Canonical coordinates on leaves of p-divisible groups: The two-slope case

Ching-Li Chai[1]

version 01/10/2005

Abstract

Let k be a perfect field of characteristic $p > 0$. Let X, Y be isoclinic Barsotti-Tate groups with Frobenius slopes μ_X, μ_Y respectively, with $\mu_X < \mu_Y$. The "extension part" $\mathcal{DE}(X, Y)$ of the equi-characteristic deformation space $\mathcal{D}ef(X, Y)$ of $X \times Y$ has a natural structure as a commutate smooth formal group over k. We show that the *central leaf* in the deformation space $\mathcal{D}ef(X, Y)$, the locus in $\mathcal{D}ef(X, Y)$ defined by the property that the fiber of the universal Barsotti-Tate group of every geometric point of the central leaf is isomorphic to $X \times Y$, is equal to the maximal p-divisible subgroup $\mathcal{DE}(X, Y)_{\text{p-div}}$ of the smooth formal group $\mathcal{DE}(X, Y)$. We also determined the Cartier module of the p-divisible formal group $\mathcal{DE}(X, Y)_{\text{p-div}}$ in terms of the Cartier modules of X and Y. A "triple Carter module" $\mathrm{BC}_p(k)$, defined to be the set of all p-typical curves of the Cartier ring functor, plays an important role.

The "two-slope case" treated in this article is an essential ingredient of a general local structure theory for central leaves, which covers both the case of a central leaf in the deformation space of an arbitrary Barsotti-Tate group over k and the formal completion at a closed point of a central leaf in a modular variety of PEL-type over k.

Contents

§1. Introduction

Let k be a perfect field of characteristic p, $p > 0$. Recently Oort defined the notion of *central leaves* in the moduli space $\mathcal{A}_g$ of g-dimensional principally polarized abelian varieties over k; see [17] for the properties of central leaves, as well as the companion notion of *isogeny leaves*. Recall that the central leaf $\mathcal{C}(x_0)$ passing through a closed point $x_0 = [(A_0, \lambda_0)] \in \mathcal{A}_g(k^{\text{alg}})$

[1]Partially supported by grants DMS01-00441 and DMS04-00482 from the National Science Foundation.

is the smooth locally closed subset such that for every algebraically closed field $K \supseteq k$, $\mathcal{C}(x_0)(K)$ consists of all points $[(A,\lambda)] \in \mathcal{A}_g(K)$ such that the quasi-polarized Barsotti-Tate group $(A[p^\infty], \lambda[p^\infty])$ is isomorphic to $(A_0[p^\infty], \lambda_0[p^\infty]) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$. The above pointwise definition of central leaves has a drawback though, because it is not easy to use for local calculation. The purpose of this article is to lay the foundation for an independent characterization, as well as a structure theory, for the formal completion of a central leaf $\mathcal{C}(x_0)$ at a closed point.

The motivation of this article came from the Hecke orbit problem, conjectured in [17], that every prime-to-p Hecke orbit is dense in the central leaf containing it. To study the Zariski closure of a Hecke orbit at a point with a large local stabilizer subgroup, it is imperative to characterize the formal completion $\mathcal{C}(x_0)^{/x_0}$ of a central leaf $\mathcal{C}(x_0)$ inside the local deformation space $\mathcal{D}ef(A_0)$ of (A_0, λ_0). The central leaves rewarded our inquiry by revealing its beautiful structure: The formal completion of a central leaf at a closed point is "built up" from many p-divisible formal groups over k through a system of fibrations, with the p-divisible formal groups as the fibers. On a fixed central leaf, the local completions at any two closed points are non-canonically isomorphic, so a central leaf is *homogeneous* in some sense. A special case of this phenomenon is the classical Serre-Tate coordinates for the local deformation space of an ordinary abelian variety, discovered forty years ago. Although there is no good notion of *canonical coordinates* on the formal completion of $\mathcal{A}_g$ at an arbitrary closed point, canonical coordinates do exist when we restrict to a central leaf. It is tempting to view each central leaf in $\mathcal{A}_g$ as a sort of Shimura variety in characteristic p, because it is "homogeneous" and exhibits many group-theoretic properties similar to Shimura varieties in characteristic 0. The modular variety $\mathcal{A}_g$ itself, being the reduction modulo p of a Shimura variety, is the union of infinitely many Shimura-like subvarieties, with continuous moduli.

We will concentrate on the essential case, when the abelian variety in question has only two slopes. It is convenient to use Barsotti-Tate groups instead of abelian varieties: According to another classical theorem of Serre and Tate, deforming an abelian variety is equivalent to deforming its Barsotti-Tate group. Let X, Y be isoclinic Barsotti-Tate groups over k with Frobenius slopes $\mu_X < \mu_Y$. In particular, Y is a p-divisible formal group, i.e. a Barsotti-Tate group all of whose Frobenius slopes are strictly positive. Let $\mathcal{DE}(X,Y)$ be the maximal closed formal subscheme of the local deformation space of $X \times_{\mathrm{Spec}(k)} Y$ such that the universal Barsotti-Tate groups over $\mathcal{DE}(X,Y)$ is an extension of the constant formal group X by the constant formal group Y; see 2.2 for the precise definition of $\mathcal{DE}(X,Y)$. The formal scheme $\mathcal{DE}(X,Y)$ is smooth over k, and has a natural structure as a commutative formal group via the Baer sum.

Every smooth formal group G over k has a maximal p-divisible smooth formal subgroup $G_{\text{p-div}}$ and a maximal unipotent smooth formal subgroup G_{unip}. In §3 we show that the maximal p-divisible formal subgroup $\mathcal{DE}(X,Y)_{\text{p-div}}$ of $\mathcal{DE}(X,Y)$ is equal to the *central leaf* in the local deformation space $\mathcal{D}ef(X \times Y)$ over k with respect to the universal Barsotti-Tate group over $\mathcal{D}ef(X \times Y)$. In the case when $A_0[p^\infty]$ is isomorphic to a product $X \times_{\mathrm{Spec}(k)} Y$ of two isoclinic Barsotti-Tate groups as above, where X and Y are Serre-dual of each other under the principal polarization λ_0, the results in §3 say that the central leaf in $\mathcal{D}ef(A_0, \lambda_0)$ is the maximal p-divisible formal subgroup of $\mathcal{DE}(X,Y)_{\text{p-div}}$ fixed by an involution ι_0, where ι_0 is the involution on $\mathcal{DE}(X,Y)_{\text{p-div}}$ induced by the principal polarization λ_0 on A_0.

The maximal unipotent smooth formal subgroup $\mathcal{DE}(X,Y)_{\text{unip}}$ of $\mathcal{DE}(X,Y)$ is also of interest. It is the intersection of $\mathcal{DE}(X,Y)$ with the isogeny leaf in the local deformation space $\mathcal{D}ef(X \times_{\mathrm{Spec}(k)} Y)$. In the case when $A_0[p^\infty]$ is isomorphic to a product $X \times_{\mathrm{Spec}(k)} Y$ of two isoclinic Barsotti-Tate groups as above, the fixed subgroup $\mathcal{DE}(X,Y)_{\mathrm{unip}}^{\iota_0}$ of $\mathcal{DE}(X,Y)_{\mathrm{unip}}$ under the involution ι_0 is the intersection of $\mathcal{DE}(X,Y)$ with the formal completion at x_0 of the isogeny leaf passing through x_0. These aspects are not addressed in this article, however the readers can find some examples in §6 and 9.9, where the Cartier module of the formal groups $\mathcal{DE}(X,Y)$, $\mathcal{DE}(X,Y)_{\text{p-div}}$, $\mathcal{DE}(X,Y)_{\mathrm{unip}}$ are computed explicitly.

The bulk of this article is devoted to the properties of the smooth formal group $\mathcal{DE}(X,Y)$, its maximal p-divisible formal group $\mathcal{DE}(X,Y)_{\text{p-div}}$, and its maximal p-divisible quotient $\mathcal{DE}(X,Y)^{\text{p-div}}$ of $\mathcal{DE}(X,Y)$. When X is étale, the formal group $\mathcal{DE}(X,Y)$ is naturally isomorphic to $\mathrm{T}_p(X)^\vee \otimes_{\mathbb{Z}_p} Y$, non-canonically isomorphic to several copies of Y; see Prop. 2.9. In the rest of this Introduction, we assume that X is a p-divisible formal group. In §2, we show that $\mathcal{DE}(X,Y)_{\text{p-div}}$ is isoclinic of Frobenius slope $\mu_Y - \mu_X$. The module $\mathrm{BC}_p(k)$ of p-typical formal curves for the functor Cart_p plays an important role in the computation of the Cartier module of the smooth formal group $\mathcal{DE}(X,Y)$ over k. Here $\mathrm{Cart}_p(R)$ denotes the Cartier ring for R, for any commutative k-algebra R with 1, and Cart_p is regarded as an infinite dimensional smooth formal group over k. The module $\mathrm{BC}_p(k)$ has a natural structure as a $(\mathrm{Cart}_p(k)\text{-}\mathrm{Cart}_p(k))$-bimodule, plus a left action of an "extra copy" of $\mathrm{Cart}_p(k)$ which commutes with the bimodule structure. The Cartier module $\mathrm{M}(\mathcal{DE}(X,Y))$ attached to the commutative smooth formal group $\mathcal{DE}(X,Y)$ is canonically isomorphic to $\mathrm{Ext}^1_{\mathrm{Cart}_p(k)}\left(\mathrm{M}(X), \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} \mathrm{M}(Y)\right)$ if X, Y are both p-divisible formal groups over k; see 5.7.3. Here $\mathrm{M}(X), \mathrm{M}(Y)$ are the left $\mathrm{Cart}_p(k)$-modules attached to X and Y respectively. The extension group is computed using the left $\mathrm{Cart}_p(k)$-module structure coming from the bimodule structure of $\mathrm{BC}_p(k)$. This extension group has a natural structure as a left $\mathrm{Cart}_p(k)$-module, which comes from the action of the "extra copy" of $\mathrm{Cart}_p(k)$ on $\mathrm{BC}_p(k)$. The basic properties of $\mathrm{BC}_p(k)$ can be found in §5. Starting from the Cartier module $\mathrm{M}(\mathcal{DE}(X,Y))$ of $\mathcal{DE}(X,Y))$, computed by the above formula involving Ext^1, the Cartier module $\mathrm{M}(\mathcal{DE}(X,Y)_{\text{p-div}})$ attached to $\mathcal{DE}(X,Y)_{\text{p-div}}$ can be characterized as a submodule of $\mathrm{M}(\mathcal{DE}(X,Y))$ in several ways; here is one: The Cartier module $\mathrm{M}(\mathcal{DE}(X,Y)_{\text{p-div}})$ consists of all elements $x \in \mathrm{M}(\mathcal{DE}(X,Y)$ such that for every $n \in \mathbb{N}$, there exists a natural number $m \in \mathbb{N}$ such that $V^m x \in F^n(\mathrm{M}(\mathcal{DE}(X,Y))$; see 4.3. In §6, the Cartier module of the formal groups $\mathcal{DE}(X,Y)$ and $\mathcal{DE}(X,Y)_{\text{p-div}}$ are computed in several examples, illustrating the use of 5.7.3 and 4.3.

In §7 we give a precise structural description of $\mathrm{BC}_p(k)$. The basic idea is that $\mathrm{BC}_p(k)$ can be approximated by $\mathrm{Cart}_p(k) \otimes_{W(k)} \mathrm{Cart}_p(k)$. In fact, $\mathrm{BC}_p(k)$ is the completion of an enlargement of $\mathrm{Cart}_p(k) \otimes_{W(k)} \mathrm{Cart}_p(k)$ in $\mathrm{Cart}_p(k) \otimes_{W(k)} \mathrm{Cart}_p(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ with respect to a suitable topology, and we obtain an identification of $\mathrm{BC}_p(k)$ with a subset of formal series of the form $\sum_{i,j\in\mathbb{Z}} a_{ij} V^i \otimes V^j$, with coefficients a_{ij} in $W(k)$, satisfying suitable growth conditions; see 7.10 for a precise statement. The above result for $\mathrm{BC}_p(k)$ in turn provides an identification of $\mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N \otimes_{\mathbb{Z}} \mathbb{Q}$ with a subset of formal series in powers of V with coefficients in $N \otimes_{\mathbb{Z}} \mathbb{Q}$. This description of $\mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N \otimes_{\mathbb{Z}} \mathbb{Q}$ allows one to compute $\mathrm{Ext}^1_{\mathrm{Cart}_p(k)}\left(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N\right) \otimes \mathbb{Q}$, using the method of [14, Appendix]: The V-isocrystal $\mathrm{M}(\mathcal{DE}(X,Y)_{\text{p-div}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is canonically isomorphic to V-isocrystal $\mathrm{Hom}_{W(k)}(\mathrm{M}(X), \mathrm{M}(Y)) \otimes_{\mathbb{Z}} \mathbb{Q}$; see §8.6.

It turns out that the Cartier modules attached to the maximal p-divisible subgroup $\mathcal{DE}(X,Y)_{\text{p-div}}$ and the maximal p-divisible quotient $\mathcal{DE}(X,Y)^{\text{p-div}}$ of $\mathcal{DE}(X,Y)$ can be expressed more directly in terms of the Cartier modules $\mathrm{M}(X), \mathrm{M}(Y)$, as follows. The $W(k)$-

submodule $H := \mathrm{Hom}_{W(k)}(\mathrm{M}(X), \mathrm{M}(Y))$ of the V-isocrystal $H_{\mathbb{Q}} := \mathrm{Hom}_{W(k)}(\mathrm{M}(X), \mathrm{M}(Y)) \otimes_{\mathbb{Z}} \mathbb{Q}$ is stable under the action of F, but not under V. Let H_1 be the maximal $W(k)$-submodule of H which is stable under F and V, and let H_2 be the minimal $W(k)$-submodule of $H_{\mathbb{Q}}$ which contains H and is stable under F and V. Thm. 9.6 says that H_1 is the Cartier module attached to $\mathcal{DE}(X,Y)_{\text{p-div}}$, and H_2 is the Cartier module attached to $\mathcal{DE}(X,Y)^{\text{p-div}}$. The $\mathrm{Cart}_p(k)$-module H_2/H_1 is the covariant Dieudonné module attached to $\mathcal{DE}(X,Y)_{\text{p-div}} \cap \mathcal{DE}(X,Y)_{\text{unip}}$; it is trivial only when Y is a formal torus.

Here are some basic numerical invariants of the formal groups $\mathcal{DE}(X,Y)$ and $\mathcal{DE}(X,Y)_{\text{p-div}}$. Suppose that $\dim(X) = r_1$, $\mathrm{codim}(X) = s_1$, $\dim(Y) = r_2$, $\mathrm{codim}(Y) = s_2$, so that the slope of X is $\frac{r_1}{r_1+s_1}$, the slope of Y is $\frac{r_2}{r_2+s_2}$, and $r_1 s_2 < r_2 s_1$. Then $\dim(\mathcal{DE}(X,Y)) = r_2 s_1$, $\dim(\mathcal{DE}(X,Y)_{\text{unip}}) = r_1 s_2)$, and $\dim(\mathcal{DE}(X,Y)_{\text{p-div}}) = r_2 s_1 - r_1 s_2$. When X, Y are Serre-dual to each other, induced by a principal quasi-polarization λ on $X \times_{\mathrm{Spec}(k)} Y$, we have an involution ι on $\mathcal{DE}(X,Y)$. Let $r = r_2 = s_1, s = s_2 = r_1$. Then $\dim(\mathcal{DE}(X,Y)^{\iota}) = \frac{r(r+1)}{2}$, $\dim(\mathcal{DE}(X,Y)^{\iota}_{\text{unip}}) = \frac{s(s+1)}{2}$, and $\dim(\mathcal{DE}(X,Y)^{\iota}_{\text{p-div}}) = \frac{(r+s+1)(r-s)}{2}$. These formulas give the dimension of the central leaves in $\mathcal{A}_g$ in the two-slope case; they were first proved by Oort, using his results on *minimal* Barsotti-Tate groups in [18].

In this article we only treat the case when the Barsotti-Tate group in question has exactly two slopes. As indicated before, in the general case, with no restriction on the number of slopes, the completion of a central leaf is built up from a system of fibrations, with the groups $\mathcal{DE}(X,Y)_{\text{p-div}}$ considered here or a subgroup of $\mathcal{DE}(X,Y)_{\text{p-div}}$ fixed by an involution, as fibers. The dimension formula for leaves generalize, using the local structural result of leaves above. These results will be documented in a planned monograph with F. Oort on Hecke orbits.

It is a pleasure to thank F. Oort for explaining his idea about the foliation structure when it was freshly conceived, for the many discussions over the years, and for his constant encouragement. I would also like to thank C. F. Yu for the enjoyable discussions during our on Hecke orbit in Hilbert modular varieties, one of which lead to the observation that the maximal p-divisible subgroup of $\mathcal{DE}(X,Y)$ is the central leaf in $\mathcal{DE}(X,Y)$. This paper would not have existed without them.

§2. The slope of $\mathcal{DE}(X,Y)_{\text{p-div}}$

In this section k denotes a field of characteristic p.

(2.1) Notation for Barsotti-Tate groups and commutative smooth formal groups

(2.1.1) Barsotti-Tate groups, or BT-groups, over a scheme are understood to be of finite height. See [11], [6] for basic properties of Barsotti-Tate groups. A BT-group $G \to S$ over a scheme S is a system $(G_n \to S)_{n\geq 1}$ of locally free group schemes of finite rank, together with homomorphisms $i_n : G_n \hookrightarrow G_{n+1}$, $[p] : G_{n+1} \to G_n$, satisfying suitable conditions; the group $G_n \to S$ is a truncated Barsotti-Tate group of level n, or a BT_n-group, over S. Here we have used the general notation that, for any sheaf of commutative groups H and any integer n, $[n]_H$ denotes the map "multiplication by n" on H. The Serre dual of a BT-group $(G_n)_{n\geq 1}$ is the system $(G_n^t)_{n\geq 1}$, where G_n^t is the Cartier dual of G_n.

(2.1.2) Every BT-group G over k of height h has a slope sequence $0 \leq \mu_1 \leq \cdots \leq \mu_h \leq h$, which depends only on $G \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^{\mathrm{alg}})$. The multiplicity of any slope μ is a multiple of the denominator of μ. The slopes of G measures the divisibility properties of iterates of the relative Frobenius of G. To avoid possible confusion, we sometimes use the term *Frobenius slope* to emphasize that the powers of relative Frobenius are compared with powers of p. Barsotti-Tate groups over an algebraic closure k^{alg} of k are determined up to isogeny by its slopes, with multiplicity; see [10], [7], [5].

A BT-group $(G_n)_{n\geq 1}$ over k is étale iff all of its Frobenius slopes are equal to 0, and it is of multiplicative type iff all of its Frobenius slopes are equal to 1. A BT-group $(G_n)_{n\geq 1}$ over k is connected, i.e. G_n is connected for every n, if and only if all slopes of $(G_n)_{n\geq 1}$ are strictly positive.

If $\mu_1 \leq \cdots \leq \mu_h$ is the slope sequence of a BT-group over k, then the slope sequence of its Serre dual is $1-\mu_h \leq \cdots \leq 1-\mu_1$.

(2.1.3) Let R be an Artinian local ring with residue field k. Denote by $\mathfrak{BT}_{R,\mathrm{conn}}$ category of BT-groups $(G_n)_{n\geq 1}$ over R such that each G_n is connected. Denote by $\mathfrak{CFG}_R$ be the category of commutative smooth formal groups over R. Then there are natural equivalence of categories

$$\xi : \mathfrak{BT}_{R,\mathrm{conn}} \longrightarrow \mathfrak{CFG}_R, \qquad \psi : \mathfrak{CFG}_R \longrightarrow \mathfrak{BT}_{R,\mathrm{conn}}\,,$$

inverse to each other. For any BT-group $(G_n)_{n\geq 1}$ in $\mathfrak{BT}_{R,\mathrm{conn}}$, $\xi((G_n)_{n\geq 1})$ is the smooth formal group $G := \varinjlim_n G_n$, where the limit is taken in the category of formal schemes over R. Conversely, for any commutative formal group G in $\mathfrak{CFG}_R$, let $G_n = \mathrm{Ker}([p] : G \to G$. Then the natural inclusions $i_n : G_n \hookrightarrow G_{n+1}$ and the homomorphism $[p] : G_{n+1} \to G_n$ induced by $[p]_G$ defines a BT-group $\psi(G) = (G_n)_{n\geq 1}$. We will freely use the above equivalence of categories to identify a connected BT-group with a smooth formal group over R, and call them *p-divisible formal groups* over R. Passing to the limit, the above equivalence of categories holds for any complete Noetherian local ring with residue field k.

(2.2) Definition Let X and Y be Barsotti-Tate groups over k.

(i) Let $\mathcal{DE}(X,Y)$ be the functor from the category of Artinian local k-algebras to the category of commutative groups, defined as follows. For every commutative Artinian local k-algebra $(R,\mathfrak{m})$ with 1, $\mathcal{DE}(X,Y)(R)$ is the set of isomorphism classes of pairs

$$(\,0 \to Y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R) \to G \to X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R) \to 0,\ \alpha\,),$$

where $0 \to Y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R) \to G \to X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R) \to 0$ is a short exact sequence of Barsotti-Tate groups over R, and

$$\alpha : G \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\mathfrak{m}) \xrightarrow{\sim} X \times_{\mathrm{Spec}(k)} Y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R/\mathfrak{m})$$

is an isomorphism which is compatible with the short exact sequence. Notice that the set $\mathcal{DE}(X,Y)(R)$ has a natural structure as a commutative group, from the Baer sum construction. The zero section in $\mathcal{DE}(X,Y)(R)$ corresponds to the direct product of $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R)$ with $Y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R)$.

(ii) The deformation functor $\mathcal{D}ef(X \times_{\mathrm{Spec}(k)} Y)$ is the functor from the category of Artinian local k-algebras to the category of sets, such that for every Artinian local k-algebra $(R, \mathfrak{m})$, $\mathcal{D}ef(X \times_{\mathrm{Spec}(k)} Y)(R)$ is the set of isomorphism classes of pairs (G, α), where $G \to \mathrm{Spec}(R)$ is a Barsotti-Tate group over $\mathrm{Spec}(R)$, and

$$\alpha : G \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\mathfrak{m}) \xrightarrow{\sim} X \times_{\mathrm{Spec}(k)} Y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R/\mathfrak{m})$$

is an isomorphism of Barsotti-Tate groups.

(2.3) Proposition *Notation as in 2.2. Let* $r_1 = \dim(X)$, $r_2 = \dim(Y)$, $s_1 = \mathrm{ht}(X) - \dim(X)$, $s_2 = \mathrm{ht}(Y) - \dim(Y)$.

(i) *The functor* $\mathcal{DE}(X, Y)$ *on the category of Artinian local* k*-algebras is formally smooth.*

(ii) *The dimension of the smooth formal group* $\mathcal{DE}(X, Y)$ *is equal to* $r_2 s_1$.

(iii) *The natural morphism of functors, which sends any pair*

$$(\, 0 \to Y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R) \to G \to X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R) \to 0,\ \alpha\,)$$

in $\mathcal{DE}(X, Y)$ *to the element* (G, α) *of* $\mathcal{D}ef(X \times_{\mathrm{Spec}(k)} Y)(R)$*, identifies* $\mathcal{DE}(X, Y)$ *as a closed subfunctor of* $\mathcal{D}ef(X \times_{\mathrm{Spec}(k)} Y)(R)$.

(2.3.1) Lemma *Let* R *be an Artinian commutative local ring. Let* $0 \to H_1 \to H \to H_2 \to 0$ *be a short exact sequence of free* R*-modules of finite rank. Let* F_i *be an* R*-submodule of* H_i *such that* H_i/F_i *is a free* R*-module,* $i = 1, 2$*. Let* $\mathfrak{n}$ *be an ideal of* R *such that* $\mathfrak{n}^2 = (0)$*. Let* $\overline{R} = R/\mathfrak{n}$*. Let* $\overline{H_i} = H_i \otimes_R \overline{R}$*, and let* $\overline{F_i} = F_i \otimes_R \overline{R} \subseteq \overline{H_i}$*,* $i = 1, 2$*. Let* $\overline{F}$ *be an* $\overline{R}$*-submodule of* $\overline{H}$ *such that* $\overline{H}$ *is a free* $\overline{R}$*-module such that* $\overline{F} \cap \overline{H_1} = \overline{F_1}$*, and the image of* $\overline{F}$ *in* $\overline{H_2}$ *is equal to* $\overline{F_2}$*.*

(i) *There exists an* R*-direct summand* F *of* H *such that* $F \cap H_1 = F_1$*, the image of* F *in* $\overline{H}$ *is equal to* $\overline{F}$*, and the image of* F *in* H_2 *is* F_2*.*

(ii) *The set* $\mathcal{L}$ *of all liftings* F *of* $\overline{F}$ *satisfying the properties in* (i) *above has a natural structure as a torsor for the group* $\Psi := \mathrm{Hom}_{\overline{R}}\,(\overline{F_2}, \mathfrak{n} \otimes_{\overline{R}} (\overline{H_1}/\overline{F_1})\,)$*. The torsor structure is given as follows. For* $F \in \mathcal{L}$ *and* $\psi \in \Psi$*, choose an* R*-linear homomorphism*

$$h' : F_2 \to \mathfrak{n} \otimes_R H_1$$

such that image of h' *under the natural map*

$$\mathrm{Hom}_R(F_2, \mathfrak{n} \otimes_R H_1) \longrightarrow \mathrm{Hom}_{\overline{R}}(\overline{F_2}, \mathfrak{n} \otimes_{\overline{R}} (\overline{H_1}/\overline{F_1}))$$

is equal to ψ*. Let* $h \in \mathrm{Hom}_R(F, \mathfrak{n} \otimes_R H_1)$ *be the composition of* $h' \circ \pi$*, where* π *is the natural surjection* $\pi : F \twoheadrightarrow F/F_1 \xrightarrow{\sim} F_2$*. Then the torsor structure sends the pair* (ψ, F) *to the* R*-submodule*

$$F' := \{x + h(x) \,|\, x \in F\}$$

in H*. Notice that the* R*-submodule* $F' \subseteq H$ *above is well-defined, independent of the choice of* h*.*

Proof. We first show that the canonical map $H \to H' := \overline{H} \times_{\overline{H_2}} H_2$ is surjective. Here H' is the pull-back of $H_2 \twoheadrightarrow \overline{H_2}$ by $\overline{H} \twoheadrightarrow \overline{H_2}$. In other words, H' is the R-submodule of $\overline{H} \oplus H_2$ consisting of all elements (u,v) in $\overline{H} \oplus H_2$ such that u and v map to the same element in $\overline{H_2}$. The surjectivity of $H \to H'$ follows from the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{n} \otimes_R H & \longrightarrow & H & \longrightarrow & \overline{H} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 0 & \longrightarrow & \mathfrak{n} \otimes_R H_2 & \longrightarrow & H' & \longrightarrow & \overline{H} & \longrightarrow & 0 \end{array}$$

with exact rows, and the five-lemma.

Let $v_1, \ldots, v_r$ be an R-basis of $F_2 \subset H_2$. Let $\bar{v}_1, \ldots, \bar{v}_r$ be the image of $v_1, \ldots, v_r$ in $\overline{F_2} \subset \overline{H_2}$. Choose elements $\bar{w}_1, \ldots, \bar{w}_r$ in $\overline{F}$ lying above $\bar{v}_1, \ldots, \bar{v}_r$ under the map $\overline{F} \twoheadrightarrow \overline{F_2}$. The surjectivity of $H \to H'$ there exist elements $w_1, \ldots, w_r$ in H, such that $w_i \mapsto v_i$ under $H \twoheadrightarrow H_2$ and $w_i \mapsto \bar{w}_i$ under $H \twoheadrightarrow \overline{H}$ for $i = 1, \ldots, r$. Then the R-submodule $F := F_1 + Rw_1 + \cdots + Rw_r$ of H satisfies the required properties in (i).

It is easy to see that the map $(\psi, F) \mapsto F'$ described in (ii) indeed defines an action of Ψ on $\mathcal{L}$. Conversely, given any two elements F, F' in $\mathcal{L}$, there exists an element $h \in \mathrm{Hom}_R(F, \mathfrak{n} \otimes_R H)$ such that $F' = \{x + h(x) \,|\, x \in F\}$. Since $F' \cap H_1 = F_1 = F \cap H_1$, $h(F_1) \subseteq \mathfrak{n} \otimes_R F_1$. So the composition

$$\bar{h} : F \xrightarrow{h} \mathfrak{n} \otimes_R H \twoheadrightarrow \mathfrak{n} \otimes_R H/F$$

factors through the natural surjection $\pi : F \twoheadrightarrow F/F_1 \xrightarrow{\sim} F_2$ to give a map $\bar{h}' : F_2 \to \mathfrak{n} \otimes_R H/F$ such that $\bar{h} = \bar{h}' \circ \pi$. Because the image of F and F' in H_2 are both equal to F_2, the map $\bar{h}'$ factors through the injection

$$\jmath : \mathfrak{n} \otimes_{\overline{R}} (\overline{H_1}/\overline{F_1}) \xleftarrow{\sim} \mathfrak{n} \otimes_R (H_1/F_1) \hookrightarrow \mathfrak{n} \otimes_R (H/F)$$

and gives us an element $\psi \in \Psi$ such that $\bar{h}' = \jmath \circ \psi$. We have $(\psi, F) \mapsto F'$ according to the definition of the action of Ψ on $\mathcal{L}$. We have proved that $\mathcal{L}$ is a Ψ-torsor under the action described in (ii). ∎

Proof of Prop. 2.3. (i) Let R be an Artinian local ring over k, and let $\mathfrak{n}$ be an ideal of R such that $\mathfrak{n}^2 = (0)$. We must show that the map $\mathcal{DE}(X,Y)(R) \to \mathcal{DE}(X,Y)(R/\mathfrak{n})$ is surjective. An element of $\mathcal{DE}(X,Y)(R/\mathfrak{n})$ corresponds to a Barsotti-Tate group G over $R/\mathfrak{n}$, which is an extension of $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R/\mathfrak{n})$ by $Y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R/\mathfrak{n})$. We need to lift G to a Barsotti-Tate group over R, as an extension of $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R)$ by $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R)$. We will use the Grothendieck-Messing theory of crystals attached to Barsotti-Tate groups; see [11, chap. IV]. Let $H = \mathbb{D}(G)_{R \twoheadrightarrow R/\mathfrak{n}}$ be the evaluation of the crystal $\mathbb{D}(G)$ at the nilpotent immersion $\mathrm{Spec}(R/\mathfrak{n}) \to \mathrm{Spec}(R)$, with the natural divided power structure on $\mathfrak{n}$. Here Y_R is short for $Y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R)$. The R-module H is free of rank $\mathrm{ht}(X) + \mathrm{ht}(Y)$. Let $\overline{F} = \omega_{G^t}$ be the cotangent space of G^t, a direct summand of $H \otimes_R R/\mathfrak{n}$. Let $(F_1, H_1) = \left(\omega_{Y_R^t} \subset \mathbb{D}(Y_R)(R)\right)$, and let $(F_2, H_2) = \left(\omega_{X_R^t} \subset \mathbb{D}(X_R)(R)\right)$. Both H_1 and H_2 are free R-modules, and F_i is a direct summand of the free R-module H_i, $i = 1, 2$. Moreover we have a short exact sequence

$$0 \to H_1 \to H \to H_2 \to 0$$

of R-modules, and a compatible short exact sequence

$$0 \to F_1 \otimes_R (R/\mathfrak{n}) \to \overline{F} \to F_2 \otimes_R (R/\mathfrak{n}) \to 0$$

of $(R/\mathfrak{n})$-modules. By [11, Thm. V.1.6], lifting the given element of $\mathcal{DE}(X,Y)(R/\mathfrak{n})$ to an element in $\mathcal{DE}(X,Y)(R)$ is equivalent to lifting $\overline{F}$ to a direct summand F of the R-module H, with the property that $F \cap H_1 = F_1$, and the image of F in H_2 is F_2. The statements (i) and (ii) follows from Lemma 2.3.1.

The statement (iii) is a consequence of the general fact that for any two Barsotti-Tate groups G_1, G_2 over an Artinian local ring $(R,\mathfrak{m})$, the natural map

$$\mathrm{Hom}_{\mathrm{Spec}(R)}(G_1,G_2) \longrightarrow \mathrm{Hom}_{\mathrm{Spec}(R/\mathfrak{m})}\big(G_1 \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\mathfrak{m}), G_2 \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\mathfrak{m})\big)$$

is injective. ∎

(2.4) For any finite dimensional commutative smooth formal group G over k, we denote by $G_{\text{p-div}}$ the maximal p-divisible formal subgroup of G. The formal subgroup $G_{\text{p-div}}$ is a smooth formal subgroup of G such that the map $\mathrm{Lie}(G_{\text{p-div}}) \to \mathrm{Lie}(G)$ induced on their Lie algebras is an injection, characterized by the property that $G_{\text{p-div}}$ is a p-divisible formal group over k, and the quotient $G/G_{\text{p-div}}$ is a unipotent commutative smooth formal group over k. See Prop. 4.3 for a description of the Cartier module of $G_{\text{p-div}}$ in terms of the Cartier module of G.

A finite dimensional commutative smooth formal group G as above also has a maximal unipotent smooth formal subgroup G_{unip} such that the map $\mathrm{Lie}(G_{\text{unip}}) \to \mathrm{Lie}(G)$ induced on their Lie algebras is an injection, characterized by the property that G/G_{unip} is a p-divisible smooth formal group; see [21, Satz 5.36].

The homomorphism $G_{\text{p-div}} \times G_{\text{unip}} \to G$ induced by the group law of G is an isogeny; the kernel of this isogeny is isomorphic to $G_{\text{p-div}} \cap G_{\text{unip}}$.

The p-divisible formal group G/G_{unip} is the maximal p-divisible quotient formal group of G, in the sense that every epimorphism $G \to G'$ from G to a p-divisible formal group G' over k factors through G/G_{unip}. Similarly, the unipotent group $G/G_{\text{p-div}}$ is the maximal unipotent quotient smooth formal group of G. The kernel of the natural homomorphisms $G_{\text{p-div}} \to G/G_{\text{unip}}$ and $G_{\text{unip}} \to G/G_{\text{p-div}}$ are both isomorphic to $G_{\text{p-div}} \cap G_{\text{unip}}$.

(2.5) Lemma (i) *Let $\alpha : Y \to Y_1$ be an isogeny of Barsotti-Tate groups over k. Then the homomorphism $\alpha_* : \mathcal{DE}(X,Y) \longrightarrow \mathcal{DE}(X,Y_1)$ induces an isogeny*

$$\alpha_* : \mathcal{DE}(X,Y)_{\text{p-div}} \longrightarrow \mathcal{DE}(X,Y_1)_{\text{p-div}}$$

between p-divisible formal groups.

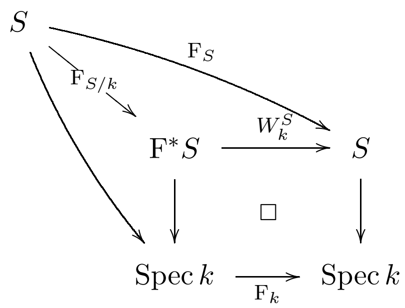
(ii) *Let $\beta : X_1 \to X$ be an isogeny of Barsotti-Tate groups over k. Then the homomorphism $\beta^* : \mathcal{DE}(X,Y) \longrightarrow \mathcal{DE}(X_1,Y)$ induces an isogeny $\beta^* : \mathcal{DE}(X,Y)_{\text{p-div}} \longrightarrow \mathcal{DE}(X_1,Y)_{\text{p-div}}$ between p-divisible formal groups.*

Proof. There exists an isogeny $\alpha' : Y_1 \to Y$ and a natural number m such that $\alpha' \circ \alpha = [p^m]_Y$ and $\alpha \circ \alpha' = [p^m]_{Y_1}$. Then $\alpha'_* \circ \alpha_* = [p^m]_{\mathcal{DE}(X,Y)}$ and $\alpha_* \circ \alpha'_* = [p^m]_{\mathcal{DE}(X,Y_1)}$. This proves (i). The proof of (ii) is similar. ∎

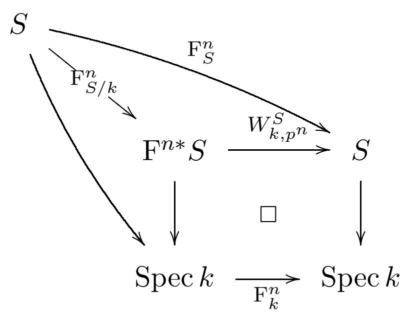
(2.6) We review some notation about absolute and relative Frobenius maps. Throughout this section k is a field of characteristic p.

(2.6.1) Denote by $\mathrm{F}_k : \operatorname{Spec} k \to \operatorname{Spec} k$ the absolute Frobenius morphism for $\operatorname{Spec} k$, induced by the ring homomorphism $x \mapsto x^p$ from k to k. More generally, for any $\mathbb{F}_p$-scheme S, denote by $\mathrm{F}_S : S \to S$ the absolute Frobenius morphism for S, induced by the ring homomorphism $x \mapsto x^p$ from $\mathcal{O}_S$ to $\mathcal{O}_S$.

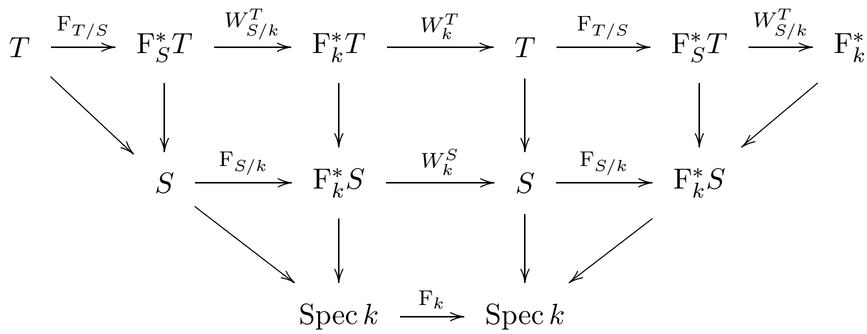
(2.6.2) For a k-scheme S, let $S^{(p)} = F_k^* S$ be the fiber product of $S \to \operatorname{Spec} k$ with $\mathrm{F}_k : \operatorname{Spec} k \to \operatorname{Spec} k$. We have a commutative diagram



The map $\mathrm{F}_{S/k} : S \to \mathrm{F}_k^* S$ is called the $S/\operatorname{Spec} k$-Frobenius, or simply S/k-Frobenius. More generally, for every natural number $n \geq 1$, denote by $\mathrm{F}_k^n : \operatorname{Spec} k \to \operatorname{Spec} k$ the morphism induced by the ring homomorphism $x \mapsto x^{p^n}$, called the n-th iterate of the absolute Frobenius for $\operatorname{Spec} k$. Let $S^{(p^n)} := (\mathrm{F}^n)^* S$, the fiber product of $S \to \operatorname{Spec} k$ and $\mathrm{F}_k^n : \operatorname{Spec} k \to \operatorname{Spec} k$. The n-th iterate $\mathrm{F}_{S/k}^n$ of the S/k-Frobenius is defined by the commutative diagram



(2.6.3) Let $T \to S$ be a morphism of k-schemes. Then we have a commutative diagram



involving the relative S/k-Frobenius $\mathrm{F}_{S/k}$ and the relative T/S-Frobenius $\mathrm{F}_{T/S}$. The following equalities hold.

- $W_k^S \circ \mathrm{F}_{S/k} = \mathrm{F}_S$, $\mathrm{F}_{S/k} \circ W_k^S = \mathrm{F}_{\mathrm{F}_k^* S}$,
- $W_{S/k}^T \circ \mathrm{F}_{T/S} = \mathrm{F}_{T/k}$,
- $W_k^T \circ \mathrm{F}_{T/k} = W_k^T \circ W_{S/k}^T \circ \mathrm{F}_{T/S} = \mathrm{F}_T$,
- $\mathrm{F}_{T/k} \circ W_k^T = W_{S/k}^T \circ \mathrm{F}_{T/S} \circ W_k^T = \mathrm{F}_{\mathrm{F}_k^* T}$,
- $W_k^T \circ W_{S/k}^T =: W_S^T$,
- $\mathrm{F}_{T/S} \circ W_S^T = \mathrm{F}_{T/S} \circ W_k^T \circ W_{S/k}^T = \mathrm{F}_{\mathrm{F}_S^* T}$.

(2.6.4) For every natural number $n \geq 1$, we have a big "winged diagram" similar to the one in 2.6.3, with the Frobenii replaced by their n-th iterates. The maps involved satisfy properties similar to those listed in 2.6.3.

(2.6.5) Given Barsotti-Tate groups G, H over a scheme S, denote by $\mathcal{EXT}(G, H)$ the category of all extensions of G by H, as sheaves on the site S_{fpqc}. Notice that every such extension of G by H is a Barsotti-Tate group over S. The set of isomorphism classes $\mathrm{Ext}(G, H)$ of objects of $\mathcal{EXT}(G, H)$ has a natural structure as an abelian group, via the Baer sum. We have a natural isomorphism $\mathrm{Ext}(G, H) \cong \mathrm{Ext}_S^1(G, H)$.

Let $\mathfrak{E} = (0 \to H \to E \to G \to 0)$ be an extension of Barsotti-Tate groups on S.

- For any homomorphism $\alpha : H \to H_1$, we have a push-forward of the extension $\mathfrak{E}$ by α, denoted by $\alpha_* \mathfrak{E}$, characterized by the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & H_1 & \longrightarrow & \alpha_* E & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

  with exact rows.

- For any homomorphism $\alpha : G_1 \to G$, we have a pull-back of the extension E by β, denoted by $\beta^* E$, characterized by the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \longrightarrow & \beta^* E & \longrightarrow & G & \longrightarrow & 0 \\ & & {\scriptstyle =}\downarrow & & \downarrow & & \downarrow{\scriptstyle\beta} & & \\ 0 & \longrightarrow & H & \longrightarrow & E & \longrightarrow & G_1 & \longrightarrow & 0 \end{array}$$

  with exact rows.

(2.6.6) Notation as in 2.6.5. For any morphism $f : S_1 \to S$ of schemes, we have a pull-back of the extension $\mathcal{E}$ by f, namely the extension of Barsotti-Tate groups

$$0 \to S_1 \times_S H \to S_1 \times_S E \to S_1 \times_S G \to 0$$

over S_1. In order to distinguish such pull-back with the previously defined pull-back by a homomorphism $G_1 \to G$, we will use the notation $f^\sharp E$ for the above extension of BT-groups on S_1.

(2.7) Proposition *Let S be a scheme over k. Let $0 \to H \to E \to G \to 0$ be an extension of Barsotti-Tate groups over S. Let n be a natural number. Then there is an isomorphism*

$$\mathrm{F}^{n\,*}_{G/S}\left(\mathrm{F}^{n\,\sharp}_{S/k}(\mathrm{F}^{n\,\sharp}_{k}E)\right) \xrightarrow{\sim} \left(\mathrm{F}^{n}_{H/S}\right)_* E$$

between extensions of Barsotti-Tate groups over S. Here

- $\mathrm{F}^{n\,\sharp}_{k}E \in \mathcal{EXT}(\mathrm{F}^{n*}_{n}G, \mathrm{F}^{n*}_{n}H)$ *is the pull-back of the extension E by the n-th iterate*

  $$\mathrm{F}^{n}_{k} : \operatorname{Spec} k \to \operatorname{Spec} k$$

  of the absolute Frobenius for $\operatorname{Spec} k$,

- $\mathrm{F}^{n\,\sharp}_{S/k}(\mathrm{F}^{n\,\sharp}_{k}E) \in \mathcal{EXT}(\mathrm{F}^{n*}_{S}G, \mathrm{F}^{n*}_{S}H)$ *is the pull-back of $\mathrm{F}^{n\,\sharp}_{k}E$ by the n-th iterate*

  $$\mathrm{F}^{n}_{S/k} : S \to \mathrm{F}^{n\,*}_{k}S$$

  of the relative Frobenius for $S/\operatorname{Spec} k$. So we have

  $$\mathrm{F}^{n\,\sharp}_{S/k}(\mathrm{F}^{n\,\sharp}_{k}E) \cong \mathrm{F}^{n\,\sharp}_{S}E\,,$$

  where $\mathrm{F}^{n\,\sharp}_{S}E$ is the pull-back of E by the n-th iterate $\mathrm{F}^{n}_{S} : S \to S$ of the absolute Frobenius for S,

- $\mathrm{F}^{n\,*}_{G/S}\left(\mathrm{F}^{n\,\sharp}_{S/k}(\mathrm{F}^{n\,\sharp}_{k}E)\right) \in \mathcal{EXT}(G, \mathrm{F}^{n*}_{S}H)$ *is the pull-back of*

  $$\mathrm{F}^{n\,\sharp}_{S/k}(\mathrm{F}^{n\,\sharp}_{k}E)$$

  by $\mathrm{F}^{n}_{G/S} : G \to \mathrm{F}^{n\,*}_{S}(G)$, the n-th iterate of the relative Frobenius for $G_{/S}$.

- $\left(\mathrm{F}^{n}_{H/S}\right)_* E \in \mathcal{EXT}(G, \mathrm{F}^{*n}_{S}H)$ *is the push-forward of E by*

  $$\mathrm{F}^{n}_{H/S} : H \to \mathrm{F}^{n\,*}_{S}H\,,$$

  the n-th iterate of the relative Frobenius for $H_{/S}$.

So both sides of the asserted isomorphism are extensions of $G_{/S}$ by $\mathrm{F}^{n\,*}_{S}(H_{/S})$.

Proof. Recall that $\mathrm{F}_{S/k}^{n\sharp}\left(\mathrm{F}_k^{n\sharp}E\right)=\mathrm{F}_S^{n\sharp}E$ is the extension

$$0\to \mathrm{F}_S^{n*}H\to \mathrm{F}_S^{n*}E\to \mathrm{F}_S^{n*}G\to 0\,,$$

and the n-th iterate of the relative Frobenii defines a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 0\\
 & & \big\downarrow {\scriptstyle \mathrm{F}^n_{H/S}} & & \big\downarrow {\scriptstyle \mathrm{F}^n_{E/S}} & & \big\downarrow {\scriptstyle \mathrm{F}^n_{G/S}} & & \\
0 & \longrightarrow & \mathrm{F}_S^{n*}H & \longrightarrow & \mathrm{F}_S^{n*}E & \longrightarrow & \mathrm{F}_S^{n*}G & \longrightarrow & 0
\end{array}$$

with exact rows. Regarding this diagram as a map from the extension E to the extension $(\mathrm{F}_S^n)^\sharp E$, it factors through the extension $(\mathrm{F}^n_{H/S})_*E$ by the defining property of the push-out. This factorization is represented by the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 0\\
 & & \big\downarrow {\scriptstyle \mathrm{F}^n_{H/S}} & & \big\downarrow & & \big\downarrow {\scriptstyle =} & & \\
0 & \longrightarrow & \mathrm{F}_S^{n*}H & \longrightarrow & (\mathrm{F}^n_{H/S})_*E & \longrightarrow & G & \longrightarrow & 0\\
 & & \big\downarrow {\scriptstyle =} & & \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{F}^n_{G/S}} & & \\
0 & \longrightarrow & \mathrm{F}_S^{n*}H & \longrightarrow & \mathrm{F}_S^{n*}E & \longrightarrow & \mathrm{F}_S^{n*}G & \longrightarrow & 0
\end{array}$$

with exact rows. The lower half of this diagram tells us that the extension $(\mathrm{F}^n_{H/S})_*E$ is isomorphic to the extension $\mathrm{F}^{n*}_{G/S}(\mathrm{F}^n_S)^\sharp E$. ∎

(2.8) Theorem *Let X, Y be isoclinic Barsotti-Tate groups over k, of Frobenius slopes μ_X, μ_Y respectively.*

(i) *If $\mu_X<\mu_Y$, then $\mathcal{DE}(X,Y)_{\text{p-div}}$ is isoclinic of Frobenius slope $\mu_Y-\mu_X$.*

(ii) *If $\mu_X\geq\mu_Y$, then $\mathcal{DE}(X,Y)_{\text{p-div}}$ is trivial.*

Proof. We may and do assume that the base field k is perfect. The key of the proof is to reinterpret Prop. 2.7 geometrically. We have a commutative diagram

$$(\dagger)\qquad \begin{array}{ccccc}
 & {\scriptstyle \mathrm{F}^n_{\mathcal{DE}(X,Y)}}\swarrow & \mathcal{DE}(X,Y) & \searrow{\scriptstyle \psi_n} & \\
 & & \big\downarrow {\scriptstyle \mathrm{F}^n_{\mathcal{DE}(X,Y)/k}} & & \\
\mathcal{DE}(X,Y) & \xleftarrow{\;W^{\mathcal{DE}(X,Y)}_{k,p^n}\;} & \mathcal{DE}(X^{(p^n)},Y^{(p^n)}) & \xrightarrow{\;\phi_n\;} & \mathcal{DE}(X,Y^{(p^n)})
\end{array}$$

The maps ψ_n, ϕ_n are described below. Let $\mathcal{E}_{X,Y}$ be the universal extension over $\mathcal{DE}(X,Y)$. Recall that $X^{(p^n)}=\operatorname{Spec}k\times_{\mathrm{F}^n_k,\operatorname{Spec}k}X$, $Y^{(p^n)}=\operatorname{Spec}k\times_{\mathrm{F}^n_k,\operatorname{Spec}k}Y$. It is clear that $\mathcal{E}^{(p^n)}_{X,Y}$ is the universal extension over $\mathcal{DE}(X,Y)^{(p^n)} = \mathcal{DE}(X^{(p^n)}, Y^{(p^n)})$, and $\mathrm{F}^{n\sharp}_{\mathcal{DE}(X,Y)/k}\mathcal{E}^{(p^n)}_{X,Y}$ is isomorphic to the pull-back of $\mathcal{E}_{X,Y} \to \mathcal{DE}(X,Y)$ by the absolute Frobenius $\mathrm{F}^n_{\mathcal{DE}(X,Y)} \to \mathcal{DE}(X,Y)$. Let $\mathcal{E}_{X,Y^{(p^n)}} \to \mathcal{DE}(X, Y^{(p^n)})$ be the universal extension over $\mathcal{DE}(X, Y^{(p^n)})$. The maps ϕ_n, ψ_n are defined by

- $\phi^\sharp_n\left(\mathcal{E}_{X,Y^{(p)}}\right) \cong \left(\mathrm{F}^n_{X\times_{\mathrm{Spec}\,k}\mathcal{DE}(X^{(p^n)},Y^{(p^n)})/\mathcal{DE}(X^{(p^n)},Y^{(p^n)})}\right)^* \mathcal{E}^{(p^n)}_{X,Y}$ as extensions of X by $Y^{(p^n)}$ over $\mathcal{DE}(X^{(p^n)}, Y^{(p^n)})$,
- $\psi^\sharp_n\left(\mathcal{E}_{X,Y^{(p)}}\right) \cong \left(\mathrm{F}^n_{Y\times_{\mathrm{Spec}\,k}\mathcal{DE}(X,Y)/\mathcal{DE}(X,Y)}\right)_* \mathcal{E}_{X,Y}$ as extensions of X by $Y^{(p^n)}$ over the formal scheme $\mathcal{DE}(X,Y)$.

Prop. 2.7 tells us that the diagram (†) commutes.

According to Lemma. 2.5, it suffices to prove the theorem with the Barsotti-Tate groups X, Y replaced by X_1, Y_1 isogenous to X and Y respectively. Therefore we may and do assume that for some $n \in \mathbb{N}_{>0}$, $a := n\mu_X$ and $b := n\mu_Y$ are both integers, and we have

$$\mathrm{Ker}\left(X \xrightarrow{\mathrm{F}^n_{X/k}} X^{(p^n)}\right) = [p^a]_X\,, \quad \text{and} \quad \mathrm{Ker}\left(Y \xrightarrow{\mathrm{F}^n_{X/k}} Y^{(p^n)}\right) = [p^b]_Y\,.$$

In other words, the homomorphism $\mathrm{F}^n_{X/k}$ is equal to p^a times an isomorphism from X to $X^{(p^n)}$. Similarly $\mathrm{F}^n_{Y/k}$ is equal to p^b times an isomorphism from Y to $Y^{(p^n)}$. Therefore ϕ_n is equal to p^a times an isomorphism from $\mathcal{DE}(X^{(p^n)}, Y^{(p^n)})$ to $\mathcal{DE}(X, Y^{(p^n)})$, and ψ_n is equal to p^b times an isomorphism from $\mathcal{DE}(X,Y)$ to $\mathcal{DE}(X, Y^{(p^n)})$. So we conclude, from $\phi_n \circ \mathrm{F}^n_{\mathcal{DE}(X,Y)/k} = \psi_n$, that $p^a \cdot \mathrm{F}^n_{\mathcal{DE}(X,Y)/k}$ is equal to p^b times an isomorphism from $\mathcal{DE}(X,Y)$ to $\mathcal{DE}(X^{(p^n)}, Y^{(p^n)})$. Passing to the induced maps between the maximal p-divisible subgroups of $\mathcal{DE}(X,Y)$ and $\mathcal{DE}(X^{(p^n)}, Y^{(p^n)})$, we deduce that the homomorphism $\mathrm{F}^n_{\mathcal{DE}(X,Y)_{\text{p-div}}/k} : \mathcal{DE}(X,Y)_{\text{p-div}} \to \mathcal{DE}(X,Y)^{(p^n)}_{\text{p-div}}$ has the property that

$$p^a \cdot \mathrm{F}^n_{\mathcal{DE}(X,Y)_{\text{p-div}}/k} = p^b \cdot \left(\text{an isomorphism } \mathcal{DE}(X,Y)_{\text{p-div}} \xrightarrow{\sim} \mathcal{DE}(X,Y)^{(p^n)}_{\text{p-div}}\right)\,.$$

Hence if $\mu_Y > \mu_X$, or equivalently $b > a$, then

$$\mathrm{F}^n_{\mathcal{DE}(X,Y)_{\text{p-div}}/k} = p^{b-a} \cdot \left(\text{an isomorphism } \mathcal{DE}(X,Y)_{\text{p-div}} \xrightarrow{\sim} \mathcal{DE}(X,Y)^{(p^n)}_{\text{p-div}}\right)\,,$$

therefore $\mathcal{DE}(X,Y)_{\text{p-div}}$ is isoclinic of Frobenius slope $\frac{b-a}{n} = \mu_Y - \mu_X$. We have proved statement (i).

Suppose that $\mu_Y \le \mu_X$, i.e. $b \le a$. Then the homomorphism

$$\mathcal{DE}(X,Y)_{\text{p-div}} \xrightarrow{p^{a-b}\cdot \mathrm{F}^n_{\mathcal{DE}(X,Y)_{\text{p-div}}}} \mathcal{DE}(X,Y)^{(p^n)}_{\text{p-div}}$$

is an isomorphism. This implies that the p-divisible formal group $\mathcal{DE}(X,Y)_{\text{p-div}}$ is trivial. The statement (ii) follows. ∎

(2.8.1) Remark (i) An isoclinic Barsotti-Tate group G over a scheme S is said to be *completely slope divisible* in [19] if there exist positive integers $r, s > 0$ such that $\mathrm{Ker}([p^r]_G) = \mathrm{Ker}(F^s_{G/S})$. The proof of Thm. 2.8 shows that $\mathcal{DE}(X,Y)_{\text{p-div}}$ is a completely slope divisible group over k if X and Y are both completely slope divisible.

(ii) A "computational proof" of Thm. 2.8 (i) will be given in 8.6.3.

(2.9) Proposition *Assume that X is an étale Barsotti-Tate group over k, and Y is a p-divisible smooth formal group over k. Then we have canonical isomorphisms*

$$\mathcal{DE}(X,Y) \cong \underline{\mathrm{Hom}}_{\mathbb{Z}_p}(\mathrm{T}_p(X), Y) = \mathrm{T}_p(X)^\vee \otimes_{\mathbb{Z}_p} Y\,,$$

where $\mathrm{T}_p(X)$ is the p-adic Tate module of X. Explicitly, the first isomorphism above attaches to every homomorphism $\alpha : \mathrm{T}_p(X) \to Y(R)$, where $(R, \mathfrak{m})$ is an Artinian local k-algebra, the push-out extension

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{T}_p(X) & \longrightarrow & \mathrm{T}_p(X)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & Y & \longrightarrow & E_\alpha & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

$E_\alpha \in \mathcal{DE}(X,Y)(R)$.

Proof. We may and do assume that k is algebraically closed. We offer two proofs.

The first proof. The explicit construction above gives an homomorphism

$$\beta : \mathrm{T}_p(X)^\vee \otimes_{\mathbb{Z}_p} \longrightarrow \mathcal{DE}(X,Y)$$

between p-divisible smooth formal groups. Both the source group and the target group have dimension $\mathrm{height}(X)\cdot\dim(Y)$. It is easy to verify that map $\mathrm{Lie}(\beta)$ between the Lie algebras induced by the homomorphism β is an isomorphism. Hence β is an isomorphism.

The second proof. We may and do assume that X is equal to the constant étale p-divisible group

$$\mathbb{Z}\left[\frac{1}{p}\right]\Big/\mathbb{Z} = \left(\varinjlim_{i\in\mathbb{N}}\mathbb{Z}\right)\Big/\mathbb{Z}$$

over k, where the transition maps in the inductive system $\varinjlim_{i\in\mathbb{N}}\mathbb{Z}$ are "multiplication by p". Let $(R,\mathfrak{m})$ be an Artinian local ring with $R/\mathfrak{m} \xrightarrow{\sim} k$. We must construct a canonical isomorphism

$$Y(R) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{Spec}(R)}\left(\varinjlim_{i\in\mathbb{N}}/\mathbb{Z}, Y\right)$$

We have a short exact sequence

$$0 \to \varprojlim_{i\in\mathbb{N}}{}^1\,(Y(R))_{i\in\mathbb{N}} \to \mathrm{Ext}^1_{\mathrm{Spec}\,R}\left(\varinjlim_{i\in\mathbb{N}}\mathbb{Z}, Y\right) \to \varprojlim_{i\in\mathbb{N}}\mathrm{Ext}^1_{\mathrm{Spec}\,R}(\mathbb{Z},Y) \to 0$$

where $(Y(R))_{i\in\mathbb{N}}$ denotes the projective system indexed by $\mathbb{N}$, all of whose terms are equal to $Y(R)$, and the transition maps are "multiplication by p". Since R is Artinian, $[p^n] : Y(R) \to Y(R)$ is the trivial homomorphism for $n \gg 0$. So $\varprojlim^1_{i\in\mathbb{N}}(Y(R))_{i\in\mathbb{N}} = (0)$, and $\varprojlim_{i\in\mathbb{N}}(Y(R))_{i\in\mathbb{N}} = (0)$ as well.

We claim that $\mathrm{Ext}^1_{\mathrm{Spec}\,R}(\mathbb{Z},Y)=(0)$. Since $\mathrm{Ext}^1_{\mathrm{Spec}\,R}(\mathbb{Z},Y)=\mathrm{H}^1(\mathrm{Spec}\,R,Y)$, the claim means that every Y-torsor over $\mathrm{Spec}\,R$ is trivial. The last statement holds because every Y-torsor Z over $\mathrm{Spec}\,R$ is formally smooth, and $(R/\mathfrak{m})$-rational points of the closed fiber of Z lifts to R-rational points of Z. The short exact sequence in the last paragraph implies that $\mathrm{Ext}^1_{\mathrm{Spec}(R)}(\varinjlim_{i\in\mathbb{N}}\mathbb{Z},Y)=0$.

Consider the short exact sequence

$$0\to\mathbb{Z}\to\varinjlim_{i\in\mathbb{N}}\mathbb{Z}\to\left(\varinjlim_{i\in\mathbb{N}}\mathbb{Z}\right)\Big/\mathbb{Z}\to 0\,.$$

The resulting long exact sequence

$$\begin{aligned}(0)=\varprojlim_{i\in\mathbb{N}}(Y(R))_{i\in\mathbb{N}}&\to\mathrm{Hom}_{\mathrm{Spec}\,R}(\mathbb{Z},Y)\\&\to\mathrm{Ext}^1_{\mathrm{Spec}\,R}\left(\left(\left(\varinjlim_{i\in\mathbb{N}}\mathbb{Z}\right)\Big/\mathbb{Z}\right),Y\right)\to\mathrm{Ext}^1_{\mathrm{Spec}\,R}\left(\varinjlim_{i\in\mathbb{N}}\mathbb{Z},Y\right)=(0)\end{aligned}$$

reduces to the desired isomorphism

$$Y\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{Spec}(R)}\left(\left(\left(\varinjlim_{i\in\mathbb{N}}\mathbb{Z}\right)\Big/\mathbb{Z}\right),Y\right)\,.$$

The Proposition follows. ∎

§3. Relation with the central leaf

In this section k denotes a perfect field of characteristic p.

(3.1) Proposition *Let X, Y be isoclinic Barsotti-Tate groups over k with Frobenius slopes λ_X, λ_Y, and $\lambda_X<\lambda_Y$. Write $\mathcal{DE}(X,Y)_{\text{p-div}}=\mathrm{Spf}R$. Then the universal p-divisible extension of X by Y over* $\mathrm{Spec}\,R$ *is canonically trivial over* $\mathrm{Spec}\,R^{\mathrm{perf}}$*, where R^{perf} is the perfection of R.*

Proof. Recall the following well-known fact about Barsotti-Tate groups. For any commutative ring A which is complete with respect to an ideal I, the functor which sends every Barsotti-Tate group G over $\mathrm{Spec}\,A$ to the compatible family $(G\times_{\mathrm{Spec}\,A}\mathrm{Spec}(A/I^n))_{n\in\mathbb{N}}$ of Barsotti-Tate groups over $\mathrm{Spec}(A/I^n)$'s is an equivalence of categories. In other words, every Barsotti-Tate group over $\mathrm{Spf}A$ comes from a Barsotti-Tate group over $\mathrm{Spec}\,A$, up to a unique isomorphism. To see this assertion, observe that for each $m\in\mathbb{N}$, $\Gamma(G[p^m],\mathcal{O}_{G[p^m]})$ is the projective limit of $\Gamma(G[p^m]\times_{\mathrm{Spec}\,A}\mathrm{Spec}(A/I^n),\mathcal{O}_{G[p^m]}\otimes_A(A/I^n))$ since $G[p^m]$ is finite and affine over A.

Let $0\to Y\times_{\mathrm{Spec}(k)}\mathrm{Spf}(R)\to\mathcal{E}^{\mathrm{univ}}\to X\times_{\mathrm{Spec}(k)}\mathrm{Spf}(R)\to 0$ be the universal extension over $\mathrm{Spf}(R)$. By definition, this extension $\mathcal{E}^{\mathrm{univ}}$ over $\mathrm{Spf}(R)$ is the direct limit of extensions $0\to Y\times_{\mathrm{Spec}(k)}\mathrm{Spec}(R/\mathfrak{m}^i)\to E^{\mathrm{univ}}_{/\,\mathrm{Spec}(R/\mathfrak{m}^i)}\to X\times_{\mathrm{Spec}(k)}\mathrm{Spec}(R/\mathfrak{m}^i)\to 0$, of BT-groups over $\mathrm{Spec}(R/\mathfrak{m}^i)$, where $\mathfrak{m}$ is the maximal ideal of R. According to the fact recalled in the previous paragraph, the extension $\mathcal{E}^{\mathrm{univ}}$ over $\mathrm{Spf}(R)$ comes from a unique extension $\mathcal{E}^{\mathrm{univ}}_{/\,\mathrm{Spec}(R)}$ of $X\times_{\mathrm{Spec}(k)}\mathrm{Spec}(R)$ by $Y\times_{\mathrm{Spec}(k)}\mathrm{Spec}(R)$ over the affine scheme $\mathrm{Spec}(R)$. We denoted this extension again by $\mathcal{E}^{\mathrm{univ}}$, shortened to $\mathcal{E}$ in the rest of the proof. This extension of BT-groups $\mathcal{E}$ over $\mathrm{Spec}(R)$ is what was meant by "the universal p-divisible extension of X by Y over $\mathrm{Spec}(R)$" in the statement of 3.1.

We may and do assume that k is perfect. Since $\mathrm{Spf}R$ is a p-divisible formal group, R^{perf} is the direct limit of the following injective system

$$R = R_0 \to R_1 \to R_2 \to \cdots \to R_n \to \cdots$$

of k-algebras, where each $R_i = R$, and the transition maps are induced by $[p]_{\mathcal{DE}(X,Y)(X,Y)_{\text{p-div}}}$.

Let $0 \to Y[p^n] \to \mathcal{E}[p^n] \to X[p^n] \to 0$ be the restriction of the extension $\mathcal{E}$ to the p^n-torsion points; $\mathcal{E}[p^n]$ is an extension of truncated BT_n groups over R. From the definition of R_n one sees that the base extension of the extension $\mathcal{E}$ from R to R_n "is" the Baer sum of $\mathcal{E}$ with itself iterated p^n times. In particular the extension $\mathcal{E}[p^n] \times_{\mathrm{Spec}\,R} \mathrm{Spec}(R_n)$ over $\mathrm{Spec}(R_n)$ is trivial.

It is well-known that there exits an $m \in \mathbb{N}$ such that $p^m \cdot \mathrm{Hom}_B(X[p^n], Y[p^n]) = 0$ for every $n \in \mathbb{N}$ and for every integral domain $B \supset k$. For each $n \in \mathbb{N}$, pick a trivialization τ_{n+m} of $\mathcal{E}[p^{n+m}]$ over R_{n+m}. Let ψ_n be the restriction of τ_{n+m} to $\mathcal{E}[p^n]$, so that ψ_n is a trivialization of $\mathcal{E}[p^n]$ over R_{n+m} for each $n \in \mathbb{N}$. Clearly ψ_n is independent of the choice of τ_{n+m}. It is easy to see that for each $N \in \mathbb{N}$, the restriction of ψ_{n+N} to $\mathcal{E}[p^n]$ is equal to the base-change of ψ_n to R_{n+m+N}, for every $n \in \mathbb{N}$. Over R^{perf}, the family of trivializations $(\psi_n)_{n\in\mathbb{N}}$ gives us a trivialization of the extension $\mathcal{E}$. We have proved the existence of a trivialization over R^{perf}. The uniqueness is obvious because $\mathrm{Hom}_{R^{\mathrm{perf}}}(X, Y) = (0)$. ∎

(3.2) Proposition *Let X, Y be as in 3.1. Let S be a complete Noetherian local domain over k. Let $\mathcal{E}$ be an extension of X by Y over $\mathrm{Spec}\,S$ such that the extension $\mathcal{E}$ over $\mathrm{Spec}\,S$ to the closed point of $\mathrm{Spec}\,S$ is trivial; let ξ be a trivialization. In other words, the extension $\mathcal{E}$ comes from an $\mathrm{Spf}S$-valued point of $\mathcal{DE}(X,Y)$. Let $\overline{S}$ be the integral closure of S in the algebraic closure of the fraction field of S. Suppose that $\mathcal{E}$ is trivial over $\overline{S}$. Then the extension $\mathcal{E}$ comes from an $\mathrm{Spf}(S)$-valued point of $\mathcal{DE}(X,Y)_{\text{p-div}}$.*

PROOF. After a purely inseparable extension of S, we may and do assume that the classifying map $f : \mathrm{Spf}(S) \to \mathcal{DE}(X,Y)$ for $\mathcal{E}$ has the form $f = f_{\text{p-div}} + f_{\mathrm{u}}$, with $f_{\text{p-div}} : \mathrm{Spf}(S) \to \mathcal{DE}(X,Y)_{\text{p-div}}$ and $f_{\mathrm{u}} : \mathrm{Spf}(S) \to \mathcal{DE}(X,Y)_{\mathrm{unip}}$. Here $f = f_{\text{p-div}} + f_{\mathrm{u}}$ is the sum of the two S-valued points $f_{\text{p-div}}$ and f_{u} in the commutative group $\mathcal{DE}(X,Y)(S)$. The assumption that $\mathcal{E}$ is trivial over $\overline{S}$ means that the extension given by f_{u} is trivial over $\overline{S}$, and we want to show that f_{u} is trivial. In other words we may and do assume that $f = f_{\mathrm{u}}$.

We know that there is a natural number N such that the unipotent commutative formal group $\mathcal{DE}(X,Y)_{\mathrm{unip}}$ over k is killed by p^N. Hence the extension $\mathcal{E}$ over S is killed by p^N. Since $\mathcal{E}$ is trivial over $\overline{S}$, there exists a homomorphism $\psi : \mathcal{E} \to X$ over $\overline{S}$ which spits the extension $\mathcal{E}$ over $\overline{S}$. Clearly $[p^N]_X \circ \psi$ is a splitting of $[p^N]_*\mathcal{E}$, therefore $[p^N]_X \circ \psi$ is equal to the base extension to $\overline{S}$ of the unique splitting of $[p^N]_*\mathcal{E}$ over S. In particular ψ is a $\overline{S}$-rational homomorphism between Barsotti-Tate groups over S, and $p^N \cdot \psi$ is rational over S. This implies that ψ itself is defined over S, because $p^N \cdot \psi$ factors through the isogeny $[p^N] : \mathcal{E} \to \mathcal{E}$. So the extension $\mathcal{E}$ is trivial over S. ∎

(3.3) In [17] Oort defined the notion of *central leaf* in the base S of a Barsotti-Tate group $G \to S$. Prop.3.1 and Prop. 3.1 say that $\mathcal{DE}(X,Y)_{\text{p-div}}$ is the central leaf in $\mathcal{DE}(X,Y)$ for the universal Barsotti-Tate group over $\mathcal{DE}(X,Y)$. The following result 3.3.1 says that the universal Barsotti-Tate group over $\mathcal{D}ef(X \times Y)$ over the central leaf in $\mathcal{D}ef(X \times Y)$ has a slope filtration. Therefore 3.1 and 3.2 implies that the central leaf in $\mathcal{D}ef(X \times Y)$ defined by the universal Barsotti-Tate group over $\mathcal{D}ef(X \times Y)$ is equal to $\mathcal{DE}(X,Y)_{\text{p-div}}$.

(3.3.1) Proposition *Let $G \to S$ be a Barsotti-Tate group over a Noetherian normal scheme over a field k, $k \supset \mathbb{F}_p$. Assume that $G \to S$ is geometrically fiberwise constant. Then there exists Barsotti-Tate groups $G_i \to S$, $i = 1,\dots,m$, $0 = G_0 \subset G_1 \subset \cdots \subset G_m = G$ such that G_i/G_{i-1} has exactly one Frobenius slope μ_i, and $\mu_1 > \mu_2 > \ldots > \mu_m$.*

PROOF. According to [19, Thm. 2.1 of], there exists a completely slope divisible Barsotti-Tate group Z over S and an isogeny $\alpha : G \to Z$ over S. The condition that Z is completely slope divisible means that there exists BT-groups $0 = Z_0 \subset Z_1 \subset \cdots Z_m = Z$ and an integer $N > 0$ with the property that $r_i := \mu_i \cdot N \in \mathbb{Z}$ for $i = 1,\dots,m$, such that each Z_i/Z_{i+1} is a Barsotti-Tate group, with the property that the N-th iterate of the relative Frobenius

$$\mathrm{Fr}^N_{(Z_i/Z_{i+1})/S} : Z_i/Z_{i+1} \longrightarrow (Z_i/Z_{i+1})^{(p^N)}$$

is equal to $[p^{r_i}]$ times an isomorphism from Z_i/Z_{i+1} to $(Z_i/Z_{i+1})^{(p^N)}$, $i = 1,\dots,m$.

By étale descent, it suffices to prove the existence of such a filtration after passing to $S \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k_1)$ for some finite separable extension field k_1 of k. In particular we may and do assume that k contains $\mathbb{F}_{p^{r_i}}$ for $i = 1,\dots,m$. Moreover, we may assume that S is irreducible, and has a k-rational point s.

By [19, Cor. 1.10], for each $i = 1,\dots,m$, there exists a BT-group B_i over k, a smooth sheaf of rank-one free right $\mathrm{End}_k(B_i)$-module $\mathcal{F}_i$ on S_{et}, and an isomorphism

$$\psi_i : \mathcal{F}_i \otimes_{\mathrm{End}_k(B_i)} (B_i \times_{\mathrm{Spec}(k)} S) \xrightarrow{\sim} Z_i/Z_{i-1}\,.$$

In other words, Z_i/Z_{i-1} is a twist of the constant BT-group $B_i \times_{\mathrm{Spec}(k)} S$ over S by the representation of $\pi_1(S, \bar{s})$ underlying the smooth p-adic étale sheaf $\mathcal{F}_i$.

Denote by $(G_m/G_{m-1})_s$ the fiber of G_m/G_{m-1} over the k-rational point s of S, and let

$$\Gamma_i = \mathcal{F}_m \otimes_{\mathrm{End}_k(B_i)} ((G_m/G_{m-1})_s \times_{\mathrm{Spec}(k)} S)\,,$$

the twist of the constant BT-group $((G_m/G_{m-1})_s \times_{\mathrm{Spec}(k)} S)$ by the smooth $\mathbb{Z}_p$-sheaf $\mathcal{F}_m$ on S_{et}. The isogeny $\alpha : G \to Z$ induces an isogeny $\alpha_m : (G_m/G_{m-1})_s \to B_m$, which induces an isogeny $\beta_m : \Gamma_m \to \mathcal{F}_i \otimes_{\mathrm{End}_k(B_i)} (B_i \times_{\mathrm{Spec}(k)} S)$. Consider the composition

$$\psi_m^{-1} \circ \mathrm{pr}_m \circ \alpha \;:\; G \xrightarrow{\alpha} Z \to Z_m/Z_{m-1} \xrightarrow{\psi_m^{-1}} \mathcal{F}_i \otimes_{\mathrm{End}_k(B_i)} (B_i \times_{\mathrm{Spec}(k)} S)$$

We claim that there exists a homomorphism $\pi : G \to \Gamma_m$, necessarily unique, such that $\beta_m \circ \pi = \psi_m^{-1} \circ \mathrm{pr}_m \circ \alpha$; moreover, $\mathrm{Ker}(\pi)$ is a Barsotti-Tate group over S. This claim is a statement about the quasi-isogeny $\beta^{-1} \circ \psi_m^{-1} \circ \mathrm{pr}_m \circ \alpha$, therefore it suffices to check it at every geometric point of S. The point-wise statement follows immediately from Lemma 3.3.3. ∎

(3.3.2) Remark The slope filtration on the universal Barsotti-Tate group on a central leaf in $\mathcal{A}_g$, guaranteed to exist by Prop. 3.3.1, gives the local moduli at any closed point of the given central leaf. See [2] for more information.

(3.3.3) Lemma *Let $G \to S$ be a geometrically fiberwise constant Barsotti-Tate group over an irreducible normal scheme S. Then there exists a scheme T, which is the inductive limit of a countable projective system of irreducible normal schemes*

$$\cdots \to T_{n+1} \to T_n \to \cdots \to T_1 \to T_0 = S\,,$$

such that all transition maps $T_{n+1} \to T_n$ are finite surjective, and the Barsotti-Tate group $G \times_S T$ over T is constant.

PROOF. We may and do assume that S has a k-rational point s. By [17, Thm. 1.3], we can find a projective system of irreducible normal schemes $\cdots \to T_{n+1} \to T_n \to \cdots \to T_1 \to T_0 = S$ such that all transition maps are finite surjective, and the truncated BT_n group $G[p^n] \times_S T_n$ is trivial for each $n \geq 1$.

Let K be the perfection of the function field of T, and let G_K be the pull-back of G to $\mathrm{Spec}(K)$. By [22], there exists a natural number N such that for every $n \geq 1$, every isomorphism $G_s[p^n] \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K) \to G_K[p^n]$ which lifts to an isomorphism

$$G_s[p^{n+N}] \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K) \xrightarrow{\sim} G_K[p^{n+N}]$$

actually lifts to an isomorphism $G_s \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K) \xrightarrow{\sim} G_K$.

We construct a compatible system of isomorphisms

$$\alpha_n : G_s[p^n] \times_{\mathrm{Spec}(k)} T \xrightarrow{\sim} G[p^n] \times_S T$$

inductively, as follows. Assume that α_n has been constructed. To construct α_{n+1}, choose any isomorphism $\beta_{n+1+N} : G_s[p^{n+1+N}] \times_{\mathrm{Spec}(k)} T \xrightarrow{\sim} G[p^{n+1+N}] \times_S T$, then adjust it by a suitable automorphism of G_s to make sure that the β_{n+1+N} is compatible with α_n. Define α_{n+1} to be the isomorphism from $G_s[p^{n+1}]$ to $G[p^n] \times_S T$ induced by β_{n+1+N}. ∎

§4. Review of Cartier theory

(4.1) We review some definitions and results in Cartier theory; see [21] for an excellent presentation. Let k be a commutative algebra with 1. Since the commutative formal groups considered in Cartier theory are not necessarily Noetherian, it is convenient to regard them as functors defined on the category $\mathfrak{Nilp}_k$ of *nilpotent* algebras over the base ring k.

(4.1.1) By definition, a nilpotent algebra over k is a commutative k-algebra N (without unit element) such that $N^m = (0)$. for some $m \in \mathbb{N}$. We say that a set-valued functor

$$G : \mathfrak{Nilp}_k \to \mathfrak{Sets}\,,$$

is *smooth* if and only if the map $G(h) : G(N_1) \to G(N_2)$ is surjective for every surjective k-linear homomorphism $h : N_1 \to N_2$ between nilpotent k-algebras. A *commutative smooth formal group* over k is a smooth functor from $\mathfrak{Nilp}_k$ to the category of commutative groups.

(4.1.2) Denote by $k[[x]]^+$ the augmentation ideal in the formal power series ring $k[[x]]$, consisting of all formal power series over k whose constant term is 0. Clearly $k[[x]]^+$ can be naturally identified with $\varprojlim_n k[[x]]^+/(x^n)$, a pro-object in $\mathfrak{Nilp}_k$. For any set-valued functor

$$G : \mathfrak{Nilp}_k \to \mathfrak{Sets}\,,$$

define $G(k[[x]]^+)$ by

$$G(k[[x]]^+) := \varprojlim_{n\geq 1} G\left(k[[x]]^+/(x^n)\right)\,.$$

Similarly, we can extend a functor G as above to the category of pro-objects in $\mathfrak{Nilp}_k$.

(4.1.3) Denote by Λ the smooth commutative formal group such that

$$\Lambda(N) = \{1 + a_1 + \cdots + a_n t^n \,|\, a_1, \ldots, a_n \in N, n \in \mathbb{N}_{\geq 1}\} \subset ((k \oplus N)[t])^\times$$

the group of "principal units" of the commutative ring $(k\oplus N)[t]$, for every nilpotent k-algebra $N \in \mathfrak{Nilp}_k$. This infinite dimensional smooth formal group Λ is a "restricted version" of the group of universal Witt vectors. The latter is the group-valued functor $\mathfrak{W}$ on the category of commutative algebras, such that $\mathfrak{W}(R)$ is the subgroup of $R[[t]]^\times$ consisting of all formal power series with coefficients in R with constant term 1.

(4.1.4) For every smooth formal group G over k, we have a Yoneda-type bijection

$$\mathrm{Hom}(\Lambda, G) \xrightarrow{\sim} G(k[[x]]^+) := \varprojlim_n G(k[[x]]^+/(x^n))\,,$$

which sends any homomorphism of group-valued functors $h : \Lambda \to G$ on the category $\mathfrak{Nilp}_k$ to the element

$$h(1 - xt) \in \varprojlim_n G(k[[x]]^+/(x^n)) = G(k[[x]]^+)\,,$$

the image of the element $1 - xt$ of $\Lambda(k[[x]]^+)$ under h. Here $k[[x]]^+$ denotes the augmentation ideal $x\, k[[x]]$ of $k[[x]]$.

(4.1.5) By definition, the "big Cartier ring" $\mathrm{Cart}(k)$ is equal to be $\mathrm{End}(\Lambda)^{\mathrm{op}}$, the opposite ring of the ring of endomorphisms of Λ. According to 4.1.4, the set underlying $\mathrm{Cart}(k)$ is identified with $\Lambda(k[[x]]^+)$. In the ring $\mathrm{Cart}(k)$ we have the following elements:

- $F_n \leftrightarrow 1 - xt^n$, $n \geq 1$,
- $V_n \leftrightarrow 1 - x^n t$, $n \geq 1$,
- $[c] \leftrightarrow 1 - cxt$, $c \in k$.

The right ideal $V^n\mathrm{Cart}(k)$ of $\mathrm{Cart}(k)$ consists of all elements of $\Lambda(k[[x]]^+)$ which maps to the unit element of $\Lambda(k[[x]]^+/(x^n))$. The right ideals $V^n\mathrm{Cart}(k)$ defines a topology on $\mathrm{Cart}(k)$, called the *V-adic topology*, and the ring $\mathrm{Cart}(k)$ is complete for the V-adic topology. The elements $F_n, V_n, [c]$ is a set of topological generators of the topological commutative group $\mathrm{Cart}(k)$.

For any commutative formal group G over k, the set $\mathrm{Hom}(\Lambda, G) = G(k[[x]]^+)$ has a natural structure as a right $\mathrm{End}(\Lambda)$-module, by pre-composition. So $G(k[[x]]^+)$ has a natural structure as a left $\mathrm{Cart}(k)$-module.

(4.1.6) A *V-reduced* Cart(k)*-module* is a left Cart(k)-module M together with a separated decreasing filtration of M

$$M = \mathrm{Fil}^1 M \supset \mathrm{Fil}^2 M \supset \cdots \mathrm{Fil}^n M \supset \mathrm{Fil}^{n+1} M \supset \cdots$$

such that each $\mathrm{Fil}^n M$ is an abelian subgroup of M and

(i) $(M, \mathrm{Fil}^\bullet M)$ is complete with respect to the topology given by the filtration $\mathrm{Fil}^\bullet M$.

(ii) $V_m \cdot \mathrm{Fil}^n M \subset \mathrm{Fil}^{mn} M$ for all $m, n \geq 1$.

(iii) The map V_n induces a bijection $V_n : M/\mathrm{Fil}^2 M \xrightarrow{\sim} \mathrm{Fil}^n M/\mathrm{Fil}^{n+1} M$ for every $n \geq 1$.

(iv) $[c] \cdot \mathrm{Fil}^n M \subset \mathrm{Fil}^n M$ for all $c \in k$ and all $n \geq 1$.

(v) For every $m, n \geq 1$, there exists an $r \geq 1$ such that $F_m \cdot \mathrm{Fil}^r M \subset \mathrm{Fil}^n M$.

A V-reduced Cart(k)-module $(M, \mathrm{Fil}^\bullet M)$ is *V-flat* if $M/\mathrm{Fil}^2 M$ is a flat k-module. The k-module $M/\mathrm{Fil}^2 M$ is called the tangent space of $(M, \mathrm{Fil}^\bullet M)$.

(4.1.7) The main theorem of Cartier theory says that the functor

$$G \rightsquigarrow \left(G(k[[x]]^+),\ \mathrm{Fil}^\bullet G(k[[x]]^+)\right) ,$$

where $\mathrm{Fil}^n := \mathrm{Ker}\left(G(k[[x]]^+) \to G(k[[x]]^+/(x^n))\right)$ for each $n \geq 1$, establishes an equivalence between the category of smooth formal groups over k and the category of V-reduced V-flat left Cart(k)-modules; see [21] for the functor giving the inverse of the equivalence of categories above.

(4.2) In this subsection, k is assumed to be an algebra over $\mathbb{Z}_{(p)}$. Then the previous equivalence can be simplified, with the big Cartier ring Cart(k) replaced by a much smaller Cartier ring $\mathrm{Cart}_p(k)$. Let

$$\epsilon_p = \prod_{\substack{(\ell,p)=1 \\ \ell \text{ prime}}} \left(1 - \tfrac{1}{\ell} V_\ell F_\ell\right) = \sum_{\substack{(n,p)=1 \\ n\geq 1}} \tfrac{\mu(n)}{n} V_n F_n \,.$$

The element ϵ_p has the property that $\epsilon_p^2 = \epsilon_p$. The Cartier ring $\mathrm{Cart}_p(k)$ is defined to be the subring

$$\mathrm{Cart}_p(k) := \epsilon_p \mathrm{Cart}(k) \epsilon_p$$

of Cart(k), with ϵ_p as its unit element.

(4.2.1) Denote by $\widehat{\mathbb{W}} = \widehat{\mathbb{W}}_p$ the smooth formal group of restricted Witt vectors over k. For every nilpotent k-algebra N, the group $\widehat{\mathbb{W}}(N)$ consists of all p-adic Witt vectors $(b_i)_{i\in\mathbb{N}}$ such that $b_i \in N$ for every $i \in \mathbb{N}$ and $b_i = 0$ for all but finitely many i's. One can identify $\widehat{\mathbb{W}}(N)$ with $\epsilon_p \Lambda(N)$, where

$$\epsilon_p = \prod_{\substack{(\ell,p)=1 \\ \ell \text{ prime}}} \left(1 - \tfrac{1}{\ell} V_\ell F_\ell\right) = \sum_{\substack{(n,p)=1 \\ n\geq 1}} \tfrac{\mu(n)}{n} V_n F_n \,.$$

Moreover $\mathrm{Cart}_p(k)$ can be identified with $\mathrm{End}(\widehat{\mathbb{W}})^{\mathrm{op}}$, the opposite ring of the endomorphism ring of $\widehat{\mathbb{W}}$.

(4.2.2) Some notable elements of the ring $\mathrm{Cart}_p(k)$ include

- $V := \epsilon_p V_p \epsilon_p$,
- $F := \epsilon_p F_p \epsilon_p$, and
- $\langle c\rangle := \epsilon_p [c] \epsilon_p$, $c \in k$.

(4.2.3) The ring $\mathrm{Cart}_p(k)$ is complete with respect to the filtration by the right ideals $(V^n \mathrm{Cart}_p(k))_{n\in\mathbb{N}}$. Every element of $\mathrm{Cart}_p(k)$ can be expressed as a convergent sum

$$\sum_{m,n\geq 0} V^m \langle a_{mn}\rangle F^n \,; \quad a_{mn} \in k \ \ \forall\, m,n \in \mathbb{N}\,, \forall\, m\ \exists\, C_m > 0 \text{ s.t. } a_{mn} = 0 \text{ if } n \geq C_m$$

in a unique way.

(4.2.4) The set of all elements of $\mathrm{Cart}_p(k)$ which can be represented as a convergent sum of the form

$$\sum_{m\geq 0} V^m \langle a_m\rangle F^m \,, \quad a_m \in k$$

is a subring of $\mathrm{Cart}_p(k)$, isomorphic to the ring of p-adic Witt vectors $W(k) = W_p(k)$ with entries in k. The element of $W(k)$ corresponding to $\sum_{m\geq 0} V^m \langle a_m\rangle F^m$ is $(a_0, a_1, a_2, \cdots)$, the Witt vector with coordinates $(a_m)_{m\in\mathbb{N}}$. Therefore $\mathrm{Cart}_p(k)$ contains $W(k)$ as a unitary subring.

(4.2.5) By definition, a *V-reduced $\mathrm{Cart}_p(k)$-module* M is a left $\mathrm{Cart}_p k$-module such that the map $V : M \to M$ is injective and the canonical map $M \to \varprojlim_n (M/V^n M)$ is an isomorphism. A V-reduced $\mathrm{Cart}_p(k)$-module M is *V-flat* if M/VM is a flat k-module. The k-module M/VM is called the *tangent space* of M.

(4.2.6) For any smooth formal group G over k, the $\mathrm{Cart}_p(k)$-module $\epsilon_p G(k[[x]]^+)$ is V-reduced and V-flat; it consists of all p-typical elements in $G(k[[x]]^+)$, that is, elements killed by F_n for all n prime to p. We call $\epsilon_p G(k[[x]]^+)$ the *Cartier module* of G in this article, denoted by $\mathrm{M}(G)$.

(4.2.7) An important fact is that the $\mathrm{Cart}(k)$-module $G(k[[x]]^+)$ can be recovered from the $\mathrm{Cart}_p(k)$-module $\epsilon_p G(k[[x]]^+)$. So we have another version of the main theorem of Cartier theory, when the base ring k is a $\mathbb{Z}_{(p)}$-algebra. It says that the functor which sends a commutative smooth formal group G to its Cartier module $\mathrm{M}(G) := \epsilon_p G(k[[x]]^+)$ establishes an equivalence from the category of commutative smooth formal groups over k to the category of V-reduced V-flat $\mathrm{Cart}_p(k)$-modules. The Lie algebra of a commutative smooth formal group G over k is canonically isomorphic to $\mathrm{M}(G)/V\mathrm{M}(G)$.

(4.2.8) Suppose that k is a perfect field of characteristic p. Under the equivalence of category in 4.2.7, a V-reduced V-flat $\mathrm{Cart}_p(k)$-module M is the Cartier module attached to a finite-dimensional p-divisible formal group over k if and only if the following conditions hold.

(i) M/VM is a finite dimensional vector space over k.

(ii) The Frobenius map $F : M \to M$ is an injection.

Equivalently, M is a free $W(k)$-module of finite rank.

If G is the p-divisible formal group with Cartier module $M = M(G)$, then $\mathrm{rk}_{W(k)}(M)$ is equal to the height of the Barsotti-Tate group $\varinjlim G[p^n]$ over k, where $G[p^n]$ is the kernel of $[p^n] : G \to G$.

(4.3) Proposition *Let k be a perfect field of characteristic $p > 0$. Let G be a finite dimensional connected smooth formal group over k. Let $M = \mathrm{M}(G)$ be the V-reduced V-flat $\mathrm{Cart}_p(k)$-module attached to G, consisting of all p-typical curves in G. Let $G_{\text{p-div}}$ be the maximal p-divisible subgroup of G. Then the $\mathrm{Cart}_p(k)$-module attached to $G_{\text{p-div}}$ is*

$$\mathrm{M}(G_{\text{p-div}}) = \{x \in M \mid \forall\, n \in \mathbb{N},\ \exists\, m \in \mathbb{N} \text{ s.t. } V^m\, x \in F^n M\}\ .$$

PROOF. Let $M_1 := \{x \in M \mid \forall\, n \in \mathbb{N},\ \exists\, m \in \mathbb{N} \text{ s.t. } V^m\, x \in F^n M\}$, the right-hand-side of the displayed formula above. It is easy to see that M_1 is a $\mathrm{Cart}_p(k)$-submodule of M. Moreover since $G_{\text{p-div}}$ is a p-divisible formal group over a perfect field k, we know that for every $x \in \mathrm{M}(G_{\text{p-div}})$ and every natural number n, there exists a natural number m such that $V^m \cdot x \in F^n \cdot \mathrm{M}(G_{\text{p-div}})$. Hence $\mathrm{M}(G_{\text{p-div}}) \subseteq M_1$. It remains to show that $M_1 \subseteq \mathrm{M}(G_{\text{p-div}})$.

By definition, for every $m \in \mathbb{N}$, we have $V^m M \cap M_1 = V^m M_1$. Hence V induces a bijection $M_1/VM_1 \xrightarrow{\sim} V^m M_1/V^{m+1} M_1$. So M_1 is a V-reduced $\mathrm{Cart}_p(k)$-submodule of M, and M_1 is the Cartier module attached to a formal group G_1 over k. The inclusion $M_1 \hookrightarrow M$ corresponds to a homomorphism $\alpha : G_1 \to G$. Since the map $M_1/VM_1 \to M/VM$ induced by the inclusion $M_1 \hookrightarrow M$ is an injection, the inclusion $M_1 \hookrightarrow M$ of V-reduced $\mathrm{Cart}_p(k)$-modules corresponds to an embedding $G_1 \hookrightarrow G$ of smooth formal groups over k. We must show that G_1 is a p-divisible formal group over k.

Let $M_2 = \{x \in M \mid \exists\, n \in \mathbb{N} \text{ s.t. } F^n x = 0\}$. Clearly M_2 is a $\mathrm{Cart}_p(k)$-submodule of M. Moreover $V^m M \cap M_2 = V^m M_2$ for every $m \in \mathbb{N}$, since $V : M \to M$ is injective. So M_2 is a V-reduced $\mathrm{Cart}_p(k)$-submodule of M, and V^m induces a bijection $M_2/VM_2 \xrightarrow{\sim} V^m M_2/V^{m+1} M_2$ for every $m \in \mathbb{N}$.

We know that $\dim_k(M_2/VM_2) \leq \dim_k(M/VM) < \infty$. Let $m_1, \ldots, m_a \in M_2$ be a finite set of elements in M_2 whose image in M_2/VM_2 is a set of generators of the k-vector space M_2/VM_2. Then every element of M_2 can be written as a convergent sum of the form

$$\sum_{n \in \mathbb{N}} \sum_{i=1}^{a} V^n \langle b_{ni} \rangle m_i\,, \qquad b_{ni} \in k\ \forall\, n \in \mathbb{N},\ \forall\, i = 1, \ldots, a.$$

So there exists a natural number $N \in \mathbb{N}$ such that $F^N \cdot M_2 = (0)$; in fact it suffices to pick an $N \in \mathbb{N}$ such that $F^N \cdot m_1 = \ldots = F^N \cdot m_a = 0$.

Now we show that $F : M_1 \to M_1$ is injective. Suppose that x is an element of M_1 such that $Fx = 0$. Then there exist $y \in M$ and $m \in \mathbb{N}$ such that $V^m x = F^N y$. Since $Fx = 0$, we get $F^{N+1} y = 0$. So $y \in M_2$, therefore $F^N y = 0$ and $V^m x = F^N y = 0$. Since $V : M \to M$ is injective, $x = 0$. We have proved that $F : M_1 \to M_1$ is injective. Hence G_1 is a p-divisible formal group. ∎

(4.3.1) Remark (i) Notation as in 4.3. Then the V-reduced $\mathrm{Cart}_p(k)$-module attached to the maximal unipotent subgroup G_{unip} of G is

$$\mathrm{M}(G_{\mathrm{unip}}) = \{x \in M \mid \exists\, n \in \mathbb{N} \text{ s.t. } F^n x = 0\} ,$$

denoted M_2 in the proof.

(ii) The $\mathrm{Cart}_p(k)$-module $\mathrm{M}(G_{\text{p-div}})$ attached to $G_{\text{p-div}}$ can also be expressed as

$$\mathrm{M}(G_{\text{p-div}}) = \{x \in M \mid \forall\, n \in \mathbb{N},\ \exists\, m \in \mathbb{N} \text{ s.t. } V^m\, x \in p^n M\} .$$

To see this, denote by M_1' the right hand side of the above displayed formula. Then $M_1' \subseteq M_1$ because $p^n M \subseteq F^n M$. On the other hand, if $V^m\, x \in F^n M$, then $V^{m+n}\, x \in p^n M$. So $M_1 \subset M_1'$.

(iii) There exists a natural number $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \mathrm{M}(G_{\text{p-div}}) &= \{x \in M \mid \exists\, m \in \mathbb{N} \text{ s.t. } V^m\, x \in F^n M\} \\ &= \{x \in M \mid \exists\, m \in \mathbb{N} \text{ s.t. } V^m\, x \in p^n M\} \end{aligned}$$

for all $n \geq n_0$. One can take n_0 to be $\dim(G_{\mathrm{unip}})$. A better choice of n_0 is

$$\mathrm{Min}\, \{n \in \mathbb{N} \,|\, F^n\, \mathrm{M}(G_{\mathrm{unip}}) = (0)\} = \mathrm{Min}\, \{n \in \mathbb{N} \,|\, F^n\, \mathrm{M}(G_{\mathrm{unip}}) = (0)\} ,$$

which is smaller than or equal to $\dim(G_{\mathrm{unip}})$.

(iv) If the field k is not perfect, then the statement of Prop. 4.3 fails. More precisely, the inclusion $M_1 \subseteq M(G_{\text{p-div}})$ still holds, by the same proof, but the inclusion $M(G_{\text{p-div}}) \subseteq M_1$ may be false. For instance if G is $\widehat{\mathbb{G}_m}$, then $M = M(G_{\text{p-div}})$ is equal to $W_p(k)$ with the usual action of F and V. In this case $M_1 = W(k_0)$, where k_0 is the largest perfect subfield of k, and $W(k_0)$ is the ring of (p-adic) Witt vectors with entries in k_0.

(v) The "sum homomorphism" $f : G_{\text{p-div}} \times_{\mathrm{Spec}(k)} G_{\mathrm{unip}} \to G$ between smooth commutative formal groups over k is faithfully flat with finite kernel. The kernel of f is isomorphic to $G_{\text{p-div}} \cap G_{\mathrm{unip}}$. The natural map $G_{\text{p-div}} \to G/G_{\mathrm{unip}}$ from the maximal p-divisible subgroup of G to the maximal p-divisible quotient of G is an isogeny, whose kernel is isomorphic to $G_{\text{p-div}} \cap G_{\mathrm{unip}}$.

(4.3.2) Lemma *Notation as in Prop. 4.3. Suppose that M' is a* $\mathrm{Cart}_p(k)$*-submodule of* $M = \mathrm{M}(G)$ *which satisfies the following conditions.*

(i) *The map $F : M' \to M'$ is injective. In other words, M_1 is the Cartier module of a p-divisible formal group.*

(ii) $\mathrm{rk}_{W(k)}(M') = \mathrm{rk}_{W(k)}(\mathrm{M}(G_{\text{p-div}}))$. *Equivalently,* $(M/M') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = 0$.

(iii) *The natural map $M'/VM' \to M/VM$ is injective. Equivalent, $M' \cap M = VM'$.*

Then $M_1 = \mathrm{M}(G_{\text{p-div}})$.

PROOF. Let $M'' = M/M'$. Since $V : M \to M$ is injective, we have an exact sequence

$$0 \to \mathrm{Ker}(V : M'' \to M'') \to M'/VM' \to M/VM \to M''/VM'' \to 0\,.$$

Condition (iii) implies that $V : M'' \to M''$ is injective, so M'' is the Cartier module of a smooth commutative unipotent p-divisible formal group by (ii). Cartier theory tells us that the short exact sequence $0 \to M' \to M \to M'' \to 0$ corresponds to a short exact sequence of smooth commutative formal groups $0 \to G' \to G \to G'' \to 0$, in which G' is p-divisible, and G' is unipotent. Therefore $G' = G_{\text{p-div}}$. ∎

Remark (i) It is clear the $\mathrm{M}(G_{\text{p-div}})$ satisfies the conditions (i)–(iii) of 4.3.2

(ii) Lemma 4.3.2 characterizes $G_{\text{p-div}}$ as the p-divisible formal subgroup G' of G of the same height as $G_{\text{p-div}}$ such the map $\mathrm{T}(G') \to \mathrm{T}(G)$ on tangent spaces is injective.

§5. A triple Cartier module

(5.1) Definition Let k be a ring over $\mathbb{Z}_{(p)}$. Let $\widehat{\mathrm{Cart}_p}$ be the commutative smooth formal group over k such that

$$\widehat{\mathrm{Cart}_p}(N) := \mathrm{Cart}_p(k \oplus N)$$

for every nilpotent k-algebra N. Define $\mathrm{BC}(k)$ to be the set of all formal curves in the functor Cart_p. In other words,

$$\begin{aligned} \mathrm{BC}(k) \;&:=\; \mathrm{Cart}_p(k[[x]]^+) = \varprojlim_n \mathrm{Ker}\left(\mathrm{Cart}_p(k[[x]]/(x^{n+1})) \longrightarrow \mathrm{Cart}_p(k)\right) \\ &=\; \varprojlim_n \epsilon_p \mathrm{Ker}\left(\mathrm{Cart}(k[[x]]^+/(x^{n+1})) \longrightarrow \mathrm{Cart}(k)\right)\epsilon_p = \epsilon_p \cdot \mathrm{Cart}(k[[x]]^+) \cdot \epsilon_p \end{aligned}$$

The set $\mathrm{BC}(k)$ carries an obvious ($\mathrm{Cart}_p(k)$-$\mathrm{Cart}_p(k)$)-bimodule structure, since $\mathrm{BC}(k)$ is an ideal in the ring $\mathrm{Cart}_p(k[[x]]) := \varprojlim_n \mathrm{Cart}_p(k[[x]]/(x^{n+1}))$ and the canonical inclusion $\mathrm{Cart}_p(k) \hookrightarrow \mathrm{Cart}_p(k[[x]])$ is a homomorphism of rings.

(5.1.1) Since the bimodule $\mathrm{BC}(k)$ is the set of all formal curves in the functor Cart_p there is a natural action of the big Cartier ring $\mathrm{Cart}(k)$ on $\mathrm{BC}(k)$, because $\mathrm{Cart}(k)$ operates naturally on the set of all formal curves of any smooth commutative formal group. Recall that $\mathrm{Cart}_p = \underline{\mathrm{End}}(\widehat{\mathbb{W}})^{\mathrm{op}}$ as a functor on the category of nilpotent k-algebras. The ($\mathrm{Cart}_p(k)$-$\mathrm{Cart}_p(k)$)-bimodule structure of $\mathrm{BC}(k)$ comes from pre-composing and post-composing with elements of $\mathrm{End}_k(\widehat{\mathbb{W}})^{\mathrm{op}}$, hence the above action of elements of $\mathrm{Cart}(k)$ on $\mathrm{BC}(k)$ are homomorphisms of ($\mathrm{Cart}_p(k)$-$\mathrm{Cart}_p(k)$)-bimodules. To avoid confusion with operators coming from the bimodule structure, we will append an "x" in superscript or subscript to the standard notation for elements in $\mathrm{Cart}(k)$ when we consider the above action of $\mathrm{Cart}(k)$ on $\mathrm{BC}(k)$ which commutes with the operations coming from the ($\mathrm{Cart}_p(k)$-$\mathrm{Cart}_p(k)$)-bimodule structure; superscript will be used if there is already a subscript in the symbol. For instance we have the operators $V_n^x, F_n^x, n \in \mathbb{N}$, and the projector

$$\epsilon_p^x := \prod_{\substack{(\ell,p)=1 \\ \ell \text{ prime}}} \left(1 - \tfrac{1}{\ell} V_\ell^x F_\ell^x\right) = \sum_{\substack{(n,p)=1 \\ n \geq 1}} \tfrac{\mu(n)}{n} V_n^x F_n^x\,.$$

to the set of all p-typical elements in $\mathrm{BC}(k)$.

(5.2) Definition Let k be an algebra over $\mathbb{Z}_{(p)}$. Define a $(\mathrm{Cart}_p(k)$-$\mathrm{Cart}_p(k))$-sub-bimodule $\mathrm{BC}_p(k)$ of $\mathrm{BC}(k)$ by

$$\mathrm{BC}_p(k) = \epsilon_p^x \mathrm{BC}(k) = \epsilon_p^x \mathrm{Cart}_p(k[[x]]^+)\,.$$

In other words, $\mathrm{BC}_p(k)$ is the Cartier module of the commutative smooth formal group $\widehat{\mathrm{Cart}_p}$ over k.

(5.2.1) Since the sub-$(\mathrm{Cart}_p(k)$-$\mathrm{Cart}_p(k))$-bimodule $\mathrm{BC}_p(k)$ of $\mathrm{BC}(k)$ is equal to the set of all p-typical elements in $\mathrm{BC}(k)$, there is a natural left action of $\mathrm{Cart}_p(k)$ on $\mathrm{BC}_p(k)$, compatible with the $(\mathrm{Cart}_p(k)$-$\mathrm{Cart}_p(k))$-bimodule structure. As before, we will append an "x" in subscript or superscript when considering this action of $\mathrm{Cart}_p(k)$ on $\mathrm{BC}_p(k)$. In particular, the elements

$$F_x := \epsilon_p^x F_p^x \epsilon_p^x, \quad V_x := \epsilon_p^x V_p^x \epsilon_p^x$$

operate on $\mathrm{BC}_p(k)$ as endomorphisms of the bimodule $BCp(k)$.

(5.3) We would like to understand $\mathrm{BC}(k)$ and $\mathrm{BC}_p(k)$ more concretely. Since the variable "x" is already occupied, we will change the notation for elements of $\Lambda(k[[x]]^+)$ in 4.1.5, and replace the variable "x" there by the variable "y". According to the newly modified notation, for any k-algebra R, the set underlying $\mathrm{Cart}(R)$ is

$$\Lambda(R[[y]]^+) := \varprojlim_n \Lambda(R[[y]]^+/(y^{n+1}))\,.$$

Both $\mathrm{BC}(k)$ and $\mathrm{BC}_p(k)$ are subsets of

$$\mathrm{Cart}(k[[x]]^+) = \Lambda(k[[x,y]]^\ddagger)\,,$$

where $k[[x,y]]^\ddagger$ denotes the subset of $k[[x,y]]$ consisting of all power series of the form

$$\sum_{m,n\geq 1} a_{mn} x^m y^n\,, \qquad a_{mn} \in k \ \forall m,n \geq 1\,.$$

(5.3.1) Inside the ring $\mathrm{Cart}(k[[x]]) = \Lambda(y \cdot k[[x,y]])$, we have elements V_n, F_n, $n \in \mathbb{N}$, where

$$F_n \leftrightarrow 1 - yt^n\,, \qquad V_n \leftrightarrow 1 - y^n t\,.$$

In the $(\mathrm{Cart}(k)$-$\mathrm{Cart}(k))$-bimodule $\mathrm{Cart}(k[[x]]^+)$, we also have elements

$$[a(x)] \leftrightarrow 1 - a(x)yt\,, \qquad a(x) \in k[[x]]^+\,.$$

In particular we have the elements

$$[x^i] \leftrightarrow 1 - x^i yt\,, \qquad i \geq 1\,.$$

In the subring $\mathrm{Cart}_p(k) \subset \mathrm{Cart}(k) \subset \mathrm{Cart}(k[[x]])$ we have elements

$$F := \epsilon_p F_p \epsilon_p, \quad V := \epsilon_p V_p \epsilon_p, \quad \langle c\rangle := \epsilon_p [c] \epsilon_p, \ c \in k\,,$$

where

$$\epsilon_p = \prod_{\substack{(\ell,p)=1\\ \ell \text{ prime}}} \left(1 - \tfrac{1}{\ell} V_\ell F_\ell\right) = \sum_{\substack{(n,p)=1\\ n\geq 1}} \tfrac{\mu(n)}{n} V_n F_n\,,$$

and $[c] \leftrightarrow 1 - cyt$ for $c \in k$.

(5.3.2) Lemma *We have the following description of the elements of* $\mathrm{BC}(k)$*.*

$$\mathrm{BC}(k) = \epsilon_p \cdot \varprojlim_{m,n\geq 0} \Lambda\left(k[[x,y]]^{\ddagger}/(x^{m+1},y^{n+1})\right)\cdot\epsilon_p$$

$$= \left\{ \sum_{\substack{n\geq 1\\ m,i\geq 0}} V^m\langle a_{mni}\rangle\langle x^n\rangle F^i \;\middle|\; \begin{array}{c} a_{mni}\in k \quad \forall\, m\geq 0, \forall\, n\geq 1, \forall\, i\geq 0 \\ \forall\, m\geq 0, \forall\, n\geq 1,\ \exists\, C(m,n) \text{ s.t. } a_{mni}=0 \text{ if } i>C(m,n) \end{array} \right\}$$

Proof. The first equality is immediate from the definition. To prove the second equality, it suffices to prove that for every $N\geq 1$, every element in $\mathrm{Cart}_p(k[[x]]^+/(x^{N+1}))$ can be written in a unique way in the form

$$\sum_{\substack{1\leq n\leq N_0\\ m,i\geq 0}} V^m\langle a_{mni}\rangle\langle\overline{x^n}\rangle F^i\,, \qquad a_{mni}\in k\ \ \forall\, m\geq 0,\ \forall\, 1\leq n\leq N_0,\ \forall\, i\geq 1\,,$$

where $\overline{x^n}$ denotes the image of x^n in $k[[x]]^+/(x^{N+1})$, and $\forall\, m\geq 0$, $\forall\, 1\leq n\leq N_0$, $\exists\, C(m,n)$ such that $a_{mni}=0$ if $i>C(m,n)$. For $N=1$, a basic property of Cartier rings says that every element u in $\mathrm{Cart}_p(k[[x]]/(x^2))$ can be written uniquely in the form

$$\sum_{m\geq 0} V^m\langle a_{mi}\rangle F^i\,, \qquad a_{m,i}\in k[[x]]/(x^2)\ \ \forall\, m,i\geq 1\,,$$

and $\forall\, m\geq 1$, $\exists\, C(m,1)$ such that $a_{mi}=0$ if $i>C(m,1)$. Since the image of u in $\mathrm{Cart}_p(k)$ is trivial, we have $a_{m,i}\in k[[x]]^+/(x^2)$ for all $m,i\geq 1$. This finishes the case when $N=1$. By induction, we can assume that this assertion holds for $N=N_0$. Then we only have to show that every element of

$$\mathrm{Ker}\left(\mathrm{Cart}_p\left(k[[x]]^+/(x^{N_0+2})\right)\to\mathrm{Cart}_p\left(k[[x]]^+/(x^{N_0+1})\right)\right)=\mathrm{Cart}_p\left(x^{N_0+1}k[[x]]/x^{N_0+2}k[[x]]\right)$$

can be written uniquely in the form

$$\sum_{m,i\geq 0} V^m\langle a_{m,N_0+1,i}\rangle\langle\overline{x^{N_0+1}}\rangle F^i\,, \qquad a_{m,N_0,i}\in k\ \ \forall\, m\geq 0,\ \ \forall\, i\geq 0\,,$$

and $\forall\, m\geq 0$, $\exists\, C(m,N_0+1)$ such that $a_{m,N_0+1,i}=0$ if $i>C(m,N_0+1)$. This follows from the case $N=1$, since the nilpotent k-algebras $x^{N_0+1}k[[x]]/x^{N_0+2}k[[x]]$ and $k[[x]]^+/(x^2)$ are isomorphic. ∎

(5.3.3) Lemma *The following equalities hold in* $\mathrm{BC}(k)$*.*

(i) $F^x_n\cdot[x^i] = r\,V_{\frac{n}{r}}\,[x^{\frac{i}{r}}]\,F_{\frac{n}{r}}$ $\forall\, n\geq 1, \forall\, i\geq 1$, $r=\text{g.c.d.}(n,i)$.

(ii) $\epsilon^x_p\cdot\langle x^i\rangle = 0$ *if* $i\geq 1$ *and* i *is not a power of* p.

(iii) $F^x_n\cdot\langle x^{p^j}\rangle = 0 \qquad \forall\, n>1$ s.t. $(n,p)=1$, $\forall\, j\geq 0$.

(iv) $\epsilon^x_p\cdot\langle x^{p^j}\rangle = \langle x^{p^j}\rangle$ *for all* $j\geq 0$.

Proof. (i) By definition, we have

$$F_n^x \cdot [x^i] = \prod_{j=0}^{n-1} \left(1 - (\zeta_n^j x^{\frac{1}{n}})^i y t\right) = \left(1 - x^{\frac{i}{r}} y^{\frac{n}{r}} t^{\frac{n}{r}}\right)^r = r\, V_{\frac{n}{r}}\, [x^{\frac{i}{r}}]\, F_{\frac{n}{r}}\,.$$

In the above, ζ_n is a "formal primitive n-th root of 1". In other words, the above equalities take place in $\mathrm{BC}(k[\zeta_n])$, where $k[\zeta_n] = k[u]/(\Phi_n(u))$, and $\Phi_n(u)$ is the n-th cyclotomic polynomial in the variable u.

(ii) Suppose that $\ell | i$, $\ell \neq p$, where ℓ is a prime number. Then $\langle x^i \rangle = V_\ell^x \cdot \langle x^{\frac{i}{\ell}} \rangle$. Since $\epsilon_p^x \cdot V_\ell^x = 0$ in $\mathrm{Cart}_p(k)$, we get $\epsilon_p^x \cdot \langle x^i \rangle = 0$.

(iii) Apply the formula in (i) to the case when $i = p^j$, $(n,p) = 1$, $n > 1$. Then $r = 1$, and we get

$$F_n^x \cdot \langle x^{p^j} \rangle = F_n^x \cdot \epsilon_p\, [x^{p^j}]\, \epsilon_p = \epsilon_p\, F_n^x\, [x^{p^j}]\, \epsilon_p = \epsilon_p\, V_n\, [x^{p^j}]\, F_n\, \epsilon_p = 0\,.$$

The last equality follows either from $\epsilon_p V_n = 0$, or from $F_n \epsilon_p = 0$.

(iv) The statement (iv) follows immediately from (iii). ∎

(5.4) Proposition *Let k be an algebra over $\mathbb{Z}_{(p)}$. For every integer $n \geq 0$, define an element $U_n \in \mathrm{BC}_p(k)$ by*

$$U_n = \langle x^{p^j} \rangle = \epsilon_p [x^{p^j}] \epsilon_p = \epsilon_p^x\, \epsilon_p [x^{p^j}] \epsilon_p\,.$$

(i) *We have the following explicit description of $\mathrm{BC}_p(k)$.*

$$\mathrm{BC}_p(k) = \left\{ \sum_{m,n,i\geq 0} V^m\ \langle a_{mni} \rangle\, U_n\, F^i \ \middle|\ \begin{array}{c} a_{mni} \in k \quad \forall\, m,n,i \geq 0 \\ \forall\, m\, \forall\, n\, \exists\, C_{mn} \text{ s.t. } a_{mni} = 0 \text{ if } i > C_{mn} \end{array} \right\}.$$

In the formula above, the element $\sum_{m,n,i\geq 0} V^m\ \langle a_{mni} \rangle\, U_n\, F^i$ represents the element

$$\epsilon_p^x \cdot \epsilon_p \cdot \left(\prod_{m,n,i\geq 0} (1 - a_{mni}\, x^{p^n} y^{p^m} t^{p^i}) \right) \cdot \epsilon_p$$

in $\Lambda(k[[x,y]]^\ddagger)$.

(ii) *Two elements of $\mathrm{BC}_p(k)$ of the form*

$$\sum_{m,n,i\geq 0} V^m\ \langle a_{mni} \rangle\, U_n\, F^i \qquad \text{and} \qquad \sum_{m,n,i\geq 0} V^m\ \langle b_{mni} \rangle\, U_n\, F^i$$

with $a_{mni}, b_{mni} \in k$ for all $m,n,i \in \mathbb{N}$, are equal if and only if

$$a_{mni} = b_{mni} \qquad \forall\, m,n,i \in \mathbb{N}\,.$$

(iii) *The copy*

$$\left\{ \sum_{m,n\geq 0} V_x^m \langle a_{mn} \rangle F_x^n \ \middle|\ \begin{array}{c} a_{mn} \in k \quad \forall\, m,n \geq 0 \\ \forall\, m \geq 0\, \exists C(m) \text{ s.t. } a_{mn} = 0 \text{ if } n > C(m) \end{array} \right\}$$

of $\mathrm{Cart}_p(k)$ operates as a ring of endomorphisms of the $(\mathrm{Cart}_p(k)$-$\mathrm{Cart}_p(k))$-bimodule $\mathrm{BC}_p(k)$ on the left of $\mathrm{BC}_p(k)$.

Proof. The statements (i) and (iii) are immediate from 5.3.3 and the definition of $\mathrm{BC}_p(k)$. The statement (ii) is left as an exercise. ∎

(5.5) Proposition *The following identities hold in* $\mathrm{BC}_p(k)$.

(i) $\langle a\rangle\, U_n = U_n\,\langle a\rangle \quad \forall\, a\in k,\ \forall\, n\geq 0.$

(ii) $F\,U_n = U_{n+1}\,F \quad \forall\, n\geq 0.$

(iii) $U_n\,V = V\,U_{n+1} \quad \forall\, n\geq 0.$

(iv) $\langle a\rangle_x\cdot U_n = \langle a^{p^n}\rangle\, U_n \quad \forall\, a\in k,\ \forall n\geq 0.$

(v) $V_x\,U_n = U_{n+1} \quad \forall\, n\geq 0.$

(vi) $F_x\,U_n = \begin{cases} p\,U_{n-1} = V U_n F & \text{if } n\geq 1\\ V U_0 F & \text{if } n=0\end{cases}$

(vii) *If* k *is an algebra over* $\mathbb{F}_p$, *then* $F_x\,U_n = V U_n F \quad \forall\, n\geq 1.$

PROOF. The statements (i)–(iv) are immediate from the standard relations in the Cartier ringCart$_p(k[[x]])$. For (v), we have

$$V_x\cdot U_n = V_x\cdot\langle x^{p^n}\rangle = \langle x^{p^{n+1}}\rangle = U_{n+1}$$

by definition. It remains to prove (vi) and (vii).

If $n\geq 1$, then by (v) we have

$$F_x\cdot U_n = F_x\cdot V_x\cdot U_{n-1} = p\,U_{n-1} = V\,U_n\,F\,.$$

For $n=0$, we have

$$F_x\cdot U_0 = \epsilon_p^x F_p^x(\epsilon_p[x]\epsilon_p) = \epsilon_p\,\epsilon_p^x\,F_p^x\,[x]\,\epsilon_p \;= \epsilon_p\,\epsilon_p^x\,V_p\,[x]\,F_p\,\epsilon_p \;= V\,\epsilon_p^x\,U_0\,F = V\,U_0\,F\,,$$

where we have used 5.3.3 (i) and (iv) in the third and the last equality. We have proved (vi). Notice that $V\,U_0\,F$ represents the element $\epsilon_p^x\,\epsilon_p\,\cdot(1-xy^pt^p)$ in $\Lambda(k[[x,y]]^\ddagger)$.

If k is an algebra over $\mathbb{F}_p$, $n\geq 1$. Then we have

$$F_x\cdot U_n = pU_{n-1} = VFU_{n-1} = VU_nF\,.$$

by (vi) and (ii). We have proved (vii). ∎

(5.6) Let k be a commutative Artinian local $\mathbb{Z}_{(p)}$-algebra. Let X,Y be a finite-dimensional smooth formal groups over k. Let M and N be the Cartier modules attached to X and Y respectively. It is well-known that the left $\mathrm{Cart}_p(k)$-module M has a resolution of the form

$$0\to \mathrm{Cart}_p(k)^n\to \mathrm{Cart}_p(k)^n\to M\to 0\,,$$

see [21, IV §8]. It is possible to choose n to be $\dim(X)$ in the resolution above, but sometimes the height of X is a more convenient choice of n. A short exact sequence as above corresponds to a short exact sequence

$$0\to\widehat{\mathbb{W}}^n\to\widehat{\mathbb{W}}^n\to X\to 0$$

of smooth formal groups over k. Suppose that X,Y are p-divisible smooth formal groups over k, we have seen that $\mathcal{DE}(X,Y)$ is a smooth formal group over k as well. Prop. 5.6.1 and Lemma 5.6.2 below say that one can compute the Cartier module of $\mathcal{DE}(X,Y)$ using a resolution of M as above, and the triple Cartier module $\mathrm{BC}_p(k)$ enters the picture in a natural way.

(5.6.1) Proposition *Let X, Y be p-divisible smooth formal groups over k as above, and let M, N be their Cartier modules. Let*

$$0 \to \widehat{\mathbb{W}}^n \xrightarrow{\mathbf{r}} \widehat{\mathbb{W}}^n \to X \to 0$$

be a short exact sequence. Assume that $\mathrm{Hom}_k(X, Y) = (0)$. *Then the smooth formal group $\mathcal{DE}(X, Y)$ over k is isomorphic to*

$$\mathrm{Coker}\left(\underline{\mathrm{Hom}}(\widehat{\mathbb{W}}^n, Y) \xrightarrow{\mathbf{r}^*} \underline{\mathrm{Hom}}(\widehat{\mathbb{W}}^n, Y)\,,\right)$$

where $\underline{\mathrm{Hom}}(\widehat{\mathbb{W}}^n, Y)$ is the smooth formal group over k, which to every nilpotent k-algebra R assigns the commutative group

$$\underline{\mathrm{Hom}}(\widehat{\mathbb{W}}^n, Y)(R) = \mathrm{Ker}\left(\mathrm{Hom}_{k\oplus R}(\widehat{\mathbb{W}}^n, Y) \to \mathrm{Hom}_k(\widehat{\mathbb{W}}^n, Y)\right)$$

as its R-valued points.

Proof. It is a standard fact that the group $\mathrm{Ext}_A(\widehat{\mathbb{W}}, Y)$ of isomorphism classes extensions of $\widehat{\mathbb{W}}$ by Y is trivial for every commutative k-algebra A with 1. So we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_{k\oplus R}(\widehat{\mathbb{W}}^n, Y) & \xrightarrow{\mathbf{r}^*} & \mathrm{Hom}_{k\oplus R}(\widehat{\mathbb{W}}^n, Y) & \longrightarrow & \mathrm{Ext}_{k\oplus R}(X, Y) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_k(\widehat{\mathbb{W}}^n, Y) & \xrightarrow{\mathbf{r}^*} & \mathrm{Hom}_k(\widehat{\mathbb{W}}^n, Y) & \longrightarrow & \mathrm{Ext}_k(X, Y) & \longrightarrow & 0
\end{array}$$

with exact rows. The Proposition follows from the snake lemma. ∎

(5.6.2) Lemma *Notation as above.*

(i) *The* $\mathrm{Cart}(k)$*-module attached to $\underline{\mathrm{Hom}}(\widehat{\mathbb{W}}, Y)$, i.e. the set of all formal curves in the smooth formal group $\underline{\mathrm{Hom}}(\widehat{\mathbb{W}}, Y)$ over k, is*

$$\underline{\mathrm{Hom}}(\widehat{\mathbb{W}}, Y)(k[[x]]^+) = \varprojlim_n \left(\underline{\mathrm{Hom}}(\widehat{\mathbb{W}}, Y)(k[[x]]^+/(x^{n+1}))\right)$$

$$= \varprojlim_n \left(\mathrm{Cart}_p(k[[x]]^+/(x^{n+1})) \otimes_{\mathrm{Cart}_p(k)} N\right) \xleftarrow[\gamma]{\sim} \left(\varprojlim_n \mathrm{Cart}_p(k[[x]]^+/(x^{n+1}))\right) \otimes_{\mathrm{Cart}_p(k)} N$$

$$= \mathrm{BC}(k) \otimes_{\mathrm{Cart}_p(k)} N\,.$$

(ii) *The* $\mathrm{Cart}_p(k)$*-module attached to $\underline{\mathrm{Hom}}(\widehat{\mathbb{W}}, Y)$, i.e. the set of all p-typical formal curves $\underline{\mathrm{Hom}}(\widehat{\mathbb{W}}, Y)$ is*

$$\epsilon_p^x \cdot \varprojlim_n \left(\mathrm{Cart}_p(k[[x]]^+/(x^{n+1})) \otimes_{\mathrm{Cart}_p(k)} N\right)$$

$$\xleftarrow{\sim} \left(\epsilon_p^x \cdot \varprojlim_n \mathrm{Cart}_p(k[[x]]^+/(x^{n+1}))\right) \otimes_{\mathrm{Cart}_p(k)} N \;=\; \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N\,.$$

PROOF. The displayed formula in (ii) follows from the formula in (i), by applying the projector ϵ_p^x. The second equality in (i) is a special case of the general statement that for any smooth formal group G and any k-algebra R with 1, $\mathrm{Hom}_R(\widehat{\mathbb{W}}, Y) = \mathrm{Cart}_p(R) \otimes_{\mathrm{Cart}_p(k)} N$, because N is a finitely generated left $\mathrm{Cart}_p(k)$-module. It remains to prove the map γ in (i) is a bijection.

Observe that the source of γ, regarded as a functor of the V-reduced left $\mathrm{Cart}_p(k)$-module N, is right exact. The target of γ, regarded as a functor of the V-reduced left $\mathrm{Cart}_p(k)$-module N, is right exact, because $N \mapsto \mathrm{Cart}_p(k[[x]]^+/(x^{n+1})) \otimes_{\mathrm{Cart}_p(k)} N$ is right exact in N and the transition maps in the projective system $\left(\mathrm{Cart}_p(k[[x]]^+/(x^{n+1})) \otimes_{\mathrm{Cart}_p(k)} N\right)_{n\in\mathbb{N}}$ are surjective. Clearly, if the V-reduced $\mathrm{Cart}_p(k)$-module N is replaced by the free module $\mathrm{Cart}_p(k)$, then the resulting map $\gamma = \gamma_N$ is an isomorphism. We deduce from the right exactness of both the source and the target implies that γ is an isomorphism. ∎

(5.7) Notation as in 5.6.1. We would like to make Prop. 5.6.1 more explicit using Lemma 5.6.2.

(5.7.1) We will represent points of $\widehat{\mathbb{W}}^n$ as row vectors with points of $\widehat{\mathbb{W}}$ as entries, so the map $\mathbf{r} : \widehat{\mathbb{W}}^n \to \widehat{\mathbb{W}}^n$ corresponds to multiplying row vectors with entries in $\widehat{\mathbb{W}}^n$ by an $n \times n$-matrix $\Gamma \in \mathrm{M}_n(\mathrm{Cart}_p(k))$, on the *right*. This convention is natural because $\mathrm{Cart}_p(k)$ acts on the right of $\widehat{\mathbb{W}}$; more explicitly, $\mathrm{Cart}_p(k) = \mathrm{End}_k(\widehat{\mathbb{W}})^{\mathrm{op}}$, and the natural left action of $\mathrm{End}_k(\widehat{\mathbb{W}})$ on $\widehat{\mathbb{W}}$ transports to a right action of $\mathrm{Cart}_p(k)$ on $\widehat{\mathbb{W}}$. The map

$$\mathbf{r}^* : \underline{\mathrm{Hom}}(\widehat{\mathbb{W}}^n, Y) \to \underline{\mathrm{Hom}}(\widehat{\mathbb{W}}^n, Y)$$

is a homomorphism of smooth formal groups over k. On the level of p-typical formal curves, the map $\mathbf{r}^*$ give the map

$$\mathbf{r}^\diamond : \mathrm{BC}_p(k)^n \otimes_{\mathrm{Cart}_p(k)} N \to \mathrm{BC}_p(k)^n \otimes_{\mathrm{Cart}_p(k)} N\,,$$

where $\mathrm{BC}_p(k)^n \otimes_{\mathrm{Cart}_p(k)} N$ is identified with the set of all *column* vectors of length n, with entries in $\mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N$, and the map $\mathbf{r}^\diamond$ is multiplication on the *left* by the same matrix $\Gamma \in \mathrm{M}_n(\mathrm{Cart}_p(k))$. Here the multiplication by Γ is performed through the left/first factor $\mathrm{Cart}_p(k)$ in the $(\mathrm{Cart}_p(k)$-$\mathrm{Cart}_p(k))$-bimodule structure of $\mathrm{BC}_p(k)$.

(5.7.2) The set $\mathrm{BC}_p(k)^n \otimes_{\mathrm{Cart}_p(k)} N$ has two mutually compatible $\mathrm{Cart}_p(k)$-module structures; one from the left/first factor in the $(\mathrm{Cart}_p(k)$-$\mathrm{Cart}_p(k))$-structure of $\mathrm{BC}_p(k)$, the other from the action of the "extra copy" of $\mathrm{Cart}_p(k)$ operating on $\mathrm{BC}_p(k)$, whose elements are decorated with "x" in subscripts. When passing to the quotient, we have "used up" the first $\mathrm{Cart}_p(k)$-module structure of $\mathrm{BC}_p(k)^n \otimes_{\mathrm{Cart}_p(k)} N$, but $(\mathrm{BC}_p(k)^n \otimes_{\mathrm{Cart}_p(k)} N)/(\Gamma\cdot(\mathrm{BC}_p(k)^n \otimes_{\mathrm{Cart}_p(k)} N))$ still has a natural structure as a $\mathrm{Cart}_p(k)$-module through the action of the "extra copy". Of course the action of this "extra copy" of $\mathrm{Cart}_p(k)$ corresponds exactly to the $\mathrm{Cart}_p(k)$-module structure of the smooth formal group $\mathcal{DE}(X, Y)$ over k. We record our discussion in the Proposition below.

(5.7.3) Proposition *Notation as in 5.6.1.*

(i) *The* $\mathrm{Cart}_p(k)$*-module of all* p*-typical formal curves in the smooth formal group* $\mathcal{DE}(X, Y)$ *over* k *is canonically isomorphic to*

$$\mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N)\,.$$

Here the extension group is computed using the $\mathrm{Cart}_p(k)$*-module structure coming from the left factor of the* $(\mathrm{Cart}_p(k)\text{-}\mathrm{Cart}_p(k))$*-bimodule structure of* $\mathrm{BC}_p(k)$*, and the Cartier ring* $\mathrm{Cart}_p(k)$ *operates on the left of* $\mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N)$ *through the action of the "extra copy" of* $\mathrm{Cart}_p(k)$ *on* $\mathrm{BC}_p(k)$*.*

(ii) *More explicitly, suppose we have a resolution*

$$0 \longrightarrow \mathrm{Cart}_p(k)^n \xrightarrow{\mathbf{r}} \mathrm{Cart}_p(k)^n \longrightarrow M \longrightarrow 0$$

of the left $\mathrm{Cart}_p(k)$*-module* M*, and the map* $\mathbf{r}$ *is given by multiplying row vectors with entries in* $\mathrm{Cart}_p(k)$ *by a matrix* Γ *on the right, with entries in* $\mathrm{Cart}_p(k)$*. Then* $\mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N)$ *is naturally isomorphic to*

$$\big(\mathrm{BC}_p(k)^n \otimes_{\mathrm{Cart}_p(k)} N\big) \,/\, \big(\Gamma \cdot (\mathrm{BC}_p(k)^n \otimes_{\mathrm{Cart}_p(k)} N)\big) \ ,$$

with the left $\mathrm{Cart}_p(k)$*-module structure coming from the action of the "extra copy" of* $\mathrm{Cart}_p(k)$ *on* $\mathrm{BC}_p(k)$*. In the above displayed formula,* $\mathrm{BC}_p(k)^n \otimes_{\mathrm{Cart}_p(k)} N$ *denotes the set of all column vectors of length* n *with entries in* $\mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N$*.*

PROOF. Compute $\mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N)$ using the finite free resolution

$$0 \longrightarrow \mathrm{Cart}_p(k)^n \xrightarrow{\mathbf{r}} \mathrm{Cart}_p(k)^n \longrightarrow M \longrightarrow 0$$

of the left $\mathrm{Cart}_p(k)$-module M, we get

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N) &\cong \big(\mathrm{BC}_p(k)^n \otimes_{\mathrm{Cart}_p(k)} N\big) \,/\, \big(\Gamma \cdot (\mathrm{BC}_p(k)^n \otimes_{\mathrm{Cart}_p(k)} N)\big) \\ &\cong \text{the Cartier module of } \mathcal{DE}(X, Y)\,.\end{aligned}$$

A standard argument shows that the isomorphism is independent of the resolution of M. ∎

(5.7.4) Remark (i) The case when $Y = \widehat{\mathbb{G}_m}$ has been extensively studied in the literature, starting with Mumford's seminal paper [12]. What we recorded in Prop. 5.7.3 is a generalization of the computation of the Cartier module of the Serre dual of a given p-divisible group.

(ii) Propositions 4.3 and 5.7.3 allows one to compute the Cartier module of the p-divisible formal group $\mathcal{DE}(X, Y)_{\text{p-div}}$ in principle. In the rest of this section we provide some properties of the module $\mathrm{BC}_p(k)$ to facilitate the computation.

§6. Examples

In this we present some examples to illustrate Prop. 5.7.3.

(6.1) Example Let k be a perfect field of characteristic $p > 0$. Let X be a one-dimensional p-divisible formal group of height three, and let Y be a one-dimensional p-divisible formal group of height two. Let M and N be the Cartier module of X and Y respectively. One knows that

$$M \cong \mathrm{Cart}_p(k)/\mathrm{Cart}_p(k) \cdot (F - V^2), \quad N \cong \mathrm{Cart}_p(k)/\mathrm{Cart}_p(k) \cdot (F - V)\,.$$

The Frobenius slopes of X and Y are $\frac{1}{3}$ and $\frac{1}{2}$ respectively. According to 5.7.3, the Cartier module of $\mathcal{DE}(X,Y)$ is

$$\big((F-V^2)\cdot \mathrm{Cart}_p(k)\backslash \mathrm{Cart}_p(k)\big)\otimes_{\mathrm{Cart}_p(k)}\mathrm{BC}_p(k)\otimes_{\mathrm{Cart}_p(k)}(\mathrm{Cart}_p(k)/\mathrm{Cart}_p(k)\cdot(F-V)) =: D\,,$$

and $\mathrm{Cart}_p(k)$ operates on D via the action of the "extra copy" of $\mathrm{BC}_p(k)$ on $\mathrm{BC}_p(k)$. Throughout this example, we denote by $\overline{B}$ the image of $B\in \mathrm{BC}_p(k)$ in D, for any element $B\in\mathrm{BC}_p(k)$. The following statements about D can be verified without difficulty; the details are omitted.

(1) The left $\mathrm{Cart}_p(k)$ module D is generated by $\overline{U_0}$ and $\overline{VU_0}$, and the tangent space D/VD of D is a two-dimensional vector space over k, generated by the image of the above two elements as a k-vector space.

(2) Every element $d\in D$ can be written in the form

$$d=\overline{\sum_{m\geq 0}\langle a_m\rangle U_m+\sum_{m\geq 0}V\langle b_m\rangle U_m}$$

with $a_m, b_m\in k\ \forall\, m\in\mathbb{N}$. Moreover the a_m's and b_m's are uniquely determined by d.

(3) The action of V_x, $\langle c\rangle_x$ and F_x is given by

$$\begin{aligned}
V_x\cdot\overline{\textstyle\sum_{m\geq 0}\langle a_m\rangle U_m+\sum_{m\geq 0}V\langle b_m\rangle U_m} &= \overline{\textstyle\sum_{m\geq 0}\langle a_m\rangle U_{m+1}+\sum_{m\geq 0}V\langle b_m\rangle U_{m+1}}\\
\langle c\rangle_x\cdot\overline{\textstyle\sum_{m\geq 0}\langle a_m\rangle U_m+\sum_{m\geq 0}V\langle b_m\rangle U_m} &= \overline{\textstyle\sum_{m\geq 0}\langle a_m c^{p^m}\rangle U_m+\sum_{m\geq 0}V\langle b_m c^{p^m}\rangle U_m}\\
F_x\cdot\overline{\textstyle\sum_{m\geq 0}\langle a_m\rangle U_m+\sum_{m\geq 0}V\langle b_m\rangle U_m} &= \overline{\textstyle\sum_{m\geq 0}V\langle a_m^{p^4}\rangle U_{m+3}+\sum_{m\geq 0}V\langle b_m^{p^6}\rangle U_{m+5}}\\
=\overline{\textstyle\sum_{m\geq 0}V\langle c_m\rangle U_m} &\qquad \text{for suitable elements } c_m\in k
\end{aligned}$$

The first equality in the displayed formulas above follows from 5.5(v), while the second equality follows from 5.5(iv).

(4) $F_x\cdot\overline{(VU_0-U_2)}=0$, and

$$D_{\mathrm{unip}} := \bigcup_{n\in\mathbb{N}}\mathrm{Ker}\,(F_x^n|_D) = \left\{\overline{\sum_{n\geq 0}(V\langle a_n^{p^n}\rangle U_n-\langle a_n^{p^{n+2}}\rangle U_{n+2})}\ \middle|\ a_n\in k\ \ \forall\, n\geq 0\right\}$$

Notice that $F_x\cdot D_{\mathrm{unip}}=(0)$, and $\overline{VU_0-U_2}$ generates D as a left $\mathrm{Cart}_p(k)$-module.

(5) The Cartier module of $\mathcal{DE}(X,Y)_{\text{p-div}}$ is

$$D_{\text{p-div}}=\left\{\sum_{m\geq 0}\overline{V\langle b_m\rangle U_m}\ \middle|\ b_m\in k\ \ \forall\, m\geq 0\right\}.$$

As a $W(k)$-module, $D_{\text{p-div}}$ is free of rank six, with $\overline{VU_0},\dots,\overline{VU_5}$ as a basis. We also have $p\,\overline{VU_m}=\overline{VU_{m+6}}$ for all $m\geq 0$.

(6) On the Cartier module $D_{\text{p-div}}$ we have

$$V_x\cdot\overline{VU_i}=\overline{VU_{i+1}},\quad 0\leq i\leq 4,\ \text{and}\ V_x\cdot\overline{VU_5}=p\overline{VU_0}\,,$$

by (3). So $\mathcal{DE}(X,Y)_{\text{p-div}}$ is a one-dimensional p-divisible formal group of height six, and its Frobenius slope is $\frac{1}{6}$.

(7) The maximal unipotent subgroup $\mathcal{DE}(X,Y)_{\rm unip}$ of $\mathcal{DE}(X,Y)$ is isomorphic to $\mathbb{G}_a$. Its Cartier module is $D_{\rm unip}$.

(8) The quotient $\mathcal{DE}(X,Y)/\mathcal{DE}(X,Y)_{\rm unip}$ is a one-dimensional p-divisible formal group of height six. The quotient $\mathcal{DE}(X,Y)/\mathcal{DE}(X,Y)_{\rm p\text{-}div}$ is isomorphic to $\mathbb{G}_a$.

(9) The covariant Diudonné module of the intersection $\mathcal{DE}(X,Y)_{\rm unip}\cap\mathcal{DE}(X,Y)_{\rm p\text{-}div}$ is not trivial. So the intersection is isomorphic to the kernel of the second iterate of the relative Frobenius $\mathrm{Fr}^2_{\mathbb{G}_a/k}$ on $\mathbb{G}_a$.

(6.2) Example In this Example, we give an explicit description of $\mathrm{BC}_p(k)\otimes_{\mathrm{Cart}_p(k)}\mathrm{M}(\widehat{\mathbb{G}_m})$, and elaborate on a Remark 5.7.4 (i) that 5.7.3 generalizes a part of [12].

Let k be a commutative ring over $\mathbb{F}_p$. Let $N=\mathrm{Cart}_p(k)/\mathrm{Cart}_p(k)\cdot(F-1)$ be the Cartier module of the formal torus $\widehat{\mathbb{G}_m}$ over k. Let

$$\overline{B}:=\mathrm{BC}_p(k)\otimes_{\mathrm{Cart}_p(k)}N=\mathrm{BC}_p(k)/\mathrm{BC}_p(k)\cdot(F-1)\,.$$

(6.2.1) The filtration $\mathrm{Fil}^\bullet_{\rm tot}$ on $\mathrm{BC}_p(k)$ induces a filtration $\mathrm{Fil}^\bullet_{\rm tot}$ on $\overline{B}=\mathrm{BC}_p(k)/\mathrm{BC}_p(k)\cdot(F-1)$, and $\overline{B}$ is complete with respect to the topology defined by the filtration $\mathrm{Fil}^\bullet_{\rm tot}$. See 7.2 for the definition of $\mathrm{Fil}^\bullet_{\rm tot}$.

Denote by $\overline{\sum_{m,n,i\geq 0}V^m\langle a_{mni}\rangle U_nF^i}$ the image in $\overline{B}=\mathrm{Cart}_p(k)/\mathrm{Cart}_p(k)\cdot(F-1)$ of the element $\sum_{m,n,i\geq 0}V^m\langle a_{mni}\rangle U_nF^i\in\mathrm{BC}_p(k)$. Clearly $\overline{\sum_{m,n,i\geq 0}V^m\langle a_{mni}\rangle U_nF^i}$ is equal to the convergent sum $\sum_{m,n,i\geq 0}\overline{V^m\langle a_{mni}\rangle U_nF^i}$.

The subgroup $\mathrm{BC}_p(k)\cdot(F-1)$ of $\mathrm{BC}_p(k)$ consists of all elements in $\mathrm{BC}_p(k)$ of the form

$$-\sum_{m,n\geq 0}V^m\langle a_{m,n,0}\rangle U_n+\sum_{m,n\geq 0}\sum_{i\geq 1}\left(V^m\langle a_{m,n,i-1}\rangle F^i-V^m\langle a_{m,n,i}\rangle U_nF^i\right)$$

where $a_{m,n,i}\in k$ for all $m,n,i\in\mathbb{N}$, and $\forall m,\forall m$, there exists $C(m,n)>0$ such that $a_{m,n,i}=0$ if $i>C(m,n)$. From the above description of $\mathrm{BC}_p(k)\cdot(F-1)$ one sees that every element of $\mathrm{BC}_p(k)/\mathrm{BC}_p(k)\cdot(F-1)$ can be expressed as an infinite sum in the form

$$\sum_{m,n\geq 0}\overline{V^m\langle a_{mn}\rangle U_n}\,,\quad a_{mn}\in k\;\forall\, m,n\in\mathbb{N}\,.$$

(6.2.2) We have

(i) $V\cdot\sum_{m,n\geq 0}\overline{V^m\langle a_{mn}\rangle U_n}=\sum_{m,n\geq 0}\overline{V^{m+1}\langle a_{mn}\rangle U_n}$

(ii) $F\cdot\sum_{m,n\geq 0}\overline{V^m\langle a_{mn}\rangle U_n}=\sum_{m,n\geq 0}\overline{V^m\langle a_{mn}^p\rangle U_{n+1}}$

(iii) $\langle c\rangle\cdot\sum_{m,n\geq 0}\overline{V^m\langle a_{mn}\rangle U_n}=\sum_{m,n\geq 0}\overline{V^m\langle c^{p^{-m}}a_{mn}\rangle U_n}$

(iv) $V_x\cdot\sum_{m,n\geq 0}\overline{V^m\langle a_{mn}\rangle U_n}=\sum_{m,n\geq 0}\overline{V^m\langle a_{mn}\rangle U_{n+1}}$

(v) $F_x\cdot\sum_{m,n\geq 0}\overline{V^m\langle a_{mn}\rangle U_n}=\sum_{m,n\geq 0}\overline{V^{m+1}\langle a_{mn}^p\rangle U_n}$

(vi) $\langle c\rangle_x\cdot\sum_{m,n\geq 0}\overline{V^m\langle a_{mn}\rangle U_n}=\sum_{m,n\geq 0}\overline{V^m\langle a_{mn}\,c^{p^n}\rangle U_n}$

(6.2.3) The structure of $\mathrm{BC}_p(k)/\mathrm{BC}_p(k)\cdot(F-1)$ can be conveniently described in terms of the ring $\mathrm{Cart}_p(k)^\wedge$, the completion of the ring $\mathrm{Cart}_p(k)$ with respect to the filtration $\mathrm{Fil}^\bullet_{\mathrm{tot}}$ on $\mathrm{Cart}_p(k)$, defined by the ideals

$$\mathrm{Fil}^N_{\mathrm{tot}}\,\mathrm{Cart}_p(k) = \left\{ \sum_{m,n\in\mathbb{N}, m+n\geq N} V^m\langle a_{mn}\rangle F^n \;\in \mathrm{Cart}_p(k) \right\}.$$

We have

$$\mathrm{Cart}_p(k)^\wedge = \left\{ \sum_{m,n\in\mathbb{N}} V^m\langle a_{mn}\rangle F^n \;\middle|\; a_{mn}\in k \;\;\forall\, m,n\in\mathbb{N} \right\}.$$

See 7.1.1 for more information about the completed Cartier ring $\mathrm{Cart}_p(k)^\wedge$.

The left action of $\mathrm{Cart}_p(k)$ on $\mathrm{BC}_p(k)$, coming from the left factor of the bimodule structure, extends to a left action of the completed Cartier ring $\mathrm{Cart}_p(k)^\wedge$ on $\mathrm{BC}_p(k)$; see 7.2.2. The action of $\mathrm{Cart}_p(k)$ on

$$\overline{B} = \mathrm{BC}_p(k)/\mathrm{BC}_p(k)\cdot(F-1)\,,$$

inherited from the bimodule structure of $\mathrm{BC}_p(k)$, extends to a left action of the completed Cartier ring $\mathrm{Cart}_p(k)^\wedge$ on the quotient module $\mathrm{BC}_p(k)/\mathrm{BC}_p(k)\cdot(F-1)$, making $\overline{B}$ a free left $\mathrm{Cart}_p(k)^\wedge$ generated by the element $\overline{U_0}\in\mathrm{BC}_p(k)/\mathrm{Cart}_p(k)\cdot(F-1)$. The action of the "extra copy" of the Cartier ring $\mathrm{Cart}_p(k)$ on $\overline{B}\xleftarrow{\sim}\mathrm{Cart}_p(k)^\wedge$ commutes with the left action of $\mathrm{Cart}_p(k)^\wedge$, and induces an ring homomorphism $*:\mathrm{Cart}_p(k)\to(\mathrm{Cart}_p(k)^\wedge)^{\mathrm{opp}}$. It is easy to see that $*$ extends to an involution on $\mathrm{Cart}_p(k)^\wedge$, i.e. an anti-automorphism of order two, such that

$$*: \sum_{m,n\in\mathbb{N}} V^m\langle a_{mn}\rangle F^n \mapsto \sum_{m,n\in\mathbb{N}} V^n\langle a_{mn}\rangle F^m\,.$$

In other words, if we identify $\overline{B}$ with $\mathrm{Cart}_p(k)^\wedge$ as above, then the left action of the "extra copy" of $\mathrm{Cart}_p(k)$ on $\overline{B}$ becomes

$$u: v\;\mapsto v\cdot u^* \qquad \forall\, u\in\mathrm{Cart}_p(k),\;\forall v\in\mathrm{Cart}_p(k)^\wedge\,.$$

(6.2.4) Remark (i) The completed Cartier ring $\mathrm{Cart}_p(k)^\wedge$, together with the involution $*$ on it, appeared in [12, p. 316], where it was called $\tilde{A}_k$. It is crucial for calculating biextensions and the Cartier module of the dual of formal groups.

(ii) In view of the identification of $\overline{B}$ with $\mathrm{Cart}_p(k)^\wedge$ explained above, when $Y=\widehat{\mathbb{G}_m}$, k is a perfect field, and X is a smooth formal group of local-local type, the explicit formula for $\mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(\mathrm{M}(X),\overline{B})$ in Prop. 5.7.3 reduces to the combination of the Theorem on p. 503 and the Proposition on p. 504 of [13]. At the same time, the fact that $\mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(\mathrm{M}(X),\overline{B})$ is the Cartier module of the Serre-dual of X for any commutative ring k over $\mathbb{Z}_{(p)}$ can be regarded as an answer to the observation in the last paragraph of the Introduction of [14].

(6.3) Example Here is a generalization of Example 6.1.

(6.3.1) Let k be a perfect field of characteristic p. Let n be a positive integer. Let

$$M = \mathrm{M}(X) = \mathrm{Cart}_p(k)/\mathrm{Cart}_p(k)\cdot(F - V^n)\,, \quad N = \mathrm{M}(Y) = \mathrm{Cart}_p(k)/\mathrm{Cart}_p(k)\cdot(F - V^{n-1})\,,$$

so that X, Y are one-dimensional smooth formal groups over k of height $n+1$ and n respectively. We have

$$\mathrm{M}(\mathcal{DE}(X,Y)) \cong (F - V^n)\cdot\mathrm{Cart}_p(k)\backslash\mathrm{BC}_p(k)/\mathrm{Cart}_p(k)\cdot(F - V^{n-1}) =: D\,.$$

(6.3.2) As before, denote by

$$\overline{\sum_{i,m} V^i\langle a_{im}\rangle U^m}$$

the image in D of the element $\sum_{i,m} V^i\langle a_{im}\rangle U^m \in \mathrm{BC}_p(k)$, $a_{im} \in k$. One can check that every element of D can be written in the form

$$\sum_{i=0}^{n-1}\sum_{m\geq 0} V^i\langle a_{im}\rangle U^m\,, \qquad a_{im} \in k \;\forall\, i = 0,\ldots,n-1,\; \forall\, m \geq 0$$

in a unique way.

(6.3.3) The action of V_x, $\langle c\rangle_x$ and F_x on D are determined by continuity and the following equalities:

$$\begin{aligned} V_x\cdot\overline{V^i\,\langle a\rangle\,U_m} &= \overline{V^i\,\langle a\rangle\,U_{m+1}} \\ \langle c\rangle_x\cdot\overline{V^i\,\langle a\rangle\,U_m} &= \overline{V^i\langle a\,c^{p^m}\rangle\,U_m} \qquad\qquad \forall\, c \in k \\ F_x\cdot\overline{V^i\,\langle a\rangle\,U_m} &= \overline{V^{n-1}\,\langle a^{p^{(i+2)n}}\rangle\,U_{m+(i+2)n-1}} \end{aligned}$$

for $i = 0\ldots,n-1$, all $m \geq 0$, and all $a \in k$.

(6.3.4) From the above formulae and Prop. 4.3, one sees that the Cartier module of the maximal p-divisible subgroup of $\mathcal{DE}(X,Y)$ is

$$\mathrm{M}(\mathcal{DE}(X,Y)_{\text{p-div}}) = \left\{ \overline{\sum_{m\geq 0} V^{n-1}\,\langle a_m\rangle\,U_m} \;\middle|\; a_m \in k\;\forall\, m \geq 0 \right\}.$$

It is generated by the element $\overline{V^{n-1}U_0} \in D$ as a left $\mathrm{Cart}_p(k)$-module; this generator gives an isomorphism

$$\mathrm{Cart}_p(k)/\mathrm{Cart}_p(k)\cdot(F - V^{n(n+1)-1}) \xrightarrow{\sim} \mathrm{M}(\mathcal{DE}(X,Y)_{\text{p-div}})\,.$$

(6.3.5) The Cartier module of the maximal unipotent subgroup of $\mathcal{DE}(X,Y)$ is generated by the subset

$$\left\{ \left(\overline{V^iU_0} - \overline{U_{in}}\right) \;\middle|\; i = 1,\ldots,n-1 \right\}$$

of $\mathrm{M}(\mathcal{DE}(X,Y)_{\mathrm{unip}})$. Each of the above generator is killed by F_x, so $\mathrm{M}(\mathcal{DE}(X,Y)_{\mathrm{unip}})$ is killed by F_x. It is easy to see that the image of the above n generators of $\mathrm{M}(\mathcal{DE}(X,Y)_{\mathrm{unip}})$ in $\mathrm{M}(\mathcal{DE}(X,Y)_{\mathrm{unip}})/V\cdot\mathrm{M}(\mathcal{DE}(X,Y)_{\mathrm{unip}})$ is a k-basis of $\mathrm{M}(\mathcal{DE}(X,Y)_{\mathrm{unip}})/V\cdot\mathrm{M}(\mathcal{DE}(X,Y)_{\mathrm{unip}})$, so $\mathrm{M}(\mathcal{DE}(X,Y)_{\mathrm{unip}})$ is isomorphic to $\mathbb{G}_a^n$.

(6.3.6) The covariant Dieudonné module of the finite group scheme

$$\mathcal{DE}(X,Y)_{\text{p-div}} \cap \mathcal{DE}(X,Y)_{\text{unip}}$$

over k is the quotient of $\mathrm{M}(\mathcal{DE}(X,Y))$ by the sum $\mathrm{M}(\mathcal{DE}(X,Y)_{\text{p-div}} + \mathcal{DE}(X,Y)_{\text{unip}}$. In the present case, it is an $n(n-1)$-dimensional k-vector space generated by the images of

$$\overline{U_0}, \overline{U_1}, \ldots, \overline{U_{n(n-1)-1}}\,.$$

As a left $\mathrm{Cart}_p(k)$-module, it is isomorphic to $\mathrm{Cart}_p(k)/(\mathrm{Cart}_p(k)F + \mathrm{Cart}_p(k)V^{n(n-1)})$, with the image of $\overline{U_0}$ as a generator. In other words, $\mathcal{DE}(X,Y)_{\text{p-div}} \cap \mathcal{DE}(X,Y)_{\text{unip}}$ is isomorphic to the kernel of the iterated relative Frobenius $\mathrm{Fr}^{n(n-1)}_{\mathbb{G}_a/k} : \mathbb{G}_a \to \mathbb{G}_a^{(p^{n(n-1)})}$ on $\mathbb{G}_a$.

(6.4) Example Let k be a perfect field of characteristic p. Let X, Y be p-divisible formal groups over k such that

$$\mathrm{M}(X) = \mathrm{Cart}_p(k)/\mathrm{Cart}_p(k)\cdot(F - V^5), \quad \mathrm{M}(Y) = \mathrm{Cart}_p(k)/\mathrm{Cart}_p(k)\cdot(F^2 - V)\,.$$

So the Cartier module of $\mathcal{DE}(X,Y)$ is isomorphic to

$$(F - V^5)\cdot \mathrm{Cart}_p(k)\backslash \mathrm{BC}_p(k)/\mathrm{Cart}_p(k)\cdot(F^2 - V) =: D\,.$$

(6.4.1) It is easy to see that every element of D can be written in the form

$$\sum_{m=0}^{4}\sum_{n=0}^{\infty}\sum_{i=0}^{1} \overline{V^m \langle a_{mni}\rangle U_n F^i} \qquad a_{mni} \in k \;\; \forall\, m = 0, \ldots, 4,\; \forall\, n \geq 0,\; \forall\, i = 0,1$$

in a unique way, where $\overline{V^m \langle a_{mni}\rangle U_n F^i}$ denotes the image of $V^m \langle a_{mni}\rangle U_n F^i$ in D, and D is given the quotient topology.

(6.4.2) The action of V_x and $\langle c\rangle_x$ on D, $c \in k$, are given by the following formulae.

$$\begin{aligned} V_x \cdot \overline{V^m \langle a\rangle U_n F^i} &= \overline{V^m \langle a\rangle U_{n+1} F^i} \\ \langle c\rangle_x \cdot \overline{V^m \langle a\rangle U_n F^i} &= \overline{V^m \langle a\cdot c^{p^n}\rangle U_n F^i} \end{aligned}$$

for all $a \in k$, $m = 0, \ldots, 4$, all $n \geq 0$, and $i = 0, 1$.

The action of F_x on D is given by the following formulae.

$$\begin{aligned} F_x \cdot \overline{V^m \langle a\rangle U_n} &= \overline{V^{m+1} \langle a^p\rangle U_n F} && m = 0, \ldots, 4,\; n \geq 0 \\ F_x \cdot \overline{V^4 \langle a\rangle U_n} &= \overline{V \langle a^{p^3}\rangle U_{n+2}} && n \geq 0 \\ F_x \cdot \overline{V^m \langle a\rangle U_n F} &= \overline{V^{m+2} \langle a^{p^2}\rangle U_{n+1}} && m = 0,1,2,\; n \geq 0 \\ F_x \cdot \overline{V^3 \langle a\rangle U_n F} &= \overline{\langle a^{p^3}\rangle U_{n+2} F} && n \geq 0 \\ F_x \cdot \overline{V^4 \langle a\rangle U_n F} &= \overline{V \langle a^{p^3}\rangle U_{n+2} F} && n \geq 0 \end{aligned}$$

(6.4.3) Using Prop. 4.3 and the above formulae, one sees that the Cartier module of the maximal p-divisible formal subgroup consists of all elements of D of the form

$$\sum_{m=1}^{4}\sum_{n=0}^{\infty}\overline{V^m\langle a_{mn0}\rangle}+\sum_{m=0}^{4}\sum_{n=0}^{\infty}\overline{V^m\langle a_{mn1}\rangle F}\qquad a_{mni}\in k\ \forall\, m,n,i\,.$$

The tangent space of $\mathcal{DE}(X,Y)_{\text{p-div}}$, canonically isomorphic to

$$\mathrm{M}(\mathcal{DE}(X,Y)_{\text{p-div}})/V\cdot\mathrm{M}(\mathcal{DE}(X,Y)_{\text{p-div}})\,,$$

is a 9-dimensional vector space, generated by the images of the following elements

$$\overline{V^mU_0}\ \ (m=1,\ldots,4);\quad \overline{V^mU_0F}\ \ (m=0,\ldots,4)$$

of $\mathrm{M}(\mathcal{DE}(X,Y)_{\text{p-div}}$.

(6.4.4) The left $\mathrm{Cart}_p(k)$-module

$$\mathrm{M}(\mathcal{DE}(X,Y)_{\text{p-div}}/(V\cdot\mathrm{M}(\mathcal{DE}(X,Y)_{\text{p-div}})+F\cdot\mathrm{M}(\mathcal{DE}(X,Y)_{\text{p-div}}))\,,$$

canonically isomorphic to the covariant Dieudonné module of the maximal α-subgroup of $\mathcal{DE}(X,Y)_{\text{p-div}}$, can be identified with

$$\left(\sum_{m=1}^{4}k\cdot\overline{\overline{V^iU_0}}+\sum_{m=0}^{4}k\cdot\overline{\overline{V^iU_0F}}\right)\Big/\left(\sum_{i=1}^{3}k\cdot\overline{\overline{V^iU_0F}}\right)\,,$$

where $\overline{\overline{u}}$ denotes the image of an element $\overline{u}\in\mathrm{M}(\mathcal{DE}(X,Y)_{\text{p-div}})\subset D$ in the quotient space

$$\mathrm{M}(\mathcal{DE}(X,Y)_{\text{p-div}}/V\cdot\mathrm{M}(\mathcal{DE}(X,Y)_{\text{p-div}})+F\cdot\mathrm{M}(\mathcal{DE}(X,Y)_{\text{p-div}})\,.$$

In particular it is a 6-dimensional vector space over k. Therefore $\mathcal{DE}(X,Y)_{\text{p-div}}$ is not minimal, i.e. $\mathcal{DE}(X,Y)_{\text{p-div}}$ is not isomorphic over $\overline{k}$ to the self-product of 9-copies of the formal group of a supersingular elliptic curve.

(6.4.5) The Cartier module of $\mathcal{DE}(X,Y)_{\text{unip}}$ is the $\mathrm{Cart}_p(k)$-submodule of $\mathrm{M}(\mathcal{DE}(X,Y))$ generated by the element $\overline{U_1}-\overline{V^4U_0F}$ of $\mathrm{M}(\mathcal{DE}(X,Y))$. This generator gives an isomorphism

$$\mathrm{Cart}_p(k)/\mathrm{Cart}_p(k)\cdot F\xrightarrow{\sim}\mathrm{M}(\mathcal{DE}(X,Y)_{\text{unip}})\,,$$

and also an isomorphism $\mathbb{G}_a\xrightarrow{\sim}\mathcal{DE}(X,Y)_{\text{unip}}$.

(6.4.6) The covariant Dieudonné module of $\mathcal{DE}(X,Y)_{\text{p-div}}\cap\mathcal{DE}(X,Y)_{\text{unip}}$ is isomorphic to $\mathrm{Cart}_p(k)/(\mathrm{Cart}_p(k)\cdot F+\mathrm{Cart}_p(k)\cdot V^2)$. In other words, $\mathcal{DE}(X,Y)_{\text{p-div}}\cap\mathcal{DE}(X,Y)_{\text{unip}}$ is isomorphic to the kernel of the second iterate of the relative Frobenius $\mathrm{Fr}^2_{\mathbb{G}_a/k}:\mathbb{G}_a\to\mathbb{G}_a^{(p^2)}$ on $\mathbb{G}_a$.

(6.4.7) The quotient $\mathrm{M}(\mathcal{DE}(X,Y))/\mathrm{M}(\mathcal{DE}(X,Y)_{\mathrm{unip}}$ is the Cartier module of the maximal p-divisible quotient of $\mathcal{DE}(X,Y)$. The quotient

$$\mathrm{M}(\mathcal{DE}(X,Y))/\left(\mathrm{M}(\mathcal{DE}(X,Y)_{\mathrm{unip}}) + V\mathrm{M}(\mathcal{DE}(X,Y)) + F\mathrm{M}(\mathcal{DE}(X,Y))\right)$$

is a 6-dimensional vector space over k. So the maximal p-divisible quotient of $\mathcal{DE}(X,Y)$ is not minimal either.

§7. The structure of $\mathrm{BC}_p(k)$

In this section we study the structure of the module $\mathrm{BC}_p(k)$, where k is a perfect field of characteristic p. A crude first approximation of $\mathrm{BC}_p(k)$ is the tensor product of $\mathrm{Cart}_p(k)$ with itself over the ring $W(k) = W_p(k)$ of p-adic Witt vectors. The module $\mathrm{BC}_p(k)$ can be identified with a completion of a suitable $W(k)$-submodule of $\mathrm{Cart}_p(k) \otimes_{W(k)} \mathrm{Cart}_p(k) \otimes_{\mathbb{Z}} \mathbb{Q}$, with respect to the topology defined by a suitable filtration.

(7.1) Definition (i) The Cartier ring $\mathrm{Cart}_p(k)$ has a decreasing filtration $\mathrm{Fil}_V^\bullet$ by right ideals, called the V-adic filtration, defined by

$$\mathrm{Fil}_V^n \mathrm{Cart}_p(k) = V^n \mathrm{Cart}_p(k)\,,$$

for all $n \geq 0$.

(ii) The Cartier ring $\mathrm{Cart}_p(k)$ has a decreasing filtration $\mathrm{Fil}_{\mathrm{tot}}^\bullet$, called the "total filtration". It is defined by

$$\mathrm{Fil}_{\mathrm{tot}}^N \mathrm{Cart}_p(k) = \left\{ \sum_{\substack{m,n\geq 0\\ m+n\geq N}} V^m \langle a_{mn}\rangle F^n \in \mathrm{Cart}_p(k) \;\middle|\; a_{m,n}\in k \;\; \forall\, m,n \right\}\,,$$

for all $N \in \mathbb{N}$.

(7.1.1) Lemma (i) $\mathrm{Fil}_V^m \mathrm{Cart}_p(k) \cdot \mathrm{Fil}_V^n \mathrm{Cart}_p(k) \subseteq \mathrm{Fil}_V^{m+n} \mathrm{Cart}_p(k)$ *for all* $m,n \in \mathbb{N}$. *In particular, each* $\mathrm{Fil}_V^m \mathrm{Cart}_p(k)$ *is an ideal of* $\mathrm{Cart}_p(k)$.

(ii) *The Cartier ring* $\mathrm{Cart}_p(k)$ *is a complete and separated topological ring for the topology defined by the V-adic filtration* $\mathrm{Fil}_V^\bullet$.

(iii) $\mathrm{Fil}_{\mathrm{tot}}^m \mathrm{Cart}_p(k) \cdot \mathrm{Fil}_{\mathrm{tot}}^n \mathrm{Cart}_p(k) \subseteq \mathrm{Fil}_{\mathrm{tot}}^{m+n} \mathrm{Cart}_p(k)$ *for all* $m,n \in \mathbb{N}$. *In particular, each* $\mathrm{Fil}_{\mathrm{tot}}^m \mathrm{Cart}_p(k)$ *is an ideal of* $\mathrm{Cart}_p(k)$.

(iv) *The Cartier ring* $\mathrm{Cart}_p(k)$ *is a separated topological ring for the topology defined by the filtration* $\mathrm{Fil}_{\mathrm{tot}}^\bullet$.

(v) *The m-th graded piece* $\mathrm{gr}_V^m \mathrm{Cart}_p(k)$ *is a k-vector space with basis* $(e_{mn})_{n\geq 0}$, *where* e_{mn} *is the image of the element* $V^m F^n \in \mathrm{Fil}_V^m \mathrm{Cart}_p(k)$ *in* $\mathrm{gr}_V^m \mathrm{Cart}_p(k)$.

(vi) *The N-th graded piece* $\mathrm{gr}_{\mathrm{tot}}^N \mathrm{Cart}_p(k)$ *is a finite-dimensional k vector space with basis*

$$\left\{ \overline{V^i F^{N-i}} \;\middle|\; i = 0,1,\ldots,N \right\}\,,$$

where $\overline{V^i F^{N-i}}$ *is the image of the element* $V^i F^{N-i} \in \mathrm{Fil}_{\mathrm{tot}}^N \mathrm{Cart}_p(k)$.

(vii) *The filtration* $\mathrm{Fil}^\bullet_{\mathrm{tot}}$ *on* $\mathrm{Cart}_p(k)$ *is coarser than the* V*-adic filtration* $\mathrm{Fil}^\bullet_V$*. We have a natural identification*

$$\mathrm{Cart}_p(k)^\wedge = \left\{ \sum_{m,n\geq 0} V^m \langle a_{mn}\rangle F^n \;\middle|\; a_{mn}\in k \quad \forall\, m,n\in\mathbb{N} \right\} ,$$

where $\mathrm{Cart}_p(k)^\wedge$ *is the completion of* $\mathrm{Cart}_p(k)$ *with respect to the filtration* $\mathrm{Fil}^\bullet_{\mathrm{tot}}$*.*

(viii) *The total filtration* $\mathrm{Fil}^\bullet_{\mathrm{tot}}\mathrm{Cart}_p(k)$ *on* $\mathrm{Cart}_p(k)$ *extends by the closure operation to a filtration* $\mathrm{Fil}^\bullet_{\mathrm{tot}}\mathrm{Cart}_p(k)^\wedge$ *on* $\mathrm{Cart}_p(k)^\wedge$*, by open-and-closed ideals, which is strictly compatible with* $\mathrm{Fil}^\bullet_{\mathrm{tot}}\mathrm{Cart}_p(k)$*.*

PROOF. Exercise. ∎

(7.2) Definition (i) The V-adic filtration on $\mathrm{BC}_p(k)$ is defined by

$$\mathrm{Fil}^m_V\mathrm{BC}_p(k) = V^m\cdot \mathrm{BC}_p(k) = \left\{ \sum_{\substack{i\geq m\\ n,j\geq 0}} V^i\,\langle a_{inj}\rangle\, U_n\, F_j \in \mathrm{BC}_p(k) \;\middle|\; \begin{array}{c} a_{inj}\in k\ \forall\, i\geq m \\ \forall\, n,j\geq 0\end{array} \right\} .$$

for all $m\geq 0$.

(ii) The filtration $\mathrm{Fil}^\bullet_{\mathrm{tot}}$ on $\mathrm{BC}_p(k)$ is defined by

$$\mathrm{Fil}^N_{\mathrm{tot}}\mathrm{BC}_p(k) = \left\{ \sum_{\substack{m+n\geq N\\ m,n,j\geq 0}} V^m\,\langle a_{mnj}\rangle\, U_n\, F_j \in \mathrm{BC}_p(k) \;\middle|\; a_{mnj}\in k \quad \forall\, m,n,j \right\}$$

for all $N\geq 0$.

(7.2.1) Lemma (i) *We have*

$$\begin{aligned}
V\cdot \mathrm{Fil}^m_V\mathrm{BC}_p(k) &\subseteq \mathrm{Fil}^{m+1}_V\mathrm{BC}_p(k)\,, \quad \forall\, m\geq 0\\
F\cdot \mathrm{Fil}^m_V\mathrm{BC}_p(k) &\subseteq \mathrm{Fil}^{m}_V\mathrm{BC}_p(k)\,, \quad \forall\, m\geq 0\\
\langle c\rangle\cdot \mathrm{Fil}^m_V\mathrm{BC}_p(k) &\subseteq \mathrm{Fil}^{m}_V\mathrm{BC}_p(k)\,, \quad \forall\, m\geq 0,\ \forall\, c\in k
\end{aligned}$$

$$\begin{aligned}
\mathrm{Fil}^m_V\mathrm{BC}_p(k)\cdot V &\subseteq \mathrm{Fil}^{m+1}_V\mathrm{BC}_p(k)\,, \quad \forall\, m\geq 0\\
\mathrm{Fil}^m_V\mathrm{BC}_p(k)\cdot F &\subseteq \mathrm{Fil}^{m}_V\mathrm{BC}_p(k)\,, \quad \forall\, m\geq 0\\
\mathrm{Fil}^m_V\mathrm{BC}_p(k)\cdot \langle c\rangle &\subseteq \mathrm{Fil}^{m}_V\mathrm{BC}_p(k)\,, \quad \forall\, m\geq 0,\ \forall\, c\in k
\end{aligned}$$

$$\begin{aligned}
V_x\cdot \mathrm{Fil}^m_V\mathrm{BC}_p(k) &\subseteq \mathrm{Fil}^{m}_V\mathrm{BC}_p(k)\,, \quad \forall\, m\geq 0\\
F_x\cdot \mathrm{Fil}^m_V\mathrm{BC}_p(k) &\subseteq \mathrm{Fil}^{m+1}_V\mathrm{BC}_p(k)\,, \quad \forall\, m\geq 0\\
\langle c\rangle_x\cdot \mathrm{Fil}^m_V\mathrm{BC}_p(k) &\subseteq \mathrm{Fil}^{m}_V\mathrm{BC}_p(k)\,, \quad \forall\, m\geq 0,\ \forall\, c\in k
\end{aligned}$$

(ii) *We have*

$$\begin{aligned}\mathrm{Fil}_V^m\mathrm{Cart}_p(k)\cdot\mathrm{Fil}_V^n\mathrm{BC}_p(k) &\subseteq \mathrm{Fil}_V^{m+n}\mathrm{BC}_p(k)\,, \quad \forall\, m,n\ge 0\\ \mathrm{Fil}_V^n\mathrm{BC}_p(k)\cdot\mathrm{Fil}_V^m\mathrm{Cart}_p(k) &\subseteq \mathrm{Fil}_V^{m+n}\mathrm{BC}_p(k)\,, \quad \forall\, m,n\ge 0\end{aligned}$$

and

$$\mathrm{Fil}_V^n\mathrm{Cart}_p(k)_x\cdot_x\mathrm{Fil}_V^m\mathrm{BC}_p(k)\subseteq\mathrm{Fil}_V^m\mathrm{BC}_p(k)\,,\quad\forall\, m,n\ge 0$$

where $\cdot_x$ *denotes the action of the "extra copy" of* $\mathrm{Cart}_p(k)$ *on the bimodule* $\mathrm{BC}_p(k)$.

(iii) *For each* $m\ge 0$, $\mathrm{Fil}_V^n\mathrm{BC}_p(k)$ *is a sub-*$(\mathrm{Cart}_p(k)$-$\mathrm{Cart}_p(k))$*-bimodule of* $\mathrm{BC}_p(k)$*, and is also a submodule with respect to the action of the "extra copy" of* $\mathrm{Cart}_p(k)$.

(iv) *The* m*-th graded piece* $\mathrm{gr}_V^m\mathrm{BC}_p(k)$ *of* $\mathrm{BC}_p(k)$ *for the* V*-adic filtration is isomorphic as a* k*-vector space to the set of all formal series of the form*

$$\sum_{n,j\ge 0} a_{mnj}\, e_{mnj}$$

where $a_{mnj}\in k$ *for all* $n,j\ge 0$*, and for each* $n\ge 0$*, there exists a constant* $C_{m,n}$ *such that* $a_{mnj}=0$ *for all* $j>C(m,n)$*. A series* $\sum_{n,j\ge 0} a_{mnj}\, e_{mnj}$ *as above represents the image of the element*

$$\sum_{n,j\ge 0} V^m\langle a_{mnj}^{p^m}\rangle U_n F^j\in\mathrm{Fil}_V^m\mathrm{BC}_p(k)$$

in $\mathrm{gr}_V^m\mathrm{BC}_p(k)$.

(v) $\mathrm{BC}_p(k)$ *is a complete and separated topological* $(\mathrm{Cart}_p(k)$-$\mathrm{Cart}_p(k))$*-bimodule, with respect to the topology defined by the V-adic filtration* $\mathrm{Fil}_V^\bullet$ *on* $\mathrm{BC}_p(k)$*, and the topology defined by the V-adic filtration on both the left and the right factor of* $\mathrm{Cart}_p(k)$ *in the bimodule structure.*

Proof. The proofs are straight-forward, therefore omitted. We illustrate an instance of (i): $V_x\cdot U_m=U_{m+1}$, $U_m,U_{m+1}\in\mathrm{Fil}_V^0\mathrm{BC}_p(k)$, but $U_{m+1}\notin\mathrm{Fil}_V^1\mathrm{BC}_p(k)$. ∎

Remark The action of the "extra copy" of $\mathrm{Cart}_p(k)$ on $\mathrm{BC}_p(k)$ is *not* continuous with respect to the V-adic filtrations on $\mathrm{Cart}_p(k)$ and $\mathrm{BC}_p(k)$.

(7.2.2) Lemma (i) *We have*

$$\begin{aligned}V\cdot\mathrm{Fil}_{\mathrm{tot}}^m\mathrm{BC}_p(k) &\subseteq \mathrm{Fil}_{\mathrm{tot}}^{m+1}\mathrm{BC}_p(k)\,,\quad\forall\, m\ge 0\\ F\cdot\mathrm{Fil}_{\mathrm{tot}}^m\mathrm{BC}_p(k) &\subseteq \mathrm{Fil}_V^{m+1}\mathrm{BC}_p(k)\,,\quad\forall\, m\ge 0\\ \langle c\rangle\cdot\mathrm{Fil}_{\mathrm{tot}}^m\mathrm{BC}_p(k) &\subseteq \mathrm{Fil}_{\mathrm{tot}}^{m}\mathrm{BC}_p(k)\,,\quad\forall\, m\ge 0,\ \forall\, c\in k\end{aligned}$$

$$\begin{aligned}\mathrm{Fil}_{\mathrm{tot}}^m\mathrm{BC}_p(k)\cdot V &\subseteq \mathrm{Fil}_{\mathrm{tot}}^{m+2}\mathrm{BC}_p(k)\,,\quad\forall\, m\ge 0\\ \mathrm{Fil}_{\mathrm{tot}}^m\mathrm{BC}_p(k)\cdot F &\subseteq \mathrm{Fil}_{\mathrm{tot}}^{m}\mathrm{BC}_p(k)\,,\quad\forall\, m\ge 0\\ \mathrm{Fil}_{\mathrm{tot}}^m\mathrm{BC}_p(k)\cdot\langle c\rangle &\subseteq \mathrm{Fil}_{\mathrm{tot}}^{m}\mathrm{BC}_p(k)\,,\quad\forall\, m\ge 0,\ \forall\, c\in k\end{aligned}$$

$$\begin{aligned}V_x\cdot\mathrm{Fil}_{\mathrm{tot}}^m\mathrm{BC}_p(k) &\subseteq \mathrm{Fil}_{\mathrm{tot}}^{m+1}\mathrm{BC}_p(k)\,,\quad\forall\, m\ge 0\\ F_x\cdot\mathrm{Fil}_{\mathrm{tot}}^m\mathrm{BC}_p(k) &\subseteq \mathrm{Fil}_{\mathrm{tot}}^{m+1}\mathrm{BC}_p(k)\,,\quad\forall\, m\ge 0\\ \langle c\rangle_x\cdot\mathrm{Fil}_{\mathrm{tot}}^m\mathrm{BC}_p(k) &\subseteq \mathrm{Fil}_{\mathrm{tot}}^{m}\mathrm{BC}_p(k)\,,\quad\forall\, m\ge 0,\ \forall\, c\in k\end{aligned}$$

(ii) *We have*

$$\begin{aligned}
\mathrm{Fil}^m_{\mathrm{tot}}\mathrm{Cart}_p(k)\cdot\mathrm{Fil}^n_{\mathrm{tot}}\mathrm{BC}_p(k) &\subseteq \mathrm{Fil}^{m+n}_{\mathrm{tot}}\mathrm{BC}_p(k)\,, \quad \forall\, m,n\geq 0\\
\mathrm{Fil}^m_{\mathrm{tot}}\mathrm{BC}_p(k)\cdot\mathrm{Fil}^n_{V}\mathrm{Cart}_p(k) &\subseteq \mathrm{Fil}^{m+2n}_{\mathrm{tot}}\mathrm{BC}_p(k)\,, \quad \forall\, m,n\geq 0\\
\mathrm{Fil}^m_{\mathrm{tot}}\mathrm{BC}_p(k)\cdot\mathrm{Fil}^n_{\mathrm{tot}}\mathrm{Cart}_p(k) &\subseteq \mathrm{Fil}^{m}_{\mathrm{tot}}\mathrm{BC}_p(k)\,, \quad \forall\, m,n\geq 0
\end{aligned}$$

and

$$\mathrm{Fil}^m_{\mathrm{tot}}\mathrm{Cart}_p(k)_x\cdot_x\mathrm{Fil}^n_{\mathrm{tot}}\mathrm{BC}_p(k)\subseteq \mathrm{Fil}^{m+n}_{\mathrm{tot}}\mathrm{BC}_p(k)\,, \quad \forall\, m,n\geq 0$$

where $\cdot_x$ *denotes the action of the "extra copy" of* $\mathrm{Cart}_p(k)$ *on the bimodule* $\mathrm{BC}_p(k)$.

(iii) *For each* $m\geq 0$, $\mathrm{Fil}^n_{\mathrm{tot}}\mathrm{BC}_p(k)$ *is a sub-*$(\mathrm{Cart}_p(k)$*-*$\mathrm{Cart}_p(k))$*-bimodule of* $\mathrm{BC}_p(k)$*, and is also a submodule with respect to the action of the "extra copy" of* $\mathrm{Cart}_p(k)$.

(iv) *For each* $N\geq 0$, $\mathrm{gr}^N_{\mathrm{tot}}\mathrm{BC}_p(k)$ *is a* k*-vector space with basis*

$$\left\{\,\overline{V^iU_{N-i}F^j}\;\middle|\; i=0,1,\ldots,N,\ j=0,1,2,\ldots\right\}\,,$$

where $\overline{V^iU_{N-i}F^j}$ *is the image of the element* $V^iU_{N-i}F^j\in\mathrm{Fil}^N_{\mathrm{tot}}\mathrm{BC}_p(k)$ *in* $\mathrm{gr}^N_{\mathrm{tot}}\mathrm{BC}_p(k)$.

(v) *The module* $\mathrm{BC}_p(k)$ *is a complete and separated topological* $(\mathrm{Cart}_p(k)$*-*$\mathrm{Cart}_p(k))$*-bimodule with respect to the topology defined by the filtration* $\mathrm{Fil}^\bullet_{\mathrm{tot}}$ *on* $\mathrm{BC}_p(k)$.

(vi) *The* $(\mathrm{Cart}_p(k)$*-*$\mathrm{Cart}_p(k))$*-bimodule action* $\mathrm{BC}_p(k)$ *is continuous, for the topology defined by the filtration* $\mathrm{Fil}^\bullet_{\mathrm{tot}}\mathrm{Cart}_p(k)$ *on the left factor of* $\mathrm{Cart}_p(k)$ *in the bimodule structure, the topology defined by the* V*-adic filtration on the right factor of* $\mathrm{Cart}_p(k)$ *in the bimodule structure, and the topology defined by the by the filtration* $\mathrm{Fil}^\bullet_{\mathrm{tot}}\mathrm{BC}_p(k)$ *on* $\mathrm{BC}_p(k)$.

(v) *The action of the "extra copy" of* $\mathrm{Cart}_p(k)$ *on* $\mathrm{BC}_p(k)$ *is continuous, for the topology defined by the filtration* $\mathrm{Fil}^\bullet_{\mathrm{tot}}\mathrm{Cart}_p(k)$ *on* $\mathrm{Cart}_p(k)$ *and the topology defined by the filtration* $\mathrm{Fil}^\bullet_{\mathrm{tot}}\mathrm{BC}_p(k)$ *on* $\mathrm{BC}_p(k)$*. Therefore* $\mathrm{BC}_p(k)$ *has a natural structure as a continuous* $(\mathrm{Cart}_p(k)^\wedge$*-*$\mathrm{Cart}_p(k))$*-bimodule, plus a continuous left action of* $\mathrm{Cart}_p(k)^\wedge$*, commuting with the* $(\mathrm{Cart}_p(k)^\wedge$*-*$\mathrm{Cart}_p(k))$*-bimodule structure.*

Proof. We only illustrate an example of (i). We have $F_xU_m = VU_mF$, $U_m\in\mathrm{Fil}^m_{\mathrm{tot}}\mathrm{BC}_p(k)$, and $U_{m+1}\in\mathrm{Fil}^{m+1}_{\mathrm{tot}}\mathrm{BC}_p(k)$, for all $m\in\mathbb{N}$. The rest is left as an exercise. ∎

Remark The right action of $\mathrm{Cart}_p(k)$ on $\mathrm{BC}_p(k)$ coming from the right factor $\mathrm{Cart}_p(k)$ of the bimodule structure is *not* continuous for topology defined by $\mathrm{Fil}^\bullet_{\mathrm{tot}}\mathrm{BC}_p(k)$ and $\mathrm{Fil}^\bullet_{\mathrm{tot}}\mathrm{Cart}_p(k)$.

(7.3) Definition (i) The V-adic filtration of $\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)$ is the tensor product filtration of the V-adic filtration of $\mathrm{Cart}_p(k)$:

$$\begin{aligned}
&\mathrm{Fil}^n_V\left(\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)\right)\\
&\quad=\sum_{\substack{i,j\geq 0\\ i+j\geq n}}\mathrm{Image}\left(V^i\mathrm{Cart}_p(k)\otimes_{W(k)}V^j\mathrm{Cart}_p(k)\to\mathrm{Cart}_p(k)\otimes_{WY(k)}\mathrm{Cart}_p(k)\right)\,,
\end{aligned}$$

for all $n\in\mathbb{N}$.

(ii) The decreasing filtration $\mathrm{Fil}^\bullet_{\mathrm{tot}}$ on $\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)$ is defined by

$$\begin{aligned}&\mathrm{Fil}^N_{\mathrm{tot}}\left(\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)\right)\\ &=\sum_{\substack{m,n\geq 0\\ m+2n\geq N}}\mathrm{Image}\left(\mathrm{Fil}^m_{\mathrm{tot}}\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Fil}^n_V\mathrm{Cart}_p(k)\to\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)\right)\,,\end{aligned}$$

for all $N\in\mathbb{N}$.

(7.3.1) Lemma (i) *There exists a unique homomorphism*

$$\alpha:\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)\to\mathrm{BC}_p(k)$$

of $(\mathrm{Cart}_p(k)$-$\mathrm{Cart}_p(k))$*-bimodules such that* $\alpha(a\otimes b)=a\cdot U_0\cdot b$ *for all* $a,b\in\mathrm{Cart}_p(k)$*. The right hand side of the above equality refers to the* $(\mathrm{Cart}_p(k)$-$\mathrm{Cart}_p(k))$*-bimodule structure of* $\mathrm{BC}_p(k)$*.*

(ii) *We have*

$$\alpha\left(\mathrm{Fil}^m_V(\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)\right)\subseteq\mathrm{Fil}^m_V\mathrm{BC}_p(k)$$

for all $m\in\mathbb{N}$*, and*

$$\alpha\left(\mathrm{Fil}^N_{\mathrm{tot}}(\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)\right)\subseteq\mathrm{Fil}^N_{\mathrm{tot}}\mathrm{BC}_p(k)$$

for all $N\in\mathbb{N}$*.*

Proof. To prove (i), it suffices to show that $\lambda\cdot U_0=U_0\cdot\lambda$ in $\mathrm{BC}_p(k)$ for every $\lambda\in W(k)$. We know that every element of $W(k)$ can be written as a convergent sum $\sum_{n\in\mathbb{N}}V^n\langle a_n\rangle F^n$ with $a_n\in k$ for all $n\geq 0$. First we show that

$$V^n\langle a\rangle F^n\cdot U_0=U_0\cdot V^n\langle a\rangle F^n\,.$$

This is clear since both sides are equal to $V^n\langle a\rangle U_nF^n$. An easy continuity argument, with $\mathrm{BC}_p(k)$ given the V-adic filtration in 7.2, finishes the proof of (i). The statement (ii) is immediate from the definition of the filtrations, Lemma 7.2.1 and Lemma 7.2.2. ∎

(7.4) Proposition *Let*

$$\mathrm{gr}^\bullet_V\alpha:\mathrm{gr}^\bullet_V\left(\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)\right)\to\mathrm{gr}^\bullet_V\mathrm{BC}_p(k)\,.$$

be the map induced by α*, between the graded pieces of* $\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)$ *and* $\mathrm{Cart}_p(k)$ *for the V-adic filtration. Let*

$$\mathrm{gr}^\bullet_{\mathrm{tot}}\alpha:\mathrm{gr}^\bullet_{\mathrm{tot}}\left(\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)\right)\to\mathrm{gr}^\bullet_{\mathrm{tot}}\mathrm{BC}_p(k)\,.$$

be the map induced by α*, between the graded pieces of* $\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)$ *and* $\mathrm{Cart}_p(k)$ *for the filtration* $\mathrm{Fil}^\bullet_{\mathrm{tot}}$*.*

(i) *The map* $\mathrm{gr}^m_V\alpha$ *is an injection for each* $m\geq 0$*.*

(ii) *The map* $\mathrm{gr}^N_{\mathrm{tot}}\alpha$ *is an injection for each* $N\geq 0$*.*

PROOF. (i) Fix a natural number $m \in \mathbb{N}$. We have seen in 7.2.1 that the group $\mathrm{gr}_V^m(\mathrm{BC}_p(k))$ is isomorphic to the k vector space of all formal linear combinations of the form

$$\sum_{n,j\geq 0} a_{nj}e_{nj}\,, \quad a_{nj}\in k\ \forall n,j\geq 0,\ \text{and}\ \forall n\,\exists\, C(n)\ \text{s.t.}\ a_{nj}=0\ \text{if}\ j>C(n)\,;$$

where a typical formal series $\sum_{n,j\geq 0} a_{nj}e_{nj}$ in the form above represents the image of the element $\sum_{n,j\geq 0} V^m\langle a_{nj}^{p^{-m}}\rangle U_n F^j$ of $\mathrm{gr}_V^m(\mathrm{BC}_p(k))$ in $\mathrm{gr}_V^m(\mathrm{BC}_p(k))$. As for the source of the map $\mathrm{gr}_V^m\alpha$, it is easy to see that $\mathrm{gr}_V^m\left(\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)\right)$ is spanned by the image of the subset

$$(\mathrm{A}) = \left\{\, F^n\otimes V^mF^j \;\middle|\; n,j\geq 0\right\}\cup\left\{\, V^i\otimes V^{m-i}F^j \;\middle|\; 1\leq i\leq m, j\geq 0\right\}$$

of $\mathrm{Fil}_V^m\left(\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)\right)$. The image of (A) under α is the following subset

$$(\mathrm{B}) = \left\{V^mU_{m+n}F^{n+j}\mid\ n,j\geq 0\right\}\cup\left\{V^mU_{m-i}F^j\mid\ 1\leq i\leq m, j\geq 0\right\}$$

of $\mathrm{Fil}_V^m(\mathrm{BC}_p(k))$. By 7.2.1 (iv), the image of the elements of (B) in $\mathrm{gr}^m(\mathrm{BC}_p(k))$ are linearly independent over k. Hence the image of the elements of (A) in the graded piece $\mathrm{gr}_V^m\left(\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)\right)$ are linearly independent, and the map gr_V^m is injective. This proves (i).

(ii) It is easy to see that the N-th graded piece $\mathrm{gr}_{\mathrm{tot}}^N\left(\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)\right)$ of the tensor product $\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)$ is spanned by the image of the subset

$$\begin{aligned}(\mathrm{C}) = &\left\{\, F^n\otimes V^mF^j \;\middle|\; n\geq 1,\ m,j\geq 0,\ n+2m=N\right\}\\ &\cup\left\{\, V^i\otimes V^rF^j \;\middle|\; i\geq 1,\ r,j\geq 0,\ i+2r=N\right\}\end{aligned}$$

of $\mathrm{Fil}_{\mathrm{tot}}^N\left(\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)\right)$. The image of (C) under α is the following subset

$$\begin{aligned}(\mathrm{D}) = &\left\{V^mU_{m+n}F^{n+j}\mid\ m\geq 1,\ n,j\geq 0,\ 2m+n=N\right\}\\ &\cup\left\{V^{i+r}U_rF^j\mid\ i\geq 1,\ r,j\geq 0,\ i+2r=N\right\}\end{aligned}$$

of $\mathrm{Fil}_V^m(\mathrm{BC}_p(k))$. By 7.2.2 (iv), the image of the set (D) in $\mathrm{gr}_{\mathrm{tot}}^N\mathrm{BC}_p(k)$ is linearly independent over k, therefore the map $\mathrm{gr}_{\mathrm{tot}}^N$ is injective. ∎

(7.4.1) Corollary *Let* $\mathrm{Cart}_p(k)\hat{\otimes}_{W(k)}^V\mathrm{Cart}_p(k)$ *and* $\mathrm{Cart}_p(k)\hat{\otimes}_{W(k)}^{\mathrm{tot}}\mathrm{Cart}_p(k)$ *be the completion of the tensor product* $\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)$ *for the V-adic filtration* $\mathrm{Fil}_V^\bullet$ *and the total filtration* $\mathrm{Fil}_{\mathrm{tot}}^\bullet$ *respectively.*

(i) *The map* $\alpha:\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)\to\mathrm{BC}_p(k)$ *extends uniquely to a continuous map*

$$\hat{\alpha}_V:\mathrm{Cart}_p(k)\hat{\otimes}_{W(k)}^V\mathrm{Cart}_p(k)\to\mathrm{BC}_p(k)$$

with respect to the V-adic filtrations. The map $\hat{\alpha}_V$ *is an injection, and is a homomorphism of* $(\mathrm{Cart}_p(k)$-$\mathrm{Cart}_p(k))$*-bimodules.*

(ii) *The map* $\alpha:\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)\to\mathrm{BC}_p(k)$ *extends uniquely to a continuous map*

$$\hat{\alpha}_{\mathrm{tot}}:\mathrm{Cart}_p(k)\hat{\otimes}_{W(k)}^{\mathrm{tot}}\mathrm{Cart}_p(k)\to\mathrm{BC}_p(k)$$

with respect to the total filtrations $\mathrm{Fil}_{\mathrm{tot}}^\bullet$*. The map* $\hat{\alpha}_{\mathrm{tot}}$ *is an injection, and is a homomorphism of* $(\mathrm{Cart}_p(k)^\wedge$-$\mathrm{Cart}_p(k))$*-bimodules.*

PROOF. The injectivity of $\hat{\alpha}_V$ and $\hat{\alpha}_{\rm tot}$ follows from 7.4 (i) and (ii) respectively. The homomorphism α is a homomorphism of ($\mathrm{Cart}_p(k)$-$\mathrm{Cart}_p(k)$)-bimodules by construction. The completion $\mathrm{Cart}_p(k)\hat{\otimes}^{\rm tot}_{W(k)}\mathrm{Cart}_p(k)$ is a ($\mathrm{Cart}_p(k)^\wedge$-$\mathrm{Cart}_p(k)$)-bimodule, because the total filtration $\mathrm{Fil}^\bullet_{\rm tot}$ on $\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)$ is the tensor product of the filtration $\mathrm{Fil}^\bullet_{\rm tot}$ on the left factor of $\mathrm{Cart}_p(k)$ and the filtration $\mathrm{Fil}^\bullet_{\rm d}$ on the right factor of $\mathrm{Cart}_p(k)$, defined by

$$\mathrm{Fil}^n_{\rm d}\mathrm{Cart}_p(k) := \mathrm{Fil}_V^{\lceil n/2\rceil}\mathrm{Cart}_p(k)\,.$$

The last sentence of statement (ii) follows from continuity. ∎

In the rest of this section $K = \mathrm{frac}(W(k))$ denotes the field of fractions of the ring $W(k)$ of p-adic Witt vectors with entries in k, and $\sigma : W(k) \to W(k)$ denotes the ring automorphism induced by $\mathrm{Fr}_p : k \to k$. The "Frobenius automorphism" σ extends to an automorphism of the fraction field K of $W(k)$, again denoted by σ. Let ord_p be the discrete p-adic valuation on K, normalized by $\mathrm{ord}_p(p) = 1$.

(7.5) Definition (i) Denote by $W(k)[V,F]$ the $W(k)$-module consisting of all finite series of the form

$$\sum_{i\in\mathbb{Z}} a_i\,V^i\,,\quad a_i\in K\,,\ \mathrm{ord}_p(a_i)\ge\max(-i,0),\ \forall i\in\mathbb{Z}\,,$$

such that $a_i = 0$ for all but finitely many i's. The set

$$\{1 = V^0, V^1, V^2, \ldots, pV^{-1}, p^2V^{-2}, \ldots\}$$

is a basis of the free $W(k)$-module $W(k)[V,F]$.

(ii) Define a ring structure on $W(k)[V,F]$ by

$$\left(\sum_i a_i\,V^i\right)\cdot\left(\sum_j b_j\,V^j\right) = \sum_{n\in\mathbb{Z}}\sum_{n=i+j} a_i b_j^{\sigma^{-j}}\ V^n$$

for elements $\sum_i a_i\,V^i$, $\sum_j b_j\,V^j$ of $W(k)[V,F]$.

(iii) Let $\mathrm{Fil}^\bullet_V W(k)[V,F]$ be the decreasing filtration on $W(k)[V,F]$, defined by

$$\mathrm{Fil}^m_V W(k)[V,F] = \left\{\sum_i a_i\,V^i \in W(k)[V,F]\ \middle|\ \mathrm{ord}_p(a_i)\ge\max(m-i,0)\ \ \forall\, i\right\}$$

for $m\ge 0$. The condition on $\mathrm{ord}_p(a_i)$ is equivalent to: $\mathrm{ord}_p(a_i)\ge\max(-i,0)$ and $\mathrm{ord}_p(a_i)+i\ge 0$. Notice that each $\mathrm{Fil}^m_V W(k)[V,F]$ is an ideal of $W(k)[V,F]$.

(iv) Let $\mathrm{Fil}^\bullet_{\rm tot} W(k)[V,F]$ be the decreasing filtration on $W(k)[V,F]$, defined by

$$\mathrm{Fil}^m_{\rm tot} W(k)[V,F] = \left\{\sum_i a_i\,V^i \in W(k)[V,F]\ \middle|\ \mathrm{ord}_p(a_i)\ge\mathrm{Max}\left(-i,\frac{m-i}{2},0\right)\ \ \forall\, i\right\}$$

for $m\ge 0$. The condition on $\mathrm{ord}_p(a_i)$ is equivalent to: $\mathrm{ord}_p(a_i)\ge\max(-i,0)$ and $2\mathrm{ord}_p(a_i)+i\ge m$. Each $\mathrm{Fil}^m_{\rm tot} W(k)[V,F]$ is an ideal of $W(k)[V,F]$.

(7.5.1) Lemma *Let $\eta : W(k)[V,F] \to \mathrm{Cart}_p(k)$ be the $W(k)$-linear homomorphism such that $\eta(V^i) = V^i$ for all $i \geq 0$, and $\eta(p^i V^{-i}) = F^i$ for all $i \geq 1$.*

(i) *We have*

$$\eta\left(\sum_{i\geq 0} a_i\, V^i + \sum_{i\geq 1} b_i p^i V^{-i}\right) = \sum_{i\geq 0}\sum_{n\geq 0} V^{n+i}\langle a_{in}^{p^i}\rangle F^n + \sum_{i\geq 1}\sum_{n\geq 0} V^n \langle b_{in}\rangle F^{n+i}$$

for all elements $\sum_{i\geq 0} a_i\, V^i + \sum_{i\geq 1} b_i p^i V^{-i} \in W(k)[V,F]$, where $a_i = (a_{in})_{n\geq 0}$, $b_i = (b_{in})_{n\geq 0}$ as Witt vectors. In other words, the a_{in}'s and the b_{in}'s are the Witt components of the Witt vectors a_i and b_i respectively.

(ii) *The map η above is an injective ring homomorphism.*

(iii) *The homomorphism η is strictly compatible for the V-adic filtrations on the source and the target, i.e. $\eta^{-1}(\mathrm{Fil}^i_V \mathrm{Cart}_p(k)) = \mathrm{Fil}^i_V W(k)[V,F]$. Similarly, $\eta^{-1}(\mathrm{Fil}^i_{\mathrm{tot}} \mathrm{Cart}_p(k)) = \mathrm{Fil}^i_{\mathrm{tot}} W(k)[V,F]$.*

Proof. Exercise. ∎

(7.5.2) Lemma (i) *The completion $W[[V,F\rangle\rangle$ of $W(k)[V,F]$ with respect to the filtration $\mathrm{Fil}^\bullet_V W(k)[V,F]$ is the set of all formal series of the form*

$$\sum_{i\in\mathbb{Z}} a_i\, V^i\,, \quad a_i \in K\,,\ \mathrm{ord}_p(a_i) \geq \max(-i,0),\ \forall i \in \mathbb{Z}\,,$$

such that $\lim_{i\to-\infty} \mathrm{ord}_(a_i) + i = \infty$.

(ii) *The V-adic filtration* $\mathrm{Fil}^\bullet_V W(k)[V,F]$ *gives rise to a filtration* $\mathrm{Fil}^\bullet_V W(k)[[V,F\rangle\rangle$ *by open- and closed ideals, via the closure operation. The filtration* $\mathrm{Fil}^\bullet_V W(k)[[V,F\rangle\rangle$ *is strictly compatible with* $\mathrm{Fil}^\bullet_V W(k)[V,F]$.

(iii) *The completion $W(k)[[V,F]]$ of $W(k)[V,F]$ with respect to the filtration* $\mathrm{Fil}^\bullet_{\mathrm{tot}} W(k)[V,F]$ *is the set of all formal series of the form*

$$\sum_{i\in\mathbb{Z}} a_i\, V^i\,, \quad a_i \in K\,,\ \mathrm{ord}_p(a_i) \geq \max(-i,0),\ \forall i \in \mathbb{Z}\,.$$

Notice that $\lim_{i\to-\infty} 2\mathrm{ord}_(a_i) + i = \infty$, *because* $\mathrm{ord}_p(a_i) \geq -i$ *for all negative integers i.*

(iv) *The total filtration* $\mathrm{Fil}^\bullet_{\mathrm{tot}} W(k)[V,F]$ *gives rise to a filtration* $\mathrm{Fil}^\bullet_{\mathrm{tot}} W(k)[[V,F\rangle\rangle$ *by open- and closed ideals, via the closure operation. The filtration* $\mathrm{Fil}^\bullet_{\mathrm{tot}} W(k)[[V,F\rangle\rangle$ *is strictly compatible with* $\mathrm{Fil}^\bullet_{\mathrm{tot}} W(k)[V,F]$.

(v) *The injective homomorphism $\eta : W(k)[V,F] \to \mathrm{Cart}_p(k)$ extends to an isomorphism*

$$\eta : W(k)[[V,F\rangle\rangle \xrightarrow{\sim} \mathrm{Cart}_p(k)$$

of rings which is strictly compatible with the V-adic filtrations.

(vi) *The homomorphism* $\eta : W(k)[V,F] \to \mathrm{Cart}_p(k)$ *extends to a ring isomorphism*

$$\eta : W(k)[[V,F]] \xrightarrow{\sim} \mathrm{Cart}_p(k)^\wedge$$

which is strictly compatible with the filtrations $\mathrm{Fil}^\bullet_{\mathrm{tot}} W(k)[[V,F]]$ *and* $\mathrm{Fil}^\bullet_{\mathrm{tot}} \mathrm{Cart}_p(k)^\wedge$.

PROOF. Exercise. ∎

(7.5.3) Corollary (i) *Multiplication in* $W(k)[[V,F\rangle\rangle$ *is given by the familiar formula*

$$\left(\sum_{i\in\mathbb{Z}} a_i\, V^i\right)\cdot\left(\sum_{j\in\mathbb{Z}} b_j\, V^j\right) = \sum_{n\in\mathbb{Z}}\left(\sum_{i+j=n} a_i\, b_j^{\sigma^{-i}}\right) V^n\,.$$

The infinite sum

$$\sum_{i+j=n} a_i\, b_j^{\sigma^{-i}} = \sum_{j\geq 0} a_{n-j}\, b_j^{\sigma^{j-n}} + \sum_{j>0} a_{n+j} b_{-j}^{\sigma^{-n-j}}$$

in the above formula is convergent because $a_{n-j}\, b_j^{\sigma^{j-n}} \equiv 0 \pmod{p^{j-n}}$ *for all* $j\in\mathbb{N}$, $j\geq n$, *and* $a_{n+j}b_{-j}^{\sigma^{-n-j}} \equiv 0 \pmod{p^j}$ *for all* $j\geq 1$. *Moreover* $\mathrm{ord}(\sum_{i+j=n} a_i\, b_j^{\sigma^{-i}})+n\geq 0$ *if* $n<0$, *so that* $\sum_{n\in\mathbb{Z}}\left(\sum_{i+j=n} a_i\, b_j^{\sigma^{-i}}\right) V^n$ *is an element of* $W(k)[[V,V^{-1}\rangle\rangle$. *Notice that* $u\cdot V = V u^\sigma$ *for every* $u\in W(k)$.

(ii) *Multiplication in* $W(k)[[V,F]]$ *is given by the same formula*

$$\left(\sum_{i\in\mathbb{Z}} a_i\, V^i\right)\cdot\left(\sum_{j\in\mathbb{Z}} b_j\, V^j\right) = \sum_{n\in\mathbb{Z}}\left(\sum_{i+j=n} a_i\, b_j^{\sigma^{-i}}\right) V^n\,.$$

Notice that the infinite sum

$$\sum_{i+j=n} a_i\, b_j^{\sigma^{-i}} = \sum_{j\geq 0} a_{n-j}\, b_j^{\sigma^{j-n}} + \sum_{j>0} a_{n+j} b_{-j}^{\sigma^{-n-j}}$$

in the above formula is again convergent, for the same reason as in (i).

(7.5.4) Corollary *The inverse* $\xi : \mathrm{Cart}_p(k) \xrightarrow{\sim} W(k)[[V,F\rangle\rangle$ *of*

$$\eta : W(k)[[V,F\rangle\rangle \xrightarrow{\sim} \mathrm{Cart}_p(k)$$

is given by the formula

$$\xi \;:\; \sum_{m,n\geq 0} V^m\langle a_{mn}\rangle F^n \;\mapsto\; \sum_{i\in\mathbb{Z}}\left(\sum_{\substack{m-n=i\\ m,n\geq 0}} p^n\,\langle a_{mn}^{p^{-m}}\rangle\right) V^i$$

Under the isomorphism ξ, *the* V*-adic filtration of* $\mathrm{Cart}_p(k)$ *corresponds to the filtration of* $W(k)[[V,V^{-1}\rangle\rangle$ *defined by*

$$\mathrm{Fil}^n_{\mathrm{V}}\left(W(k)[[V,V^{-1}\rangle\rangle\right) = \left\{\sum_{i\in\mathbb{Z}} a_i\, V^i \in W(k)[[V,V^{-1}\rangle\rangle \;\middle|\; \mathrm{ord}_p(a_i)+i\geq n\ \forall\, i\in\mathbb{Z}\right\}.$$

(7.6) Definition (i) Let $\mathrm{Fil}_V^\bullet(W(k)[V,F]\otimes_{W(k)} W(k)[V,F])$ be the filtration on the tensor product $W(k)[V,F]\otimes_{W(k)} W(k)[V,F]$ induced by the filtration $\mathrm{Fil}_V^\bullet$ on

$$\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)$$

and the natural injection

$$W(k)[V,F]\otimes_{W(k)} W(k)[V,F]\hookrightarrow \mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)$$

Denote by $W(k)[V,F]\widehat{\otimes}^V_{W(k)}W(k)[V,F]$ the completion of $W(k)[V,F]\otimes_{W(k)} W(k)[V,F]$ for the filtration $\mathrm{Fil}_V^\bullet(W(k)[V,F]\otimes_{W(k)} W(k)[V,F])$.

(ii) Let $\mathrm{Fil}_{\mathrm{tot}}^\bullet(W(k)[V,F]\otimes_{W(k)} W(k)[V,F])$ be the filtration on $W(k)[V,F]\otimes_{W(k)} W(k)[V,F]$ induced by the filtration $\mathrm{Fil}_{\mathrm{tot}}^\bullet$ on $\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)$ and the natural injection

$$W(k)[V,F]\otimes_{W(k)} W(k)[V,F]\hookrightarrow \mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)$$

Denote by $W(k)[V,F]\widehat{\otimes}^{\mathrm{tot}}_{W(k)}W(k)[V,F]$ the completion of $W(k)[V,F]\otimes_{W(k)} W(k)[V,F]$ for the filtration $\mathrm{Fil}_{\mathrm{tot}}^\bullet(W(k)[V,F]\otimes_{W(k)} W(k)[V,F])$.

(7.6.1) Lemma (i) *For every $n\ge 0$, $\mathrm{Fil}_V^n(W(k)[V,F]\otimes_{W(k)} W(k)[V,F])$ is the $W(k)$-submodule of $W(k)[V,F]\otimes_{W(k)} W(k)[V,F]$ generated by elements of the form*

$$aV^i\otimes V^j\,,\quad a\in W(k)\,,$$

such that

$$\mathrm{ord}_p(a)\ge \max(0,-i)+\max(0,-j),\quad\text{and}\quad \mathrm{ord}_p(a)+i+j\ge n\,.$$

(ii) *For every $n\ge 0$, $\mathrm{Fil}_{\mathrm{tot}}^n(W(k)[V,F]\otimes_{W(k)} W(k)[V,F])$ is the $W(k)$-submodule of the tensor product $W(k)[V,F]\otimes_{W(k)} W(k)[V,F]$ generated by elements of the form*

$$aV^i\otimes V^j\,,\quad a\in W(k)\,,$$

such that

$$\mathrm{ord}_p(a)\ge \max(0,-i)+\max(0,-j),\quad\text{and}\quad 2\mathrm{ord}_p(a)+i+2j\ge n\,.$$

Proof. Omitted. ∎

(7.6.2) Lemma (i) *The V-adic completion $W(k)[V,F]\widehat{\otimes}^V_{W(k)}[V,F]$ of the tensor product $W(k)[V,F]\otimes_{W(k)}[V,F]$ can be naturally identified with the set of all formal double series of the form*

$$\sum_{i,j\in\mathbb{Z}} a_{ij}\,V^i\otimes V^j$$

satisfying the following conditions:

- $a_{ij}\in W(k)\ \forall\, i,j\in\mathbb{Z}$,
- $\mathrm{ord}_p(a_{ij})\ge\max(0,-i)+\max(0,-j)\ \ \forall\, i,j\in\mathbb{Z}$,
- $\lim_{|i|+|j|\to\infty}(\mathrm{ord}_p(a_{ij})+i+j)=\infty$.

Clearly $W(k)[V,F]\widehat{\otimes}^V_{W(k)}[V,F]$ *is a* $(W(k)[[V,F\rangle\rangle$-$W(k)[[V,F\rangle\rangle)$*-bimodule.*

(ii) *The natural embedding*

$$W(k)[V,F]\otimes_{W(k)}[V,F] \hookrightarrow \mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)$$

extends to an isomorphism

$$W(k)[V,F]\widehat{\otimes}^V_{W(k)}[V,F] \xrightarrow{\sim} \mathrm{Cart}_p(k)\widehat{\otimes}^V_{W(k)}\mathrm{Cart}_p(k)\,.$$

Proof. Omitted. ∎

(7.6.3) Lemma (i) *The completion* $W(k)[V,F]\widehat{\otimes}^{\mathrm{tot}}_{W(k)}[V,F]$ *of* $W(k)[V,F]\otimes_{W(k)}[V,F]$ *with respect to the filtration* $\mathrm{Fil}^\bullet_{\mathrm{tot}}(W(k)[V,F]\otimes_{W(k)}W(k)[V,F])$ *can be naturally identified with the set of all formal double series of the form*

$$\sum_{i,j\in\mathbb{Z}} a_{ij}\,V^i\otimes V^j$$

satisfying the following conditions:

- $a_{ij}\in W(k)\ \forall\, i,j\in\mathbb{Z}$,
- $\mathrm{ord}_p(a_{ij})\geq \max(0,-i)+\max(0,-j)\quad \forall\, i,j\in\mathbb{Z}$,
- $\lim_{|i|+|j|\to\infty}(2\,\mathrm{ord}_p(a_{ij})+i+2j)=\infty$.

Moreover, $W(k)[V,F]\widehat{\otimes}^V_{W(k)}[V,F]$ *is a* $(W(k)[[V,F]]$-$W(k)[[V,F\rangle\rangle)$*-bimodule.*

(ii) *The natural embedding*

$$W(k)[V,F]\otimes_{W(k)}[V,F] \hookrightarrow \mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)$$

extends to an isomorphism

$$W(k)[V,F]\widehat{\otimes}^{\mathrm{tot}}_{W(k)}[V,F] \xrightarrow{\sim} \mathrm{Cart}_p(k)\widehat{\otimes}^{\mathrm{tot}}_{W(k)}\mathrm{Cart}_p(k)\,.$$

Proof. Exercise. ∎

(7.7) We would like to give a more explicit description of $\mathrm{BC}_p(k)$, similar to what 7.6.2 and 7.6.3 provided for $\mathrm{Cart}_p(k)\widehat{\otimes}^V_{W(k)}\mathrm{Cart}_p(k)$ and $\mathrm{Cart}_p(k)\widehat{\otimes}^{\mathrm{tot}}_{W(k)}\mathrm{Cart}_p(k)$ respectively. We have seen that $W(k)[V,F]\otimes_{W(k)}W(k)[V,F]$ is in bijection with a subset of $\mathrm{BC}_p(k)$ under the map

$$\alpha' : W(k)[V,F]\otimes_{W(k)}W(k)[V,F]\longrightarrow \mathrm{BC}_p(k)\,,\qquad a\otimes b\ \mapsto\ a{\cdot}U_0{\cdot}b\,.$$

The subset

$$\alpha'(W(k)[V,F]\otimes_{W(k)}W(k)[V,F])$$

is a first approximation to $\mathrm{BC}_p(k)$. We will enlarge it inside $\alpha'(W(k)\otimes_{W(k)}W(k)[V,F])\otimes_{\mathbb{Z}_p}\otimes\mathbb{Q}_p$ to get a dense subset of $\mathrm{BC}_p(k)$.

(7.7.1) Definition Let $\mathrm{BC}_p(k)'$ be the subset of $\mathrm{BC}_p(k)$, defined by

$$\mathrm{BC}_p(k)' := \sum_{n\geq 0} V_x^n \cdot \alpha'(W(k)[V,F]\otimes_{W(k)} W(k)[V,F])\,.$$

Let $\mathrm{Fil}^{\bullet}_{\mathrm{tot}}\mathrm{BC}_p(k)'$ be the filtration on $\mathrm{BC}_p(k)'$ induced by the filtration $\mathrm{Fil}^{\bullet}_{\mathrm{tot}}\mathrm{BC}_p(k)$ on $\mathrm{BC}_p(k)$.

(7.7.2) Lemma *The endomorphisms $[p]$ on $\mathrm{Cart}_p(k)\hat{\otimes}_{W(k)}\mathrm{Cart}_p(k)$ and $\mathrm{BC}_p(k)$ defined by multiplication with p are both injections. Therefore all arrows in the commutative diagram*

$$\begin{array}{ccccc}
W(k)[V,F]\otimes_{W(k)}W(k)[V,F] & \longrightarrow & \mathrm{Cart}_p(k)\widehat{\otimes}^{\mathrm{tot}}_{W(k)}\mathrm{Cart}_p(k) & \xrightarrow{\widehat{\alpha}_{\mathrm{tot}}} & \mathrm{BC}_p(k) \\
\Big\downarrow{\scriptstyle [p]} & & \Big\downarrow{\scriptstyle [p]} & & \Big\downarrow{\scriptstyle [p]} \\
W(k)[V,F]\otimes_{W(k)}W(k)[V,F]\otimes_{\mathbb{Z}}\mathbb{Q} & \longrightarrow & \mathrm{Cart}_p(k)\widehat{\otimes}^{\mathrm{tot}}_{W(k)}\mathrm{Cart}_p(k)\otimes_{\mathbb{Z}}\mathbb{Q} & \xrightarrow{\widehat{\alpha}_{\mathrm{tot}}} & \mathrm{BC}_p(k)\otimes_{\mathbb{Z}}\mathbb{Q}
\end{array}$$

are injections, where $\alpha_{\mathbb{Q}} = \alpha \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof. The injectivity of $\mathrm{BC}_p(k)\xrightarrow{[p]}\mathrm{BC}_p(k)$ is easy, since

$$\mathrm{BC}_p(k) \subseteq \mathrm{Cart}_p(k[[x]]^+) = \Lambda(k[[x,y]]^{\ddagger})\,,$$

and $[p] : \mathrm{Cart}_p(k[[x]]^+) \to \mathrm{Cart}_p(k[[x]]^+)$ is injective. The injectivity of

$$\mathrm{Cart}_p(k)\widehat{\otimes}^{\mathrm{tot}}_{W(k)}\mathrm{Cart}_p(k)\xrightarrow{[p]}\mathrm{Cart}_p(k)\widehat{\otimes}^{\mathrm{tot}}_{W(k)}\mathrm{Cart}_p(k)$$

is immediate from Lemma 7.6.2. ∎

(7.8) With Lemma 7.7.2, we can identify $\mathrm{BC}_p(k)'$ with a suitable subset of

$$W(k)[V,F] \otimes_{W(k)} W(k)[V,F] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\,.$$

This subset $\mathrm{BC}_p(k)'$ of $\mathrm{BC}_p(k)$ is dense in $\mathrm{BC}_p(k)$ with respect to the filtration $\mathrm{Fil}^{\bullet}_{\mathrm{tot}}\mathrm{BC}_p(k)$. From this we will obtain an explicit description of the completion $\mathrm{BC}_p(k)$ of $\mathrm{BC}_p(k)'$, to be worked out in the rest of this section.

(7.8.1) Lemma *The following operators on the tensor product $\mathrm{Cart}_p(k) \otimes_{\mathbb{Z}} \mathrm{Cart}_p(k)$ passes to $\mathrm{Cart}_p(k) \otimes_{W(k)} \mathrm{Cart}_p(k)$, $\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k) \otimes_{\mathbb{Z}} \mathbb{Q}$, as well as $\mathrm{Cart}_p(k)\widehat{\otimes}^{V}_{W(k)}\mathrm{Cart}_p(k)$ and $\mathrm{Cart}_p(k)\widehat{\otimes} \otimes^{\mathrm{tot}}_{W(k)} \mathrm{Cart}_p(k)$, and define operators on these modules.*

$$\begin{array}{ccccll}
F_B & : & \mathrm{Cart}_p(k)\otimes_{\mathbb{Z}}\mathrm{Cart}_p(k) & \longrightarrow & \mathrm{Cart}_p(k)\otimes_{\mathbb{Z}}\mathrm{Cart}_p(k) & \\
 & & u\otimes v & \mapsto & u\cdot V\otimes F\cdot v & \forall\, u,v\in \mathrm{Cart}_p(k)
\end{array}$$

$$\begin{array}{ccccll}
\langle a\rangle_B & : & \mathrm{Cart}_p(k)\otimes_{\mathbb{Z}}\mathrm{Cart}_p(k) & \longrightarrow & \mathrm{Cart}_p(k)\otimes_{\mathbb{Z}}\mathrm{Cart}_p(k) & \\
 & & u\otimes v & \mapsto & u\cdot \langle a\rangle\otimes v & \forall\, a\in k,\ \forall\, u,v\in \mathrm{Cart}_p(k)
\end{array}$$

$$\begin{array}{ccccll}
w_B & : & \mathrm{Cart}_p(k)\otimes_{\mathbb{Z}}\mathrm{Cart}_p(k) & \longrightarrow & \mathrm{Cart}_p(k)\otimes_{\mathbb{Z}}\mathrm{Cart}_p(k) & \\
 & & u\otimes v & \mapsto & u\cdot w\otimes v & \forall\, w\in W(k),\ \forall\, u,v\in \mathrm{Cart}_p(k)
\end{array}$$

Similarly, the operator

$$\begin{array}{rcccl} V_B & : & \mathrm{Cart}_p(k)\otimes_{\mathbb Z}\mathrm{Cart}_p(k)\otimes_{\mathbb Z}\mathbb Q & \longrightarrow & \mathrm{Cart}_p(k)\otimes_{\mathbb Z}\mathrm{Cart}_p(k)\otimes_{\mathbb Z}\mathbb Q \\ & & u\otimes v\otimes 1 & \mapsto & u\cdot F\otimes V\cdot v\otimes p^{-1} \qquad \forall\, u,v\in \mathrm{Cart}_p(k) \end{array}$$

passes to the modules $\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)\otimes_{\mathbb Z}\mathbb Q$, $\mathrm{Cart}_p(k)\widehat{\otimes}^{V}_{W(k)}\mathrm{Cart}_p(k)\otimes_{\mathbb Z}\mathbb Q$*, and* $\mathrm{Cart}_p(k)\widehat{\otimes}^{\mathrm{tot}}_{W(k)}\mathrm{Cart}_p(k)\otimes_{\mathbb Z}\mathbb Q$.

PROOF. The main issue here is to verify that these operators preserve the subgroup generated by elements of the form $uc\otimes v-u\otimes cv$, where $u,v\in\mathrm{Cart}_p(k)$, $c\in W(k)$, or $c\in W(k)\otimes_{\mathbb Z}\mathbb Q$ in the last case. This is clear for the $\langle a\rangle_B$'s. The rest follows from the following easy calculations:

$$\begin{aligned} F_B(uc\otimes v-u\otimes cv) &= ucV\otimes Fv-uV\otimes Fcv = uVc^{\sigma}\otimes Fv-uV\otimes c^{\sigma}Fv\\ w_B(uc\otimes v-u\otimes cv) &= ucw\otimes v-uw\otimes cv = uwc\otimes v-uw\otimes cv\\ V_B(uc\otimes v-u\otimes cv) &= p^{-1}ucF\otimes Vv-p^{-1}uF\otimes Vcv = p^{-1}uFc^{\sigma^{-1}}\otimes Vv-p^{-1}uF\otimes c^{\sigma^{-1}}Vv \end{aligned}$$

Therefore these operators pass to $\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)$, or $\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)\otimes_{\mathbb Z}\mathbb Q$ in the case of V_B. Each operator is continuous with respect to the topology defined by $\mathrm{Fil}^{\bullet}_V$ and $\mathrm{Fil}^{\bullet}_{\mathrm{tot}}$, therefore they extend uniquely to $\mathrm{Cart}_p(k)\hat\otimes^{V}_{W(k)}\mathrm{Cart}_p(k)$ and $\mathrm{Cart}_p(k)\hat\otimes^{\mathrm{tot}}_{W(k)}\mathrm{Cart}_p(k)$, or $\mathrm{Cart}_p(k)\hat\otimes^{V}_{W(k)}\mathrm{Cart}_p(k)\otimes_{\mathbb Z}\mathbb Q$ and $\mathrm{Cart}_p(k)\hat\otimes^{\mathrm{tot}}_{W(k)}\mathrm{Cart}_p(k)\otimes_{\mathbb Z}\mathbb Q$ in the case for V_B. ∎

(7.8.2) Lemma *Let* $\alpha_{\mathbb Q}=\alpha\otimes_{\mathbb Z}\mathbb Q:\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)\otimes_{\mathbb Z}\mathbb Q\to\mathrm{BC}_p(k)\otimes_{\mathbb Z}\mathbb Q$ *be the map induced by* α*. Then*

$$\begin{array}{rcl} \alpha_{\mathbb Q}\circ F_B &=& F_x\circ\alpha_{\mathbb Q}\\ \alpha_{\mathbb Q}\circ V_B &=& V_x\circ\alpha_{\mathbb Q}\\ \alpha_{\mathbb Q}\circ\langle a\rangle_B &=& \langle a\rangle_x\circ\alpha_{\mathbb Q} \end{array}$$

for every $a\in k$*, where* F_x*,* V_x *and* $\langle a\rangle_x$ *appearing at the right of the displayed equalities denote the endomorphisms of* $\mathrm{BC}_p(k)\otimes_{\mathbb Z}\mathbb Q$ *induced by the endomorphisms* F_x*,* V_x *and* $\langle a\rangle_x$ *of* $\mathrm{BC}_p(k)$ *respectively.*

PROOF. The source and the target of the maps

$$F_B,\ V_B,\ \langle a\rangle_B,\ F_x, V_x,\ \langle x\rangle_x$$

are $(\mathrm{Cart}_p(k)$-$\mathrm{Cart}_p(k))$-bimodules, and the above maps are all endomorphisms of bimodules. Hence it suffices to check the assertions when both sides of the equalities are applied to the element $1\otimes 1\in\mathrm{Cart}_p(k)\otimes\mathrm{Cart}_p(k)\otimes_{\mathbb Z}\mathbb Q$. which is immediate from Prop. 5.5. ∎

(7.8.3) Lemma *The operators* F_B *and* w_B *on* $Wk)[V,F]\otimes_{W(k)}Wk)[V,F]$*,* $w\in W(k)$*, given by the formulae below, are well-defined.*

$$\begin{array}{rcl} F_B\ :\ W(k)[V,F]\otimes_{W(k)}W(k)[V,F] & \longrightarrow & W(k)[V,F]\otimes_{W(k)}W(k)[V,F]\\ u\otimes v & \mapsto & uV\otimes pV^{-1}v \qquad \forall\,u,v\in W(k)[F,V]\\ w_B\ :\ W(k)[V,F]\otimes_{W(k)}W(k)[V,F] & \longrightarrow & W(k)[V,F]\otimes_{W(k)}W(k)[V,F]\\ u\otimes v & \mapsto & uw\otimes v \qquad \forall\,u,v\in W(k)[F,V] \end{array}$$

Similarly, the operator V_B *on* $Wk)[V,F]\otimes_{W(k)}Wk)[V,F]\otimes\mathbb{Z}\mathbb{Q}$*, given by*

$$\begin{array}{rcl} V_B\ :\ W(k)[V,F]\otimes_{W(k)}W(k)[V,F]\otimes_{\mathbb Z}\mathbb Q & \longrightarrow & W(k)[V,F]\otimes_{W(k)}W(k)[V,F]\otimes_{\mathbb Z}\mathbb Q\\ u\otimes v\otimes 1 & \mapsto & u\,pV^{-1}\otimes V\,v\otimes p^{-1} \end{array}$$

for all $u,v\in W(k)[F,V]$*, is well-defined.*

Proof. The same calculation as in 7.8.1 work. ∎

(7.8.4) Lemma *Let $\alpha'_{\mathbb{Q}} = \alpha \otimes_{\mathbb{Z}} \mathbb{Q} : W(k)[V,F] \otimes_{W(k)} W(k)[V,F] \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{BC}_p(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ be the map induced by α'. Then*

$$\begin{array}{rcl} \alpha_{\mathbb{Q}} \circ F_{B} & = & F_x \circ \alpha'_{\mathbb{Q}} \\ \alpha_{\mathbb{Q}} \circ V_{B} & = & V_x \circ \alpha'_{\mathbb{Q}} \\ \alpha_{\mathbb{Q}} \circ \langle a \rangle_{B} & = & \langle a \rangle_x \circ \alpha'_{\mathbb{Q}} \end{array}$$

for every $a \in k$, where F_x, V_x and $\langle a \rangle_x$ appearing at the right of the displayed equalities denote the endomorphisms of $\mathrm{BC}_p(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ induced by the endomorphisms F_x, V_x and $\langle a \rangle_x$ of $\mathrm{BC}_p(k)$ respectively, while F_B, V_B, $\langle a \rangle_B$ are defined in 7.8.3.

Proof. Exercise. ∎

(7.8.5) Definition Define $B' \subset W(k[V,F] \otimes_{W(k)} W(k[V,F] \otimes_{\mathbb{Z}} \mathbb{Q}$ by

$$B' := \sum_{i \geq 0} V_{B}^{i} \big(W(k[V,F] \otimes_{W(k)} W(k[V,F], \big)$$

the algebraic sum of the $V_{B}^{i}\big(W(k)[V,F] \otimes_{W(k)} W(k)[V,F]\big)$'s. It is easy to see that B' is stable under the actions of F_B, V_B, w_B on $W(k)[V,F] \otimes_{W(k)} W(k)[V,F] \otimes_{\mathbb{Z}} \mathbb{Q}$, and is also sub-bimodule for the natural $(W(k)[V,F]$-$W(k)[V,F]$-bimodule structure on $W(k)[V,F] \otimes_{W(k)} W(k)[V,F]$. Clearly the map α' induces an isomorphism $B' \xrightarrow{\sim} \mathrm{BC}_p(k)'$.

(7.8.6) Lemma (i) *The map $\alpha' : W(k[V,F] \otimes_{W(k)} W(k[V,F] \to \mathrm{BC}_p(k)$ induces and isomorphism*

$$\alpha' : B' \xrightarrow{\sim} \mathrm{BC}_p(k)',$$

under which the operators F_B, V_B, w_B on B'

(ii) *Under the isomorphism α' in* (i)*, the operators F_B, V_B, w_B on B', $w \in W(k)$, corresponds to the operators F_x, V_x w_x on $\mathrm{BC}_p(k)'$.*

Proof. Exercise. ∎

(7.8.7) Definition Denote by $\tilde{B}$ the space of all formal linear combinations of the form

$$\sum_{i,j \in \mathbb{Z}} a_{ij} \, V^i \otimes V^j, \qquad a_{ij} \in K = \mathrm{frac}(W(k)) \; \forall \, i,j \in \mathbb{Z}.$$

Define $V_{\tilde{B}} : \tilde{B} \to \tilde{B}$ to be the operator

$$V_{\tilde{B}} \; : \; \sum_{i,j \in \mathbb{Z}} a_{ij} \, V^i \otimes V^j \; \mapsto \; \sum_{i,j \in \mathbb{Z}} a_{ij} \, V^{i-1} \otimes V^{j+1}$$

on $\tilde{B}$. Let $F_{\tilde{B}} \tilde{B} \to \tilde{B}$ be the operator

$$F_{\tilde{B}} \; : \; \sum_{i,j \in \mathbb{Z}} a_{ij} \, V^i \otimes V^j \; \mapsto \; \sum_{i,j \in \mathbb{Z}} p \, a_{ij} \, V^{i+1} \otimes V^{j-1}$$

(7.9) Consider the following commutative diagram

$$\begin{array}{ccccc} W(k)[V,F](k)\otimes_{W(k)}W(k)[V,F] & \hookrightarrow & B' & \longrightarrow & W(k)[V,F](k)\otimes_{W(k)}W(k)[V,F](k)\otimes_{\mathbb{Z}}\mathbb{Q} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k) & \xrightarrow{\alpha} & \mathrm{BC}_p(k) & \overset{\imath}{\dashrightarrow} & \tilde{B} \end{array}$$

of injections, where the vertical arrows are the obvious maps, and the dotted arrow $\imath$ is an embedding of $\mathrm{BC}_p(k)$ into $\tilde{B}$, to be constructed in the remaining part of this section, with the property that the action of $V_{\tilde{B}}$ (resp. $F_{\tilde{B}}$) on $\tilde{B}$ extends the action of V_x (resp. F_x on $\mathrm{BC}_p(k)$. We will also obtain an explicit description of the image $\imath(\mathrm{BC}_p(k)$ of $\imath$ in $\tilde{B}$.

Recall that in 7.6.2 and 7.6.3 we constructed embeddings

$$\mathrm{Cart}_p(k)\widehat{\otimes}^{V}_{W(k)}\mathrm{Cart}_p(k) \hookrightarrow \tilde{B}$$

and

$$\mathrm{Cart}_p(k)\widehat{\otimes}^{\mathrm{tot}}_{W(k)}\mathrm{Cart}_p(k) \hookrightarrow \tilde{B}\,.$$

These two embeddings are compatible with the action of the completion of V_B on the source and the action of $V_{\tilde{B}}$ on the target $\tilde{B}$. We will see that the restriction of these two embeddings to $\mathrm{Cart}_p(k)\otimes_{W(k)}\mathrm{Cart}_p(k)$ are both equal to the composition $\imath\circ\alpha$.

In order to construct the embedding $\imath$, we will first compute the restriction to B' of the total filtration $\mathrm{Fil}^{\bullet}_{\mathrm{tot}}\mathrm{BC}_p(k)$ on $\mathrm{BC}_p(k)$. Then we can reconstruct $\mathrm{BC}_p(k)$ as the completion of B' with respect to the total filtration on B'.

(7.9.1) Lemma *For any integers $i,j\in\mathbb{Z}$, define a function $f_{i,j}:\mathbb{N}\to\mathbb{N}$ by*

$$f_{i,j}(n)=\max(-i-n,0)+\max(-j+n,0),\qquad n\in\mathbb{N}$$

Then

$$\mathrm{Min}_{n\geq 0}\, f_{i,j}(n)=\max(-i-j,-j,0)$$

Proof. We evaluate both sides for (i,j) in different regions of $\mathbb{Z}^2$ and show that they are equal. Consider first the case when $j\geq 0$ and $i+j\geq 0$. Then $f_{i,j}(j)=0$. So $\mathrm{Min}_{n\geq 0}\, f_{i,j}(n)=0$. On the other hand, $\max(-i-j,-j,0)=0$ as well.

Consider the second case: $j\geq 0$ and $i+j\leq 0$. Then

$$f_{i,j}(n)=\max(-i-n,0)+\max(-j+n,0)\geq -i-n-j+n=-i-j$$

for all $n\in\mathbb{N}$, and $f_{i,j}(j)=-i-j$. So $\mathrm{Min}_{n\geq 0}\, f_{i,j}(n)=-i-j$. On the other hand, $\max(-i-j,-j,0)=-i-j$.

Consider the third case: $j\leq 0$ and $i\geq 0$. Then $f_{i,j}(n)=\max(n-j,0)$ for all $n\in\mathbb{N}$, so $\mathrm{Min}_{n\geq 0}\, f_{i,j}(n)=0$. On the other hand, $\max(-i-j,-j,0)=0$.

Finally we consider the case when $j\leq 0$ and $i\leq 0$. Then $f_{i,j}(n)=-i-j$ if $0\leq n\leq -i$, and $f_{i,j}(n)=-j+n$ if $n>-i$. So $\mathrm{Min}_{n\geq 0}\, f_{i,j}(n)=-i-j$. On the other hand, $\max(-i-j,-j,0))=-i-j$, and again both sides are equal. ∎

(7.9.2) Corollary *Recall that B' is the algebraic sum*

$$B' = \sum_{n\geq 0} V_B^n \cdot W(k)[V,F] \otimes_{W(k)} W(k)[V,F] \subset W(k)[V,F] \otimes_{W(k)} W(k)[V,F] \otimes_{\mathbb{Z}} \mathbb{Q}\,.$$

Then

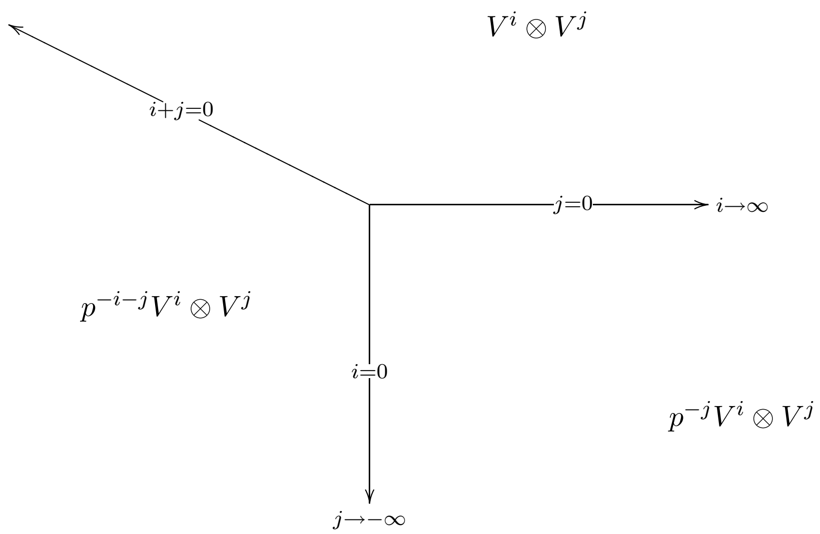
$$B' = \left\{ \sum_{\substack{i,j\in\mathbb{Z} \\ \text{finite}}} a_{ij} V^i \otimes V^j \;\middle|\; \begin{array}{l} a_{ij} \in W(k) \quad \forall\, i,j \in \mathbb{Z} \\ \mathrm{ord}_p(a_{ij}) \geq \max(-j, -i-j, 0) \quad \forall\, i,j \in \mathbb{Z} \end{array} \right\}.$$

Proof. Immediate from Lemma 7.9.1. ∎

Remark The following diagram depicts a $W(k)$-basis of B',

$$B' \subset W(k)[V,F] \otimes_{W(k)} W(k)[V,F] \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{i,j\in\mathbb{Z}} K \cdot V^i \otimes V^j\,,$$

shown for pairs (i,j) at different locations in $\mathbb{Z}^2$:



(7.10) Proposition *Consider the composition*

$$\beta:\; B' \xrightarrow{\cong} \mathrm{BC}_p(k)' \hookrightarrow \mathrm{BC}_p(k)$$

of injections. The following statements hold.

(i) *The inverse image of the subset*

$$\begin{aligned} &\mathrm{Fil}^N_{\mathrm{tot}}\left(\mathrm{BC}_p(k)\right) \\ &= \left\{ \sum_{\substack{m,n,j\geq 0 \\ m+n\geq N}} V^m \langle a_{mnj}\rangle\, U_n\, F^j \;\middle|\; \begin{array}{l} a_{mnj} \in k \quad \forall\, m,n,j \in \mathbb{N} \text{ with } m+n \geq N \\ \forall\, m,n \geq 0\; \exists\, C(m,n) \text{ s.t. } a_{mnj} = 0\; \forall\, j > C(m,n) \end{array} \right\} \end{aligned}$$

of $\mathrm{BC}_p(k)$ *under* β *is*

$$\mathrm{Fil}^N_{\mathrm{tot}} B' = \left\{ \sum_{i,j\in\mathbb{Z}} a_{ij}\, V^i \otimes V^j \;\middle|\; \begin{array}{l} a_{ij} \in W(k) \quad \forall\, i,j \in \mathbb{Z} \\ a_{ij} = 0 \text{ for almost all } i,j \in \mathbb{Z} \\ \mathrm{ord}_p(a_{ij}) \geq \max(-i-j, -j, 0)\ \forall\, i,j \\ 2\mathrm{ord}_p(a_{ij}) + i + 2j \geq N \end{array} \right\}.$$

(ii) *The image* $\beta(B')$ *of* β *is dense in* $\mathrm{BC}_p(k)$ *with respect to the topology of* $\mathrm{BC}_p(k)$ *defined by the filtration* $\mathrm{Fil}^\bullet_{\mathrm{tot}}\mathrm{BC}_p(k)$.

(iii) *The map* β *induces a bijection* $\hat\beta$ *from the set*

$$(B')^\wedge = \left\{ \sum_{i,j\in\mathbb{Z}} a_{ij}\, V^i \otimes V^j \;\middle|\; \begin{array}{l} a_{ij} \in W(k) \quad \forall\, i,j \in \mathbb{Z} \\ \mathrm{ord}_p(a_{ij}) \geq \max(-j, -i-j, 0)\ \forall\, i,j \in \mathbb{Z} \\ \lim_{|i|+|j|\to\infty} 2\mathrm{ord}_p(a_{ij}) + i + 2j = \infty \end{array} \right\}$$

to $\mathrm{BC}_p(k)$.

Proof. The point of (i) is that

$$\beta^{-1}(V^m\langle a\rangle U_n F^j) = p^j V^{m-n}\langle a^{p^{-n}}\rangle \otimes V^{n-j} \qquad \forall\, m,n,j \geq 0,\ \forall\, a \in k\,.$$

The rest of (i) follows from the definitions.

The statement (ii) follows from the fact that the subgroup of $\mathrm{BC}_p(k)$ generated by the subset

$$\left\{ V^m\langle a\rangle U_n F^i \;\middle|\; a \in k,\ m,n,i \geq 0 \right\}$$

is dense in $\mathrm{BC}_p(k)$, and this subset is contained in the image of β.

According to statement (ii), $\mathrm{BC}_p(k)$ can be naturally identified with the completion of B' with respect to the filtration $\mathrm{Fil}^\bullet_{\mathrm{tot}}B'$, since $\mathrm{BC}_p(k)$ is complete with respect to the total filtration. The statement (iii) follows from the description in (i) of the fundamental system of neighborhoods

$$\left\{\ \beta^{-1}\left(\mathrm{Fil}^N_{\mathrm{tot}}\mathrm{BC}_p(k)\right) \;\middle|\; N \in \mathbb{N}\,\right\}$$

for the induced topology on B'. ∎

(7.10.1) Remark Under the bijection in 7.10 (iii), the ($\mathrm{Cart}_p(k)$-$\mathrm{Cart}_p(k)$)-bimodule structure on $\mathrm{BC}_p(k)$ corresponds to the obvious actions of the ring

$$W(k)[[V,F\rangle\rangle = \left\{ \sum_{i\in\mathbb{Z}} a_i V^i \;\middle|\; \begin{array}{l} a_i \in W(k) \quad \forall\, i \in \mathbb{Z} \\ \mathrm{ord}_p(a_i) + i \geq 0\ \forall\, i \leq 0 \\ \lim_{|i|\to\infty} \mathrm{ord}_p(a_i) + i = \infty \end{array} \right\}$$

on both the right and on the left of the set $(B')^\wedge$ above. We have

$$V^b \cdot \left(\sum_{i,j\in\mathbb{Z}} a_{ij} V^i \otimes V^j \right) \cdot V^c = \sum_{i,j\in\mathbb{Z}} a_{ij}^{\sigma^{-b}} V^{b+i} \otimes V^{\sigma^{j+c}}\,.$$

The action of the "extra copy" of $\mathrm{Cart}_p(k)$ corresponds to the continuous action of $\mathrm{Cart}_p(k)$ on $(B')^\wedge$, such that V_x, F_x, $\langle c\rangle_x$ acts via the following formulas

$$\begin{array}{rcl} V_{\tilde B} &:& \sum_{i,j\in\mathbb{Z}} a_{ij}\, V^i \otimes V^j \;\mapsto\; \sum_{i,j\in\mathbb{Z}} a_{ij}\, V^{i-1} \otimes V^{j+1} \\ F_{\tilde B} &:& \sum_{i,j\in\mathbb{Z}} a_{ij}\, V^i \otimes V^j \;\mapsto\; \sum_{i,j\in\mathbb{Z}} p a_{ij}\, V^{i+1} \otimes V^{j-1} \\ \langle c\rangle_{\tilde B} &:& \sum_{i,j\in\mathbb{Z}} a_{ij} V^i \otimes V^j \;\mapsto\; \sum_{i,j\in\mathbb{Z}} a_{ij} \cdot \langle c^{p^{-i}}\rangle\, V^i \otimes V^j \qquad \forall\, c \in k \end{array}$$

§8. Computation up to isogeny

(8.1) Notation Let k be a perfect field of characteristic $p > 0$. Let $N_{\mathbb{Q}_p}$ be an isoclinic V-isocrystal of dimension h over $K = \mathrm{frac}(W(k))$ with V-slope μ, $0 < \mu \leq 1$. Let $h = \dim_K(N_{\mathbb{Q}_p})$. If $N_{\mathbb{Q}_p}$ is the V-isocrystal attached to the Cartier module N of a p-divisible formal group Y, then Y is isoclinic of Frobenius slope μ, $\mathrm{height}(Y) = h$, $\dim(Y) = \mu\, h$.

Let $N_{\mathbb{Z}_p}$ be a $W(k)$-lattice in N. Since $N_{\mathbb{Q}_p}$ is isoclinic of slope μ, there exist constants $c_1, c_2 \geq 0$ such that

$$\begin{array}{rcccl} p^{\lfloor n\mu \rfloor + c_1} \cdot N_{\mathbb{Z}_p} & \subseteq & V^n \cdot N_{\mathbb{Z}_p} & \subseteq & p^{\lfloor n\mu \rfloor - c_2} \cdot N_{\mathbb{Z}_p} \\ p^{\lfloor n(1-\mu) \rfloor + c_1} \cdot N_{\mathbb{Z}_p} & \subseteq & F^n \cdot N_{\mathbb{Z}_p} & \subseteq & p^{\lfloor n(1-\mu) \rfloor - c_2} \cdot N_{\mathbb{Z}_p} \end{array}$$

for all $n \in \mathbb{N}$.

(8.2) We formulate two combinatorial lemmas which will be used for an explicit description of the tensor product $\mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}$.

(8.2.1) Lemma *Let f be the function from $\mathbb{Z}^2$ to $\mathbb{N}$ defined by*

$$f(i,j) := \max(-i-j, -j, 0) \qquad \forall\, i,j \in \mathbb{Z}\,.$$

Define a function $g : \mathbb{Z} \to \mathbb{Z}$ by

$$g(i) := \mathrm{Min}\,\{\, f(i,j) + j\mu \,|\, j \in \mathbb{Z}\,\} \qquad \forall\, i \in \mathbb{Z}\,.$$

Then $g(i) = \max(-i\mu, 0)$ for all $i \in \mathbb{Z}$.

Proof. Let b be the denominator of μ. Then $f(i,j) + j\mu \in \frac{1}{b}\mathbb{N}$ for all $j \in \mathbb{N}$. Therefore the minimum in the definition of $g(i)$ exists.

We have $f(i,j) + j\mu = (f(i,j) + j)\mu + (1-\mu) f(i,j) \geq \max(-i\mu, 0)$ for all $j \in \mathbb{Z}$, since $f(i,j) + j \geq \max(-i, 0)$ for all $j \in \mathbb{Z}$. The minimum is attained at $j = 0$ if $i \geq 0$, and at $j = -i$ if $i < 0$. ∎

(8.2.2) Lemma *For each natural number $N \geq 0$, define a function $g_N : \mathbb{Z} \to \mathbb{Z}$ by*

$$g_N(i) := \mathrm{Min}\,\{\, m + j\mu \,|\, m \geq \max(-i-j, -j, 0),\ 2m + i + 2j \geq N \,\}\ .$$

Then $g_N(i) = \max\left(\lceil \frac{N-i}{2} \rceil, -i\mu, 0\right)$ for all $i \in \mathbb{Z}$.

Proof. The constraints on (m,j) are $m \geq 0$, $m + j \geq \max(-i, 0)$, and $2(m+j) \geq N - i$. Therefore $m + j \geq \max\left(\lceil \frac{N-i}{2} \rceil, -i, 0\right)$, and $m + j\mu \geq \max\left(\lceil \frac{N-i}{2} \rceil \mu, -i\mu, 0\right)$. The minimum is attained at $(m,j) = \left(0, \max\left(\lceil \frac{N-i}{2} \rceil\right), -i, 0\right)\big)$. ∎

(8.3) Consider the map

$$\alpha_{N_{\mathbb{Q}_p}} : W(k)[V,F] \otimes_{W(k)} W(k)[V,F] \otimes_{W(k)[V,F]} N_{\mathbb{Q}_p} \longrightarrow \mathrm{BC}_p(k)' \otimes_{W(k)[V,F]} N_{\mathbb{Q}_p}$$

defined by

$$\alpha_{N_{\mathbb{Q}_p}} : a \otimes b \otimes w \ \mapsto\ a \cdot U_0 \cdot b \otimes w \qquad \forall\, a,b \in W(k)[V,F],\ \forall\, w \in N_{\mathbb{Q}_p}\,.$$

It is clear that $\alpha_{N_{\mathbb{Q}_p}}$ induces an isomorphism from $B' \otimes_{W(k)[V,F]} N_{\mathbb{Q}_p}$ to $\mathrm{BC}_p(k)' \otimes_{W(k)[V,F]} N_{\mathbb{Q}_p}$. We have a canonical isomorphism

$$\mathrm{can}_{N_{\mathbb{Q}_p}} : W(k)[V,F] \otimes_{W(k)} W(k)[V,F] \otimes_{W(k)[V,F]} N_{\mathbb{Q}_p} \xrightarrow{\sim} W(k)[V,F] \otimes_{W(k)} N_{\mathbb{Q}_p}$$

Therefore we have an isomorphism

$$\gamma' : \mathrm{BC}_p(k)' \otimes_{W(k)[V,F]} N_{\mathbb{Q}_p} \xrightarrow{\sim} W(k)[V,F] \otimes_{W(k)} N_{\mathbb{Q}_p}$$

induced by $\mathrm{can}_{N_{\mathbb{Q}_p}} \circ \alpha^{-1}_{N_{\mathbb{Q}_p}}$.

(8.3.1) Let $K[V,V^{-1}] := W(k)[V,F] \otimes_{W(k)} K$, so that we have a canonical isomorphism

$$W(k)[V,F] \otimes_{W(k)} N_{\mathbb{Q}_p} \xrightarrow{\sim} K[V,V^{-1}] \otimes_K N_{\mathbb{Q}_p}\,.$$

It is immediate from the definition of $W(k)[V,F]$ that $K[V,V^{-1}]$ can be naturally identified with the set of all finite K-linear combinations of monomials V^i, $i \in \mathbb{Z}$. If we choose a K-basis of $w_1, \ldots, w_h$ of $N_{\mathbb{Q}_p}$, then every element of $K[V,V^{-1}] \otimes_K N_{\mathbb{Q}_p}$ can be written as a finite series of the form

$$\sum_{\substack{i\in\mathbb{Z}\\ 1\le r\le h}} a_{ir} V^i \otimes w_r\,, \qquad a_{ir} \in K\ \forall\, i \in \mathbb{Z},\ \forall r = 1,\ldots,h;\quad a_{ir} = 0 \text{ for } |i| \gg 0$$

in a unique way.

(8.3.2) A coordinate-free way to describe elements of $K[V,V^{-1}] \otimes_K N_{\mathbb{Q}_p}$ is as follows. Define $[V,V^{-1}]N_{\mathbb{Q}_p}$ to be the set of all finite series of the form

$$\sum_{i\in\mathbb{Z}} A_i\, V^i$$

where $A_i : K \to N_{\mathbb{Q}_p}$ is a σ^i-linear map from K to $N_{\mathbb{Q}_p}$, i.e. $A_i(b) = b^{\sigma^i} A_i(1)$ for every $b \in K$, and $A_i = 0$ for $|i| \gg 0$. Then there is a canonical isomorphism from $K[V,V^{-1}] \otimes_K N_{\mathbb{Q}_p}$ to $[V,V^{-1}]N_{\mathbb{Q}_p}$. An element $\sum_{\substack{i\in\mathbb{Z}\\ 1\le r\le h}} a_{ir} V^i \otimes w_r$ of $K[V,V^{-1}] \otimes_K N_{\mathbb{Q}_p}$ as in 8.3.1 above corresponds to the element $(A_i)_{i\in\mathbb{Z}}$ with

$$A_i(c) = \sum_{r=1}^{h} c^{\sigma^i}\, a_{ir}^{\sigma^i}\, w_r\,.$$

(8.3.3) We have a natural left action of $K[V,V^{-1}]$ on $[V,V^{-1}]N_{\mathbb{Q}_p}$, such that

$$(b\,V^j) \cdot (A_i)_{i\in\mathbb{Z}} = \left(A'_i\right)_{i\in\mathbb{Z}} \qquad b \in K,\ j \in \mathbb{Z}$$

with the A'_i's defined by

$$A'_i(c) = c^i\, b^i A_{i-j}(1) \qquad \forall\, b,c \in K,\ \forall\, i \in \mathbb{Z}\,.$$

(8.3.4) There is an action of an "extra copy" of $K[V, V^{-1}]$ on the left of $[V, V^{-1}]N_{\mathbb{Q}_p}$ which commutes with the action of $K[V, V^{-1}]$ defined in 8.3.3; it comes from the action of the "extra copy" of $W(k)[V, F]$ on B'. We will use a subscript "B" when referring to this action in formulas. This action is given by

$$(b\,V_j)_B \cdot_B (A_i)_{i\in\mathbb{Z}} = \left(\tilde{A}_i\right)_{i\in\mathbb{Z}} \qquad b \in K,\ j \in \mathbb{Z}$$

where the $\tilde{A}_j$'s are given by

$$\tilde{A}_j(c) = c^{\sigma^i}\, b\, V^j \cdot A_{i+j}(1)\,.$$

(8.3.5) Let $N_{\mathbb{Z}_p}$ be a $W(k)$-lattice of $N_{\mathbb{Q}_p}$. Then the total filtration $\mathrm{Fil}^{\bullet}_{\mathrm{tot}}$ on $\mathrm{BC}_p(k)'$ induces a filtration $\mathrm{Fil}^{\bullet}_{\mathrm{tot},N_{\mathbb{Z}_p}}$ on $\mathrm{BC}_p(k)' \otimes_{W(k)[V,F]} N_{\mathbb{Q}_p}$, defined by

$$\mathrm{Fil}^{m}_{\mathrm{tot},N_{\mathbb{Z}_p}}\big(\mathrm{BC}_p(k)' \otimes_{W(k)[V,F]} N_{\mathbb{Q}_p}\big) := \mathrm{Image}\begin{pmatrix} \mathrm{Fil}^m_{\mathrm{tot}}\mathrm{BC}_p(k)' \otimes_{W(k)[V,F]} N_{\mathbb{Z}_p} \\ \longrightarrow \mathrm{BC}_p(k)' \otimes_{W(k)[V,F]} N_{\mathbb{Q}_p} \end{pmatrix},$$

$m \in \mathbb{N}$.

(8.3.6) Let $\mathrm{Fil}^{\bullet}_{\mathrm{tot},N_{\mathbb{Z}_p}} K[V, V^{-1}] \otimes_K N_{\mathbb{Q}_p}$ be the filtration on $K[V, V^{-1}] \otimes_{W(k)} N_{\mathbb{Q}_p}$ defined by

$$\mathrm{Fil}^{m}_{\mathrm{tot},N_{\mathbb{Z}_p}} K[V, V^{-1}] \otimes_K N_{\mathbb{Q}_p} := \gamma'\left(\mathrm{Fil}^{m}_{\mathrm{tot},N_{\mathbb{Z}_p}} \mathrm{BC}_p(k)' \otimes_{W(k)[V,V^{-1}]} N_{\mathbb{Q}_p}\right),$$

where $\gamma' : \mathrm{BC}_p(k)' \otimes_{W(k)[V,F]} N_{\mathbb{Q}_p} \xrightarrow{\sim} K[V, V^{-1}] \otimes_K N_{\mathbb{Q}_p}$ is the canonical isomorphism from $\mathrm{BC}_p(k)' \otimes_{W(k)[V,V^{-1}]} N_{\mathbb{Q}_p}$ to $K[V, V^{-1}] \otimes_K N_{\mathbb{Q}_p}$ explained above. It is easy to see that the topology on $K[V, V^{-1}] \otimes_K N_{\mathbb{Q}_p}$ attached to the filtration $\mathrm{Fil}^{\bullet}_{\mathrm{tot},N_{\mathbb{Z}_p}} K[V, V^{-1}] \otimes_K N_{\mathbb{Q}_p}$ is independent of the choice of the $W(k)$-lattice $N_{\mathbb{Z}_p}$ in $N_{\mathbb{Q}_p}$.

(8.3.7) Lemma *The tensor product* $\mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}$ *is naturally isomorphic to the completion of* $K[V, V^{-1}] \otimes_K N_{\mathbb{Q}_p}$ *with respect to the filtration* $\mathrm{Fil}^{\bullet}_{\mathrm{tot},N_{\mathbb{Z}_p}} K[V, V^{-1}] \otimes_K N_{\mathbb{Q}_p}$

PROOF. This is an easy consequence of the fact that $\mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}$ is complete with respect to the tensor product of the filtration $\mathrm{Fil}^{\bullet}_{\mathrm{tot}}\mathrm{BC}_p(k)$ on $\mathrm{BC}_p(k)$ and the trivial filtration on $N_{\mathbb{Q}_p}$. ∎

(8.4) Lemma *Let* $N_{\mathbb{Q}_p}$ *be as in 8.1. Let* $N_{\mathbb{Z}_p}$ *be a* $W(k)$*-lattice for* $N_{\mathbb{Q}_p}$*. Let* $w_1, \ldots, w_h$ *be a* K*-basis of* $N_{\mathbb{Q}_p}$*. Then there exist positive constants* C_1, C_2 *such that*

$$\mathrm{Fil}^{N}_{\mathrm{tot}}\left(K[V, V^{-1}] \otimes_K N_{\mathbb{Q}_p}\right) \subseteq \left\{ \sum_{\substack{i\in\mathbb{Z},\,\text{finite}\\ 1\le r\le h}} a_{ir} V^i \otimes w_r \;\middle|\; \begin{array}{c} \mathrm{ord}_p(a_{ir}) \ge \mu \cdot \mathrm{Max}\left(\lceil \frac{N-i}{2} \rceil, -i, 0\right) - C_2 \\ \forall\, i \in \mathbb{Z},\ \forall r = 1, \ldots, h \end{array} \right\}$$

and

$$\mathrm{Fil}^{N}_{\mathrm{tot}}\left(K[V, V^{-1}] \otimes_K N_{\mathbb{Q}_p}\right) \supseteq \left\{ \sum_{\substack{i\in\mathbb{Z},\,\text{finite}\\ 1\le r\le h}} a_{ir} V^i \otimes w_r \;\middle|\; \begin{array}{c} \mathrm{ord}_p(a_{ir}) \ge \mu \cdot \mathrm{Max}\left(\lceil \frac{N-i}{2} \rceil, -i, 0\right) + C_1 \\ \forall\, i \in \mathbb{Z},\ \forall r = 1, \ldots, h \end{array} \right\}.$$

PROOF. Immediate from Lemma 8.2.2. ∎

(8.5) Proposition *Notation as in 8.1 and 8.3, so that $N_{\mathbb{Q}_p}$ be a V-isoclinic isocrystal of dimension h over K, $N_{\mathbb{Z}_p}$ is a $W(k)$-lattice for $N_{\mathbb{Q}_p}$, and $w_1, \ldots, w_h$ is a K-basis of $N_{\mathbb{Q}_p}$.*

(i) *The choice of basis $w_1, \ldots, w_h$ leads to a bijection*

$$\gamma : \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p} \xrightarrow{\sim} \left\{ \sum_{\substack{i\in\mathbb{Z}\\ 1\le r\le h}} a_{ir} V^i \otimes w_r \;\middle|\; \begin{array}{l} a_{ir} \in K \;\forall\, i \in \mathbb{Z}, \quad \forall\, r = 1, \ldots, \dim\left(N_{\mathbb{Q}_p}\right) = h \\ \exists\, C \in \mathbb{Z} \text{ s.t. } \mathrm{ord}_p(a_{ir}) - \max(-i, 0)\mu \ge C \\ \quad \forall i \in \mathbb{Z},\ \forall\, r = 1, \ldots, h \end{array} \right\}$$

In the formula above the target of the isomorphism γ is a subset of $N_{\mathbb{Q}_p}[[V, V^{-1}]]$, with growth conditions on the coefficients. Here $N_{\mathbb{Q}_p}[[V, V^{-1}]] := K[[V, V^{-1}]] \otimes_K N_{\mathbb{Q}_p}$, and is identified with the set of all formal series of the form

$$\sum_{\substack{i\in\mathbb{Z}\\ 1\le r\le h}} a_{ir} V^i \otimes w_r\,, \qquad a_{ir} \in K \ \forall i \in \mathbb{Z},\ \forall r = 1, \ldots, h\,.$$

via the chosen basis $w_1, \ldots, w_h$. The bijection γ extends the bijection

$$\gamma' : \mathrm{BC}_p(k)' \otimes_{W(k)[V,F]} N_{\mathbb{Q}_p} \xrightarrow{\sim} W(k)[V, F] \otimes_{W(k)} N_{\mathbb{Q}_p}$$

in 8.3, and γ is a homomorphism of left $\mathrm{Cart}_p(k)$-modules.

(ii) *There is a canonical isomorphism from $\mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}$ to the set $[[V, V^{-1}\rangle\rangle N_{\mathbb{Q}_p}$ of all sequences $(A_i)_{i\in\mathbb{Z}}$ indexed by $\mathbb{Z}$, where $A_i : K \to N$ is a σ^i-linear map for each $i \in \mathbb{Z}$, and there exists $C \in \mathbb{N}$, such that*

$$A_i(W(k)) \subset p^{\lfloor \max(-i,0)\cdot\mu \rfloor - C} N_{\mathbb{Z}_p} \qquad \forall\, i \in \mathbb{Z}\,.$$

The set $[[V, V^{-1}\rangle\rangle N_{\mathbb{Q}_p}$ is the completion of $[V, V^{-1}]N_{\mathbb{Q}_p}$ with respect to the filtration $\mathrm{Fil}^\bullet_{\mathrm{tot}}$. This canonical isomorphism is compatible with the commuting action of two copies of $\mathrm{Cart}_p(k)$: The actions on the source, $\mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}$, comes from the first copy of $\mathrm{Cart}_p(k)$ in the bimodule structure of $\mathrm{BC}_p(k)$ and the action of the extra copy of $\mathrm{Cart}_p(k)$; the actions on the target, $[[V, V^{-1}\rangle\rangle N_{\mathbb{Q}_p}$, was described in 8.3.3 and 8.3.4.

PROOF. We saw in 8.3.7 that $\mathrm{BC}_p(k) \otimes_{W(k)} N_{\mathbb{Q}_p}$ is naturally isomorphic to the completion of $K[V, V^{-1}] \otimes_K N_{\mathbb{Q}_p}$ with respect to the topology defined by the filtration $\mathrm{Fil}^\bullet_{\mathrm{tot},N_{\mathbb{Q}_p}} K[V, V^{-1}] \otimes_K N_{\mathbb{Q}_p}$. That filtration was computed up to multiplication by $p^{\pm C}$ in Lemma 8.4, which determines the topology on $K[V, V^{-1}] \otimes_K N_{\mathbb{Q}_p}$ attached to the filtration. It is not difficult to convince oneself that the completion of $K[V, V^{-1}] \otimes_K N_{\mathbb{Q}_p}$ with respect to this topology is exactly the set of all formal series with growth conditions as in the target of γ in the displayed formula in (i). The rest of the statement (i) are immediate.

The statement (ii) is a reformulation of (i), using the model $[V, V^{-1}]N_{\mathbb{Q}_p}$ of the tensor product $K[V, V^{-1} \otimes_K N_{\mathbb{Q}_p}$ as in 8.3.2. ∎

(8.6) Theorem *Let $N_{\mathbb{Q}_p}$ be as in 8.5. Let M be a V-reduced $\mathrm{Cart}_p(k)$-module attached to a finite dimensional p-divisible group over k, all of whose Frobenius slopes are strictly less than μ. Then there is a natural isomorphism*

$$\phi_{M,N_{\mathbb{Q}_p}} : \mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}) \xrightarrow{\sim} \mathrm{Hom}_{W(k)}(M, N_{\mathbb{Q}_p})\,.$$

Moreover $\phi_{M,N_{\mathbb{Q}_p}}$ is an isomorphism of $\mathrm{Cart}_p(k)$-modules, where

- *$\mathrm{Cart}_p(k)$ operates on $\mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p})$ via the action of the "extra copy" of $\mathrm{Cart}_p(k)$ on $\mathrm{BC}_p(k)$,*
- *the $\mathrm{Cart}_p(k)$-action on $\mathrm{Hom}_{W(k)}(M, N_{\mathbb{Q}_p})$ is given by*

$$(u \cdot h)(m) = \sum_{i\in\mathbb{Z}} a_i\, V^i \cdot h(V^{-i}m)$$

for all $h \in \mathrm{Hom}_{W(k)}(M, N_{\mathbb{Q}_p}) = \mathrm{Hom}_K(M \otimes_{W(k)} K, N_{\mathbb{Q}_p})$, for all $m \in M$, and for all $u = \sum_{i\in\mathbb{Z}} a_i\, V^i \in \mathrm{Cart}_p(k) \cong W(k)[[V, F\rangle\rangle$, where $a_i \in W(k)$ for all $i \in \mathbb{Z}$, $\mathrm{ord}_p(a_i) \geq \max(-i, 0)$ for all $i \in \mathbb{Z}$, and $\lim_{|i|\to\infty} \mathrm{ord}_p(a_i) + i = \infty$.

Note that the series $\sum_{i\in\mathbb{N}} a_i\, V^i \cdot h(V^{-i}m)$ converges because $a_i \in W(k)$ for all $i \in \mathbb{N}$ and the Frobenius slopes of M are all strictly smaller than μ, while the series $\sum_{i<0} a_i\, V^i \cdot h(V^{-i}m)$ converges because $\lim_{i\to-\infty} \mathrm{ord}_p(a_i) + i = \infty$. Hence the infinite sum $\sum_{i\in\mathbb{Z}} a_i\, V^i \cdot h(V^{-i}m)$ converges for every $m \in M$.

Proof. Step 1. One analyzes extensions of M by $\mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}$ as follows. Suppose we are given any such extension

$$0 \longrightarrow \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p} \longrightarrow E \xrightarrow{\pi} M \longrightarrow 0$$

of left $\mathrm{Cart}_p(k)$-modules. Choose a $W(k)$-linear splitting $\epsilon : M \to E$ such that $\pi \circ \epsilon = \mathrm{id}_M$; such splittings exist because M is a free $W(k)$-module. Then one obtains a σ^{-1}-linear map

$$v : M \longrightarrow \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}\,,$$

defined by

$$v(m) = V \cdot \epsilon(m) - \epsilon(V\, m)\,, \qquad \forall\, m \in M\,.$$

If we change the splitting ϵ to $\epsilon' = \epsilon + g$ for an element $g \in \mathrm{Hom}_{W(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p})$, then the resulting σ^{-1}-linear map $v' = V \circ \epsilon' - \epsilon' \circ V$ is related to v by

$$v'(m) = v(m) + V \cdot g(m) - g(V \cdot m)\,, \qquad \forall\, m \in M\,.$$

The above construction defines a map

$$\phi'_{M,N_{\mathbb{Q}_p}} : \mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}) \longrightarrow$$
$$\left\{ \begin{array}{c} \sigma^{-1}\text{-linear maps} \\ M \xrightarrow{v} \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p} \end{array} \right\} \Big/ \left(\begin{array}{c} v \sim v' \text{ if } \exists\, g \in \mathrm{Hom}_{W(k)}(M, \mathrm{BC}_p(k) \otimes N_{\mathbb{Q}_p}) \\ \text{s.t. } v' = v + V \circ g - g \circ V \end{array} \right)$$

STEP 2. Given any σ^{-1}-linear map $v : M \to \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}$, we construct an extension E_v of M by $\mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}$ as follows. The $W(k)$-module underlying E_v is $M \oplus \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}$ by definition. Define an action of V on E_v by

$$V \cdot (m,b) = (Vm, Vb + v(m)) \qquad \forall\, m \in M,\ \forall\, b \in \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}\,.$$

More generally, for every element of $W(k)$, written in the form $\sum_{i\in\mathbb{Z}} a_i\, V^i$ with $a_i \in W(k)$ for all $i \in \mathbb{Z}$, $\mathrm{ord}_p(a_i) + i \geq 0$ for all $i \leq 0$, and $\lim_{|i|\to\infty} (\mathrm{ord}_p(a_i) + i) = \infty$, define

$$\left(\sum_{i\in\mathbb{Z}} a_i V^i\right) \cdot (m,b) =$$
$$\left(\sum_{i\in\mathbb{Z}} a_i V^i\, m,\ \sum_{i\in\mathbb{Z}} a_i(V^i b) + \sum_{r,s\geq 0} a_{r+s+1}\, V^r \cdot v(V^s m) - \sum_{r,s\leq -1} a_{r+s+1}\, V^r \cdot v(V^s m)\right)$$

for all $m \in M$ and all $b \in \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}$. This construction gives us an extension E_v of M by $\mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}$ as $\mathrm{Cart}_p(k)$-modules.

Suppose that $v \sim v'$ in the sense that

$$v' - v = V \circ g - g \circ V$$

for some $g \in \mathrm{Hom}_{W(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p})$. Then the map $(m,b) \mapsto (m, b - g(m))$ defines an isomorphism $E_v \xrightarrow{\sim} E_{v'}$ of extensions of M by $\mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}$ as left $\mathrm{Cart}_p(k)$-modules. So we have shown that

$$\phi'_{M,N_{\mathbb{Q}_p}} : \mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}) \xrightarrow{\sim}$$
$$\left\{ \begin{array}{c} \sigma^{-1}\text{-linear maps} \\ M \xrightarrow{v} \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p} \end{array} \right\} \Big/ \left(\begin{array}{rl} v \sim v' & \text{if } \exists\, g \in \mathrm{Hom}_{W(k)}(M, \mathrm{BC}_p(k) \otimes N_{\mathbb{Q}_p}) \\ & \text{s.t. } v' = v + V \circ g - g \circ V \end{array} \right)$$

is a bijection. It is clear from our constructions that $\phi'_{M,N_{\mathbb{Q}_p}}$ is an isomorphism of left $\mathrm{Cart}_p(k)$-modules, for the actions coming from the "extra copy" of $\mathrm{Cart}_p(k)$ on $\mathrm{BC}_p(k)$.

STEP 3. We choose a K-basis $(w_1, \ldots, w_r)$ of $N_{\mathbb{Q}_p}$ and identify $\mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}$ with the set $\mathrm{BC}_{N_{\mathbb{Q}_p}}$ via the bijection γ in Prop. 8.5. Given a σ^{-1}-linear map $v : M \to \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}$, we write

$$v(m) = \sum_{r=1}^{h} \sum_{j\in\mathbb{Z}} b_{jr}(m)^{\sigma^{-1}}\, V^j \otimes w_r \qquad \forall\, m \in M\,,$$

where $b_{jr} \in \mathrm{Hom}_{W(k)}(M, K)$ for all $j \in \mathbb{Z}$ and all $r = 1, \ldots, h$. Of course the homomorphisms b_{jr} depend on v. Define a homomorphism $h \in \mathrm{Hom}_{W(k)}(M, N_{\mathbb{Q}_p})$, depending on $v \in \mathrm{Hom}_{W(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p})$, by

$$h(m) = - \sum_{\substack{j\in\mathbb{Z} \\ 1\leq r\leq h}} b_{jr}(V^{j-1}m)^{\sigma^{j-1}} \cdot w_r\,;$$

We remark that the series defining $h(m)$ converges:

(a) There exists a constant $C_1 > 0$ such that $\operatorname{ord}_p(b_{jr}(m)) + \mu j \geq -C_1$ for all $j \in \mathbb{Z}_{\leq 0}$ and for all $r = 1, \ldots, h$ and all $m \in M$. Since all Frobenius slopes of M are strictly smaller than μ, so the series

$$\sum_{j \leq 0} b_{jr}(V^{j-1}m)^{\sigma^{j-1}}$$

converges for $r = 1, \ldots, h$.

(b) There exists a constant $C_2 > 0$ such that $\operatorname{ord}_p(b_{jr}(m)) \geq -C_2$ for all $j \geq 0$, all $m \in M$, and $r = 1, \ldots, h$. Since all Frobenius slopes of M are strictly positive, the series

$$\sum_{j > 0} b_{jr}(V^{j-1}m)^{\sigma^{j-1}}$$

converges for $r = 1, \ldots, h$.

It is routine to verify that if $v = V \circ g - g \circ V$ for some $g \in \operatorname{Hom}_{W(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p})$, then

$$b_{jr}(m) = a_{j-1,r}(m)^{\sigma^{-1}} - a_{ir}(Vm) \qquad \forall\, j \in \mathbb{Z},\ \forall\, r = 1, \ldots, h\,,$$

where $a_{jr} \in \operatorname{Hom}_{W(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p})$ are defined by

$$g(m) = \sum_{j \in \mathbb{Z}} \sum_{r=1}^{h} a_{jr}(m)\, V^j \otimes w_r \qquad \forall\, m \in M\,,$$

and the homomorphism attached to $V \circ g - g \circ V$ is defined by a telescoping series, therefore equal to 0. We have constructed a well-defined homomorphism

$$\left\{ \begin{array}{c} \sigma^{-1}\text{-linear maps} \\ M \xrightarrow{v} \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p} \end{array} \right\} \Big/ \left(\begin{array}{c} v \sim v' \ \text{ if } \exists\, g \in \operatorname{Hom}_{W(k)}(M, \mathrm{BC}_p(k) \otimes N_{\mathbb{Q}_p}) \\ \text{s.t. } \ v' = v + V \circ g - g \circ V \end{array} \right)$$

$$\xrightarrow{\phi''_{M,N_{\mathbb{Q}_p}}} \operatorname{Hom}_{W(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p})\,.$$

STEP 4. We show that $\phi''_{M,N_{\mathbb{Q}_p}}$ is surjective. Given $h \in \operatorname{Hom}_{W(k)}(M, N_{\mathbb{Q}_p})$, we define a σ^{-1}-linear map $v_h : M \to \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}$ by

$$v_h(m) = -V \otimes h(m) \qquad \forall\, m \in M\,.$$

It follows immediately from the definition of $\phi''_{M,N_{\mathbb{Q}_p}}$ that $\phi''_{M,N_{\mathbb{Q}_p}}(v_h) = h$.

STEP 5. The last step is to show that for any σ^{-1}-linear map $v : M \to \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}$, if we set $h = \phi''_{M,N_{\mathbb{Q}_p}}(v)$, then $v \sim v_h$. In other words, we must find a homomorphism $g \in \operatorname{Hom}_{W(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p})$ such that

$$v(m) - v_h(m) = V \cdot g(m) - g(Vm) \qquad \forall\, m \in M\,.$$

We write the given map $v : M \to \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}$ as

$$v(m) = \sum_{r=1}^{h} \sum_{j \in \mathbb{Z}} b_{jr}(m)^{\sigma^{-1}}\, V^j \otimes w_r \qquad m \in M\,.$$

Then we have

$$v_h(m) = \sum_{r=1}^{h} \sum_{j\in\mathbb{Z}} b_{jr} (V^{j-1}m)^{\sigma^{j-2}}\, V \otimes w_r \qquad m \in M\,.$$

Define an element $g \in \mathrm{Hom}_{W(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p})$ by

$$g(m) = \sum_{r=1}^{h} \sum_{\substack{j\geq 1\\ 1\leq i\leq j-1}} b_{jr}(V^{j-i-1}m)^{\sigma^{j-i-1}}\, V^i \otimes w_r - \sum_{r=1}^{h} \sum_{\substack{j\leq 0\\ j\leq i\leq 0}} b_{jr}(V^{j-i-1}m)^{\sigma^{j-i-1}}\, V^i \otimes w_r$$

for all $m \in M$. We leave it to the readers to check that $g(m)$ is well-defined, that is the sums indeed are convergent and defines an element of $\mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}$. A formal calculation shows that

$$V\, g(m) - g(Vm) = v(m) - v_h(m) \qquad \forall\, m \in M\,.$$

So we have shown that $\phi''_{M,N_{\mathbb{Q}_p}}$ is an injection by constructing its inverse explicitly. This completes the proof of Theorem 8.6, with $\phi_{M,N_{\mathbb{Q}_p}} = \phi''_{M,N_{\mathbb{Q}_p}} \circ \phi'_{M,N_{\mathbb{Q}_p}}$. ∎

Remark (i) The proof of 8.6 is a generalization of the appendix of [14], where the case $Y = \widehat{\mathbb{G}_m}$ is treated, and the author used the ring $W(k)[F,V]$ instead of the Cartier ring $\mathrm{Cart}_p(k)$.

(ii) Thm. 8.6 also determines the action of $\mathrm{End}_k(X)^{\mathrm{opp}} \otimes_{W(k)} \mathrm{End}_k(Y)$ on the Cartier module of $\mathcal{DE}(X,Y)_{\text{p-div}}$, up to isogeny.

(8.6.1) Remark It may be of some interest to reformulate some part of the proof of Thm. 8.6 using the coordinate-free description $[[V, V^{-1}\rangle\rangle N_{\mathbb{Q}_p}$ of $\mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p}$ in 8.5. Denote by $\mathrm{Hom}^{\sigma^i}(M, N_{\mathbb{Q}_p})$ the set of all σ^i-linear maps from M to N. A σ^{-1}-linear map $v : M \to [[V, V^{-1}\rangle\rangle N_{\mathbb{Q}_p}$ corresponds to a sequence $\mathbf{b} = (b_i)_{i\in\mathbb{Z}}$, with $b_i \in \mathrm{Hom}^{\sigma^{i-1}}(M, N)$ for all $i \in \mathbb{Z}$, satisfying the growth condition that there exists a constant $C > 0$ such that $B_i(M) \subseteq p^{\max(-i,0)\mu - C} N_{\mathbb{Z}_p}$ for all $i \in \mathbb{Z}$.

A $W(k)$-linear map $g : M \to [[V, V^{-1}\rangle\rangle N_{\mathbb{Q}_p}$ corresponds to a sequence $\mathbf{a} = (a_i)_{i\in\mathbb{Z}}$, with $a_i \in \mathrm{Hom}^{\sigma^i}(M, N_{\mathbb{Q}_p})$ for each $i \in \mathbb{Z}$, satisfying the growth condition that there exists a constant $C > 0$ such that $a_i(M) \subseteq p^{\max(-i,0)\mu - C} N_{\mathbb{Z}_p}$ for all $i \in \mathbb{Z}$. Then $V{\cdot}g$ corresponds to the sequence $V{\cdot}\mathbf{g} = (b_i)_{i\in\mathbb{Z}}$ with $b_i = a_{i-1} \forall\, i \in \mathbb{Z}$, and $g \circ V$ corresponds to the sequence $\mathbf{g}{\circ}V = (b'_i)_{i\in\mathbb{Z}}$ with $b'_i = b_i \circ V \in \mathrm{Hom}^{\sigma^{i-1}}(M, N_{\mathbb{Q}_p})$ for all $i \in \mathbb{Z}$.

Using the above coordinate-free description, the construction in Step 3 of the proof of Thm. 8.6, which produces an element $h \in \mathrm{Hom}_{W(k)}(M, N_{\mathbb{Q}_p})$ from a σ^{-1}-linear map v, can be described as follows. Given a sequence $\mathbf{b} = (b_i)_{i\in\mathbb{Z}}$ satisfying the growth condition, with $b_i \in \mathrm{Hom}^{\sigma^{i-1}}(M, N_{\mathbb{Q}_p})\ \forall\, i \in \mathbb{Z}$, we define an element h of $\mathrm{Hom}_{W(k)}(M, N_{\mathbb{Q}_p})$ by

$$h(m) = -\sum_{i\in\mathbb{Z}} b_i(V^{i-1}m) \qquad \forall\, m \in M\,.$$

The growth condition on $\mathbf{b}$ ensures that the infinite series defining $h(m)$ converges. We can think of h as the *trace* of $\mathbf{b}$ in $\mathrm{Hom}_{W(k)}(M, N_{\mathbb{Q}_p})$. It is easy to see that, given a sequence $\mathbf{g}$ with growth condition as in the previous paragraph, the trace of $V{\cdot}\mathbf{g} - \mathbf{g}{\circ}V$ is equal to zero.

Finally, we reformulate the construction in Step 5 of the proof of Thm. 8.6. Start with a σ^{-1}-linear map v, which corresponds to a sequence $\mathbf{b} = (b_i)_{i\in\mathbb{Z}}$ with growth condition, $b_i \in \mathrm{Hom}^{\sigma^{i-1}}(M, N_{\mathbb{Q}_p})$ for all $i \in \mathbb{Z}$. Let h be the trace of $\mathbf{b}$ in $\mathrm{Hom}_{W(k)}(M, N_{\mathbb{Q}_p})$. Denote by $\mathbf{b}'$ the sequence $(b'_i)_{i\in\mathbb{Z}}$ such that $b'_1 = -h$, while $b'_i = 0$ if $i \neq 0$. We need a sequence $\mathbf{g} = (a_i)_{i\in\mathbb{Z}}$ satisfying the growth condition, with $a_i \in \mathrm{Hom}^{\sigma^i}(M, N_{\mathbb{Q}_p})$ for all $i \in \mathbb{Z}$, such that

$$V\cdot \mathbf{g} - \mathbf{g}\circ V = \mathbf{b} - \mathbf{b}' .$$

This sequence is given by

$$a_i(m) = \sum_{j\geq i+1} b_j(V^{j-i-1}m) \quad \text{for } i \geq 1 ,$$

and

$$a_i(m) = -\sum_{j\leq i} b_j(V^{j-i-1}m) \quad \text{for } i \leq 0 .$$

There exists a constant $C > 0$ such that $b_j(V^{j-i-1}M) \subseteq p^{\mu_X^{\min}\cdot(j-i-1)-C} N_{\mathbb{Z}_p}$ if $j-1 \geq i \geq 1$, and $b_j(V^{j-i-1}M) \subseteq p^{-j\mu+\mu_X^{\max}\cdot(j-i-1)-C} N_{\mathbb{Z}_p}$ if $j \leq i \leq 0$, where $\mu_X^{\min}$ (resp. $\mu_X^{\max}$) is the smallest (resp. the biggest) V-slope of M. It follows that the above infinite series converge, and define a sequence $\mathbf{g}$ satisfying the growth condition. We leave it to the reader to verify, as a routine exercise, that the sequence $\mathbf{g}$ has all the required properties.

(8.6.2) Corollary *The statement of Thm. 8.6 holds for $M = \mathrm{M}(X)$, $N := \mathrm{M}(Y)$, where X, Y are finite-dimensional p-divisible formal groups such that each Frobenius slope of X is strictly smaller than any Frobenius slope of Y.*

Proof. There exists an isogeny from Y to a direct product of finite dimensional isoclinic p-divisible formal groups. ∎

(8.6.3) Corollary *Let X, Y be p-divisible formal groups over k, where k is a field of characteristic p. Assume that X, X are isoclinic of Frobenius slopes μ_X, μ_Y respectively, and $\mu_X < \mu_Y$. Then the p-divisible formal group $\mathcal{DE}(X,Y)_{\text{p-div}}$ is isoclinic of Frobenius slope $\mu_Y - \mu_X$, its height is equal to* $\mathrm{height}(X)\cdot\mathrm{height}(Y)$*, and its dimension is* $(\mu_Y - \mu_X)\cdot\mathrm{height}(X)\cdot\mathrm{height}(Y)$.

Proof. According to Prop. 5.7.3, the Cartier module of $\mathcal{DE}(X,Y)$ is

$$\mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N) ,$$

where M, N are the Cartier modules of X and Y respectively. Let E be the Cartier module of $\mathcal{DE}(X,Y)_{\text{p-div}}$. Then the natural inclusion $E \hookrightarrow \mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N)$ induces an isomorphism $E \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N) \otimes_{\mathbb{Z}} \mathbb{Q}$. Combined with the natural isomorphism $\phi_{M,N_{\mathbb{Q}_p}} : \mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \mathrm{Hom}_{W(k)}(M, N) \otimes_{\mathbb{Z}} \mathbb{Q}$, we obtain a natural isomorphism $E \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \mathrm{Hom}_{W(k)}(M, N \otimes_{\mathbb{Z}} \mathbb{Q})$. Hence the height of $\mathcal{DE}(X,Y)_{\text{p-div}}$ is equal to the product of the heights of X and Y. It is well-known that the V-slope of the V-isocrystal $\mathrm{Hom}_{W(k)}(M, N) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the difference of the V-slopes of M and N, Since the relative Frobenius of a smooth formal group corresponds to the operator V on Cartier modules, we see that $\mathcal{DE}(X,Y)_{\text{p-div}}$ is isoclinic of Frobenius slope $\mu_Y - \mu_X$. ∎

(8.6.4) Remark Cor. 8.6.3 provides an independent proof of $\mathcal{DE}(X,Y)_{\text{p-div}}$ is isoclinic of slope $\mu_Y - \mu_X$ was proved in Thm. 2.8, by "pure thought". The proof here, based on Thm. 8.6, is more complicated. But we do not have a "pure thought" proof that the height of $\mathcal{DE}(X,Y)_{\text{p-div}}$ is equal to the product of the heights of X and Y.

(8.7) In this subsection we consider the effect of a quasi-polarization on $\mathcal{DE}(X,Y)_{\text{p-div}}$.

(8.7.1) Let X be a p-divisible formal group over a perfect field k of characteristic p such that every Frobenius slope μ of X satisfies $0 < \mu < \frac{1}{2}$. Let $Y = X^t$ be the Serre-dual of X. Let λ be a quasi-polarization on $X \times Y$, so that λ induces an isogeny $\beta : X \to X^t$ such that $\beta^t : X = (X^t)^t \to X^t$ is equal to α. Let $g = \text{height}(X) = \dim(X \times_{\text{Spec}(k)} X^t)$.

The symmetric isogeny $\beta : X \to X^t$ induces an involution ι on the smooth formal group $\mathcal{DE}(X,X^t)$, as follows. For any Artinian local ring R over k, an R-point of $\mathcal{DE}(X,X^t)$ corresponds to an extension

$$0 \to X \times_{\text{Spec}(k)} \text{Spec}(R) \to E \to X^t \times_{\text{Spec}(k)} \text{Spec}(R) \to 0$$

of p-divisible formal groups over R, together with a isomorphism $\alpha : E \times_{\text{Spec}(R)} \text{Spec}(k) \xrightarrow{\sim} X \times_{\text{Spec}(k)} X^t$. The dual of a pair (E,α) as above is the extension

$$0 \to X \times_{\text{Spec}(k)} \text{Spec}(R) = (X^t \times_{\text{Spec}(k)} \text{Spec}(R))^t \to E^t \to X^t \times_{\text{Spec}(k)} \text{Spec}(R) \to 0\,,$$

together with the isomorphism

$$(\alpha^{-1})^t : E^t \times_{\text{Spec}(R)} \text{Spec}(k) \xrightarrow{\sim} (X \times_{\text{Spec}(k)} X^t)^t = X \times_{\text{Spec}(k)} X^t\,.$$

By definition, the involution ι on $\mathcal{DE}(X,X^t)$ sends an R-point $[(E,\alpha)]$ above to the R-point $[(E^t,(\alpha^{-1})^t)]$. It is clear that $\mathcal{DE}(X,X^t)_{\text{p-div}}$ is stable under ι.

Denote by $\mathcal{DE}(X,X^t)^{\text{sym}}$ the maximal formal subgroup of $\mathcal{DE}(X,X^t)$ fixed under the involution ι. It is easy to see that $\mathcal{DE}(X,X^t)^{\text{sym}}$ is formally smooth, and it is the maximal formal subgroup of $\mathcal{DE}(X,X^t)$ such that the quasi-polarization λ on $X \times_{\text{Spec}(k)} X^t$ extends to a quasi-polarization of the universal p-divisible formal group over $\mathcal{DE}(X,X^t)^{\text{sym}}$.

Let $\mathcal{DE}(X,X^t)^{\text{sym}}_{\text{p-div}}$ be the maximal p-divisible formal subgroup of $\mathcal{DE}(X,X^t)^{\text{sym}}$. It is clear that $\mathcal{DE}(X,X^t)^{\text{sym}}_{\text{p-div}}$ is equal to the maximal p-divisible formal subgroup of $\mathcal{DE}(X,X^t)_{\text{p-div}}$ fixed under ι, and is also the maximal p-divisible formal subgroup of $\mathcal{DE}(X,X^t)_{\text{p-div}}$ such that the quasi-polarization λ on $X \times_{\text{Spec}(k)} X^t$ extends to a quasi-polarization of the universal p-divisible formal group over $\mathcal{DE}(X,X^t)^{\text{sym}}_{\text{p-div}}$.

We identify the Cartier module of X^t with $M^\vee = \text{Hom}_{W(k)}(M,W(k))$, the linear dual of M. Denote by $\text{Hom}_K^{\text{sym}}(M_{\mathbb{Q}_p}, M^\vee_{\mathbb{Q}_p})$ the K-vector space consisting of all symmetric elements of $\text{Hom}_K(M_{\mathbb{Q}_p}, M^\vee_{\mathbb{Q}_p})$.

(8.7.2) Proposition *Notation as in 8.7.1 above.*

(i) *The V-isocrystal* $\mathrm{M}(\mathcal{DE}(X,X^t)^{\text{sym}}_{\text{p-div}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ *is isomorphic to* $\text{Hom}_K^{\text{sym}}(M_{\mathbb{Q}_p}, M^\vee_{\mathbb{Q}_p})$ *under the isomorphism* $\phi_{M,M^\vee_{\mathbb{Q}_p}}$ *in Thm. 8.6.*

(ii) *The height of* $\mathcal{DE}(X,X^t)^{\mathrm{sym}}_{\text{p-div}}$ *is equal to* $\frac{g(g+1)}{2}$. *In particular, if* X *is isoclinic of Frobenius slope* μ, $0<\mu<\frac{1}{2}$, *then*

$$\dim(\mathcal{DE}\left(X,X^t)^{\mathrm{sym}}_{\text{p-div}}\right)=\frac{1}{2}\,(1-2\mu)\,g(g+1)\,.$$

PROOF. Statement (ii) follows from (i), because $\dim_K(\mathrm{Hom}^{\mathrm{sym}}_K(M_{\mathbb{Q}_p},M^\vee_{\mathbb{Q}_p}))=\frac{g(g+1)}{2}$, and the dimension of an isoclinic BT-group is equal to the slope times the height.

The honest way to prove (i) would be to compute, on the nose, the effect of the involution ι on the Cartier module of $\mathcal{DE}(X,Y)_{\text{p-div}}$. That will involve chasing through the construction of the isomorphism $\phi_{M,M^t_{\mathbb{Q}_p}}$ in Thm. 8.6 and verify the commutativity of certain diagrams. Here we give an argument which is enough to prove (i).

We may and do assume that k is algebraically closed. Let $E=\mathrm{End}_{\mathrm{Cart}_p(k)}(M)\otimes_{\mathbb{Z}}\mathbb{Q}$, so that E is a central simple algebra over $\mathbb{Q}_p$, $\dim_{\mathbb{Q}_p}(E)=g^2$. It is well-known that E is isomorphic to a matrix algebra with entries in a central division algebra over $\mathbb{Q}_p$ with Brauer invariant μ — or $-\mu$, depending on the normalization one uses. The group of automorphisms $\mathrm{End}(X)^\times=(\mathrm{End}_{\mathrm{Cart}_p(k)}(M))^\times$ operates on $\mathcal{DE}(X,X^t)_{\text{p-div}}$, $\mathrm{M}(\mathcal{DE}(X,X^t)_{\text{p-div}})$, and $\mathrm{Hom}_{W(k)}(M,M^\vee)$. Moreover the canonical isomorphism $\mathrm{M}(\mathcal{DE}(X,X^t)_{\text{p-div}})\xrightarrow{\sim}\mathrm{Hom}_{W(k)}(M,M^\vee)$ is compatible with the natural action of $(\mathrm{End}_{\mathrm{Cart}_p(k)}(M))^\times$. The group $(\mathrm{End}_{\mathrm{Cart}_p(k)}(M))^\times$ is a compact open subgroup of $E^\times$, and the linear action of $(\mathrm{End}_{\mathrm{Cart}_p(k)}(M))^\times$ on $\mathrm{Hom}_{W(k)}(M,M^\vee)\otimes_{\mathbb{Z}}\mathbb{Q}$ extends to a linear action of $E^\times$ on $\mathrm{Hom}_{W(k)}(M,M^\vee)\otimes_{\mathbb{Z}}\mathbb{Q}$. We know that the $W(k)$-submodule $\mathrm{M}(\mathcal{DE}(X,Y)^{\mathrm{sym}}_{\text{p-div}})$ of $\mathrm{M}(\mathcal{DE}(X,Y)_{\text{p-div}}$ is stable under the action of $(\mathrm{End}_{\mathrm{Cart}_p(k)}(M))^\times$, therefore $\mathrm{M}(\mathcal{DE}(X,Y)^{\mathrm{sym}}_{\text{p-div}})\otimes_{\mathbb{Z}}\mathbb{Q}$ corresponds, under the canonical isomorphism, to a subspace of $\mathrm{Hom}_K(M\otimes_{\mathbb{Z}}\mathbb{Q},M^\vee\otimes_{\mathbb{Z}}\mathbb{Q})$ which is stable under the natural action of $E^\times$.

We have a decomposition of the vector space $\mathrm{Hom}_K(M\otimes_{\mathbb{Z}}\mathbb{Q},M^\vee\otimes_{\mathbb{Z}}\mathbb{Q})$ as a direct sum of its symmetric and skew-symmetric part:

$$\mathrm{Hom}_K(M\otimes_{\mathbb{Z}}\mathbb{Q},M^\vee\otimes_{\mathbb{Z}}\mathbb{Q})=\mathrm{Hom}^{\mathrm{sym}}_K(M\otimes_{\mathbb{Z}}\mathbb{Q},M^\vee\otimes_{\mathbb{Z}}\mathbb{Q})\oplus\mathrm{Hom}^{\mathrm{skew}}_K(M\otimes_{\mathbb{Z}}\mathbb{Q},M^\vee\otimes_{\mathbb{Z}}\mathbb{Q})\,.$$

Both direct summands are stable under the action of $E^\times$. It a standard fact that the action of $E^\times$ on $M^\vee\otimes_{\mathbb{Z}}\mathbb{Q}$, regarded as a linear representation of an algebraic group, is isomorphic to the *standard representation* of GL_g after passing to the algebraic closure of K. Moreover the action of $E^\times$ on $\mathrm{Hom}^{\mathrm{sym}}_K(M\otimes_{\mathbb{Z}}\mathbb{Q},M^\vee\otimes_{\mathbb{Z}}\mathbb{Q})$ (resp. $\mathrm{Hom}^{\mathrm{skew}}_K(M\otimes_{\mathbb{Z}}\mathbb{Q},M^\vee\otimes_{\mathbb{Z}}\mathbb{Q})$) is isomorphic to the second symmetric product of the standard representation (resp. the second exterior product of the standard representation) after passing to K^{alg}; both representations are absolutely irreducible. So there are only four possibilities for the $E^\times$-invariant subspace $\mathrm{M}(\mathcal{DE}(X,Y)^{\mathrm{sym}}_{\text{p-div}})\otimes_{\mathbb{Z}}\mathbb{Q}$: it can be (0), $\mathrm{Hom}^{\mathrm{sym}}_K(M\otimes_{\mathbb{Z}}\mathbb{Q},M^\vee\otimes_{\mathbb{Z}}\mathbb{Q})$, $\mathrm{Hom}^{\mathrm{skew}}_K(M\otimes_{\mathbb{Z}}\mathbb{Q},M^\vee\otimes_{\mathbb{Z}}\mathbb{Q})$, or $\mathrm{Hom}_K(M\otimes_{\mathbb{Z}}\mathbb{Q},M^\vee\otimes_{\mathbb{Z}}\mathbb{Q})$ itself. We claim that

(A) $\mathrm{M}(\mathcal{DE}(X,Y)^{\mathrm{sym}}_{\text{p-div}})\otimes_{\mathbb{Z}}\mathbb{Q}\not\subseteq\mathrm{Hom}^{\mathrm{skew}}_K(M\otimes_{\mathbb{Z}}\mathbb{Q},M^\vee\otimes_{\mathbb{Z}}\mathbb{Q})$.

(B) $\mathrm{M}(\mathcal{DE}(X,Y)^{\mathrm{sym}}_{\text{p-div}})\otimes_{\mathbb{Z}}\mathbb{Q}\neq\mathrm{Hom}_K(M\otimes_{\mathbb{Z}}\mathbb{Q},M^\vee\otimes_{\mathbb{Z}}\mathbb{Q})$.

It is clear that the statement (i) follows from (A) and (B).

To prove (A), we may and do assume that X is minimal. Choose an embedding of $W(\mathbb{F}_{p^g})\hookrightarrow\mathrm{End}(X)$, denote by $\mathcal{O}$ the image of $W(\mathbb{F}_{p^g})$, and consider the maximal closed reduced formal subscheme of $\mathcal{DE}(X,Y)^{\mathcal{O}}_{\text{p-div}}$ such that the natural action of $\mathcal{O}$ on $X\times_{\mathrm{Spec}(k)}X^t$

extends to an action of $\mathcal{O}$ on the restriction to $\mathcal{DE}(X,Y)^{\mathcal{O}}_{\text{p-div}}$ of the universal extension $\mathcal{E}$ of X by X^t. Choose a totally real number field F such that $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathcal{O}$. Let $\mathcal{M}$ be the Hilbert Blumenthal modular variety $\mathcal{M}$ over k, attached to F. There exists a closed point x_0 of $\mathcal{M}$ such that the $A_0[p^\infty]$ together with the action by $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathcal{O}$ is isomorphic to $X \times_{\mathrm{Spec}(k)} X^t$ with the $\mathcal{O}$ action. Then $\mathcal{DE}(X,Y)^{\mathcal{O}}_{\text{p-div}}$ is the formal completion at x_0 of the central leaf $\mathcal{C}_{\mathcal{M}}(x_0)$ in $\mathcal{M}$.

Suppose that $\mathrm{M}(\mathcal{DE}(X,Y)^{\mathrm{sym}}_{\text{p-div}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is contained in $\mathrm{Hom}^{\mathrm{skew}}_K(M \otimes_{\mathbb{Z}} \mathbb{Q}, M^\vee \otimes_{\mathbb{Z}} \mathbb{Q})$. It is easy to see that $\mathcal{DE}(X,Y)^{\mathcal{O}}_{\text{p-div}} \subseteq \mathrm{Hom}_{K,\mathcal{O}}(M \otimes_{\mathbb{Z}} \mathbb{Q}, M^\vee \otimes_{\mathbb{Z}} \mathbb{Q})$, where $\mathrm{Hom}_{K,\mathcal{O}}(M \otimes_{\mathbb{Z}} \mathbb{Q}, M^\vee \otimes_{\mathbb{Z}} \mathbb{Q})$ is the subset of all $\mathcal{O}$-equivariant elements in $\mathrm{Hom}_K(M \otimes_{\mathbb{Z}} \mathbb{Q}, M^\vee \otimes_{\mathbb{Z}} \mathbb{Q})$. But the intersection of $\mathrm{Hom}^{\mathrm{skew}}_K(M \otimes_{\mathbb{Z}} \mathbb{Q}, M^\vee \otimes_{\mathbb{Z}} \mathbb{Q})$ with $\mathrm{Hom}_{K,\mathcal{O}}(M \otimes_{\mathbb{Z}} \mathbb{Q}, M^\vee \otimes_{\mathbb{Z}} \mathbb{Q})$ is (0). Hence the central leaf $\mathcal{C}_{\mathcal{M}}(x_0)$ in $\mathcal{M}$ is zero-dimensional. This is impossible, because the prime-to-p Hecke orbit of x_0 in $\mathcal{M}$ is not finite, by the argument in [1, Prop. 1, p. 448]. We have proved claim (A).

Suppose that $\mathrm{M}(\mathcal{DE}(X,Y)^{\mathrm{sym}}_{\text{p-div}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is equal to $\mathrm{Hom}_K(M \otimes_{\mathbb{Z}} \mathbb{Q}, M^\vee \otimes_{\mathbb{Z}} \mathbb{Q})$. This implies that *every* principal quasi-polarization on $X \times X^t$ extends to the universal BT-group $\mathcal{G}$ over $\mathcal{DE}(X,X^t)_{\text{p-div}}$. In particular, the image of $\mathrm{Aut}_{\mathcal{DE}(X,X^t)_{\text{p-div}}}(\mathcal{G})$ in $\mathrm{Aut}_k(X \times X^t)$ contains the subset consisting of all elements of the form (β, β^t), $\beta \in \mathrm{Aut}_k(X)$. We know that $E \ncong E^{\mathrm{opp}}$, i.e. $\mathrm{End}_k(X) \otimes_{\mathbb{Z}} \mathbb{Q} \ncong \mathrm{End}_k(X^t) \otimes_{\mathbb{Z}} \mathbb{Q}$, because $0 < \mu < \frac{1}{2}$. Using the standard theory of semisimple modules, it is easy to see that the $\mathbb{Q}$-subalgebra of $\mathrm{End}_k(X \times_{\mathrm{Spec}(k)} X^t)$ generated by the above automorphisms is equal to $(\mathrm{End}_k(X) \otimes_{\mathbb{Z}} \mathbb{Q}) \times (\mathrm{End}_k(X^t) \otimes_{\mathbb{Z}} \mathbb{Q}) \cong E \times E^{\mathrm{opp}}$.

We may and do assume that k is algebraically closed, and that there exists a principally polarized abelian variety A_0 over k such that $A_0[p^\infty] \cong X \times_{\mathrm{Spec}(k)} X^t$. Write $\mathcal{DE}(X,X^t)_{\text{p-div}} = \mathrm{Spf}(R)$. By Serre-Tate we get a formal abelian scheme $\tilde{A}$ over $\mathrm{Spf}(R)$, together with an isomorphism $\tilde{A}[p^\infty] \cong \mathcal{G}$. Since every principal quasi-polarization on $A_0[p^\infty]$ extends to $\mathrm{Spf}(R)$, the formal abelian scheme $\tilde{A}$ over $\mathrm{Spf}(R)$ is algebraic, i.e. there exists an abelian scheme A over $\mathrm{Spec}(R)$ whose formal completion is isomorphic to $\tilde{A}$. Since A_0 is defined over a finite field, $\mathrm{End}(A_0) \otimes \mathbb{Q}_p \xrightarrow{\sim} \mathrm{End}(A_0[p^\infty]$. The conclusion of the previous paragraph implies that there exists an integer N such that every element of $p^N \,\mathrm{End}(A_0)$ extends to an endomorphism of A over $\mathrm{Spec}(R)$. Since $A \to \mathrm{Spec}(R)$ is of finite presentation, there exists a finitely generated k-subalgebra $S \subset R$ and an abelian scheme A_S over $\mathrm{Spec}(S)$ and an isomorphism $A_S \times_{\mathrm{Spec}(S)} \mathrm{Spec}(R) \xrightarrow{\sim} A$. By [15, Thm. 2.1], there exists a finite surjective base change map $T \to \mathrm{Spec}(S)$ such that $A_S \times_{\mathrm{Spec}(S)} T$ is isogenous to a constant abelian scheme over T. Hence the BT-group $\mathcal{G} \times_{\mathrm{Spec}(R)} (\mathrm{Spec}(R) \times_{\mathrm{Spec}(S)} T)$ is isogenous to a constant BT-group over the scheme $\mathrm{Spec}(R) \times_{\mathrm{Spec}(S)} T$ which is finite over $\mathrm{Spec}(R)$, a contradiction! We have finished the proof of (B) and the statement (i) of 8.7.2.

(8.7.3) Corollary *Let $x_0 = [(A_0, \lambda_0)]$ be a closed point of $\mathcal{A}_g$ over a field of characteristic p. Suppose that A_0 has only two Frobenius slopes, μ and $1-\mu$, with $0 < \mu < \frac{1}{2}$. Then the dimension of the central leaf $\mathcal{C}(x_0)$ in $\mathcal{A}_g$ passing through x_0 is equal to $\frac{1}{2}\,(1-2\mu)\, g(g+1)$.*

Proof. Immediate from Prop. 8.7.2 and 3.3. ∎

Remark The statement of 8.7.3 is a special case of [17, 3.17]. Oort's original proof, mentioned in [17, 3.17], uses the main result of [18] and deformation theory of abelian varieties.

§9. The integral structure

(9.1) Notation Let X, Y be finite-dimensional p-divisible formal groups over a perfect field k of characteristic p such that every Frobenius slope of X is strictly smaller than any Frobenius slope of Y. Let $K = B(k)$ be the fraction field of $W(k)$. Let $M = \mathrm{M}(X)$, $N = \mathrm{M}(Y)$ be the Cartier modules of X, Y respectively. Let $r_1 = \dim(X) = \dim_k(M/VM)$, $r_2 = \dim(Y) = \dim_k(N/VN)$, $s_1 = \dim_k(VM/pM)$, $s_2 = \dim_k(VN/pN)$.

Let $H = \mathrm{Hom}_{W(k)}(M, N)$. The K-module $H_K = H \otimes_{W(k)} K$ has a natural structure as a V-isocrystal, such that

$$(V \cdot h)(m) = V\, h(V^{-1}m), \quad (F \cdot h)(m) = F\, h(Vm) \quad \forall\, m \in M\,.$$

Notice that the $W(k)$-lattice $H \subseteq H_K$ is stable under F.

Let H_1 be the maximal $W(k)$-submodule of H such that $F(H_1) \subseteq H_1$ and $V(H_1) \subseteq H_1$. Similarly, let H_2 be the minimal $W(k)$-submodule of H_K containing H such that $F(H_2) \subseteq H_2$, $V(H_2) \subseteq H_2$. It is easy to see that

$$H_1 = \bigcap_{i \geq 0} V^{-i}H, \quad H_2 = \sum_{i \geq 0} V^i H\,.$$

(9.1.1) Lemma *Notation as above.*

(i) *The natural map*

$$H_1/VH_1 \to H/(H \cap VH)$$

is injective.

(ii) *Let $\overline{VM} = VM/pM$ be the image of VM in $\overline{M} = M/pM$, and let $\overline{VN} = VN/pN$ be the image of VN in $\overline{N} = N/pN$. Then the natural map $H \to \mathrm{Hom}_k(\overline{VM}, \overline{N}/\overline{VN})$ induces an isomorphism*

$$H/(H \cap VH) \xrightarrow{\sim} \mathrm{Hom}_k(\overline{VM}, \overline{N}/\overline{VN})\,.$$

In particular, $\dim_k(H/(H \cap VH)) = r_2\, s_1$.

PROOF. (i) Suppose that $x \in H_1 \cap VH$, so that $x = Vy$ with $y \in H$. We must show that $y \in H_1$. Consider the $W(k)$-submodule

$$H' := H_1 + \sum_{i \geq 0} W(k) \cdot F^i y$$

of H_K. Clearly $H_1 \subseteq H$ because $F(H) \subseteq H$, and $F(H') \subset H'$ by construction. Moreover $V(F^i y) = F^i(x) \in H_1 \subseteq H'$ for all $i \geq 0$. So $V(H') \subseteq H'$, and $H_1 = H' \ni y$.

(ii) The natural map $\alpha : H \to \mathrm{Hom}_k(\overline{VM}, \overline{N}/\overline{VN})$ is a composition

$$H \twoheadrightarrow \mathrm{Hom}_k(\overline{M}, \overline{N}) \twoheadrightarrow \mathrm{Hom}_k(\overline{VM}, \overline{N}/\overline{VN})$$

of two surjections, hence is surjective. It is clear that $\mathrm{Ker}(\alpha)$ consists of all elements $h \in \mathrm{Hom}_{W(k)}(M, N)$ such that $h(VM) \subseteq VN$, which means that the element $V^{-1}h$ belongs to H. In other words, $\mathrm{Ker}(\alpha) = H \cap VH$. ∎

(9.2) Lemma *Let M be the Cartier module of a finite-dimensional p-divisible formal group over a perfect field k of characteristic p. Let B be a V-reduced left $\mathrm{Cart}_p(k)$-module.*

(i) *There is a natural bijection between the following two sets:*

- *the set of all isomorphism classes of pairs*

$$(\,0 \to B \to E \to M \to 0,\ \mathrm{sp}\,)$$

where $0 \to B \to E \to M \to 0$ is a short exact sequence of left $\mathrm{Cart}_p(k)$-modules, and $\mathrm{sp} : M \to E$ is a $W(k)$-linear splitting of the above exact sequence;

- *the set of pairs (f, v), where $f : M \to B$ is a σ-linear map, $v : M \to B$ is a σ^{-1}-linear map such that*

$$f(Vm) + Fv(m) = 0, \quad Vf(m) + v(Fm) = 0 \ \ \forall\, m \in M\,.$$

(ii) *There is a natural bijection between the following two sets:*

- *the set of all isomorphism classes of extensions $(0 \to B \to E \to M \to 0)$ of left $\mathrm{Cart}_p(k)$-modules;*
- *the set of pairs (f, v), where $f : M \to B$ is a σ-linear map, $v : M \to B$ is a σ^{-1}-linear map such that*

$$f(Vm) + Fv(m) = 0 \ \text{ and } \ Vf(m) + v(Fm) = 0 \ \ \forall\, m \in M\,,$$

modulo the following equivalence relation: $(f', v') \sim (f, v)$ iff there exists a $W(k)$-linear map $g : M \to B$ such that

$$f'(m) - f(m) = Fg(m) - g(Fm) \ \text{ and } \ v'(m) - v(m) = Vg(m) - g(Vm) \quad \forall\, m \in M\,.$$

Proof. The bijection in (i) is given as follow. For any given extension

$$0 \to B \to E \to M \to 0$$

of left $\mathrm{Cart}_p(k)$-modules together with a splitting $\mathrm{sp} : M \to E$ of the short exact sequence of the underlying $W(k)$-modules, the corresponding pair is (f, v), where

$$f(m) := F{\cdot}\mathrm{sp}(m) - \mathrm{sp}(Fm), \quad v(m) := V{\cdot}\mathrm{sp}(m) - \mathrm{sp}(Vm), \quad \forall\, m \in M\,.$$

It is easy to check that $f : M \to B$ is σ-linear, $v : M \to B$ is σ^{-1}-linear, and

$$f(Vm) + Fv(m) = 0 \ \text{ and } \ Vf(m) + v(Fm) = 0 \ \ \forall\, m \in M\,.$$

Conversely, for any pair (f, v) which satisfies the above conditions, we use the maps f and v to define the actions of F and V on $B \oplus M$ as follows.

$$F \cdot (b, m) = (Fb + f(m), Fm), \quad V \cdot (b, m) = (Vb + v(m), Vm), \quad \forall\, b \in B,\ \forall\, m \in M\,.$$

It is easy to check that $FV = VF = [p]$ on $B \oplus M$, so that the action of F and V extends to an action of $W(k)[F, V]$ on $B \oplus M$. We claim that the action of $W(k)[F, V]$ on $B \oplus M$ extends to an action of $\mathrm{Cart}_p(k)$. Assuming the claim, then $B \oplus M$ is an extension of M by B as a left $\mathrm{Cart}_p(k)$-module, endowed with the $W(k)$-linear splitting $m \mapsto (0, m)$, $m \in M$. It is easy to check that these two constructions are inverse to each other, and the bijection (i) will be established.

To prove the claim, we recall that any element of $\mathrm{Cart}_p(k)$ can be written as a convergent sum

$$\sum_{j\geq 1} c_j\, F^j + \sum_{i\geq 0} d_i V^i\,, \quad c_j, d_i \in W(k)\ \forall\, i\geq 0\ \forall\, j\geq 1,\ \textit{and}\ \mathrm{ord}_p(c_j)\to\infty\,.$$

A simple computation shows that

$$F^j(b,m) = (F^j b + \sum_{j_1=0}^{j-1} F^{j_1} f(F^{j-j_1-1}m),\ F^j m) \quad j\geq 1$$

and

$$V^i(b,m) = (V^i b + \sum_{i_1=0}^{i-1} V^{i_1} v(V^{i-i_1-1}m),\ V^i m) \quad i\geq 1$$

for all $(b,m)\in B\oplus M$. Notice that there exists $\delta\geq 0$ and a constant C such that $V^iM\subseteq p^{\lfloor \delta i - C\rfloor}M$ for $i\gg 0$, and the map $v: M\to B$ is continuous for the V-adic topology. Hence $V^i(b,m)$ converges to 0 in $B\oplus M$ as $i\to\infty$, where $B\oplus M$ is given the product topology. Therefore a sum of the form

$$\sum_{j\geq 1} c_j\, F^j(b,m) + \sum_{i\geq 0} d_i V^i(b,m)$$

converges in $B\oplus M$ if $c_j, d_i\in W(k)\ \forall\, i,j\geq 1$, and $\mathrm{ord}_p(c_j)\to\infty$. This shows that the action of $W(k)[F,V]$ on $B\oplus M$ can be extended to an action of $\mathrm{Cart}_p(k)$ by continuity.

To prove statement (ii), we only have to examine the effect on the pair (f,v) from a different choice of the splitting. The set of all splittings of a short exact sequence of $W(k)$-modules $0\to B\to E\to M$ is a torsor under $\mathrm{Hom}_{W(k)}(M,B)$: The difference $\mathrm{sp}'-\mathrm{sp}$ is a $W(k)$-linear map from M to B. If $\mathrm{sp}'-\mathrm{sp}=g$, $g\in\mathrm{Hom}_{W(k)}(M,B)$, and (v',f') is the pair attached to the splitting sp', then an easy computation shows that

$$f'(m)-f(m) = Fg(m)-g(Fm),\ v'(m)-v(m) = Vg(m)-g(Vm) \quad \forall\, m\in M\,.$$

We have proved (ii). ∎

(9.3) Lemma *Let X be a finite dimensional p-divisible formal group over a perfect field k of characteristic. Let N be a free $W(k)$-module of finite rank. Then there is a natural isomorphism*

$$\phi : \mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(\mathrm{M}(X), \mathrm{Cart}_p(k)\otimes_{W(k)} N) \xrightarrow{\sim} \mathrm{Hom}_{W(k)}(\mathrm{M}(X), N)$$

of $W(k)$-modules, where the $W(k)$-module structure on $\mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(\mathrm{M}(X), \mathrm{Cart}_p(k)\otimes_{W(k)} N)$ comes from the $W(k)$-module structure of N. Under the bijection in Lemma 9.2, for any element $h\in\mathrm{Hom}_{W(k)}(\mathrm{M}(X),N)$, the element

$$\phi^{-1}(h)\in \mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(\mathrm{M}(X), \mathrm{Cart}_p(k)\otimes_{W(k)} N)$$

corresponds to the pair (f_h, v_h), where the σ-linear function $f_h : M\to \mathrm{Cart}_p(k)\otimes_{W(k)} N$ and the σ^{-1}-linear function $v_h : M\to \mathrm{Cart}_p(k)\otimes_{W(k)} N$ are defined by

$$f_h(m) := F\otimes h(m)\,, \quad v_h(m) = -1\otimes h(Vm) \quad \forall\, m\in M\,.$$

PROOF. In [14, p. 617], the authors treated the case $N = W(k)$, and they used the ring $W(k)[F, V]$ instead of $\mathrm{Cart}_p(k)$. Their proof works in the present situation without change, except that the sums in the displayed formulae in pp. 619–620 of [14] should be understood as infinite sums, which all converge. ∎

(9.3.1) Remark On the other hand, in the statement of Lemma 9.3, if we replace the ring $\mathrm{Cart}_p(k)$ by $W(k)[[F, V]]$, the completion of $\mathrm{Cart}_p(k)$ with respect to the total filtration $\mathrm{Fil}^\bullet_{\mathrm{tot}}$, in the statement of Lemma 9.3, then the statement is false for $X = \widehat{\mathbb{G}_m}$.

(9.4) Lemma *Notation as in 9.1. Let*

$$\psi : H := \mathrm{Hom}_{W(k)}(M, N) \to \mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N)$$

be the map which sends each element $h \in \mathrm{Hom}_{W(k)}(M, N)$ the element

$$\psi(h) = [(f_h, v_h)] \in \mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N)$$

attached to the pair (f_h, v_h) under the bijection in Lemma 9.2, where the σ-linear map $f_h : M \to \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N$ and the σ^{-1}-linear map $v_h : M \to \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N$ are given by

$$f_h(m) = FU_0 \otimes h(m), \quad g_h(m) = -U_0 \otimes h(m) \quad \forall\, m \in M\,.$$

Recall that $\mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N)$ is a left $\mathrm{Cart}_p(k)$-module via the action of the "extra copy" of $\mathrm{Cart}_p(k)$ on $\mathrm{BC}_p(k)$, and $H \otimes_{W(k)} K$ has a natural structure as a V-isocrystal such that $F(H) \subseteq H$.

(i) *Let $\phi_{M,N_{\mathbb{Q}_p}} : \mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N_{\mathbb{Q}_p} \to H_K$ be the map constructed in Thm. 8.6. Then $\phi_{M,N_{\mathbb{Q}_p}}(\psi(h)) = h$ for every $h \in H$. In particular, the map ψ is injective.*

(ii) *Suppose that h is an element of H such that $h_1 := Vh \in H$. Then $\psi(h_1) = V_x \cdot \psi(h)$.*

(iii) *We have $\psi(Fh) = F_x \cdot \psi(h)$ for every $h \in H$.*

(iv) *The map ψ is $W(k)$-linear. Here the $W(k)$-structure of the target*

$$\mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p \otimes_{\mathrm{Cart}_p(k)} N)$$

of the map ψ comes from the canonical embedding $W(k) \hookrightarrow \mathrm{Cart}_p(k)$ and the action of the "extra copy" of $\mathrm{Cart}_p(k)$ on $\mathrm{BC}_p(k)$.

PROOF. The statement (i) is an immediate consequence of the definition of maps $\phi_{M,N_{\mathbb{Q}_p}}$ and ψ in 8.6 and 9.4 respectively.

(ii) By definition, $\psi(h_1)$ corresponds to the pair

$$(\,FU_0 \otimes h_1(m), -U_0 \otimes h_1(Vm)\,)\,,$$

while $V_x \cdot \psi(h)$ corresponds to the pair

$$(\,V_x \cdot FU_0 \otimes h(m), -V_x \cdot U_0 \otimes h(Vm)\,) = (\,FU_1 \otimes h(m), -U_1 \otimes h(Vm)\,)\,.$$

We must produce an element $g_h \in \mathrm{Hom}_{W(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N)$ such that

$$Fg_h(m) - g_h(Fm) = FU_1 \otimes h(m) - FU_0 \otimes h_1(m)$$

and

$$Vg_h(m) - g_h(Vm) = U_0 \otimes h_1(Vm) - U_1 \otimes h(Vm)$$

for all $m \in M$. Here in the requirement for g_h to be $W(k)$-linear, the $W(k)$-module structure on $\mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N$ comes from left multiplication by elements of $W(k) \subset \mathrm{Cart}_p(k)$ on $\mathrm{BC}_p(k)$. Recall that $h_1 = Vh$ means that $Vh(m) = h_1(Vm)$ for all $m \in M$, which implies that $h(Fm) = Fh_1(m)$ for all $m \in M$. A simple computation, using the commutation relations $VU_1 = U_0V, FU_0 = U_1F$ in $\mathrm{BC}_p(k)$, shows that the function $g_h : M \to \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N$ defined by

$$g_h(m) := U_1 \otimes h(m) \qquad \forall\, m \in M$$

satisfies the required conditions.

(iii) Let h_2 be the element Fh in H, so that $h_2(m) = Fh(Vm)$ for all $m \in M$. The element $\psi(Fh)$ corresponds to the pair

$$\begin{aligned}(\, FU_0 \otimes h_2(m), -U_0 \otimes h_2(Vm)\,) &= (\, FU_0 \otimes Fh(Vm), -U_0 \otimes Fh(V^2m)\,) \\ &= (\, U_1F^2 \otimes h(Vm), -U_0F \otimes h(V^2m)\,)\,,\end{aligned}$$

while the element $F_x \cdot \psi(h)$ corresponds to the pair

$$\begin{aligned}(\, F_x \cdot FU_0 \otimes h(m), -F_x \cdot U_0 \otimes h(Vm)\,) &= (\, FVU_0F \otimes h(m), -VU_0F \otimes h(Vm)\,) \\ &= (\, VU_1F^2 \otimes h(m), -VU_0F \otimes h(Vm)\,)\,.\end{aligned}$$

Let $g : M \to \mathrm{BC}_p \otimes_{\mathrm{Cart}_p(k)} N$ be the function defined by

$$g(m) = U_0F \otimes h(Vm) \quad \forall\, m \in M\,.$$

It is easy to see that g is $W(k)$-linear, and a simple computation shows that

$$Fg(m) - g(Fm) = U_1F^2 \otimes h(Vm) - VU_1F^2 \otimes h(m)$$

and

$$Vg(m) - g(Vm) = VU_0F \otimes h(Vm) - U_0F \otimes h(V^2m)$$

for all $m \in M$. We have proved (iii).

Recall that a Witt vector $u = (c_0, c_1, c_2, \ldots)$ in $W(k)$ goes to the element $\sum_{i\geq 0} V^i\langle c_i\rangle_x F^i$ under the canonical embedding $W(k) \hookrightarrow \mathrm{Cart}_p(k)$. To prove statement (iv), it suffices to check that $\psi(\langle c\rangle \cdot h) = \langle c\rangle_x \cdot \psi(h)$ for every $c \in k$. It is clear that $\langle c\rangle_x \cdot \psi(h)$ is the class represented by the pair $(\langle c\rangle_x \cdot f_h, \langle c\rangle_x \cdot v_h)$. Using the commutation relations for $\mathrm{BC}_p(k)$, we see that

$$\langle c\rangle_x \cdot f_h(m) = \langle c\rangle_x \cdot FU_0 \otimes h(m) = FU_0 \otimes \langle c\rangle h(m) = f_{\langle c\rangle h}(m)$$

and

$$\langle c\rangle_x \cdot v_h(m) = -\langle c\rangle_x \cdot U_0 \otimes h_1(Vm) = -U_0 \otimes \langle c\rangle h(Vm) = v_{\langle c\rangle h}(m)$$

for all $m \in M$. We have shown that $\langle c\rangle_x \cdot \psi(h)$ is equal to $[(f_{\langle c\rangle h}, v_{\langle c\rangle h})] = \psi(\langle c\rangle h)$. ∎

(9.5) Lemma *Notation as in Lemma 9.4. Then* $\mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N)$ *is topologically generated as a left* $\mathrm{Cart}_p(k)$*-module by the subset* $\psi(H)$*, where* $H := \mathrm{Hom}_{W(k)}(M, N)$*. In particular, the embedding* $\psi : \mathrm{Hom}_{W(k)}(M, N) \hookrightarrow \mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N) = \mathrm{M}(\mathcal{DE}(X, Y))$ *induces a surjection*

$$H/(H \cap VH) \twoheadrightarrow \mathrm{M}(\mathcal{DE}(X, Y))/V\mathrm{M}(\mathcal{DE}(X, Y))\,.$$

PROOF. Let $0 \to \mathrm{Cart}_p(k)^n \xrightarrow{\mathbf{r}} \mathrm{Cart}_p(k)(k) \to M \to 0$ be a finite free finite free resolution of the left $\mathrm{Cart}_p(k)$-module M of length one as in Prop. 5.7.3. We saw in 5.7.3 that

$$\mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N) \cong \left(\mathrm{BC}_p(k)^n \otimes_{\mathrm{Cart}_p(k)} N\right) / \left(\Gamma \cdot (\mathrm{BC}_p(k)^n \otimes_{\mathrm{Cart}_p(k)} N)\right)\,,$$

where Γ is the matrix representation of $\mathbf{r}$.

Recall that we have a injection

$$\alpha : \mathrm{Cart}_p(k) \otimes_{W(k)} \mathrm{Cart}_p(k) \to \mathrm{BC}_p(k)$$

such that $\alpha(u \otimes v) = u \cdot U_0 \cdot v$ for all $u, v \in \mathrm{Cart}_p(k)$. The above injection α induces a map

$$\bar{\alpha} : \mathrm{Cart}_p(k) \otimes_{W(k)} N = \mathrm{Cart}_p(k) \otimes_{W(k)} \mathrm{Cart}_p(k) \otimes_{\mathrm{Cart}_p(k)} N \longrightarrow \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N\,.$$

It is clear that $\bar{\alpha}(\mathrm{Cart}_p(k) \otimes_{W(k)} N)$ is stable under left multiplication by Γ. Let

$$\tilde{\alpha} : \mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{Cart}_p(k) \otimes_{W(k)} N) \xrightarrow{\tilde{\alpha}} \mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N)$$

be the map induced by $\bar{\alpha}$.

Consider the following commutative diagram.

$$\begin{array}{ccc}
H = \mathrm{Hom}_{W(k)}(M, N) & \xrightarrow{\quad = \quad} & \mathrm{Hom}_{W(k)}(M, N) = H \\
\psi \downarrow \cong & & \downarrow \psi \\
\mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{Cart}_p(k) \otimes_{W(k)} N) & \xrightarrow{\quad \tilde{\alpha} \quad} & \mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k) \otimes_{\mathrm{Cart}_p(k)} N) \\
\cong \downarrow & & \downarrow \cong \\
\mathrm{Cart}_p(k)^n \otimes_{W(k)} N / \Gamma \cdot (\mathrm{Cart}_p(k)^n \otimes_{W(k)} N) & \xrightarrow{\bar{\alpha}} & \mathrm{BC}_p(k)^n \otimes_{\mathrm{Cart}_p(k)} N / \Gamma \cdot (\mathrm{BC}_p(k)^n \otimes_{\mathrm{Cart}_p(k)} N)
\end{array}$$

We know from the structure of $\mathrm{BC}_p(k)$ that

$$\sum_{n \geq 0} V_x^n \cdot \alpha(\mathrm{Cart}_p(k) \otimes_{W(k)} \mathrm{Cart}_p(k))$$

is dense in $\mathrm{Cart}_p(k)$ with respect to the topology defined by the total filtration. Therefore the sum

$$\sum_{n \geq 0} V_x^n \psi(H)$$

is dense in $\mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k))$ with respect to the V-adic topology on the left $\mathrm{Cart}_p(k)$-module $\mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k))$. This finishes the proof of the first assertion of Lemma 9.5. The second assertion of Lemma 9.5 follows. ∎

(9.6) Theorem *Notation as in 9.1 and 9.4. Let* $r_1 = \dim(X)$, $r_2 = \dim(Y)$, $s_1 = \mathrm{ht}(X) - \dim(X)$, $s_2 = \mathrm{ht}(Y) - \dim(Y)$.

(i) *We have*

$$\dim(\mathcal{DE}(X,Y)) = r_2 s_1,\ \dim(\mathcal{DE}(X,Y)_{\text{p-div}}) = r_2 s_1 - r_1 s_2,\ \dim(\mathcal{DE}(X,Y)_{\text{unip}} = r_1 s_2\,.$$

(ii) *The image of the Cartier module* $\mathrm{M}(\mathcal{DE}(X,Y)$ *of* $\mathcal{DE}(X,Y)$ *under the isomorphism*

$$\phi_{M,N_{\mathbb{Q}_p}} : \mathrm{M}(\mathcal{DE}(X,Y))\otimes_{W(k)} K = \mathrm{Ext}^1_{\mathrm{Cart}_p(k)}(M, \mathrm{BC}_p(k)\otimes_{\mathrm{Cart}_p(k)} N)\otimes_{W(k)} K$$
$$\xrightarrow{\sim} \mathrm{Hom}_{W(k)}(M,N)\otimes_{W(k)} K = H\otimes_{W(k)} K =: H_K\,,$$

is equal to H_2*, the* $\mathrm{Cart}_p(k)$*-span of* H *in* H_K*. In other words, the map* $\phi_{M,N_{\mathbb{Q}_p}}$ *induces an isomorphism*

$$\mathrm{M}(\mathcal{DE}(X,Y)^{\text{p-div}}) = \mathrm{M}(\mathcal{DE}(X,Y)/\mathcal{DE}(X,Y)_{\text{unip}}) \xrightarrow{\sim} H_2\,,$$

where $\mathrm{M}(\mathcal{DE}(X,Y)^{\text{p-div}}) := \mathrm{M}(\mathcal{DE}(X,Y)/\mathcal{DE}(X,Y)_{\text{unip}})$ *is the maximal* p*-divisible quotient of* $\mathcal{DE}(X,Y)$*.*

(iii) *The map* $\phi_{M,N_{\mathbb{Q}_p}}$ *induces an isomorphism*

$$\mathrm{M}(\mathcal{DE}(X,Y)_{\text{p-div}}) \xrightarrow{\sim} H_1\,,$$

where H_1 *is the maximal* $W(k)$*-submodule of* H *which is stable under the action of* $\mathrm{Cart}_p(k)$*. In other words, under the isomorphism* $\mathrm{M}(\mathcal{DE}(X,Y)^{\text{p-div}} \xrightarrow{\sim} H_2$ *in* (ii) *above, the natural isogeny* $\mathcal{DE}(X,Y)_{\text{p-div}} \to \mathcal{DE}(X,Y)^{\text{p-div}}$ *corresponds to the inclusion* $H_1 \subset H_2$*.*

Proof. The statement (i) has been proved in Prop. 2.3 (ii) and Thm. 8.6.

From the short exact sequence

$$0 \to \mathrm{M}(\mathcal{DE}(X,Y)_{\text{unip}}) \to \mathrm{M}(\mathcal{DE}(X,Y)) \to \mathrm{M}(\mathcal{DE}(X,Y)^{\text{p-div}}) \to 0$$

we see that

$$\phi_{M,N_{\mathbb{Q}_p}}(\mathrm{M}(\mathcal{DE}(X,Y)) = \phi_{M,N_{\mathbb{Q}_p}}(\mathrm{M}(\mathcal{DE}(X,Y)^{\text{p-div}}) = H_2\,;$$

the last equality follows from Lemma 9.5. We have proved statement (ii).

We know from Lemma 9.1.1 (ii) that $\dim_k(H/(H\cap VH)) = r_2\, s_1$. So the source and the target of the surjection

$$H/(H\cap VH) \twoheadrightarrow \mathrm{M}(\mathcal{DE}(X,Y))/V\mathrm{M}(\mathcal{DE}(X,Y)$$

in Lemma 9.5 have same dimension. Therefore the above surjection is an isomorphism.

Lemma 9.4 gives us a natural injection $\jmath : H_1 \to \mathrm{M}(\mathcal{DE}(X,Y))$ of $\mathrm{Cart}_p(k)$-modules. The map

$$\bar{\jmath} : H_1/VH_1 \to \mathrm{M}(\mathcal{DE}(X,Y))/V\mathrm{M}(\mathcal{DE}(X,Y))$$

induced by $\jmath$ on the tangent spaces is equal to the following composition

$$H_1/VH_1 \hookrightarrow H/(H\cap VH) \xrightarrow{\sim} \mathrm{M}(\mathcal{DE}(X,Y))/V\mathrm{M}(\mathcal{DE}(X,Y)$$

of canonical maps. The map $H_1/VH_1 \hookrightarrow H/(H\cap VH)$ is an injection by Lemma 9.1.1 (i). Hence $\bar{\jmath} : H_1/VH_1 \to \mathrm{M}(\mathcal{DE}(X,Y))/V\mathrm{M}(\mathcal{DE}(X,Y))$ is an injection. We conclude by Lemma 4.3.2 that $\jmath(H_1)$ is the Cartier module of the maximal p-divisible subgroup of $\mathcal{DE}(X,Y)_{\text{p-div}}$. We have proved statement (iii). ∎

(9.6.1) Remark (i) The canonical map $H_1 \to H_2$ is an injection of left $\mathrm{Cart}_p(k)$-modules, and the quotient H_2/H_1 is the covariant Dieudonné module of the finite group scheme $\mathcal{DE}(X,Y)_{\text{p-div}} \cap \mathcal{DE}(X,Y)_{\text{unip}}$.

(ii) The proof of Thm. 9.6 shows that the tangent space of $\mathcal{DE}(X,Y)$ is canonically isomorphic to $H/(H \cap VH)$, and the tangent space of $\mathcal{DE}(X,Y)^{\text{p-div}}$ is canonically isomorphic to

$$H_2/VH_2 = \left(\sum_{i=0}^{\infty} V^i H\right) \Big/ \left(\sum_{i=1}^{\infty} V^i H\right) \xleftarrow{\sim} H \Big/ \left(H \cap \sum_{i=1}^{\infty} V^i H\right) .$$

(iii) The tangent space of $\mathcal{DE}(X,Y)/\mathcal{DE}(X,Y)_{\text{p-div}}$ is canonically isomorphic to

$$H/(H_1 + (H \cap VH)) .$$

Hence

$$\dim_k(H/(H_1+(H\cap VH))) = \dim(\mathcal{DE}(X,Y)/\mathcal{DE}(X,Y)_{\text{p-div}}) = \dim(\mathcal{DE}(X,Y)_{\text{unip}}) = r_1 s_2 .$$

(iv) Suppose that $H_1 = H_2 = H$. Then $r_1 s_2 = 0$ by (iii), therefor $s_2 = 0$, because $r_1 > 0$. In other words, the natural map $\mathcal{DE}(X,Y)_{\text{p-div}} \to \mathcal{DE}(X,Y)^{\text{p-div}}$ is an isomorphism if and only if Y is a formal torus.

(9.7) Proposition *Notation as in 8.7.1. Let M be the Cartier module of X, and identify the Cartier module of X^t with $M^\vee := \mathrm{Hom}_{W(k)}(M, W(k))$. Let $H' = \mathrm{Hom}^{\text{sym}}_{W(k)}(M, M^\vee)$ be the module of all symmetric elements in $\mathrm{Hom}_{W(k)}(M, M^\vee)$; $H' \otimes_{W(k)} K$ is a sub-V-isocrystal of $\mathrm{Hom}_{W(k)}(M, M^\vee) \otimes_{W(k)} K$. Let H_1' be the maximal $W(k)$-linear submodule of H' such that $V(H_1') + F(H_1') \subseteq H_1$. Then the map $\phi_{M,M^\vee_{\mathbb{Q}_p}}$ in 8.6 induces an isomorphism from the Cartier module of $\mathcal{DE}(X,X^t)^{\text{sym}}_{\text{p-div}}$ to H_1'.*

Proof. Immediate from Prop. 8.7.2 and Thm. 9.6. ∎

(9.8) Continuation of Example 6.3

Notation as in 6.3. The Cartier module $M = \mathrm{M}(X) = \mathrm{Cart}_p(k)/\mathrm{Cart}_p(k)\cdot(F - V^n)$, is a free $W(k)$-module of rank $n+1$, with basis

$$u_r = \text{the image of } V^r \text{ in } \mathrm{Cart}_p(k)/\mathrm{Cart}_p(k)\cdot(F - V^n) , \quad 0 \le r \le n .$$

The Cartier module $N = \mathrm{M}(Y) = \mathrm{Cart}_p(k)/\mathrm{Cart}_p(k)\cdot(F - V^{n-1})$, is a free $W(k)$-module of rank n, with basis

$$w_i = \text{the image of } V^i \text{ in } \mathrm{Cart}_p(k)/\mathrm{Cart}_p(k)\cdot(F - V^{n-1}) , \quad 0 \le i \le n-1 .$$

For any $0 \le i \le n-1$, $0 \le r \le n$, let e_{ir} be the element of $H := \mathrm{Hom}_{W(k)}(M, N)$ such that

$$e_{ir}(u_s) = \delta_{rs}\, w_i \qquad \forall\, s = 0, \ldots, n .$$

The e_{ir}'s form a $W(k)$-basis of the free $W(k)$-module $H := \mathrm{Hom}_{W(k)}(M, N)$.

The action of F and V on $H_K = H \otimes_{W(k)} K$ can be expressed as follows:

$$V\, e_{ir} = e_{i+1,r+1}\,, \quad F\, e_{ir} = e_{i+n-1,r-1} = p\, e_{i-1,r-1}$$

where we have used the convention that $e_{i,r+n+1} = p^{-1}e_{ir}$ and $e_{i+n,r} = p\, e_{ir}$ to define elements $e_{ir} \in H_K$ for all $i, r \in \mathbb{Z}$.

The submodule $H_2 = \sum_{i\geq 0} V^i H$ is equal to the $W(k)$-span of

$$e_{ir}\ \ (0 \leq i \leq r \leq n)\,; \quad p^{-1}\, e_{ir}\ \ (0 \leq r < i \leq n-1)\,.$$

The submodule $H_1 = \bigcap_{i\geq 0} V^{-i} H$ is equal to the $W(k)$-span of

$$e_{ir}\ \ (0 \leq r \leq i+1 \leq n\,,\ i \geq 0)\,; \quad p\, e_{ir}\ \ (0 \leq i < r-1 \leq n-1)\,.$$

We have $\mathrm{length}_{W(k)}(H_2/H_1) = n(n-1)$, compatible with what we saw in 6.3.6.

(9.9) Continuation of Example 6.4

Notation as in 6.4. The Cartier module $M = \mathrm{M}(X) = \mathrm{Cart}_p(k)/\mathrm{Cart}_p(k)\cdot(F - V^5)$, is a free $W(k)$-module of rank 6, with basis

$$u_r = \text{the image of } V^r \text{ in } \mathrm{Cart}_p(k)/\mathrm{Cart}_p(k)\cdot(F - V^5)\,, \quad r = 0, \ldots, 5\,.$$

The Cartier module $N = \mathrm{M}(Y) = \mathrm{Cart}_p(k)/\mathrm{Cart}_p(k)\cdot(F^2 - V)$, is a free $W(k)$-module of rank 3, with basis

$$w_i = \text{the image of } F^i \text{ in } \mathrm{Cart}_p(k)/\mathrm{Cart}_p(k)\cdot(F^2 - V)\,, \quad i = 0, 1, 2\,.$$

For any $0 \leq i \leq 5$, $0 \leq r \leq 2$, let e_{ir} be the element of $H := \mathrm{Hom}_{W(k)}(M, N)$ such that

$$e_{ir}(u_s) = \delta_{rs}\, w_i \qquad \forall\, s = 0, \ldots, 5\,.$$

The e_{ir}'s form a $W(k)$-basis of the free $W(k)$-module $H := \mathrm{Hom}_{W(k)}(M, N)$.

The action of F and V on $H_K = H \otimes_{W(k)} K$ can be expressed as follows:

$$V\, e_{ir} = e_{i+2,r+1}\,, \quad F\, e_{ir} = e_{i+1,r-1} = p\, e_{i-1,r-1}$$

where we have used the convention that $e_{i,r+6} = p^{-1}e_{ir}$ and $e_{i+3,r} = p\, e_{ir}$ to define elements $e_{ir} \in H_K$ for all $i, r \in \mathbb{Z}$.

It is easy to see that $H_2 := \sum_{i\geq 0} V^i H$ is equal to $H + W(k)\, p^{-1}e_{20}$. Similarly the $\mathrm{Cart}_p(k)$-module $H_1 := \bigcap i \geq 0\, V^{-i} H$ is the $W(k)$-span of

$$\{\, p\, e_{05}\,\} \cup \{\, e_{ir} \mid i \neq 0 \text{ or } r \neq 5,\ 0 \leq i \leq 2,\ 0 \leq r \leq 5\,\}\,.$$

We have $\mathrm{length}_{W(k)}(H_2/H_1) = 2$, compatible with what we saw in 6.4.6.

(9.9.1) A simple calculation shows that $H_1/(VH_1 + FH_1)$ is a 6-dimensional vector space over k, generated by the images of e_{01}, e_{02}, e_{03}, e_{04}, e_{14}, and e_{15}, which is compatible with what we saw in 6.4.4. Similarly, $H_2/(VH_2 + FH_2)$ is a 6-dimensional vector space over k, generated by the images of e_{01}, e_{02}, e_{03}, e_{04}, e_{05}, and e_{15}, compatible with 6.4.7

References

[1] C.-L. Chai. Every ordinary symplectic isogeny class in positive characteristic is dense in the moduli. *Invent. Math.*, 121:439–479, 1995.

[2] C.-L. Chai. Hecke orbits on Siegel modular varieties. To appear in *Geometric Methods in Algebra and Number Theory.* Birkhäuser, 2004. 35 pages. Available from `http://www.math.upenn.edu/~chai`

[3] C.-L. Chai. Monodromy of Hecke-invariant subvarieties. Preprint, 2003, 10 pages. Available from `http://www.math.upenn.edu/~chai`

[4] C.-L. Chai. A rigidity result for p-divisible formal groups. Preprint, 2003, 11 pages. Available from `http://www.math.upenn.edu/~chai`

[5] M. Demazure. *Lectures on p-divisible groups.* LNM 302, 1972.

[6] L. Illusie. Déformation de groupes de Barsotti-Tate. *Astérisque*, **127** (1985), 151–198.

[7] N. M. Katz Slope filtration of F-isocrystals. *Astérisque*, **63** (1979), 113–164.

[8] A. Grothendieck, *Groupes de Monodromie en Géométrie Algébrique (SGA 7) I.* LNM **288**, Springer-Verlag, 1972.

[9] A. Grothendieck, *Groupes de Barsotti-Tate et Cristaux de Dieudonné.* Sém. Math. Sup. **45**, Presses de l'Université de Montreal, 1974.

[10] Y. Manin. The theory of commutative formal groups over fields of finite characteristic. *Russian Math. Surveys*, **18** (1963), 1–80.

[11] W. Messing, *The Crystals Associated to Barsotti-Tate Groups: with Applications to Abelian Schemes.* LNM **264**, Springer-Verlag, 1972.

[12] D. Mumford, Biextensions of formal groups. In *Algebraic Geometry*, Proc. Intl. Colloq., Bombay 1968, 307–322. Oxford University Press, 1969.

[13] P. Norman, An algorithm for computing local moduli of abelian varieties. *Ann. Math.* **101** (1975), 499–509.

[14] T. Oda and F. Oort, Supersingular abelian varieties. In *Intl. Symp. on Algebraic Geometry, Kyoto 1977*, 595–621 Kinokuniya Book Store, 1978.

[15] F. Oort. Subvarieties of moduli spaces. *Invent. Math.*, 24:95–119, 1974.

[16] F. Oort, A stratification of a moduli space of polarized abelian varieties. In *Moduli of Abelian Varieties*, 345–416. Progress in Math. **195**, Birkhäuser, 2001.

[17] F. Oort, Foliation in moduli spaces of abelian varieties. *J. Amer. Math. Soc.* **17** (2004), 267–296.

[18] F. Oort. Minimal p-divisible groups. To appear in *Ann. Math.* Available from `http://www.math.uu.nl/people/oort/`

[19] F. Oort and T. Zink. Families of p-divisible groups with constant Newton polygon. To appear in *Documenta Mathematica.*

[20] K. Ribet. p-adic interpolation via Hilbert modular forms. *Proc. Symp. Pure Math.*, vol. 29, *Algebraic Geometry–Arcata 1974*, pp. 581–592, Amer. Math. Soc., 1975.

[21] T. Zink. *Cartiertheorie kommutativer former Gruppen.* Teubner, 1984.

[22] T. Zink *On the Oort foliation.* Letter to C.-L. Chai, May 1, 1999

[23] T. Zink. On the slope filtration. *Duke Math. J.*, 109:79–95, 2001.

Ching-Li Chai
Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104-6395
U. S. A.
email: chai@math.upenn.edu