spec}((R_W, \mathfrak{m}_W)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]})$ to its self-product.

Because the formal subvariety $W \subseteq E$ is assumed to be stable under G, therefore stable under $\exp(p^{na}v)$ for all $n \geq n_4$. Hence the congruence relations (10.6.3.1) implies that

$$(10.6.3.6) \qquad (\phi^{nr}\otimes 1)\big((\tau^\flat\otimes\tau^\flat)(f_\chi)\big) \equiv 0 \pmod{\mathrm{Fil}_\flat^{ns-i_1}} \qquad \forall\, n \geq n_4,$$

where $\phi^{nr}\otimes 1$ is the ring homomorphism

$$\phi^{nr}\otimes 1\colon\ (R_W,\mathfrak{m}_W)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}\,\hat{\otimes}\,(R_W,\mathfrak{m}_W)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} \longrightarrow (R_W,\mathfrak{m}_W)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}\,\hat{\otimes}\,(R_W,\mathfrak{m}_W)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}\,.$$

Applying theorem 10.5.6 on hypocoptyl elongation for tempered virtual functions, we conclude that

$$(\tau^\flat\otimes\tau^\flat)(f_\chi) = 0$$

in $(R_W,\mathfrak{m}_W)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}\,\hat{\otimes}\,(R_W,\mathfrak{m}_W)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}$, for every element $\chi\in I_W$, which is the statement (C) in step 5. As we have seen, this implies that

$$(C|^*_{Z_1'}\otimes 1)(\Delta_1'(\chi) = 0$$

in $R_{Z_1'}\hat{\otimes}R_W$ for every element χ of the ideal I_W. We have proved theorem 10.6.2. $\square$

10.6.3.1. Remark. In the situation of 10.6.2.2, the conclusion of 10.6.2.2 implies that the natural formal morphism

$$\bar{\pi}|_{W/Z_1''} : (W/Z_1'') \longrightarrow E/Z_1 = (Z \twoheadrightarrow Z/Z_1)_*E$$

is finite, because the closed fiber of $\bar{\pi}|_{W/Z_1''}$ is finite. However we will need the stronger statement 10.6.4.4 that the formal morphism from W/Z_1 to the schematic image of $\bar{\pi}|_{W/Z_1''}$ induced by $\bar{\pi}|_{W/Z_1''}$ is finite and *purely inseparable*.

This stronger statement will follow from the method used in the proof of theorem 10.6.2. More precisely, we will further exploit the Z_1-equivariant "virtual morphisms"

$$\tilde{\mathfrak{z}}[v] : E \to Z_1,$$

which correspond to continuous ring homomorphisms

$$R_{Z_1} \xrightarrow{\tilde{\mathfrak{z}}[v]^*} (R_E,\mathfrak{m}_E)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}\,.$$

See 10.6.4.1 for the terminology "virtual morphism". The precise meaning that the virtual morphisms $\tilde{\mathfrak{z}}[v] : E \to Z_1$ are Z_1-equivariant is spelled out in lemma 10.6.3.2 below.

10.6.3.2. Lemma. *We continue with the notation 10.6.2 and 10.6.3. For every element $v = (A,B,C)$ in the Lie algebra of G with components $A \in \mathrm{End}(X)$, $B \in \mathrm{End}(Y)$ and*

$C \in \mathrm{End}(C)$, *the diagram*

$$\begin{tikzcd}
R_{Z_1} \arrow[r, "\Delta_{Z_1}"] \arrow[d, "\tilde{\mathfrak{z}}[v]^*"'] & R_{Z_1}\hat{\otimes}R_{Z_1} \arrow[r, "(j_{R_Z}\circ \mathrm{pr}^*_{Z_1}\circ C|^*_{Z_1})\hat{\otimes}\tilde{\mathfrak{z}}[v]^*"] & (R_Z)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}\,\hat{\otimes}\,(R_E)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} \arrow[d, "j"] \\
(R_E)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} \arrow[rr, "\Delta^\flat"] & & \big(R_Z\hat{\otimes}R_E\big)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} \\
R_E \arrow[u, "j_{R_E}"] \arrow[rr, "\Delta"] & & R_Z\hat{\otimes}R_E \arrow[u, "j_{R_Z\hat{\otimes}R_E}"']
\end{tikzcd}$$

commutes. The arrows $\mathrm{pr}_{Z_1}, \Delta_{Z_1}, \Delta, \Delta^\flat, j_{R_Z}, j_{R_E}, j_{R_Z\hat{\otimes}R_E}, j$ *are as follows.*

- *The homomorphism* $\mathrm{pr}^*_{Z_1} : R_{Z_1} \to R_Z$ *corresponds to the projection* pr_{Z_1} *from* $Z = Z_1 \times (U_1 \times U_c)$ *to* Z_1*, where* $U_1, \ldots, U_c$ *are isoclinic* p*-divisible groups with slopes strictly smaller than* μ_1*,*
- Δ_{Z_1} *corresponds to the group law of the* p*-divisible group* Z_1*,*
- $\Delta : R_E \to R_Z \otimes R_E$ *corresponds to the* Z*-torsor structure* $Z \times E \to E$ *on* E*, which induces a ring homomorphism* $\Delta^\flat : (R_E)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} \longrightarrow \big(R_Z\hat{\otimes}R_E\big)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}$ *between tempered perfections*
- j_{R_Z}, j_{R_E} *and* $j_{R_Z\hat{\otimes}R_E}$ *are the inclusions maps from* R_Z, R_E *and* $R_Z\hat{\otimes}R_E$ *to their respective tempered perfections, and*
- *the downward vertical arrow* j *on the right is the natural ring homomorphism, from the tensor product* $(R_Z)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}\,\hat{\otimes}\,(R_E)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}$ *of tempered perfections of* R_Z *and* R_E*, to the tempered perfection* $\big(R_Z\hat{\otimes}R_E\big)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}$ *of* $R_Z\hat{\otimes}R_E$*,*

The proof of 10.6.3.2 is left as an exercise.

10.6.4. Further consequences of the proof of 10.6.2.

The proof of theorem 10.6.2 shows more than the statement of 10.6.2. We will review the assumptions and make some definitions before stating other consequences of the argument.

10.6.4.1. We will use the notation in step 2 of 10.6.3. In particular X, Y, Z are p-divisible groups over a perfect field k of characteristic p. Let $\pi : E \to X \times Y$ be a biextension of $X \times Y$ by Z. Let G be a closed subgroup of $\mathrm{Aut}_{\mathrm{biext}}(E)$. Let W be a reduced irreducible closed formal subscheme of E stable under the action of G. Let $v = (A, B, C)$ be an element of the Lie algebra of G with components $A \in \mathrm{End}(X)$, $B \in \mathrm{End}(Y)$ and $C \in \mathrm{End}(Z)$. We make the following assumptions.

- The largest slope μ_1 of Z is strictly bigger than every slope of $X \times Y$.
- The maximal isoclinic p-divisible subgroup Z_1 with slope μ_1 is a direct factor of Z, so that $Z = Z_1 \times Z_0$ where Z_0 is a p-divisible subgroup of Z all of whose slopes are strictly smaller than μ_1.
- There exist positive integers a, r, s, n_0, n_4 such that

- $0 < a \leq r < s$,
- $\mu_1 = \frac{a}{r}$, $Z_1[p^a] = Z[F^r]$,
- condition (d) in step 1 of 10.6.3 holds, and
- the congruence relation (10.6.3.1) in step 2 of 10.6.3 holds.

In step 3 of 10.6.3 we picked a regular system of parameters $u_1, \ldots, u_b$ of the complete local ring R_{Z_1} with $[p^a]^*_{Z_1}(u_i) = u_i^{p^r}$ for all $i = 1, \ldots, b$, and constructed a continuous ring homomorphism

$$\tilde{\mathfrak{z}}[v]^* \colon R_{Z_1} \longrightarrow (R_E, \mathfrak{m}_E)^{\mathrm{perf},\,\#}_{s:\phi^r;[i_1]}\ .$$

We will say that $\tilde{\mathfrak{z}}[v]^*$ corresponds to a "virtual morphism with tempered coefficient" $\tilde{\mathfrak{z}}[v]$ from E to Z_1. There are obvious benefits from this geometric view. However we do not have a fully developed theory of virtual morphisms with tempered coefficients at this moment, and allusions to virtual morphisms are completely formal.

Define the schematic image $\mathrm{Im}\big(\tilde{\mathfrak{z}}[v]\big|_W\big)$ of the restriction to W of $\tilde{\mathfrak{z}}[v]$ by

$$\begin{aligned}\mathrm{Im}\big(\tilde{\mathfrak{z}}[v]\big|_W\big) &:= \mathrm{Spf}\big(R_{Z_1}/\mathrm{Ker}(\tau^\flat \circ \tilde{\mathfrak{z}}[v]^*)\big)\\ &= \mathrm{Spf}\Big(R_{Z_1}/\mathrm{Ker}\big(R_{Z_1} \xrightarrow{\tilde{\mathfrak{z}}[v]^*} (R_E, \mathfrak{m}_E)^{\mathrm{perf},\,\#}_{s:\phi^r;[i_1]} \xrightarrow{\tau^\flat} (R_W, \mathfrak{m}_W)^{\mathrm{perf},\,\#}_{s:\phi^r;[i_1]}\big)\Big).\end{aligned}$$

10.6.4.2. Proposition. *We use the notations and make the assumptions in 10.6.4.1.*

(a) *The formal subvariety W of E is stable under the translation action by the smallest p-divisible subgroup of Z_1 which contains the schematic image $\mathrm{Im}\big((\tilde{\mathfrak{z}}[v])\big|_W\big)$ of the restriction to W of the virtual morphism $\tilde{\mathfrak{z}}[v] : E \to Z_1$, for every element $v \in \mathrm{Lie}(G) \cap (\mathrm{End}(X) \oplus \mathrm{End}(Y) \oplus \mathrm{End}(Z))$.*

(b) *Let $Z_{1,\tilde{\mathfrak{z}}}$ be the smallest p-divisible subgroup of Z_1 which contains the schematic image $\mathrm{Im}\big((\tilde{\mathfrak{z}}[v])\big|_W\big)$ for every $v \in \mathrm{Lie}(G) \cap (\mathrm{End}(X) \oplus \mathrm{End}(Y) \oplus \mathrm{End}(Z))$. Then W is stable under the translation action by $Z_{1,\tilde{\mathfrak{z}}}$.*

Proof. We will show W is stable under the translation action of $\mathrm{Im}\big((\tilde{\mathfrak{z}}[v])\big|_W\big)$. The statement (a) follows easily from this apparently weaker statement.

Let $I_W := \mathrm{Ker}\big(\tau : R_E \to R_W\big)$ be the ideal of R_E consisting of all formal functions on E which vanish on W. Let

$$J[v] := \mathrm{Ker}\big(\tau^\flat \circ \tilde{\mathfrak{z}}[v]^* : R_{Z_1} \longrightarrow (R_W, \mathfrak{m}_W)^{\mathrm{perf},\,\#}_{s:\phi^r;[i_1]}\ .$$

We need to show that the kernel of the composition

$$R_E \xrightarrow{\Delta_1} R_{Z_1} \otimes R_E \xrightarrow{q_{[v]} \otimes \tau} (R_{Z_1}/J[v]) \otimes R_W$$

contains I_W, where $q_{[v]} : R_{Z_1} \twoheadrightarrow J[v]$ is the quotient map. Let

$$\jmath_{[v]} : R_{Z_1}/J[v] \longrightarrow (R_W, \mathfrak{m}_W)^{\mathrm{perf},\,\#}_{s:\phi^r;[i_1]}$$

be the injective ring homomorphism such that

$$\tau^{\flat} \circ \tilde{\mathfrak{z}}[v]^{*} = \jmath_{[v]} \circ q_{[v]}.$$

We have a commutative diagram

$$\begin{tikzcd}
R_E \arrow[r, "\Delta_1"] & R_{Z_1} \otimes R_E \arrow[r, "q_{[v]}\otimes\tau"] \arrow[d, "\tilde{\mathfrak{z}}[v]^{*}\otimes j_{R_E}"'] & (R_{Z_1}/J[v]) \otimes R_W \arrow[d, "\jmath_{[v]}\otimes j_{R_W}"] \\
& (R_E, \mathfrak{m}_E)^{\text{perf},\#}_{s:\phi^r;[i_1]} \hat{\otimes} (R_E, \mathfrak{m}_E)^{\text{perf},\#}_{s:\phi^r;[i_1]} \arrow[r, "\tau^{\flat}\otimes\tau^{\flat}"] & (R_W, \mathfrak{m}_W)^{\text{perf},\#}_{s:\phi^r;[i_1]} \hat{\otimes} (R_W, \mathfrak{m}_W)^{\text{perf},\#}_{s:\phi^r;[i_1]}
\end{tikzcd}$$

In step 6 of 10.6.3 we proved that

$$I_W \subseteq \operatorname{Ker}\big((\tau^{\flat} \otimes \tau^{\flat}) \circ \tilde{\mathfrak{z}}[v]^{*} \otimes j_{R_E} \circ \Delta_1\big).$$

Therefore

$$I_W \subseteq \operatorname{Ker}\big((q_{[v]} \otimes \tau) \circ \Delta_1\big)$$

because $\jmath_{[v]} \otimes j_{R_W}$ is an injective ring homomorphism. We have proved the statement (a). The statement (b) follows from (a). $\square$

10.6.4.3. Corollary. *In 10.6.4.2, assume in addition that G operates strongly nontrivially on Z_1. Then the intersection $W \cap Z$ with reduced structure is equal to $Z_{1,\tilde{\mathfrak{z}}}$, the smallest p-divisible subgroup of Z which contains all schematic images $\operatorname{Im}\big((\tilde{\mathfrak{z}}[v])\big|_W\big)$, where v runs through all elements of $\operatorname{Lie}(G) \cap (\operatorname{End}(X) \oplus \operatorname{End}(Y) \oplus \operatorname{End}(Z))$.*

10.6.4.4. Proposition. *Let $\pi : E \to X \times Y$ be a biextension of $X \times Y$ by Z over k. Let μ_1 be the largest slope of Z, and let Z_1 be the largest isoclinic p-divisible subgroup of Z with slope μ_1. Let G be a closed subgroup of $\operatorname{Aut}_{\text{biext}}(E)$ such that the action of G on Z_1 is strongly nontrivial. Let W be a reduce irreducible subscheme of E stable under G. Assume that μ_1 is strictly bigger than every slope of $X \times Y$.*

(a) *The closed formal subscheme $Z_1'' := (W \cap Z_1)_{\text{red}}$ is a p-divisible subgroup of Z, and W is stable under the translation action by Z_1''.*

Let $W_2 := W/Z_1''$, a reduced irreducible closed formal subscheme of the biextension $E/Z_1'' = (Z \twoheadrightarrow Z/Z_1'')_* E$ of $X \times Y$ by Z/Z_1''.

(b) *The natural map*

$$q_{W_2} : W_2 \longrightarrow E/Z_1 = (Z \twoheadrightarrow Z/Z_1)_* E$$

is a finite purely inseparable *formal morphism. In other words the affine coordinate ring R_{W_2} of W_2 is a finite module over the subring $R_{\operatorname{Im}(q_{W_2})}$, the affine coordinate ring of the schematic image of q_{W_2}, and there exists a natural number m such that $x^{p^m} \in R_{\operatorname{Im}(q_{W_2})}$ for every $x \in R_{W_2}$.*

Proof. The statement (a) is 10.6.2.2. We only need to prove (b).

Extending the perfect base field k if necessary, we may and do assume that the base field k is algebraically closed. Replacing E by $E/Z_1'' = (Z \twoheadrightarrow Z/Z_1'')_*E$ and W by W/Z_1'', we may and do assume also that $Z_1'' = 0$.

Let $\bar{E} := E/Z_1 = (Z \twoheadrightarrow Z/Z_1)_*E$. Corollary 10.6.2.2 tells us that the closed fiber of the formal morphism $\pi|_W : W \to \bar{E}$ is finite over k, therefore $\pi|_W$ is finite. Denote by $\bar{W}$ be schematic image of $\pi|_W$, a reduced irreducible formal subscheme of $\bar{E}$ stable under the action of G. We need to show that W is purely inseparable over $\bar{W}$.

Let R_W and $R_{\bar{W}}$ be the coordinate rings of W and $\bar{W}$ respectively, and let $\jmath : R_{\bar{W}} \to R_W$ be the continuous injective ring homomorphism induced by $\pi|_W$. We know that R_W is finite over $R_{\bar{W}}$, and must show that there exists $N \in \mathbb{N}$ such that $x^{p^N} \in R_W$ for all $x \in R_{\bar{W}}$. Suppose no such natural number N exists. Then there exist continuous ring homomorphisms $h_1, h_2 : R_W \to k[[u]]$ from R_W to the power series ring $k[[u]]$ in one variable u, such that $h_1 \circ \jmath = h_2 \circ \jmath$ but $h_1 \neq h_2$. Since the projection $E \to E/Z_1$ is a Z_1-torsor, there exists a continuous k-linear ring homomorphism $\delta : R_{Z_1} \to k[[u]]$ such that

$$\mu_{k[[u]]} \circ (\delta \otimes h_1) \circ \Delta_1 = h_2,$$

where

- $\Delta_1 : R_E \to R_{Z_1} \hat{\otimes} R_E$ corresponds to the action of Z_1 on E,
- $\mu_{k[[u]]} : k[[u]] \hat{\otimes} k[[u]] \to k[[u]]$ is the multiplication map on $k[[u]]$, and
- $\mathrm{Ker}(\delta) \subsetneq \mathfrak{m}_{Z_1}$, or equivalently $k[[u]]$ is a finite module over the subring $\mathrm{Im}(\delta)$, because $h_1 \neq h_2$.

We know from 10.6.4.3 that for every element (A, B, C) of the Lie algebra of G with components $A \in \mathrm{End}(X)$, $B \in \mathrm{End}(Y)$ and $C \in \mathrm{End}(Z)$, the kernel of the composition $\tau^\flat \circ \tilde{\mathfrak{z}}[v]^*$ of continuous ring homomorphism

$$R_{Z_1} \xrightarrow{\tilde{\mathfrak{z}}[v]^*} (R_E, \mathfrak{m}_E)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} \xrightarrow{\tau^\flat} (R_W, \mathfrak{m}_W)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}$$

contains the maximal ideal $\mathfrak{m}_{Z_1}$ of R_{Z_1}. In other words $\tau^\flat \circ \tilde{\mathfrak{z}}[v]^*$ is the equal to the composition $R_{Z_1} \twoheadrightarrow k \hookrightarrow (R_W, \mathfrak{m}_W)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}$, the trivial k-linear ring homomorphism.

Consider the following diagram, an expansion of the top half of the diagram in 10.6.3.2.

$$\begin{tikzcd}
R_{Z_1} \arrow[r, "\Delta_{Z_1}"] \arrow[d, "\tilde{\mathfrak{z}}[v]^*"'] & R_{Z_1}\hat{\otimes} R_{Z_1} \arrow[r, "(j_{R_{Z_1}} \circ C|^*_{Z_1}) \otimes \tilde{\mathfrak{z}}[v]^*"] & (R_{Z_1})^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} \hat{\otimes} (R_E)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} \arrow[d, "j"] \\
(R_E)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} \arrow[rr, "\Delta_1^\flat"] \arrow[d, "(h_2 \circ \tau)^\flat"'] & & \big(R_{Z_1} \hat{\otimes} R_E\big)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} \arrow[d, "(1 \otimes \tau)^\flat"] \\
(k[[u]])^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} & \big(k[[u]] \hat{\otimes} k[[u]]\big)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} \arrow[l, "\mu^\flat_{k[[u]]}"'] & \big(R_{Z_1} \hat{\otimes} R_W\big)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} \arrow[l, "(\delta \otimes h_1)^\flat"']
\end{tikzcd}$$

The commutativity of the top half of the diagram follows from 10.6.3.2, while the bottom half commutes because $\mu_{k[[u]]} \circ (\delta \otimes h_1) \circ \Delta_1 = h_2$. The homomorphism

$$R_{Z_1} \xrightarrow{(h_2 \circ \tau)^\flat \circ \tilde{\mathfrak{z}}[v]^*} \big(k[[u]] \hat{\otimes} k[[u]]\big)^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]}$$

is the trial k-linear ring homomorphism because $\tau^\flat \circ \tilde{\mathfrak{z}}[v]^*$ is. On the other hand, we have

$$(h_2 \circ \tau)^\flat \circ \tilde{\mathfrak{z}}[v]^* = \mu^\flat_{k[[u]]} \circ (\delta \otimes h_1)^\flat \circ (\delta \otimes h_1)^\flat \circ \circ (1 \otimes \tau)^\flat \circ j \circ ((j_{R_E} \circ C|^*_{Z_1}) \otimes \tilde{\mathfrak{z}}[v]^*) \circ \Delta_{Z_1}.$$

The right hand side of the above equality is equal to the following composition

$$R_{Z_1} \xrightarrow{C|^*_{Z_1}} R_{Z_1} \xrightarrow{\delta} k[[u]] \xrightarrow{j_{k[[u]]}} (k[[u]])^{\mathrm{perf},\flat}_{s:\phi^r;[i_1]} .$$

Therefore the non-trivial $k[[u]]$-point δ^* of Z_1 lies in the kernel of the endomorphism $C|_{Z_1}$ for every element $v = (A, B, C) \in (\mathrm{End}(X) \oplus \mathrm{End}(Y) \oplus \mathrm{End}(Z)) \cap \mathrm{Lie}(G)$. Since the action of G on Z_1 is strongly non-trivial, the point $\delta^* \in Z_1(k[[u]])$ is 0. This is a contradiction. We have proved that W is purely inseparable over $\bar{W}$. $\square$

10.6.5. Proposition. *Let $\pi : E \to X \times Y$ be a biextension of $X \times Y$ over k. Let G be a closed subgroup of $\mathrm{Aut}_{\mathrm{biext}}(E)$. Let μ_2 be a slope of Z, and let Z_2 be the largest p-divisible subgroup of Z will all slopes $\geq \mu_2$. Let W be a reduced irreducible closed formal subscheme of E stable under the action of G. Suppose that the action of G on Z_2 is strongly nontrivial, and μ_2 is strictly bigger than every slope of $X \times Y$.*

(a) *The reduced formal subscheme $Z_2'' := (W \cap Z_2)_{\mathrm{red}}$ is a p-divisible subgroup of Z, and W is stable under the translation action by Z_1''.*

Let $W_3 := W/Z_2''$, a reduced irreducible closed formal subschem of the biextension $E/Z_2'' = (Z \twoheadrightarrow Z_2'')_* E$ of $X \times Y$ by Z/Z_2''.

(b) *The natural map*

$$q_{W_3} : W_3 \longrightarrow E/Z_2 = (Z \twoheadrightarrow Z/Z_2)_* E$$

is a finite purely inseparable formal morphism.

Proof. The case when Z_2 is isoclinic is proposition 10.6.4.4.

Consider next the case when Z_2 has exactly two slopes, μ_1, μ_2 with $\mu_1 > \mu_2$. The largest isoclinic p-divisible subgroup of Z with slope μ_1 is contained in Z_2.

Let $Z_1'' := (W \cap Z_1)_{\mathrm{red}}$, a p-divisible subgroup of Z_1. We know that W is stable under translation by Z_1'', and the natural map $W_2 := W/Z_1'' \longrightarrow E/Z_1$ is finite and purely inseparable. Let $\bar{W}_2 = \mathrm{Im}(q_{W_2})$ be the schematic image of $q_{W_2} : W_2 \to E/Z_1$. The intersection $W \cap Z_2$ with reduced structure has a finite number of irreducible components, and each irreducible component is a p-divisible subgroup of Z_2, by orbital rigidity for p-divisible groups. Since $W \cap Z_2$ is stable under translation by Z_1'', each irreducible components of $(W \cap Z_2)_{\mathrm{red}}$ is stable under translation by Z_1''. Let U be one of the irreducible components of $(W \cap Z_2)_{\mathrm{red}}$.

Claim. The formal subscheme $W \subseteq E$ is stable under the translation action of the p-divisible group $U \subseteq W \cap Z_2$.

PROOF OF CLAIM. We may and do assume that $i = 1$. Changing Z by an isogeny, we may and do assume that $U = Z_2'' \times U_2$, where U_2 is an isoclinic p-divisible subgroup of Z_2 of slope μ_2. Replacing W by W/Z_1'', we may and do assume that $Z_1'' = 0$.

In our simplified situation, $(W \cap Z_1)_{\text{red}} = 0$, the p-divisible subgroup U is an irreducible component of $(W \cap Z_2)_{\text{red}}$, U is isoclinic of slope μ_2. We need to show that W is stable under the translation action of U.

Let $q_W : W \to E/Z_1$ be the composition $W \hookrightarrow E \twoheadrightarrow E/Z_1$. Let $\bar{W}$ be the schematic image of q_W. Since q_W is finite dominant and purely inseparable, there exist a natural number N and a morphism $\zeta : \bar{W} \to W^{(p^N)}$ such that the relative Frobenius $\mathrm{Fr}_{W/k} : W \to W^{(p^N)}$ is equal to the composition $\zeta \circ q_W$. On the other hand 10.6.4.4 tells us that the reduced irreducible formal subscheme $\bar{W} \subseteq E/Z_1$ is stable under the translation action of U on E/Z_1. Consider the two morphisms

$$\alpha, \beta : U \times \bar{W} \longrightarrow E^{(p^N)},$$

defined by

$$\alpha(u, \bar{w}) = \zeta(u * \bar{w}), \quad \beta(u, \bar{w}) = \mathrm{Fr}^r_{U/k}(u) * \zeta(\bar{w}).$$

for all functorial points $(u, \bar{w})$ of $U \times \bar{W}$. For every functorial point w of W, we have

$$\alpha(u, q_W(w)) = \zeta(q_W(u * w)) = \mathrm{Fr}^N_{E/k}(u * w) = \mathrm{Fr}^N_{U/k}(u) * \mathrm{Fr}^N_{E/k}(w)$$

and

$$\beta(u, q_W(w)) = \mathrm{Fr}^N_{U/k}(u) * \zeta(q_W(w)) = \mathrm{Fr}^N_{U/k}(u) * \mathrm{Fr}^N_{E/k}(w),$$

i.e. $\alpha \circ (1_U \times q_W) = \beta \circ (1_U \times q_W)$. So $\alpha = \beta$ because $1_U \times q_W$ is faithfully flat. The equality $\alpha = \beta$ implies that the schematic image of ζ is stable under translation by the schematic image of $\mathrm{Fr}^N_{U/k} : U \to U^{(p^N)}$. It follows that W is stable under translation by U. We have proved the claim.

We go back to the situation in the paragraph before the claim. Since W is stable under translation by every irreducible component of $(W \cap Z_2)_{\text{red}}$, W is stable under the smallest p-divisible subgroup containing $(W \cap Z_2)_{\text{red}}$. It follows that $(W \cap Z_2)_{\text{red}}$ is a p-divisible group. We have proved statement (a) in the case when Z_2 has two slopes.

We turn to the statement (b). We may and do assume that $Z_2 = Z_1 \times U'$, where U' is isoclinic with slope μ_2. Let $W_2 = W/(W \cap Z_1)_{red}$, let $\bar{W}_2$ be the schematic image of W_2 in E/Z_1. Proposition 10.6.4.4 tells us that the map $W_2 \to \bar{W}_2$ is purely inseparable, and also that the map from $W = \bar{W}_2/(\bar{W}_2 \cap U_2)_{\text{red}}$ to E/Z_2 is purely inseparable. The statement (b) follows. We have proved proposition 10.6.5 when Z_2 has two slopes.

An easy induction on the number of slopes of Z_2, using the argument for the two slope case above, proves the general case. $\square$

10.6.6. Proposition. *Let X, Y, Z be p-divisible formal groups over k, let $E \to X \times Y$ be a biextension of $X \times Y$ by Z, and let G be a closed subgroup of $\mathrm{Aut}_{\mathrm{biext}}(E)$ operating strongly nontrivially on E. Let W be a reduced irreducible formal subscheme of E stable under the action of G. Let μ_1 be the maximum among the slopes of Z, and let Z_1 be the largest isoclinic p-divisible subgroup of Z with slope μ_1.*

(a) *The reduced formal subscheme $(W \cap Z_1)_{\mathrm{red}}$ is a p-divisible subgroup Z_1' of Z_1.*
(b) *The formal subscheme W of E is stable under the translation action by Z_1'.*
(c) *The composition $q_{W/Z_1'} : W/Z_1' \hookrightarrow E/Z_1' \twoheadrightarrow E/Z_1$ is a finite purely inseparable formal morphism from W/Z' to $E/Z_1 = (Z \twoheadrightarrow Z/Z_1)_* E$.*

PROOF. If μ_1 is strictly bigger than every slope of $X \times Y$, the statements (a)–(c) follow from 10.6.2.2 and 10.6.4.4.

Suppose that some slopes of $X \times Y$ are bigger than or equal to μ_1. Modifying X, Y by suitable isogenies, we may and do assume that X and Y are products of the form $X = X_1 \times X_2$, $Y = Y_1 \times Y_2$, such that all slopes of $X_1 \times Y_1$ are bigger than or equal to μ_1, and all slopes of $X_2 \times Y_2$ are strictly smaller than μ_1. We know from 10.2.4.4 that the Weil pairings $\theta_n^E : X[p^n] \times Y[p^n] \to Z[p^n]$ vanishes on $X_1[p^n] \times Y[p^n]$ and $X[p^n] \times Y_1[p^n]$ for all n. Apply lemma 10.2.5.10, we get a new biextension structure

$$(\pi' : E \to X_2 \times Y_2,\ +_1' : E \times_{Y_2} E \to E,\ +_1' : E \times_{X_2} E \to E,\ \epsilon_1' : Y_2 \to E,\ \epsilon_2' : X_2 \to E)$$

on E, of $X_2 \times Y_2$ by $Z' := X_1 \times Y_2 \times Z$, which satisfies properties (1)–(8) in 10.2.5.10. In particular the Z-torsor structure on E associated to the old biextension structure is compatible with the Z'-torsor structure associated to the new biextension structure, i.e. $z *_E e = z *_{E'} e$ for all functorial points (z, e) of $Z \times E$. So it suffices to prove statement (a)–(c) for the new biextension structure on E.

Apply proposition 10.6.5 with the biextension structure $\pi' : E \to X_2 \times Y_2$ and $Z_2 = X_1 \times Y_1 \times Z_1$, we see that $(W \cap Z_2)_{\mathrm{red}}$ is a p-divisible subgroup of Z', W is stable under translation $(W \cap Z_2)_{\mathrm{red}}$ with respect to the Z'-torsor structure attached to the biextension $\pi' : E \to X_2 \times Y_2$, and the natural formal morphism

$$W/(W \cap Z_2)_{\mathrm{red}} \longrightarrow E/Z_2$$

is finite and purely inseparable. Since $(W \cap Z_1)_{\mathrm{red}} \subseteq (W \cap Z_2)_{\mathrm{red}})$, and the Z-torsor structure for the biextension $\pi : E \to X \times Y$ is compatible with the $(X_1 \times Y_1 \times Z)$-torsor structuer for the biextension $\pi' : E \to X_2 \times Y_2$, W is stable under translation by $(W \cap Z_1)_{\mathrm{red}}$, $(W \cap Z_1)_{\mathrm{red}}$ is stable under the group law of Z_1. It follows that $(W \cap Z_1)$ is a p-divisible subgroup of Z_1. We have proved statements (a) and (b).

Consider the commutative diagram

$$\begin{tikzcd}
W/(W\cap Z_1)_{\mathrm{red}} \arrow[r, hook, "j_1"] \arrow[d] \arrow[rr, bend left, "q_{W/(W\cap Z_1)_{\mathrm{red}}}"] \arrow[dr, phantom, "\square"] & E/(W\cap Z_1)_{\mathrm{red}} \arrow[r, "q_1"] \arrow[d] \arrow[dr, phantom, "\boxdot"] & E/Z_1 \arrow[d] \\
W/(W\cap Z_2)_{\mathrm{red}} \arrow[r, hook, "j_2"] \arrow[drr, "q_{(W\cap Z_2)_{\mathrm{red}}}"'] & E/(W\cap Z_2)_{\mathrm{red}} \arrow[r, "q_1"] & E/(Z_1+(W\cap Z_2)_{\mathrm{red}}) \arrow[d, "q_3"] \\
& & E/Z_2\,,
\end{tikzcd}$$

where the vertical arrows are the obvious ones. The square $\square$ on the left is Cartesian. The square $\boxdot$ on the right is not Cartesian, but it induces a finite purely inseparable morphism

$$E/(W\cap Z_1)_{\mathrm{red}} \longrightarrow E/(W\cap Z_2)\times_{E/(Z_1+(W\cap Z_2)_{\mathrm{red}})} E/Z_1$$

from $E/(W\cap Z_1)_{\mathrm{red}}$ to the fiber product, over $E/(Z_1+(W\cap Z_2)_{\mathrm{red}})$, of $E/(W\cap Z_2)$ and E/Z_1. Therefore $q_1\circ j_1$ is finite and purely inseparable if and only if $q_2\circ j_2$ is. Since $q_{(W\cap Z_2)_{\mathrm{red}}} = q_3\circ(q_2\circ j_2)$ is finite purely inseparable, $q_2\circ j_2$ is also finite and purely inseparable. It follows that $q_1\circ j_1$ is finite and purely inseparable as well. We have proved the statement (c). □

10.6.7. Theorem. *Let X, Y, Z be p-divisible formal groups over k, let $E\to X\times Y$ be a biextension of $X\times Y$ by Z, and let G be a closed subgroup of $\mathrm{Aut}_{\mathrm{biext}}(E)$ acting strongly nontrivially on E. Let W be a reduced irreducible formal subscheme of E stable under the action of G.*

(1) *The formal subscheme W of E is a special formal subvariety.*
(2) *If the slopes of X, Y, Z are pairwise disjoint, then W is a sub-biextension of E.*

Proof. The statement (1) follows, by induction on the height of Z, from 10.6.6, and 10.3.4.6. The statement (2) is a corollary of (a); see 10.3.4.7. □

10.7. Appendix: Tempered perfections of complete local domains

In this appendix we define a class a complete augmented commutative local domains over a perfect field κ of characteristic $p>0$, *completed tempered perfections* of complete augmented Noetherian local domains $(R,\mathfrak{m})$ over κ, and document some of their basic properties used in 10.5–10.6. These rings are completions of suitable subrings sandwiched between R and the perfection R^{perf} of R. We will often shorten "completed tempered perfection" to "tempered perfection".

The method of hypocotyl prolongation, first established in 7.2.1 and 7.2.2 for augmented Noetherian completed local domains, also holds for their tempered perfections; see 10.5.3 and 10.5.6. This method, enhanced by the adoption of tempered perfections, provides the critical ingredient in the proof of orbital rigidity for p-divisible formal groups. It should also be useful in studying the orbital rigidity phenomenon for sustained deformation spaces of p-divisible groups.

Since the main body of this appendix is pretty dry and technical, we provide a long introductory subsection 10.7.1 with examples and motivations. The examples 10.7.1.3 and 10.7.1.4 both involve the Poincaré biextension of a supersingular elliptic curve. The generalization of 10.7.1.3–10.7.1.4 to biextensions of p-divisible formal groups, which is the genesis of the notion of tempered perfections, is explained in 10.7.2–10.7.3.2.

Only the easy properties of tempered perfections, explained in 10.7.5, are used in the proof of orbital rigidity for biextensions of p-divisible formal groups. We include an analog of Weierstrass preparation theorem in 10.7.7, as an example among a host of questions one may ask about these rings. These questions and their potential applications are left to the interested readers.

10.7.1. What are completed tempered perfections?

10.7.1.1. An impressionistic sketch.

Throughout 10.7.1 the base field κ is a perfect of characteristic p. A completed tempered perfection R' of a complete augmented Noetherian local domain R over κ is sandwiched between R and the completion $(R^{\mathrm{perf}})^\wedge$ of the perfection of R. In general R' is not Noetherian, except for trivial cases such as the field κ itself. But R' is not as big as $(R^{\mathrm{perf}})^\wedge$, and it retains some weak versions of the finiteness properties valid for complete Noetherian local domains. These properties are illustrated in the example in 10.7.1.2.

Elements of a tempered perfections R' of R can be viewed as limits of functions on a suitable projective system $\mathcal{T}$ of purely inseparable covers of the formal spectrum $\mathrm{Spf}(R)$ of R. Such a tower $\mathcal{T}$ is substantially smaller than the projective family of all purely inseparable covers of $\mathrm{Spf}(R)$, except in the trivial case when $R = \kappa$. This point is partly reflected in the weak finiteness properties of R'.

Inspired by the analogy with tempered distributions as generalized functions, we propose to call elements of a tempered perfection R' of R *tempered virtual functions* on $\mathrm{Spf}(R)$, and elements of R^{perf} *virtual functions* on $\mathrm{Spf}(R)$.

10.7.1.2. Tempered perfections of $\kappa[[t]]$.

A simple example of tempered perfections is the following family $\big(\kappa\langle\langle t^{p^{-\infty}}\rangle\rangle^E_{C;d}\big)_{C,d,E}$ of tempered perfections of the power series ring $\kappa[[t]]$, parametrized by triples (C,d,E) of real numbers with $C>0, d\geq b, E>0$. By definition $\kappa\langle\langle t^{p^{-\infty}}\rangle\rangle^E_{C;d}$ is the completed semigroup algebra attached to the semigroup

$$\mathbb{N}[\tfrac{1}{p}]^E_{C;d} = \Big\{ i\in\mathbb{Z}[\tfrac{1}{p}]_{\geq 0} \,\Big|\, |i|_p \leq \max\big(C(|i|+d)^E, 1)\big) \Big\} \;\subseteq\; (\mathbb{Z}[\tfrac{1}{p}]_{\geq 0}, +),$$

where $|\cdot|_p = p^{-\mathrm{ord}_p(\cdot)}$ is the normalized p-adic absolute value on $\mathbb{Q}$. In other words

$$\kappa\langle\langle t^{p^{-\infty}}\rangle\rangle^E_{C;d} = \Big\{ \textstyle\sum_{i\in\mathbb{N}[1/p]^E_{C;d}} b_i\, t^i \,\Big|\, b_i\in\kappa\ \forall\, i \Big\},$$

consisting of all formal power series of the form $\sum_{i\in\mathbb{N}[1/p]^E_{C;d}} b_i\, t^i$ with coefficients in κ. Note that the product of any two elements of $\kappa\langle\langle t^{p^{-\infty}}\rangle\rangle^E_{C;d}$ is well-defined because

$$\mathrm{Card}\big(\{i\in\mathbb{N}[\tfrac{1}{p}]^E_{C;d}\,\big|\, i\le M\}\big)<\infty$$

for every $M\in\mathbb{R}_{\ge 0}$.

Tempered perfections of the power series ring $\kappa[[t_1,\dots,t_m]]$ are defined similarly, but there are at least two versions, corresponding to two archimedean norms

$$|(i_1,\dots,i_m)|_\infty := \max(|i_1|,\dots,|i_m|) \quad\text{and}\quad |(i_1,\dots,i_m)|_\sigma := |i_1|+\cdots+|i_m|$$

on $\mathbb{Q}^m$. As the parameters (C,d,E) vary, these two versions of tempered perfections of $\kappa[[t_1,\dots,t_m]]$ give rise to two filtered inductive systems of subrings of the completion of the perfection of $\kappa[[t_1,\dots,t_m]]$, which are cofinal to each other in the obvious sense.

10.7.1.3. The Weil pairings on a supersingular elliptic curve as a toy model. Let κ be an algebraically closed field of characteristic p as before and let A be a supersingular elliptic curve over κ. For each positive integer $n\ge 1$, let

$$\omega_n : A[p^n]\times A[p^n]\longrightarrow \mu_{p^n}=\mathbb{G}_{\mathrm{m}}[p^n],$$

be the p-adic Weil pairing on $E[p^n]$. The family $\big(\omega_n\big)_{n\ge1}$ satisfies the compatibility condition

$$\omega_{n+1}(x_{n+1},py_{n+1})=\omega_n(px_{n+1},py_{n+1})=\omega_{n+1}(px_{n+1},y_{n+1})$$

for all functorial points x_{n+1},y_{n+1} of $A[p^{n+1}]$.

Let $\hat{A}$ be the formal completion of A. We pose the following question.

Question. Is there a "formula", in terms of a single function on $\hat{A}\times\hat{A}$, which gives all ω_n's?

On the face of it, this is a stupid and unmotivated question. The obvious answer is "no", because the ω_n's do *not* glue to a map from $\hat{A}\times\hat{A}$ to the formal completion of $\mathbb{G}_{\mathrm{m}}$. The restriction of ω_{n+1} to $A[p^n]\times A[p^n]$ is not equal to ω_n; instead

$$\omega_{n+1}|_{A[p^n]\times A[p^n]}=[p]_{\mu_{p^n}}\circ\omega_n.$$

But let's continue this foolhardy pursuit undeterred. Let $\kappa[x,x^{-1}]$ be the coordinate ring of $\mathbb{G}_{\mathrm{m}}$, let $t=x-1\in\kappa[x,x^{-1}]$, so $\widehat{\mathbb{G}_{\mathrm{m}}}=\mathrm{Spf}(\kappa[[t]])$ and $[p]^*_{\mathbb{G}_{\mathrm{m}}}t=t^p$. Pick a uniformizer u of the coordinate ring of $\hat{A}$ such that $[p]^*_A(u)=u^{p^2}$. So $A[p^n]=\mathrm{Spec}\big(\kappa[u]/(u^{p^{2n}})\big)$ for each $n\ge1$, and ω_n is encoded by an element

$$w_n := \omega_n^*(t)\ \in\ \kappa[u,v]/(u^{p^{2n}},v^{p^{2n}}).$$

The equality $\omega_{n+1}|_{E[p^n]\times A[p^n]}=[p]_{\mu_{p^n}}\circ\omega_n$ means that

$$w_{n+1}\,\mathrm{mod}\,(u^{p^{2n}},v^{p^{2n}})=w_n^p\ \text{ in }\ \kappa[u,v]/(u^{p^{2n}},v^{p^{2n}}) \tag{10.7.1.3.a}$$

for all $n\ge1$. An easy induction shows that

$$w_n\equiv 0\,\mathrm{mod}\,(u^{p^{2\lfloor n/2\rfloor}},v^{p^{2\lfloor n/2\rfloor}}). \tag{10.7.1.3.b}$$

This estimate of w_n also follows from the fact that
$$\omega_n \circ ([p^{\lceil n/2 \rceil}]_{A[p^n]} \times [p^{\lceil n/2 \rceil}]_{A[p^n]}) = 0.$$
Consider the element
$$f_n := w_n^{p^{-n}} \ \in \ \kappa[u^{p^{-n}}, v^{p^{-n}}]\big/(u^{p^n}, v^{p^n})\, \kappa[u^{p^{-n}}, v^{p^{-n}}]$$
The congruence (10.7.1.3.a) implies that

(10.7.1.3.c) $$f_n \bmod (u^{p^{n-1}}, u^{p^{n-1}}) = f_{n+1} \bmod (u^{p^{n-1}}, u^{p^{n-1}})$$

in the ring $\kappa[u^{p^{-n-1}}, v^{p^{-n-1}}]/(u^{p^{n-1}}, v^{p^{n-1}})\kappa[u^{p^{-n-1}}, v^{p^{-n-1}}]$. As $n \to \infty$, the f_n's converges to a formal power series
$$f_\infty = \sum_{i,j \in \mathbb{N}[1/p]} b_{i,j} u^i v^j,$$
and the estimate (10.7.1.3.c) implies that for every pair (i,j) with $b_{i,j} \neq 0$ we have
$$\max\bigl(|i|_p,\, |j|_p\bigr) \leq p^2 \cdot \max\bigl(|i|,\, |j|\bigr).$$
So $f_\infty \in \kappa\langle\langle u^{p^{-\infty}}, v^{p^{-\infty}}\rangle\rangle_{C;d}^{E}$ with $C = p^2$, $d = 0$, $E = 1$, for either of the two archimedian norms on $\mathbb{Q}^2$.

The whole family of Weil pairings $\bigl(\omega_n\bigr)_n$ is encoded in the power series f_∞: for each $n \geq 1$, ω_n is the unique element of $\kappa[u,v]\big/(u^{p^{2n}}, v^{p^{2n}}$ such that $\omega_n \equiv f_\infty^{p^n}$ modulo $(u^{p^{2n}}, v^{p^{2n}})$. This gives a positive answer to the question at the beginning of 10.7.1.3 if the power series f_∞, a "generalized function" on the formal scheme $\hat{A} \times \hat{A}$ is deemed admissible.

10.7.1.4. Splitting a Poincaré biextension with tempered virtual functions.

Let A be a supersingular elliptic curve over κ as in 10.7.1.3. Let $\pi : P \to A \times A$ be the Poincaré biextension of $A \times A$ by $\mathbb{G}_{\mathrm{m}}$. There are two relative group laws
$$+_1 : P \times_{(\mathrm{pr}_2 \circ \pi, A, \mathrm{pr}_2 \circ \pi)} P \to P, \quad +_2 : P \times_{(\mathrm{pr}_1 \circ \pi, A, \mathrm{pr}_1 \circ \pi)} P \to P$$
on P, with zero sections ϵ_1, ϵ_2 respectively. The group law $+_1$ comes from the theorem of the square, while the group law $+_2$ corresponds to the group structure of the Jacobian of A, i.e. tensor product of invertible sheaves which are algebraically equivalent to 0.

For each integer $n > 0$, multiplication by p^n for the relative group law $+_1$ defines a morphism $[p^n]_{+_1} : P \to P$ over $[p^n]_A \times 1_A : A \times A \to A \times A$. The restriction of $[p^n]_{+_1}$ to $P_n := \pi^{-1}(A[p^n] \times A[p^n])$ defines a morphism
$$\eta_n : P_n \to \mathbb{G}_{\mathrm{m}}$$
such that
$$[p^n]_{+_1}\big|_{P_n} = \eta_n * \bigl(\epsilon_1\big|_{A[p^n]}\bigr),$$
where $* : \mathbb{G}_{\mathrm{m}} \times P \to P$ denotes the structural map of the $\mathbb{G}_{\mathrm{m}}$-torsor $P \to A \times A$. The η_n's satisfy the compatibility condition

(10.7.1.4.a) $$\eta_{n+1}\big|_{P_n} = [p]_{\mathbb{G}_{\mathrm{m}}} \circ \eta_n,$$

which is hardly a surprise given that $\eta_n|_{\mathbb{G}_{\mathrm{m}}} = [p^n]_{\mathbb{G}_{\mathrm{m}}}$.

As before the map η_n corresponds to a function $\eta_n^*(t)$ on P_n. For each $m \geq 1$, let $\mathrm{Fr}_P^m : P \to P^{(p^m)}$ be the m-th iterate of the relative Frobenius map for P. Denote by $P[F^m]$ the inverse image of the 0-point 0_P of P abover $(0_A, 0_A) \in A \times A$. Let $\hat{P} = \cup_m P[F^m]$ be the formal completion of P.

Let h_n be the restriction of $\eta_n^*(t)$ to $P[F^{2n}]$. The same argument in 10.7.1.3 shows that $h_n^{p^{-n}}$ converges to an element g_∞^* of a tempered perfection of the coordinate ring of the formal completion $\hat{P}$ with parameters $(C, d, E) = (p^2, 0, 1)$.

It is helpful to regard g_∞^* as the coordinate function of a map g_∞ from a tower $\mathcal{T}_{(C,d,E)}$ of inseparable covers of $\hat{P}$ to $\mathbb{G}_{\mathrm{m}}$, so that g_∞ is a "virtual tempered map" from $\hat{P}$ to $\mathbb{G}_{\mathrm{m}}$. The tower $\mathcal{T}_{(C,d,E)}$ is substantially smaller then the projective family of all inseparable covers of $\hat{P}$. This accounts for the weak finiteness properties of tempered perfections, which make "virtual tempered retractions" such as g_∞ useful in proving orbital rigidity for biextensions.

10.7.1.5. Remarks on different families of tempered perfections. There are several families of completed tempered perfections of a complete augmented Noetherian local domain $(R, \mathfrak{m})$ over a perfect field κ of characteristic p. Each of the families listed below has a $\flat$-version and a $\sharp$-version, depending on whether one uses the filtration defined by the ideals $(\mathfrak{m}^N)$ or the ideals $(\mathfrak{m}^{(p^n)})$, where $\mathfrak{m}^{(p^n)}$ denotes the ideal generated by $\{y^{p^n} \mid y \in \mathfrak{m}\}$.

(a) In 10.7.2–10.7.3.2 we define a family of tempered perfections of $R \cong \kappa[[t_1, \ldots, t_m]]$ as rings of limits of *ϕ_r-compatible sequences*. The case when R is the coordinate ring of the completion of a biextension of p-divisible formal groups is what led to the notion of tempered perfections.

(b) The family $\big(\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;\,d}^{E,\,\#}\big)_{C,d,E}$ and $\big(\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;\,d}^{E,\,\flat}\big)_{C,d,E}$ of tempered perfections of $\kappa[[t_1, \ldots, t_m]]$ are defined in 10.7.3.6 as completed semigroup algebras of suitable sub-semigroups S of $(\mathbb{N}[1/p]^m, +)$ containing $\mathbb{N}^m$. Such a sub-semigroup S of "allowed exponents" consist of all elements $I = (i_1, \ldots, i_m) \in \mathbb{N}[1/p]^m$ satisfying an inequality involving parameters C, d, E which bounds the p-adic norm of I in terms of the archimedean norms of I. This family include those in (a) defined through ϕ_r-compatible sequences.

(c) For a general augmented complete Noetherian local domain R over κ, we have two families of tempered perfections of R. The family $\big((R,\mathfrak{m})_{s:\phi^r;[i_0]}^{\mathrm{perf},\,\#}\big)_{r,s,i_0}$ and $\big((R,\mathfrak{m})_{s:\phi^r;[i_0]}^{\mathrm{perf},\,\flat}\big)_{r,s,i_0}$ defined in 10.7.4.1 is close in spirit to (a), while the family $\big((R,\mathfrak{m})_{A,b;d}^{\mathrm{perf},\,\flat}\big)_{A,b,d}$ and $\big((R,\mathfrak{m})_{A,b;d}^{\mathrm{perf},\,\#}\big)_{A,b,d}$ defined in 10.7.4.2–10.7.4.3 is close to (b) above and include the family $\big((R,\mathfrak{m})_{s:\phi^r;[i_0]}^{\mathrm{perf},\,\flat}\big)_{r,s,i_0}$ and $\big((R,\mathfrak{m})_{s:\phi^r;[i_0]}^{\mathrm{perf},\,\#}\big)_{r,s,i_0}$ defined in 10.7.4.1

Each of the above families of tempered perfections of R forms a filtered inductive system as the parameters vary. For a general complete augmented Noetherian local domain R, the two families of tempered perfections defined in 10.7.4.1 and 10.7.4.2 are mutually cofinal.

Similarly for $R \cong \kappa[[t_1,\ldots,t_m]]$, the filtered inductive systems of tempered perfections of R described in (a)–(c) above are mutually confinal to each other.

We sketch the definition of the tempered perfection $(R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}$ of an augmented complete Noetherian local domain $(R,\mathfrak{m})$ over a perfect field κ of characteristic p, where A,b,d are real numbers, $A,b>0$, and $d\geq b$; see 10.7.4.2 for details. First we define a decreasing filtration $\mathrm{Fil}^\bullet$ of the perfection R^{perf} of R with $\mathrm{Fil}^0 = R^{\mathrm{perf}}$, by

$$\mathrm{Fil}^u = \left\{ x \in R^{\mathrm{perf}} \;\middle|\; \exists j \in \mathbb{N} \text{ s.t. } x^{p^j} \in \mathfrak{m}^{\lceil u\cdot p^j \rceil} \right\}$$

for any $u\geq 0$. Note that restriction to R of this filtration is essentially the $\mathfrak{m}$-adic filtration of R.

Next we define a subring $\big((R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}\big)_{\mathrm{fin}}$ of R^{perf} by

$$\big((R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}\big)_{\mathrm{fin}} := \sum_{n\in\mathbb{N}} \left(\phi^{-n} R \cap \mathrm{Fil}^{b\cdot p^{An}-d}\right),$$

consisting of the linear span of the set of all elements $y\in R^{\mathrm{perf}}$ such that there exists an $n\in\mathbb{N}$ with $y^{p^n}\in R$ and $y\in \mathrm{Fil}^{b\cdot p^{An}-d}$. Note that in the case $R=\kappa[[t_1,\ldots,t_m]]$, such an element y has the form

$$y = \textstyle\sum_{I\in S} c_I\, \underline{t}^I$$

with $c_I\in\kappa$ for all $I\in S$, where S is the subset of $\mathbb{N}[1/p]^m$ consisting of all elements $(i_1,\ldots,i_m)\in p^{-n}\mathbb{N}^m$ such that $b\cdot p^{An}-d\leq i_1+\ldots+i_m$. In particular

$$|I|_p := \max(|i_1|_p,\ldots,|i_m|_p) \;\leq\; p^n \;\leq\; b^{-1/A}\left[(i_1+\cdots+i_m)+d\right]^{1/A}$$

for every element $I\in S$.

The completed tempered perfection $(R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}$ of R is by definition the completion of $\big((R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}\big)_{\mathrm{fin}}$ with respect to the filtration of $\big((R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}\big)_{\mathrm{fin}}$ induced by the filtration $\mathrm{Fil}^\bullet$ of R^{perf}.

10.7.2. ϕ_r-compatible sequences.

In 10.2.6.1 we defined a compatible sequence of morphisms $\left\{\eta_n : \pi^{-1}E =: E_n \to Z\right\}_{n\in\mathbb{N}}$ for any biextension of E of p-divisible groups X,Y by another p-divisible group Z, over an arbitrary base scheme S. In this section we will consider the special case when S is the spectrum of a perfect field $k\supset\mathbb{F}_p$. An interesting phenomenon reveals itself in the special case described in 10.7.2.1, and the compatible sequence of morphisms (η_n) lead us to families commutative rings, whose elements consists of formal series of the form

$$\sum_{(i_1,\ldots,i_m)\in\mathbb{Z}[1/p]^m_{\geq 0}} a_{i_1,\ldots,i_m}\, t_1^{i_1} t_2^{i_2}\cdots t_m^{i_m}$$

with coefficients $a_{i_1,\ldots,i_m}\in k$, subject to the condition roughly of the following form

$$|I|_p \leq C\cdot |I|^E_{\infty,\max}$$

for every I such that $a_I \neq 0$, where $C, E > 0$ are parameters which define the ring. Here for any multi-index $I = (i_1, \ldots, i_m) \in \mathbb{Z}[1/p]^m_{\geq 0}$, $|I|_p$ is the p-adic norm of I and $|I|_{\infty,\max}$ is the archimedean norm of I, defined by

$$|I|_p := \max(p^{-\mathrm{ord}_p(i_1)}, \ldots, p^{-\mathrm{ord}_p(i_1)}), \quad \text{and} \quad |I|_{\infty,\max} := \max(i_1, i_2, \ldots, i_m).$$

These rings do not seem to have appeared in the literature, but they hold the key to the orbital rigidity for biextensions of p-divisible groups. In this section we give the motivation and definition of these new rings.

10.7.2.1. Definition. Let κ be a perfect field of characteristic p. Let R be an augmented complete Noetherian local domain over κ, and let $\mathcal{Q} = \mathrm{Spf}(R)$. Let Z be a p-divisible formal group over κ. Let $a, s > 0$ be positive integers. A sequence of morphisms

$$\tau_n : \mathcal{Q}[F^{ns}] \to Z, \quad n \in \mathbb{N},\, n \geq n_0$$

is said to be $[p^a]$-*compatible with respect to* ϕ^s if

$$\tau_{n+1}\big|_{\mathcal{Q}[F^{ns}]} = [p^a]_Z \circ \tau_n \quad \forall n \geq n_0.$$

Here for every positive integer j, $\mathcal{Q}[F^j] = \mathrm{Spf}\left(R/\mathfrak{m}^{(p^j)}\right)$ is the inverse image of $\mathrm{Spf}(k)$ under the relative Frobenius morphism $\mathrm{Fr}^j_{\mathcal{Q}/k} : \mathcal{Q} \to \mathcal{Q}^{(p^j)}$, and $\mathfrak{m}^{(p^j)}$ is the ideal of R generated by $\{x^{p^j} \big| x \in \mathfrak{m}\}$. If the integer s is clear from the context, we will shorten "$[p^a]$-compatible with respect to ϕ^s" to "$[p^a]$-compatible".

Clearly if $\left(\mathcal{Q}[F^{ns}] \xrightarrow{\tau_n} Z \right)$ is $[p^a]_Z$-compatible sequence and $h : Z \to Z'$ is a homomorphism of p-divisible groups, then the sequence $\left(\mathcal{Q}[F^{ns}] \xrightarrow{h\circ\tau_n} Z' \right)$ is $[p^a]_{Z'}$-compatible. Similarly if $\mathcal{Q}'$ is a reduced irreducible formal subscheme of $\mathcal{Q}$, $\mathcal{Q}' \xhookrightarrow{\mathrm{inc}} \mathcal{Q}$ is the inclusion map, and $\left(\mathcal{Q}[F^{ns}] \xrightarrow{\tau_n} Z \right)$ is $[p^a]_Z$-compatible, then $\left(\mathcal{Q}'[F^{ns}] \xrightarrow{\tau_n\circ\mathrm{inc}} Z \right)$ is also $[p^a]_Z$-compatible.

Remark. Given a $[p^a]$-compatible sequence of morphisms $\left(\mathcal{Q}[F^{ns}] \xrightarrow{\tau_n} Z \right)_{n\geq n_0}$, one might wish to encode such a compatible sequence in a formal morphism from $\mathcal{Q}$ to Z by a suitable limit process. A moment's reflection shows that this is wishful thinking, and counter-examples abound. So if this wish is to have any chance of being partially realized, some sort of "generalized morphism" from $\mathcal{Q}$ to Z, whose coordinates are "generalized functions" on $\mathcal{Q}$, must be allowed. This is indeed the case, as we will see soon.

10.7.2.2. $[p^a]$-compatible sequences of maps in applications to biextensions

In applications to orbital rigidity of biextensions, the $[p^a]_Z$-compatible sequences

$$\left(\mathcal{Q}[F^{ns}] \xrightarrow{\tau_n} Z \right)_{n\geq n_0}$$

we encounter satisfy the conditions (1)–(3) below after some preliminary maneuver.

(1) The formal scheme $\mathcal{Q} = \mathrm{Spf}(R)$ is a reduced irreducible formal subscheme of a biextension $\pi : E \to X \times Y$ of p-divisible formal groups of $X \times Y$ over κ by Z.
(2) All slopes of X, Y are strictly smaller than the biggest slope μ_1 of Z.
(3) There exist maps $\left(E_m := \pi^{-1}(X[p^m] \times Y[p^m]) \xrightarrow{\tilde{\lambda}_m} Z \right)_{m\geq 1}$ such that

$$\tilde{\lambda}_{m+1}|_{E_m} = [p]_Z \circ \tilde{\lambda}_m \quad \forall\, m \geq 1$$

and

$$\tau_n = \tilde{\lambda}_{na}\big|_{\mathcal{Q}[F^{ns}]} \quad \forall\, n \geq n_0.$$

Here s is chosen as in R2, and $n_0 = \lceil \frac{m_0}{a} \rceil$ is chosen/defined also in R2 below. Note that $\mathcal{Q}[F^{ns}] \subseteq E_{na}$ for all $n \geq n_0$ according to the estimate (2a) in R2.

Reduction steps and consequences of the above assumptions.

R1. After modifying Z be a suitable isogeny, one may assume $Z = Z_1 \times \cdots \times Z_c$ is a product of isoclinic p-divisible groups $Z_1, \ldots, Z_c$, with distinct slopes $\mu_1 > \cdots > \mu_c$.

R2. Choose and fix a positive rational number $\mu_0 < \mu_1$ such that μ_0 is strictly bigger than every slope of $Z_2 \times \cdots \times Z_c \times X \times Y$. Write μ_0, μ_1 in the form

$$\mu_1 = \frac{a}{r},\ \mu_0 = \frac{a}{s}, \quad s > r, \quad a, b, s, r \in \mathbb{N}_{>0}.$$

From general properties of slopes we know that there exists a $m_0 \in \mathbb{N}$ such that

$$X[p^m] \supset \mathrm{Ker}(\mathrm{Fr}_{X/k}^{\lfloor m/\mu_0 \rfloor}) \quad \text{and} \quad Y[p^m] \supset \mathrm{Ker}(\mathrm{Fr}_{Y/k}^{\lfloor m/\mu_0 \rfloor}) \tag{2a}$$

for all $m \geq m_0$. Therefore

$$X[p^{na}] \supset \mathrm{Ker}(\mathrm{Fr}_{X/k}^{ns}) \quad \text{and} \quad Y[p^{na}] \supset \mathrm{Ker}(\mathrm{Fr}_{Y/k}^{ns}) \tag{2b}$$

for all $n \geq n_0 := \lceil \frac{m_0}{a} \rceil$.

R3. After extending the base field κ we may assume that κ is algebraically closed. Modifying the isoclinic p-divisible group Z_1 by an isogeny if necessary, we may assume that

$$\mathrm{Ker}([p^a]_{Z_1}) = \mathrm{Fr}_{Z_1/\kappa}^r\,,$$

where $\mathrm{Fr}_{Z_1/\kappa}^r : Z_1 \to Z_1^{(p^r)}$ is the r-th iterate of the relative Frobenius morphism for Z_1/κ. Equivalently, there exist local parameters $u_1, \ldots, u_b$ of the formal scheme Z such that

$$Z_1 = \mathrm{Spf}(k[[u_1, \ldots, u_b]]) \ \text{ and } \ [p^a]_{Z_1}^*(u_i) = u_i^{p^r} \quad \forall\, i = 1, \ldots, b.$$

R4. Passing to the isoclinic component Z_1 of Z, and consider the $[p^a]_{Z_1}$-compatible sequence

$$\left(\mathcal{Q}[F^{ns}] \xrightarrow{\tilde{\lambda}_n} Z \xrightarrow{\mathrm{pr}_1} Z_1 \right)_{n\geq n_1}$$

with coordinates

$$(\mathrm{pr}_1 \circ \tilde{\lambda}_n)^* u_i =: \tilde{f}_{i,n} \ \in \ R/\mathfrak{m}^{p^{(ns)}}, \quad i = 1, \ldots, b,\ n \geq n_0.$$

For each $i = 1, \ldots, b$, the sequence $\big(\tilde{f}_{i,n}\big)_{n \geq n_0}$ satisfy

(4a) $$\tilde{f}_{i,n}^{p^r} \equiv \tilde{f}_{i,n+1} \pmod{\mathfrak{m}^{(p^{ns})}} \quad \forall\, n \geq n_0.$$

We may assume that $s - r \geq 2$. Choose an integer s_1 such that $r < s_1 < s$. Let $f_{i,n}$ be the image of $\tilde{f}_{i,n}$ in $R/\mathfrak{m}^{(p^{ns_1})}$ for all $i = 1, \ldots, b$ and all $n \geq \max(n_0, r) := n_1$. Then

(4b) $$f_{i,n}^{p^r} \equiv f_{i,n+1} \pmod{\mathfrak{m}^{(p^{ns_1+r})}} \quad \forall\, n \geq n_1,\ i = 1, \ldots, b.$$

Remark. (i) In practice we will choose μ_0 to be "just a tiny bit bigger than the maximum of the slopes of X and Y".

(ii) If we choose μ_0 to be the maximum of the slopes of X and Y, then the estimate (2b) needs to be changed to: there exists a constant e (depending on X and Y) such that

(10.7.2.2.3) $$X[p^{na}] \supset \mathrm{Ker}(\mathrm{Fr}_{X/k}^{ns-e}), \quad \text{and} \quad Y[p^{na}] \supset \mathrm{Ker}(\mathrm{Fr}_{Y/k}^{ns-e})$$

for all $n \geq n_1 := \lceil \frac{m_0}{a} \rceil$.

(iii) The congruences relations (4b) means that for each $i = 1, \ldots, b$, the sequence

$$\big(f_{i,n} \in R/\mathfrak{m}^{(p^{ns_1})}\big)_{n \geq n_1}$$

is ϕ^r-$\sharp$-compatible in the sense of definition 10.7.2.3 (a).

As one sees in R4 above, the difference in the compatibility conditions (4a) and (4b) is essentially one of appearance rather than substance.

10.7.2.3. Definition. Let $(R, \mathfrak{m})$ be an augmented complete Noetherian local ring over a field κ of characteristic p. Let $r, s > 0$ be positive integers with $r < s$.

(a1) A sequence of elements $(f_n)_{n \geq n_0}$ with $f_n \in R/\mathfrak{m}^{(p^{ns})}$ for all n is *ϕ^r-$\sharp$-compatible with respect to ϕ^s* if

$$f_n^{p^r} \equiv f_{n+1} \pmod{\mathfrak{m}^{(p^{ns+r})}} \quad \forall\, n \geq n_0$$

(a2) A sequence of elements $(f_n)_{n \geq n_0}$ with $f_n \in R/\mathfrak{m}^{(p^{ns})}$ for all n is weakly *ϕ^r-$\sharp$-compatible with respect to ϕ^s* if

$$f_n^{p^r} \equiv f_{n+1} \pmod{\mathfrak{m}^{(p^{ns})}} \quad \forall\, n \geq n_0$$

(b1) A sequence of elements $(g_n)_{n \geq n_0}$ with $g_n \in R/\mathfrak{m}^{p^{ns}}$ for all n is *ϕ^r-$\flat$-compatible with respect to ϕ^s* if

$$g_n^{p^r} \equiv g_{n+1} \pmod{\mathfrak{m}^{p^{ns+r}}} \quad \forall\, n \geq n_0$$

(b2) A sequence of elements $(g_n)_{n \geq n_0}$ with $g_n \in R/\mathfrak{m}^{p^{ns}}$ for all n is weakly *ϕ^r-$\flat$-compatible with respect to ϕ^s* if

$$g_n^{p^r} \equiv g_{n+1} \pmod{\mathfrak{m}^{p^{ns}}} \quad \forall\, n \geq n_0$$

If the context makes confusion unlikely, we will shorten both "ϕ^r-$\sharp$-compatible with respect to ϕ^s " and "ϕ^r-$\flat$-compatible" to "ϕ^r-compatible with respect to ϕ^s ". Similarly, if the integer s is clear in the context, we will omit the part "with respect to ϕ^s.

Remark. (i) We have used the fact that the r-th power of Frobenius induces well-defined maps

$$R/\mathfrak{m}^{(p^{ns})} \to R/\mathfrak{m}^{(p^{ns+r})} \quad \text{and} \quad R/\mathfrak{m}^{p^{ns}} \to R/\mathfrak{m}^{p^{ns+r}}$$

in the statements (a1) and (b1) above.

(ii) The $\flat$-version (b1), (b2) is different from the $\sharp$-version (a1), (a2) in that the element g_n is in the congruence class modulo the ideal $\mathfrak{m}^{p^{ns}}$, which is bigger than the ideal $\mathfrak{m}^{(p^{ns})}$.

(iii) Suppose that $\big(f_n\big)_{n\geq n_0}$ with $f_n \in R/\mathfrak{m}^{(p^{ns})}$ is a ϕ^r-$\sharp$-compatible (respectively weakly ϕ^r-$\sharp$-compatible) sequence. Let g_n be the image of f_n in $R/\mathfrak{m}^{p^{ns}}$. Then $\big(g_n\big)_{n\geq n_0}$ is a ϕ^r-$\flat$-compatible (respectively weakly ϕ^r-$\sharp$-compatible) sequence.

The following lemma 10.7.2.4 is obvious.

10.7.2.4. Lemma. *Let $r, s, s' > 0$ are positive integers with $r < s < s'$. Let $(\tilde{f}_n)_{n\geq n_0}$ be a sequence of elements with $\tilde{f}_n \in R/\mathfrak{m}^{(p^{ns'})}$ for each n, and let $(\tilde{g}_n)_{n\geq n_0}$ be a sequence of elements with $\tilde{g}_n \in R/\mathfrak{m}^{p^{ns'}}$ for each n. Let f_n be the image of $\tilde{f}_n$ in $R/\mathfrak{m}^{(p^{ns'})}$, and let g_n be the image of $\tilde{g}_n$ in $R/\mathfrak{m}^{p^{ns'}}$. Let n_1 be a natural number such that $n_1 \geq n_0$ and $n_1(s'-s) \geq r$.*

(a) *If $(\tilde{f}_n)_{n\geq n_0}$ is weakly ϕ^r-$\sharp$-compatible then $(f_n)_{n\geq n_1}$ is ϕ^r-$\sharp$-compatible.*
(b) *If $(\tilde{g}_n)_{n\geq n_0}$ is weakly ϕ^r-$\flat$-compatible then $(g_n)_{n\geq n_1}$ is ϕ^r-$\flat$-compatible.*

The following lemma 10.7.2.5, which relates the notion of $[p^a]$-compatible sequences to the notion of ϕ^r-compatible sequences, follows easily from the definitions.

10.7.2.5. Lemma. *Let R be an augmented complete Noetherian local domain over a field κ of characteristic p, and let $\mathfrak{Q} = \mathrm{Spf}(R)$. Let Z, Z_1 be p-divisible groups over κ. Let $h : Z \to Z_1$ be a κ-homomorphism. Suppose that $Z_1 = \mathrm{Spf}(\kappa[[u_1, \ldots, u_b]])$ and*

$$[p^a]_{Z_1}^*(u_i) = u_i^{p^r} \quad \text{for} \quad i = 1, \ldots, b,$$

where $a, r > 0$ are positive integers. Let $\tau_n : \mathfrak{Q}[F^{n(s+1)}] \to Z$, $n \geq n_0$ be maps from $\mathfrak{Q}[F^{ns}]$ to Z, where $s > r$ is a positive integer. If the sequence $\big(\tau_n\big)_{n\geq n_0}$ is $[p^a]_Z$-compatible, then for each $i = 1, \ldots, b$, the sequence

$$\Big(f_{i,n} := (h \circ \tau_n)^*(u_i)\big|_{\mathcal{Q}[F^{ns}]}\Big)_{n\geq n_0}$$

of elements of $R/\mathfrak{m}^{(p^{ns})}$ is weakly ϕ^r-$\sharp$-compatible.

10.7.3. Tempered perfections of formal power series rings.

In this subsection κ is a perfect field of characteristic p, and m is a positive integer. We will define several families of completed tempered perfections of the power series ring $\kappa[[t_1, \ldots, t_m]]$.

10.7.3.1. Notations.

(i) Let $\kappa[[\underline{t}]] := \kappa[[t_1, \ldots, t_m]]$, let $\underline{t}$ be the m-tuple with entries $t_1, \ldots, t_m$, and let $(\underline{t}) := t_1\kappa[[\underline{t}]] + \cdots + t_m\kappa[[\underline{t}]]$, the maximal ideal of $\kappa[[\underline{t}]]$.

(ii) Denote by ϕ the Frobenius map on $\kappa[[\underline{t}]]$ which sends every element of $\kappa[[\underline{t}]]$ to its p-th power.

(iii) For each element $I = (i_1, \ldots, i_m) \in \mathbb{N}^m$, let
$$\underline{t}^I := t_1^{i_1} \cdots t_m^{i_m}$$
be the corresponding monomial in the variables $t_1, \ldots, t_m$.

(iv) For every natural number j, let $(\underline{t})^{p^j}$ be the p^j-th power of the maximal ideal $(\underline{t})$ as usual. Note that $(\underline{t})^{p^j}$ is the ideal generated by all monomials $\underline{t}^I := t_1^{i_1} \cdots t_m^{i_m}$ such that $I = (i_1, \ldots, i_m) \in \mathbb{N}^m$ satisfies $i_1 + \cdots + i_m \geq p^j$.

Let
$$(\underline{t}^{p^j}) = (\underline{t})^{(p^j)} := (t_1^{p^j}, \ldots, t_m^{p^j})$$
be the ideal of $\kappa[[\underline{t}]]$ generated by $\phi^j(t_1\kappa[[\underline{t}]] + \cdots + t_m\kappa[[\underline{t}]])$; i.e. $(\underline{t}^{p^j}) = (\underline{t})^{(p^j)}$ is the completion of the κ-linear span of all monomials $\underline{t}^I$ with $I \equiv 0 \,(\mathrm{mod}\, p^j)$.

(v) We will use the following two archimedean norms on $\mathbb{Q}^m$:
$$|J|_\infty := \max(|j_1|, \ldots, |j_m|), \qquad |J|_\sigma := |j_1| + \cdots + |j_m|$$
for every element $J = (j_1, \ldots, j_m) \in \mathbb{Q}^m$. Obviously
$$|J|_\infty \leq |J|_\sigma \leq m \cdot |J|_\infty \qquad \forall\, J \in \mathbb{Q}^m$$

(vi) There is also the following p-adic norm on $\mathbb{Q}^m$:
$$|J|_p := \max\left(|j_1|_p, \ldots, |j_m|_p\right)$$
where $|\cdot|_p$ is multiplicative p-adic absolute value on $\mathbb{Q}$, defined by $|x|_p = p^{-\mathrm{ord}_p(x)}$ for all $x \in \mathbb{Q}$, so that $|p| = \frac{1}{p}$ and $|x|_p = 1$ if both the numerator and denominator of x are prime to p. Define
$$\mathrm{ord}_p(J) := \mathrm{Min}\left(\mathrm{ord}_p(j_1), \ldots \mathrm{ord}_p(j_m)\right),$$
hence
$$|J|_p = p^{-\mathrm{ord}_p(J)}.$$
We will use the restriction of these norms to $\mathbb{N}[1/p]^m := \mathbb{Z}[1/p]^m_{\geq 0}$, the additive semigroup of exponents with p-power denominators.

10.7.3.2. Limits of ϕ_r-compatible sequences. Let $r < s$ be positive integers. Let $(a_n)_{n \geq n_0}$ be a sequence with $a_n \in \kappa[[\underline{t}]]/(\underline{t}^{p^{ns}})$ for all $n \geq n_0$. Suppose that this sequence is ϕ^r-$\sharp$-compatible in the sense of 10.7.2.3 (a), i.e.

$$(\dagger) \qquad a_{j,n}^{p^r} \equiv a_{j,n+1} \pmod{\mathfrak{m}_{\kappa[[\underline{t}]]}^{(p^{ns+r})}}$$

for all $n \geq n_0$. We will construct a "limit" of such a ϕ^r-$\sharp$-compatible sequence in a lowbrow fashion.

For each $n \geq n_0$, write the element $a_n \in \kappa[[\underline{t}]]/(\underline{t}^{p^{ns}})$ as

$$a_n = \sum_{J \in \mathbb{N}^m,\, |J|_\infty < p^{ns}} a_{n,J}\, \underline{t}^J \mod (\underline{t}^{p^{ns}}).$$

Clearly the coefficients $a_{n,J} \in \kappa$ with $|J|_\infty < p^{ns}$ are uniquely determined by a_n. The compatibility relation $a_n^{p^r} \equiv a_{n+1} \pmod{(\underline{t}^{p^{ns+r}})}$ means that $a_{n+1,J} = a_{n,p^{-r}J}$ for all $J \in \mathbb{N}^m$ with $|J|_\infty < p^{ns+r}$ and all $n \geq n_0$. More precisely,

$$(\ddagger) \qquad a_{n+1,J} = \begin{cases} 0 & \text{if} \quad |J|_\infty < p^{ns+r}, \quad p^{-r}J \notin \mathbb{N}^m \\ a_{n,p^{-r}J}^{p^r} & \text{if} \quad |J|_\infty < p^{ns+r}, \quad p^{-r}J \in \mathbb{N}^m \end{cases}$$

for all $n \geq n_0$. Thus the among the coefficients $a_{n,J}$ for a fixed natural number $n \geq n_0 + 1$, those with $|J|_\infty < p^{(n-1)s+r}$ arises from coefficients $a_{n',J'}$ with $n' < n$. More precisely suppose that $n \geq n_0 + 1$, then the following statements hold.

- If $|J|_\infty < p^{(n-1)s+r}$ and J is not divisible by p^r, then $a_{n,J} = 0$.
- Suppose that $|J|_\infty < p^{(n-1)s+r}$ and $J = p^{(n-n')r}J'$, where $n' < n$ and J' is not divisible by p^r. Then $a_{n,J} = a_{n',J'}^{p^{(n-n')r}}$.

There is no constraint for those $a_{n,J}$'s with $|J|_\infty \geq p^{(n-1)s+r}$; these coefficients will be propagated to coefficients of $a_{n'',J}$'s with $n'' > n$.

Construction of the limit. For each element $I \in \mathbb{N}[1/p]^m$, define $b_I \in \kappa$ by

$$b_I := (a_{n,p^{nr}J})^{p^{-rn}} = \phi^{-rn}(a_{n,p^{nr}J}),$$

where $n \in \mathbb{N}$ is sufficiently large such that $p^{nr}I \in \mathbb{N}^m$ and $|p^{nr}I|_\infty < p^{sn}$, so that $a_{n,p^{nr}I}$ makes sense. The compatibility relation for the $a_{n,J}$'s immediately implies that the above definition does not depend on the choice of n, as long as

$$n \geq \operatorname{Max}\left(\frac{-\operatorname{ord}_p(J)}{r},\ \frac{\log_p(|J|_\infty)}{s-r}\right).$$

The formal series

$$\sum_{I \in \mathbb{N}[1/p]^m} b_I\, \underline{t}^I = \sum_{(i_1,\ldots,i_m) \in \mathbb{N}[1/p]^m} b_{i_1,\ldots,i_m}\, t_1^{i_1} \cdots t_m^{i_m}$$

attached to a given ϕ^r-compatible sequence of elements $\left(a_n \in \kappa[[\underline{t}]]/(\underline{t}^{p^{sn}})\right)_{n \geq n_0}$ according to the above construction will be called the *limit* of the ϕ^r-compatible sequence $(a_n)_{n \geq n_0}$.

10.7.3.3. Proposition. (a) *The construction described in 10.7.3.2 establishes a bijection, from the set of all ϕ^r-$\sharp$-compatible sequences of elements $\left(a_n \in \kappa[[\underline{t}]]/(\underline{t}^{p^{sn}})\right)_{n \geq n_0}$, to the set of all formal series*

$$\sum_{I \in \mathbb{N}[1/p]^m} b_I\, \underline{t}^I$$

such that $b_I \in \kappa$ for all $I \in \mathbb{N}[1/p]^m$, and

$$(*) \qquad -\mathrm{ord}_p(I) \le \mathrm{Max}\left\{ n_0,\ r \cdot \left(\left\lfloor \frac{\log_p(|I|_\infty)}{s-r} \right\rfloor + 1 \right) \right\}$$

for every $I \in \mathbb{N}[1/p]^m$ with $b_I \neq 0$.

(b) *A similar construction, with the archimedean norm $|\cdot|_\infty$ replaced by $|\cdot|_\sigma$, gives a bijection from set of all ϕ^r-$\flat$-compatible sequences of elements $\left(a_n \in \kappa[[\underline{t}]]/(\underline{t})^{p^{sn}}\right)_{n\ge n_0}$, to the set of all formal series*

$$\sum_{I\in\mathbb{N}[1/p]^m} b_I\ \underline{t}^I$$

such that $b_I \in \kappa$ for all $I \in \mathbb{N}[1/p]^m$, and

$$(**) \qquad -\mathrm{ord}_p(I) \le \mathrm{Max}\left\{ n_0,\ r \cdot \left(\left\lfloor \frac{\log_p(|I|_\sigma)}{s-r} \right\rfloor + 1 \right) \right\}$$

for every $I \in \mathbb{N}[1/p]^m$ with $b_I \neq 0$.

Proof. We will prove the statement (a) only. After replacing $|\cdot|_\infty$ by $|\cdot|_\sigma$, the construction of the limits of ϕ^r-$\flat$-compatible sequences works verbatim. So does the proof of (b).

Although the estimate in the statement (a) of 10.7.3.3 looks complicated, its proof is completely straight-forward from the construction explained in 10.7.3.2.

Suppose that $\sum_{I\in\mathbb{N}[1/p]^m} b_I\ \underline{t}^I$ is attached to a ϕ^r-$\sharp$-compatible sequence $(a_n)_{n\ge n_0}$, $a_n \in \kappa[[\underline{t}]]/(\underline{t}^{p^{ns}})$ for all $n \ge n_0$. Let $I \in \mathbb{N}[1/p]^m$ be an index in the support of the above formal series, i.e. $b_I \neq 0$. We need to show that the inequality $(*)$ holds. Let n_1 be the smallest natural number such that $p^{n_1 r} I \in \mathbb{N}^m$. There is nothing to prove if $n_1 \le n_0$, so we may assume that $n_1 \ge n_0 + 1$. In particular $\mathrm{ord}_p(I) < 0$, and $n_1 = \lceil \frac{-\mathrm{ord}_p(I)}{r} \rceil$.

From the definition of n_1 we know that $p^{n_1 r} I$ is not divisible by p^r. If $|p^{n_1 r} I|_\infty < p^{(n_1-1)s+r}$, we get from 10.7.3.2 (‡) that $b_I = 0$, a contradiction. We have shown that

$$|p^{n_1 r} I|_\infty \ge p^{(n_1-1)s+r}.$$

The last inequality is equivalent to

$$\left\lceil \frac{-\mathrm{ord}_p(I)}{r} \right\rceil = n_1 \le \frac{\log_p |I|_\infty}{s-r} + 1,$$

which is easily seen to be equivalent to the inequality $(*)$.

It remains to show that every formal series $\sum_{I\in\mathbb{N}[1/p]^m} b_I\ \underline{t}^I$ whose support satisfies the inequality $(*)$ arises from a ϕ^r-compatible sequence $(a_n)_{n\ge n_0}$. One verifies using the inequality $(*)$ that for every natural number $n \ge n_0$, the truncated series

$$c_n := \sum_{I\in\mathbb{N}[1/p]^m,\, |p^{nr}I|_\infty < p^{ns}} b_I^{p^{nr}}\ \underline{t}^{p^{nr}I} \in \kappa[[\underline{t}]]$$

Let $a_n := c_n \mod (\underline{t}^{p^{ns}})$. It is easily verified that $(a_n)_{n\geq n_0}$ is a ϕ^r-$\sharp$-compatible sequence, whose limit is the given formal series $\sum_{I\in\mathbb{N}[1/p]^m} b_I\ \underline{t}^I$. $\square$

10.7.3.4. Definition. Let κ be a perfect field of characteristic p, and let $t_1,\dots,t_m$ be m variables, $m\geq 1$. Let $r,s\in\mathbb{Z}_{>0}$ be two positive integers with $r<s$, and let n_0 be a natural number.

(a) $\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle^{\#}_{s:\phi^r,\,\geqslant n_0}$ is the commutative κ-algebra consisting of all formal series

$$\sum_{I\in\mathbb{N}[1/p]^m} b_I\ \underline{t}^I$$

such that $b_I\in\kappa$ for all $I\in\mathbb{N}[1/p]^m$, and

$$(*)\qquad -\mathrm{ord}_p(I)\leq \mathrm{Max}\left\{n_0,\ r\cdot\left(\left\lfloor\frac{\log_p(|I|_\infty)}{s-r}\right\rfloor+1\right)\right\}$$

for every $I\in\mathbb{N}[1/p]^m$ such that $b_I\neq 0$.

Denote by $\mathrm{supp}\bigl(\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle^{\#}_{s:\phi^r,\,\geqslant n_0}\bigr)$ the subset of $\mathbb{N}[1/p]^m$ consisting of all multi-indices $I\in\mathbb{N}[1/p]^m$ such that the inequality $(*)$ holds.

(b) $\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle^{\flat}_{s:\phi^r,\,\geqslant n_0}$ is the commutative κ-algebra consisting of all formal series

$$\sum_{I\in\mathbb{N}[1/p]^m} b_I\ \underline{t}^I$$

such that $b_I\in\kappa$ for all $I\in\mathbb{N}[1/p]^m$, and

$$(**)\qquad -\mathrm{ord}_p(I)\leq \mathrm{Max}\left\{n_0,\ r\cdot\left(\left\lfloor\frac{\log_p(|I|_\sigma)}{s-r}\right\rfloor+1\right)\right\}$$

for every $I\in\mathbb{N}[1/p]^m$ such that $b_I\neq 0$.

Denote by $\mathrm{supp}\bigl(\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle^{\flat}_{s:\phi^r,\,\geqslant n_0}\bigr)$ the subset of $\mathbb{N}[1/p]^m$ consisting of all multi-indices $I\in\mathbb{N}[1/p]^m$ such that the inequality $(**)$ holds.

Remark. (i) The two support sets

$$\mathrm{supp}\bigl(\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle^{\#}_{s:\phi^r,\,\geqslant n_0}\bigr)\quad\text{and}\quad\mathrm{supp}\bigl(\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle^{\flat}_{s:\phi^r,\,\geqslant n_0}\bigr)$$

are sub-semigroups of $(\mathbb{N}[1/p]^m,+)$ containing the 0-element. Moreover for every $M>0$, there are only a finite number elements I in either sub-semigroup such that $|I|_\infty\leq M$. The last property implies that for each I, there are only a finite number of pairs (I_1,I_2) of elements in either sub-semigroup such that $I_1+I_2=I$. Therefore multiplication is well-defined on both $\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle^{\#}_{s:\phi^r,\,\geqslant n_0}$ and $\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle^{\flat}_{s:\phi^r,\,\geqslant n_0}$, via the standard formula for multiplication of formal series.

(ii) It is easy to see that the rings $\kappa\langle\langle t_1,\ldots,t_m\rangle\rangle^{\#}_{s:\phi^r,\geqslant n_0}$ and $\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{\flat}_{s:\phi^r,\geqslant n_0}$ are non-Neotherian local domains. It is easy to see that neither of the two local domains is normal. Moreover the integral closure of

$$\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{\#}_{s:\phi^r,\geqslant n_0}\quad\big(\text{respectively}\quad\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{\flat}_{s:\phi^r,\geqslant n_0}\big)$$

in its own fraction field is *not* a finitely generated module over $\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{\#}_{s:\phi^r,\geqslant n_0}$ (respectively $\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{\flat}_{s:\phi^r,\geqslant n_0}$), because the normalizations of both rings contain t_i^j for all $j\in\mathbb{N}[1/p]$ and all $i=1,\ldots,m$.

Below is a slightly different version of the rings defined in 10.7.3.4.

10.7.3.5. Definition. Let κ be a perfect field of characteristic p. Let $r<s$ be two positive integers, and let $i_0\in\mathbb{N}$ be a natural number. The perfection of the formal power series $\kappa[[t_1,\ldots,t_m]]$ is naturally isomorphic to

$$\bigcup_{n\in\mathbb{N}}\kappa[[t_1^{p^{-n}},\ldots,t_m^{p^{-n}}]].$$

Denote by ϕ the Frobenius automorphism of this perfect ring.

(a) Consider the following subring

$$\big(\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{\#}_{s:\phi^r;[i_0]}\big)_{\mathrm{fin}}:=\sum_{n\in\mathbb{N}}\phi^{-nr}\big((\underline{t})^{(p^{ns-i_0})}\big)$$

of the perfection of the formal power series ring $\kappa[[t_1,\ldots,t_m]]$, where our convention is that $(\underline{t})^{(p^{ns-i_0})}=R$ if $ns-i_0\le 0$.

(a1) Define a decreasing filtration $\Big(\mathrm{Fil}^{\#,p^\bullet}_{s:\phi^r,[i_0]}\Big)_{\bullet\in\mathbb{Z}}$ on $\big(\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{\#}_{s:\phi^r;[i_0]}\big)_{\#,\,\mathrm{fin}}$ by ideals

$$\mathrm{Fil}^{\#,p^j}_{s:\phi^r,[i_0]}:=\Big\{x\in\big(\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{\#}_{s:\phi^r;[i_0]}\;\Big|\;\exists n\in\mathbb{N}_{>0}\ \text{ s. t. }\ n+j\ge 0\text{ and }\ x^{p^n}\in(\underline{t})^{(p^{n+j})}\Big\},$$

of $\big(\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{\#}_{s:\phi^r;[i_0]}\big)_{\#,\mathrm{fin}}$, where $(\underline{t})$ is the maximal ideal of $\kappa[[t_1,\ldots,t_m]]$.

(a2) Define $\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{\#}_{s:\phi^r;[i_0]}$ to be the completion of the ring

$$\big(\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{\#}_{s:\phi^r;[i_0]}\big)_{\mathrm{fin}}$$

with respect to the filtration $\mathrm{Fil}^{\#,p^\bullet}_{s:\phi^r,[i_0]}$.

(b) Consider the following subring

$$\big(\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{\flat}_{s:\phi^r;[i_0]}\big)_{\mathrm{fin}}:=\sum_{n\in\mathbb{N}}\phi^{-nr}\big((\underline{t})^{p^{ns-i_0}}\big)$$

of the perfection of the formal power series ring $\kappa[[t_1,\ldots,t_m]]$. In the above our convention is that $(\underline{t})^{p^{ns-i_0}}=R$ if $ns-i_0\le 0$.

(b1) Define a decreasing filtration $\left(\mathrm{Fil}^{\flat,\bullet}_{s:\phi^r,[i_0]}\right)_{\bullet\in\mathbb{Z}[1/p]_{\geqslant 0}}$ on $\left(\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{\flat}_{s:\phi^r;[i_0]}\right)_{\mathrm{fin}}$ by

$$\mathrm{Fil}^{\flat,u}_{s:\phi^r,[i_0]} := \left\{x\in\left(\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{\flat}_{s:\phi^r;[i_0]}\,\middle|\,\exists n\in\mathbb{N}_{>0}\text{ such that } p^n u\in\mathbb{N}\text{ and } x^{p^n}\in(\underline{t})^{u\cdot p^n}\right\}.$$

(b2) Define $\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{\flat}_{s:\phi^r;[i_0]}$ to be the completion of the ring

$$\left(\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{\flat}_{s:\phi^r;[i_0]}\right)_{\mathrm{fin}}$$

with respect to the filtration $\mathrm{Fil}^{\flat,\bullet}_{s:\phi^r,[i_0]}$.

We will introduce in 10.7.3.6 two other families,

$$\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{E,\#}_{C;\,d}\qquad\text{and}\qquad\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d},$$

of completed tempered perfections of the power series ring $\kappa[[t_1,\ldots,t_m]]$, related to the rings defined in 10.7.3.4 and 10.7.3.5. We will also see in 10.7.4.1 and 10.7.4.2 that the notion of completed tempered perfection in 10.7.3.4 and 10.7.3.5 can be extended to general complete Noetherian local domains of equi-characteristic $p>0$ with perfect residue fields.

10.7.3.6. Definition. Let κ be a perfect field of characteristic p, and let $t_1,\ldots,t_m$ be variables. Let $C>0, d\geq 0, E>0$ be real numbers.

(a) Define a commutative κ-algebra

$$\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{E,\#}_{C;\,d}$$

whose underlying abelian group is the set of all formal series $\sum_I b_I\,\underline{t}^I$ with $b_I\in\kappa$ for all I, where I runs through all elements in $\mathbb{N}[1/p]^m$ such that

$$(\sharp)\qquad |I|_p\leq\mathrm{Max}\big(C\cdot(|I|_\infty+d)^E,1\big)\,.$$

The ring structure is given by the standard formula for product of power series.

The inequality ($\sharp$) defines a subset $\mathrm{supp}(m\!:\!\sharp\!:\!E;C,d)$ of $\mathbb{N}[1/p]^m$:

$$\mathrm{supp}(m\!:\!\sharp\!:\!E;C,d) := \left\{I\in\mathbb{N}[1/p]^m\,\middle|\,|I|_p\leq\mathrm{Max}\big(C\cdot(|I|_\infty+d)^E,1\big)\right\}.$$

(b) Define a commutative κ-algebra

$$\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d}$$

whose underlying abelian group is the set of all formal series $\sum_I b_I\,\underline{t}^I$ with $b_I\in\kappa$ for all I, where I runs through all elements in $\mathbb{N}[1/p]^m$ such that

$$(\flat)\qquad |I|_p\leq\mathrm{Max}\big(C\cdot(|I|_\sigma+d)^E,1\big)\,.$$

The above condition on the support (of elements of this subset) shows that the standard formula for multiplication makes sense and gives $\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d}$ a natural structure as an augmented commutative algebra over κ.

The inequality ($\flat$) defines a subset $\operatorname{supp}(m{:}\flat{:}E;C,d) \subseteq \mathbb{N}[1/p]^m$:

$$\operatorname{supp}(m{:}\flat{:}E;C,d) := \Big\{ I \in \mathbb{N}[\tfrac{1}{p}]^m \;\Big|\; |I|_p \le \operatorname{Max}\big(C\cdot(|I|_\sigma + d)^E, 1\big)\Big\}$$

10.7.3.7. Lemma. *Denote by* $\operatorname{Fil}^{\bullet}_{\mathrm{t.deg}}$ *the decreasing filtration on* $\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d}$ *such that*

$$\operatorname{Fil}^u_{\mathrm{t.deg}}\big(\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d}\big) := \left\{ \sum_{I\in \operatorname{supp}(m:E;C,d),\,|I|_\sigma\ge u} b_I\,\underline{t}^I \;\middle|\; b_I\in\kappa \;\;\forall\, I\right\}$$

for every $u\in\mathbb{R}$. *Let*

$$\operatorname{Fil}^{u+}_{\mathrm{t.deg}}\big(\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d}\big) := \bigcup_{\epsilon>0}\operatorname{Fil}^{u+\epsilon}_{\mathrm{t.deg}}(\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d})$$

(i) *Both* $\operatorname{Fil}^u_{\mathrm{t.deg}}\big(\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d}\big)$ *and* $\operatorname{Fil}^{u+}_{\mathrm{t.deg}}\big(\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d}\big)$ *are ideals of the ring* $\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d}$, *for every* $u\in\mathbb{R}$.

(ii) *Let* $\operatorname{gr}^{\bullet}\big(\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d}\big)$ *be the graded ring attached to the filtration* $\operatorname{Fil}^{\bullet}_{\mathrm{t.deg}}$ *of the ring* $\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d}$. *This graded ring is naturally isomorphic to the graded subring*

$$\bigoplus_{I\in\operatorname{supp}(m:E;C,d)} \kappa\cdot\underline{t}^I$$

of the perfection

$$\kappa[t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}] = \bigoplus_{I\in\mathbb{N}[1/p]^m}\kappa\cdot\underline{t}^I$$

of the polynomial ring $\kappa[t_1,\dots,t_m]$, *where the latter is graded by the total degree* $|I|_\sigma$ *of monomials* $\underline{t}^I$.

The proof is easy, therefore omitted. □

10.7.4. Tempered perfections of general augmented Noetherian local domains.

10.7.4.1. Definition. Let $(R,\mathfrak{m})$ be an augmented complete Noetherian local domain over a perfect field κ of characteristic p. Let R^{perf} be the perfection of R, and let ϕ be the Frobenius automorphism on R. Let r,s,n_0 be natural numbers, $0<r<s$, $i_0\ge 0$.

(a) Consider the following subset

$$\big((R,\mathfrak{m})^{\mathrm{perf},\,\#}_{s:\phi^r;[i_0]}\big)_{\mathrm{fin}} := \sum_{n\ge 0}\phi^{-nr}\big(\mathfrak{m}^{(p^{ns-i_0})}\big)$$

of the perfect domain R^{perf}. In the above $\mathfrak{m}^{(p^{ns-i_0})} = R$ by convention if $ns-i_0\le -1$. It is easy to see that this subset is a subring of R^{perf}. Define a decreasing filtration

$\big(\mathrm{Fil}^{\#,p^\bullet}_{s:\phi^r;[i_0]}\big)_{\bullet\in\mathbb{Z}}$ on $\big(\,(R,\mathfrak{m})^{\mathrm{perf},\,\#}_{s:\phi^r;[i_0]}\,\big)_{\mathrm{fin}}$ by ideals

$$\mathrm{Fil}^{\#,p^j}_{s:\phi^r;[i_0]} := \Big\{ x \in \big(\,(R,\mathfrak{m})^{\mathrm{perf},\,\#}_{s:\phi^r;[i_0]}\,\big)_{\mathrm{fin}} \;\Big|\; \exists n\in\mathbb{N} \text{ s.t. } x^{p^n} \in \mathfrak{m}^{p^{n+j}} \Big\}.$$

Define a complete augmented local domain $(R,\mathfrak{m})^{\mathrm{perf},\,\#}_{s:\phi^r;[i_0]}$ over κ by

$$(R,\mathfrak{m})^{\mathrm{perf},\,\#}_{s:\phi^r;[i_0]} := \text{ the completion of } \big(\,(R,\mathfrak{m})^{\mathrm{perf},\,\#}_{s:\phi^r;[i_0]}\,\big)_{\mathrm{fin}} \text{ with respect to } \big(\mathrm{Fil}^{\#,p^\bullet}_{s:\phi^r;[i_0]}\big)_\bullet .$$

(b) Consider the following subset

$$\big(\,(R,\mathfrak{m})^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_0]}\,\big)_{\mathrm{fin}} := \sum_{n\geq 0} \phi^{-nr}\big(\mathfrak{m}^{p^{ns-i_0}}\big)$$

of the perfect domain R^{perf}. Here $\mathfrak{m}^{p^{ns-i_0}} = R$ if $ns - i_0 \leq -1$. It is easy to see that this subset is a subring of R^{perf}. Define a decreasing filtration $\big(\mathrm{Fil}^{\flat,p^\bullet}_{s:\phi^r;[i_0]}\big)_{j\in\mathbb{Z}}$ on $\big(\,(R,\mathfrak{m})^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_0]}\,\big)_{\mathrm{fin}}$ by ideals

$$\mathrm{Fil}^{\flat,p^j}_{s:\phi^r;[i_0]} := \Big\{ x \in \big(\,(R,\mathfrak{m})^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_0]}\,\big)_{\mathrm{fin}} \;\Big|\; \exists n\in\mathbb{N} \text{ s.t. } x^{p^n} \in \mathfrak{m}^{p^{n+j}} \Big\}.$$

Define a complete augmented local domain $(R,\mathfrak{m})^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_0]}$ over κ by

$$(R,\mathfrak{m})^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_0]} := \text{the completion of } \big(\,(R,\mathfrak{m})^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_0]}\,\big)_{\mathrm{fin}} \text{ with respect to } \big(\mathrm{Fil}^{\flat,p^\bullet}_{s:\phi^r;[i_0]}\big)_\bullet .$$

10.7.4.2. Definition. Let $(R,\mathfrak{m})$ be an augmented complete Noetherian local domain over a perfect field κ of characteristic p. Let R^{perf} be the perfection of R, and let ϕ be the Frobenius automorphism on R. Let A,b,d be real numbers, $A,b>0$, and $d\geq b$.

(i) Define a decreasing filtrations $\big(\mathrm{Fil}^\bullet_{R^{\mathrm{perf}},\deg}\big)_{\bullet\in\mathbb{R}\,0}$ on R^{perf} indexed by real numbers u by

$$\mathrm{Fil}^u_{R^{\mathrm{perf}},\deg} := \begin{cases} \Big\{ x\in R^{\mathrm{perf}} \;\big|\; \exists j\in\mathbb{N} \text{ s.t. } x^{p^j} \in \mathfrak{m}^{\lceil u\cdot p^j\rceil} \Big\} & \text{if } u\geq 0 \\ R^{\mathrm{perf}} & \text{if } u\leq 0 \end{cases}$$

It is easy to see that $\mathrm{Fil}^u_{R^{\mathrm{perf}},\deg}$ is an ideal of R^{perf} for every $u\in\mathbb{R}$.

(ii) Define a subring $\big(\,(R,\mathfrak{m})^{\mathrm{perf},\,\flat}_{A,b;d}\,\big)_{\mathrm{fin}}$ of R^{perf} by

$$\big(\,(R,\mathfrak{m})^{\mathrm{perf},\,\flat}_{A,b;d}\,\big)_{\mathrm{fin}} := \sum_{n\in\mathbb{N}} \big(\phi^{-n}R \cap \mathrm{Fil}^{b\cdot p^{An}-d}_{R^{\mathrm{perf}},\deg}\big)$$

It is not difficult to see that $\big(\,(R,\mathfrak{m})^{\mathrm{perf},\,\flat}_{A,b;d}\,\big)_{\mathrm{fin}}$ is a subring of R^{perf}.

(iii) Define

$$(R,\mathfrak{m})^{\mathrm{perf},\,\flat}_{A,b;d}$$

to be the completion of $\big(\,(R,\mathfrak{m})^{\mathrm{perf},\#}_{A,b;d}\,\big)_{\mathrm{fin}}$ with respect to the filtration induced by the filtration $\big(\mathrm{Fil}^{\bullet}_{R^{\mathrm{perf}},\mathrm{deg}}\big)$ of R^{perf}:

$$(R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d} = \lim_{u\to\infty}\big(\,(R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}\,\big)_{\mathrm{fin}}\Big/\Big(\mathrm{Fil}^{u}_{R^{\mathrm{perf}},\mathrm{deg}}\cap\big(\,(R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}\,\big)_{\mathrm{fin}}\Big).$$

(iv) Define a filtration $\Big(\mathrm{Fil}^{\bullet}_{(R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}}\Big)_{\bullet}$ on $(R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}$ by

$$\mathrm{Fil}^{u}_{(R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}} := \lim_{v\to\infty}\Big(\mathrm{Fil}^{u}_{R^{\mathrm{perf}},\mathrm{deg}}\cap\big(\,(R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}\,\big)_{\mathrm{fin}}\Big)\Big/\Big(\mathrm{Fil}^{v}_{R^{\mathrm{perf}},\mathrm{deg}}\cap\big(\,(R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}\,\big)_{\mathrm{fin}}\Big).$$

10.7.4.3. Definition. Let $(R,\mathfrak{m})$ be an augmented complete Noetherian local domain over a perfect field κ of characteristic p. Let R^{perf} be the perfection of R, and let ϕ be the Frobenius automorphism on R. Let A,b,d be real numbers, $A,b>0$, and $d\geq\max(b-1,0)$.

(i) Define a decreasing filtrations $\big(\mathrm{Fil}^{\bullet}_{R^{\mathrm{perf}},\mathrm{fr}}\big)_{\bullet\in\mathbb{R}\,0}$ on R^{perf} by ideals of R^{perf}, indexed by real numbers u as follows.

$$\mathrm{Fil}^{u}_{R^{\mathrm{perf}},\mathrm{fr}} := \begin{cases}\Big\{x\in R^{\mathrm{perf}}\;\Big|\;\exists j\in\mathbb{N}\text{ s.t. } x^{p^j}\in\mathfrak{m}^{(p^{j+\lceil\log u/\log p\rceil})}\Big\} & \text{if } u\geq 1\\ \\ R^{\mathrm{perf}} & \text{if } u\leq 1\end{cases}$$

(ii) Define a subring $\big(\,(R,\mathfrak{m})^{\mathrm{perf},\#}_{A,b;d}\,\big)_{\mathrm{fin}}$ of R^{perf} by

$$\big(\,(R,\mathfrak{m})^{\mathrm{perf},\#}_{A,b;d}\,\big)_{\mathrm{fin}} := \sum_{n\in\mathbb{N}}\big(\phi^{-n}R\cap\mathrm{Fil}^{b\cdot p^{An}-d}_{R^{\mathrm{perf}},\mathrm{fr}}\big)$$

(iii) Define

$$(R,\mathfrak{m})^{\mathrm{perf},\#}_{A,b;d}$$

to be the completion of $\big(\,(R,\mathfrak{m})^{\mathrm{perf},\#}_{A,b;d}\,\big)_{\mathrm{fin}}$ with respect to the filtration induced by the filtration $\big(\mathrm{Fil}^{\bullet}_{R^{\mathrm{perf}},\mathrm{fr}}\big)$ of R^{perf}:

$$(R,\mathfrak{m})^{\mathrm{perf},\#}_{A,b;d} = \lim_{u\to\infty}\big(\,(R,\mathfrak{m})^{\mathrm{perf},\#}_{A,b;d}\,\big)_{\mathrm{fin}}\Big/\Big(\mathrm{Fil}^{u}_{R^{\mathrm{perf}},\mathrm{fr}}\cap\big(\,(R,\mathfrak{m})^{\mathrm{perf},\#}_{A,b;d}\,\big)_{\mathrm{fin}}\Big).$$

(iv) Define a filtration $\Big(\mathrm{Fil}^{\bullet}_{(R,\mathfrak{m})^{\mathrm{perf},\#}_{A,b;d}}\Big)_{\bullet}$ on $(R,\mathfrak{m})^{\mathrm{perf},\#}_{A,b;d}$ by

$$\mathrm{Fil}^{u}_{(R,\mathfrak{m})^{\mathrm{perf},\#}_{A,b;d}} := \lim_{v\to\infty}\Big(\mathrm{Fil}^{u}_{R^{\mathrm{perf}},\mathrm{fr}}\cap\big(\,(R,\mathfrak{m})^{\mathrm{perf},\#}_{A,b;d}\,\big)_{\mathrm{fin}}\Big)\Big/\Big(\mathrm{Fil}^{v}_{R^{\mathrm{perf}},\mathrm{fr}}\cap\big(\,(R,\mathfrak{m})^{\mathrm{perf},\#}_{A,b;d}\,\big)_{\mathrm{fin}}\Big).$$

10.7.5. How various tempered perfections compare.

In 10.7.3.2 we defined six families of rings. Each ring in these families consist of formal series of the form $\sum_{I\in\mathbb{N}[1/p]} b_I\,\underline{t}^I$, where $b_I\in\kappa\ \forall\, I$, subject uniform constraint (depending on parameters) on the support of such series. The six families are:

(1♯) $\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{\#}_{s:\phi^r,\geqslant n_0}$

(1♭) $\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{\flat}_{s:\phi^r,\geqslant n_0}$

(2♯) $\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^{\#}_{s:\phi^r;[i_0]}$

(2♭) $\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^{\flat}_{s:\phi^r;[i_0]}$

(3♯) $\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^{E,\#}_{C;\,d}$

(3♭) $\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d}$

We have defined four additional family of rings, attached to any given equi-characteristic-p complete Noetherian local domain $(R, \mathfrak{m})$:

(4♯) $(R, \mathfrak{m})^{\mathrm{perf},\#}_{s:\phi^r;[i_0]}$

(4♭) $(R, \mathfrak{m})^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]}$

(5♯) $(R, \mathfrak{m})^{\mathrm{perf},\#}_{A,b;d}$

(5♭) $(R, \mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}$

Each rings in the families (1♯)–(3♭) is naturally embedded in the completion of the perfection of $\kappa[[t_1, \ldots, t_m]]$. We will explain how they compare. Similarly each ring in the families (4♯)–(5♭) is naturally embedded in the completion of the perfection of R, and we will compare them.

10.7.5.1. Remark. (i) The families (1♯) and (1♭) are motivated by the notion of $[p^a]$-compatible sequence of maps; c.f. 10.7.2.1 and 10.7.2.2. The family (1♯) is directly tied with $[p^a]$-compatible families of maps; see 10.7.2.2 R4. With r, s fixed, the ring increases as the second parameter n_0 increases. The ♭-version results from the # version when one replaces congruences modulo $(t_1^{p^n}, \ldots, t_m^{p^n})$ by the coarser congruences modulo $(t_1, \ldots, t_m)^{p^n}$.

(ii) The families (2♯) and (2♭) are slight variants of (1♯) and (1♭) and somewhat more convenient than the family (1♯) and (1♭). It is straight forward to generalize them to tempered perfections of augmented complete Noetherian local domains; see 10.7.4.2.

(iv) In (3♯) and (3♭) the parameters $E, C > 0$ and $d \geq 0$ are real numbers. The most significant parameter is the "exponent" E; it is written as a superscript in the notation, to indicate that it serves as an exponent in the estimate of p-adic absolute value in terms of archimedean absolute value for elements in the support of formal series in family (3).

The "multiplicative constant" C is secondary, while the parameter d is of least importance among the three. When E is fixed while C and d vary, the #-version and the ♭-version are interlaced; see 10.7.5.4 (1). Rings in the family (2♯) (respectively (2♭)) with primary parameters $s > r > 0$ are closely related to rings in the family (3♯) (respectively (3♭) with $E = \frac{r}{s-r}$; see 10.7.5.3 (3) and 10.7.5.4 (2).

(iv) Clearly the families (2♯) and (2♭) are special cases of the families (4♯) and (4♭). This is reflected in the notation for (2) and (4).

(v) The families (5♯) and (5♭) with real parameters $A > 0, b > 0, d \geq b$ generalize the families (3♯) and (3♭). When R is the formal power series ring $\kappa[[t_1, \ldots, t_m]]$, the parameter triple (A_1, b_1, d_1) corresponding to a given parameter triple (E, C, d) is

$$A_1 = \tfrac{1}{E}, \quad b_1 = C^{1/E}, \quad d_1 = d.$$

When the parameters are related as above, the rings

$$\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;\,d}^{E,\,\flat} \quad \text{and} \quad (\kappa[[\underline{t}]], (\underline{t}))_{A_1,b_1;d_1}^{\mathrm{perf},\,\flat}$$

are "quite close".

10.7.5.2. Below is a summary of the relation between this myriad of tempered perfections.

(i) With the primary parameters $r < s$ fixed, the inductive family of rings in ($1\sharp$) is interlaces with the inductive family of rings in ($1\flat$) as their respective secondary parameters n_0 and i_0 vary, i.e. the inductive family ($1\sharp$) is co-final with the family ($1\flat$). Similarly the inductive family of rings in ($2\sharp$) interlaces with the inductive family of rings in ($2\flat$). See 10.7.5.3 (1)–(2).

However, with r, s fixed, the family of rings in ($1\sharp$) is in general not co-final with the family of rings in ($2\sharp$); c.f. 10.7.5.3 (3).

(ii) When all parameters r, s, n_0, i_0 vary, the four families ($1\sharp$), ($1\flat$), ($2\sharp$) and ($2\flat$) are mutually co-final; they are also co-final with the families ($3\sharp$) and ($3\flat$). See lemmas 10.7.5.4 and 10.7.5.5.

(iii) With parameters r, s fixed the inductive families ($4\sharp$) and ($4\flat$) with varying i_0 are co-final to each other. Similarly with parameters A fixed while b, d vary, the inductive families ($5\sharp$) and ($5\flat$) are co-final to each other. See 10.7.5.6.

The main takeaway of the above comparison is: given an augmented noetherian local domain $(R, \mathfrak{m})$, in any of the eligible family tempered perfections of R, the union of all tempered perfections in the chosen family is independent of the family you happen to choose. We propose to call elements in such unions *tempered virtual functions* on the formal scheme $\mathrm{Spf}(R)$.

10.7.5.3. Lemma. *Let $s > r > 0$ be positive integers. and let $i_0 \geq 0$ be a natural number. Let $\kappa \supseteq \mathbb{F}_p$ be a perfect field, and let $t_1, \ldots, t_m$ be variables.*

(1) *Let $i_0 \geq 0$ be a natural number. We have inclusions*

$$\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{s:\phi^r;[i_0]}^{\#} \subseteq \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{s:\phi^r;[i_0]}^{\flat}$$

and

$$\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{s:\phi^r;[i_0]}^{\flat} \subseteq \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{s:\phi^r;[i_0+\lceil\log_p m\rceil]}^{\#}$$

as sets of formal series.

(2) *Let n_0 be a natural number. If i_1 is a natural number such that*

$$i_1 \geq \max\bigl(s - r, s \cdot \lceil \tfrac{n_0}{r} \rceil\bigr),$$

then

$$\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^{\#}_{s:\phi^r, \geqslant n_0} \subseteq \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^{\#}_{s:\phi^r;[i_1]}$$

and

$$\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^{\flat}_{s:\phi^r, \geqslant n_0} \subseteq \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^{\flat}_{s:\phi^r;[i_1]}.$$

(3) *Let i_0 be a natural number. We have*

$$\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^{\#}_{s:\phi^r;[i_0]} \subseteq \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^{r/(s-r),\#}_{p^{i_0\,r/(s-r)};\,0}$$

and

$$\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^{\flat}_{s:\phi^r;[i_0]} \subseteq \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^{r/(s-r),\flat}_{p^{i_0\,r/(s-r)};\,0}$$

Proof. The first inclusion in (1) is obvious. The second inclusion in (1) holds because

$$(t_1, \ldots, t_m)^{p^{j+\lceil \log_p m\rceil}} \subseteq (t_1^{p^j}, \ldots, t_m^{p^j})$$

for all $j \in \mathbb{N}$. The statements (2), (3) are easy exercises. $\square$

10.7.5.4. Lemma. *Let $\kappa \supseteq \mathbb{F}_p$ be a perfect field. Let $E > 0, C > 0$ be positive real numbers. Let $d \geq 0$ be a non-negative real number as in 10.7.3.6.*

(1) *We have natural inclusions*

$$\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^{E,\#}_{C;\,d} \;\subseteq\; \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d}$$

and

$$\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d} \;\subseteq\; \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^{E,\#}_{C\cdot m^E;\,d/m}.$$

(2) *Let $r < s$ be positive integers such that*

$$E < \frac{r}{s-r}.$$

Suppose that i_2 a sufficiently natural number such that

$$p^{\lceil m/r\rceil\cdot(s-r)-i_2} \leq C^{-1/E}\cdot p^{m/E} - d$$

for every integer $m \geq \frac{r\cdot i_2}{s-r}$. Note that such an integer i_2 exists because $\frac{s-r}{r} < \frac{1}{E}$. Then

$$\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^{E,\#}_{C;\,d} \;\subseteq\; \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^{\#}_{s:\phi^r;[i_2]}$$

and

$$\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d} \;\subseteq\; \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^{\flat}_{s:\phi^r;[i_2]}.$$

10.7.5.5. Lemma. *Let $0 < r < s$ be positive integers, and let $i_0 \in \mathbb{N}$ be a natural number. For any pair of positive integers $0 < r' < s'$ with $\frac{r'}{s'-r'} > \frac{r}{s-r}$, there exists a natural number n_0, depending on r', s' and i_0, such that*

$$\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^{\#}_{s:\phi^r;[i_0]} \;\subseteq\; \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^{\#}_{s:\phi^r, \geqslant n_0}$$

and

$$\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^\flat_{s:\phi^r;[i_0]} \subseteq \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle^\flat_{s:\phi^r, \geqslant n_0}.$$

The proofs of lemmas 10.7.5.4, 10.7.5.5, and 10.7.5.6–10.7.5.7 below are omitted.

10.7.5.6. Lemma. *Let $(R, \mathfrak{m})$ be an augmented complete Noetherian local ring over a perfect field κ of characteristic p. Suppose that the maximal ideal $\mathfrak{m}$ can be generated by m elements.*

(1) *Let $s > r > 0$ be positive integers. Let i_0 be a natural number. We have*

$$(R, \mathfrak{m})^{\mathrm{perf},\#}_{s:\phi^r;[i_0]} \subseteq (R, \mathfrak{m})^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]}$$

and

$$(R, \mathfrak{m})^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]} \subseteq (R, \mathfrak{m})^{\mathrm{perf},\#}_{s:\phi^r;[i_0+\lceil\log_p m\rceil]}.$$

(2) *Let A, b, d be real numbers, $A, b > 0$, $d \geq b$. We have*

$$(R, \mathfrak{m})^{\mathrm{perf},\#}_{A,b;d} \subseteq (R, \mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}$$

and

$$(R, \mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d} \subseteq (R, \mathfrak{m})^{\mathrm{perf},\#}_{A,b/m;d/m}.$$

10.7.5.7. Lemma. *Let $(R, \mathfrak{m})$ be an augmented complete Noetherian local ring over a perfect field κ of characteristic p.*

(1) *Let r, s, i_0 be natural numbers with $r < s$. We have natural inclusions*

$$(R, \mathfrak{m})^{\mathrm{perf},\#}_{s:\phi^r;[i_0]} \subseteq (R, \mathfrak{m})^{\mathrm{perf},\#}_{(s-r)/r,1;0}$$

and

$$(R, \mathfrak{m})^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]} \subseteq (R, \mathfrak{m})^{\mathrm{perf},\flat}_{(s-r)/r,1;0}.$$

(2) *Let A, b, d be real numbers with $A, b > 0$ and $d \geq b-1$. Suppose that r, s are positive integers with $s > r > 0$ such that $\frac{s-r}{r} < A$. There exists a natural number i_0, depending only on the parameters A, b, d, r, s, such that we have natural inclusions*

$$(R, \mathfrak{m})^{\mathrm{perf},\#}_{A,b;d} \subseteq (R, \mathfrak{m})^{\mathrm{perf},\#}_{s:\phi^r;[i_0]}$$

and

$$(R, \mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d} \subseteq (R, \mathfrak{m})^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]}.$$

10.7.6. Functoriality of tempered perfections. Every local homomorphism h between two equi-characteristic-p complete Noetherian local domains induces a ring homomorphism between their completed tempered perfections. It is clear that surjections induce surjections between completed tempered perfections. We show that injective local homomorphisms induce injections on completed tempered perfections.

10.7.6.1. Lemma. *Let $(R_1, \mathfrak{m}_1)$, $(R_2, \mathfrak{m}_2)$ equi-characteristic-p complete Noetherian local domains with perfect residue fields κ_1 and κ_2. Let $h : R_1 \to R_2$ be a ring homomorphism such that $h(\mathfrak{m}_1) \subseteq \mathfrak{m}_2$.*

(a) *Let A, b, d be real numbers, $A, b > 0$, $d \geq b$. Let $\iota_i : R_i \to (R_i, \mathfrak{m}_i)^{\mathrm{perf},\flat}_{A,b;d}$ be the natural ring homomorphisms from R_i to its completed tempered perfection $(R_i, \mathfrak{m}_i)^{\mathrm{perf},\flat}_{A,b;d}$ for $i = 1, 2$. The ring homomorphism h induces a homomorphism*

$$\tilde{h}^\flat : (R_1, \mathfrak{m}_1)^{\mathrm{perf},\flat}_{A,b;d} \to (R_2, \mathfrak{m}_2)^{\mathrm{perf},\flat}_{A,b;d}$$

such that $\tilde{h} \circ \iota_1 = \iota_2 \circ h$. Similarly h induces a continuous ring homomorphism

$$\tilde{h}^\sharp : (R_1, \mathfrak{m}_1)^{\mathrm{perf},\#}_{A,b;d} \to (R_2, \mathfrak{m}_2)^{\mathrm{perf},\#}_{A,b;d}$$

(b) *Let $r, s, i_0 \in \mathbb{N}$, $r, s > 0$, $i_0 \geq 0$ Let $\iota_1 : R_1 \to (R_1, \mathfrak{m}_1)^{\mathrm{perf},\#}_{b:\phi^A;[d]}$ be the natural ring homomorphism from R_1 to its completed tempered perfection $(R_1, \mathfrak{m}_1)^{\mathrm{perf},\#}_{s:\phi^r;[i_0]}$. Similarly we have a ring homomorphism $\iota_2 : R_2 \to (R_2, \mathfrak{m}_1)^{\mathrm{perf},\#}_{s:\phi^r;[i_0]}$.*

The ring homomorphism h induces a homomorphism

$$h^\# : (R_1, \mathfrak{m}_1)^{\mathrm{perf},\#}_{s:\phi^r;[i_0]} \to (R_2, \mathfrak{m}_2)^{\mathrm{perf},\#}_{s:\phi^r;[i_0]}$$

such that $h^\# : \circ \iota_1 = \iota_2 \circ h$. Similarly h extends naturally to a ring homomorphism

$$h^\flat : \ (R_1, \mathfrak{m}_1)^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]} \to (R_1, \mathfrak{m}_1)^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]} .$$

The proof is easy, therefore omitted. □

10.7.6.2. Proposition. *Let $(R, \mathfrak{m})$ be a Noetherian local domain. Assume that the integral closure S of R in the field of fraction of R is a finite R-module. There exists a natural number n_0 such that such that*

$$\{x \in R \mid x^a \in \mathfrak{m}^n\} \subseteq \mathfrak{m}^{\lfloor \frac{n}{a} - n_0 \rfloor} \quad \forall\, a \in \mathbb{N}_{>0},\ \forall n \geq a \cdot n_0.$$

PROOF. Let $\mathrm{Bl}_{\mathfrak{m}}(R) = \mathrm{Spec}\big(\oplus_{j\in\mathbb{N}} \mathfrak{m}^j\big)$ be the blow-up of $\mathrm{Spec}(R/\mathfrak{m}) \subseteq \mathrm{Spec}(R)$, and let Y be the normalization of $\mathrm{Bl}_{\mathfrak{m}}(R)$. The Noetherian normal domain S is a semi-local finite R-algebra. The natural morphism $\pi : Y \to \mathrm{Spec}(R)$ factors through a unique morphism $f : Y \to \mathrm{Spec}(S)$: $\pi = g \circ f$, where $g : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ corresponds to the inclusion $R \hookrightarrow S$. We know that $\Gamma(Y, \mathcal{O}_Y) = S$ because S is normal.

Let $\mathcal{L} = \pi^*\mathfrak{m} = \mathfrak{m} \cdot \mathcal{O}_{Y_i}$ be the pull-back to Y of the maximal ideal $\mathfrak{m} \subseteq R$; it is an invertible sheaf of $\mathcal{O}_Y$-ideals on Y and is an ample invertible $\mathcal{O}_Y$-module. The closed subset $\mathcal{S}pec_Y(\mathcal{O}_Y/\mathfrak{m}\mathcal{O}_Y)$ of Y is the union of irreducible Weil divisors $E_1, \ldots, E_r$, where r is a positive integer. There exist positive integers $e_1, \ldots, e_r \in \mathbb{N}_{>0}$ such that

$$\mathcal{L} = \mathcal{O}_Y\big(-(e_1E_1 + \cdots + e_rE_r)\big) .$$

Define for each $n \in \mathbb{N}$ an ideal $J_n \subseteq S$ by

$$J_n := \Gamma(Y, \mathcal{L}^n) \subseteq \Gamma(Y, \mathcal{O}_Y) = S.$$

It is clear that $\mathfrak{m}^nS \subseteq J_n$ for all $n \in \mathbb{N}$.

Claims.

1. There exist a positive natural number $n_1 \in \mathbb{N}$ such that $J_{n+1} = \mathfrak{m} J_n$ for all integers $n \geq n_1$. In particular $J_n \subseteq \mathfrak{m}^{n-n_1} S$ for all $n \geq n_1$
2. There exists a natural number $n_2 \in \mathbb{N}$ such that $R \cap (\mathfrak{m}^{n+n_2} S) \subseteq \mathfrak{m}^n$ for all $n \in \mathbb{N}$.
3. We have $J_{n+n_1+n_2} \cap R \subseteq \mathfrak{m}^n$ for all $n \in \mathbb{N}$, with the constants n_1, n_2 in claims 1 and 2 respectively.
4. If $y \in S$, $a \in \mathbb{N}_{>0}$, $n \in \mathbb{N}$ and $y^a \in J_n$, then $y \in J_{\lfloor n/a \rfloor}$.
5. If $x \in R$, $a \in \mathbb{N}_{>0}$, $n \in \mathbb{N}$, and $x^a \in \mathfrak{m}^n$, then $x \in \mathfrak{m}^{\lfloor n/a \rfloor - n_1 - n_2}$ for all $n \geq a(n_1 + n_2)$.

Obviously proposition 10.7.6.2 follows from claim 5, with $n_0 = n_1 + n_2$. $J_1 \subseteq \tilde{\mathfrak{m}}_1 \cap \cdots \cap \tilde{\mathfrak{m}}_s$ and S is Noetherian.

The general finiteness property for proper morphism [EGA III, §5, Cor. 3.3.2], applied to the proper morphism $Y \to \operatorname{Spec}(R)$ and the coherent sheaf $\mathcal{L} = \mathfrak{m}\mathcal{O}_Y$, implies that the graded $\oplus_{i \geq 0} \mathfrak{m}^i$-module

$$\oplus_{i \geq 0} \Gamma(Y, \mathfrak{m}^i \mathcal{O}_Y) = \oplus_{i \geq 0} J_i$$

is a finitely generated as a graded module. The claim 1 follows.

Claim 2 is the Artin–Rees lemma applied to the finite R-module S. Claim 3 is a formal consequence of claims 1 and 2, while claim 5 is a formal consequence of claims 3 and 4.

It remains to prove claim 4. Given an element $y \in S$ such that $y^a \in J_n$. For each $i = 1, \ldots, s$, let S_i be the localization of S at the generic point of the exceptional divisor E_i. Each E_i is a discrete valuation ring; let $\operatorname{ord}_{E_i}(\cdot)$ be associated normalized valuation with value group $\mathbb{Z}$. The assumption that $y^a \in J_n$ implies that $\operatorname{ord}_{E_i}(y^a) \geq n \cdot e_i$ for all i, therefore

$$\operatorname{ord}_{E_i}(y) \geq \tfrac{n\, e_i}{a} \geq \lfloor \tfrac{n}{a} \rfloor e_i$$

for $i = 1, \ldots, s$. Therefore there exists an open subset $U \subseteq Y$ such that U contains $Y \smallsetminus (E_1 \cup \cdots \cup E_s)$ and also the maximal points of $E_1 \cup \cdots \cup E_s$, and y defines a section y_U of $\mathcal{L}^{\lfloor \frac{n}{a} \rfloor}$ over U. Because Y is normal and the codimension of U in Y is at least 2, y_U extends uniquely to a section of $\mathcal{L}^{\lfloor \frac{n}{a} \rfloor}$ over Y. Therefore $y \in J_{\lfloor n/a \rfloor}$. We have proved claim 4 and proposition 10.7.6.2. □

10.7.6.3. Corollary. *Let $(R, \mathfrak{m})$ be a complete Noetherian local domain of characteristic $p > 0$, with perfect residue field κ.*

(i) *Let $A, b > 0$, $d \geq b$ be real numbers. The linear topology on the ring*

$$\big((R, \mathfrak{m})^{\mathrm{perf}, \flat}_{A,b;d} \big)_{\mathrm{fin}}$$

defined by the filtration on $\big((R, \mathfrak{m})^{\mathrm{perf}, \flat}_{A,b;d} \big)_{\mathrm{fin}}$ induced by the filtration $\big(\mathrm{Fil}^u_{R^{\mathrm{perf}}, \deg}\big)$ of R^{perf} is separated. Therefore the natural ring homomorphism

$$\big((R, \mathfrak{m})^{\mathrm{perf}, \flat}_{A,b;d} \big)_{\mathrm{fin}} \longrightarrow (R, \mathfrak{m})^{\mathrm{perf}, \flat}_{A,b;d}$$

from $\big((R, \mathfrak{m})^{\mathrm{perf}, \flat}_{A,b;d} \big)_{\mathrm{fin}}$ to its completion $(R, \mathfrak{m})^{\mathrm{perf}, \flat}_{A,b;d}$ is an injection.

(ii) *Let r, s, n_0 be natural numbers, $0 < r < s$. The natural ring homomorphism*

$$\big(\, (R,\mathfrak{m})^{\mathrm{perf},\,\#}_{s:\phi^r;[i_0]}\,\big)_{\mathrm{fin}} \longrightarrow (R,\mathfrak{m})^{\mathrm{perf},\,\#}_{s:\phi^r;[i_0]}$$

and

$$\big(\, (R,\mathfrak{m})^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_0]}\,\big)_{\mathrm{fin}} \longrightarrow (R,\mathfrak{m})^{\mathrm{perf},\,\flat}_{s:\phi^r;[i_0]}$$

are injections.

Proof. The statements (i) and (ii) are easy consequences of 10.7.6.2. We note that the statements (i) and (ii) are in fact equivalent. □

10.7.6.4. Corollary. *Notation as in 10.7.6.1. In particular $h : (R_1,\mathfrak{m}_1) \to (R_2,\mathfrak{m}_2)$ is a ring homomorphism between equi-characteristic-p complete Noetherian local domains. Suppose that h is an injection. Then the induced homomorphisms $\tilde{h}$, $h^{\#}$ and $h^{\flat}$ in 10.7.6.1 are also injections.*

Proof. This statement is a corollary of 10.7.6.3. We explain the proof for $\tilde{h}$. The same argument in general topology also proves the statement for $h^{\#}$ and $h^{\flat}$.

The injective ring homomorphism $h : R_1 \to R_2$ induces a injective ring homomorphism

$$h' : \big(\, (R_1,\mathfrak{m}_1)^{\mathrm{perf},\,\flat}_{A,b;d}\,\big)_{\mathrm{fin}} \longrightarrow \big(\, (R_2,\mathfrak{m}_2)^{\mathrm{perf},\,\flat}_{A,b;d}\,\big)_{\mathrm{fin}}.$$

According to 10.7.6.3, we can identify $\big(\, (R_2,\mathfrak{m}_2)^{\mathrm{perf},\,\flat}_{A,b;d}\,\big)_{\mathrm{fin}}$ as a subring of $(R_2,\mathfrak{m}_2)^{\mathrm{perf},\,\flat}_{A,b;d}$. The injection h' identifies $\big(\, (R_1,\mathfrak{m}_1)^{\mathrm{perf},\,\flat}_{A,b;d}\,\big)_{\mathrm{fin}}$ also as a subring of $(R_2,\mathfrak{m}_2)^{\mathrm{perf},\,\flat}_{A,b;d}$. (It is actually contained in $\big(\, (R_2,\mathfrak{m}_2)^{\mathrm{perf},\,\flat}_{A,b;d}\,\big)_{\mathrm{fin}}$.) Let $\big(\, (R_2,\mathfrak{m}_2)^{\mathrm{perf},\,\flat}_{A,b;d}\,\big)^{\wedge}_{\mathrm{fin}}$ be the closure of $\big(\, (R_2,\mathfrak{m}_2)^{\mathrm{perf},\,\flat}_{A,b;d}\,\big)_{\mathrm{fin}}$ in the topological ring $(R_2,\mathfrak{m}_2)^{\mathrm{perf},\,\flat}_{A,b;d}$.

The topology on $\big(\, (R_1,\mathfrak{m}_1)^{\mathrm{perf},\,\flat}_{A,b;d}\,\big)_{\mathrm{fin}}$ induced by the filtration $\big(\mathrm{Fil}^u_{R_1^{\mathrm{perf}},\mathrm{deg}}\big)$ is stronger than the topology $(R_2,\mathfrak{m}_2)^{\mathrm{perf},\,\flat}_{A,b;d}$. The closure of $\big(\, (R_1,\mathfrak{m}_1)^{\mathrm{perf},\,\flat}_{A,b;d}\,\big)_{\mathrm{fin}}$ with respect to this stronger topology is naturally identified with a subset of $\big(\, (R_2,\mathfrak{m}_2)^{\mathrm{perf},\,\flat}_{A,b;d}\,\big)^{\wedge}_{\mathrm{fin}}$. We have shown that $\tilde{h}$ is an injection. □

Let κ be a perfect field. Denote by σ the Frobenius automorphism on κ, which sends every element $x \in \kappa$ to x^p. Let $u_1,\ldots,u_a$ and $t_1,\ldots,t_m$ be variables, and let $\kappa[u_1^{p^{-\infty}},\ldots,u_a^{p^{-\infty}}]$ be the perfection of the polynomial ring $\kappa[u_1,\ldots,u_m]$. Elements of $\kappa[u_1^{p^{-\infty}},\ldots,u_a^{p^{-\infty}}]$ are finite sums of the form

$$\sum_{J\in\mathbb{N}[1/p]^a} b_J\,\underline{u}^J,$$

where $b_J \in \kappa$ for all $J \in \mathbb{N}[1/p]^a$, and all $b_J = 0$ for all J outside of a finite subset of $\mathbb{N}[1/p]^a$.

We observe that for each element $i \in \mathbb{N}[1/p]$, the i-th power of an element

$$\sum_{J\in\mathbb{N}[1/p]^a} b_J\,\underline{u}^J \;\in\; \kappa[u_1^{p^{-\infty}},\ldots,u_a^{p^{-\infty}}]$$

is well-defined: write $i = \frac{r}{p^s}$ with $r \in \mathbb{Z}$ and $s \in \mathbb{N}$, and define

$$\Big(\sum_{J\in\mathbb{N}[1/p]^a} b_J\,\underline{u}^J\Big)^{r/p^s} := \Big(\sum_{J\in\mathbb{N}[1/p]^a} b_J^{\sigma^{-s}}\cdot \underline{u}^{p^{-s}J}\Big)^r.$$

Therefore if $f \in \kappa[u_1^{p^{-\infty}},\dots,u_a^{p^{-\infty}}]$ and $g_1,\dots,g_a \in \kappa[t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}]$, the composition $f(g_1,\dots,g_a)$ is a well-defined element of $\kappa[t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}]$. It is not difficult to show that the operation "composition" extends to completed tempered perfections of power series rings.

10.7.6.5. Lemma (Functoriality of composition)**.** *Let $\kappa \supset \mathbb{F}_p$ be a perfect field. Let $u_1,\dots,u_a$ and $t_1,\dots,t_m$ be variables. Suppose that $f \in \kappa\langle\langle u_1^{p^{-\infty}},\dots,u_a^{p^{-\infty}}\rangle\rangle^{E_1,\flat}_{C_1;\,d_1}$, and $g_i \in \kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle^{E_2,\flat}_{C_2;\,d_2}$ for $i=1,\dots,a$. Assume for simplicity that $C_1,C_2,d_1,d_2 \geq 1$. There exists a positive real number d_3 such that*

$$f(g_1(\underline{t}),\dots,g_a(\underline{t})) \in \kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle^{E_3,\flat}_{C_3;\,d_3}$$

where

- $E_3 = E_1\cdot E_2 + E_1 + E_2$,
- $C_3 = C_2\cdot C_1^{1+E_2}\cdot(\frac{1}{e_2})^{E_1(1+E_2)}$, *and*
- $e_2 := \mathrm{Min}\Big\{|J|_\sigma :\ J\neq 0 \ \text{ and } \ \underline{t}^J \in \kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle^{E_2,\flat}_{C_2;\,d_2}\Big\}$.

A trivial lower bound for e_2 is

$$e_2 \geq C_2^{-1}(1+d_2)^{-E_2}.$$

PROOF. Let $S_2 \subseteq \mathbb{N}[1/p]^m$ be the set of supports of all formal power series in the ring $\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle^{E_2,\flat}_{C_2;\,d_2}$ whose constant terms are 0. Similarly let $S_1 \subseteq \mathbb{N}[1/p]^a$ be the set of supports of all formal series in $\kappa\langle\langle u_1^{p^{-\infty}},\dots,u_a^{p^{-\infty}}\rangle\rangle^{E_1,\flat}_{C_1;\,d_1}$ whose constant terms are 0. By definition $e_2 = \mathrm{Min}\{|J|_\sigma\colon J\in S_2\}$. Every non-zero element K in the support of $f(g_1(\underline{t}),\dots,g_a(\underline{t}))$ can be written in the following form

$$K = p^{-r}\left(J_{1,1}+\cdots+J_{1,i_1}+\cdots+J_{a,i}+\cdots+J_{a,i_a}\right),$$

where

- $(i_1,\dots,i_a)\in\mathbb{N}^a$, $r = \max(-\mathrm{ord}_p(i_1),\dots,-\mathrm{ord}_p(i_1),0)$,
- $I := p^{-r}(i_1,\dots,i_a) \in S_1$, and
- $J_{\nu,\mu}\in S_2$ for all $\nu=1,\dots,a$ and all $\mu = 1,\dots,i_a$.

Clearly the following inequalities hold.

$$|K|_\sigma \geq e_2\cdot p^{-r}(i_1\,e+\cdots+i_s\,e) = e_2\cdot|I|_\sigma \tag{10.7.6.5.1}$$

$$M_\sigma := \mathrm{Max}\left\{|J_{\nu,\mu}|_\sigma :\ 1\leq\mu\leq i_\nu,\ 1\leq\nu\leq a\right\} \leq p^r\cdot|K|_\sigma \tag{10.7.6.5.2}$$

$$p^{-r}\cdot|K|_p \leq \mathrm{Max}\left\{|J_{\nu,\mu}|_p :\ 1\leq\mu\leq i_\nu, 1\leq\nu\leq a\right\} =: M_p \tag{10.7.6.5.3}$$

From the definitions of the rings $\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle_{C_2;\,d_2}^{E_2,\flat}$ and $\kappa\langle\langle u_1^{p^{-\infty}},\dots,u_a^{p^{-\infty}}\rangle\rangle_{C_1;\,d_1}^{E_1,\flat}$ we know that

$$p^r \le C_1(|I|_\sigma + d_1)^{E_1} \le C_1\cdot(\tfrac{1}{e_2}\,|K|_\sigma + d_1)^{E_1} \tag{10.7.6.5.4}$$

$$M_p \le C_2(M_\sigma + d_2)^{E_2} \tag{10.7.6.5.5}$$

Combining the above inequalities, we see that

$$|K|_p \le p^r\cdot C_2\cdot(p^r\,|K|_\sigma + d_2)^{E_2} \le C_1(e_2^{-1}\cdot|K|_\sigma + d_1)^{E_1}\cdot C_2\left(C_1(e_2^{-1}\cdot|K|_\sigma + d_1)^{E_1}|K|_\sigma + d_2\right)^{E_2}$$

The last term in the above displayed inequality is a polynomial in $|K|_\sigma$ of degree

$$E_3 := E_1 + E_2 + E_1\cdot E_2$$

whose leading term is

$$C_3 := C_1^{1+E_2}\cdot C_2\cdot(\tfrac{1}{e_2})^{E_1(1+E_2)}.$$

Hence for a sufficiently large constant d_3 it is bounded above by $C_3(|K|_\sigma + d_3)^{E_3}$ for all $|K|_\sigma \ge 0$. We have proved the main assertion of lemma 10.7.6.5.

To see the trivial lower bound for e_2, we only have to observe that if $J \in S_2$ and $|J|_\sigma \le 1$ and $J \ne \mathbb{N}^m$, then

$$|J|_\sigma \ge |J|_p^{-1} \ge \left(C_2(1+d_2)^{E_2}\right)^{-1}. \qquad \square$$

Remark. Composition can be formulated for completed tempered perfections of general equi-characteristic-p complete Noetherian local rings.

10.7.7. Weierstrass preparation theorem for tempered perfections. Let $\kappa \supset \mathbb{F}_p$ be a perfect field. We will generalize the Weierstrass preparation theorem to completed tempered perfections $\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle_{C;\,d}^{E,\flat}$ of power series rings.

10.7.7.1. Definition. Let $\kappa \supset \mathbb{F}_p$ be a perfect field of characteristic $p > 0$.

(i) Let $\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle$ be the set of all formal series of the form

$$\sum_{i_1,\dots,i_m\in\mathbb{N}[1/p]} b_{i_1,\dots,i_m}\, t_1^{i_1}\cdots t_m^{i_m}$$

where $b_{i_1,\dots,i_m} \in \kappa$ for all $(i_1,\dots,i_m) \in \mathbb{N}[1/p]^m$. Note that $\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle$ has a natural structure as a module over the perfection $\kappa[t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}]$ of the polynomial ring $\kappa[t_1,\dots,t_m]$.

(ii) Let $e \in \mathbb{Z}[1/p]_{>0}$ be a positive rational number whose denominator is a powere of p. An non-zero element $F(t_1,\dots,t_m)$ in $\kappa\langle\langle t_1^{p^{-\infty}},\dots,t_m^{p^{-\infty}}\rangle\rangle$ is *regular of order e* in the variable t_m if the formal series $F(0,\dots,0,t_m)$ in one variable t_m has order

e. In other words when $F(t_1,\ldots,t_m)$ is expanded in powers of t_m with coefficients in formal series of $t_1,\ldots,t_{m-1}$,

$$F(t_1,\ldots,t_m) = \sum_{j\in\mathbb{N}[1/p]} F_j(t_1,\ldots,t_{m-1})\, t_m^j$$

we have

$$F_j(0,\ldots,0) = 0 \quad \forall\, j < e, \quad \text{and} \quad F_e(0,\ldots,0) \in \kappa^\times.$$

10.7.7.2. Proposition. *Let $F(t_1,\ldots,t_m)$ be a non-zero element of the ring*

$$\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle_{C;\,d}^{E,\flat}$$

which is regular of order $e > 0$ in the variable t_m.

(1) *There exist constants $C' > 0, d' > 0$ depending only on the parameters C, d, E, m such that for every element $G \in \kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle_{C;\,d}^{E,\flat}$, there exist elements $U, R \in \kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle_{C';\,d'}^{E,\flat}$ such that*

$$G = U\cdot F + R$$

and for every element $I = (i_1,\ldots,i_m) \in \mathbb{N}[1/p]^m \in \operatorname{supp}(R)$ in the support of R, the inequalities

$$i_m < e, \quad i_1+\cdots+i_{m-1} > 0$$

hold. Moreover the quotient U and the remainder R are uniquely determined by G and F. The constants C' and d' can be taken to be

$$C' = C\cdot(1+\epsilon_0^{-1})^E, \quad d' = \tfrac{d+e}{1+\epsilon_0^{-1}},$$

where ϵ_0 is defined in 10.7.7.6.

(2) *Suppose that $e = \operatorname{Min}\{|I|_\sigma : I \in \operatorname{supp}(F)\}$. Then*

$$U, R \in \kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle_{C;\,d+2e}^{E,\flat}.$$

10.7.7.3. The uniqueness part 10.7.7.2 (1) is easy: suppose that

$$G = U'\cdot F + R'$$

with $U', R' \in \kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle_{C';\,d'}^{E,\flat}$ and R' satisfies the same condition as R. Then $(U'-U)\cdot F = R - R'$. Examine the degree in t_m of monomials appearing on both sides, we see that $R'-R = 0$. Therefore $(U'-U)\cdot F = 0$. Hence $U'-U = 0$ because $\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle_{C';\,d'}^{E,\flat}$ is an integral domain.

Our proof of the existence part of 10.7.7.2 is a generalization of the constructive proof of the Weierstrass preparation theorem in [**134**, p. 139]. The actual proof is in 10.7.7.5–10.7.7.8 below; the crucial estimates are in lemma 10.7.7.7. We will review the argument in [**134**, p. 139] after recalling the definition of the linear operators used in [**134**, p. 139].

10.7.7.4. Definition. Let κ be a perfect field of characteristic p. Let $t_1, \ldots, t_m$ be variables. Let $e > 0$ be a positive rational number in $\mathbb{N}[1/p]$. Let $F \in \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle$ be a formal series which is regular of order e in the variable t_m.

(i) Define κ-linear operators

$$\eta, \rho : \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle \longrightarrow \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle$$

depending on e, by

$$f = t_m^e \cdot \eta(f) + \rho(f)$$

for every element $f \in \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle$. Clearly for every monomial $t_1^{i_1} \cdots t_m^{i_m}$ with exponent $(i_1, \ldots, i_m) \in \mathbb{N}[1/p]^m$, $\eta(t_1^{i_1} \cdots t_m^{i_m})$ and $\rho(t_1^{i_1} \cdots t_m^{i_m})$ are given by

$$\eta(t_1^{i_1} \cdots t_m^{i_m}) = \begin{cases} t_1^{i_1} \cdots t_{m-1}^{i_{m-1}} \cdot t_m^{i_m - e} & \text{if } i_m \geq e \\ 0 & \text{if } i_m < e \end{cases}$$

$$\rho(t_1^{i_1} \cdots t_m^{i_m}) = \begin{cases} 0 & \text{if } i_m \geq e \\ t_1^{i_1} \cdots t_m^{i_m} & \text{if } i_m < e \end{cases}.$$

For a general element $f = \sum_{i_1,\ldots,i_m \in \mathbb{N}[1/p]} b_{i_1,\ldots,i_m}\, t_1^{i_1} \cdots t_m^{i_m} \in \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle$, we have

$$\eta(f) = \sum_{i_1,\ldots,i_m \in \mathbb{N}[1/p]} b_{i_1,\ldots,i_m}\ \eta(t_1^{i_1} \cdots t_m^{i_m})$$

$$\rho(f) = \sum_{i_1,\ldots,i_m \in \mathbb{N}[1/p]} b_{i_1,\ldots,i_m}\ \rho(t_1^{i_1} \cdots t_m^{i_m})).$$

Note that if $f \in \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;\,d}^{E,\flat}$ for some parameter $E, C > 0$ and $d \geq 0$, then $\rho(f) \in \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;\,d}^{E,\flat}$ and $\eta(f) \in \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;\,d+e}^{E,\flat}$.

(ii) Suppose that the formal series F is in $\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;\,d}^{E,\flat}$ for some constants $E > 0$, $C > 0$, $d \geq 0$. Define a κ-linear operator

$$\mu : \bigcup_{C'', d'' > 0} \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C'';\,d''}^{E,\flat} \longrightarrow \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle$$

depending on e and F, by

$$\mu(f) := \eta(-\eta(F)^{-1} \cdot \rho(F) \cdot f)$$

for all $C'', d'' > 0$ and every element $f \in \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C'';\,d''}^{E,\flat}$. Note that $\eta(F)$ is a formal series whose contant term is in $\kappa^\times$, therefore

$$\eta(F)^{-1} \in \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;\,d+e}^{E,\flat}$$

because $\eta(F) \in \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;\,d+e}^{E,\flat}$. The product $\eta(F)^{-1} \cdot \rho(F) \cdot f$ on the right hand side of the above displayed formula makes sense because both formal series $\rho(F)$ is also an element of $\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;\,d+e}^{E,\flat}$.