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Conjectures.

1. RH. GRH

$$\sum_{n\le x}\Lambda(n) = x + O(x^{\frac{1}{2}}(\log x)^2)$$

where $\Lambda(n) = \begin{cases} \log p & \text{if } n=p^m \\ 0 & \text{otherwise} \end{cases}$

2. Tate's conjecture (poles orders of L-functions)

3. Special values of L-functions
(Birch, Sw-Dyer. Deligne. Beilinson. Bloch Kato)

4 Langlands' program.
(generalization of CFT to non-abelian ext.)

5. Others

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<u>Example</u>. $K = \mathbb{Q}(\sqrt{-5}) \supseteq \mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\cdot\sqrt{-5}$.

$$\begin{array}{c} K \\ | \\ \mathbb{Q} \end{array}$$

$\mathcal{O}_K^\times = \{\pm 1\}$ ⟵ root of unity. and in $\mathcal{O}_K$

let $\alpha = 1 + 2\sqrt{-5}$. $\quad \bar{\alpha} = 1 - 2\sqrt{-5}$.

$\alpha\bar{\alpha} = 21 = 3 \cdot 7$. ⟵ not uniquely factorized

$$\alpha^2 = -19 + 4\sqrt{-5} = \underbrace{(2+\sqrt{-5})}_{=\lambda}\underbrace{(-2+3\sqrt{-5})}_{=\mu}$$

$\bar{\alpha}^2 = \bar{\lambda}\bar{\mu}$. ⇝ jump to "Hilbert class field". etc

⇝ the "genesis" of "ideal numbers" $\sqrt{\lambda}.\sqrt{\mu}.\sqrt{\bar{\lambda}}.\sqrt{\bar{\mu}}$.

Dedekind domains

<u>Dedekind domain</u> A.

$\iff$ noetherian integral domain. integrally closed.
$\dim \leq 1$.

<u>Def</u>. fractional ideals : f.g. A-submodule of $K = \mathrm{frac}(A)$

<u>Facts</u>.

1. $\forall\, \mathfrak{p} \neq 0$, prime ideal $\subseteq A$, then $A_{\mathfrak{p}} = $ D.V.R
2. Unique factorization of fractional ideals.
$I = \prod_{\text{finite}} \mathfrak{p}_i^{e_i}$. $\quad e_i \in \mathbb{Z}$.

<u>Basic examples</u>

$$\begin{array}{c} K \\ |\ \text{finite} \\ \mathbb{Q} \end{array}$$

then $\overset{=A}{\mathcal{O}_K}$ is a Dedekind domain

⇝ $\forall\, \mathfrak{p} \neq 0$ prime $\quad A_{\mathfrak{p}}$

$\hat{A}_{\mathfrak{p}}$: again a Dedekind domain.

$K_{\mathfrak{p}} = \mathrm{frac}(\hat{A}_{\mathfrak{p}})$ a finite extension of $\mathbb{Q}_p$, $p = \mathrm{char}(\mathcal{O}_K/\mathfrak{p})$

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$$\begin{array}{ccc} B & \subseteq & L \\ | & & |\ \text{finite sep.} \\ A & \subseteq & K = \mathrm{frac}(A) \end{array}$$

A : Dedekind
B = integral closure of A in L.

Properties.

- B is a finite A-module (using $tr_{L/K}$) (bound the denom.)
- (1.⇒) $\forall\ \wp^{\neq 0}$ prime ideal in A.

Detour : localization vs. completion

$A_\wp$: localization.

$\hat{A}_\wp$: completion $= \varprojlim_n A/\wp^n = \varprojlim_n A_\wp/\wp^n A_\wp$

$A_\wp \longrightarrow \hat{A}_\wp$ (henselian ?)

Notation: we'll write $A_\wp$ for $\hat{A}_\wp$.

$$B_\wp = B \otimes_A A_\wp = \prod_{\mathfrak{P}_i} B_{\mathfrak{P}_i} \qquad \mathfrak{P}_i \mid \wp$$

$$\begin{array}{c} B_\wp \\ | \\ A_\wp \end{array} \quad \text{D.V.R.}$$

"completion might make $B_\wp$ not an integral domain."

$$B_\wp/\wp B_\wp = B/\wp B \cong (B/\mathfrak{P}_1)^{e_1} \times \cdots (B/\mathfrak{P}_r)^{e_r} \qquad e_i \geq 1$$

$e_i = e(\mathfrak{P}_i/\wp)$ ramification degree.

defined by: $\wp B_{\mathfrak{P}_i} = \mathfrak{P}_i^{e_i} B_{\mathfrak{P}_i}$.

$$\Longrightarrow [B_\wp : A_\wp] = [L : K]$$

$\|$

length of $B_\wp/\wp B_\wp$ over $A_\wp/\wp A_\wp$

$\|$

$\sum_{i=1}^{r} e_i f_i$.

$f_i = [B/\mathfrak{P}_i : A/\wp]$
$= [B_{\mathfrak{P}_i}/\mathfrak{P}_i B_{\mathfrak{P}_i} : A_\wp/\wp A_\wp]$

Now assume L/K is Galois.

$\Rightarrow$ $G = \mathrm{Gal}(L/K)$ operates on L, B, $\{\mathfrak{P}_1, \ldots, \mathfrak{P}_r\}$

- G acts on $\{\mathfrak{P}_1, \ldots, \mathfrak{P}_r\}$ transitively.
- $e_1 = e_2 = \cdots = e_r = e$
- $f_1 = \cdots = f_r = f$.

<u>Def.</u> $D_{\mathfrak{P}_i}$ = the fixer of $\mathfrak{P}_i$ in G = decomposition group at $\mathfrak{P}_i$
(or the valuation defined by $\mathfrak{P}_i$)

$\bigcup\!|$

$I_{\mathfrak{P}_i}$ - the inertia group at $\mathfrak{P}_i$.

$D_{\mathfrak{P}_i} \circlearrowright B_{\mathfrak{P}_i}$.

$\circlearrowright B_{\mathfrak{P}_i}/\mathfrak{P}_i B_{\mathfrak{P}_i}$. too.

check the surjectivity $\rightarrow$ $I_{\mathfrak{P}_i} := \mathrm{Ker}\left(D_{\mathfrak{P}_i} \twoheadrightarrow \mathrm{Aut}\left((B_{\mathfrak{P}_i}/\mathfrak{P}_i B_{\mathfrak{P}_i}) / (A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})\right)\right)$

$\Rightarrow \{D_{\mathfrak{P}_i}\}$ are conjugate, so are $\{I_{\mathfrak{P}_i}\}$.

<u>Caution:</u> $\mathfrak{P}_i/\mathfrak{p}$ <u>unramified</u> $\iff$ $e_i = 1$ & $\kappa_{\mathfrak{P}_i}/\kappa_{\mathfrak{p}}$ is separable.
$\iff$ $I_{\mathfrak{P}_i}$ is trivial. (check!)

If $\kappa_{\mathfrak{P}_i}$ is finite and $\mathfrak{P}_i/\mathfrak{p}$ unramified,

$\rightsquigarrow$ define $\varphi_{\mathfrak{P}_i/\mathfrak{p}} \in D_{\mathfrak{P}_i}$. the Frobenius $x \mapsto x^{\#\kappa_{\mathfrak{p}}}$ $\leftarrow q_{\mathfrak{p}}$

(arithmetic Frobenius)

= (geometric Frobenius)$^{-1}$

<u>Examples:</u> $A = A_{\mathfrak{p}}$, $B = B_{\mathfrak{p}}$ $\mathfrak{P} \mid \mathfrak{p}$

Two opposite situations:

1. unramified: $B = A[x]/(f(x))$ $f(x)$: monic, and $\bar{f}(x)$ is irred. and sep. in $\kappa_{\mathfrak{p}}[x]$

2. totally ramified: $B = A[x]/(f(x))$ $f(x)$: Eisenstein polynomial

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In general.

$$\begin{tikzcd} B \arrow[dd, dash] \arrow[dr, dash, "\leftarrow \text{totally ram}"] & \\ & C \arrow[dl, dash, "\leftarrow \text{unramified}"] \\ A & \end{tikzcd}$$

A : <u>complete</u> d.v.r. $\qquad K = \mathrm{frac}(A)$

$$\begin{tikzcd} & L \arrow[d, dash, "\text{finite}"] \\ A \subseteq & K \end{tikzcd}$$

κ = residue field of A.
(perfect , finite)

<u>unramified</u> <u>extensions</u>

<u>Fact</u>. $\begin{array}{c} L \\ \big|\,{\scriptstyle \text{unram.}} \\ K \end{array} \qquad \begin{array}{c} \kappa_L \\ \big|\,{\scriptstyle \text{sep}} \\ \kappa \end{array} \implies [L : K] = [\kappa_L : \kappa]$.

roughly speaking. $\left\{ \begin{array}{l} \text{finite} \\ \text{unram'd ext.} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{finite (separable)} \\ \text{extension of } \kappa \end{array} \right\}$

$\rightsquigarrow$ Have an iso. $\mathrm{Gal}(K^{ur}/K) \xrightarrow{\sim} \mathrm{Gal}(\kappa^{sep}/\kappa)$.

$$\mathrm{Gal}(\kappa^{sep}/\kappa) \xrightarrow[\text{when } \kappa \text{ is finite}]{\sim} \hat{\mathbb{Z}}$$

<u>proof</u>. exercise !?

"Essential ingredient", structure of finite unram ext'n
\+ Hensel's lemma.

- Examples of finite unram ext'n:

$K[x]/(f(x))$, $\quad f(x) \in A[x]$. monic. irreducible.
$\qquad \bar{f}(x) \in \kappa[x]$ irr. sep.

<u>proof</u>. L/K. $\quad e = 1$?

We know. $f(L/K) \geq \deg(f)$, but we have equality.

- Conversely, must show: L/K unramified, have that form

proof. Take $\bar{\alpha} \in \kappa_L$ s.t. $\kappa_L = \kappa(\bar{\alpha})$.
Take $\alpha \in \mathcal{O}_L$ lifting $\bar{\alpha}$.
Then the minimal polynomial $f(x)$ of α is integral
$\deg f(x) = [L : K]$; since $\bar{f}(\bar{\alpha}) = 0$. so $\bar{f}(x)$ min. poly. of $\bar{\alpha}$.
done. □

K : local field. (i.e. fraction field of complete d.v.r. with finite residue field $\Leftrightarrow$ locally compact nonarchimedian field)

Prop. Fix $n \in \mathbb{N}$ There is only a finite number of ext'n of K of degree $= n$. if $(\operatorname{char} K, n) = 1$.

proof. Two ingredients
(1) Structure of totally ram. ext'n of a local field
(2) Krasner's lemma. (proved)

$$\begin{array}{c} L \\ | \\ (L/K)^{ur} \\ | \\ K \end{array}$$

$[L : K] \leq n$.
$\exists$ only finite number of $(L/K)^{ur}$ (1 for each $m \mid n$).
So: suffices to show Prop. for totally ram. ext'n.
Totally ram'd ext'n are given by Eisenstein poly
These are parametrized by $(\mathfrak{m})^{d-1} \times A^\times$ $(d = n/m)$.
$\mathfrak{m}^{d-1} \times A^\times$ is compact, $\therefore$ by Kranser's lemma done □.

Back to totally ram. ext'n:
$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ Eisenstein. $\Rightarrow$ $f(x)$ irred.
Let $f(\alpha) = 0$ $\Rightarrow$ $v(a_{n-1}\alpha^{n-1} + \cdots + a_0) = v(a_0) = v(\alpha^n)$
$\Longrightarrow e = n$.

Conversely. $\begin{array}{c} L \\ n \,|\, \text{tot. ram.} \\ K \end{array}$ π_L $\operatorname{Irred}(\pi_L, x, L/K) = g(x)$
$\deg g(x) \leq n$

Say $g(\alpha) = \underset{\substack{\downarrow \\ \text{val} \equiv m \\ (\text{mod } n)}}{\alpha^m} + \underset{\substack{\downarrow \\ \text{val} \equiv m-1 \\ (\text{mod } n)}}{b_{m-1}\alpha^{m-1}} + \cdots + \underset{\substack{\downarrow \\ \text{val} \equiv 0 \\ (\text{mod } n)}}{b_0}$

$\therefore m = n$.

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Different and Discriminant

Good for · 1 bound ramification

2 Compute $\mathcal{O}_K$.

$$\begin{array}{ccc} L & \supseteq & \mathcal{O}_L \\ \text{sep.}\Big| & & \Big| \\ K & \supseteq & \mathcal{O}_K \end{array} \qquad \begin{array}{c} \mathrm{Spec}(\mathcal{O}_L) \supseteq Z \ni P_1, \cdots, P_m \quad \swarrow \text{different} \\ \Big\downarrow \pi \\ \mathrm{Spec}(\mathcal{O}_K) \supseteq W \ni \pi(P_1), \cdots \pi(P_m) \quad \swarrow \text{discriminant} \end{array}$$

π : étale ? Say π is not étale at $P_1 \cdots\cdots P_m$

Say $\mathcal{O}_L = \mathcal{O}_K[\alpha] \cong \mathcal{O}_K[x]/(f(x)) \rightsquigarrow$ different $f'(\alpha)$

$\mathrm{Spec}(\mathcal{O}_L[f'(\alpha)^{-1}])$ $\qquad \mathrm{Spec}(\mathcal{O}_L/(f'(\alpha)))$

$f'(\alpha)$ gives a closed subset in $\mathrm{Spec}(\mathcal{O}_L)$. tells where π is not étale

different $:= f'(\alpha)\,\mathcal{O}_L$. (defined by Jacobian ideal)

In general, although $\mathcal{O}_L$ is not generated by a single element over $\mathcal{O}_K$, but this is true after localization !

proof. exercise. (hint : true if K is complete, then true for Zariski localization.)

i.e. locally over $\mathrm{Spec}\,\mathcal{O}_K$. this statement is true

a posteori, true locally over $\mathrm{Spec}\,\mathcal{O}_L$.

Usual definition (idea: trace pairing is strongly nondegenerate for finite étale extensions.)

$$\begin{array}{ccc} L & \supseteq & \mathcal{O}_L \\ \text{sep}\Big| & & \Big| \\ K & \supseteq & \mathcal{O}_K \end{array}$$

$\mathcal{D}_{L/K}^{-1} := \{\alpha \in L \mid tr_{L/K}(\alpha \cdot x) \in \mathcal{O}_K \ \forall x \in \mathcal{O}_L\}$

is an $\mathcal{O}_L$-submodule, f.g. $\supseteq \mathcal{O}_L$

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<u>Exercise</u>. If $\mathcal{O}_L = \mathcal{O}_K[\alpha]$. then $\mathfrak{D}_{L/K} = f'(\alpha)\,\mathcal{O}_L$.

<u>Discriminants</u>

$$\mathrm{disc}_{L/K} = N_{L/K}(\mathfrak{D}_{L/K})$$

Now try to define N. N is multiplicative

$$\begin{array}{ccc} \mathfrak{P} & & L \\ | & & \text{Galois}\,| \\ \mathfrak{p} & & K \end{array}$$

Want to define $N_{L/K}(\mathfrak{P})$

$\mathfrak{P} = \mathfrak{P}_1 \cdots \mathfrak{P}_r$ have e. f. , $[L:K] = ref$.

$$\prod_{\sigma \in \mathrm{Gal}(L/K)} \sigma\mathfrak{P}_1 = (\mathfrak{P}_1 \cdots \mathfrak{P}_r)^{ef} = \mathfrak{p}^f = N_{L/K}(\mathfrak{P})$$

$$\mathfrak{p}\mathcal{O}_L = (\mathfrak{P}_1 \cdots \mathfrak{P}_r)^e$$

Also, suppose $\{\alpha_1 \ldots \alpha_n\}$ is an $\mathcal{O}_K$-basis of $\mathcal{O}_L$.

$\Longrightarrow$ $\mathrm{disc}_{L/K} = \det\left(\mathrm{tr}(\alpha_i\alpha_j)_{1\le i,j\le n}\right)$.

<u>Exercise</u>. Verify that these two def. are equivalent when $\mathcal{O}_L = \mathcal{O}_K[\alpha]$.

Now assume all ext'ns are separable.

$$\begin{array}{c} L \\ | \\ K' \\ | \\ K \end{array}$$

<u>exer</u>. $\mathfrak{D}_{L/K} = \mathfrak{D}_{L/K'} \cdot (\mathfrak{D}_{K'/K} \cdot \mathcal{O}_L)$

$\mathrm{disc}_{L/K} = N_{K'/K}(\mathrm{disc}_{L/K'}) \cdot (\mathrm{disc}_{K'/K})^{[L:K']}$

<u>Example</u> $K = \mathbb{Q}(\sqrt[3]{5})\,/\,\mathbb{Q}$. , $\alpha = \sqrt[3]{5} \in \mathbb{R}$

min. poly. of $\alpha = x^3 - 5 = f(x)$

$f'(\alpha) = 3\alpha^2$. $\rightsquigarrow$ $\mathfrak{D}_{K/\mathbb{Q}} \mid (3\alpha^2)$ $\left(\text{since } \begin{array}{c} \mathcal{O}_K \supseteq \mathbb{Z}[\alpha] \\ | \\ \mathbb{Z} \end{array}\right)$

conclude. $K/\mathbb{Q}$ is unram. outside 3 & 5.

over (5) : totally ram.

$$\begin{tikzcd} L = \mathbb{Q}(\alpha, \omega) \arrow[r, dash] \arrow[d, dash] & \mathbb{Q}(\alpha) = K \arrow[d, dash] \\ K' = \mathbb{Q}(\omega) \arrow[r, dash] & \mathbb{Q} \end{tikzcd} \qquad \omega = \zeta_3 = e^{2\pi\sqrt{-1}/3}$$

over (5) :

$$\begin{tikzcd} & L \arrow[dl, dash, "\text{tot. ram}"'] \arrow[dr, dash, "\text{unram.}"] & \\ K' \arrow[dr, dash, "\text{unram}"'] & & K \arrow[dl, dash, "\text{tot. ram.}"] \\ & \mathbb{Q} & \end{tikzcd}$$

over (3) :

$$\begin{tikzcd} & L \arrow[dl, dash] \arrow[dr, dash] & \\ K' \arrow[dr, dash] & & K' \arrow[dl, dash] \\ & \mathbb{Q}. & \end{tikzcd}$$

look at $\mathbb{Q}_3(\sqrt[3]{5})$. $\qquad x^3 - 5$ over $\mathbb{Q}_3$.

$\underline{\text{mod } 9}$: $\quad x^3 \equiv \pm 1 \pmod 9$ if $x \not\equiv 0 \pmod 9$

conclusion. $x^3 - 5$ is irreducible. (mod 3 is not enough by strong Hensel's lemma.)

so now

$$\begin{tikzcd} \mathbb{Q}_3(\sqrt[3]{5}) \arrow[d, dash] \\ \mathbb{Q}_3 \end{tikzcd}$$

first thing $\quad x^3 - 5 \equiv 0$ in $\mathbb{Z}_3$

$\Rightarrow \quad x^3 - 5 \equiv x^3 + 1 \equiv (x+1)^3$

$\therefore \quad (3) = (\alpha + 1)^3$.

or let $\beta = \alpha + 1$. $\Rightarrow 0 = (\beta - 1)^3 - 5 = \beta^3 - 3\beta^2 + 3\beta - 6$

Eisenstein

$\therefore$ totally ramified.

$$\therefore \begin{tikzcd} K = \mathbb{Q}(\sqrt[3]{5}) \arrow[d, dash] & \mathfrak{p} \arrow[d, dash] & \mathfrak{q}. \arrow[d, dash] \\ \mathbb{Q} & (3) & (5) \end{tikzcd}$$

$\Rightarrow \quad \mathfrak{p}^2 \cdot \mathfrak{q}^2 \mid \mathfrak{D}_{K/\mathbb{Q}}$

tamely ram. : char $\kappa \nmid e$, otherwise wild.

$\therefore \quad \mathfrak{q}^2 \,\|\, \mathfrak{D}_{K/\mathbb{Q}}$.

and $\quad \mathfrak{p}^3 \mid \mathfrak{D}_{K/\mathbb{Q}}$

but $\mathfrak{D}_{K/\mathbb{Q}} \mid (3\alpha^2) = \mathfrak{p}^3 \cdot \mathfrak{q}^2 \quad \therefore \mathfrak{D}_{K/\mathbb{Q}} = \mathfrak{p}^3 \cdot \mathfrak{q}^2$

and $\quad \mathrm{disc}_{K/\mathbb{Q}} = \pm 3^3 \cdot 5^2$

$\Rightarrow \quad \mathcal{O}_K = \mathbb{Z}[\sqrt[3]{5}] \qquad$ (dual under $\mathrm{tr}_{K/\mathbb{Q}}$ are the same !)

__Exercise.__ $\Lambda \subseteq \mathcal{O}_K$ such that $\mathrm{disc}(\Lambda) = \mathrm{disc}_{K/\mathbb{Q}}$
$\Longrightarrow \Lambda = \mathcal{O}_K$

Go back to Galois picture:

$$\begin{tikzcd}[column sep=small]
 & L = \mathbb{Q}(\sqrt[3]{5}, \omega) \arrow[dl, dash] \arrow[dr, dash] & \\
K' = \mathbb{Q}(\omega) \arrow[dr, dash] & & K = \mathbb{Q}(\sqrt[3]{5}) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

unram. outside 3.5.

over (5):

$$\begin{tikzcd}[column sep=small]
 & L \arrow[dl, dash, "\therefore\ \text{tot. ram}"'] \arrow[dr, dash, "\text{unram} \to \text{inert}"] & \\
K' \arrow[dr, dash, "\text{unram.}"'] & & K' \arrow[dl, dash, "\text{tot. ram}"] \\
 & \mathbb{Q} &
\end{tikzcd}$$

$e = 3, \; f = 2$

split or inert ? look at Frob : $\mathrm{Frob}_5(\zeta_3) = \zeta_3^5 = \zeta_3^{-1}$.
non-trivial !

$\therefore$ (5) is inert !

over (3)

$$\begin{tikzcd}[column sep=small]
 & L \arrow[dl, dash] \arrow[dr, dash] & \\
K' \arrow[dr, dash, "e=2"'] & & K \arrow[dl, dash, "e=3"] \\
 & \mathbb{Q} &
\end{tikzcd}$$

$\Rightarrow e_{L/\mathbb{Q}} = 6$!
totally ramified.

__Exercise.__ Find $\mathcal{O}_L$. $(\supsetneq \mathbb{Z}[\omega, \sqrt[3]{5}])$

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Higher ramification groups

$$\begin{array}{c} L \\ \text{Gal}\ \Big| \\ K \end{array}$$

L, K : local fields
$\mathrm{Gal}(L/K) = G$.
(need only: κ_L/κ : sep.; usual assumption: κ perfect.)

Have a sequence of subgroups which filter:

$$G \geq G_0 \geq G_1 \geq \cdots \cdots \geq G_N = \{e\}$$

2 indexing schemes $\left\{ \begin{array}{l} \text{lower numbering — uses } \Gamma_L \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}. \\ \text{upper numbering. — uses } \Gamma_K \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}. \end{array} \right.$

(Γ_L : value group of L $(\cong \mathbb{Z})$)
(lower vs. upper : depending on the ext'n situation, i.e fixing L or fixing K)

$\exists$ correspondence : $G^v = G_u$ call $v = \varphi(u)$, $u = \psi(v)$
$= \varphi_{L/K}(u)$ $= \psi_{L/K}(v)$

in fact, ψ, φ are increasing, piecewise linear, and $\varphi = \psi^{-1}$

<u>Theorem</u> If L/K is abelian, then "interesting" upper numbering occurs only in integers.

<u>Definition</u>. $G_i = \{ s \in G \mid s : \mathcal{O}_L/\mathfrak{p}_L^{i+1}\cdot\mathcal{O}_L \longrightarrow \mathcal{O}_L/\mathfrak{p}_L^{i+1}\cdot\mathcal{O}_L$ is the identity$\}$

<u>Remark</u> $G_0 = I$, the inertia group.
$G_{-1} = G$ by convention.

<u>numerical functions</u> $i = i_G : G \longrightarrow \mathbb{N}$
$j = j_G : G \longrightarrow \mathbb{N}$

<u>classical def</u>. may assume : $\mathcal{O}_L = \mathcal{O}_K[\alpha]$
then s operates trivially on $\mathcal{O}_L/\mathfrak{p}_L^{i+1}\mathcal{O}_L \Leftrightarrow s(\alpha) \equiv \alpha \pmod{\mathfrak{p}_L^{i+1}}$

$$\Longleftrightarrow \underbrace{v_L(s(\alpha) - \alpha)}_{= i_G(s)} \geq i + 1$$

In other words, $s \in G_i \iff i_G(s) \geq i+1$

$j_G(s) = i_G(s) - 1. \qquad \iff j_G(s) \geq i$

$$\begin{array}{ll} & L \\ \text{Gal} & \mid\ )\, G' \\ G & K' \\ & \mid \\ & K \end{array} \qquad \Rightarrow \quad G_i' = G_i \cap G'.$$

$i_{G'}(s) = i_G(s) \quad \forall s \in G'$. same for $j_{G'}$.

<u>Prop</u>
$$\begin{array}{lcc} & L & \{e\} \\ & \mid & \mid \\ \text{Gal} & K' & H \\ & \mid & /\Delta \\ & K & G \end{array}$$
then $i_{G/H}(\sigma) = \dfrac{1}{e(L/K')} \displaystyle\sum_{\substack{s \in G \\ s \mapsto \sigma}} i_G(s)$

<u>pf.</u> may assume $\mathcal{O}_L = \mathcal{O}_K[\alpha]$, $\mathcal{O}_{K'} = \mathcal{O}_K[\beta]$

Given $\sigma \in G/H$. compare:

$e(L/K')\cdot v_{K'}(\sigma\beta - \beta)$ and $\displaystyle\sum_{t\in H}(st\alpha - \alpha)$ (some $s \mapsto \sigma$)

Let $f(X) =$ min. poly. of α over K'

$= \displaystyle\prod_{t\in H}(X - tH).$

$\Longrightarrow \quad f(s\alpha) \equiv \displaystyle\prod_{t\in H}(st\alpha - \alpha) \mod \mathcal{O}_L^\times.$

$\qquad \|$

$f(s\alpha) - f^s(s\alpha) = f(s\alpha) - f^\sigma(s\alpha). = (f - f^\sigma)(s\alpha) \equiv 0 \pmod{\sigma\beta - \beta}$

since $f \equiv f^\sigma \pmod{\sigma\beta - \beta}.$

Write $\beta = g(\alpha). \qquad g(X) \in \mathcal{O}_K[X]$

$\sigma(\beta) = g(s\alpha)$

$\Rightarrow \quad f(X) \mid (g(X) - \beta)$

$\Rightarrow \quad f(s\alpha) \mid (g(s\alpha) - \beta) = \sigma\beta - \beta.$

$\therefore \quad \sigma\beta - \beta \equiv \displaystyle\prod_{t\in H}(st\alpha - \alpha) \pmod{\mathcal{O}_L^\times}$

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$$\begin{array}{cc} L & \{e\} \\ | & | \\ K' & H \\ | & \triangle \\ K & G \end{array}$$

<u>Showed</u>

$$i_{G/H}(\sigma) = \frac{1}{e(L/K')} \sum_{\substack{s \mapsto \sigma \\ s \in G}} i_G(s)$$

$$= \frac{1}{e(L/K')} \sum_{\substack{s \mapsto \sigma \\ s \in G_0}} i_G(s)$$

$$\Longrightarrow \quad j_{G/H}(s) = \frac{1}{e(L/K')} \sum_{\substack{s \mapsto \sigma \\ s \in G_0}} j_G(s). \qquad (e(L/K') = \#|H \cap G_0|).$$

Take $\mathcal{O}_L = \mathcal{O}_K[\alpha]$ $\quad f(X) = \mathrm{irr}_K(\alpha). = \prod_{s \in G} (X - s(\alpha))$

$$f'(\alpha) = \prod_{\substack{s \in G \\ s \neq id}} (\alpha - s(\alpha))$$

$$\Longrightarrow \quad v_L(\mathcal{D}_{L/K}) = v_L(f'(\alpha))$$

$$= \sum_{\substack{s \in G \\ s \neq id}} v_L(\alpha - s(\alpha)) = \sum_{\substack{s \in G \\ s \neq id}} i_G(s) = \sum_{\substack{s \in G_0 \\ s \neq id}} i_G(s)$$

Let δ_{G_i} be the delta function for G_i $\quad \left(G_i(s) = \begin{cases} 0 & \text{if } s \notin G_i \\ 1 & \text{if } s \in G_i \end{cases} \right)$

$$\Rightarrow \quad i_G(s) = \sum_{i \geq 0} \delta_{G_i}(s).$$

$$\Longrightarrow \quad v_L(\mathcal{D}_{L/K}) = \sum_{\substack{s \in G_0 \\ s \neq id}} i_G(s) = \sum_{\substack{s \in G_0 \\ s \neq id}} \sum_{i \geq 0} \delta_{G_i}(s)$$

$$= \sum_{i \geq 0} [\mathrm{Card}(G_i) - 1]$$

therefore : $\quad v_K(\mathrm{disc}_{L/K}) = [G : G_0] \sum_{\substack{s \in G_0 \\ s \neq id}} i_G(s) = [G : G_0] \sum_{i \geq 0} (\mathrm{Card}(G_i) - 1$

Next : what v satisfies $\quad (G/H)_u = G_vH/H$?

$$\sigma \in (G/H)_u \iff j_{G/H}(\sigma) \geq u$$

$$\iff \frac{1}{\#(H_0)} \sum_{\substack{s \mapsto \sigma \\ s \in G_0}} j_G(s) \geq u.$$

Pick s_0 s.t. $\quad j_G(s_0) = \max\{ j_G(s) \mid s \mapsto \sigma \} =: m$, i.e $s_0 \in G_m \setminus G_m$

$\forall s, \ s \mapsto \sigma. \quad \Rightarrow \exists t \in H_0 \quad s = s_0 t$

if $j_G(t) < m$. then $t \notin G_m \Rightarrow s_0 t \notin G_m \Rightarrow j_G(s_0 t) = j_G(t)$

if $j_G(t) \geq m \Rightarrow j_G(s_0 t) \geq m$ therefore $j_G(s_0 t) = m$

since m is the maximum.

Conclusion: $j_G(s_0 t) = \min\big(j_G(t),\ \underset{m}{\overset{\shortparallel}{j_G^{\#}(\sigma)}} \big)$

Therefore $\sigma \in (G/H)_u \iff \frac{1}{\#(H_0)} \sum_{t \in H_0} \min(j_G(t).\ \underset{j_G^{\#}(\sigma)}{\overset{\shortparallel}{m}}) \geq u$

<u>Define</u> $\varphi_H(x) = \int_0^x \frac{dt}{(H_0 : H_t)} = \frac{1}{\#(H_0)} \int_0^x \#(H_t)\, dt$

$\sigma \in H_t \overset{\lfloor t \rfloor}{\iff} \sigma \in H_{\lceil t \rceil}$. $\lceil t \rceil$: round-up of t.

$\Rightarrow \varphi_H(x) = \frac{1}{\#(H_0)} \Big(\sum_{k=1}^{\lfloor t \rfloor} \#(H_k) + (t - \lfloor t \rfloor) \# (H_{\lceil t \rceil}) \Big)$

$\varphi_H(u)$ is supposed to be v
where $G_u = G^v$

$\varphi_H(u) = \frac{1}{g_0}(g_0 + g_1 + \dots g_u)$

where $u \in \mathbb{N}$ + $g_i = \# G_i$

so I'm not sure of the exact formula but somehow you want to use $(G/H)_u = (G/H)^v = G^v_u H/H$

9/23/96

$$\begin{array}{cc} L & \{1\} \\ | & \\ K' & H \\ | & \Delta \\ K & G \end{array}$$

Define $\varphi_{L/K}(x) = \int_0^x \frac{dy}{(G_0 : G_y)}$

$$= \frac{1}{\#(G_0)} \int_0^x \underbrace{\#(G_y)}_{\displaystyle \| \atop \displaystyle \sum_{s\in G} \delta_{G_y}(s) = \sum_{s\in G_0} \delta_{G_y}(s)} dy$$

$$= \frac{1}{\#(G_0)} \sum_{s\in G_0} \int_0^x \delta_{G_y}(s)\, ds.$$

$$\delta_{G_y}(s) = \begin{cases} 1 & \text{if } y \le j_G(s) \\ 0 & \text{if } y > j_G(s). \end{cases}$$

$$= \frac{1}{\#(G_0)} \sum_{s\in G_0} \min(x, j_G(s)).$$

Last time $\sigma \in (G/H)_u \iff \frac{1}{\#(H_0)} \sum_{t\in H_0} \min(j_H(t), j^{\#}(\sigma)) \ge u$

$$\iff \varphi_{L/K}(j^{\#}(\sigma)) \ge u$$
$$\iff j^{\#}(\sigma) \ge \psi_{L/K'}(u).$$
$$\iff \sigma \in G_{\psi_{L/K'}(u)} \cdot H/H$$

So:

Proposition. $(G/H)_u = G_{\psi_{L/K'}(u)} \cdot H/H$

or equivalently $(G/H)_{\varphi_{L/K'}(v)} = G_v H/H$

Remark. $u \in \Gamma_{K'} \otimes \mathbb{R}$. $\psi_{L/K'}(u) \in \Gamma_L \otimes \mathbb{R}$

$\varphi_{L/K'} : \Gamma_L \otimes \mathbb{R} \longrightarrow \Gamma_{K'} \otimes \mathbb{R}$

$\psi_{L/K'} = (\varphi_{L/K})^{-1}$

$$\begin{tikzcd} \Gamma_L \otimes \mathbb{R} \arrow[rr, "\varphi_{L/K}"] \arrow[dr, "\varphi_{L/K'}"'] & \circlearrowright & \Gamma_K \otimes \mathbb{R} \\ & \Gamma_{K'} \otimes \mathbb{R} \arrow[ur, "\varphi_{K'/K}"'] & \end{tikzcd}$$

← Have to show that this diagram commutes!

$\rightsquigarrow$ $G_u = G^{\varphi(u)}$

$$(G/H)_u = G_{\psi_{L/K'}(u)} \cdot H/H \overset{\text{def}}{=} G^{\varphi_{L/K}(\psi_{L/K'}(u))} \cdot H/H = G^{\varphi_{K'/K}(u)} \cdot H/H$$

$\|$

$(G/H)^{\varphi_{K'/K}(u)}$

9/25/96

Need to show: $\varphi_{L/K} = \varphi_{K'/K} \circ \varphi_{L/K'}$

since $LHS(0) = 0 = RHS(0)$.

the only thing to show now is:

$$\varphi'_{L/K} = (\varphi_{K'/K} \circ \varphi_{L/K'})'$$

$$(*) \qquad \varphi'_{L/K}(x) = \varphi'_{K'/K}(\varphi_{L/K'}(x)) \cdot \varphi'_{L/K'}(x)$$

(*): $LHS = \dfrac{\#(G_x)}{\#(G_0)}$

$$RHS = \frac{\#\left((G/H)_{\varphi_{L/K'}(x)}\right)}{\#\left((G/H)_0\right)} \cdot \frac{\# H_x}{\# H_0}$$

while $\# G_0 = \#\left((G/H)_0\right) \cdot \# H_0$.

and $(G/H)_{\varphi_{L/K'}(x)} = G_x \cdot H/H \Rightarrow$ numerators are the same.

done. $\#$

$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_N = \{e\}$ $\qquad L/K$: Galois

$G_i \trianglelefteq G$ since they are all the kernels of some maps.

<u>Fact</u> G_1 is a p-group. G_0/G_1 is abelian, and of order prime-to-p. ($p = \operatorname{char} k$)

$(G_i = \ker(G \longrightarrow \operatorname{Aut}(\mathcal{O}_L/\mathfrak{p}_L^{i+1})))$.

Study G_i/G_{i+1} . $i \geq 0$:

Choose a uniformizer $\pi = \pi_L$. $\quad \mathcal{O}_L = \mathcal{O}_K[\pi]$

$s \in G_i \implies s\pi \equiv \pi \pmod{\mathfrak{p}^{i+1}}$

$\implies \frac{s\pi}{\pi}$ is a unit in $\mathcal{O}_L$ if $j(s) \geq 1$ i.e. $s \in G_1$.

$s\pi = \pi v$ for some $v \in \mathcal{O}^\times$

But, if $s \in G_i$, then $\frac{s\pi}{\pi} \equiv 1 \pmod{\mathfrak{p}^i}$

<u>Define</u> $\mathcal{O}_{L,i}^\times = \{ u \in \mathcal{O}_L^\times \mid u \equiv 1 \pmod{\mathfrak{p}^i} \}$ — principal unit.

i.e. $s\pi/\pi \in \mathcal{O}_{L,i}^\times \quad \forall s \in G_i$.

so: have a map $\quad G_i/G_{i+1} \longrightarrow \mathcal{O}_{L,i}^\times/\mathcal{O}_{L,i+1}^\times$

(indep. of choice of π) $\qquad s \longmapsto s\pi/\pi$. (also preserve the group law)

So: $\operatorname{gr}.(G.\operatorname{fil}) := \bigoplus_{i\geq 0} (G_i/G_{i+1}) \longrightarrow \operatorname{gr}.(\mathcal{O}_L^\times)$

After choosing a uniformizer π, $\quad 1 + \pi^i a \longmapsto \bar{a} \in k_L$.

$\implies \mathcal{O}_{L,i}^\times/\mathcal{O}_{L,i+1}^\times \xrightarrow{\sim} k_L \qquad i > 0$

When $i = 0$. $\quad \mathcal{O}_L^\times/\mathcal{O}_{L,1}^\times \xrightarrow{\sim} k_L^\times$

9/27/96

$$\mathrm{gr.}(\mathcal{O}_L^\times) \overset{\text{non-canonical}}{\cong} \kappa_L^\times \oplus \kappa_L \oplus \kappa_L \oplus \cdots$$
$$= \big(\underbrace{S^\cdot \kappa_L(1)}_{\text{graded algebra.}}\big)^\times \qquad (\text{"Tate twist"})$$

one maximal ideal $\xrightarrow{\text{namely}}$ augmented ideal.

$$s \longmapsto s\pi/\pi = 1 + \pi^i \cdot u \qquad u \in \mathcal{O}_L$$

conjugation. $\quad tst^{-1} \longmapsto tst^{-1}(t\pi)/t\pi = t(s\pi)/t\pi$

$$= \frac{t\pi \cdot (t(1+\pi^i u))}{t\pi} = t(1+\pi^i u) = 1 + (t\pi)^i \cdot (tu)$$
$$= 1 + \pi^i \underline{\left(\tfrac{t\pi}{\pi}\right)^i tu}$$

$t \longmapsto \frac{t\pi}{\pi}$ is a 1-dim character of G with values in $\kappa_L^\times$ (cyclotomic char.)

$\kappa_L(1)$ twists by the char. $t \longmapsto \frac{t\pi}{\pi}$!

$G_0 \geq G_1 \geq G_2 \geq \cdots \quad \rightsquigarrow \mathrm{gr.}(G)$

commutator: $\quad [x.y] = xyx^{-1}y^{-1} \quad \rightsquigarrow$ get a bracket structure $\downarrow$ turns out to be trivial

if $x \in G_i$, $y \in G_j \to [x\ y] \in G_{i+j}$.

<u>Examples.</u> $\quad \ell$: prime number

$\mathbb{Q}(\sqrt{\ell})$

$\mathbb{Q}(\sqrt{-\ell})$

$$\begin{array}{c} \mathbb{Q}(\sqrt{\ell}) \\ | \\ \mathbb{Q} \end{array} \quad \rightsquigarrow$$

$\mathbb{Q}[\mu_{p^n}]$

9/30/96

Adèles. Idèles. ← 1 dim'l global objects.

Setup K = global field (i.e a finite ext'n of $\begin{cases}\mathbb{Q} \\ \mathbb{F}_p(t)\end{cases}$)

We're trying to define $\mathbb{A}_K$. the ring of adèles of K

Def. $\mathbb{A}_K = \prod'_{\substack{v \\ \text{places of } K}} K_v$ $\qquad v$: place of K.

1) finite . non-archimedian

2) infinite . archimedian

v: finite $\longleftrightarrow$ $\mathfrak{p}_v \subseteq \mathcal{O}_K$. $\quad K_v$. $\mathfrak{p}$-adic completion

v. infinite $\longleftrightarrow$ $K_v \hookrightarrow \mathbb{C}$

sometimes it factors thru $\mathbb{R}$. ← real place.

complex places come in pairs (conjugate). $\# = 2r_2$

real places $\# = r_1$.

$K \overset{\text{dense}}{\hookrightarrow} K_v$.

business of $\prod'$. $= \{(x_v) \mid x_v \in \mathcal{O}_v$ for almost all $\mathcal{O}_v\}$

topology on $\mathbb{A}_K$: fundamental system of open nbhds of 0:

$\{\text{inf places}\} \subseteq S$: finite set of places, take $\prod_{v \notin S} \mathcal{O}_v \times \prod_{v} U_v$.

U_v: open nbhd of $0 \in K_v$

$\rightsquigarrow$ locally compact for $\mathbb{A}_K$.

exer. 1) $K \hookrightarrow \mathbb{A}_K$ discrete

2) $\mathbb{A}_K / K$ is compact.

3) $\mathbb{A}_\mathbb{Q} \otimes_\mathbb{Q} K \xrightarrow{\sim} \mathbb{A}_K$. K # field

4). $\mathbb{A}_K^\times = \mathbb{G}_m(\mathbb{A}_K)$? Describe it.

5) describe $GL_n(\mathbb{A}_K)$.

K: global field. $\mathbb{A}_K$: K-adèles.

$\mathbb{A}_K^\times = \{ (x_v) \mid x_v \neq 0 \ \forall v, \ x_v^{-1} \in \mathcal{O}_v$ for almost all $v \}$

$= \{ (x_v) \mid x_v \in K_v^\times, \ x_v \in \mathcal{O}_v^\times$ for almost all $v \}$

$=:$ K-idèles $= \mathbb{G}_m(\mathbb{A}_K)$

$\mathbb{A}_K = \mathbb{G}_a(\mathbb{A}_K)$

exer. 0) (absolute values)

$\forall v$. define a norm $\| \cdot \|_v$ on K_v; μ_{K_v}: Haar measure.

$\mu_{K_v}(ax) = \|a\|_v \cdot \mu_{K_v}(x)$

Make it explicit!

(non-arch case: $\|a\| = c^{v(a)}$; decide c

arch case: $\begin{cases} \mathbb{R} \to * dx \\ \mathbb{C} \to * dx\,dy \end{cases}$)

1) (product formula) $\prod_v \|a\|_v = 1 \quad \forall a \in K^\times$.

2) $\mathbb{A}_{K,1}^\times = \{ x = (x_v) \in \mathbb{A}_K^\times \mid \|x\| = \prod_v (x_v) = 1 \} \geq K^\times$.

Show that $\mathbb{A}_{K,1}^\times / K^\times$ is compact.

Integration on Locally Compact Topological Groups

G: loc. comp. ab. top. gp. $\rightsquigarrow \exists$ a Haar measure μ_G on G.

$H < G$. H closed $\rightsquigarrow \mu_H \cdot \mu_{G/H} = \mu_G$

Normalization.

- G compact. $\rightsquigarrow \mu_G(G) = 1$ after normalization. dual cases.
- G discrete $\rightsquigarrow \mu_G$ is the counting measure.

(Pontrjagin duality). for loc. comp. abelian top. gps. G.

$G \rightsquigarrow \hat{G} = \mathrm{Hom}_{\mathrm{cont}}(G, \mathbb{C}_1^\times)$. exer. top. on $\hat{G}$?

is again a loc. comp. top. gp.

Th'm. $G \xrightarrow{\sim} (\hat{G})^\wedge$ is an isom between top. gps.

Prop. G: compact $\Longrightarrow \hat{G}$ discrete

G: discrete $\Longrightarrow \hat{G}$ compact.

$$\begin{tikzcd} G \arrow[d, dash, "\text{discrete}"'] \\ H \arrow[d, dash, "\text{compact}"'] \\ \{e\} \end{tikzcd}$$

example. v : non arch

$$\begin{tikzcd} K_v \arrow[d, dash] \\ I \arrow[d, dash] \\ (0) \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[d, dash, "\text{compact}"'] \\ H \arrow[d, dash, "\text{discrete}"'] \\ \{e\} \end{tikzcd}$$

example. $\mathbb{R}$. $\mathbb{R}^\times$. $\mathbb{C}$. $\mathbb{C}^\times$

$\mathbb{A}_K$ $\mathbb{A}^\times_{K,1}$.

$$\begin{tikzcd} & G \arrow[dl, dash, "\text{disc.}"'] \arrow[dr, dash, "\text{disc}"] & \\ H_1 \arrow[dr, dash, "\text{comp}"'] & & H_2 \arrow[dl, dash, "\text{comp.}"] \\ & \{e\} & \end{tikzcd}$$

$\rightsquigarrow$ $\mu_{G/H_1} \cdot \mu_{H_1} = C \cdot \mu_{G/H_2} \cdot \mu_{H_2}$. $C = ?$

Evaluate the measure of $H_1 \cdot H_2$.

LHS $= [H_1 H_2 : H_1]$ $\Longrightarrow C = \dfrac{[H_1 H_2 : H_1]}{[H_1 H_2 : H_2]}$

RHS $= [H_1 H_2 : H_2]$

(or $[H_2 : H_1]$ symbolically)

$$\begin{tikzcd} & G \arrow[dl, dash, "\text{comp.}"'] \arrow[dr, dash, "\text{comp.}"] & \\ H_1 \arrow[dr, dash, "\text{disc.}"'] & & H_2 \arrow[dl, dash, "\text{disc}"] \\ & \{e\} & \end{tikzcd}$$

$\rightsquigarrow$ $\mu_{G/H_1} \cdot \mu_{H_1} = C' \cdot \mu_{G/H_2} \cdot \mu_{H_2}$.

exer. Find C'. (first. understand $H_1 \cap H_2$).

10/4/96

Fourier Transforms

G: locally compact (Hausdorff) abelian group

f: function on $G \longrightarrow \hat{f}$: function on $\hat{G}$

$$\hat{f}(y) = \int_G f(x) \langle -x, y \rangle \, dx$$

Fourier inversion: $f(x) = \left(\int_{\hat{G}} \hat{f}(y) \langle x, y \rangle \, dy \right) \cdot c$

if $c = 1$ then dx (on G) and dy (on $\hat{G}$) are dual to each other

Example. G: compact $\Longrightarrow$ $\hat{G}$ discrete.

f: function on G. $\hat{\mu} \in \hat{G}$

$$\hat{f}(\hat{\mu}) = \int_G f(g) \, \hat{\mu}(g)^{-1} \, dg \qquad \left(\int_G dg = 1 \right)$$

$$\Longrightarrow f(g) = c \cdot \sum_{\hat{\mu}} \hat{f}(\hat{\mu}) \cdot \hat{\mu}(g) \qquad \text{(counting measure on } \hat{G}\text{)}$$

Assertion: $c = 1$!

Peter-Weyl G compact (not nec. abel.) top group

$\Longrightarrow L^2(G)$ = orthogonal direct sum of fin dim'l unitary rep. of G

$$= \bigoplus_{\pi \in \hat{G}} (V_\pi \otimes V_\pi^*)$$

equivariant for both the left and the right actions.

- Every central function f on G is a linear combination of characters of G.

- $\int_G \chi_1(g) \, \bar{\chi}_2(g) \, dg = \begin{cases} 1 & \text{if } \chi_1 \cong \chi_2 \\ 0 & \text{otherwise} \end{cases}$

Apply to G commutative (+ compact):

Every $f \in L^2(G)$ can be written as $f(x) = \sum_{\hat{\mu} \in \hat{G}} a_{\hat{\mu}} \cdot \hat{\mu}(x)$

$$\Rightarrow \int_G f(x)\,\hat{\nu}(x)^{-1}\,dx = \sum_{\hat{\mu}} a_{\hat{\mu}} \int \hat{\mu}(x)\hat{\nu}(x)^{-1}\,dx = a_{\hat{\nu}}$$

<u>Example of dual Haar measures</u>

self-dual Haar measures:

$$G \times G \longrightarrow \mathbb{C}_1^\times \quad \text{perfect pairing}$$

e.g. $\mathbb{R}^n$.

$$\begin{aligned} \mathbb{R}^n \times \mathbb{R}^n &\longrightarrow \mathbb{C}_1^\times \\ (x, \ y) &\longmapsto e(\langle x, y \rangle) \end{aligned}$$

Lebesgue measure on $\mathbb{R}^n/\mathbb{Z}^n$ and counting measure on $\mathbb{Z}^n$ give rise to the usual Lebesgue measure on $\mathbb{R}^n$.

<u>Conclusion</u> $dx_1 \cdots dx_n$ is self-dual w.r.t. the above pairing.

e.g. $\mathbb{C}$ or in general K (global field)

$$\begin{aligned} K \times K &\longrightarrow \mathbb{C}_1^\times \\ (x, \ y) &\longmapsto e(\mathrm{Tr}_{K/\mathbb{Q}}(xy)) \end{aligned}$$

10/7/96

$K_v = \mathbb{C}$. want a "right" pairing ?

$$\mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}_1^\times$$
$$(z_1, z_2) \longmapsto e(\mathrm{Tr}_{\mathbb{C}/\mathbb{R}}\, z_1 z_2).$$

What is the self-dual measure under this pairing ?

$$(0) \subseteq \overbrace{\mathbb{Z}[\sqrt{-1}]}^{dxdy \to 1} \subseteq \mathbb{C}. \qquad dx\,dy$$

$$\mathbb{Z}[\sqrt{-1}]^\perp = \tfrac{1}{2}\cdot \mathbb{Z}[\sqrt{-1}]. \qquad \text{(computed from definition.)}$$

so the dual picture:

$$(0) \subseteq \tfrac{1}{2}\mathbb{Z}[\sqrt{-1}] \subseteq \mathbb{C} \qquad 4\,dx\,dy.$$

$\Longrightarrow$ $2|dx\cdot dy|$ is the self-dual measure (w.r.t. that pairing.)

$$2|dx\cdot dy| = |dz \wedge d\bar{z}|$$

Now for local field. K = local field char. 0. complete at a non-arch place. (char $K = p > 0$)

pairing $K \times K \longrightarrow \mathbb{C}_1^\times$.

$\mathbb{Q} \times \mathbb{Q} \longrightarrow \mathbb{C}^\times$. $(x, y) \longrightarrow e(xy)$,

$\mathbb{Q}_p \times \mathbb{Q}_p \longrightarrow \mathbb{C}^\times$. want to extend from the above one continuously via <u>the p-adic topology</u>.

BUT the above pairing on $\mathbb{Q}$ is not continuous !

Idea. look at $\mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}] \longrightarrow \mathbb{C}^\times$ first.

then extends by continuity to a pairing $\mathbb{Q}_p \times \mathbb{Q}_p \longrightarrow \mathbb{C}^\times$.

$(x_1, x_2) \longmapsto e_p(x_1 x_2)$.

$e_p : \mathbb{Q}_p \longrightarrow \mathbb{C}^\times$ s.t. $e_p(\mathbb{Z}_p) = 1$, $e_p(\frac{1}{p}) = \exp(2\pi i/p)$.

done.

$K \times K \longrightarrow \mathbb{C}^\times$

$(x, y) \longmapsto e_p(\mathrm{Tr}_{K/\mathbb{Q}_p}(x_1 x_2))$

<u>exer</u> This identifies K with its dual.

Self-dual measure ?

$(0) \subseteq \mathcal{O}_K \subseteq K$ $\qquad$ $\mu_{naive}(\mathcal{O}) = 1$

$\updownarrow$

dual $\quad (0) \subseteq \mathcal{D}_{K/\mathbb{Q}_p}^{-1} \subseteq K$ $\qquad$ $\mu'(\mathcal{D}_{K/\mathbb{Q}_p}^{-1}) = 1$

$\Rightarrow \mu' = N\mathcal{D}_{K/\mathbb{Q}_p}^{-1} \cdot \mu_{naive}$

$\therefore$ self-dual measure :

$$\mu(\mathcal{O}_K) = (N\mathcal{D}_{K/\mathbb{Q}_p})^{-\frac{1}{2}} = \|\mathcal{D}_{K/\mathbb{Q}_p}\|_p^{\frac{1}{2}}$$

Cautious when talking about $\mathbb{F}_q((t))$ ——

No canonical guy like $\mathbb{Q}_p$. $\mathbb{R}$ —, ; just take the naive Haar measure.

K : number field. $\quad \mathbb{A}_K / K$: compact.

<u>Reason</u>. $\quad \mathbb{A}_\mathbb{Q}/\mathbb{Q} \longleftarrow [0.1[\times \prod_p \mathbb{Z}_p$

surjection (bijection) $\qquad$ fundamental domain.

$\therefore \mathbb{A}_\mathbb{Q}/\mathbb{Q}$ compact

$\mu_{\mathbb{A}_K} = \bigotimes_v' \mu_{K_v}$ $\qquad$ $\mu_{\mathbb{A}_\mathbb{Q}}(\mathbb{A}_\mathbb{Q}/\mathbb{Q}) = 1$

$\Rightarrow \quad \mathbb{A}_K = \mathbb{A}_\mathbb{Q} \otimes_\mathbb{Q} K.$

$\qquad K = \mathbb{Q} \otimes_\mathbb{Q} K$ $\qquad \rightsquigarrow \quad \mathbb{A}_K/K = \bigoplus_{i=1}^{N} (\mathbb{A}_\mathbb{Q}/\mathbb{Q})$

10/9/96

K : number field.

$$\begin{array}{c} \mathbb{A}_K \\ \cup\!| \\ K \end{array} \qquad \mathbb{A}_K = \mathbb{A}_{\mathbb{Q}} \otimes_{\mathbb{Q}} K$$

$$K \cap \prod_v \mathcal{O}_{K_v} = \mathcal{O}_K$$

$K \leq \mathbb{A}_K$ discrete $\iff$ $\mathcal{O}_K \hookrightarrow \prod_v K_v$ is discrete

While $x \in \mathcal{O}_K$ $\quad |N_{K/\mathbb{Q}}(x)| \geq 1$

$$|N_{K/\mathbb{Q}}(x)| = \prod_{v \text{ arch}} \|x\|_v$$

(holds as a consequence of product formula)

Also $\quad \mathbb{A}_K / K \cong (\mathbb{A}_{\mathbb{Q}}/\mathbb{Q})^{[K:\mathbb{Q}]}$ $\leftarrow$ compact.

$\forall\, x \in K^\times$ $\quad x : \mathbb{A}_K/K \xrightarrow{\sim} \mathbb{A}_K/K$ (multiplication by)

$\rightsquigarrow$ multiplication by x fixes every Haar measure on $\mathbb{A}_K$.

(Automorphism of compact groups has modulus 1.)

$\Downarrow$

product formula

i.e.

$$\operatorname{mod}(x) = \prod_v \operatorname{mod}_{K_v}(x) = \prod_v \|x\|_v = 1$$

$$\mu_{\mathbb{A}_K} = \prod_v{}' \underset{\substack{\uparrow \\ \text{self-dual.}}}{\mu_{K_v}} \quad \text{normalized Haar measure on } \mathbb{A}_K.$$

<u>exer.</u> $\mu_{\mathbb{A}_K}$ is the self-dual measure on $\mathbb{A}_K$.
w.r.t. $\mathbb{A}_K \times \mathbb{A}_K \longrightarrow \mathbb{C}_1^\times$
$((x_v), (y_v)) \longmapsto \prod_v e_v(x_v y_v)$

Compute : $\mu_{\mathbb{A}_K}(\mathbb{A}_K / K) = ?$

$$\mathbb{A}_K / K = \bigoplus \underset{\substack{\uparrow \\ \text{represented by } [0,1[\times \prod_p \mathbb{Z}_p.}}{(\mathbb{A}_{\mathbb{Q}} / \mathbb{Q})} \xi_i \qquad \{\xi_i\} \ \mathbb{Z}\text{-basis of } \mathcal{O}_K$$

$\Longrightarrow$ $\mathbb{A}_K / K$ is represented by $\bigoplus ([0,1[\times \prod_p \mathbb{Z}_p) \cdot \xi_i$

volume of $\bigoplus ([0,1[\times \prod_p \mathbb{Z}_p) \cdot \xi_i$

$$= \underbrace{\prod_p \Big(\prod_{v|p} \mu_v\Big)(\mathbb{Z}_p \xi_1 \oplus \cdots \oplus \mathbb{Z}_p \xi_n)}_{\substack{\| \\ \prod_{v:\text{finite}} \mu_{K_v}(\mathcal{O}_v) \\ \| \\ |\mathrm{disc}_{K/\mathbb{Q}}|^{-\frac{1}{2}}}} \times \Big(\prod_{v \text{ arch}} \mu_{K_v}\Big)([0,1)\xi_1 \oplus \cdots \oplus [0,1)\xi_n)$$

10/11/96

$$\mu_{\mathbb{A}_K}(\mathbb{A}_K/K) = \left(\prod_v{}' \mu_{K_v}\right)\left(\left(\prod_{v\text{ finite}} \mathcal{O}_{K_v}\right) \times \left(\sum_{i=1}^{n} [0,1)\xi_i\right)\right)$$

$$= \left(\prod_{v\text{ finite}} \mu_{K_v}(\mathcal{O}_{K_v})\right) \times \prod_{v|\infty} \mu_{K_v}\left(\sum [0,1)\xi_i\right)$$

$$= \prod_p \cdot \prod_{v|p} \left(N\mathfrak{D}_{K_v/\mathbb{Q}_p}\right)^{-\frac{1}{2}} \times \quad ''$$

$$= |\mathrm{disc}_{K/\mathbb{Q}}|^{-\frac{1}{2}} \times \quad ''$$

Now, what is $\prod_{v|\infty} \mu_{K_v}\left(\sum [0,1)\cdot \xi_i\right)$?

$$K\otimes\mathbb{R} \longrightarrow \prod_{v|\infty} K_v$$

$$x\in K \longmapsto (\iota_v(x))_v. \qquad \iota_v : K \hookrightarrow K_v \text{ canonical}$$

<u>Want</u>: to relate $\mu_{K\otimes\mathbb{R}}\left(K\otimes\mathbb{R}/\sum\mathbb{Z}\xi_i\right)$ with $\mathrm{disc}_{K/\mathbb{Q}}$.

$$|\mathrm{disc}_{K/\mathbb{Q}}| = |N_{K/\mathbb{Q}}(\mathfrak{D}_{K/\mathbb{Q}})|$$

$$= \#(\mathcal{O}_K/\mathfrak{D}_{K/\mathbb{Q}})$$

$$= \#(\mathfrak{D}^{-1}_{K/\mathbb{Q}}/\mathcal{O}_K) = |\det(\mathrm{tr}_{K/\mathbb{Q}}(\xi_i\xi_j))|$$

(ξ_i) : $\mathbb{Z}$-basis of $\mathcal{O}_K$.

$\sigma_i : K \hookrightarrow \mathbb{C}$
$1\le i\le n$

$$\Rightarrow \mathrm{tr}_{K/\mathbb{Q}}(\xi_i\xi_j) = \begin{pmatrix} \sigma_1\xi_1 & \sigma_2\xi_1 & \sigma_3\xi_1 & \cdots & \sigma_n\xi_1 \\ \sigma_1\xi_2 & & & & \\ & & & & \\ & & & & \end{pmatrix}\begin{pmatrix} \sigma_1\xi_1 & \sigma_1\xi_2 & \\ \sigma_2\xi_1 & & \\ \vdots & & \\ & & \end{pmatrix}$$

$$\Rightarrow |\det(\mathrm{tr}_{K/\mathbb{Q}}(\xi_i\xi_j))| = |\det(\sigma_i\xi_j)|^2$$

Let $\sigma_1 \cdots \sigma_{r_1}$ real embedding

$\underbrace{\sigma_{r_1+1}, \sigma_{r_1+2}}_{\text{conj.}} \cdots \underbrace{\sigma_{r_1+2r_2-1}, \sigma_{r_1+2r_2}}_{\text{conj}}$: cx embedding.

matrix $(\sigma_i \xi_j)$ looks like

$$\left|\begin{matrix} \sigma_1\xi_1 & \cdots & \sigma_1\xi_n \\ | & & \\ \sigma_{r_1}\xi_1 & \cdots & \sigma_{r_1}\xi_n \\ \sigma_{r_1+1}\xi_1 & \cdots & \sigma_{r_1+1}\xi_n \\ \overline{\sigma_{r_1+1}\xi_1} & \cdots & \overline{\sigma_{r_1+1}\xi_n} \\ & & \\ \overline{\sigma_{r_1+2r_2-1}\xi_1} & \cdots & \overline{\sigma_{r_1+2r_2-1}\xi_n} \end{matrix}\right| \quad |\det *|$$

$$\begin{matrix} dx_1 \cdots dx_{r_1} & dz_{r_1+1}\, d\bar z_{r_1+1} \cdots dz_{r_1+r_2}\, d\bar z_{r_1+r_2} \\ \underbrace{\mathbb{R}\times \cdots \times\mathbb{R}}_{r_1} & \times\ \underbrace{\mathbb{C}\times\cdots\times\mathbb{C}}_{r_2} \end{matrix}$$

have: $\mathbb{Z}\xi_1 + \cdots + \mathbb{Z}\xi_n$. want its covolume

using x y coord. instead of $z, \bar z$ coord. get

$$\underset{\substack{z\,\bar z \\ \text{coord.}}}{|\det *|} = 2^{r_2} \left|\underset{\substack{x\,y \\ \text{coord}}}{\det \Delta}\right|$$

$$= 2^{r_2} \left(dx_1 \cdots dx_{r_1}\, dx_{r_1+1} \cdots\right) \left(\prod_{v|\infty} K_v / \mathbb{Z}\xi_i\right)$$

Conclude: $\prod_{v|\infty} \mu_{K_v}(\quad)$

$$= \left|\det |\sigma_i \xi_j|\right| = (\text{disc}_{K/\mathbb{Q}})^{\frac{1}{2}}$$

Conclude: $\mu_{\mathbb{A}_K}(\mathbb{A}_K/K) = 1$.

2^{nd} way of computing

$\mu_{\mathbb{A}_K}(\mathbb{A}_K/K) =$

$$(0) \subseteq K \subseteq \mathbb{A}_K$$

$$(0) \subseteq K^{\perp} \subseteq \mathbb{A}_K$$

Question $K = K^{\perp}$?

$\Rightarrow$ self-dual measure $\mu_{\mathbb{A}_K}$

$\mu_{\mathbb{A}_K}(\mathbb{A}_K/K) = 1$.

$K \subseteq K^{\perp}$. by product formula

$\Rightarrow K^{\perp}/K$ is compact + discrete $\Rightarrow$ finite

Also by product formula

$K^{\perp}$ is also a $\cancel{K}$ $\mathbb{Q}$-module,

10/18/96

K : global field

Theorem $\mathbb{A}_{K,1}^\times / K^\times$ is compact.

$$\begin{array}{l} \mathbb{A}_{K,1}^\times / K^\times \\ \quad \cap \\ \mathbb{A}_K^\times / K^\times \text{ (idèle class group of } K) \end{array}$$

non-compact since the norm map has a non-compact image.

Corollary (Dirichlet) The class number of K is finite.

e.g. K : number field. $\mathcal{O}_K$ ring of integers.

$\rightsquigarrow$ the group of invertible ideals.

= finite product of prime ideals in $\mathcal{O}_K$ and their inverses.

$=: I$

$P = \{\text{principal invertible ideals}\} \subseteq I$

$\quad \| $

$\{x \cdot \mathcal{O}_K \mid x \in K^\times\}$

Finiteness of class number h_K of K :

$$h_K = |P \backslash I| < \infty$$

the ideal class group of K.

exercises 1. Every element of I/P is represented by an ideal of $\mathcal{O}_K$

2. $h_K = 1 \iff \mathcal{O}_K$ is a PID.

proof of corollary There is a map $\mathbb{A}_K^\times \longrightarrow I$,

$$x = (x_v) \longmapsto \prod_{v \text{ finite}} \mathfrak{p}_v^{v(x_v)} = \prod_{v \text{ finite}} x_v \mathcal{O}_v \text{ ; it's surjective.}$$

$$\text{Ker} = \left(\prod_{v|\infty} K_v^\times \times \prod_{v \text{ finite}} \mathcal{O}_v^\times\right) =: U_{\Sigma_\infty}$$

$\therefore I \cong \mathbb{A}_K^\times / U_{\Sigma_\infty}$; $I/P \cong \mathbb{A}_K^\times / K^\times \cdot U_{\Sigma_\infty}$

Also $\mathbb{A}_{K,1}^\times \twoheadrightarrow I$ $\qquad \cong \mathbb{A}_{K,1}^\times / (K^\times \cdot U_{\Sigma_\infty} \cap \mathbb{A}_{K,1}^\times)$

= a quotient of $\mathbb{A}_{K,1}^\times / K^\times$ by open subgroup

$\therefore I/P$ is compact (from $\mathbb{A}_{K,1}^\times / K^\times$) and discrete

$\Longrightarrow I/P$ is finite. Q.E.D.

proof of theorem

$\mathrm{Spec}\,\mathbb{Q}[x,x^{-1}] = \mathbb{G}_m \hookrightarrow \mathbb{A}_1 \times \mathbb{A}_1$ closed embedding.

$\mathrm{Spec}\,\mathbb{Q}[x,y]/(xy-1) \quad x \longmapsto (x, x^{-1})$.

Similarly $\mathbb{A}_K^\times \hookrightarrow \mathbb{A}_K \times \mathbb{A}_K$ closed embedding.

WTS. $\exists$ compact subsets $U, V \subseteq \mathbb{A}_K$ s.t. $\forall x \in \mathbb{A}_{K,1}^\times$, $\exists a \in K^\times$ s.t. $ax \in U$ and $(ax)^{-1} \in V$.

Recall $\mathbb{A}_K/K$ is compact therefore has finite volume

Take $W = \prod_{v \text{ finite}} \mathcal{O}_v \times \prod_{v|\infty} \{\|x\|_v \leq M\}$. $\mathrm{vol}(W) = M' \gg 1$

Given $x \in \mathbb{A}_{K,1}^\times$, $\mu_{\mathbb{A}_K}(x^{-1}W) = \mu_{\mathbb{A}_K}(W) = M'$

$\Longrightarrow \exists a \in K^\times$ s.t. $a \in x^{-1}W$. $\Rightarrow ax \in W$

Similarly, $\exists b \in K^\times$ s.t. $b \in ax\,W$.

i.e. $(ax)^{-1} \in b^{-1}W$

need to estimate b: $b \in W \cdot W$ = compact.

but $b \in K$ which is discrete, $\therefore$ there are only a finite number of possibilities for b. q.e.d.

S: a finite number of finite places.

$\mathcal{O}_{K,S}$ = S-integers in $\mathcal{O}_K$

$= \{x \in K \mid x \in \mathcal{O}_v \ \forall v \notin S \cup \Sigma_\infty\}$

$\mathcal{O}_{K,S}^\times$: units in $\mathcal{O}_{K,S}$

Dirichlet unit theorem $\mathcal{O}_K^\times \cong \underbrace{W_K}_{\text{roots of unity}} \times \mathbb{Z}^{r_1+r_2-1}$

More generally. $\mathcal{O}_{K,S}^\times \cong W \times \mathbb{Z}^{r_1+r_2+\#(S)-1}$

10/21/96

K : number field. S : places of K s.t. $S \supseteq \Sigma_{K,\infty}$

$\mathcal{O}_{K,S}$ = the ring of S-integers in K.

$\rightsquigarrow$ $\mathcal{O}_S$ - ideals.

$\rightsquigarrow$ (ideal) class group for $\mathcal{O}_S$ = fractional $\mathcal{O}_S$-ideals / (principal)

if finite. card. $= h_S$.

<u>Want</u>. $\mathbb{A}_K^\times \longrightarrow \{$fractional $\mathcal{O}_S$-ideals$\}$

$$(x_v) \longmapsto \Big(\prod_{v \notin S} x_v \mathcal{O}_v\Big) \cap K = \prod_{v \notin S} \mathfrak{p}_v^{v(x_v)}, \quad \text{surjective.}$$

$$\rightsquigarrow \quad 0 \longrightarrow \underbrace{\Big(\prod_{v \in S} K_v^\times\Big) \times \Big(\prod_{v \notin S} \mathcal{O}_v^\times\Big)}_{= U_S} \longrightarrow \mathbb{A}_K^\times \longrightarrow\!\!\!\!\to \{\text{fractional } \mathcal{O}_S\text{-ideals}\}$$

$$\mathbb{A}_K^\times \supseteq \mathbb{A}_{K,1}^\times$$

$$0 \longrightarrow U_S \cap \mathbb{A}_{K,1}^\times \longrightarrow \mathbb{A}_{K,1}^\times \longrightarrow \{\text{fractional } \mathcal{O}_S\text{-ideals}\}$$

$\therefore$ ideal class group for $\mathcal{O}_S \cong \mathbb{A}_{K,1}^\times / (U_S \cap \mathbb{A}_{K,1}^\times) \cdot K^\times$

is a quotient of $\mathbb{A}_{K,1}^\times / K^\times$ (compact) by $U_S \cap \mathbb{A}_{K,1}^\times$ (open)

$\rightarrow$ compact + discrete $\rightarrow$ finite.

<u>General class groups</u> $\pi_0(\mathbb{A}_K^\times / K^\times)$: "big group"

instead, we take $(\mathbb{A}_K^\times / K^\times) / (\text{open})$ $\leftarrow$ always finite

<u>e.g.</u>

("ray class groups") $\mathfrak{f}$: finite subset of $\Sigma_{K,\text{fin}}$ with multiplicity,

$(\mathfrak{f} = \sum m_v \cdot v,\ m_v \geq 1)$ $\quad T$: subset of $\Sigma_{K,\text{real}}$

Take the open subgroup : $\Big(\prod_{v \in T} K_{v,+}^\times\Big) \times \Big(\prod_{v:\, m_v \geq 1} \underbrace{\mathcal{O}_{v,m_v}^\times}_{= 1 + \mathfrak{p}_v^{m_v}}\Big) \times \Big(\prod_{\substack{v \text{ finite} \\ m_v = 0}} \mathcal{O}_v^\times\Big)$

$\rightsquigarrow$ get the ray class group w.r.t. $\mathfrak{f}$.

CFT ("naive form") Given a class group $\mathbb{A}_K^\times / K^\times \cdot U = Cl_U$, U open, there exists a (unique) finite abelian ext'n K_U (the class field attached to Cl_U) of K s.t.

1) $\forall$ unramified v in K_U/K, fr_v $(= Fr^{-1}) \in Gal(K_U/K)$ is trivial iff $(\cdots 1, \pi_v^{-1}, 1, \cdots, 1)$ is 0 in Cl_U.

2) $Cl_U \cong Gal(K_U/K)$

3) Artin map :

$$\begin{array}{ccccc} Cl_U & \xrightarrow{\sim} & Gal(K_U/K) & & \\ \uparrow & & \uparrow & & \\ & & Fr_v & & \\ \mathbb{A}_K^\times & \ni & (1, 1, \cdots, \pi_v, 1 \cdots 1) & & \forall \text{ unramified } v \text{ in } K_U/K \end{array}$$

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Dirichlet's unit theorem

K : number field ; $\mathbb{A}_{K,1}^\times / K^\times$: compact.

$\Sigma_{K,\infty} \subseteq S \subseteq \Sigma_K$ S : finite

What is S-unit $\mathcal{O}_S^\times$?

(global elements which are units outside S.)

DUT : $\mathcal{O}_S^\times$ is a f.g. abelian group.

$(\mathcal{O}_S^\times)_{tor} = \mu(\overline{\mathbb{Q}}) \cap K =: W$

$\mathcal{O}_S^\times / W \cong \mathbb{Z}^{\#(S)-1}$

Proof
$$\mathcal{O}_S^\times = \underbrace{\left(\prod_{v\in S} K_v^\times \times \prod_{v\notin S} \mathcal{O}_v^\times\right)}_{U_S} \cap K^\times .$$

$$= (\underbrace{U_S \cap \mathbb{A}_{K,1}^\times}_{U_{S,1}}) \cap K^\times$$

$U_{S,1}$: open & closed in $\mathbb{A}_K$.

(lattice : co-compact in something)

$$U_{S,1} / (\underbrace{U_{S,1} \cap K^\times}_{\mathcal{O}_S^\times}) \longrightarrow \mathbb{A}_{K,1}^\times / K^\times$$

$\therefore$ LHS is compact.

$$\begin{array}{ccccc}
U_{S,1} & \xrightarrow{\alpha} & \left(\prod_{v\in S} K_v^\times\right)_1 & \xrightarrow{\beta = \|\cdot\|_v} & \{x\in(x_v)\in \prod_{v\in S} \mathbb{R}_{v,+}^\times \mid \prod_{v\in S}\|x_v\|_v = 1\} \cong (\mathbb{R}_{>0}^\times)^{\#(S)-1} \\
\cup\!| & & \cup\!| & & \cup\!| \\
\mathcal{O}_S^\times & \xrightarrow{\sim} & \alpha(\mathcal{O}_S^\times) & \longrightarrow & \beta\alpha(\mathcal{O}_S^\times) \\
\text{co-compact} & \Longrightarrow & \text{co-compact (lattice ; discrete)} & \Longrightarrow & \text{co-compact.}
\end{array}$$

$\ker \beta$ is compact $\Longrightarrow$ $\ker \beta|_{\alpha(\mathcal{O}_S^\times)}$ is finite

therefore : $\beta\alpha(\mathcal{O}_S^\times)$ is a f.g. abelian group of rank $\#(S)-1$

$\Rightarrow$ same for $\mathcal{O}_S^\times$

Have computed the volume of $\mathbb{A}_K / K$:

- $\mu_{\text{self-dual}}(\mathbb{A}_K / K) = 1$

Tamagawa measure:

$G = \mathbb{G}_a$ (over K)

$\mathcal{L}ie(G)^\vee$ = translation-invariant 1-form on G.

$\rightsquigarrow \Lambda^{top}(\mathcal{L}ie(G)^\vee) \cong K \cdot \omega$ (if $G \cong \mathbb{G}_a$ can take $\omega = $ "dx")

(not a canonical choice.)

$G(\mathbb{A}_K) \left(= \mathrm{Hom}_{K\text{-alg}}(K[G], \mathbb{A}_K) \right)$

(adèlization of an algebraic group)

($GL_n(\mathbb{A}_K)$: $n \times n$ matrix with entries in $\mathbb{A}_K$ whose determinant is in $\mathbb{A}_K^\times$)

unimodular

Pick ω. $\rightsquigarrow$ one can get a "measure on $G(\mathbb{A}_K)$".

$\forall v$. ω_v gives a measure on $G(K_v)$.

$v | \infty$

- v real : $G(K_v)$. Lie group. $\dim_{\mathbb{R}} G(K_v) = \dim(G)$.
  $|\omega|$ volume form on $G(K_v)$
- v complex : $G(K_v)$ Lie group. $\dim_{\mathbb{R}} G(K_v) = 2 \cdot \dim(G)$
  $|\omega_v \wedge \bar{\omega}_v|$: volume form on $G(K_v)$

$v \nmid \infty$: locally. $G(K_v) \cong K_v^d$ by choosing coordinates.

measure : requires $\mu(\mathcal{O}_v^d) = 1$

again get a measure on $G(K_v)$

$|\omega_v|$: Haar measure on $G(K_v)$.

$\Rightarrow$ $|\omega|$ on $G(\mathbb{A}_K)$ = "$\prod_v |\omega_v|$"

but sometimes it's not convergent on compact-open subsets!

what's needed is a modified measure: $|\omega_v|' = a_v \cdot |\omega_v|$.

Homework. measure on idèles? ($\mathbb{G}_m$: take $\frac{dx}{x}$)

10/25/96

(Digression)

G : alg. group over K . K . number field

ω : a generator of $\Lambda^{top}(\mathcal{Lie}(G)^\vee)$

"a gauge form". → the space of (left) translation-invariant top differential forms on G, rational over K.

Last time : this allows us to put a measure $|\omega_v|$ on $G(K_v)$.

"Often", $\exists\ (\lambda_v)_{v\in\Sigma_K}$ s.t $\prod_v (\lambda_v^{-1}|\omega_v|)$ defines a measure on $G(\mathbb{A}_K)$.

Then, define the Tamagawa measure $\mu_{G/K}$ on $G(\mathbb{A}_K)$ as:

$$\mu_{K.G} = |\mathrm{disc}_{K/\mathbb{Q}}|^{\frac{-\dim(G)}{2}} \prod_v (\lambda_v^{-1}|\omega_v|)$$

<u>Example</u> $G = \mathbb{G}_a$ $\omega = dx$

<u>Take</u> $\lambda_v = 1$ $\forall v \in \Sigma_K$. $\Rightarrow$ ω_v on $G(K_v) = K_v$ is

$\left.\begin{array}{l} |dx| \text{ or } |dz\wedge d\bar{z}| \ , \ v|\infty \\ \mathcal{O}_v \text{ has measure } 1 \ , \ v\nmid\infty \end{array}\right.$

$$\Big(\prod_v \omega_v\Big)(\mathbb{A}_K/K) = |\mathrm{disc}_{K/\mathbb{Q}}|^{\frac12}$$

<u>Example</u>. $G = \mathbb{G}_m$ $\omega = \dfrac{dx}{x}$. $|\omega_v| = \dfrac{|dx_v|}{\|x_v\|}$

$\rightsquigarrow$ $|\omega_v|(\mathcal{O}_v^\times) = \displaystyle\int_{\mathcal{O}_v^\times} dx$ (Here $dx(\mathcal{O}_v) = 1$)

Here, $\mathcal{O}_v^\times = \mathcal{O}_v \setminus \mathfrak{p}_v$.

$dx(\mathcal{O}_v) = 1$. $|dx|(\mathfrak{p}_v) = |dx|(\pi_v\cdot\mathcal{O}_v) = \|\pi_v\|$
$= \mathfrak{f}_v^{-1}$

$\therefore |\omega_v|(\mathcal{O}_v^\times) = 1 - \mathfrak{f}_v^{-1}$

Put : $\lambda_v = 1 - \mathfrak{f}_v^{-1}$ (otherwise, get an infinite product which will 0.)

<u>Remark</u> The choice of λ_v is NOT arbitrary.

$\zeta_K(s) = \prod_v (1-\mathfrak{f}_v^{-s})^{-1}$ ← local zeta factor $\zeta_v(s)$.

$\zeta_K(s)$ has a simple pole at $s=1$; look at $(s-1)\cdot\zeta_K(s)$

$\rightsquigarrow$ get " $\left[\lim_{s\to 1}(s-1)\zeta_K(s)\right]^{-1}\cdot|\mathrm{disc}_{K/\mathbb{Q}}|^{-\frac{1}{2}}\cdot\prod_v \zeta_v(1)\cdot\omega_v$ "

= the Tamagawa measure for $\mathbb{G}_m/K$ $= \mu_{\mathbb{G}_m,K}$

take $\mu_{\mathbb{G}_m,K}(\mathbb{A}_{K,1}^\times/K^\times) = \tau(\mathbb{G}_m/K)$

<u>Prop</u> $\tau(\mathbb{G}_m/K) = 1.$

Classically.

<u>exer.</u> $\left(\prod_v \zeta_v(1)\cdot\omega_v\right)\cdot(\mathbb{A}_{K,1}^\times/K^\times)$

$$= \frac{2^{r_1}\cdot(2\pi)^{r_2}\cdot h_K\cdot R_K}{w}$$

$w = \#(\mu \cap K^\times)$

R_K = regulator of K

= $\left|\det \log|\sigma_i(u_j)|\right|$

$\sigma_i : K \longrightarrow K_v$

$1 \le i \le r_1+r_2$

$\sigma_1 \cdots \sigma_{r_1}$ real embedding

$\sigma_{r_1+1} - \sigma_{r_1+r_2}$ complex non-conjugate embedding

$u_1, \cdots, u_{r_1+r_2-1}$: basis for $\mathcal{O}_K^\times$/torsion

10/28/96

L-functions

abelian L-functions:

K: number field

$\chi : \mathbb{A}_K^\times / K^\times \xrightarrow{\text{cont.}} \mathbb{C}^\times$ idèle class character (grossen character, etc)

terminology:

we / now	classical
characters	quasicharacters
unitary characters (values in $\mathbb{C}_1^\times$)	characters

L-function

$$L(\chi, s) = \prod_{v \text{ finite}} (1 - \chi(\pi_v) \cdot q_v^{-s})^{-1}$$

χ: character, $s \in \mathbb{C}$

well-defined only if

$$\begin{tikzcd} K_v^\times \arrow[r, "\chi_v"] \arrow[d] & \mathbb{C}^\times \\ \mathbb{A}_K^\times \arrow[ur, "\chi"'] & \end{tikzcd}$$

is unramified i.e. $\chi_v(\mathcal{O}_v^\times) = 1$

examples

(1) $K = \mathbb{Q}$. $\chi : (\mathbb{Z}/n)^\times \longrightarrow \mathbb{C}_1^\times$. Dirichlet char.

$h_{\mathbb{Q}} = 1 \implies \mathbb{A}_{\mathbb{Q}}^\times = \mathbb{Q}^\times \times \left(\prod_p \mathbb{Z}_p^\times \times \mathbb{R}^\times\right)$

$\implies \mathbb{A}_{\mathbb{Q}}^\times / \mathbb{Q}^\times = \left(\mathbb{R}^\times \times \prod_p \mathbb{Z}_p^\times\right) / (\pm 1)$

$= \mathbb{R}_+^\times \times \prod_p \mathbb{Z}_p^\times$

So

$$\begin{tikzcd} (\mathbb{Z}/n\mathbb{Z})^\times \arrow[r, "\chi"] & \mathbb{C}^\times \\ \mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times \arrow[u] \arrow[ur, "\chi"'] & \end{tikzcd}$$

← get this but only finite order!

Note that $\chi(\mathbb{R}_+^\times)$ is trivial, i.e. $\chi(\mathbb{R}_+^\times) = 1$.

Exercise If $\chi : \mathbb{A}_K^\times / K^\times \longrightarrow \mathbb{C}^\times$ is trivial on $(K \otimes_{\mathbb{Q}} \mathbb{R})^{\times,0}$, is χ necessarily of finite order?

The right way to think about χ.

χ : an automorphic representation of $GL_1(\mathbb{A}_K) = \mathbb{A}_K^\times$

nowadays. π : automorphic representation of $GL_n(\mathbb{A}_K)$

$\rightsquigarrow L(\pi, s)$

So far. L-functions come from harmonic analysis.

Another source of L-functions : Galois group.

(Artin - Hecke)

abelian case :

$$\begin{array}{c} L \\ \Big| \text{ abel extension} \\ K \end{array} \qquad Gal(L/K) = G \xrightarrow{\chi} \mathbb{C}^\times \quad \text{say } G \text{ cycl}$$

$$\rightsquigarrow L(\chi, s) = \prod_{\substack{v \\ v:\text{ unramified} \\ \text{finite in } L}} \left(1 - \chi(Fr_v)\, q_v^{-s}\right)^{-1}$$

<u>CFT</u> : get the same L-function.

Finite order question. use W_K = Weil group for K to replace $Gal(L/K)$.

can get the L-function on the other side.

<u>Artin</u> <u>L-functions</u> :

$$\begin{array}{c} L \\ \Big| \; G = Gal(L/K) \\ K \end{array} \qquad \rho : G \longrightarrow GL(V) \leftarrow \text{fin. dim'l vector space} / \mathbb{C}$$

$$\rightsquigarrow L_S(\rho, s) = \prod_{v \notin S} \det \Big(\underbrace{1 - \rho(Fr_v)\, q_v^{-s}}_{\text{operator on } V} \Big)^{-1}$$

$S \supset \Sigma_{K,\infty} \cup \{\text{ram'd places}\}$

Goals : Functional equations. meromorphic ext'n poles/zeros.

10/30/96

Tate's thesis

Local theory

Main statement: local functional equation:

f: Schwartz function on K_v (not $K_v^\times$!)

("locally constant with compact support")

$$\Longrightarrow \frac{\int_{K_v^\times} \hat{f}_\psi(x)\cdot(\omega_1\chi^{-1})(x)\,d^\times x}{L_v(\omega_1\chi^{-1})} = \varepsilon_v(\chi.\psi\,dx)\cdot\frac{\int_{K_v^\times} f(x)\,\chi(x)\,d^\times x}{L_v(\chi)}$$

where χ: a character of $K_v^\times$

$$L_v(\chi) = \begin{cases} (1-\chi(\pi_v))^{-1} & \text{if } \chi \text{ is ramified} \\ 1 & \text{if } \chi \text{ is unramified} \end{cases}$$

$\omega_s = \|\cdot\|^s$. $(\omega_1 = \|\cdot\|)$

$$\rightsquigarrow L_v(\chi.\omega_s) = (1-\chi(\pi_v)\cdot q_v^{-s})^{-1}.$$

<u>Remark</u> χ "varies" in the families $\chi\cdot\omega_s$, parametrized by $s\in\mathbb{C}$.

Looking at the numerators: $\chi \longleftrightarrow \omega_1\chi^{-1}$

(classical: $s \longleftrightarrow 1-s$)

$\psi : (K,+) \longrightarrow \mathbb{C}_1^\times$. non-trivial unitary character.

then identify $K_v^\times$ with its own dual $(K_v^\times)^\vee$

$\varepsilon_v(\chi.\psi.dx)$ ε-factors, "local root numbers"

$=1$ for almost all places v. global root number is the product of all local root numbers

"Knowing ε_v for all χ will give something about automorphic rep'n."

Global functional equation:

$$\chi : \mathbb{A}_K^\times / K^\times \longrightarrow \mathbb{C}^\times$$

f : Schwartz function on $\mathbb{A}_K$ (not $\mathbb{A}_K^\times$!)

$$\int_{\mathbb{A}_K^\times} f(x)\,\chi(x)\,d^\times x = \int_{\mathbb{A}_K^\times} \hat{f}_\psi(x)\,(\omega_1 \cdot \chi^{-1})(x)\,d^\times x$$

" functional equation for global zeta integrals."

(Follows from Poisson summation formula.)

Poisson Summation Formula

G · locally compact abelian group

H closed subgroup of G.

Fourier transform : $\underset{\text{function on } G}{f} \longrightarrow \underset{\text{function on } \hat{G}}{\hat{f}}$

$$G \geq H \geq (e) \qquad \mu_G = \mu_{G/H} \cdot \mu_H$$

$$\hat{G} \geq \underset{(G/H)^\wedge}{\overset{}{H^\perp}} \geq (e) \qquad \mu_{\hat{G}} = \mu_{H^\perp} \cdot \mu_{\hat{H}} \qquad \hat{H} = \hat{G}/H^\perp$$

Poisson summation formula $\displaystyle\int_H f(h)\,dh = \int_{H^\perp} \hat{f}(\hat{h})\,d\hat{h}$?

proof. f : function on G.

$F(\bar{g}) = \displaystyle\int_H f(gh)\,dh$ function on G/H.

Applying Fourier inversion on G/H :

$$F(\bar{o}) = \int_{(G/H)^\wedge = H^\perp} \hat{F}(\lambda) \cdot \langle \overset{1}{\overline{g}}, \bar{o} \rangle\, d\lambda$$

$$\therefore \int_H f(h)\,dh = \int_{H^\perp} \int_{G/H} F(\bar{g}) \langle \bar{g}, \lambda \rangle^{-1}\, d\bar{g}$$

$$= \int_{H^\perp} \int_{G/H} \int_H f(g\lambda) \langle \bar{g}, \lambda \rangle^{-1}\, dh\, d\bar{g}$$

$$= \int_{H^\perp} \int_G f(gh) \langle g, \lambda \rangle^{-1} dg$$

$$= \int_{H^\perp} \hat{f}(\lambda) \, d\lambda. \qquad \text{q.e.d.}$$

idea for GFE :

$$\int_{\mathbb{A}_K^\times} f(x) \chi(x) \, d^\times x \qquad \chi : \text{idèle class character}$$

$$= \int_{\mathbb{A}_K^\times / \mathbb{A}_K^{\times,1}} \underbrace{\left[\int_{\mathbb{A}_K^{\times,1}} f(tu) \chi(tu) \, d^\times u \right]}_{= \, \zeta_t(f, \chi)} \frac{dt}{t} \qquad \begin{array}{l} t \in \mathbb{R}_+^\times \\ u \in \mathbb{A}_K^{\times,1} \end{array}$$

$$\mathbb{A}_K^\times / \mathbb{A}_K^{\times,1} \xrightarrow[\ \|\cdot\|\ ]{\sim} \mathbb{R}_+^\times$$

Hint : use Poisson summation formula (letting $H = K$)
on $\zeta_t(f, \chi)$. (K : cocompact in $\mathbb{A}_K$)

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Functional Equation

$\chi : \mathbb{A}_K^\times / K^\times \longrightarrow \mathbb{C}^\times$

$f \in \mathcal{S}(\mathbb{A}_K)$. Schwartz function on $\mathbb{A}_K$ (not $\mathbb{A}_K^\times$!)

$$(*) \quad \int_{\mathbb{A}_K^\times} f(x)\, \chi(x)\, d^\times x = \int_{\mathbb{A}_K^\times} \hat{f}_\psi(x)\, (\omega_1 \chi^{-1})(x)\, d^\times x .$$

$d^\times x = \frac{dx}{\|x\|}$

dx . self-dual measure on $\mathbb{A}_K$.

$\psi : \mathbb{A}_K \times \mathbb{A}_K \longrightarrow \mathbb{C}_1^\times$ given by $e \circ \mathrm{tr}$

proof of $(*)$:

$$\mathrm{LHS} = \int_{\mathbb{R}_+^\times = \mathbb{A}_K^\times/\mathbb{A}_K^{\times,1}} \frac{dt}{t} \int_{\mathbb{A}_K^{\times,1}} f(tu) \cdot \chi(tu)\, d^\times u$$

$$= \int_{\mathbb{R}_+^\times} \frac{dt}{t} \underbrace{\int_{E \sim \mathbb{A}_K^{\times,1}/K^\times} d^\times v \left[\sum_{\xi \in K^\times} f(t v \xi)\, \overset{\chi(tv)}{\overset{\|}{\chi(t v \xi)}} \right]}_{\zeta_t(f, \chi)} \quad \text{since } \chi(\xi) = 1.$$

$$(\dagger) \quad \sum_{\xi \in K^\times} f(tv\xi) = \sum_{\xi \in K} f(tv\xi) - f(0).$$

$$(0) \overset{\text{discrete}}{\subseteq} K \overset{\text{compact}}{\subseteq} \mathbb{A}_K$$

dualize: $$(0) \overset{\text{discrete}}{\subseteq} K^\perp \overset{\text{compact}}{\subseteq} (\mathbb{A}_K)^\vee \cong \mathbb{A}_K$$
$$K^\perp \supseteq K$$

$K \subseteq K^\perp$ discrete. compact $\Rightarrow$ $K^\perp / K$ is finite.

$K^\perp$: stable under multiplication by K $\Rightarrow$ $K^\perp$ is a K-vector sp.

$\therefore$ $K^\perp / K = (0)$. i.e. $K^\perp = K$.

$g(x) = f(tvx)$

The Fourier transform of g :

$$\hat{g}(\lambda) = \int g(x)\, e(\underbrace{-\mathrm{tr}(x \cdot \lambda)}_{})\, dx \qquad e(-\mathrm{tr}(x\cdot\lambda)) = \langle x, \lambda \rangle^{-1}$$

$$= \int f(tvx)\, e(-\mathrm{tr}(x \cdot \lambda))\, dx$$

$$= \int f(y)\, e\left(-\mathrm{tr}\left(\frac{y}{tv} \cdot \lambda\right)\right) \frac{dy}{\|tv\|} = \|tv\|^{-1} \cdot \hat{f}\left(\frac{\lambda}{tv}\right)$$

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$\therefore (\dagger) = \sum_{\eta \in K^{\perp}=K} \|tv\|^{-1} \hat{f}(\frac{\eta}{tv}) - f(0)$ $\qquad \|tv\|^{-1} = \|t\|^{-1} = t^{-1}$ since $v \in \mathbb{A}_K^{\times,1}$, $t \in \mathbb{R}_+^{\times}$

$= \sum_{\eta \in K^{\times}} t^{-1} \hat{f}(\frac{\eta}{tv}) + \frac{\hat{f}(0)}{t} - f(0)$

$$\therefore \zeta_t(f,\chi) = \int_E t^{-1} \chi(tv) \sum_{\eta \in K^{\times}} \hat{f}(\frac{\eta}{tv})\, d^{\times}v + \hat{f}(0) \cdot \int_E t^{-1} \cdot \chi(tv)\, d^{\times}v - f(0) \cdot \int_E \chi(tv)\, d^{\times}v.$$

$$\int_E (\omega_{-1}\chi)(tv) \cdot \Big(\sum_{\eta \in K^{\times}} \hat{f}(\frac{\eta}{tv})\Big)\, d^{\times}v \qquad w = v^{-1}$$

$$= \int_E (\omega_1 \chi^{-1})(t^{-1}w) \cdot \Big(\sum_{\eta \in K^{\times}} \hat{f}(t^{-1}w\eta)\Big)\, d^{\times}w = \zeta_{t^{-1}}(\hat{f}, \underbrace{\omega_1\chi^{-1}}_{\hat{\chi}})$$

$$\therefore \zeta_t(f,\chi) = \zeta_{t^{-1}}(\hat{f},\hat{\chi}) + \hat{f}(0) \int_E \hat{\chi}(t^{-1}w)\, d^{\times}w - f(0) \int_E \chi(tv)\, d^{\times}v.$$

$$\Longrightarrow \text{l.h.s.} = \text{r.h.s.} + \hat{f}(0) \int_{\mathbb{R}_+^{\times}} \frac{dt}{t} \int_E \hat{\chi}(tv)\, d^{\times}v - f(0) \int_{\mathbb{R}_+^{\times}} \frac{dt}{t} \int_E \chi(tv)\, d^{\times}v.$$

Have shown:

$$\zeta(f,\chi) = \begin{cases} \int_1^{\infty} \zeta_t(f,\chi)\, d^{\times}t + \int_1^{\infty} \zeta_t(\hat{f},\hat{\chi})\, d^{\times}t & \text{when } \chi|_{\mathbb{A}_K^{\times,1}/K^{\times}} \neq \text{trivial} \\ \text{"} + \text{"} + \hat{f}(0) \int_{\mathbb{A}_K^{\times,1}/K^{\times}} d^{\times}u \cdot \int_0^1 t^{s-1}\, dt - f(0) \int_{\mathbb{A}_K^{\times,1}/K^{\times}} d^{\times}u \int_0^1 t^{s}\, d^{\times}t & \text{when } \chi|_{\mathbb{A}_K^{\times,1}/K^{\times}} = \text{trivial} \end{cases}$$

$\Rightarrow \chi = \omega_s$ for some $s \in \mathbb{C}$.

Remarks (exer.) $\int_{\mathbb{A}_K^{\times}} f(x)\, \chi(x)\, d^{\times}x$ is absolutely convergent if $\mathrm{Re}(|\chi|) > 1$.
(i.e. $|\chi| = \omega_s \Rightarrow \mathrm{Re}(|\chi|) = \mathrm{Re}\, s$)

Assume $\mathrm{Re}\, s > 1$. $\quad \int_0^1 t^{s-1}\, d^{\times}t = \frac{1}{s-1} t^{s-1} \Big|_0^1 = \frac{1}{s-1}$

$\int_0^1 t^{s}\, d^{\times}t = \frac{1}{s}$

$\Rightarrow$ case 2:

$$\zeta(f,\chi) = \int_1^\infty \zeta_t(f,\chi)\,d^\times t + \int_1^\infty \zeta_t(\hat{f},\hat{\chi})\,d^\times t - \hat{f}(0)\frac{c}{1-s} - f(0)\frac{c}{s}$$

$$c = \int_{\mathbb{A}_K^{\times,1}/K^\times} d^\times u$$

this is unchanged if $\begin{cases} f \longleftrightarrow \hat{f} \\ \chi \longleftrightarrow \hat{\chi} \\ (s \longleftrightarrow 1-s) \end{cases}$

Conclusions

- Analytic continuation. $\zeta(f,\chi)$ is a meromorphic function of χ with at most simple poles at $\chi = \omega$ & $\chi =$ trivial, and we know their residues.
- $\zeta(f,\chi) = \zeta(\hat{f},\hat{\chi})$ Functional equation.

exer. get functional equation for ζ from functional equation for Jacobi theta function

$$\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{2\pi\sqrt{-1}\,n^2\tau}$$

(hint: $n^{-s}\,\Gamma(\frac{s}{2}) = \int_{\mathbb{R}_+^\times} e^{-n^2 w}\cdot w^{\frac{s}{2}}\,d^\times w$).

Local Functional Equation

Prop. $\zeta(f,\chi)\cdot\zeta(\hat{g},\hat{\chi}) = \zeta(\hat{f},\hat{\chi})\cdot\zeta(g,\chi)$. if $0 < \mathrm{Re}|\chi| < 1$

K: local field ; f, g: Schwartz functions ; fixed Haar measure

$$\zeta(f,\chi) = \int_{K^\times} f(x)\,\chi(x)\,d^\times x.$$

exer. This integral converges for $\mathrm{Re}(|\chi|) > 0$

Corollary $\dfrac{\zeta(f,\chi)}{\zeta(\hat{f},\hat{\chi})} = \rho(\chi)\big(= \rho(\chi,\psi,dx)\big)$ independent of f.

ρ is meromorphic in χ.

rewrite: $\dfrac{\zeta(\hat{f},\hat{\chi})}{L(\hat{\chi})} = \varepsilon(\chi,\psi,dx)\,\dfrac{\zeta(f,\chi)}{L(\chi)}$

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K : non-arch. $L(\chi) = \begin{cases} (1-\chi(\pi) q^{-s})^{-1} & \text{when } \chi|_{\mathcal{O}_K^\times} = 1 \\ 1 & \text{otherwise} \end{cases}$

K : arch. $L_{\mathbb{R}}(x^{-N}\omega_s) = \Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$. $N = 0$ or 1.

$L_{K \cong \mathbb{C}}(z^{-N}\omega_s) = \Gamma_{\mathbb{C}}(s) = 2\cdot(2\pi)^{-s}\Gamma(s) \quad (= \Gamma_{\mathbb{R}}(s)\cdot\Gamma_{\mathbb{R}}(s+1))$, $N \geq 0$.

<u>Proposition</u>. K local field. then
$$\zeta(f,\chi)\ \zeta(\hat{g},\hat{\chi}) = \zeta(\hat{f},\hat{\chi})\ \zeta(g,\chi).$$

<u>Proof</u> LHS $= \int_{K^\times} f(x)\chi(x)\,d^\times x \int_{K^\times} \hat{g}(y)\hat{\chi}(y)\,d^\times y$

$= \int_{K^\times\times K^\times} f(x)\cdot\hat{g}(y)\cdot\chi(xy^{-1})\cdot\|y\|\,d^\times x\,d^\times y$ let $u = xy^{-1}$

$= \int_{K^\times\times K^\times} f(uy)\cdot\hat{g}(y)\cdot\chi(u)\cdot\|y\|\,d^\times u\cdot d^\times y$

$= \int_{K^\times\times K^\times} f(uy)\chi(u)\cdot\|y\|\,d^\times u\,d^\times y \int_K g(v)\psi(-vy)\,dv$.

$= \int_{K^3} f(uy) g(v)\psi(-vy)\chi(u)\|u\|^{-1}\,du\,dy\,dv$

$w = uy \qquad dw = \|u\|\,dy$
$v = uz \qquad \|u\|\,dz = dv$.

$= \int_{K^3} f(w)\cdot g(uz)\cdot\psi(-wz)\cdot\chi(u)\ \|u\|^{-1}\,du\,dw\,dz =$ RHS. $\square$

$\chi : \mathbb{A}_K^\times / K^\times \longrightarrow \mathbb{C}$

$\rightsquigarrow$ $L(\chi, s) = \prod_v L_v(\chi, \omega_s)$ is meromorphic.

L is holomorphic if $\chi|_{\mathbb{A}_K^{\times,1}} \neq$ trivial ;

If $\chi|_{\mathbb{A}_K^{\times,1}} =$ trivial, then L has simple poles at $s = -s_0$ or $s = 1 - s_0$. ("known" residues)

+ functional equation. $L(\chi, s) = \varepsilon(\chi, s)\; L(\chi^{-1}, 1-s)$.

$$\varepsilon(\chi, s) = \prod_v \varepsilon_v(\chi\omega_s, \psi_v, dx_v).$$

Class Field Theory (Takagi Artin)

K : number field.

CFT : $\exists$ a map $\mathrm{rec}_K : \mathbb{A}_K^{\times,1} / K^\times \twoheadrightarrow \mathrm{Gal}(\bar{K}/K)^{ab}$.

(reciprocity map)

such that $\pi_0(\mathbb{A}_K^\times / K^\times) \xrightarrow{\sim} \mathrm{Gal}(\bar{K}/K)^{ab}$.

or equiv. $(\mathbb{A}_K^{\times,1} / K^\times)^{\text{profinite completion}} \xrightarrow{\sim} \mathrm{Gal}(\bar{K}/K)^{ab}$.

Artin. $\begin{array}{c} L \\ | \\ K \end{array}$ $\quad G = \mathrm{Gal}(L/K)$ abelian. $S \supseteq \Sigma_{K,\infty} \cup \Sigma_{K,\mathrm{ram}}$.

$\chi : G \longrightarrow \mathbb{C}^\times$.

$\rightsquigarrow$ Artin L-function: $L(\chi, s) = \prod_{v \notin S} (1 - \chi(\mathrm{Fr}_v) \cdot q_v^{-s})^{-1}$.

He thought these abelian (Artin) L-functions should be Hecke L-functions

i.e. come from characters $c_\chi : \mathbb{A}_K^\times / K^\times \longrightarrow \mathbb{C}^\times$ with finite image.

exer. This means:

$\exists$ surjection $h : \mathbb{A}_K^\times / K^\times \twoheadrightarrow G$. which sends π_v to Fr_v for almost all v's.

$\rightsquigarrow$ $\forall\, x \in K^\times$, $\quad x \equiv 1 \pmod{\mathfrak{m}}$.

$\prod_v h_v(x) = e \in G$. $\qquad$ Now, find $\mathfrak{m}$.

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Fixing notation.

$$L(\chi, s) = \prod_{\substack{\mathfrak{p} \\ \text{finite}}} L_{\mathfrak{p}}(\chi \omega_s)$$

(when χ = trivial, get $\zeta_K(s)$).

$$\Lambda(\chi, s) = \Big(\prod_{v|\infty} L_v(\chi\omega_s)\Big) \cdot L(\chi, s)$$

<u>Dirichlet density</u> K: number field $\quad A \subseteq \Sigma_{K,\text{fin}}$

$$d(A) = \lim_{s \to 1^+} \frac{\log\big(\prod_{\mathfrak{p}\in\Sigma}(1-N\mathfrak{p}^{-s})^{-1}\big)}{\log \zeta_K(s)}$$ if this limit exists.

<u>remark</u>. $d(\text{finite set}) = 0$.

$\zeta_K(s)$ has a simple pole at $s=1$, so

$$\log \zeta_K(s) \sim -\log(s-1) + O(1)$$

<u>Proposition</u> $A = \{\mathfrak{p} \in \Sigma_{K,\text{fin}} \mid \kappa_{\mathfrak{p}} \cong \mathbb{F}_p\}$ — degree 1 primes.

$\Longrightarrow d(A) = 1$.

<u>proof</u>. $\prod_p (1-p^{-2s})^{-1} = \zeta(2s) \Rightarrow \log \prod_p (1-p^{-2s})^{-1} = O(1)$ near $s=1$

$$\Longrightarrow \log \prod_{\substack{\mathfrak{p} \\ \kappa_{\mathfrak{p}} \cong \mathbb{F}_{p^n},\ n\geq 2}} (1-N\mathfrak{p}^{-s})^{-1} \leq [K:\mathbb{Q}] \log\Big(\prod_p (1-p^{-2s})^{-1}\Big) = O(1)$$

near $s=1$. q.e.d.

<u>Proposition</u> L/K: Galois, finite.

$A_{L/K} = \{\mathfrak{p} \in \Sigma_{K,\text{fin}} \mid \mathfrak{p}$ splits (completely) in $L/K\}$.

$\Longrightarrow d(A_{L/K}) = [L:K]^{-1}$.

<u>proof</u> $$\log \prod_{\mathfrak{p}\in A_{L/K}} (1-N\mathfrak{p}^{-s})^{-1} = [L:K]^{-1} \cdot \log \prod_{\substack{\mathfrak{q}\in\Sigma_L \\ \deg(\mathfrak{q}/\mathfrak{p})=1}} (1-N\mathfrak{q}^{-s})^{-1} + O(1)$$

$$= -[L:K]^{-1} \log(s-1) + O(1).$$

(from the proposition above).

q.e.d.

Proposition Suppose L_1, L_2 are finite, Galois extensions of K, and $d((A_{L_1/K} \setminus A_{L_2/K}) \cup (A_{L_2/K} \setminus A_{L_1/K})) = 0$ then $L_1 = L_2$

proof $L = L_1 \cdot L_2$, their compositum

$$\begin{tikzcd} & L \arrow[dl, dash] \arrow[dr, dash] & \\ L_1 \arrow[dr, dash] & & L_2 \arrow[dl, dash] \\ & K & \end{tikzcd}$$

$\Longrightarrow \ \forall \mathfrak{p} \in A_{L_1/K} \cap A_{L_2/K}, \quad \mathfrak{p} \in A_{L/K}$

$$\rightsquigarrow \underset{\substack{\| \\ [L:K]^{-1}}}{d(A_{L/K})} \geq \underset{\substack{\| \\ [L_1:K]^{-1}}}{d(A_{L_1/K})} = \underset{\substack{\| \\ [L_2:K]^{-1}}}{d(A_{L_2/K})}$$

$\Longrightarrow \ L_1 = L_2 = L.$ q.e.d.

The 1^{st} inequality

Theorem L/K: finite Galois extension of # fields, then
$$[\mathbb{A}_K^\times : K^\times N_{L/K}(\mathbb{A}_L^\times)] \leq [L:K]$$

Remark "=" holds iff L/K is abelian (from the 2^d inequality.) in which case: $\text{rec}: \mathbb{A}_K^\times / K^\times N_{L/K}(\mathbb{A}_L^\times) \xrightarrow{\sim} \mathrm{Gal}(L/K)$

proof. $B_{L/K} = \{ \mathfrak{p} \in \Sigma_{K,\text{fin}} \mid (\mathfrak{p} \text{ unram}); \ (1, \cdots, 1, \pi_\mathfrak{p}, 1, \cdots) \in K^\times \cdot N_{L/K}(\mathbb{A}_L^\times) \}$.
(i.e. $Fr_\mathfrak{p} = \text{id.} \in \mathrm{Gal}(L/K)$)

$$A_{L/K} = \{ \mathfrak{p} \in \Sigma_{K\,\text{fin}} \mid \mathfrak{p} \text{ splits completely in } L/K \}$$

lemma if L_w/K_v is unramified, then $N_{L_w/K_v}(\mathcal{O}_w^\times) = \mathcal{O}_v^\times$.
by lemma, the condition in $B_{L/K}$ is independent of the choice of $\pi_\mathfrak{p}$

Clearly, $A_{L/K} \subseteq B_{L/K}$

lemma $d(B_{L/K}) = [\mathbb{A}_K^\times : K^\times N_{L/K}(\mathbb{A}_L^\times)]^{-1}$.

Cor. $\mathbb{A}_K^\times / K^\times N_{L/K}(\mathbb{A}_L^\times)$ is finite. ($\because \mathbb{A}_K^1 / K^\times$ is compact.)

$\rightsquigarrow$ look at Hecke char. ; Hecke L-functions ; etc.

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First inequality.

$$\begin{array}{l} L \\ \big| \text{Galois.} \\ K \end{array} \Rightarrow [\mathbb{A}_K^\times : K^\times N_{L/K}(\mathbb{A}_L^\times)] \leq [L:K]$$

only need to prove : Lemma

$$d(B_{L/K}) = [\mathbb{A}_K^\times : K^\times N_{L/K}(\mathbb{A}_L^\times)]^{-1}$$

" $K^\times N_{L/K}(\mathbb{A}_L^\times)$ norm subgroup in $\mathbb{A}_K^\times / K^\times$ "

(classical) Dirichlet theorem

Check!

$$\boxed{\begin{array}{l} \mathbb{A}_\mathbb{Q}/\mathbb{Q} = \prod_p \mathbb{Z}_p \times (\mathbb{R}/\mathbb{Z}) \\ \mathbb{A}_\mathbb{Q} = \mathbb{Q} + (\mathbb{R} \times \prod_p \mathbb{Z}_p) \\ \mathbb{A}_\mathbb{Q}^\times = \mathbb{Q}^\times \cdot (\mathbb{R}^\times \times \prod_p \mathbb{Z}_p^\times) \\ \mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times = \mathbb{R}_+ \cdot \prod_p \mathbb{Z}_p^\times \end{array}}$$

$$\begin{array}{c} \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}/n\hat{\mathbb{Z}} \\ \big\downarrow \\ \mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times \twoheadrightarrow (\mathbb{Z}/n\mathbb{Z})^\times \end{array}$$

D's thm. $d(\{p \mid p \equiv k \pmod n\}) = \varphi(n)^{-1}$ for $(k,n) = 1$.

$$\begin{array}{l} \mathbb{Q}(\mu_n) \\ \big| \; \Big) (\mathbb{Z}/n\mathbb{Z})^\times \\ \mathbb{Q} \end{array}$$

p. $(p,n) = 1$

$\mathrm{Fr}_p = \bar{p} \in (\mathbb{Z}/n\mathbb{Z})^\times$ ← exer

D's thm : equidistribution of primes in $(\mathbb{Z}/n\mathbb{Z})^\times = \mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})$.

Recall D's proof. Consider the char. $\chi : (\mathbb{Z}/n\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$

Look at :
$$\sum_{(m,n)=1} \frac{\chi(m)}{m^s} = \prod_{(p,n)=1} (1 - \chi(p) \cdot p^{-s})^{-1}$$
$$= L(\chi, s).$$

For $\chi \neq$ trivial, $L(\chi, s)$ is entire. and holomorphic around 1. , get $L(\chi, 1) \neq 0$.

($\because$ $\prod_\chi L(\chi, s) = \underbrace{(\text{euler factors})}_{\text{for primes} \mid n} \zeta_{\mathbb{Q}(\mu_n)}(s)$).

Artin L-function

$$\begin{array}{c} L \\ \Big| \, G.\,\mathrm{Gal} \\ K \end{array} \qquad \rho : G \longrightarrow GL(V)$$

$$\rightsquigarrow \; L(\rho, s) = \prod_{\mathfrak{p}} \det\left(1_V - \mathfrak{f}^{-s} \cdot \rho(\pi_{\mathfrak{p}})\big|_{V^{I(\mathfrak{p})}}\right)^{-1}.$$

exer

$$G \left(\begin{array}{c} L \\ | \,)H \\ M \\ | \\ K \end{array} \right.$$

1. $\rho : H \longrightarrow GL(V)$

$\mathrm{Ind}(\rho) \longleftrightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} V.$ (cf. Serre. local fields)

induced representation.

Frobenius reciprocity law

$$\mathrm{Hom}_G(\mathrm{Ind}(\rho), W) \xrightarrow[\text{canonical}]{\sim} \mathrm{Hom}_H(\rho, W)$$

"representation of G is a linear combination of 1-dim'l representations of H" (Brauer's theorem.)

2. $L(\rho, s) = L(\mathrm{Ind}(\rho), s)$ where ρ. $\mathrm{Ind}(\rho)$ are as above

Corollary

$$G \left| \begin{array}{c} L \\ \\ K \end{array} \right.$$

take $H = \{e\}$. ρ = trivial representation of $H = \{e\}$

then $L(\rho, s) = \zeta_L(s)$

$$\mathrm{Ind}_{\{e\}}^{G}(\text{trivial}) = \text{regular representation of } G = \bigoplus_{\rho:\text{ irred rep of } G} \rho^{\chi_\rho(1)}.$$

$$L(\rho_1 \oplus \rho_2, s) = L(\rho_1, s) \cdot L(\rho_2, s)$$

$$\rightsquigarrow \; \zeta_L(s) = \prod_{\rho \text{ irred of } G} L(\rho, s)^{\chi_\rho(1)}$$

$$\prod_{\chi} L(\chi, s) = \zeta_{\mathbb{Q}(\mu_m)}(s) \cdot (\text{euler factors})$$

$$\| $$

$$\zeta_{\mathbb{Q}}(s) \cdot \prod_{\chi \neq \text{trivial}} L(\chi, s)$$

since $\zeta_{\mathbb{Q}}$. $\zeta_{\mathbb{Q}(\mu_m)}$ both have simple poles at $s = 1$.

$\therefore$ $L(\chi, 1) \neq 0$ for all $\chi \neq$ trivial holomorphic around 1

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$$\begin{array}{l} L \\ \big| \text{ finite Galois} \qquad\qquad G = \mathrm{Gal}(L/K) \\ K \end{array}$$

$$\mathbb{A}_K^\times \supseteq K^\times \cdot N_{L/K}(\mathbb{A}_L^\times)$$

We consider $B_{L/K} = \{ \mathfrak{p} \in \Sigma_{K\,fin} \mid (\dots, 1, \pi_{\mathfrak{p}}, 1 \dots) \in K^\times N_{L/K}(\mathbb{A}_L^\times) \}$

<u>WTS</u> $d(B_{L/K}) = [\mathbb{A}_K^\times : K^\times N_{L/K}(\mathbb{A}_L^\times)]^{-1}$

Back to the sketch of Dirichlet's prime progression thm.

(†) $\chi : (\mathbb{Z}/n\mathbb{Z})^\times \longrightarrow \mathbb{C} \quad \rightsquigarrow \quad L(\chi, 1) \neq 0 \quad \forall \chi$

Look at primes $p \equiv 1 \pmod n$. (WTS (*) $d(\text{such primes}) = \varphi(n)^{-1}$).

$$\lim_{s \to 1^+} \log \prod_{p \equiv 1 (n)} (1 - p^{-s})^{-1} \sim \varphi(n)^{-1} \log \frac{1}{s-1} \quad ?$$

$$\log L(\chi, s) = \sum_p \log (1 - \chi(p) p^{-s})^{-1}$$

$$= \sum_p \sum_m \frac{\chi(p^m) p^{-ms}}{m} \qquad \text{Taylor expansion.}$$

Target : $\log \prod_{p \equiv 1 (n)} (1 - p^{-s})^{-1} = \sum_{p \equiv 1 (n)} \sum_m \frac{p^{-ms}}{m} = \sum_{p \equiv 1 (n)} p^{-s} + O(1)$ as $s \to 1^+$

Consider : $\sum_\chi \log L(\chi, s)$

$$= \varphi(n) \cdot \sum_{\substack{p, m \\ p^m \equiv 1 (n)}} \frac{p^{-ms}}{m} = \varphi(n) \sum_{p \equiv 1 (n)} p^{-s} + O(1)$$

On the other hand, from (†),

$$\sum_\chi \log L(\chi, s) = \log L(\text{trivial}, s) + O(1) \quad \text{as } s \to 1^+$$

$$= \log \frac{1}{s-1} + O(1).$$

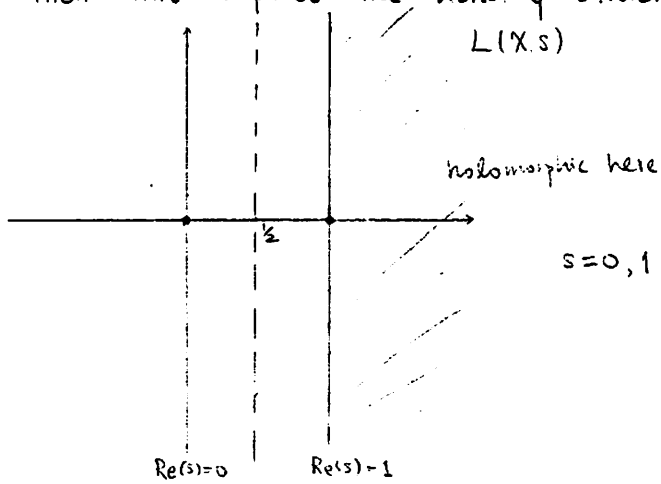
$\therefore$ (*) is shown. $\square$

For similar reason, we want

$\forall\ \chi : \mathbb{A}_K^\times / K^\times \cdot N_{L/K}(\mathbb{A}_L^\times) \longrightarrow \mathbb{C}^\times$, $\quad L(\chi, 1) \neq 0$

Hecke character ; is of finite order

then this implies the density statement. $d(B_{L/K}) = [L : K]^{-1}$



$s = 0, 1$: possible poles (only when χ = trivial)

Argument : No zeroes on the boundaries $Re(s) = 0$ $Re(s) = 1$

$\longleftrightarrow \quad \pi(x) = \sum_{p \le x} 1 \sim \frac{x}{\log x}$.

<u>Rapid Review of Dirichlet Series</u> $\quad \sum_{n \ge 1} a(n) \cdot n^{-s}$. $\quad s = \sigma + it$.

- $\exists$ a half-plane of absolute convergence.

- <u>Proposition</u> If $\sum_{n \ge 1} a(n) \cdot n^{-s}$ converges for $s = s_0$, then it converges uniformly on compact subsets of $\{ Re(s) \ge \sigma_0 = Re(s_0) \}$

  <u>proof</u> <u>Key lemma</u> Suppose $\left| \sum_{1 \le n \le m} a(n) \cdot n^{-s_0} \right| \le M \quad \forall m$

  then $\left| \sum_{A \le n \le B} a(n)\, n^{-s} \right| \le 2M \cdot A^{\sigma_0 - \sigma} \left(1 + \frac{|s - s_0|}{\sigma - \sigma_0} \right)$. $\quad$ <u>exercise</u> (w/ Abel summation) $\square$

- <u>Theorem</u> (Landau) Suppose $f(s) = \sum_{n \ge 1} a(n) \cdot n^{-s}$ converges for $Re(s) > c$ and $a(n) \ge 0$ for almost all n, and $f(s)$ extends to a holo. function in a neighborhood of $s = c$ then : $\exists\ \varepsilon > 0$ s.t. $\sum_n a(n) \cdot n^{-s}$ converges for $Re(s) > c - \varepsilon$.

  <u>proof</u> $\quad f(s) = \sum_k \frac{f^{(k)}(c+1)}{k!} (s - c - 1)^k$

$$\frac{f^{(k)}(s)}{k!} = \sum \frac{(-1)^k}{k!} a(n) \cdot (\log n)^k \cdot n^{-s}$$

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may assume: $a(n) \geq 1 \quad \forall n.$

$$\Longrightarrow f(s) = \sum_{k\geq 0} \sum_{n\geq 1} \frac{1}{k!} (-1)^k \cdot a(n) \cdot (\log n)^k \cdot n^{-c-1} \cdot (s-c-1)^k$$

$\rightsquigarrow$ can evaluate at $s = c - \varepsilon$:

$$f(c-\varepsilon) = \sum_{k=0} \sum_{n\geq 1} \frac{1}{k!} \cancel{(-1)^k} a(n) (\log n)^k \cdot n^{-c-1} \cdot \cancel{(-1)^k} \cdot (\varepsilon+1)^k$$

converges with <u>nonnegative</u> terms.

$$= \sum_{n\geq 1} a(n) \cdot n^{-c-1} \cdot e^{(\log n)(1+\varepsilon)}$$

$$= \sum_{n\geq 1} a(n) \, n^{-(c-\varepsilon)} \quad \text{converges} \qquad \square.$$

=

<u>Nonvanishing of L-functions</u> (Method of Hadamard – de la Valée-Pousin)

<u>Theorem</u> Suppose that χ is a Hecke character, then $L(\chi, s)$ does not vanish at $\mathrm{Re}(s) = 1$ (also $\mathrm{Re}(s) > 1$)

<u>proof.</u> By multiplying suitable ωs, we may assume χ is unitary.

Suppose $s_0 = \sigma_0 + i t_0$ is a zero for L. $(\sigma_0 = 1)$

Case (I) $\chi^2 \neq$ trivial when $t_0 = 0$.

crucial inequality: $3 + 4\cos\theta + \cos 2\theta \geq 0$: $\forall \theta \in \mathbb{R}$.

have: $L(\chi, s) = \exp\left(\sum_{\mathfrak{p}, m} \frac{\chi(\mathfrak{p}^m) \cdot N(\mathfrak{p})^{-ms}}{m}\right)$.

$$= \exp\left(\sum_{\mathfrak{p}, m} \frac{\chi(\mathfrak{p}^m) \cdot N(\mathfrak{p})^{-m\sigma} \cdot e^{-imt \log N(\mathfrak{p})}}{m}\right) \qquad \chi(\mathfrak{p}) = e^{-i\theta_{\mathfrak{p}}}$$

$$\Longrightarrow |L(\chi, s)| = \exp\left(\sum_{\mathfrak{p}, m} \frac{N(\mathfrak{p})^{-m\sigma}}{m} \cdot \cos\left[m \cdot (\theta_{\mathfrak{p}} + t \log N(\mathfrak{p}))\right]\right)$$

By crucial. $\Rightarrow \left| L^3(\mathbb{1}, s) \cdot L^4(\chi, s) \cdot L(\chi^2, \sigma + i2t) \right| \geq 1$

$\forall \sigma > 1.$

Let $\sigma \longrightarrow 1+$, $t \longrightarrow t_0$

$\Rightarrow$ by assumption $\left.\begin{array}{l} L^4(\chi, s) = o(|s-s_0|^4) \\ \text{but } L^3(\mathbb{1}, s) \sim c \cdot \left(\frac{1}{s-s_0}\right)^3 \\ L(\chi^2, \sigma+it) = O(1) \end{array}\right\}$ around s_0

$\longrightarrow\!\!\!\!\!\!\times$

Case (II) $\chi^2 = \text{trivial}$ $\chi \neq \text{trivial}$ (i.e. range of $\chi = \{\pm 1\}$).

(Landau). $L(\chi, s)$ has a zero at $s=1$

<u>Consider</u> $\zeta_K(s) \cdot L(\chi, s)$ ($\leftarrow$ this is a holo. function)

$$= \left(\sum_{\mathfrak{a}} N(\mathfrak{a})^{-s}\right) \cdot \left(\sum_{\mathfrak{a}} \chi(\mathfrak{a}) \cdot N(\mathfrak{a})^{-s}\right).$$

$$= \sum_{\mathfrak{a}} N(\mathfrak{a})^{-s} \left(\sum_{\mathfrak{b}|\mathfrak{a}} \chi(\mathfrak{b})\right) \qquad (\dagger)$$

$$\sum_{\mathfrak{b}|\mathfrak{a}} \chi(\mathfrak{b}) = \prod_{\mathfrak{p}^m \| \mathfrak{a}} (1 + \chi(\mathfrak{p}) + \cdots + \chi(\mathfrak{p}^m)) \geq 0$$ since range$(\chi) = \{\pm 1\}$

$$\Longrightarrow (\dagger) = \sum_n n^{-s} \Bigg(\underbrace{\sum_{N(\mathfrak{a})=n} \Big(\prod_{\mathfrak{p}^m \| \mathfrak{a}} (1 + \chi(\mathfrak{p}) + \cdots + \chi(\mathfrak{p}^m)\Big)}_{a(n)} \Bigg)$$

By Landau's theorem on Dirichlet's series:

$$\Longrightarrow \sum_n a(n) \cdot n^{-s} \text{ converges everywhere} \qquad (\text{esp. } \mathrm{Re}(s) > 0)$$

However, $\displaystyle\sum_n a(n) \cdot n^{-s} \geq \sum_{\substack{\mathfrak{a} = \mathfrak{b}^2 \\ \text{unramified}}} N(\mathfrak{a})^{-s} = \sum_{\substack{\mathfrak{b} \\ \text{unramified}}} N(\mathfrak{b})^{-2s} \sim \zeta_K(2s) \to \infty$ for $s = \frac{1}{2}$. $\times$.

q.e.d.

11/18/96

Theorem (Wiener - Ikehara) (Tauberian theorem)

$$f(s) = \sum_m \frac{a_m}{m^s} \qquad g(s) = \sum_m \frac{b_m}{m^s}.$$

Both converge for $\mathrm{Re}(s) > 1$, regular on $\mathrm{Re}(s) = 1$ except (possibly) a simple pole at $s = 1$.

$\mathrm{Res}_{s=1} f = 1$. $\quad \mathrm{Res}_{s=1}\, g = \eta$.

Also assume that $\exists\, c > 0$ s.t. $|b_m| \le c \cdot a_m \quad \forall m$.

Then: $\displaystyle\sum_{m \le x} b_m \sim \eta x$.

Application.

$$f = \frac{-\zeta_K'(s)}{\zeta_K(s)} \qquad \zeta_K(s) = \exp\Big(\sum_{\mathfrak{p}, m} \frac{N(\mathfrak{p})^{-ms}}{m}\Big)$$

$$= \sum_{\mathfrak{p}\, m} N(\mathfrak{p})^{-ms} \cdot \log N(\mathfrak{p}).$$

$$g = \frac{-L'(\chi, s)}{L(\chi, s)} = \sum_{\mathfrak{p}, m} \chi(\mathfrak{p}^m) \cdot N(\mathfrak{p})^{-ms} \cdot \log N(\mathfrak{p}) \qquad \begin{array}{c} c = 1 \\ (\chi \text{ unitary}) \end{array}$$

$$= \sum_n \Big(\sum_{\substack{\mathfrak{p}\, m \\ N(\mathfrak{p})^m = n}} \log N(\mathfrak{p}) \cdot \chi(\mathfrak{p}^m) \Big) \cdot n^{-s}$$

$$= \sum_n \sum_{\substack{\mathfrak{p} \\ N(\mathfrak{p}) = n}} \log n \cdot \chi(\mathfrak{p}) \cdot n^{-s} + O(1)$$

$$\rightsquigarrow \sum_{N(\mathfrak{p}) \le x} \log N(\mathfrak{p}) \cdot \chi(\mathfrak{p}) \sim \eta_\chi \cdot x.$$

$$\eta_\chi = \begin{cases} 1 & \text{if } \chi\restriction_{\mathbb{A}_K^{\times 1}} = \text{trivial} \\ 0 & \text{otherwise.} \end{cases}$$

exercise. Reformulate: $\displaystyle\sum_{N(\mathfrak{p}) \le x} \chi(\mathfrak{p}) \sim \eta_\chi \cdot \frac{x}{\log x}$

Jensen's theorem

$f(z)$: holomorphic on $\overline{D(0;R)}$. $f(0) \neq 0$. $f(Re^{i\theta}) \neq 0$

$$\Longrightarrow \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta = \log|f(0)| + \log\left|\frac{R^n}{\rho_1 - \rho_n}\right|.$$

$\rho_1 - \rho_n$ zeroes repeated with multicipity.

Entire functions of finite order

Def. f : entire. f is of finite order if

$M(R) = \sup_{|z|\le R} |f|$ $\quad$ $\log M(R) \sim O(R^\alpha)$ for some $\alpha > 0$.

order : $\inf\{\alpha\}$

ζ order 1 doesn't mean $\log M(R) = O(R)$
but rather $\log M(R) = O(R^{1+\varepsilon})$ $\forall\, \varepsilon > 0$.

Proposition. Let $n(r) = \#$ of zeroes of $f(z)$ in $\{|z| \le r\}$

Jensen's theorem $\Longrightarrow$

$$(*) \quad \log M(R) \ge \log|f(0)| + \log\left|\frac{R^n}{\rho_1 - \rho_n}\right|$$

$$= \log|f(0)| + \int_0^R r^{-1}\cdot n(r)\, dr$$

Then if $\log M(R) = O(R^\alpha)$ then $n(r) = O(R^\alpha)$.

$\therefore (*) \Rightarrow \quad n(R)\cdot \log 2 \le \int_R^{2R} r^{-1} n(r)\, dr \le \log M(2R) = O(R^\alpha)$ $\square$

Weierstrass product. $\{r_i\}$: sequence in $\mathbb{C}$. $|r_i| \to \infty$.

$n(r) := \#\{i : |r_i| \le r\}$

Assume $n(r) = O(r^\alpha)$ for some $\alpha > 0$.

pick $a-1 \le \alpha < a$. $a \in \mathbb{N}$.

Consider $F(z) = \prod \left[\left(1 - \frac{z}{r_i}\right) \exp\left(\frac{z}{r_i} + \frac{1}{2}\left(\frac{z}{r_i}\right)^2 + \cdots + \frac{1}{a-1}\cdot\left(\frac{z}{r_i}\right)^{a-1}\right)\right.$

$\Rightarrow$ (1) Converges absolutely on compact sets. and $F(z)$ order at most α.

(2) $\exists$ seq. of $\{R_i\}$. $\lim_{n\to\infty} R_i = \infty$. s.t.

$\log m(R_i, F) = \log \inf_{|z|\le R_i} |F(z)| \gg -R_i^{\alpha'}$ $\quad \forall\, \alpha' > \alpha$.

11/20/96

<u>Weierstrass product</u> $\{r_i\}$ a sequence in $\mathbb{C}$. $|r_i| \to \infty$

(for convenience, assume $0 < |r_1| \le |r_2| \le \cdots\cdots$)

$n(r) = \#\{i \mid |r_i| \le r\}$

Assume $n(r) = O(r^\alpha)$ for some $\alpha \geq 0$; $\exists a \in \mathbb{N}$, $a-1 \le \alpha < a$

For the product

$$F(z) = \prod \underbrace{\left(1 - \frac{z}{r_i}\right) \exp\left(\frac{z}{r_i} + \frac{1}{2}\left(\frac{z}{r_i}\right)^2 + \frac{1}{3}\left(\frac{z}{r_i}\right)^3 + \cdots + \frac{1}{a-1}\left(\frac{z}{r_i}\right)^{a-1}\right)}_{E(z, r_i, a)}$$

<u>Assertions.</u>

1. F converges absolutely on any compact subset of $\mathbb{C}$.

2. F is an entire function of order $\le \alpha$.

3. $\frac{1}{F(z)}$ doesn't grow too fast, i.e. $O((e^{r^{\alpha+\varepsilon}}))$ on selected circles.

<u>Corollary</u> (Hadamard) $f(z)$: entire function of order $\alpha \geq 0$

$a-1 \le \alpha < a$ $a \in \mathbb{N}$. $f(0) \neq 0$ $F(z)$ as above.

<u>Then</u>, $f(z) = e^{P(z)} \cdot F(z)$. $\sup_{|z| \le r} |\operatorname{Re} P(z)| \le O(r^{\alpha+\varepsilon})$ $\forall \varepsilon > 0$

<u>claim</u> $P(z)$ is a polynomial of order $\le a-1$
(to be shown later.)

<u>example</u>.

Find an entire function f s.t the zeroes of f are $\mathbb{Z} - \{0\}$

solution : take $\prod_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \left(1 - \frac{z}{n}\right) \cdot e^{\frac{z}{n}}$

$$= \prod_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \left(1 - \frac{z^2}{n^2}\right)$$

$\{n^2\}$: growth at $\alpha = \frac{1}{2}$

$\Rightarrow g(z) = \prod_{n \neq 0} \left(1 - \frac{z}{n^2}\right)$ converges.

$$= g(z^2).$$

candidate : $\frac{\sin \pi z}{\pi z}$: entire of order 1, even function $\Rightarrow h(z^2)$

$h(z)$ is entire.

$\rightsquigarrow h(z)$ is entire of order $\frac{1}{2}$.

$\Rightarrow h(z) = \text{const} \cdot g(z)$ $\Rightarrow$ const $= 1$. by evaluating both sides at $z=0$.

conclusion. $$\frac{\sin \pi z}{\pi z} = \prod_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \left(1 - \frac{z^2}{n^2}\right) = \prod_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \left(1 - \frac{z}{n}\right) \cdot e^{\frac{z}{n}}$$

<u>exercise</u> $$\Gamma(z) = \int_{\mathbb{R}^\times} e^{-t}\, t^{z}\, d^\times t.$$

$(\Gamma(n+1) = n! \quad \forall n \geq 0, \quad z\Gamma(z) = \Gamma(z+1))$

$\rightsquigarrow$ poles: $-\mathbb{N}$.

$\rightsquigarrow \frac{1}{\Gamma(z)}$: entire ! (of order 1)

zeroes: $-\mathbb{N}$. simple zeroes.

ask $\frac{1}{\Gamma(z)\cdot\Gamma(1-z)}$. zeroes at $\mathbb{Z}$.

$\|$

$\sin \pi z \cdot c'$.

or: $\Gamma(z)\cdot\Gamma(1-z) = \frac{c\pi}{\sin \pi z}$. from the residues. we know $c = 1$.

<u>exercise</u> $\Gamma(\frac{z}{m})\,\Gamma(\frac{z+1}{m}) \cdots \Gamma(\frac{z+m-1}{m}) = ?$

especially, $\Gamma(z)\cdot\Gamma(z+\frac{1}{2}) = \sqrt{\pi}\,\Gamma(2z)$.

11/22/96

Poisson-Jensen formula

$f(0) \neq 0$ on $|z| = R$

$\rho_1 \ldots \rho_n$: zeroes of f. $|\rho_i| \neq R$

$$\rightsquigarrow \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t) \log|f(Re^{it})|\, dt$$

$$= \log|f(z)| + \sum_j \log\left|\frac{R^2 - \bar{\rho}_j z}{R(z-\rho_j)}\right|$$

(same proof as in Jensen's formula)

Question How to recover $f(z)$ from ∂-values of $u = \operatorname{Re} f$?

Consider :

$$g(z) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\frac{2Re^{it} - 1}{Re^{it} - z}}_{\substack{\text{Real part} \\ \text{is the Poisson kernel}}} \cdot \log|f(Re^{it})|\, dt$$

$$- \sum_j \log \frac{R^2 - \rho_j \bar{z}}{R(z-\rho_j)}$$

well-defined up to $2\pi n i$.

$\rightsquigarrow g(z) = \log f(z)$

take $\left(\frac{d}{dz}\right)^m$: $\rightsquigarrow$ get a bound. done

12/2/96

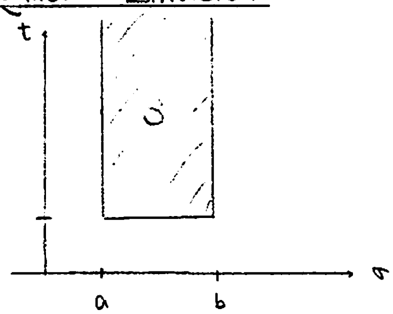
Last Time

Gave the density of zeroes of $\zeta(s)$:

$N(T) = $ # of zeroes of $\zeta(s)$ in $0 \le \sigma \le 1$, $0 \le t \le T$.
with multiplicities.

Conclusion: $N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$.

Phragmen - Lindelöf (a:)



$f(z)$. holomorphic on $\bar{S}$.

Assume. 1) $\exists A$, $|f(z)| \ll e^{|z|^A}$.

2) $|f(z)| \le M \quad \forall z \in \partial S$

Then : $|f(z)| \le M \quad \forall z \in S$.

proof Choose an m s.t. $m > A$ and $m \equiv 2 \pmod 4$.
(then in S, $\arg(z^m) \sim \frac{\pi}{2} \cdot m \equiv \pi \pmod{2\pi}$ as $z \to \infty$)

Now consider $f_\varepsilon(z) = e^{\varepsilon \cdot z^m} \cdot f(z)$. for $\varepsilon > 0$.

$|f_\varepsilon(z)| \longrightarrow 0$ as $z \to \infty$ in S (by assumption.)

Then by maximal principle,

$$\sup_{z \in S} |f_\varepsilon(z)| \le \sup_{z \in \partial S} |f_\varepsilon(z)|$$

Now let $\varepsilon \to 0$. $f_\varepsilon(z) \to f(z)$ uniformly. q.e.d.

Variants: Others unchanged, yet

* 2)' $|f(z)| = O(|z|^\nu)$ $\nu > 0$ on ∂S

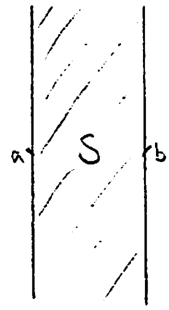
$\Longrightarrow |f(z)| = O(|z|^\nu)$ on S.

* 2)'' $\log |f(z)| = O(|z|^\nu)$. $\nu > 0$ on ∂S

$\Longrightarrow \log |f(z)| = O(|z|^\nu)$ on S.

* for the entire strip (instead of half-strips).

Hadamard's 3 circles theorem



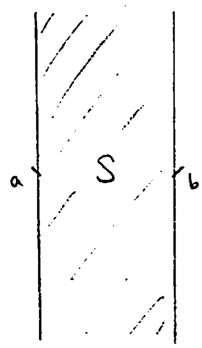
Assume $|f(z)|$ is bounded on S.

Let $M(\sigma) = \sup_{Re(z)=\sigma} |f(z)|$

$\Rightarrow$ $M(\sigma)$ is a convex function of σ

(exer.)

(compare M with e^{az+b} . $a \in \mathbb{R}$)



Assume $|f(z)| = O((1+|t|)^N)$

(as usual , $z = \sigma + it$)

Then $\forall$ $a \le \sigma \le b$, let

$\psi(\sigma)$ be the smallest non-negative number so that $|f(\sigma+it)| = O((1+|t|)^{\psi+\varepsilon})$

Conclusion : $\psi(\sigma)$ is a convex function of σ

(exer.)

Examples. $\Gamma(z)$.

* no zeroes . simple poles at $0, -1, -2, \cdots$
* $\Gamma(z)^{-1}$ is an entire function of order 1.

Stirlings formula :

$$\log \Gamma(z) = (z - \tfrac{1}{2}) \log z - z + \tfrac{1}{2} \log 2\pi + O(|z|^{-1})$$

$$\therefore \Gamma(z)^{-1} = e^{az+b} \cdot z \cdot \prod_{n \in \mathbb{N}} (1 + \tfrac{z}{n}) e^{-\frac{z}{n}}$$

(at $z=0$ $\Rightarrow$ $b=0$, $a \to$ Euler's constant.)

* $L(\chi, s)$
  $\zeta(s)$.

$\Big\}$ integral representation after removing the simple pole.

$\rightsquigarrow$ functional equation.

12/6/96

$$\lim_{T\to\infty} \sum_{|\gamma|<T} \Phi(\omega)$$

$$= \delta_\chi \cdot \int_{-\infty}^{\infty} F(x)\,(e^{\frac{x}{2}} + e^{-\frac{x}{2}})\,dx + 2F(0)\log A$$

$$- \sum_{\mathfrak{p},\, n\ge 1} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{n/2}} \left[\chi(\mathfrak{p})^n F(\log N(\mathfrak{p})^n) + \chi(\mathfrak{p})^{-n} F(\log N(\mathfrak{p})^{-n}) \right]$$

$$- \sum_{v|\infty} \lim_{\lambda\to\infty} \left[\int_{-\infty}^{\infty} (1-e^{-\lambda|x|})\, K_v(x)\, F(x)\, e^{-i\varphi_v(x)}\,dx - 2F(0)\log\lambda \right]$$

Weil's explicit formula

Notations. $\chi : \mathbb{A}_K^\times / K^\times \longrightarrow \mathbb{C}_1^\times$ unitary idèle class char.

(Remarks on normalizations:

$[K:\mathbb{Q}] = n$, $\chi : \mathbb{A}_K^\times / K^\times \longrightarrow \mathbb{C}_1^\times$

$v|\infty$. $\rightsquigarrow$ $\chi(a_v) = \left(\frac{a_v}{|a|}\right)^{m_v} \cdot |a|^{i n_v \varphi_v}$. $a_v \in K_v \hookrightarrow \mathbb{C}$

$m_v \in \mathbb{Z}/(n\mapsto -n)$ (if $K_v \cong \mathbb{C}$) or $\mathbb{Z}/2\mathbb{Z}$. (if $K_v \cong \mathbb{R}$)

$n_v = [K_v : \mathbb{R}]$.

classical normalization $\boxed{\sum_{v|\infty} n_v \varphi_v = 0}$

(up to translation.)

"pole of L?"

$\rightsquigarrow L(s,\chi)$, if has poles, will be at $s=0$ and $s=1$.)

Functional Equation

(completing L-function to Λ as in 11/8/96.)

Z definitions of $\Lambda(s,\chi)$ changes a little here:

$$\Lambda(s,\chi) = A^s \cdot \left(\prod_{v|\infty} \Gamma\left(\tfrac{s_v}{2}\right)\right) \cdot L(s,\chi).$$

where $s_v = n_v(s + i\varphi_v) + |m_v|$

$A = 2^{-r_2} \cdot \pi^{-\frac{n}{2}} \cdot d_\chi^{\frac{1}{2}}$; $d_\chi = d_{K/\mathbb{Q}} \cdot N_{K/\mathbb{Q}}(f_\chi)$

f_χ : conductor of χ.

classical FE.

$$W(\chi)\ \Lambda(s,\chi) = \Lambda(1-s,\bar{\chi})$$

(W : artin root number, ε-factors, etc. ⋯)

or $\quad W(\chi)\,\Lambda(\bar{s},\chi) = \overline{\Lambda(1-s,\chi)}$. $\rightsquigarrow W(\chi)\in \mathbb{C}_1^\times$.

($\bar{\chi} = \chi^{-1}$ because it's unitary.)

Back to notations,

$\omega = \beta + i\gamma$; in LHS of WEF, we're summing zeroes of $L(s,\chi)$ (or $\Lambda(s,\chi)$) in the critical strip $0 < \mathrm{Re}(s) < 1$.

$F(x)$: function on $\mathbb{R}$, piecewise C^1 with finite number of discountinuity, $F(\alpha_i) = \frac{1}{2}(F(\alpha_i^+) + F(\alpha_i^-))$ at those points. and $\exists\, b>0$ s.t. $F(x)$ and $F'(x)$ are both $O\left(e^{-(\frac{1}{2}+b)|x|}\right)$

Φ : Mellin transform of F

$$\Phi(s) = \int_{-\infty}^{\infty} F(x)\, e^{(s-\frac{1}{2})x}\, dx \quad ; \text{ uniformly } O\left(\tfrac{1}{|t|}\right)$$

in some strip $-\alpha \le \sigma \le 1+\alpha$. $\alpha>0$.

(follows from properties of F.)

$$K_v = \begin{cases} \dfrac{e^{(\frac{1}{2}-|m_v|)\cdot|x|}}{e^{x}-e^{-x}}, & \text{if } n_v = 1 \\[2ex] \dfrac{e^{-\frac{1}{2}|m_v|\cdot|x|}}{\left|e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right|}, & \text{if } n_v = 2 \end{cases}$$

$$\delta_\chi = -\mathrm{ord}_{s=1} L(s,\chi) = \underset{\text{hol at } s=1}{0} \text{ or } \underset{\text{simple pole at } s=1}{1}.$$

EF : Both sides define a distribution

last term : Fourier transform of log. der. of Γ-factors

$1^{st}, 2^{nd}$ term from poles & A

3^{rd} term : distribution of primes (weighted sum)

LHS : summing the Mellin transform at zeroes of L-function.

Exercise. (due to Weil)

This distribution T is positive definite.

$\Longleftrightarrow$ GRH is true for $L(s, \chi)$

Definition. G: locally compact abelian group.

$f : G \longrightarrow \mathbb{C}$ cont

f positive definite. $\Longleftrightarrow$ $\forall\, x_1, \dots, x_k \in G$, $c_1 \dots c_k \in G$

$$\sum_{m,n} f(x_m \cdot x_n)\, c_m \cdot \bar{c}_n \geq 0$$

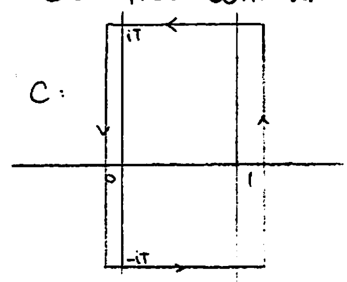
$\Longleftrightarrow$ (by continuity) $\langle f, h * h^* \rangle \geq 0$

$\Longleftrightarrow$ f is the Fourier transform of a bounded positive measure on $\hat{G}$.

Structure of the proof of EF: Do the contour integral.

$$\frac{1}{2\pi\sqrt{-1}} \int_C \Phi(s)\, d \log \Lambda(s, \chi)$$

C :



$\rightleftarrows$: contributions on these two horizontal lines tend to 0.

We'll need to estimate $\frac{\Lambda'}{\Lambda}(s, \chi)$: use Weierstraß product formula

$$\Lambda(s, \chi) = \prod_{\omega} \left(1 - \frac{s}{\omega}\right) e^{\frac{s}{\omega}}$$

$$\rightsquigarrow \frac{\Lambda'}{\Lambda}(s, \chi) = \sum_{\omega} \left(\frac{1}{s-\omega} + \frac{1}{\omega} \right)$$

12/9/96

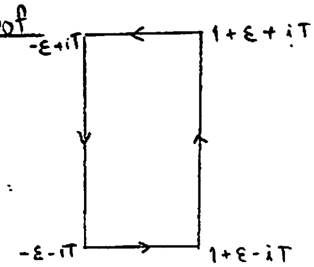
Weil's explicit formula

$$\Phi(s) = \int_{-\infty}^{\infty} F(x)\, e^{(s-\frac{1}{2})x}\, dx \qquad \text{Mellin's transform.}$$

$\rightsquigarrow$ E.F

Sketch of proof

Integrate
$\Phi(s)\, d\log \Lambda(s)$
on the path:



LHS : comes from non-trivial zeroes of Λ

RHS : $\delta_x \int$ —— : possible poles

remaining terms:
"explicitly write down" the integral
on the contour:

$$\Phi(s)\, d\log \Lambda(s) = \Phi(s)\, d\log A^s + \Phi(s)\, d\log G_\chi(s) + \Phi(s)\, d\log L(s, \chi)$$

$$(\Lambda(s,\chi) = A^s \cdot G_\chi(s) \cdot L(s,\chi))$$

via Euler product & F.E.

contributions of these two horizontal sides $\longrightarrow 0$ as $T \to \infty$.

using F.E., shift integration on the left vertical path to one on the right vertical path.

F.E.: $W(\chi) \cdot \Lambda(s, \chi) = \Lambda(1-s, \chi^{-1}) = \overline{\Lambda(1-\bar{s}, \chi)}$

"The horizontal edges might run into the zeros of $\Lambda(s)$":

Strategy Select good values of T to "stay away from zeroes".

Need Estimates:

(1) # of zeroes of $\Lambda(s,\chi)$ in a box: $\{T \le t \le T+1\}$.
(answer : $O(\log|T|)$)

$\rightsquigarrow$ $\exists\, \alpha > 0$. $\forall m \in \mathbb{Z}$, $|m| \geq N$ $\exists\, T_m$, $m < T_m < m+1$
$\Lambda(s,\chi)$ has no zeroes in $|t - T_m| \le \dfrac{\alpha}{\log|m|}$

(2) $\dfrac{\Lambda'(\sigma + iT_m)}{\Lambda(\sigma + iT_m)} = O(\log^2 |m|)$

$\rightsquigarrow$ vanishing of integrals on horizontal edges.

These estimates justify $\qquad O\left(\dfrac{\log^N |m|}{|m|}\right) \quad N = 3n+4$

① contribution from horizontal sides $\to 0$ as $m \to \infty$.

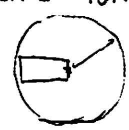
② improper integrals converge as $|m| \to \infty$

(3) $\dfrac{\Gamma'(s)}{\Gamma(s)}$ estimate as $t \to \infty$.

How to do (1) : to estimate $|L(\sigma+it, \chi)| = O(|t|^N)$

(use Phragmen - Lindelöf) $\quad a \le \sigma \le b$

then use Jensen's formula on



How to do (2) : Weierstraß product factorization:

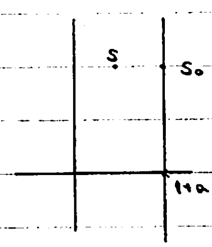
$$\Lambda(s, \chi) = a \cdot e^{bs} \cdot \prod_{\omega} \left(1 - \frac{s}{\omega}\right) \cdot e^{\frac{s}{\omega}}$$

(pretending no poles.)

$$\frac{\Lambda'}{\Lambda}(s, \chi) = b + \sum_{\omega} \left(\frac{1}{s-\omega} + \frac{1}{\omega} \right)$$

Take a difference :

$$\frac{\Lambda'}{\Lambda}(s, \chi) - \frac{\Lambda'}{\Lambda}(s_0, \chi) = \sum_{\omega} \left(\frac{1}{s-\omega} - \frac{1}{s_0-\omega} \right)$$



Take $\left.\begin{array}{l} s_0 = (1+a) + it \\ s = \sigma + it \end{array}\right\}$ same imaginary parts

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Non-abelian reciprocity . Langlands program

* abelian reciprocity law

(-1) Quadratic reciprocity (omitted)

(0) Power reciprocity (omitted)

(1) Artin reciprocity. F : global field

$$\begin{tikzcd}
\mathrm{rec}_F : \mathbb{A}_F^\times / F^\times \arrow[rr] \arrow[dr] & & \mathrm{Gal}(F^{ab}/F) \\
& \pi_0(\mathbb{A}_F^\times/F^\times) \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

(profinite) ← Artin

(2) ① $\chi : \mathrm{Gal}(\bar{F}/F) \longrightarrow \mathbb{C}^\times$. 1-dim character. (finite order)

$$\begin{tikzcd}
\mathrm{Gal}(\bar{F}/F) \arrow[rr] \arrow[dr] & & \mathbb{C}^\times \\
& \mathrm{Gal}(F^{ab}/F) \arrow[ur] &
\end{tikzcd}$$

⤳ defines an L-function : $L(\chi, s)$

② $\rho : \mathbb{A}_F^\times/F^\times \longrightarrow \mathbb{C}^\times$. assuming finite order

⤳ defines an L-function : $L(\rho, s) = L(\rho \cdot \omega_s)$

(harmonic analysis)

claim : ①, ② get the same L-functions.

Question: non-abelian (generalized) reciprocity ?

F : global. $\quad Gal(\bar{F}/F)$??

(not interested in itself, but its "representation").

V : finite dim'l vector space over $\mathbb{C}$ look at
$\rho : Gal(\bar{F}/F) \longrightarrow GL(V) \quad \sim L(\rho, s)$
How do we "understand" this representation ?

Or ? in terms of objects from harmonic analysis ?

Artin's Conjecture: If $(\rho \mid \text{trivial}) = 0$, i.e.
ρ doesn't contain the trivial representation
of its image. then $L(\rho, s)$ is an entire function
on the whole $\mathbb{C}$-plane.

? Answer ? $L(\rho, s)$ should be a(n) (product of)
automorphic L-function.

$\sim \rho$ comes from automorphic representations. (but
not all auto. repr. give ρ.)

Further generalization. F : global field
$X_{/F}$: algebraic variety.
$\sim$ $\underbrace{\text{defines}}_{\text{a zeta function}}$ $Z_{X,S}(s) = \exp\left(\sum_{\mathfrak{p} \notin S} \sum_{m \geq 1} \frac{\# X_{\mathfrak{p}}(\mathbb{F}_{\mathfrak{p}^m})}{m} \mathfrak{p}^{-ms} \right)$
S : finite set of primes s.t X has good reductions
outside S.

$\longrightarrow$ Hasse-Weil zeta function.

Conjecture $Z_{X,S}$ should be quotients of (a product) of
automorphic L-functions.

$$\rho : \mathrm{Gal}(\bar{F}/F) \longrightarrow GL(V)$$

can take V : finite dimensional vector space / E_λ.
where E · number field. λ : finite place (e.g. $\mathbb{Q}_p$)

With "reasonable" restrictions · (e.g. abs. irred., $\det = 1$)
$\rightsquigarrow$ still get automorphic L-functions.

<u>Local</u> <u>situation:</u> F · local field (locally compact)

Local abelian reciprocity :

$$\mathrm{rec}_F : F^\times \longrightarrow \mathrm{Gal}(F^{ab}/F).$$ (abelianization of Weil group?)

and the image is dense. (topology of RHS ?)

non-abelian reciprocity suppose F is non-archimedean.

Weil group

$$\mathrm{Gal}(\bar{F}/F)$$
$$\cup\!|$$
$$W_F := \{ \sigma : \sigma|_{F^{ur}} = \mathrm{Frob}^n \text{ for some } n \}$$

(NB.
$$\begin{tikzcd} \mathrm{Gal}(\bar{F}/F) \arrow[r] & \mathrm{Gal}(F^{ur}/F) \cong \hat{\mathbb{Z}} \\ W_F \arrow[r] \arrow[u, hook] \arrow[ur, phantom, "\square"] & \mathbb{Z} \arrow[u, hook] \end{tikzcd}$$
);

suppose F is archimedean.

$F \cong \mathbb{C} \Rightarrow W_F = F^\times$.

an extension.

$F \cong \mathbb{R} \Rightarrow 1 \to \bar{F}^\times \to W_F \to \mathbb{Z}/2\mathbb{Z} \to 0$ exact

$\bar{F}^\times \cong \mathbb{C}^\times$ (non-split)

$\Downarrow$

always $W_F \to \mathrm{Gal}(\bar{F}/F)$ set-theoretically.

$W_F = \bar{F}^\times \amalg \bar{F}^\times j$. $j^2 = -1$

$\mathrm{ad}(j)$ on $\mathbb{C}$ is the complex conjugation

adjoint by j

"Weil group has more representations"

Now.

$$\begin{array}{ccc} \rho : W_F & \longrightarrow & GL(V) \\ \Updownarrow & & \\ \pi : GL_n(F) & \longrightarrow & Aut(U) \end{array}$$

admissable irred. $\qquad$ U : complex vector space

$n = \dim V$ $\qquad$ probably infinite dimensional

The correspondence is more or less 1-1 but with nice functorial properties.

or π is parametrized by ρ. finite dimensional repre. of a more complicated group W_F.

$$\begin{tikzcd} \rho : W_F \arrow[r] \arrow[rr, dashed, bend right] & GL(V) \arrow[r, "\psi"] & GL_m \end{tikzcd}$$

then $\psi(\pi) : GL_m(F) \longrightarrow U'$.

moreover, same holds for automorphic representations.

F : locally compact non-archimedian local field.
$G = GL_2$; $G(F) = GL_2(F) \supseteq K = GL_2(\mathcal{O}_F)$.
(Hecke alg. at only one prime). "good max compact subgp"

Goal. Characterize all irred. representations of $G(F)$.

$\psi_F : (F, +) \longrightarrow \mathbb{C}_1^\times$. fix a non-trivial unitary char.

<u>Def</u> An <u>admissible</u> representation π of $G(F)$ is a (continuous) representation of $G(F)$ on a $\mathbb{C}$-vector space V s.t.
(1) π is <u>smooth</u> (i.e stabilizers are open)
(2) $\pi|_K = \bigoplus_{\omega \in \hat{K}} V(\omega)$.
(alg. direct sum)
and $\dim_{\mathbb{C}} V(\omega) < \infty$ (Peter-Weyl Th'm.)

$d^\times y$: Haar measure $\left(d^\times y = \dfrac{\prod dy_{ij}}{\det y} \right)$
$\mathcal{H}_F = \mathcal{H}_{G(F)}$
= locally constant functions ($\mathbb{C}$-value) with compact support
(Hecke algebra for $G(F)$) with convolution product (depending on $d^\times y$)

$V \rightsquigarrow$ dual $\check{V} = \bigoplus_{\omega \in \hat{K}} V(\omega)^\vee$ ($\not\cong$ naive dual $\mathrm{Hom}(V, \mathbb{C})$)

pairing : $V \times \check{V} \longrightarrow \mathbb{C}$
$$\begin{array}{c} \cup\!| \\ W \rightsquigarrow W^\perp \subseteq \check{V} \end{array}$$
$(\check{V})^\vee \xleftarrow{\sim} V$ + $(W^\perp)^\perp \xleftarrow{\sim} W$.

<u>NB</u> (exercise) An irreducible finite dimensional admissible representati[on] of $G(F)$ has the form
$$G(F) \xrightarrow{\det} F^\times \longrightarrow \mathbb{C}^\times$$
(i.e. 1-dim'l)

Theorem (Existence of Kirillov models)

Let (π, V) be an infinite dimensional admissible irreducible representation of $G(F)$. Then there exists a unique (depends on ψ_F) subspace $V' \subseteq \{$ locally const. functions on $F^\times \}$ with action π' by $G(F)$ s.t.

(1) $(\pi', V') \cong (\pi, V)$

(2) $\forall \xi \in V'$, $\forall a \in F^\times$, $\forall b \in F$.

$$\left(\pi'\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \xi \right)(x) = \psi_F(bx)\ \xi(ax)$$

Moreover,

(3) $\forall \xi \in V'$, $\xi(x) = 0$ if $|x| \gg 0$.

(4) $V' \supseteq \mathcal{S}(F^\times)$ Schwartz space with finite codimension

Such model will be labeled as $\mathcal{K}(\pi)$.

exer $\mathcal{S}(F^\times)$ is irreducible under $\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G(F) \right\}$.

exer. $\psi_F(bx)\,\xi(x) - \xi(x) \in \mathcal{S}(F^\times)$.

Relation with Whittaker models.

Given $\xi \in \mathcal{K}(\pi)$. define $W_\xi : G(F) \longrightarrow \mathbb{C}$

$$g \in G(F) \longmapsto W_\xi(g) := (\pi(g)\ \xi)(1_F)$$

$$W_{\pi(g)\xi}(x) = (\pi(x)\ \pi(g)\ \xi)(1) = W_\xi(xg)$$

i.e On $\{ W_\xi : \xi \in \mathcal{K}(\pi) \}$, $G(F)$ acts by right translation

K(2) Also. $W_\xi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) = \left(\pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \pi(g)\ \xi \right)(1)$

$= \psi_F(x)\,(\pi(g)\,\xi)(1) = \psi_F(x) \cdot W_\xi(g)$. ← another func. equ.

K(2) Also. W_ξ determines ξ:

$$\xi(x) = W_\xi\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \right)$$

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Denote: $\mathcal{W}(\pi) = \{ W_\xi : \xi \in \mathcal{K}(\pi) \}$ with an action by $G(F)$

Point. $\mathcal{W}(\pi)$ determines $\mathcal{K}(\pi)$, and vice versa.

<u>Theorem</u> (Existence of Whittaker models)
Let π be an infinite dimensional admissible irreducible representation of $G(F)$. Then $\exists$ a unique subrepresentation $(\mathcal{W}(\pi), \pi)$ of $\{$ locally constant functions W on $G(F)$: $W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi_F(x) W(g)$, $\forall x \in F$, $g \in G$ under right translation

<u>Def</u> (π, V) (adm. inf. irr) is <u>supercuspidal</u>
if $\mathcal{K}(\pi) = \mathcal{S}(F^\times)$.

(meaning. supercuspidal representations can't be picked up from induction.)

<u>Fact</u> $\mathcal{K}(\pi) \supseteq \mathcal{S}(F^\times)$.
$\mathcal{S}(F^\times) + w\,\mathcal{S}(F^\times) = \mathcal{K}(\pi)$. $\left(w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right.$, Weyl elemen
(for the proofs of the existence of K. W models.)

<u>Supercuspidal</u>
equiv. conditions. (1) in def.
(2) $\forall \xi \in \mathcal{K}(\pi)$, $\displaystyle\int_{p^{-n}} \pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \xi \, dx = 0$ if $n >> 0$

$$\left(\int_{p^{-n}} \pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \xi \, dx \right)(y) := \int_{p^{-n}} \left(\pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \xi\right)(y) \, dx$$
$$= \left(\int_{p^{-n}} \psi_F(xy) \, dx \right) \cdot \xi(y) \quad)$$

$\sim$ (1) $\Rightarrow$ (2)
(if $n >> 0$, $x \mapsto \psi_F(xy)$ is not trivial
(with $y \in \operatorname{supp}(\xi)$.)

(2) $\Rightarrow$ (1) vice versa

(3) $\forall\ \xi \in V \quad \forall\ \check{\eta} \in \check{V}$

$g \mapsto \langle \pi(g)\xi, \check{\eta} \rangle$ "matrix coefficient"

has compact support mod $Z(F)$ (center of $G(F)$)

$$Z(F) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad a \in F^\times$$

exer. (1). (2) $\Longleftrightarrow$ (3)

hint Use the Cartan decomposition

$$G(F) = K \cdot A \cdot K$$

$K = GL_2(\mathcal{O}_F)$, $A(F) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \middle| a, d \in F^\times \right\}$

(only have to look at $g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ $a \in F^\times$, $b \in F$)

Fact (π, V) adm. irr.

$\Longrightarrow$ $\exists$ central char. i.e. $Z(F) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \middle| a \in F^\times \right\}$

operates by a char. $\omega_\pi(a)$

Induced representations

$B(F) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G(F) \right\}$ Borel subgroup of $G(F)$.

$$B(F) \xrightarrow{\mu_1, \mu_2} \mathbb{C}^\times, \quad \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \mu_1(a) \cdot \mu_2(d), \quad \mu_1, \mu_2 : F^\times \longrightarrow \mathbb{C}^\times \text{ characters.}$$

Def $\mathrm{Ind}_{B(F)}^{G(F)}(\mu_1, \mu_2)$

$$= \left\{ f : G(F) \longrightarrow \mathbb{C} \text{ locally constant} \ \middle| \ f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} x \right) = \mu_1(a) \cdot \mu_2(d) \cdot \left| \frac{a}{d} \right|^{\frac{1}{2}} f(x) \quad \forall \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B(F),\ x \in G(F) \right\}$$

NB $\mathrm{supp}(f) \subseteq B(F) \backslash G(F) \underset{\text{exer.}}{\cong} \mathbb{P}^1(F)$

- Why $\left|\frac{a}{d}\right|^{\frac{1}{2}}$?

(Partial answer: if μ_1, μ_2 are unitary, then $\mathrm{Ind}(\mu_1, \mu_2)$ is also unitary.)

Roughly speaking, $\mathrm{Ind}(\mu_1, \mu_2)$ is the space of sections of $\mathbb{C} \overset{B(F)}{\times} G(F)$ vector bundle over $B(F)\backslash G(F)$ $\quad (v, bg) \sim (vb, g) = (b^{-1}v, g)$
twist by half density bundle

$B\backslash G \sim$ (Lie) density : left translation invariant ----
$\rightsquigarrow$ top differential form

More generally, given μ_1, μ_2 characters on $F^\times$, then
$$\mathrm{Ind}(\mu_1, \mu_2) \times \mathrm{Ind}(\mu_1^{-1}, \mu_2^{-1}) \longrightarrow \mathbb{C}$$
is a perfect pairing.

<u>Haar measures</u> $\quad G$: locally compact group.

$d_\ell g$. left invariant measure.
$\rightsquigarrow \int f(g)\, d_\ell g$ makes sense now.
and $\int f(xg)\, d_\ell g = \int f(g)\, d_\ell g$

Translate on the right : $d_\ell(g g_0^{-1})$
(meaning : $\int f(g)\, d_\ell(g g_0^{-1}) = \int f(g g_0)\, d_\ell g$)

$d_\ell(g g_0^{-1})$ is also a left invariant Haar measure
$\Rightarrow d_\ell(g g_0^{-1})$ is a multiple of $d_\ell g$.

$\therefore \exists$ a modular function m_G s.t.
$$d_\ell(g g_0^{-1}) = m_G(g_0)\, d_\ell g.$$

$m_G : G \longrightarrow \mathbb{R}_+^\times$ continuous homomorphism.

- $m_G(g)\, d_\ell(g)$ is invariant under <u>right</u> translation. denote it by $d_r(g)$

- $d_r(g_0 g) = m_G(g_0)\, d_r(g)$ $\left(\text{cf. } d_\ell(g g_0^{-1}) = m_G(g_0)\, d_\ell g\right.$ (inverse) $\quad \overset{\|}{d_\ell(g_0 g g_0^{-1})})$

<u>exercise</u> $d_\ell(g^{-1})$ is right invariant. Prove that $d_\ell g^{-1} = 1\, d_r g$.

Look at $L(G\,B) = \left\{ f : G(F) \to \mathbb{C} \;\middle|\; f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \left|\frac{a}{d}\right| f(g) \quad \forall \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B,\ g \in G(F) \right\}$

$\exists \displaystyle\int_{B\backslash G} d^{\cdot}g : L(G\,B) \longrightarrow \mathbb{C}$
right invariant linear functional

(if the factor $\left|\frac{a}{d}\right|^{\frac{1}{2}}$ is omitted. Later there'll be a normalization.)

$$\begin{array}{ccc} \mathcal{C}_c(G) & \longrightarrow & L_c(G\,P) \\ \psi & & \psi \\ f & \longmapsto & \varphi_f(g) = \displaystyle\int_P f(pg)\, d_\ell p \end{array}$$
$P \subseteq G$ closed subgroup., $d_\ell p$.

$$\Longrightarrow \quad \varphi_f(p_1 g) = \int_P f(p p_1 g)\, d_\ell p = m_P(p_1)\, \varphi_f(g).$$

$L_c(G\,P)$: function φ on G s.t. $\varphi(p_1 g) = m_P(p_1)\, \varphi(g)$ $\forall p_1 \in P,\ \forall g \in G$ with compact support.

General situation. G : unimodular (i.e $m_G \equiv 1$)

$$\begin{tikzcd} \mathcal{C}_c(G) \arrow[r, two heads, "①"] \arrow[d, "\int_G"'] & L_c(G\,P) \arrow[dl, dashed, "②"] \\ \mathbb{C} & \end{tikzcd}$$

② . kills the kernel in ①

$$\begin{tikzcd} P \arrow[r] & G \arrow[d] \\ & P\backslash G \end{tikzcd}$$

Then $\int_{P\backslash G} d^{\cdot}g \int_P f(pg)\, d_\ell p = \int_G f(g)\, dg$

(after normalization of Haar measures.)
(unique up to an $\mathbb{R}_+^\times$ factor)

Properties of $\int_{P\backslash G} \cdot \, d^{\cdot}g$:

(1) Uniqueness

(2) If $\exists$ a closed subgroup $M \leq G$ s.t. $P \cap M$ compact and $PM = G$ up to a measure-zero set, then (up to $\mathbb{R}_+^\times$)

$$\int_{P\backslash G} \varphi(g)\, d^{\cdot}g = \int_M \varphi(mx)\, \underset{\text{right invariant}}{\underbrace{d_r(m)}} \qquad \forall x \in G$$

$\exists ! \int_{P\backslash G} \cdot \, d^{\cdot}g$. $\rightsquigarrow$ done

Back to our situation $\quad G = G(F), \quad P = B(F)$
$M = GL_2(\mathcal{O}) = K$

if $\varphi \in L(G, B)$, then

$$\int_{B\backslash G} \varphi(g)\, d^{\cdot}g = \int_K \varphi(x)\, dx$$

($G = BK = ANK$ Iwasawa decomposition for G)
$N(F) = \{ \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \}$.

Since $B\backslash G \cong \mathbb{P}^1$ we tend to represent φ in one variable
$\exists w$. $x \longmapsto \varphi(w^{-1}\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right))$ represents φ.
(because $B \cdot w^{-1} \cdot N$, or equiv. $B w^{-1} N w$ is dense in G
therefore

$$\int_{B\backslash G} \varphi(\dot{g})\, d\dot{g} = \int_F \varphi(w^{-1}\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right))\, dx$$

$$= \int_{GL_2(\mathcal{O}_F)} \varphi(xg)\, d^\times x$$ Check they are all translation invariant.

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<u>Exercise</u> Show that $\mathrm{Ind}(\mu_1, \mu_2)^\vee = \mathrm{Ind}(\mu_1^{-1}, \mu_2^{-1})$.

<u>Prop</u> If (π, V) is not supercuspidal but infinite dimensional, irreducible, admissible, then it is isomorphic to a subrepresentation of an induced representation

<u>proof</u>. $\mathcal{K}(\pi) \supsetneqq \mathcal{S}(F^\times)$, so we consider $X = \mathcal{K}(\pi)/\mathcal{S}(F^\times)$ which is finite dimensional.

Note. $\mathcal{S}(F^\times)$ is stable under $B(F)$.

(actually it's irreducible under $\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\}$ <u>exer.</u>)

Moreover $N(F) := \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\}$ operates trivially on X.

(i.e. $x \longmapsto \xi(x) - \psi(bx)\xi(x) \in \mathcal{S}(F^\times)$, $\forall \xi \in \mathcal{K}(\pi)$)

$\rightsquigarrow$ $B(F)$ acts via $B(F) \longrightarrow B(F)/N(F) \xleftarrow{\sim} A(F)$ commutative.

$\rightsquigarrow$ $\exists$ $\mathcal{K}(\pi) \longrightarrow X \longrightarrow\!\!\!\!\rightarrow \mathbb{C}$ $B(F)$-equivariant.

$\begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \longmapsto \mu_1(a)\mu_2(d)\left|\frac{a}{d}\right|^{\frac{1}{2}}$

$$\text{Our goal : } \quad (\pi, V) \xrightarrow{G(F)\text{-equivariant}} \mathrm{Ind}_{B(F)}^{G(F)}(\mu_1, \mu_2)$$

$$\Big\updownarrow \text{exer.}$$

$$\pi|_{B(F)} \xrightarrow[\cup]{B(F)\text{-equivariant}} \mathbb{C} \qquad \text{(Frobenius reciprocity.)}$$

$\square$

(From representation, say $H \subseteq G$, (π, V), (ρ, W) repr. of G, H.
then $\mathrm{Hom}_G((\pi, V), \mathrm{Ind}_H^G(\rho, W)) \cong \mathrm{Hom}((\pi, V)_H, (\rho, W))$.)

<u>Main Results on Induced Representations.</u>

(for non-archimedian local field)

(1) $\rho(\mu_1, \mu_2)$ is irreducible unless $\mu := \mu_1/\mu_2 = |\cdot|^{\pm 1}$

(2) If $\mu = |\cdot|$, then the space $\rho(\mu_1, \mu_2)$ contains an irreducible representation of codimension 1.

($\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ acts by $\mu_1(a) \cdot \mu_2(a)$.)

consisting of $\{ f \in B(\mu_1, \mu_2) \mid \int_{B\backslash G} f(g)\, \mu^{-1}(\det(g))\, |\det g|^{\frac{1}{2}}\, dg = 0$

$= \int_F f(w^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix})\, dx. \}$

(3) (or $(2)^\vee$) If $\mu = |\cdot|^{-1}$, then $\rho(\mu_1, \mu_2)$ contains a 1-dim'l subrepresentation generated by $g \longmapsto \mu_1(\det(g)) \, |\det(g)|^{\frac{1}{2}}$, and the quotient is irreducible.

$$\underset{\substack{\mathcal{B}(\mu_1,\mu_2) \\ \| \\ \{f: G\to\mathbb{C} \mid f\left(\begin{pmatrix} a & b \\ 0 & d\end{pmatrix} g\right) = \mu_1(a)\mu_2(d)\left|\frac{a}{d}\right|^{\frac{1}{2}} f(g) \\ \forall \begin{pmatrix} a & b \\ 0 & d\end{pmatrix}\in B,\ \forall g\in G\}}}{f \in} \longmapsto \underset{\text{function on } F.}{\left(\Phi_f : x \longmapsto f\left(w^{-1}\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\right)\right)} \qquad w = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} \text{ Weyl ele.}$$

$$c \neq 0 \quad \begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} c^{-1}\det(g) & a \\ 0 & c\end{pmatrix} w^{-1} \begin{pmatrix} 1 & \frac{d}{c} \\ 0 & 1\end{pmatrix}$$

$$\Longrightarrow f\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \mu_1(c^{-1}\det(g))\,\mu_2(c)\,\left|\frac{\det(g)}{c^2}\right|^{\frac{1}{2}} \Phi_f\left(\frac{d}{c}\right)$$

$$= \mu(c)^{-1}\,\mu_1(\det(g))\,|\det(g)|^{\frac{1}{2}}\,|c|^{-1}\,\Phi_f\left(\frac{d}{c}\right) \qquad (*)$$

f : like function on $\mathbb{P}^1$. Φ_f : function on $\mathbb{A}^1$.

$\rightsquigarrow$ need to analyze or "pin down" the asymptotic behavior of f.

$$w^{-1}\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} = \begin{pmatrix} 1 & -x^{-1} \\ 0 & 1\end{pmatrix}\begin{pmatrix} x^{-1} & 0 \\ 0 & x\end{pmatrix}\begin{pmatrix} 1 & 0 \\ x^{-1} & 1\end{pmatrix}$$

$$\rightsquigarrow f\left(w^{-1}\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\right) = \mu^{-1}(x)\,|x|^{-1}\, \underbrace{f\begin{pmatrix} 1 & 0 \\ x^{-1} & 1\end{pmatrix}}_{\substack{\| \\ f(e) \text{ for } |x|>>0}}$$

$\rightsquigarrow \quad \Phi_f(x)\,\mu(x)\,|x| = \text{constant} \quad \text{for } |x| >> 0$

<u>Exercise</u> This property characterizes Φ_f's.

(i.e. using $(*)$, we define $f\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ for $c \neq 0$.
claim can extend to the case $c = 0$ and
$f \in \mathcal{B}(\mu_1, \mu_2)$)

<u>Fourier transform</u>

$$\xi_f(x) = \mu_2(x)\,|x|^{\frac{1}{2}} \int_F \underbrace{f\left(w^{-1}\begin{pmatrix} 1 & y \\ 0 & 1\end{pmatrix}\right)}_{\substack{\| \\ \Phi_f(y)}} \overline{\psi_F(xy)}\, dy$$

$\rightsquigarrow \quad \xi_{\rho_{\mu_1,\mu_2}\left(\begin{smallmatrix} a & b \\ 0 & 1\end{smallmatrix}\right) f}(x) = \psi_F(bx)\,\xi_f(ax)$ as in Kirillov's model.

proof. $\xi_{\rho_{\mu_1,\mu_2}\left(\begin{smallmatrix}a&b\\0&1\end{smallmatrix}\right)f}(x)$

$$= \mu_2(x)\,|x|^{\frac12} \int_F \rho_{\mu_1,\mu_2}\begin{pmatrix}a&b\\0&1\end{pmatrix} f\left(w^{-1}\begin{pmatrix}1&y\\0&1\end{pmatrix}\right) \overline{\psi_F(xy)}\, dy$$

right action here

$$= \mu_2(x)\,|x|^{\frac12} \int_F f\left(w^{-1}\begin{pmatrix}1&y\\0&1\end{pmatrix}\begin{pmatrix}a&b\\0&1\end{pmatrix}\right) \overline{\psi_F(xy)}\, dy$$

$$= \mu_2(x)\,|x|^{\frac12} \int_F f\left(w^{-1}\begin{pmatrix}a&b+y\\0&1\end{pmatrix}\right) \overline{\psi_F(xy)}\, dy.$$

let $z=b+y$

$$= \mu_2(x)\,|x|^{\frac12} \int_F f\left(w^{-1}\begin{pmatrix}a&z\\0&1\end{pmatrix}\right) \overline{\psi_F(xz)\cdot\psi_F(xb)^{-1}}\, dz$$

$$= \psi_F(bx)\,\mu_2(x)\,|x|^{\frac12} \int_F f\left(w^{-1}\begin{pmatrix}a&y\\0&1\end{pmatrix}\right) \overline{\psi_F(xy)}\, dy$$

$$= \psi_F(bx)\,\mu_2(x)\cdot|x|^{\frac12} \int_F f\left(\begin{pmatrix}1&0\\0&a\end{pmatrix} w^{-1}\begin{pmatrix}1&\frac{y}{a}\\0&1\end{pmatrix}\right) \overline{\psi_F(xy)}\, dy$$

$$= \psi_F(bx)\,\mu_2(x)\,|x|^{\frac12}\cdot\mu_2(a)\,|a|^{-\frac12} \int_F f\left(w^{-1}\begin{pmatrix}1&\frac{y}{a}\\0&1\end{pmatrix}\right) \overline{\psi_F(xy)}\, dy$$

let $z=\frac{y}{a}$

$$= \psi_F(bx)\,\mu_2(x)\cdot|x|^{\frac12}\cdot\mu_2(a)\cdot|a|^{-\frac12} \int_F f\left(w^{-1}\begin{pmatrix}1&z\\0&1\end{pmatrix}\right) \overline{\psi_F(axz)}\, |a|\, dz$$

$$= \psi_F(bx)\,\mu_2(ax)\cdot|ax|^{\frac12} \int_F f\left(w^{-1}\begin{pmatrix}1&z\\0&1\end{pmatrix}\right) \overline{\psi_F(ax\cdot z)}\, dz$$

$$= \psi_F(bx)\,\xi_f(ax).$$

q.e.d

<u>Lemma</u> Let μ be a character of $F^\times$, and $\mathcal{F}_\mu = \{$ locally const. functions Φ on F s.t. $\mu(x)\,|x|\cdot\Phi(x) = \text{const}$ for $|x| >> 0 \}$

For $\Phi \in \mathcal{F}_\mu$, consider the sum:

$$\hat{\Phi}(x) := \sum_{n\in\mathbb{Z}} \int_{v(y)=n} \Phi(y)\,\overline{\psi_F(xy)}\,dy, \qquad x\in F^\times.$$

Then (i) sum converges uniformly on compact sets of $F^\times$

(ii) $\Phi \longmapsto \hat{\Phi}$ is injective unless $\mu = |\cdot|^{-1}$

(iii) If $\mu = |\cdot|^{-1}$, then $\mathrm{Ker}\,(\Phi \longmapsto \hat{\Phi})$ consists of constant function

(iv) $\mathrm{Im}\,(\Phi \longrightarrow \hat{\Phi}) =$

$\{$ locally const. functions Ψ on $F^\times$ s.t.

(a) $\Psi(x)$ vanishes for $|x| >> 0$.

(b) The behavior of $\Psi(x)$ near 0 is

$$\Psi(x) = \begin{cases} a\,\mu(x) + b & \text{if } \mu \neq \mathbb{1}, |\cdot|^{-1} \\ a\,v_F(x) + b & \text{if } \mu = \mathbb{1} \\ b & \text{if } \mu = |\cdot|^{-1} \end{cases} \quad \}$$

<u>exer</u> (1) Show that if $\mu = |\cdot|^{-1}$ then Fourier transform of constant function is 0.

(2) F.T. $(\mathcal{S}(F)) \xrightarrow{\sim} \mathcal{S}(F)$

<u>Define</u> $\Phi_\mu(x) = \begin{cases} \mu(x)\cdot|x|^{-1}. & \text{if } |x| \geq 1 \\ 0 & \text{if } |x| < 1 \end{cases}$

<u>Claim</u> $\mathcal{F}_\mu = \mathcal{S}(F) \oplus \mathbb{C}\cdot\Phi_\mu(x)$

To verify (i), needs only to check for Φ_μ.

case (A) μ is ramified : we're looking at:

$$\sum_{n\leq 0} \int_{v(y)=n} \mu(y)^{-1}\cdot|y|^{-1}\cdot\overline{\psi_F(xy)}\,dy \qquad |y|^{-1}dy = d^\times y$$

essentially the ε-factors.

(at most one of the terms is not zero.)

<u>Claim</u> $\displaystyle\int_{v(y)=n} \mu(y)^{-1}\, \overline{\psi_F(xy)}\, d^\times y = 0$ unless

$v(x) + n = -d - f$. d = conductor of μ ≥ 0

f = conductor of ψ

(clear.)

$\rightsquigarrow$ verify (i) . (iii). (iv) directly

case (B) . μ is unramified. i.e. $\mu = |\cdot|^s$ for some $s \in \mathbb{C}$.

Then $\displaystyle\int_{v(y)=n} \mu(y)^{-1}\, \overline{\psi_F(xy)}\, d^\times y$

x^n ? or the previous one is xz.

$$= q^{-ns} \left(\int_{\mathcal{O}} \overline{\psi_F}(x^n z)\, dz - \int_{\mathfrak{p}} \overline{\psi_F}(\pi^n x z)\, dz \right)$$

$\rightsquigarrow$ $\hat{\Phi}_\mu(x) = F_s(x) - |\pi|\, F_s(\pi x)$

where $F_s(x) = \displaystyle\sum_{-d-v(x) \leq n \leq 0} q^{-ns}$, a finite geom. series.

$TFT_{\mu_1\mu_2}: \mathcal{B}(\mu_1, \mu_2) \longrightarrow \mathcal{K}_{\mu_1,\mu_2}$ (twisted fourier transform)

$f \longmapsto \xi_f : x \longmapsto \mu_2(x)|x|^{\frac{1}{2}} \int_F f(w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}) \overline{\psi_F(xy)}\, dy.$

$$\mathcal{K}_{\mu_1,\mu_2} \supseteq \mathcal{S}(F^\times)$$

$\|$

{ functions on $F^\times$ vanishing for $|x| \gg 0$, & for x in a nbd of 0,

given by $\begin{cases} \cdot\ |x|^{\frac{1}{2}}(a\mu_1(x) + b\mu_2(x)) & \text{if } \mu \neq \mathbb{1},\ |\cdot|^{-1} \\ \cdot\ |x|^{\frac{1}{2}}(a\mu_1(x)\cdot v_F(x) + b\mu_2(x)) & \text{if } \mu = \text{trivial} \\ \cdot\ b\,|x|^{\frac{1}{2}}\mu_2(x) & \text{if } \mu = |\cdot|^{-1} \end{cases}$ }

$$\mathrm{Ker}(TFT_{\mu_1,\mu_2}) = \begin{cases} 0 & \text{if } \mu \neq |\cdot|^{-1} \\ \mathbb{C}\cdot \mu_1(\det(g))\,|\det(g)|^{\frac{1}{2}} & \text{if } \mu = |\cdot|^{-1} \end{cases} \qquad (\mu = \mu_1\mu_2^{-1})$$

<u>Note</u> $\mathcal{S}(F^\times)$ is irreducible under $\{\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}\ a \in F^\times,\ b \in F\}$

(A) Suppose $\mu \neq |\cdot|^{-1}$. Then TFT_{μ_1,μ_2} is an isomorphism.

Given a subrepresentation $0 \neq V \subseteq \mathcal{B}(\mu_1, \mu_2)$.

$\rightsquigarrow V \supseteq \{ f - \rho(\mu_1,\mu_2)\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} f \mid f \in V \}$ by definition.

i.e. $V^\perp$ is fixed under all $\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$, $b \in F$.

$V^\perp \subseteq \mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$.

A function $f \in \mathcal{B}(\mu_1^{-1}\mu_2^{-1})$ fixed under all $\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$, $b \in F$

has the property that

$$f\left(\begin{pmatrix}t_1 & *\\ 0 & t_2\end{pmatrix} w^{-1} \begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix}\right) = \mu_1(t_1)^{-1}\mu_2(t_2)^{-1} \cdot \left|\frac{t_1}{t_2}\right|^{\frac{1}{2}} f(w^{-1})$$

$\rightsquigarrow \dim V^\perp \leq 1$ (only constant: $f(w^{-1})$!)

On the other hand, $w^{-1}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & -b^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}b^{-1} & 0\\ 0 & b\end{pmatrix}\begin{pmatrix}1 & 0\\ b^{-1} & 1\end{pmatrix}$ $b \neq 0$.

$\rightsquigarrow f(w^{-1}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}) = \mu(b)\,|b|^{-1} \cdot f\begin{pmatrix}1 & 0\\ b^{-1} & 1\end{pmatrix}$ $b \neq 0$.

$\Longrightarrow$ if $\mu \neq |\cdot|$, then $V^\perp = (0)$. $\rightsquigarrow V = \mathcal{B}(\mu_1, \mu_2)$.

<u>exercise</u>. How about $\mu = |\cdot|$?

<u>exercise</u> $\mu = |\cdot|^{-1}$? (Hint: By duality look at $\mu = |\cdot|$.)

<u>Names</u>. If $\mu \neq |\cdot|^{\pm 1}$. write $\pi(\mu_1, \mu_2) = \rho(\mu_1\ \mu_2)$ (principal series)

If $\mu = |\cdot|^{\pm 1}$. write $\pi(\mu_1\ \mu_2) =$ the 1 dim'l subquotient.

$\sigma(\mu_1\ \mu_2) =$ the infinite dim'l subquotient.

(also called special representations)

<u>e.g.</u> Use $\pi(\mu_1 . \mu_2) = \pi(\mu_2 . \mu_1)$ (to be proved by understanding $\check{\pi}$)

& $\rho(\pi_1 \chi . \pi_2 \chi) = \rho(\mu_1 , \mu_2)(\chi \circ \det)$.

(twisted by a char. of $F^\times$ (or . GL_1))

(same for π, σ.)

$\Rightarrow$ all special representations are twists of $\sigma(|\cdot|^{\frac{1}{2}} . |\cdot|^{-\frac{1}{2}})$

<u>Remark</u> π : irreducible infinite dimensional representation

$\rightsquigarrow \dim_{\mathbb{C}}\left(\mathcal{K}(\pi)/\mathcal{S}(F^\times) \right) = \begin{cases} 0 & \text{if } \pi \text{ is supercuspidal} \\ 1 & \text{if } \pi \text{ is special} \\ 2 & \text{if } \pi \text{ is a principal series representation} \end{cases}$

2/3/97

How to express the contragradient representation of an irred. adm. repr. concretely?

Theorem Let (π, V) be an admissible irreducible representation. ω_π = central character of π. Then:

(1) $\check{\pi}$ is isomorphic to (π', V') where $V = V'$ and

$$\pi'(g) : v \longmapsto \omega_\pi(\det(g))^{-1} \cdot \pi(g)(v)$$

(2) $\mathcal{K}(\check{\pi}) = \{ (x \longrightarrow \omega_\pi(x)^{-1} \xi(x)) \mid \xi \in \mathcal{K}(\pi) \}$
(only the underlying space.)

(3) Define $\mathcal{K}(\pi) \xrightarrow{\Phi} \mathcal{K}(\check{\pi})$ by (2). i.e.

$$\eta \longmapsto \eta' = (x \longmapsto \omega_\pi(x)^{-1} \eta(x)) \quad \text{a bijection}$$

making

$$\begin{tikzcd} \mathcal{K}(\pi) \arrow[r, "\Phi"] \arrow[d, "\omega_\pi(\det(g))^{-1} \cdot \pi(g)"'] & \mathcal{K}(\check{\pi}) \arrow[d, "\check{\pi}(g)"] \\ \mathcal{K}(\pi) \arrow[r, "\Phi"] & \mathcal{K}(\check{\pi}) \end{tikzcd}$$

commutative. $\forall g \in G(F)$ (equivalent to (1))

(4) The pairing $\mathcal{K}(\pi) \times \mathcal{K}(\check{\pi}) \longrightarrow \mathbb{C}$ is :

$\xi = \xi_1 + \pi(w)\xi_2$, $\xi_i \in \mathcal{S}(F^\times)$

$$\Rightarrow \langle \xi, \eta \rangle = \int_{F^\times} \xi_1(x)\, \eta(-x)\, d^\times x + \int_{F^\times} \xi_2(x) (\check{\pi}(w^{-1})\eta)(-x)\, d^\times x.$$

Equivalently, the following bilinear form on $\mathcal{K}(\pi)$.

$\xi = \xi_1 + \pi(w)\xi_2$, $\xi_i \in \mathcal{S}(F^\times)$.

$$\langle \xi, \eta \rangle' = C\left[\int_{F^\times} \xi_1(x)\, \eta(-x)\, \omega_\pi(x)^{-1} d^\times x + \int_{F^\times} \xi_2(x) (\pi(w^{-1})\eta)(-x)\, \omega_\pi(x)^{-1} d^\times x \right]$$

The pairing is well-defined. and

$$\langle \pi(g)\xi, \eta \rangle' = \langle \xi, \pi'(g^{-1})\eta \rangle' \qquad \forall\, \xi, \eta \in \mathcal{K}(\pi)$$

where $\pi'(g) = \omega_\pi(\det(g))^{-1} \cdot \pi(g)$.

" The Ultimate Truth ".

Local Langlands correspondence

F: non-arch. local field. $G = GL_2$

$$\left\{\begin{array}{l}\text{admissible}\\ \text{irred repr.}\\ \text{of } G(F)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}2 \text{ dim'l repr. of } Gal(F^{sep}/F)\\ \text{i.e.}\\ \quad Gal(F^{sep}/F) \longrightarrow GL_2(\mathbb{C})\end{array}\right\}$$

↑ this will fail ; to be modified

Problem $Gal(F^{sep}/F)$ is profinite.

1st modification Replace $Gal(F^{sep}/F)$ by W_F, the Weil group

$$\begin{tikzcd}
0 \arrow[r] & I_F \arrow[r] & Gal(F^{sep}/F) \arrow[r, two heads] & Gal(K_F^{sep}/K_F) \cong \hat{\mathbb{Z}} \\
0 \arrow[r] & I_F \arrow[r] \arrow[u, equal] & W_F \arrow[r] \arrow[u, phantom, "\bigcup"] & \mathbb{Z} \arrow[u, phantom, "\bigcup"]
\end{tikzcd}$$

the topology of W_F is given so that I_F is open.

Problem No correspondence of special repr.

2ND modification Replace W_F by either $W_F \times SL_2(\mathbb{C})$ or $W_F \rtimes \mathbb{G}_a$ (here $\mathbb{G}_a = \{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \}$

($w \in W_F$ $x \in \mathbb{G}_a$, then $w x w^{-1} = \|w\| x$. ; $w \longmapsto F_r^n \Rightarrow \|w\| = q^{n/2}$
geometric Frobenius.)

Going from one to the other: given repr. of $W_F \times SL_2(\mathbb{C})$

i.e. $W_F \times SL_2(\mathbb{C}) \xrightarrow{\rho} GL_2(\mathbb{C})$.

then for $(w, x) \in W_F \rtimes \mathbb{G}_a$,

$$(w, x) \longmapsto \rho(w) \ \rho\begin{pmatrix} \|w\|^{\frac{1}{2}} & 0 \\ 0 & \|w\|^{-\frac{1}{2}} \end{pmatrix} \cdot \rho\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

What have we known?

supercuspidal repr. $\longleftrightarrow$ 2-dim'l irred. repr. of W_F
($SL_2(\mathbb{C})$ operates trivially)

principal series $\pi(\mu_1, \mu_2)$ $\longleftrightarrow$ $\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ ($SL_2(\mathbb{C})$ operates trivially)

special repr. $\longleftrightarrow$ $SL_2(\mathbb{C})$ operates non-trivially.
$SL_2(\mathbb{C})$ has only $\underline{1}$ non-trivial repr. . W_F commutes with it.
$\Longrightarrow$ W_F operates by a 1-dim'l character (Schur's lemma)

This correspondence preserves the <u>L-factors</u>

$$\left\{ \begin{array}{l} \text{admissible irred} \\ \text{representations of} \\ GL_2(F) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{2 dim'l } \mathbb{C} \\ \text{representations} \\ \text{of } W_F \times SL_2(\mathbb{C}) \end{array} \right\}$$

How to define an L-factor? (when $SL_2(\mathbb{C})$ acts trivially)

$$L(\underset{\substack{\uparrow \\ \text{repr. of } W_F \\ (\rho, V)}}{\rho}, s) = \det\left(1 - \rho(Fr) q^{-s} \big|_{V^{I_F}}\right)^{-1}$$

Fr geometric Frobenius.
I_F : inertia group. V^{I_F} : fixed space
$1 -$: $1 \rightsquigarrow$ identity.

L-factors don't determine the representations.
(e.g 1 dim'l ramified $\rightsquigarrow$ $L = 1$)

The correspondence preserves the twist

$$\pi \longleftrightarrow \rho \qquad \Rightarrow$$
$$\pi \circ \chi \circ \det \longleftrightarrow \rho \cdot \chi \qquad (\chi \in W_F)$$

ω_π : central character. $\rightsquigarrow$ a character of W_F. (or Galois gp.)

$$(\underset{\substack{\uparrow \\ \text{repr.}}}{\pi} \rightsquigarrow \underset{\substack{\uparrow \\ \text{central char.}}}{\omega_\pi}) \longleftrightarrow (\underset{\substack{\uparrow \\ \text{2 dim}}}{\rho} \rightsquigarrow \underset{\substack{\uparrow \\ \text{1 dim.}}}{\det \rho})$$

unitary repr.
$\Downarrow$
unitary central char. $\rightsquigarrow$ det will be unitary as well

$$\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \xrightarrow{\text{L-factors}} L(\mu_1, s) \cdot L(\mu_2, s)$$
(we expect so.)

Functional Equation

π : admissible irred infinite dim'l repr. of $GL_2(F)$

we know Whittaker model $\mathcal{W}(\pi) \ni W$

Kirillov model $\mathcal{K}(\pi) \ni \xi$

For $W \in \mathcal{W}(\pi)$, define

Local ζ-function $\zeta(\underset{\substack{\uparrow \\ G(F)}}{g}, \underset{\substack{\uparrow \\ \text{char of } F^\times}}{\chi}, W, s) = \int_{F^\times} \underbrace{W\left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right]}_{\pi(g)\,\xi(x)} \chi(x) |x|^{s-\frac{1}{2}} d^\times x$

(twisting by χ)

<u>Theorem</u> (1) $\zeta(g, \chi, W, s)$ converges for $|x| \gg 0$

(2) $\exists$ an Euler factor $L(\underbrace{\pi, \chi}_{\pi\otimes\chi}, s)$

(a polynomial in q^{-s}. constant term 1, take the reciprical)

such that $\dfrac{\zeta(g, \chi, W, s)}{L(\pi, \chi, s)}$ is an entire function in s $\forall \chi$. $\forall W$ (indep. of g, W)

(common denominator so to speak)

<u>and</u> such that $\exists W^0$ with $\zeta(1, \chi, W^0, s) = L(\pi, \chi, s)$

(3) $\zeta(g, \chi, W, s)$ has a meromorphic continuation to $\mathbb{C}s$, and satisfies the functional equation: Weyl element, $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

$$\frac{\zeta(g, \chi, W, s)}{L(\pi, \chi, s)} \cdot \varepsilon(\pi, \chi, s) = \frac{\zeta(wg, \chi^{-1}\omega_\pi^{-1}, W, 1-s)}{L(\pi, \omega_\pi^{-1}\chi^{-1}, 1-s)}$$

Through the proof, we can take $g = 1$. $\rightsquigarrow$ pick a suitable function W in effect.

<u>Remarks</u> ① $\dfrac{\zeta(g, \chi, W, s)}{\zeta(wg, \chi^{-1}\omega_\pi^{-1}, W, 1-s)}$ is independent of W.

② common denominator ----

③ $\varepsilon(\pi, \chi, s)$ assumes some special form.

Assume FE. $\rightsquigarrow$ Find L-factors.

(1) Supercuspidal : ξ is a Schwartz function !

conclusion: local zeta function

$$\zeta_{\xi,\chi}(s) = \int_{F^\times} \xi(x)\, \chi(x)\, |x|^{s-\frac{1}{2}}\, d^\times x \qquad (\text{take } g = 1)$$

is always ENTIRE !

Moreover, if we take $\xi_0(x) = \begin{cases} \chi^{-1}(x) & \text{if } x \in \mathcal{O}_F^\times \\ 0 & \text{if } x \notin \mathcal{O}_F^\times . \end{cases}$

then $\zeta_0(x)$ = non-zero constant. $\rightsquigarrow$ normalize it to be 1

i.e. $L(\pi, \chi, s) = 1$.

(2) Principal series. $\pi(\mu_1, \mu_2)$ $\mu_1 \cdot \mu_2^{-1} \neq |\cdot|^{\pm 1}, \mathbb{1}$

$\mathcal{K}(\mu_1, \mu_2)$ $\rightarrow$ look at notes on 1/31/97

$\Longrightarrow$ $\xi \in \mathcal{K}(\mu_1, \mu_2)$ has the form

$$\xi(x) = |x|^{\frac{1}{2}} \mu_1(x) f_1(x) + |x|^{\frac{1}{2}} \mu_2(x) f_2(x) \qquad f_i \in \mathcal{S}(F) \quad (\text{not } F^\times!)$$

So do the integral :

$$\zeta_{\xi,\chi}(s) = \int_{F^\times} \xi(x) \cdot \chi(x) \cdot |x|^{s-\frac{1}{2}}\, d^\times x$$

$$= \zeta(f_1, \mu_1\chi, s) + \zeta(f_2, \mu_2\chi, s)$$

where $\zeta(f, \chi, s) = \int_{F^\times} f(x)\, \chi(x) |x|^s\, d^\times x$.

Linear algebra says $L(\pi, \chi, s) = L(\mu_1, \chi, s)\, L(\mu_2, \chi, s)$

(common denominator)

(partial fraction) when $\mu_1 \mu_2^{-1} \neq |\cdot|^{\pm 1}, \mathbb{1}$

(3) $\longrightarrow$ 2/10/97

2/7/97

"Structural constants" of adm irred. (inf dim'l) repr of $G(F)$
$\rightsquigarrow$ a bunch of numbers which determine the adm irred repr. of $G(F)$ completely

<u>formal Mellin transform</u>

$\varphi : F^{\times} \longrightarrow \mathbb{C}$ locally constant. $\varphi(x) = 0$ when $|x| >> 0$.

<u>Fix</u> a uniformizer $\tilde{\omega}$ in $\mathfrak{p}_F$

for $\nu : \mathcal{O}_F^{\times} \longrightarrow \mathbb{C}^{\times}$ character, $n \in \mathbb{Z}$. define

$$\hat{\varphi}_n(\nu) = \int_{\mathcal{O}_F^{\times}} \varphi(\varepsilon \tilde{\omega}^n)\, \nu(\varepsilon)\, d^{\times}\varepsilon$$

normalized Haar measure for $\mathcal{O}_F^{\times}$.

Knowing $\varphi \longleftrightarrow$ knowing $\hat{\varphi}_n$

Make the formal sum:

$$\hat{\varphi}(\nu, t) = \sum_{n \in \mathbb{Z}} \hat{\varphi}_n(\nu)\, t^n. \longrightarrow \text{Laurent series in } t$$

(because $\varphi(x) = 0$ for $|x| >> 0$.)

we've known central character. the action of Borel subgp on Kirillov model.

$\rightsquigarrow$ remaining data : action of w. the Weyl element!

<u>Gauss sums</u> $\mu : \mathcal{O}_F^{\times} \longrightarrow \mathbb{C}$ character. $x \in F^{\times}$.

$$\rightsquigarrow \eta(\mu, x) = \int_{\mathcal{O}_F^{\times}} \mu(\varepsilon)\, \psi(\varepsilon x)\, d^{\times}\varepsilon.$$

Fact: $\eta(\mu, x) = 0$ unless $v(x) = -\mathrm{cond}(\mu) - \mathrm{cond}(\psi)$.;
in that case, $\eta(\mu, x) \neq 0$.

For $\nu : \mathcal{O}_F^{\times} \longrightarrow \mathbb{C}^{\times}$ character, $n \in \mathbb{Z}$.

Define $\varphi_\nu(x) := \begin{cases} (\omega_\pi \nu)(x) & \text{if } x \in \mathcal{O}_F^{\times} \\ 0 & \text{if } x \notin \mathcal{O}_F^{\times} \end{cases}$

$\varphi_\nu(x) \in \mathcal{S}(F^{\times}) \subseteq \mathcal{K}(\pi)$

Compute the number $(\pi(w)\varphi_\nu)\hat{}_n(\nu) =: C_n(\nu)$

Form the formal sum: $C(\nu, t) = \sum_{n \in \mathbb{Z}} C_n(\nu)\, t^n$.

$\rightsquigarrow C_n$, C determine π

Discussion: $I = \int_{O_F^\times} \pi(w)\varphi_\nu(\varepsilon\cdot\tilde{\omega}^n)\,\mu(\varepsilon)\,d^\times\varepsilon.$

$$\begin{pmatrix}\varepsilon & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \begin{pmatrix}0 & \varepsilon\\ -1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \varepsilon\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\varepsilon^{-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon\end{pmatrix}$$

$$\rightsquigarrow\ \pi\begin{pmatrix}\varepsilon & 0\\ 0 & 1\end{pmatrix}\pi(w)\varphi_\nu = \omega_\pi(\varepsilon)\cdot\omega_\pi(\varepsilon)^{-1}\nu(\varepsilon)^{-1}\cdot\pi(w)\varphi_\nu$$
$$= \nu(\varepsilon)^{-1}\pi(w)\varphi_\nu.$$

$\Rightarrow\ I = 0$ unless $\mu = \nu$. $\rightsquigarrow$ this determines $\pi(w)$.

<u>Prop</u> (1) for $\delta \in O_F^\times$, $l \in \mathbb{Z}$.

$$\left(\pi\begin{pmatrix}\delta\tilde{\omega}^l & 0\\ 0 & 1\end{pmatrix}\varphi\right)^\wedge(\nu,t) = t^{-l}\,\nu^{-1}(\delta)\cdot\hat{\varphi}(\nu,t).$$

(2) $$\left(\pi\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\varphi\right)^\wedge(\nu,t) = \sum_n t^n\Big(\sum_{\mu \text{ char of } O_F^\times}\eta(\mu^{-1}\nu,\tilde{\omega}^n x)\cdot\hat{\varphi}_n(\mu)\Big).$$

(3) $$(\pi(w)\varphi)^\wedge(\nu,t) = C(\nu,t)\,\hat{\varphi}\big(\nu^{-1}\cdot\underbrace{w_0^{-1}}_{=\,\omega_\pi^{-1}|_{O_F^\times}},\ t^{-1}\cdot\omega_\pi(\tilde{\omega})^{-1}\big).$$

<u>proof.</u> (1) $(\quad)^\wedge_n(\nu)$

$$= \int_{O_F^\times}\varphi(\varepsilon\delta\cdot\tilde{\omega}^{n+l})\,\nu(\varepsilon)\cdot d^\times\varepsilon$$
$$= \nu(\delta)^{-1}\int_{O_F^\times}\varphi(\varepsilon'\tilde{\omega}^{n+l})\,\nu(\varepsilon^\times)\cdot d^\times\varepsilon = \nu(\delta)^{-1}\cdot\hat{\varphi}_{n+l}(\nu). \quad \checkmark$$

(2) $(\quad)^\wedge_n(\nu)$

$$\varphi(\varepsilon\tilde{\omega}^n) = \sum_\mu \hat{\varphi}_n(\mu)\,\mu^{-1}(\varepsilon)$$
(fourier inversion)

$$= \int_{O_F^\times}\psi(x\varepsilon\tilde{\omega}^n)\cdot\varphi(\varepsilon\tilde{\omega}^n)\cdot\nu(\varepsilon)\,d^\times\varepsilon.$$

$$= \sum_\mu\hat{\varphi}_n(\mu)\,\underbrace{\int_{O_F^\times}\psi(x\varepsilon\tilde{\omega}^n)\,(\nu\mu^{-1})(\varepsilon)\,d^\times\varepsilon}_{=\ \eta(\nu\mu^{-1},\,x\tilde{\omega}^n)} \quad \leftarrow \text{Gauss sum} \quad \checkmark$$

(3) it suffices to check for those φ_ν's and their translates.
if translated by a local unit, it gets multiplied by a constant.
$\therefore$ suffices to check for translating by $\tilde{\omega}^l$.
for φ_ν, it's true by definition of $C(\nu,t)$.

Only need: if true for φ, then true for $\pi\begin{pmatrix}\tilde{\varpi}^{\ell} & 0 \\ 0 & 1\end{pmatrix}\varphi$

LHS: $\pi(w)\,\pi\begin{pmatrix}\tilde{\varpi}^{\ell} & 0 \\ 0 & 1\end{pmatrix}$

$$w\begin{pmatrix}\tilde{\varpi}^{\ell} & 0 \\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 \\ 0 & \tilde{\varpi}^{\ell}\end{pmatrix} w = \begin{pmatrix}\tilde{\varpi}^{\ell} & 0 \\ 0 & \tilde{\varpi}^{\ell}\end{pmatrix}\begin{pmatrix}\tilde{\varpi}^{\ell} & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}.$$

$$\rightsquigarrow \pi(w)\,\pi\begin{pmatrix}\tilde{\varpi}^{\ell} & 0 \\ 0 & 1\end{pmatrix} = \omega_\pi(\tilde{\varpi}^{\ell}) \cdot \pi\begin{pmatrix}\tilde{\varpi}^{-\ell} & 0 \\ 0 & 1\end{pmatrix} \cdot \pi(w).$$

$$\Rightarrow \pi(w)\,\pi\begin{pmatrix}\tilde{\varpi}^{\ell} & 0 \\ 0 & 1\end{pmatrix}\cdot\hat{\varphi}(\nu,t) = t^{\ell}\cdot\omega_\pi(\tilde{\varpi}^{\ell}) \cdot \pi(w)\,\hat{\varphi}(\nu,t).$$

RHS: $\hat{\varphi}(\nu,t) \longmapsto t^{-\ell}\,\hat{\varphi}(\nu,t)$.

$\Rightarrow$ multiplied by $t^{\ell}\,\omega_\pi(\tilde{\varpi}^{\ell})$. $\square$

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Relations between structural constants and F.E.

Local zeta functions $\xi \in \mathcal{K}(\pi)$,

$$\zeta_{\xi,\chi}(s) = \int_{F^\times} \xi(x)\,\chi(x)\cdot |x|^{s-\frac{1}{2}}\, d^\times x.$$

on the other hand. $\varphi_n(\nu) = \int_{\mathcal{O}_F^\times} \varphi(\varepsilon\cdot\tilde{\omega}^n)\,\nu(\varepsilon)\, d^\times\varepsilon.$

$$\Longrightarrow \zeta_{\xi,\chi}(s) = \sum_n \int \xi(\varepsilon\,\tilde{\omega}^n)\ \chi(\varepsilon\cdot\tilde{\omega}^n)\cdot q^{-n(s-\frac{1}{2})}\, d^\times\varepsilon$$

$$= \sum_n \xi_n(\chi)\cdot\left(\chi(\tilde{\omega})\cdot q^{-(s-\frac{1}{2})}\right)^n$$

$$= \hat{\xi}\left(\chi,\ \chi(\tilde{\omega})\, q^{-(s-\frac{1}{2})}\right). \qquad \leftarrow \chi = \chi|_{\mathcal{O}_F^\times}$$

$\zeta_{\xi,\chi}(s)$ is a meromorphic function. with analytic continuation, etc. from abelian case.

$$\zeta_{\pi(w)\xi,\ \chi^{-1}\omega_\pi^{-1}}(1-s)$$

$$= (\pi(w)\,\xi)^\wedge\left(\chi^{-1}\omega_\pi^{-1},\ \chi^{-1}(\tilde{\omega})\cdot\omega_\pi^{-1}(\tilde{\omega})\ q^{(s-\frac{1}{2})}\right)$$

F.E.: $$\frac{\zeta_{\xi,\chi}(s)}{L(\pi\otimes\chi,s)}\ \varepsilon(\pi\otimes\chi,s) = \frac{\zeta_{\pi(w)\xi,\ \chi^{-1}\omega_\pi^{-1}}(1-s)}{L(\pi\otimes\omega_\pi^{-1}\otimes\chi^{-1},\ 1-s)}$$

$$\rightsquigarrow \frac{\zeta_{\pi(w)\xi,\ \chi^{-1}\omega_\pi^{-1}}(1-s)}{\zeta_{\xi,\chi}(s)} = \frac{L(\pi\otimes\omega_\pi^{-1}\otimes\chi^{-1},\ 1-s)}{L(\pi\otimes\chi,s)}\cdot\varepsilon(\pi\otimes\chi,s) =: \gamma(\pi\otimes\chi,s)$$

independent of ξ !

$$\frac{\zeta_{\pi(w)\xi,\ \chi^{-1}\omega_\pi^{-1}}(1-s)}{\zeta_{\xi,\chi}(s)} = C\left(\chi^{-1}\omega_\pi^{-1}\big|_{\mathcal{O}_F^\times},\ \chi^{-1}(\tilde{\omega})\cdot\omega_\pi^{-1}(\tilde{\omega})\cdot q^{s-\frac{1}{2}}\right)$$

If we allow χ to vary, it's to know all of the structura[l] constants C.

Also showed. ① $\gamma(\pi\otimes\chi,s)$ exists. !

② $\varepsilon(\pi\otimes\chi,s)$ exists !

Corollary (Assuming the existence of F.E.) If $L(\pi\otimes\chi, s) = L(\pi'\otimes\chi, s)$, $\varepsilon(\pi\otimes\chi, s) = \varepsilon(\pi'\otimes\chi, s)$ $\forall\chi$, then $\pi\cong\pi'$.

L-factors of special representations.

$$L(\sigma(\mu_1,\mu_2)\otimes\chi, s) = \begin{cases} L(\mu_1\chi, s) & \text{if } \mu_1\mu_2^{-1} = |\cdot| \\ L(\mu_2\chi, s) & \text{if } \mu_1\mu_2^{-1} = |\cdot|^{-1} \end{cases}$$

Always to think about: $L(\sigma(|\cdot|^{\frac12}, |\cdot|^{-\frac12}), s) = L(|\cdot|^{\frac12}, s) = (1-q^{-s-\frac12})^{-1}$

exercise. show above (hint: same method as in P.S.)

special repr $\longleftrightarrow$ $\underset{\substack{\uparrow \\ L,\ \text{up to shifting}}}{W_F} \times \underset{\substack{\uparrow \\ \text{standard}}}{SL_2} \longrightarrow GL_2$

$$\therefore\ \zeta_{\pi(w)\xi,\,\chi^{-1}\omega_\pi^{-1}}(1-s) = \gamma(\pi\otimes\chi, s)\cdot\zeta_{\xi,\chi}(s) \qquad \xi = \pi(w)\pi(w^{-1})\xi$$

$$= \omega_\pi(-1)\gamma(\pi\otimes\chi, s)\,\gamma(\pi\otimes\chi^{-1}\omega_\pi^{-1}, 1-s)\cdot\zeta_{\pi(w)\xi,\,\chi^{-1}\omega_\pi^{-1}}(1-s) \qquad = \omega_\pi(-1)\pi(w)\underbrace{(\pi(w)\xi)}_{\xi_1}$$

$$\Longrightarrow\quad \gamma(\pi\otimes\chi, s)\cdot\gamma(\pi\otimes\chi^{-1}\omega_\pi^{-1}, 1-s) = \omega_\pi(-1) = \pm 1 \quad !!$$

Compare: classical case: $f\in\mathcal{S}(F)$

$$\frac{\zeta(f,\chi,s)}{L(\chi,s)}\,\varepsilon(\chi,s,\psi,dx) = \frac{\zeta(\hat f,\chi^{-1},1-s)}{L(\chi^{-1},1-s)}$$

$$\frac{\zeta(\hat f,\chi^{-1},1-s)}{\zeta(f,\chi,s)} = \frac{L(\chi^{-1},1-s)}{L(\chi,s)}\cdot\varepsilon(\chi,s) =: \gamma(\chi,s)$$

$$\rightsquigarrow\ \gamma(\chi,s)\cdot\gamma(\chi^{-1},1-s) = \ ? \qquad (\ \chi(-1)\ ?\ )$$

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ε - factors.

Remarks. 1. in abelian case. $\varepsilon(\chi, s, \psi_F, dx)$

Here, ε doesn't depend on the Haar measure $d^\times x$.

but ε depends on the choice of ψ_F

(hidden in $\mathcal{K}(\pi)$, or the action of $\pi(w)$)

2. When π is supercuspidal, $L \equiv 1$ because ζ is entire.

$\rightsquigarrow$ $L(\pi \otimes \chi, s) = 1$. and a priori we know:

① $\varepsilon(\pi, \chi, s)$ is a finite Laurent series in q^{-s}.

Also. ② $\varepsilon(\pi \otimes \chi, s) \cdot \varepsilon(\pi \otimes \omega_\pi^{-1} \otimes \chi^{-1}, 1-s) = \omega_\pi(-1) (= \pm 1)$

$\overset{①+②}{\Longrightarrow}$ both factors in ② have exactly 1 term.

i.e. $\varepsilon(\pi \otimes \chi, s) = A\, q^{-ns}$, some $n \in \mathbb{Z}$.

$\rightsquigarrow$ A. n determine the supercuspidal representation !!

Computation of ε-factors for $\pi(\mu_1, \mu_2)$

Answers: $\varepsilon(\pi(\mu_1, \mu_2) \otimes \chi, s)$

$= \varepsilon(\mu_1 \chi, s) \cdot \varepsilon(\mu_2 \chi, s)$

$\varepsilon(\mu_i \chi, s, \psi_F, dx)$

$\longrightarrow$ self-dual Haar measure of F w.r.t. ψ_F

"same" for special representation.

(if one believes in Langlands' correspondence:

special $\longleftrightarrow$ $[W_F \times SL_2(\mathbb{C}) \to GL_2(\mathbb{C})]$)

Recall: F.E. for the abelian local ζ-function. $f \in \mathcal{S}(F)$

$$\zeta(f, \chi, s) = \int_F f(x) \cdot \chi(x)\, |x|^s \cdot d^\times x.$$

Local F.E:

$$\frac{\zeta(f, \chi, s)}{L(\chi, s)} \cdot \varepsilon(\chi, s, \psi, dx) = \frac{\zeta(\hat{f}_{\psi, dx}, \chi^{-1}, 1-s)}{L(\chi^{-1}, 1-s)}$$

Pick $\xi \in \mathcal{S}(F^\times) \cap \pi(w)\,\mathcal{S}(F^\times)$

$$\longleftrightarrow \quad \underset{\cap\, B(\mu_1,\mu_2)}{\varphi}\left(w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right) = \int_F \mu_2^{-1}(x)\cdot |x|^{-\frac{1}{2}}\ \xi(x)\cdot\psi_F(xy)\,dx.$$

$$\xi(x) = \mu_2(x)\cdot|x|^{\frac{1}{2}}\int_F \varphi\left(w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right)\overline{\psi_F(xy)}\,dy.$$

$$\Longrightarrow \varphi\left(w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}w\right) = \int_F \mu_2(x)\cdot|x|^{-\frac{1}{2}}\,(\pi(w)\xi)(x)\cdot\psi_F(xy)\,dx$$

$$w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}w = \begin{pmatrix}1 & -\frac{1}{y}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{y} & 0\\ 0 & y\end{pmatrix}\cdot w\cdot\begin{pmatrix}1 & -\frac{1}{y}\\ 0 & 1\end{pmatrix}.$$

$$\rightsquigarrow \mu_1(y^{-1})\cdot\mu_2(y)\cdot|y|^{-1}\ \varphi\left(w\begin{pmatrix}1 & -\frac{1}{y}\\ 0 & 1\end{pmatrix}\right)$$

first, $\quad \zeta_{\xi,\chi}(s) = \int_{F^\times}\xi(x)\cdot\chi(x)\cdot|x|^{s-\frac{1}{2}}\cdot d^\times x$

$$\boxed{\begin{aligned} f(x) &:= \mu_2^{-1}(x)\cdot|x|^{-\frac{1}{2}}\cdot\xi(x)\\ &= \int_F\varphi\left(w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right)\overline{\psi_F(xy)}\,dy\end{aligned}}$$

$$\begin{aligned} &= \int_{F^\times} f(x)\cdot(\chi\mu_2)(x)\ |x|^s\ d^\times x.\\ &= \zeta(f,\mu_2\chi,s)\\ &= \gamma(\mu_2\chi,s)^{-1}\cdot\zeta(\hat{f},\chi^{-1}\cdot\mu_2^{-1},1-s).\end{aligned}$$

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$$w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}w = \begin{pmatrix}1 & 0\\ -y & 1\end{pmatrix} = \begin{pmatrix}1 & -\frac{1}{y}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{y} & 0\\ 0 & y\end{pmatrix} w \cdot \begin{pmatrix}1 & -\frac{1}{y}\\ 0 & 1\end{pmatrix}.$$

$$\Longrightarrow \pi(w)\,\varphi\left(w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right) = \varphi\left(w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}w\right)$$

$$= \mu_1(y^{-1})\cdot\mu_2(y)\,|y|^{-1}\cdot\varphi\left(w\begin{pmatrix}1 & -\frac{1}{y}\\ 0 & 1\end{pmatrix}\right)$$

$$= \omega_\pi(-1)\cdot\mu_1(y^{-1})\cdot\mu_2(y)\,|y|^{-1}\,\varphi\left(w^{-1}\begin{pmatrix}1 & -\frac{1}{y}\\ 0 & 1\end{pmatrix}\right)$$

So we let.

$$\eta(x) = \mu_2^{-1}(x)\,|x|^{-\frac{1}{2}}\cdot\pi(w)\,\xi(x) = \omega_\pi(-1)\int_F \mu(y^{-1})\,|y|^{-1}\cdot\varphi\left(w^{-1}\begin{pmatrix}1 & -\frac{1}{y}\\ 0 & 1\end{pmatrix}\right)\overline{\psi(xy)}\,dy$$

$$\Longrightarrow \zeta_{\xi,\chi}(s) = \int_{F^\times}\xi(x)\cdot\chi(s)\cdot|x|^{s-\frac{1}{2}}\,d^\times x$$

$$= \int_{F^\times} f(x)\cdot(\chi\mu_2)(x)\cdot|x|^{s}\,d^\times x = \zeta(f,\chi\mu_2,s).$$

But f itself is the fourier transform of something,

$$\rightsquigarrow \zeta(f,\chi\mu_2,s) = \gamma(\mu_2^{-1}\cdot\chi^{-1},1-s)\cdot\int_{F^\times}\varphi\left(w^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right)\mu_2^{-1}(y)\,\chi^{-1}(y)\cdot|y|^{1-s}\,d^\times y$$

The other thing:

$\zeta_{\pi(w)\xi,\,\omega_\pi^{-1}\chi^{-1}}(1-s)$ $\quad(\omega_\pi = \mu_1\mu_2$ principal series case.)

$$= \int_{F^\times}(\pi(w)\xi)(x)\cdot\omega_\pi^{-1}(x)\cdot\chi^{-1}(x)\cdot|x|^{\frac{1}{2}-s}\,d^\times x$$

$$= \int_{F^\times}\eta(x)\cdot\cancel{\mu_2(x)}\cdot\mu_1^{-1}(x)\cdot\cancel{\mu_2^{-1}(x)}\cdot\chi^{-1}(x)\cdot|x|^{1-s}\,d^\times x$$

$$= \zeta(\eta,\mu_1^{-1}\chi^{-1},1-s)$$

$\mu_1^{-1}(y)\cdot\mu_2(y)$

$$= \omega_\pi(-1)\cdot\gamma(\mu_1\chi,s)\cdot\int_{F^\times}\varphi\left(w^{-1}\begin{pmatrix}1 & -\frac{1}{y}\\ 0 & 1\end{pmatrix}\right)\cdot\mu(y^{-1})\cdot|y|^{-1}\cdot\mu_1(y)\cdot\chi(y)\cdot|y|^{s}\,d^\times y$$

$$= \omega_\pi(-1)\cdot\gamma(\mu_1\chi,s)\cdot\int_{F^\times}\varphi\left(w^{-1}\begin{pmatrix}1 & -\frac{1}{y}\\ 0 & 1\end{pmatrix}\right)\cdot(\chi\mu_2)(y)\cdot|y|^{s-1}\,d^\times y$$

Let $z = -\frac{1}{y}$.

$$\rightsquigarrow \zeta_{\pi(w)\xi,\, \omega_\pi^{-1}\chi^{-1}}(1-s)$$

$$= \gamma(\mu_1\chi, s)\cdot \mu_1(-1)\cdot \chi(-1)\cdot \int_{F^\times} \varphi\left(w^{-1}\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\right) \mu_2(z^{-1})\cdot \chi(z)^{-1}\, |z|^{1-s}\, d^\times z$$

$$\therefore\ \gamma(\pi(\mu_1,\mu_2)\otimes\chi, s) = \frac{\zeta_{\pi(w)\xi,\, \omega_\pi^{-1}\chi^{-1}}(1-s)}{\bar{\zeta}_{\xi,\chi}(s)}$$

$$= \frac{\gamma(\mu_1\chi, s)\quad \mu_1(-1)\cdot\chi(-1)}{\gamma(\mu_2^{-1}\chi^{-1}, 1-s)}$$

oops - - - - -

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Spherical representations

<u>Definition</u> π : adm. irred. inf. dim'l repr. of $G(F)$ is a <u>spherical representation</u> if $V^{GL_2(O_F)} \neq (0)$

($\exists$ a non-zero vector in V fixed by the maximal compact subgroup $GL_2(O_F) \leq G(F)$.)

<u>Fact / Exer.</u> π is of the form $\pi(\mu_1, \mu_2)$ with both μ_1, μ_2 unramified ($\rightarrow$ π is neither supercuspidal nor special.) and $\dim V^{GL_2(O_F)} = 1$

Hint. $GL_2(F) = B(F) \cdot GL_2(O_F)$

if $\varphi \in \mathcal{B}(\mu_1, \mu_2)$, $\varphi(b \cdot k) = \varphi(b)$ $(b \in B(F),\ k \in GL_2(O_F))$

$= \mu_1(a) \cdot \mu_2(d) \left|\frac{a}{d}\right|^{\frac{1}{2}} \cdot \varphi(e)$ $\quad \left(b = \begin{pmatrix} a & * \\ 0 & d \end{pmatrix}\right)$

and μ_1, μ_2 both unramified !

Unitary representations (just for GL_2 !)

<u>Theorem</u> The irreducible (pre-)unitary representations of $G(F)$ are

(1) Supercuspidal repr. with unitary central character

(2) Special repr. with unitary central character.

(3) $\pi(\mu_1, \mu_2)$ with μ_1, μ_2 both unitary.

(4) $\pi(\mu_1, \mu_2)$ with $\mu_1(x) \cdot \overline{\mu_2(x)} = 1$ and $(\mu_1 \cdot \mu_2^{-1})(x) = |x|^{\sigma}$, with $0 < \sigma < 1$.

("complementary series".)

(0) unitary characters of $F^{\times}$

<u>proof.</u> (1) Hermitian form : $\langle v | w \rangle = \int_{G(F)/Z(F)} \langle \pi(g) v, \lambda_0 \rangle \cdot \overline{\langle \pi(g) w, \lambda_0 \rangle}\, dg$

with fixed $\lambda_0 \in \check{V} - \{0\}$

First, $\langle \pi(g) v, \lambda_0 \rangle$: matrix coefficient with compact support

$\rightsquigarrow$ can be integrated over a Haar measure.

Well-defined (up to $Z(F)$), invariant.

≥ 0, $\langle v | v \rangle = 0 \Rightarrow \langle \pi(g) v, \lambda_0 \rangle = 0 \quad \forall g \Leftrightarrow \langle V, \lambda_0 \rangle = 0$

$\Leftrightarrow \lambda_0 = 0$. $\divideontimes$

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(3). (4). $\pi(\mu_1, \mu_2)^\vee \simeq \pi(\bar\mu_1, \bar\mu_2)$

$\|$

$\pi(\mu_1^{-1}, \mu_2^{-1})$.

(3) $\mu_i^{-1} = \bar\mu_i \Rightarrow$ both μ_i are unitary

(4) $\mu_1^{-1} = \bar\mu_2$, $\mu_2^{-1} = \bar\mu_1$. $\rightsquigarrow$ with extra condition to write down a hermitian form.

=

"invariant hermitian pairing on (π, V)" is a mapping $\bar\pi \longrightarrow \check\pi$ which is G-invariant.

When we have a principal series $\pi(\mu_1, \mu_2)$,
$\overline{\pi(\mu_1, \mu_2)} = \pi(\bar\mu_1, \bar\mu_2)$ (on the level of functions.)
question: when is $\pi(\bar\mu_1, \bar\mu_2) \cong \pi(\mu_1, \mu_2)^\vee = \pi(\mu_1^{-1}, \mu_2^{-1})$?
the only possibilities: ① $\mu_i \cdot \bar\mu_i = 1$, $i = 1, 2$. $\Rightarrow$ (3)
② $\mu_1 \bar\mu_2 = 1$ & $\mu_2 \cdot \bar\mu_1 = 1$. $\Longrightarrow$ (4).

<u>Assume</u>. $\mu(x) = \mu_1(x) \cdot \mu_2^{-1}(x) = |\mu_1(x)|^2 = |x|^\sigma$. & $\sigma > 0$.
($\sigma = 0 \Rightarrow \mu_1 = \mu_2$ are both unitary.)

Consider the intertwining operator:

$A: B(\mu_1, \mu_2) \longrightarrow B(\mu_2, \mu_1) = B(\bar\mu_1^{-1}, \bar\mu_2^{-1})$

$(A(\varphi))(g) = \int_F \varphi\left(w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) dx$

<u>exer</u>. check the integral above converges.
(<u>hint</u>. $\varphi\left(w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right)$ has the same order of growth as $|x|^{-\sigma-1}$ as $|x| >> 0$.)
(<u>hint</u>. $w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & -x^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}-x^{-1} & 0\\ 0 & -x\end{pmatrix}\begin{pmatrix}1 & 0\\ x^{-1} & 1\end{pmatrix}$).

<u>exer</u>. $A(\varphi) \in B(\bar\mu_1^{-1}, \bar\mu_2^{-1})$

Now, write down a hermitian pairing —

$$\langle \varphi_1 | \varphi_2 \rangle = c \int_{B(F)\backslash G(F)} (A\varphi_1)(g) \, \overline{\varphi_2(g)} \, dg$$

$$= c \int_F (A\varphi_1)(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}) \cdot \overline{\varphi_2 (w^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix})} \, dx$$

some $c \in \mathbb{C}^\times$.

Since the pairing is G-invariant, there is essentially only one such by Schur's lemma.

<u>exer.</u> $\langle \varphi_1 | \varphi_2 \rangle = c \iint \varphi_1 (w^{-1} \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix}) \, \overline{\varphi_2 (w^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix})} \, |y|^\sigma \, dx \, d^\times y$. $(*)$

$\pi(\mu_1, \mu_2)$ is unitary
$\iff \exists \; c \in \mathbb{C}^\times$ s.t. $\langle \varphi_1 | \varphi_2 \rangle$ is positive definite!

<u>Bochner's theorem</u> ← look at some book
for positive definiteness. $\sum a_n \bar{a}_m f(x_n - x_m) \geq 0$

B' thm:
continuous positive definite functions
$\longleftrightarrow$ Fourier transforms of positive semi definite measures on the dual space.

the integral $(*)$ is in fact $\int f_1(x+y) \overline{f_2(x)} \, dx$

$\rightsquigarrow f_1 * f_2^\vee \quad f_2^\vee(x) = \overline{f_2(-x)}$.

therefore $\pi(\mu_1, \mu_2)$ is unitary
$\iff$ F.T. $(|y|^\sigma d^\times y)$ is a positive def. measure

($(f_1 * f_2)^\wedge = \hat{f_1} \cdot \hat{f_2} \rightsquigarrow$ by Bochner's theorem)

Fourier transform of $|y|^{\sigma} d^{\times}y$ $= ?$

Think about functional equation for abelian local zeta functio[n]

$$\int f(x)\, |x|^{s}\, d^{\times}x \quad \longleftrightarrow \quad \int \hat{f}(x)\, |x|^{1-s}\, d^{\times}x.$$

$$\Longrightarrow \quad \text{F.T. of } |y|^{\sigma} d^{\times}y = C \cdot |y|^{1-\sigma} d^{\times}y$$

Now $1-\sigma > 0$ is exactly the condition for $|x|^{1-\sigma} d^{\times}x$ to extend to a (finite) measure on F.

2/21/97

<u>Archimedian cases</u> : <u>$GL_2(\mathbb{R})$</u> <u>$GL_2(\mathbb{C})$</u>
(reductive Lie groups.)

* NO supercuspidal representations !
→ no compactly supported (real analytic) functions unless the space is compact ! (by identity principle.)

$GL_2(\mathbb{R}) \longrightarrow$ modulo center, ∃ discrete (countably many) series (matrix coeff are in L^2)
of representations

$GL_2(\mathbb{C})$ doesn't have ↑.

<u>admissible representation</u> G: reductive real Lie group
($G(\mathbb{R})^\circ$: G reductive alg group/$\mathbb{R}$

$G \supseteq K$: maximal compact subgroup.

An <u>admissible representation</u> of G on a Banach space V is one such that each irreducible representation of K occurs with <u>finite multiplicity</u>.

Then : (i) V_K = K-finite vectors in V. (restrict to K-rep the space is finite dimensional.) $V_K \subseteq V^\infty$:= smooth vector.
(i.e the vectors are differentible.)
(weak / strong def. for differentibility.)
$\rightsquigarrow V_K$ is a $(\mathfrak{g}, K)$-module.

V_K : direct sums of finite spaces. V is in some sense a completion of V_K. V_K essentially catches V.

V_K has both $\mathfrak{g}$- and K-actions which are compatible

(ii) $\begin{pmatrix} G\text{-invariant} \\ \text{closed subspaces} \\ \text{of } V \end{pmatrix} \underset{\xrightarrow[\text{vectors}]{K\text{-finite}}}{\overset{\xleftarrow{\text{closure}}}{\longleftrightarrow}} \begin{pmatrix} (\mathfrak{g}, K)\text{-invariant} \\ \text{subspaces of } V_K \end{pmatrix}$

Facts.

(i) Any <u>irreducible</u> <u>unitary</u> representation of G is automatically admissible.

(ii) If (π_i, V_i) $i = 1, 2$ are irreducible unitary representations of G s.t.

$$(\pi_1, V_1)_K \cong (\pi_2, V_2)_K \quad \text{as } (\mathfrak{g}, K)\text{-modules},$$

then $\pi_1 \cong \pi_2$ unitarily.

Thm.

(iii) Let X be a $(\mathfrak{g}, K)$-module (so it is locally K-finite etc.) Then

ⓐ X is admissible. (structure of rep. of red. real Lie gp.)

ⓑ X is isomorphic to the Harish-Chandra module of an irred. rep. of G on a Hilbert space s.t elements of K operate via unitary operators.

Back to $GL_2(\mathbb{R})$. $GL_2(\mathbb{C})$. $\rightsquigarrow$ Hecke algebras

$\mathfrak{g}$ · left invariant diff. operators.

$F = \mathbb{R}$ or $\mathbb{C}$. the Hecke algebra $\mathcal{H}_F$ = distributions supported of $\begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & F = \mathbb{C} \\ \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix} & F = \mathbb{R} \end{cases}$ $U(\mathfrak{g})$. ← rep. the 2 connected components of GL

(diff. never gives the inform. outside any given conn. component.)

$\rightsquigarrow$ $\mathcal{H}_F$-module.

$F = \mathbb{R} / \mathbb{C}$

$B(\mu_1, \mu_2) = \{ f : G(F) \longrightarrow \mathbb{C}$: smooth K-finite functions
$$f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \mu_1(a) \cdot \mu_2(d) \cdot \left|\frac{a}{d}\right|_F^{\frac{1}{2}} f(g) \}$$

induced representation $\rho(\mu_1, \mu_2)$

$$| \ |_F = \begin{cases} | \ | & \text{if } F = \mathbb{R} \\ \| \ \|^2 & \text{if } F = \mathbb{C} \end{cases}$$

<u>Theorem</u> Every irreducible admissible representation of $\mathcal{H}_F$ is isomorphic to a subrepresentation of a $\rho(\mu_1, \mu_2)$

<u>Theorem</u> $F = \mathbb{R}$. $\rho(\mu_1, \mu_2)$ is irreducible, except when $\mu := \mu_1 \cdot \mu_2^{-1}$ has the form $\mu(t) = t^n \operatorname{sgn}(t)$, $n \neq 0$; if μ has the form above, then two cases (dual to each other):

① $n > 0$. then $\rho(\mu_1, \mu_2)$ contains exactly 1 non-trivi irreducible subrepresentation which is infinite dimensional. The quotient is finite dimensional and irreducible of dim. n.

② $n < 0$ then $\rho(\mu_1, \mu_2)$ contains exactly 1 non-trivial irreducible subrepresentation of dim $(-n)$. In fact, this subrepresentation consists of $\{ \varphi : G(F) \to \mathbb{C} \mid$
$$\varphi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mu_1 \circ \det(g) \cdot |\det g|^{\frac{1}{2}} \cdot f(c\ d).$$
f: homogeneous polynomial of deg $|n| - 1$. $\}$.
The quotient is infinite dimensional.

finite dim'l guys are called $\pi(\mu_1 \otimes \mu_2)$
$\wr$
Langlands program correspondence.

① $\rightsquigarrow$ infinite dimensional repr. are called <u>discrete series</u>.

$\mathbb{R}$: $\pi(\omega_{\frac{1}{2}} \cdot t^n, \omega_{-\frac{1}{2}})$. $n > 0$.
$= \operatorname{Sym}^n(\text{std})$ symmetric product.

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<u>Remarks</u> <u>on</u> <u>$\mathcal{B}(\mu_1, \mu_2)$</u> in $GL_2(\mathbb{R})$ case :

$$\mathcal{B}(\mu_1, \mu_2) = \left\{ f : GL_2(\mathbb{R}) \longrightarrow \mathbb{C} \ : \ \text{smooth}, \ K\text{-finite}, \quad f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \mu_1(a)\, \mu_2(d) \cdot \left|\frac{a}{d}\right|_{\mathbb{R}}^{\frac{1}{2}} f(g) \right\}$$

$\because$ $\underset{\text{Borel sb.}}{B(\mathbb{R})}$ $\underset{K^+}{SO_2(\mathbb{R})} = GL_2(\mathbb{R})$ Iwasawa decomposition.

($K = O_2(\mathbb{R})$ in this case : max. compact subgp of $G(\mathbb{R})$)

K-finite $\equiv$ K^+-finite.

$\therefore$ f is determined by $f|_{K^+}$.

<u>e.g.</u> $f_n\left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right) = e^{in\theta}$ (in fact, characters.)

<u>and</u> every $f \in \mathcal{B}(\mu_1, \mu_2)$ is a (finite) linear combination of f_n's.

(classical way : write down f_n as basis of Lie algebra. look at their actions.)

In $\mathbb{C}$ case. $K^+ = SU(2)$ ------

For $F = \mathbb{C}$,

"<u>Theorem</u>" $\rho(\mu_1, \mu_2)$ is irreducible except when $\mu = \mu_1 \cdot \mu_2^{-1} = t^p \cdot \bar{t}^q$, $p \cdot q > 0$.

(0) Every irreducible admissible $\underbrace{\text{representation}}_{\text{inf. dim'l}}$ of $GL_2(\mathbb{C})$ is a $\rho(\mu_1, \mu_2)$.

$F = \mathbb{R}$
$\rho(\mu_1, \mu_2)$ is reducible
$\Rightarrow \sigma(\mu_1, \mu_2)$ denotes its inf. dim'l subq.

(1) $\mu = t^p \cdot \bar{t}^q$, $p < 0$ $q < 0$. $\rightsquigarrow$ $\mathcal{B}(\mu_1, \mu_2)$ contains an irred. finite dim'l subrepr, $\{ \varphi : G(F) \longrightarrow \mathbb{C} \mid$
$\varphi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mu_1(\det(g)) \cdot |\det(g)|_{\mathbb{C}}^{\frac{1}{2}} \, \Phi_1(c, d) \cdot \overline{\Phi_2(c, d)}$,
Φ_1 : homog. poly. of degree $|p| - 1$
Φ_2 : homog. poly. of degree $|q| - 1$ $\}$
and the quotient is irreducible.

(2) If $p > 0$, $q > 0$, then $\rho(\mu_1, \mu_2)$ contains an irred. inf. dim'l subrepr, the quotient is finite dim'l and irred.

Write, in (1), (2), these inf. dim'l subquotients as $\sigma(\mu_1, \mu_2)$.

<u>Answer to mystery</u>. $\sigma(\mu_1, \mu_2) = \rho(\nu_1, \nu_2)$ with
$\nu_1(t) := \bar{t}^{-\delta} \mu_1(t)$, $\nu_2(t) = \bar{t}^{\delta} \mu_2(t)$
check. $\nu_1 \cdot \nu_2^{-1} = t^{p} \cdot \bar{t}^{-\delta}$ $\Rightarrow$ $\rho(\nu_1, \nu_2)$ irred. by (o).

$$\mathrm{Sym}^{\otimes n}(\mathrm{std}) \otimes \mathrm{Sym}^{\otimes m}(\overline{\mathrm{std}}) = \pi\left(|\cdot|_{\mathbb{C}}^{\frac{1}{2}} z^n \bar{z}^m, \ |\cdot|_{\mathbb{C}}^{-\frac{1}{2}}\right)$$
(π: finite dimensional subrepr.)

<u>Equivalent relations</u>.

$F = \mathbb{R}$: $\pi(\mu_1, \mu_2) \simeq \pi(\mu_2, \mu_1)$.
$\sigma(\mu_1, \mu_2) \simeq \sigma(\mu_2, \mu_1) \simeq \sigma(\mu_1 \cdot \mathrm{sgn}, \mu_2 \cdot \mathrm{sgn})$
discrete series $\simeq \sigma(\mu_2 \cdot \mathrm{sgn}, \mu_1 \cdot \mathrm{sgn})$
} only ones.

$F = \mathbb{C}$: $\pi(\mu_1, \mu_2) \simeq \pi(\mu_2, \mu_1)$ that's all

<u>Remarks</u> on discrete series and their parameters for $GL_2(\mathbb{R})$

$$GL_2(\mathbb{R}) = \underbrace{SL_2(\mathbb{R})^{\pm}}_{\{g \in M_2(\mathbb{R}) : \det(g) = \pm 1\}} \times \mathbb{R}_+^{\times}.$$

D_ℓ^{+} : a repr. of $SL_2(\mathbb{R})$. $\ell \in \mathbb{N}$.

<u>space</u>. : $\{ f : \mathfrak{H} \longrightarrow \mathbb{C}$: f holomorphic, s.t. $\|f\|_2^2 := \iint_{\mathfrak{H}} |f(z)|^2 \cdot |y|^{\ell-1} \, dx\, dy < \infty \}$.

$\hookrightarrow$ " modular form of weight $\ell+1$ " ("Hardy" space)

<u>action</u>. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$.
$$f \longmapsto \left(z \longmapsto (bz+d)^{-\ell-1} \cdot f\left(\frac{az+c}{bz+d}\right)\right)$$
"act on the <u>left</u>"

$D_\ell = \mathrm{Ind}_{SL_2(\mathbb{R})}^{SL_2(\mathbb{R})^{\pm}} D_\ell^{+}$ index 2 induction.

$\rightsquigarrow$ $D_\ell \otimes \omega_t$ is an irred. discrete series rep.,
and every irred. discrete series rep. is of this form.

$\omega_t = |\det(\cdot)|^{t}$

The central character of $D_\ell \otimes \omega_t$ is
$(\mathrm{sgn})^{\ell+1} \cdot |\cdot|^{2t}$

induced rep: $D_\ell^+ \otimes \mathcal{H}(SL_2(\mathbb{R})^{\pm})$
$\mathcal{H}(SL_2(\mathbb{R}))$ ← index 2

$$D_\ell^+ \oplus \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} D_\ell^+$$

$(\mathrm{sgn})^{\ell+1}$ comes from the action $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

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<u>More on discrete series</u>:

$D_\ell \otimes \omega_t \rightsquigarrow$ relates this with $\sigma(\mu_1 . \mu_2)$.

<u>known</u> $D_\ell \otimes \omega_t \cong \sigma(\mu_1 . \mu_2)$ for some $\mu_1 . \mu_2$. $(\ell > 0)$.

Look at the central char. :

LHS : $(\mathrm{sgn})^{\ell+1} \cdot |\cdot|^{2t}$

RHS : $\mu_1 \cdot \mu_2$; $\mu_1 \cdot \mu_2^{-1} = t^n \cdot \mathrm{sgn}$. $n > 0$.

$\Longrightarrow n = \ell$; i.e. $\sigma(\mu_1 . \mu_2) = \sigma(|\cdot|^{t+\frac{\ell}{2}} \mathrm{sgn}^{\ell+1}, |\cdot|^{t-\frac{\ell}{2}})$.

<u>Philosophy</u> / <u>Langlands correspondence</u>.

(In 2-dim'l case :).

$$\left(\underset{\text{Weil group}}{W_F} \longrightarrow GL_2(\mathbb{C}) \right) \longleftrightarrow \left(\text{representations of } GL_2(F) \right)$$

More general situation :

$$\left(W_F \xrightarrow[\text{admissible}]{} {}^L G \right) \longleftrightarrow \left(\text{(packets of) rep. of } G(F) \right)$$

Langlands' dual group

<u>e.g.</u> if $G = \mathbb{H}^\times$ hamiltonian quaternions

$\Rightarrow {}^L G$ is essentially $GL_2(\mathbb{C})$ with a twist.

<u>Upshot</u> : Given an irred. rep of $\mathbb{H}^\times$, we have an admissible rep. of $W_F \longrightarrow {}^L G \rightsquigarrow$ get $W_F \longrightarrow GL_2(\mathbb{C})$

$\rightsquigarrow$ attach an admissible irreducible representation of $GL_2(F)$.

$\mathbb{H}^\times /$ center $=$ compact symplectic group complexified $SL_2(\mathbb{C})$

topologically, it's $\cong S^3$

However, we understand all rep. of SL_2. period.

Another parameter.

For every representation σ of $\mathbb{H}^\times$. get a representation $\pi(\sigma)$ of $GL_2(\mathbb{R})$ in this discrete series.

Question. what is $\pi(\sigma)$?

σ has a central character $\omega_\sigma(t)$, $t \in \mathbb{R}^\times$

if $\pi(\sigma) \cong \pi(\mu_1, \mu_2)$ they have the same central char.

i.e. $\omega_\sigma = \mu_1 \cdot \mu_2$

$\mu = \mu_1 \cdot \mu_2^{-1} = t^n \cdot \mathrm{sgn}$.

σ is parametrized up to a central character by

$\mathbb{H} \otimes \mathbb{C} \cong M_2(\mathbb{C})$

	0.	1	2.	3.	4	...
$\deg(\sigma)$	1	2	3	4	5	...
	trivial represent.					

$\Longrightarrow$ both parameters determine $D_k \otimes \omega_t$.

Weil group: $W_F = F^\times$ if $F \cong \mathbb{C}$

$W_\mathbb{R}$: we have a non-split extension

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow W_\mathbb{R} \longrightarrow \{1, -1\} \longrightarrow 0$$

set theoretically, $W_\mathbb{R} = \mathbb{C}^\times \amalg j \cdot \mathbb{C}^\times$

$j \in W_\mathbb{R}$ s.t. $j^2 = -1 \in \mathbb{C}^\times$.

and $jzj^{-1} = \bar{z}$ for $z \in \mathbb{C}^\times$

$$W_\mathbb{R}^{ab} \cong \mathbb{R}^\times$$
$$[j] \longmapsto -1$$
$$[\sqrt{a}] \longmapsto a$$
$a > 0$

Langlands parametrization

$F \cong \mathbb{C}$: every 2-dim'l rep. of $W_F = F^\times$ is isom. to $\mu_1 \oplus \mu_2$, where $\mu_1 . \mu_2$ are char. of $W_F = F^\times$

Langlands correspondence : $\pi(\mu_1, \mu_2)$
(the irred one ; or the finite dim'l subquotient if $\rho(\mu_1 . \mu_2)$ is reducible.)

$F \cong \mathbb{R}$: $W_{\mathbb{R}}$ has a large normal subgroup $\mathbb{C}^\times$ $\rightsquigarrow$ Mackey's thm. applies (sp?)
Suppose σ is a 2-dim'l rep.
$\sigma : W_{\mathbb{R}} \longrightarrow GL_2(V)$. $\dim_{\mathbb{C}} V = 2$.

$\Longrightarrow$ only possibilities.

① $\sigma \cong \mu_1 \oplus \mu_2$ $\quad \mu_i : W_{\mathbb{R}} \longrightarrow W_{\mathbb{R}}^{ab} \xleftarrow{\sim} \mathbb{R}^\times \longrightarrow \mathbb{C}^\times$
(local Artin reciprocity.)

② $\sigma|_{\bar{F}^\times} \cong \theta_1 \oplus \theta^\tau$ $\quad$ θ^τ . conjugation by j. (or complex conj.)
and $\sigma = \mathrm{Ind}_{\bar{F}^\times}^{W_F}(\theta)$
$\cong \mathrm{Ind}_{\bar{F}^\times}^{W_F}(\theta^\tau)$
and $\pi(\sigma)$ is a discrete series rep.

Write $\theta = |\cdot|_{\mathbb{C}}^{r} \; z^m \cdot \bar{z}^n$. $\quad$ z : an embedding of $\bar{F} \xrightarrow{\sim} \mathbb{C}$
$r \in \mathbb{C}$, $m\ n \in \mathbb{Z}_{\geq 0}$. $\min(m.n) = 0$
$\max(m,n) > 0$.

then $\pi(\sigma) = \pi(\mu_1, \mu_2)$
central character, $\mu_1 \cdot \mu_2 = \det(\sigma) \overset{?}{=} |\cdot|_{\mathbb{R}}^{2r+m+n} \cdot \mathrm{sgn}^{m+n+1}$
$\mu = \mu_1 \cdot \mu_2^{-1} = |\cdot|_{\mathbb{R}}^{m+n} \cdot \mathrm{sgn}^{m+n+1}$.

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F : archimedian local field

Whittaker models

Theorem Let π be an adm. irred. inf. dim'l repr. (i.e. adm irred $(\mathfrak{g}, K)$-modules) of $GL_2(F)$. Let $\psi : F \longrightarrow \mathbb{C}^\times$. be a non-trivial add. char. of F.
Then $\exists!$ space $W_\psi(\pi)$ of functions on $GL_2(F)$ s.t.

(1) $\forall\ W \in W_\psi(\pi)$, $W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi(x) W(g)$ $\forall x \in F$ $g \in GL_2(F)$.

(2) Every element $W \in W_\psi(\pi)$ is smooth K-finite, and $W\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right) = O\left(|t|_F^N\right)$ as $|t|_F \longrightarrow \infty$ (N may depend on W.)

(3) $\mathfrak{g}$ operates on $W_\psi(\pi)$ via right translation ; making $W_\psi(\pi)$ a $(\mathfrak{g}, K)$-module isomorphic to π.

$\rightsquigarrow$ existence of Kirillov models.

Functional Equations

General form $\forall\ W \in W_\psi(\pi)$. $\chi : F^\times \longrightarrow \mathbb{C}^\times$.

local ζ-function : $\zeta(g.\chi.W.s) = \int_{F^\times} W\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right) \chi(x) |x|_F^{s-\frac{1}{2}} d^\times x$

Then F.E :

$$\frac{\zeta(g.\chi.W.s)}{L(\pi\otimes\chi.s)}\ \varepsilon(\pi\otimes\chi.\psi,s) = \frac{\zeta(wg.\chi^{-1}.W.1-s)}{L(\omega_\pi^{-1}\otimes\chi^{-1}\otimes\pi.1-s)}$$

$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ Weyl ele.

For Kirillov model, let $\xi(x) = W\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right) \in \mathcal{K}_\psi(\pi)$.

$\zeta_{\xi,\chi}(s) = \int_{F^\times} \xi(x) \cdot \chi(x)\ |x|_F^{s-\frac{1}{2}} d^\times x$.

Then F.E:
$$\frac{\zeta_{\xi,\chi}(s)}{L(\pi\otimes\chi,s)}\ \varepsilon(\pi\otimes\chi,\psi,s) = \frac{\zeta_{\pi(w)\xi,\chi^{-1}}(1-s)}{L(\pi\otimes\omega_\pi^{-1}\otimes\chi^{-1}, 1-s)}$$

"similar proof as in the previous cases".

Remark : $\pi\otimes\omega_\pi^{-1} = \check{\pi}$.

Explicit forms for L- ε-factors.

Recall: abelian case.

i) $F = \mathbb{R}$. $\psi : F \longrightarrow \mathbb{C}^\times$. $\psi(x) = \exp(2\pi\sqrt{-1}\,u\,x)$.

$$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\,\Gamma\left(\tfrac{s}{2}\right).$$

$$\Gamma_{\mathbb{C}}(s) = 2\cdot(2\pi)^{-s}\,\Gamma(s) = \Gamma_{\mathbb{R}}(s)\,\Gamma_{\mathbb{R}}(s+1)$$

$\rightsquigarrow$ compatible with induction $\mathbb{R} \to \mathbb{C}$

$\mu : \mathbb{R}^\times \longrightarrow \mathbb{C}^\times$. $\mu(t) = |t|_{\mathbb{R}}^{r}\,\mathrm{sgn}(t)^m$. $m = 0$ or 1

$$\Longrightarrow \quad L(\mu, s) = \Gamma_{\mathbb{R}}(s + r + m).$$

ii) $F \cong \mathbb{C}$. $\mu : F^\times \longrightarrow \mathbb{C}^\times$. $\mu(z) = |z|_{\mathbb{C}}^{r}\, z^m\, \bar{z}^n$, $m, n \geq 0$, $m\cdot n = 0$.

$z : F \xrightarrow{\sim} \mathbb{C}$

$$\Longrightarrow \quad L(\mu, s) = \Gamma_{\mathbb{C}}(s + r + m + n)$$

i) $\varepsilon(\mu, \psi, s) = (\sqrt{-1}\,\mathrm{sgn}\,u)^m \cdot |u|_{\mathbb{R}}^{s+r-\frac{1}{2}}$

(using the self-dual measure w.r.t. ψ)

ii) $\varepsilon(\mu, \psi\circ \mathrm{tr}_{F/\mathbb{R}}, s) = \mu(u)\,|u|_{\mathbb{C}}^{s-\frac{1}{2}} \cdot (\sqrt{-1})^{m+n}$.

$$L(\pi(\mu_1, \mu_2), s) = L(\mu_1, s)\,L(\mu_2, s) \qquad \text{for} \quad F \cong \mathbb{R} \text{ or } F \cong \mathbb{C}.$$

$$\varepsilon(\pi(\mu_1, \mu_2), \psi, s) = \varepsilon(\mu_1, \psi, s)\cdot\varepsilon(\mu_2, \psi, s) \qquad \text{"}$$

for discrete series. $(F \cong \mathbb{R})$

$$L(\pi(\mathrm{Ind}_{W_{\bar F}}^{W_F}(\theta)), s) = L(\theta, s).$$

$$\varepsilon(\pi(\mathrm{Ind}_{W_{\bar F}}^{W_F}(\theta)), \psi, s) = (\sqrt{-1}\,\mathrm{sgn}\,u)\cdot\varepsilon(\theta, \psi\circ \mathrm{tr}_{\bar F/F}, s).$$

Automorphic representations.

F : number field. $\mathbb{A} = \mathbb{A}_F$. adèle ring of F.
$= \prod'_v F_v$.

$G(\mathbb{A}) = GL_2(\mathbb{A}_F)$ (2×2-matrix whose det. is a unit in $\mathbb{A}$)
$= \prod'_v GL_2(F_v)$

We're talking about auto. rep. of $GL_2(\mathbb{A}_F)$.
spectral decomposition of $L^2(G(F)\backslash G(\mathbb{A}_F))$.

$\rightsquigarrow$ take a top differential form . get a Haar measure,
then get L^2 functions.

$$L^2(G(F)\backslash G(\mathbb{A}_F)) = \int_\omega L^2(G(F)\backslash G(\mathbb{A}_F), \omega)$$ (like Fourier inversion)

direct integral.

center $Z(\mathbb{A}_F)$ operates via
$\omega : \mathbb{A}_F^\times / F^\times \longrightarrow \mathbb{C}_1^\times$

3/3/97

Automorphic representations of $GL_2(F)$

F : number field or global field. "ω : Hecke char."

$$L^2(G(F)\backslash G(\mathbb{A}_F)) = \int_{\substack{\omega \in (F^\times\backslash \mathbb{A}_F^\times)^\wedge \\ \text{unit char.}}} L^2(G(F)\backslash G(\mathbb{A}_F), \omega)\, d\omega.$$

direct integral

$G(\mathbb{A}_F)$ operates via right translation.

Classical rep. theory: $\exists$ a subspace

$$L^2_0(G(F)\backslash G(\mathbb{A}_F), \omega) \subseteq L^2(G(F)\backslash G(\mathbb{A}_F), \omega)$$

$$\| $$

$$\{ f \in G(F)\backslash G(\mathbb{A}_F) \longrightarrow \mathbb{C} \mid \int_{F\backslash \mathbb{A}_F} T\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) f \, dx = 0 \}$$

cuspidal functions / forms

i.e. $\int_{F\backslash \mathbb{A}_F} f(xg)\, dx = 0$ a.e.

The rest is given by Eisenstein series.

General facts

(1) For every unitary idèle class character ω, every irred. rep of K (maximal compact subgroup),

$$L^2_0(G(F)\backslash G(\mathbb{A}_F), \omega)(\vartheta) \text{ is } \underline{\text{finite}} \text{ dimensional.}$$

Especially, every irred. subrep. of $L^2_0(G(F)\backslash G(\mathbb{A}_F), \omega)$ is admissib
(proof used classical reduction theory.)

(2) $L^2_0(G(F)\backslash G(\mathbb{A}_F), \omega)$ is a discrete direct sum of irred. subrepresentations.
(proof: Relies on $T(f)$, f : Schwaartz function on $G(\mathbb{A}_F)$, being compact.)

Def An L^2-automorphic form is an L^2-function on $G(F)\backslash G(\mathbb{A}_F)$ which is K-finite and $Z(\mathbb{A}_F)$-finite (may + $Z(\mathcal{U}(\mathfrak{g}_v))$-finite $\forall$ arch. v)

(3) F : global function field.
Every cuspidal automorphic form in $L^2_0(G(F)\backslash G(\mathbb{A}_F), \omega)$ has compact support mod $G(F)\,Z(\mathbb{A}_F)$

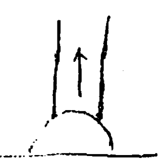
(4) F : number field
Every cuspidal automorphic form in $L^2(G(F)\backslash G(\mathbb{A}_F), \omega)$ decreases rapidly :
$\mathfrak{S}$: a Siegel set (a fundamental domain up to finite multiplicity

$$\left\{ \begin{array}{l} ① \bigcup_{g \in G(F)} g\,\mathfrak{S} = G(\mathbb{A}_F). \\ \text{and } ② \ \# \{ g \in G(F) \mid g\mathfrak{S} \cap \mathfrak{S} \neq \phi \} \text{ is finite.} \end{array} \right.$$
)

Iwasawa decomposition $\mathfrak{S} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} k \;\middle|\; |b| \leq c_0,\ \left|\frac{a}{d}\right| \geq c_1,\ k \in K \right\}$

$c_0 >> 1$, $c_1 << 1$ $\Rightarrow$ ① is satisfied.

(allow to go to infinity only in one direction.)



rapidly $\Longrightarrow$ $f\Big(\underset{\substack{\cap \\ G(F)}}{g_0} \underbrace{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} k}_{\substack{\cap \\ \mathfrak{S}}}\Big) = O\left(\left|\frac{a}{d}\right|^{-N}\right)$ as $\left|\frac{a}{d}\right| \longrightarrow \infty$ $\quad \forall N$.

Fact / Theorem : Every irred. admissible rep. π of $G(\mathbb{A}_F)$,
$\pi = \bigotimes_v' \pi_v$ restricted tensor product of adm. irr. rep.
(proof involves too much functional analysis.)
What is $\bigotimes_v' \pi_v$? need for almost all v, π_v is spherical
(i.e. v : non arch. $\pi_v = \pi(\mu_{1v}, \mu_{2v})$, μ_{iv} unramified.
$\mu_{1v} \cdot \mu_{2v}^{-1} \neq |\cdot|_v^{\pm 1}$)

(unramified principal series.
See 2/17/97.)

Suppose that S is a finite set of primes s.t. π_v is spherical $\forall\, v \notin S$.

$\rightsquigarrow$ $\forall\, v \notin S$. choose a vector $w_v \in \pi_v - \{0\}$ fixed under $K_v = G(\mathcal{O}_v)$ (for unitary rep, choose $\|w_v\| = 1$.)

Then the space of $\bigotimes_v' \pi_v$ is generated by $\bigotimes_v u_v$. s.t $u_v \in \pi_v$ and $u_v = w_v$ for almost all v.

$$\bigotimes_v{}' \pi_v = \bigcup \left\{ u \otimes \bigotimes_{v \notin S} w_v \;\middle|\; u \in \bigotimes_{v \in T} V_v \;.\; T \supseteq S.\ \#(T) < \infty \right\}$$

(this is a vector space on which the adèle group acts.)

This is an irred admis. rep. of $G(\mathbb{A}_F)$.

Conversely, all irred. admis. rep of $G(\mathbb{A}_F)$ are of this form.

<u>Fourier expansion</u>

$f \in L^2(G(F)\backslash G(\mathbb{A}_F), \omega)$. Consider the function

$x \longmapsto f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)$ for some fixed g

on $F\backslash \mathbb{A}_F$ (compact.)

Look at the Fourier transform, fix $\psi : F\backslash \mathbb{A}_F \longrightarrow \mathbb{C}_1^\times$ non-trivial.

$$\rightsquigarrow\ f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \sum_{\xi \in F} \varphi_\xi(g) \cdot \psi(\xi x).$$

$$\text{where } \varphi_\xi(g) = \int_{F\backslash \mathbb{A}_F} f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \overline{\psi(\xi x)}\, dx.$$

Whittaker model: $W = W_f = \varphi_1$.

i.e. $W(g) = \displaystyle\int_{F\backslash \mathbb{A}_F} f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \overline{\psi(x)}\, dx$

$\rightsquigarrow$ $W\left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} g\right) = \psi(y)\, W(g)$ $\leftarrow$ functional equation satisfied by Whittaker model.

$\varphi_0(g) = \displaystyle\int_{F\backslash \mathbb{A}_F} f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx$ $\leftarrow$ vanishes if f is cuspidal

<u>Exer</u> For $\xi \in F^\times$. $\varphi_\xi(g) = W\left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g\right)$.

<u>Consequence</u>. $f(g) = \varphi_0(g) + \displaystyle\sum_{\xi \in F^\times} W_f\left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g\right)$. $\rightsquigarrow$ global W. model

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Remarks about Whittaker models

(1) F : archimedean case. In fact,
$\forall\ W \in \mathcal{W}(\pi),\ W\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right) = O\left(|x|^{-N}\right) \quad \forall N > 0.$

(2) Spherical representations. F : non-arch, $d = d(\psi)$ conductor of ψ
(unram. prin. series repr. : $\pi(\mu_1, \mu_2)$. μ_i unramified.
$\mu = \mu_1 \mu_2^{-1} \neq |\cdot|_F^{\pm 1}$)
$W(\pi)$ contains a unique function right invariant under $K = G(\mathcal{O}_F)$, such that the value at $\begin{pmatrix} \tilde{\omega}^{-d} & 0 \\ 0 & 1 \end{pmatrix}$ is 1.
It is:

$$W^\circ\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{cases} 0 & \text{if } x \notin \mathfrak{p}^{-d} \\ |x \cdot \tilde{\omega}^d|^{\frac{1}{2}} \displaystyle\sum_{\substack{i, j \geq 0 \\ i + j = v(x) + d}} \mu_1(\tilde{\omega}^i)\, \mu_2(\tilde{\omega}^j) & \text{if } x \in \mathfrak{p}^d \end{cases}$$

in the global case, $d = 0$ for almost all places v of F.)

Global Whittaker models

admissible rep. of $G(\mathbb{A}_F)$: $\pi = \bigotimes_v{}' \pi_v$.
$\psi : \mathbb{A}_F/F \longrightarrow \mathbb{C}^\times$
$\rightsquigarrow \mathcal{W}_\psi(\pi) = \bigotimes_v{}' \mathcal{W}_{\psi_v}(F_v) \quad$ (w.r.t. $W^\circ_{\psi_v}$'s)

Proposition $\mathcal{W}_\psi(\pi)$ is the unique model of π in the space of functions on $G(\mathbb{A}_F)$ which are real analytic, moderate growth (when restricted to arch. places), K-finite, such that
$W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi(x)\, W(g),\ \forall x \in \mathbb{A}_F,\ g \in G(\mathbb{A}_F)$ and
$W\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right) = \omega_\pi(x) \cdot W(g)$.

In fact, $W\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right) = O\left(|x|^{-N}\right) \ \forall N. \quad \forall W \in \mathcal{W}(\pi)$
(restricted to Siegel sets $\mathfrak{S}$)

proof Directly from local calculations.

<u>Recall</u>

$$f(g) = \varphi_0(g) := \int_{F\backslash \mathbb{A}_F} f\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) dx + \sum_{\xi\in F^\times} W_f\left(\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix} g\right)$$

$$\text{where } W_f(g) = \int_{F\backslash \mathbb{A}_F} f\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right)\overline{\psi(x)}\, dx$$

$$\rightsquigarrow W_f\left(\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix} g\right) = \psi(y) W_f(g).$$

<u>Theorem</u> (Multiplicity one) Every irreducible cuspidal automorphic representation occurs with multiplicity exactly 1 in $L^2_0(G(F)\backslash G(\mathbb{A}_F), \omega)$

<u>proof</u>. From $\left(\begin{matrix}\text{the existence}\\ \text{\& uniqueness}\end{matrix}\right)$ of global Whittaker models.
& Fourier transforms —

<u>Euler product</u> <u>and</u> <u>L-functions</u>.

$\pi = \bigotimes_v{}' \pi_v$ unitary admissible representation
$\rightsquigarrow$ each π_v is unitary. ($\because$ restricted to the component —)

one knows : For almost all v, $\pi_v = \pi(\mu_{1v}, \mu_{2v})$
<u>where</u> μ_{1v}, μ_{2v} : unramified. and

$$|\mu_{1,v}(x)| = |x|_v^{\sigma_v/2} \qquad 0 \le \sigma_v \le 1$$
$$|\mu_{2,v}(x)| = |x_v|_v^{-\sigma_v/2}$$

$$L(\pi\otimes\chi, s) := \prod_v L(\pi_v\otimes\chi_v, s)$$

product converges for $\mathrm{Re}(s) \gg 0$.

<u>Theorem</u> Let $\pi = \bigotimes_v{}' \pi_v$ be a cuspidal automorphic representation of $G($
Then for every Hecke character $\chi : F^\times\backslash\mathbb{A}_F^\times \longrightarrow \mathbb{C}^\times$. the L-function $L(\pi\otimes\chi, s)$ extends to an entire function and it is bounded in every vertical strip. and satisfies :

$$L(\pi\otimes\chi, s) = \underbrace{\varepsilon(\pi\otimes\chi, s)}_{\prod_v \varepsilon(\pi_v\otimes\chi_v, \psi_v, s) \text{ finite product}} L(\check{\pi}\otimes\chi^{-1}, 1-s).$$

Proof $f \in \pi$. realized as a cusp form.

$f \longleftrightarrow W_f \in \mathcal{W}(\pi)$

$$f(g) = \sum_{\xi \in F^\times} W_f\left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

Define $$\zeta_f(g, \chi, s) := \int_{F^\times \backslash \mathbb{A}_F^\times} f\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right) \chi(x) \cdot |x|^{s-\frac{1}{2}} d^\times x$$

$\|$

$$\zeta(g, \chi W, s) = \int_{\mathbb{A}_F^\times} W_f\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right) \chi(x) \cdot |x|^{s-\frac{1}{2}} d^\times x$$

from the integral rep. $\Rightarrow \zeta_f(g, \chi, s)$ is entire in $s \in \mathbb{C}$.

(because f decays rapidly at two ends.)

and bounded on every vertical strip. (only $\mathrm{Re}(s)$ matters)

$\rightsquigarrow$ same for L-functions $L(\pi \otimes \chi, s)$.

Now only need to show one more thing:

(*) $\zeta_f(g, \chi, s) = \zeta_f(wg, \chi^{-1} \otimes \omega_\pi^{-1}, 1-s)$.

then from local F.E. $\rightsquigarrow$ get relations for L.

$$\text{RHS of } (*) = \int_{F^\times \backslash \mathbb{A}_F^\times} f\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} wg\right) \chi(x)^{-1} \omega_\pi^{-1}(x) \cdot |x|^{\frac{1}{2}-s} d^\times x$$

$$= \int f(\underbrace{w^{-1}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} w}_{\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}} g) - - -$$

$$= \int_{F^\times \backslash \mathbb{A}_F^\times} \cancel{\omega_\pi(x)} \cdot f\left(\begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix} g\right) \chi(x)^{-1} \cancel{\omega_\pi^{-1}(x)} |x|^{\frac{1}{2}-s} d^\times x$$

$= \text{LHS of } (*)$ (change of variable $x \mapsto x^{-1}$) $\square$.

3/7/97

Converse theorem of Weil

Natural question: Given an admissible irred rep. $\pi = \bigotimes_v' \pi_v$, when is it automorphic?

Necessary condition. (for π to be cuspidal automorphic).
$L(\pi, s)$ is entire and bounded on every vertical strip and satisfies the functional equation. $L(\pi, s) = \varepsilon(\pi, s)\, L(\check{\pi}, 1-s)$

Converse theorem for GL_2 (à la Weil).
If $\pi = \bigotimes_v' \pi_v$ is an admis. irred. rep. of $GL_2(\mathbb{A}_F)$ such that
$L(\pi \otimes \chi, s)$ is entire, bounded in every vertical strip
($\rightsquigarrow$ same for $\check{\pi}$) and satisfies F.E.
$L(\pi \otimes \chi, s) = \varepsilon(\pi \otimes \chi, s) \cdot L(\check{\pi} \otimes \chi^{-1}, 1-s)$, $\forall\ \chi$: Hecke char
then: π is a cuspidal automorphic representation.

Example. (Will do later.)

$$\begin{array}{ll} K_v & \theta_v : K_v^\times \longrightarrow \mathbb{C}^\times \\ \Big| \text{quad.} & \\ F_v & \end{array}$$

$K_v^\times$: $F_v^\times$-rational point of a linear algebraic group
$K_v^\times = (\mathrm{Res}_{K_v/F_v} \mathbb{G}_m)(F_v)$ - , a 2-dim'l
$= T^{K_v/F_v}(F_v) =: T_v$ alg. torus / F_v.

dual group ${}^L T_v$: $1 \longrightarrow {}^L T_v^\circ \longrightarrow {}^L T_v \longrightarrow W_F \longrightarrow 1$. @
charactor group of ${}^L T_v^\circ$
= co-char. group of T_v

i.e. $X_*({}^L T_v^\circ) = X^*(T_v)$.

@ semi-direct product with the canonical action of W_F
${}^L T_v^\circ$: connected dual group of $T_v / \mathbb{C}$.

$\rightsquigarrow$ θ_v corresponds to a homomorphism $W_F \longrightarrow {}^L T_v$
(Langlands' philosophy.)

$$\begin{tikzcd} W_F \arrow[r] \arrow[dr] & {}^L T_v \arrow[d] \\ & GL_2(\mathbb{C}) \times W_F \end{tikzcd}$$ dual group of GL_2

$\rightsquigarrow$ $W_F \longrightarrow GL_2(\mathbb{C}) \times W_F$ over W_F gives a representation $\pi(\theta_v)$ of $GL_2(F_v)$, which is supercuspidal.

With "good" understanding of its local L-factors and ε-factor (relating to L and ε factor of θ).

if K_v/F_v quad split $\rightsquigarrow$ tends to get a principal series rep.

Similarly, $\theta : \mathbb{A}_K^\times / K^\times \longrightarrow \mathbb{C}^\times$ to start with,
$\longrightarrow$ get an irred adm. repr. of $G(\mathbb{A}_F)$.

question When is this rep. $\pi(\theta)$ automorphic ?
To check $\pi(\theta)$ is cuspidal automorphic, one can use the converse theorem.

<u>Proof of the converse theorem</u>: Let $\mathcal{W} = \mathcal{W}(\pi)$ be the global Whittaker model for π. Verify these functions in $\mathcal{W}$ have the right properties. $\rightsquigarrow$ reconstruct the corresponding automorphic forms from its 1st Fourier coefficients.

Ⓐ $\forall\, W \in \mathcal{W}$, let

$$f_W(g) = \sum_{\xi \in F^\times} W\left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g \right)$$ (candidate for auto. form)

"Exer." ① (with transformation properties) The formal sum can be replaced by $\sum_{\xi \in F^\times \cap \mathfrak{a}}$ where

$\mathfrak{a}$ is a fractional ideal (from F. E. of Whittaker model.)

② convergence (do some estimates.).

Ⓑ Consider "some-sort-of" ζ - integral.
(F.E. $\iff$ translated by the Weyl element.)
(global F.E. , the translation of w . a global element, doesn't have effect.)

Check : $f_W(g)$ is invariant under the left translation by elements in G
, suffices to check the Borel subgroup
(*) F.E. $\Rightarrow$ invariant under w. from the left.

Consider $\zeta(g, \chi, W, s) = \int_{\mathbb{A}_F^\times} W\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right) \chi(x) \cdot |x|^{s-\frac{1}{2}} d^\times x$

$$= \int_{F^\times \backslash \mathbb{A}_F^\times} f_W\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right) \chi(x) \cdot |x|^{s-\frac{1}{2}} d^\times x. \qquad (\dagger)$$

ζ : a product of local ζ - functions.

This differs from $L(\pi \otimes \chi, s)$ by at most finitely many factors.
$\Rightarrow$ $\zeta(g, \chi, W, s)$ is entire, bounded in every vertical strip. and $\zeta(g, \chi, W, s) = \zeta(wg, \chi^{-1}\omega_\pi^{-1}, W, 1-s)$.

Ⓒ (†) : sort of Mellin transform.
$\leadsto$ 2 functions having the same Mellin transform with right analytic conditions (bdd, ...)
$\Rightarrow$ 2 functions are the same.
<u>Conclusion</u> $f_W(wg) = f_W(g)$
$\therefore f \in L_0^2\left(G(F) \backslash G(\mathbb{A}_F), \omega_\pi\right)$

$\therefore$ f automorphic form

□.

3/17/97

The Shale - Weil Representation

cf Weil. 1964(?) Sur certaines groupes des -----
(collected work. vol. III, 1^{st} paper.)

Heisenberg group

V : a free module over k of finite rank,
k = local field / adèle ring of a global field.
(generally locally compact abelian group)

Look at $V \times V^*$ $V^* = \mathrm{Hom}_{cont}(V, \mathbb{C}_1^\times)$
Pontrjagin dual

(e.g. $V = \mathbb{R}^n$ V^* is also $\cong \mathbb{R}^n$)

Heisenberg group : $1 \longrightarrow \mathbb{C}_1^\times \longrightarrow H(V) \longrightarrow V \times V^* \longrightarrow 1$.
$H(V) = \{ (u, u^*, t) \mid u \in V, u^* \in V^*, t \in \mathbb{C}_1^\times \}$ as a set.

$L^2(V)$. $\Phi \in L^2(V)$. $H(V)$ operates on $L^2(V)$:
$(u, 0, 1) : \Phi \longmapsto (x \longmapsto \Phi(x+u))$ translation
$(0, u^*, 1) . \Phi \longmapsto (x \longmapsto \langle x, u^* \rangle \Phi(x))$ "Fourier transform".

$(u, u^*, t) : \Phi \longmapsto (x \longmapsto t \langle x, u^* \rangle \Phi(x))$. general action.
call this action ρ.

Group law :
$(u_1, u_1^*, t_1)(u_2, u_2^*, t_2)\,\Phi(x)$
$= t_1 \langle x, u_1^* \rangle \left[(u_2, u_2^*, t_2)\Phi(x+u_1) \right]$
$= t_1 \cdot \langle x, u_1^* \rangle \cdot t_2 \cdot \langle x+u_1, u_2^* \rangle \cdot \Phi(x+u_1+u_2)$
$= (u_1+u_2,\ u_1^*+u_2^*,\ t_1 t_2 \langle u_1, u_2^* \rangle)\ \Phi(x)$

$(0, 0, t) \in Z(H(V)) \rightsquigarrow H(V)$: Central extension.
("almost smallest non-abelian group")

Commutator : $[(u_1, u_1^*, t_1), (u_2, u_2^*, t_2)]$
$= (0, 0, \langle u_1, u_2^* \rangle \cdot \langle u_2, u_1^* \rangle^{-1})$
$= F(w_1, w_2) \cdot F(w_2, w_1)^{-1}$

$w_i = (u_i, u_i^*)$, $F(w_1, w_2) = \langle u_1, u_2^* \rangle$.

On $V \times V^*$ (cotangent bundle of V)
$\exists$ a canonical symplectic pairing: $(w_1, w_2) \longmapsto F(w_1, w_2)\, F(w_2, w_1)$
(canonical 2-form.)

<u>Theorem</u> (Stone-von Neumann, ...) $H(V)$ has exactly 1 unitary irreducible representation such that $\mathbb{C}_1^\times$ operates via $\mathbb{C}_1^\times \xhookrightarrow{id} \mathbb{C}^\times$.

So the representation ρ we just wrote down is the unique one.

$B(V) = \mathrm{Aut}(H(V)) \supseteq B_0(V) = \{\alpha \in B(V) \mid \alpha|_{\mathbb{C}_1^\times} = id\}$
Clearly, any element $\alpha \in B_0(V)$ induces a symplectic automorphism of $V \times V^*$ via quotient. $[B(V) : B_0(V)] \leq 2$.

Conversely, every element of $Sp(V \times V^*)$ $(= Sp(V))$ can be lifted to an element of $B_0(V)$ (group theoretic fact.)

$\forall\, \alpha \in B_0(V)$, consider the representation ρ^α (ρ twisted by α)
i.e. $\rho^\alpha : x \longmapsto \rho(x^\alpha)$
By theorem, $\exists\, U$ s.t. $U \rho(x) U^{-1} = \rho^\alpha(x)$.
unitary operator on L^2, unique up to $\mathbb{C}_1^\times$
$\therefore$ this gives a <u>projective</u> $\overset{\text{unitary}}{\vee}$ representation of $Sp(V)$ on $L^2(V)$ = space of ρ
(not a genuine rep. because "up to $\mathbb{C}_1^\times$") $(\alpha \rightsquigarrow U)$
if passing to the universal cover of $sp(V)$, descending down to (a double) cover, get a genuine rep. : the metapletic representation
(Shale-Weil oscillator)

<u>Weil observed</u>: $Mp(V) \xrightarrow{2\text{-}1} Sp(V)$ double cover
adèlic version:

$$\begin{array}{ccc} Mp(V_{\mathbb{A}_k}) & \xrightarrow{2\text{-}1} & Sp(V_{\mathbb{A}_k}) \\ \uparrow & & \uparrow \\ Mp(V_{k_v}) & \xrightarrow{2\text{-}1} & Sp(V_{k_v}) \end{array}$$

$$\begin{tikzcd}
\text{Splitting} & & \mathrm{Sp}(V)(k) \arrow[dll, "r"'] \arrow[d] \ \text{rational points} \\
\mathrm{Mp}(V_{\mathbb{A}_k}) \arrow[rr, "2\text{-}1"] & & \mathrm{Sp}(V_{\mathbb{A}_k}) \\
\mathrm{Mp}(V_{k_v}) \arrow[u] \arrow[rr, "2\text{-}1"] & & \mathrm{Sp}(V_{k_v}) \arrow[u]
\end{tikzcd}$$

<u>and</u> $\exists$ a linear functional

$$\Theta : V(\rho) = L^2(V_{\mathbb{A}_k}) \longrightarrow \mathbb{C}$$

which is invariant under $r(\mathrm{Sp}(V)(k))$.

$\rightsquigarrow$ get lots of functions :

e.g. choose v, consider $f_v : g \longmapsto \Theta(gv)$.

f_v is invariant under $r(\mathrm{Sp}(V)(k))$.

$\rightsquigarrow$ tightly related to automorphic forms.

<u>cf.</u> Leon-Vergne (for $\mathbb{R}$).

Gelbart (for $\mathrm{Mp}(2)$).

$$\ulcorner \quad \begin{tikzcd}
1 \arrow[r] & \mathbb{C}_1^\times \arrow[r] & H(V) \arrow[r] & V \times V^* \arrow[r] & 1 \\
1 \arrow[r] & k \arrow[u, "\psi"] \arrow[r] & H'(V) \arrow[r] & V \times V^* \arrow[r] & 1
\end{tikzcd} \quad \lrcorner$$

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"Details".

<u>Definition</u> "character of 2^{nd} order" on a locally compact abelian group G is a continuous function $f : G \longrightarrow \mathbb{C}^\times$ s.t. $(x, y) \longmapsto f(x+y) \cdot f(x)^{-1} \cdot f(y)^{-1}$ is bimultiplicative on $G \times G$. i.e. $f(x_1 + x_2, y) = f(x_1, y) \cdot f(x_2, y)$.

$$f(x, y_1 + y_2) = f(x, y_1) \cdot f(x, y_2)$$

Why are we interested in these?

$Bo(V) \subseteq Aut(H(V))$ index 2 (almost, not sure)

$s \in B_0(V)$. $(w, t) \in H(V)$. $w = (x, x^*) \in V \times V^*$.

$(w, t)\, s = (w\sigma, f(w) \cdot t)$ $\quad$ $B_0(V)$ acts on $H(V)$

$\sigma \in Sp(V)$. "$s = (\sigma, f)$" $\quad$ from the right.

Condition for $s = (\sigma, f)$ to induce an element of $Bo(V)$:

$(w_1, t_1) \longrightarrow (w_1\sigma, f(w_1)\, t_1)$

$(w_2, t_2) \longrightarrow (w_2\sigma, f(w_2)\, t_2)$

$(w_1, t_1) \cdot (w_2, t_2) = (w_1 + w_2, F(w_1, w_2)\, t_1 t_2)$

$\longrightarrow ((w_1 + w_2)\sigma, f(w_1 + w_2)\, F(w_1, w_2)\, t_1 t_2)$.

<u>Need</u>: $F(w_1\sigma, w_2\sigma) \cdot f(w_1)\, f(w_2)$

$= F(w_1, w_2)\, f(w_1 + w_2)$

<u>i.e.</u> $\boxed{f(w_1 + w_2)\, f(w_1)^{-1} \cdot f(w_2)^{-1} = F(w_1, w_2)^{-1} \cdot F(w_1\sigma, w_2\sigma)}$

$\Longrightarrow$ • f is a character of 2^{nd} order. (from the def'n of F.

• $1 \longrightarrow V^* \times V \longrightarrow B_0(V) \xrightarrow{\alpha} Sp(V) \longrightarrow 1$

$\alpha : (\sigma, f) \longmapsto \sigma$

$\ker \alpha = \{\sigma = 1\} =$ char of $V \times V^* = V^* \times V$.

(from $\square$)

Group law for $Bo(V)$:

$(w, t) \xrightarrow{(\sigma_1, f_1)} (w\sigma_1, f_1(w)t) \xrightarrow{(\sigma_2, f_2)} (w\sigma_1\sigma_2, f_2(w\sigma_1)\, f_1(w)\, t)$.

$\therefore (\sigma_1, f_1) \cdot (\sigma_2, f_2) = (\sigma_1\sigma_2, f_1(w)\, f_2(w\sigma_1))$

<u>Matricial</u> <u>notation</u> $\sigma \in Sp(V)$.

$$(x, x^*) \xmapsto{\sigma} (x, x^*)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where $\alpha : V \longrightarrow V$, $\beta : V \longrightarrow V^*$
$\gamma : V^* \longrightarrow V$, $\delta : V^* \longrightarrow V^*$.

Weyl element $w \longleftrightarrow$ "$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$" symbolically.

interpretation : $(x, x^*) \xrightarrow{w} (-x^*, x)$
$\overset{\Cap}{V \times V^*} \xrightarrow{w} \overset{\Cap}{V^* \times V}$. canonically dual to $V \times V^*$

<u>Define</u> $\sigma^* = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} : V^* \times V \longrightarrow V^* \times V$.

$$\sigma^{I} := w\,\sigma^*\,w^{-1} = \begin{pmatrix} \delta^* & -\beta^* \\ -\gamma^* & \alpha^* \end{pmatrix} : V \times V^* \longrightarrow V \times V^*$$

$$\ulcorner \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} \lrcorner.$$

" symplectic form produces an involution of automorphism. ".

Check: $\sigma^{II} = \sigma$. $\sigma \longmapsto \sigma^{I}$ is an involution

$[\ \sigma \cdot \sigma^{I} = Id \iff \sigma \in Sp(V) \iff \sigma^{I} \cdot \sigma = Id.\ ]$.

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$Sp(V)$. symplectic transformation on $V \times V^*$

$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Sp(V)$

Condition for $(\sigma, f) \in B_0(V)$ in explicit form.

1. $f(w_1 + w_2)\, f^{-1}(w_1)\, f^{-1}(w_2) = F(w_1, w_2)^{-1}\; F(w_1\sigma, w_2\sigma)$ (*)

say $w_i = (u_i, u_i^*)$, $i = 1, 2$

Let $f'(u, u^*) = f(u, u^*)\; \langle u^*\gamma, -u\beta \rangle$.

(*) becomes:

$$f'(u_1+u_2,\, u_1^*+u_2^*)\cdot f'(u_1, u_1^*)^{-1}\; f'(u_2, u_2^*)^{-1}$$
$$\cdot \langle (u_1^*+u_2^*)\gamma,\, (u_1+u_2)\beta \rangle \cdot \langle u_1^*\gamma, -u_1\beta \rangle \cdot \langle u_2^*\gamma, -u_2\beta \rangle$$
$$= \langle u_1\alpha + u_1^*\gamma,\, u_2\beta + u_2^*\delta \rangle \cdot \langle u_1, u_2^* \rangle^{-1}.$$

$$\Longrightarrow f'(u_1+u_2,\, u_1^*+u_2^*)\cdot f'(u_1, u_1^*)^{-1}\cdot f'(u_2, u_2^*)^{-1}$$
$$= \langle u_1\alpha + u_1^*\gamma,\, u_2\beta + u_2^*\delta \rangle \cdot \langle u_1, u_2^* \rangle^{-1}$$
$$\cdot \langle u_1^*\gamma, -u_2\beta \rangle \cdot \langle u_2^*\gamma, -u_1\beta \rangle$$
$$= \langle u_1\alpha, u_2\beta \rangle \cdot \langle u_1\alpha, u_2^*\delta \rangle\; \langle u_1^*\gamma, u_2^*\delta \rangle \cdot \langle u_2^*\gamma, -u_1\beta \rangle$$
$$\langle u_1, -u_2^* \rangle$$
$$= \langle u_1, u_2\beta\alpha^* \rangle \cdot \cancel{\langle u_1\alpha\delta^*, u_2^* \rangle}\; \langle u_1^*, u_2^*\delta\gamma^* \rangle \cdot \underset{\langle -u_1\beta\gamma^*,\, u_2^* \rangle}{\underset{\parallel}{\cancel{\langle -u_1, u_2^*\gamma\beta^* \rangle}}}$$
$$\cdot \cancel{\langle u_1, -u_2^* \rangle}$$
$$= \langle u_1, u_2\beta\alpha^* \rangle \cdot \langle u_1^*, u_2^*\delta\gamma^* \rangle$$
$$= \langle u_1, u_2\alpha\beta^* \rangle \cdot \langle u_1^*\gamma\delta^*, u_2^* \rangle$$

$\left(\text{use } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \delta^* & -\beta^* \\ -\gamma^* & \alpha^* \end{pmatrix} = \begin{pmatrix} \delta^* & -\beta^* \\ -\gamma^* & \alpha^* \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right.$

i.e. $f'(u_1+u_2\; u_1^*+u_2^*)\cdot f'(u_1, u_1^*)^{-1}\cdot f'(u_2, u_2^*)^{-1}$
$$= \langle u_1, u_2\alpha\beta^* \rangle \cdot \langle u_1^*\gamma\delta^*, u_2^* \rangle.$$

Write $g(u) = f'(u, 0)$. $h(u^*) = f'(0, u^*)$
(both are char. of 2^d order.)

$\rightsquigarrow$ $g(u_1+u_2)\cdot g(u_1)^{-1}\cdot g(u_2)^{-1} = \langle u_1, u_2\alpha\beta^* \rangle$
$h(u_1^*+u_2^*)\cdot h(u_1^*)^{-1}\cdot h(u_2^*)^{-1} = \langle u_1^*\gamma\delta^*, u_2^* \rangle.$

$\therefore$ let $w_1 = (u, 0)$. $w_2 = (0, u^*)$

$\Rightarrow$ $f'(u, u^*) = g(u)\cdot h(u^*)$.

Together, we get:

$$\boxed{f(u.u^*) = g(u)\cdot h(u^*)\cdot \langle u^*\gamma, -u\beta\rangle}$$

where $g(u)$ is associated with $\alpha\beta^*$
$h(u^*)$ is associated with $\gamma\delta^*$.

g: symmetric homomorphism from V to V^*
h: symmetric homomorphism from $\tilde{V}^*$ to V
(symmetric : $g = g^*$. $h = h^*$.)

(most of the time, $2: V \longrightarrow V$ is invertible, then
$f(u.u^*) = \langle u, 2^{-1}u\alpha\beta^*\rangle\cdot\langle u^*, 2^{-1}u^*\gamma\delta^*\rangle\cdot\langle u^*\gamma, u\beta\rangle.$ (†)
The right hand side is a formula, which gives a (set-theoretic) lifting of $B_0(V) \underset{r}{\rightleftarrows} Sp(V)$.

<u>Fact</u>. This lifting is a group-homomorphism. ← checkable.
(Need Explanation.)
$\Longrightarrow B_0(V) \cong Sp(V) \ltimes (V \times V^*)$ when 2 is invertible)

(in general, there are lots of char. of 2^d order associated with a symmetric homomorphism. We just picked two chars in (†).)

($H^1(Sp(V), V \times V^*) \overset{?}{=} (0)$)

(Remark on (†). made on 3/26/97) To check r is a homomorphism when V is a vector space over k, $\mathrm{char}(k) \neq 2$, and $V \xrightarrow{\sim} V^*$:
$\forall \sigma \in Sp(V)$, $r(\sigma) = (\sigma, f)$ with f of the form $\chi\circ\tilde{f}$.
where $\tilde{f}: V \longrightarrow k$ is a quadratic form. ($\chi: k \longrightarrow \mathbb{C}_1^{\times}$ fixed.)
⇝ fixes the quadratic form in the lifting.
⇝ proved by "pure thought."

3/24/97

<u>Various Special Liftings</u>

(1) $\sigma = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix} \in Sp(V)$

then take : $f = 1$ trivial. !

get a lifting $d_0 : GL(V) \longrightarrow B_0(V)$

$d_0(\alpha) = \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix}, 1 \right)$.

(2) $\sigma = \begin{pmatrix} 0 & -\gamma^{*-1} \\ \gamma & 0 \end{pmatrix}$. $\gamma : V^* \xrightarrow{\sim} V$.

(like a family of Weyl elements.)

then take $f(u, u^*) = \langle u^*\gamma, -u\gamma^{*-1} \rangle = \langle u, -u^* \rangle$

get a lifting $d_0'(\gamma) = \left(\begin{pmatrix} 0 & -\gamma^{*-1} \\ \gamma & 0 \end{pmatrix}, \langle u, -u^* \rangle \right)$

(3) $\sigma = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$ $\beta : V \longrightarrow V^*$ sym $(\beta = \beta^*)$

then take $f(u, u^*) = \langle u, 2^{-1} u\beta \rangle$ (if 2 is invertible.)

more generally, if β is attached to f. i.e.

$f(x+y) \cdot f(x)^{-1} \cdot f(y)^{-1} = \langle x, y\beta \rangle$.

so attach to $f : V \longrightarrow \mathbb{C}_1^\times$. char. of 2^d order.

$t_0(f) = \left(\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, f \right)$ (take f, a more natural parameter.)

(4) $\gamma : V^* \longrightarrow V$. $\gamma = \gamma^*$ (symmetric.)

attached to $f' : V^* \longrightarrow \mathbb{C}_1^\times$ char. of 2^d order.

then. $t_0'(f') = \left(\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}, f' \right)$.

<u>Rmk.</u> if $\gamma = \gamma^*$ in (2) $\sigma^2 = -\mathrm{Id}$ ← real analog to Weyl elem.

Goal. To lift $B_0(V)$ to the normalizer of $\rho(H(V))$ in $\mathrm{Aut}(L^2(V))$.

"normalization" (or convention) (we use right action)

For any $s \in B_0(V)$. $\exists\ \tilde{s} \in U(L^2(V))$ unitary automorphism

(†) s.t. $\tilde{s}^{-1}\rho(w,t)\tilde{s} = \rho((w,t)s)$ $(s = (\sigma, f))$

$$\rho((w,t)s) = \rho(w\sigma,\ t\cdot f(w))$$

$(x, x^*, t) = (w, t)$ (cf. 3/17/97 Stone - von N.)

$\widetilde{s_1 s_2} = \lambda(s_1, s_2)\ \tilde{s}_1\tilde{s}_2$. $\lambda(s_1, s_2) \in \mathbb{C}_1^\times$ cocycle

i.e. get a <u>projective unitary representation</u> on $L^2(V)$.

Explicit liftings

(0) $\sigma = 1$ look at $1 \to V^* \times V \to B_0(V) \twoheadrightarrow Sp(V) \to 1$.

$\rightsquigarrow$ $s = (1, f)$. f: a char. of $V \times V^*$.

$\to$ $\rho((w,t)s) = f(w)\ \rho((w,t))$. when $s = (1, f)$.

Assertion. $\exists\ \tilde{s} \in U(L^2(V))$ s.t. $\tilde{s}^{-1}\rho(x, x^*, t)\tilde{s} = f(x, x^*)\ \rho(x, x^*, t)$.

$\exists\ v \in V,\ v^* \in V^*$ s.t. $f(x, x^*) = \langle x, v^*\rangle \cdot \langle v, x^*\rangle$.

$= [\tilde{s}^{-1}, \rho(x, x^*)]$

relates to Heisenberg commutator relation. (cf. 3/17/97).

can take $\tilde{s} = \rho(-v, v^*)$

Correspondingly in those special liftings, $\Phi \in L^2(V)$.

(1) $s = \left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix}, 1\right)$

Let $\tilde{d}_0(\alpha): \Phi \longmapsto (y \longmapsto |\alpha|^{1/2}\,\Phi(y\alpha))$ $\quad |\alpha| \Rightarrow d(x^\alpha) = |\alpha|\,dx$

Check (†): $\rho\left((x, x^*)\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix}, 1\right)\Phi$

$= \tilde{d}_0(\alpha)^{-1}\ \rho((x, x^*, 1))\ \tilde{d}_0(\alpha)\ \Phi$??

equivalently. $\tilde{d}_0(\alpha)\ \rho(x\alpha, x^*\alpha^{*-1}, 1)\ \Phi = \rho(x, x^*, 1)\ \tilde{d}_0(\alpha)\Phi$?

(the factor $|\alpha|^{\frac{1}{2}}$ comes in to preserve the L^2-norm

i.e. $\tilde{d}_0(\alpha)$ is unitary on $L^2(V)$.)

3/26/97

<u>Proof</u>. Evaluate both sides at y:

LHS: $|\alpha|^{\frac{1}{2}} \cdot (\rho(x\alpha, x^*\alpha^{*-1}, 1)\Phi)(y\alpha)$

$= |\alpha|^{\frac{1}{2}} \; \Phi(x\alpha + y\alpha) \cdot \langle y\alpha, x^*\alpha^{*-1} \rangle$ (cf. 3/17/97)

RHS: $(\tilde{d}_0(\alpha)\Phi)(x+y) \cdot \langle y, x^* \rangle$

$= |\alpha|^{\frac{1}{2}} \cdot \Phi((x+y)\alpha) \cdot \langle y, x^* \rangle$. OKAY.

$(\tilde{2})$ $s = \left(\begin{pmatrix} 0 & -\gamma^{*-1} \\ \gamma & 0 \end{pmatrix}, \langle u, -u^* \rangle \right)$

$\Rightarrow \tilde{d}_0'(s) \cdot \Phi \longmapsto \left(x \longmapsto |\gamma|^{-\frac{1}{2}} \hat{\Phi}(-x\gamma^{*-1}) \right)$

$\hat{\Phi}$: Fourier transform of Φ. ← ambiguity.

but it's an intrinsic quantity. (the ambiguity above cancels with the choice of $|\gamma|^{-\frac{1}{2}}$)

(3) $\tilde{t}_0(f)$ $\Phi \longmapsto (x \longmapsto f(x)\Phi(x))$

(f. char. of 2^d order on V)

<u>Remark</u> $1 \longrightarrow V^* \cdot V \longrightarrow B_0(V) \longrightarrow Sp(V) \longrightarrow 1$. (most of the time)

$\cup$|

big cell $\Omega_0(V) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \gamma: V^* \xrightarrow{\sim} V \right\}$

open subset (not a subgroup!)

* Have an explicit lift for elements in $\Omega_0(V)$.

(We are actually lifting a double cover of $Sp(V)$.)

* Will have other models for ρ, the Heisenberg representation.

e.g. $\Lambda \subseteq V$. Λ a closed subgp of V, realize ρ in a space of functions on $V \times V^*$ satisfying transformation laws for $\Lambda \times \Lambda^*$.

Then, will be able to lift a subgroup $Sp(V, \Lambda) \subseteq Sp(V)$

(e.g. $V = \mathbb{A}_K$. $\Lambda = K \hookrightarrow \mathbb{A}_K$.

K: # field. , ρ preserves global elements, blah blah blah.

when restricted to $Sp(V, \Lambda)$, have a lifting. like blah blah blah.)

P.S. Every element in $\Omega_0(V)$ has the form
$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

P.S.S. Compare to lifting of product and product of lifting. have a 2-cocycle. blah blah blah ...

Proof (or Check) of (3): Need to check:

(‡) $\quad \tilde{t}_0(f)^{-1} \cdot U((u,u^*,1)) \cdot \tilde{t}_0(f) = U((u,u^*,1)\, t_0(f))$.

everything is projective. fixed L^2-norm. claim: done after this

(‡) $\quad t_0(f) = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$ where β appears in $f(x+y) \cdot f(x)^{-1} \cdot f(y)^{-1} = \langle x, y\beta \rangle$

$\therefore$ RHS $= U((u, u\beta + u^*, f(u)))$.

$$\begin{aligned}
\therefore (\text{RHS})(\Phi)(x) &= U(u, u\beta + u^*, f(u))\, \Phi(x) \\
&= f(u) \cdot \langle x, u\beta + u^* \rangle\, \Phi(x+u) \\
&= \cancel{f(u)} \cdot f(x+u) \cdot f(x)^{-1} \cdot \cancel{f(u)^{-1}} \cdot \langle x, u^* \rangle\, \Phi(x+u)
\end{aligned}$$

$$\begin{aligned}
\therefore (\text{LHS})\, \Phi(x) &= f(x)^{-1} \cdot \langle x, u^* \rangle\, (\tilde{t}_0(f)\, \Phi)(x+u) \\
&= f(x)^{-1} \langle x, u^* \rangle \cdot f(x+u) \cdot \Phi(x+u) \qquad \text{OKAY :).}
\end{aligned}$$

Proof of (2): Need to check
$$\tilde{d}_0'(\gamma)^{-1}\, U((u,u^*,1))\, \tilde{d}_0'(\gamma) = U\left((u,u^*,1)\left(\begin{pmatrix} 0 & -\gamma^{*-1} \\ \gamma & 0 \end{pmatrix}, \langle u, -u^* \rangle\right)\right).$$

$$\begin{aligned}
(\text{RHS})\Phi(x) &= U((u^*\gamma, -u\gamma^{*-1}, \langle u, -u^* \rangle))\, \Phi(x) \\
&= \langle u, -u^* \rangle \cdot \langle x, -u\gamma^{*-1} \rangle \cdot \Phi(x + u^*\gamma)
\end{aligned}$$

$$\tilde{d}_0'(\gamma)(\text{RHS})\,\Phi(x) = |\gamma|^{-\frac{1}{2}} \int_V \langle u, -u^* \rangle \cdot \langle y, -u\gamma^{*-1} \rangle \cdot \Phi(y + u^*\gamma) \cdot \langle y, -x\gamma^{*-1} \rangle\, dy.$$

let $y = z - u^*\gamma$

$$\begin{aligned}
\tilde{d}_0'(\gamma)(\text{LHS})\,\Phi(x) &= U((u,u^*,1))\, (\tilde{d}_0'(\gamma)\, \Phi)(x) \\
&= \langle x, u^* \rangle\, (\tilde{d}_0'(\gamma)\Phi)(x+u) \\
&= \langle x, u^* \rangle \cdot |\gamma|^{-\frac{1}{2}} \int_V \Phi(z)\, \langle z, -(x+u)\gamma^{*-1} \rangle\, dz
\end{aligned}$$

One more thing: $\tilde{d}_0'(\gamma)$ preserves the L^2-norm. q.e.d.

3/28/97

<u>Points</u>. k global. $V_{/k}$. $V_{A_k} = V \otimes A_k$

(1) $\exists$ lifting of $Sp(V)$, say r.

(2) $\exists$ Θ. $L^2(V_{A_k}) \longrightarrow \mathbb{C}$.

(3) $\forall$ $s \in Sp(V)$. $\Theta(r(s)\,\Phi) = \Theta(\Phi)$

(generalized Poisson summation formula.)

(e.g. $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\rightsquigarrow$ proved by PSF. now we put the situation in more general case.)

Let $\Omega_0(V) = \left\{ \left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, f \right) \in B_0(V) \;\middle|\; \gamma : V^* \xrightarrow{\sim} V \right\}$

<u>formulae</u>: $$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & \alpha\gamma^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\gamma^{*-1} \\ \gamma & 0 \end{pmatrix} \begin{pmatrix} 1 & \gamma^{-1}\delta \\ 0 & 1 \end{pmatrix} \qquad (*)$$

(corresponds to "big cell.")

For any $s = \left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, f \right) \in \Omega_0(V)$. s can be written as

(Δ) $\boxed{s = t_0(f_1) \cdot d_0'(\gamma) \cdot t_0(f_2)}$ uniquely.

where $f_1(u) = f(u, -u\alpha\gamma^{-1})$. char. of 2^d order.

$f_2(u) = f(0, u\gamma^{-1})$.

(check f component, σ component is automatic from $(*)$.)

So, assume these.

<u>Consequence</u>. $\rho : V \xrightarrow{\sim} V^*$ symmetric homomorphism,

non-degenerate in the strong sense.

& ρ is attached to $f : V \longrightarrow \mathbb{C}_1^\times$ a char. of 2^d order of V

Define. $f' : V^* \longrightarrow \mathbb{C}_1^\times$ by $f'(u^*) = f(-u^*\rho^{-1})$

the symmetric homomorphism from V^* to V attached to f' is ρ^{-1}

We'll write $t_0'(f') = \left(\begin{pmatrix} 1 & 0 \\ \rho^{-1} & 1 \end{pmatrix}, f \right)$ in two ways:

(1) $t_0'(f') = d_0'(\rho^{-1}) \cdot t_0(f^{-1}) \cdot d_0'(-\rho^{-1})$ by direct cal $f^{-1}(v) := (f(v))^{-1}$

$\downarrow$ project down to σ component.

$$\begin{pmatrix} 1 & 0 \\ \rho^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\rho^* \\ \rho^{-1} & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & -\rho \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & \rho^* \\ -\rho^{-1} & 0 \end{pmatrix}$$

(2) From (Δ), we get

$t_0'(f') = t_0(f_1) \cdot d_0'(\rho^{-1}) \cdot t_0(f_2)$, where $f_1(u) = f(u)$

$f_2(u) = f(-u)$

notation: $f^{-}(x) := f(-x)$

Combine (1) & (2), we get:

$$d_0'(\rho^{-1})\cdot t_0(f^{-1})\cdot d_0'(-\rho^{-1}) = t_0(f)\cdot d_0'(\rho^{-1})\cdot t_0(f^{-})$$

$$\rightsquigarrow \quad \boxed{t_0(f^{-1})\cdot d_0'(-\rho^{-1}) = d_0'(-\rho^{-1})\cdot t_0(f)\, d_0'(\rho^{-1})\cdot t_0(f^{-})}$$

Rmk. We write down generators of symplectic group.

i.e. $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix}$ $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$. $\begin{pmatrix} 0 & -\gamma^{*-1} \\ \gamma & 0 \end{pmatrix}$

We just wrote down one of the relations, namely $\square$.

Lift $\square$ to $B_0(V)$:

$\exists\ \gamma(f) \in \mathbb{C}_1^\times$ st.

(*) $\quad \gamma(f)\, \tilde{t}_0(f^{-1})\cdot \tilde{d}_0'(-\rho^{-1}) = \tilde{d}_0'(-\rho^{-1})\, \tilde{t}_0(f)\cdot \tilde{d}_0'(\rho^{-1})\, \tilde{t}_0(f^{-})$.

$\rightsquigarrow$ Apply to all Φ in either $\mathcal{S}(V)$ or $L^2(V)$

$$(LHS)\Phi(x) = \gamma(f)\cdot f(x)^{-1}\,(\tilde{d}_0'(-\rho^{-1})\Phi)(x)$$

$$= \gamma(f)\cdot f(x)^{-1}\cdot |\rho|^{\frac{1}{2}}\ \hat{\Phi}(x\rho) \qquad (\text{using } \rho = \rho^*).$$

$$(RHS)\Phi(x) = |\rho|^{\frac{1}{2}}\ (\tilde{t}_0(f)\cdot \tilde{d}_0'(\rho^{-1})\ \tilde{t}_0(f^{-})\,\Phi)^{\wedge}(x\rho)$$

$$(\tilde{t}_0(f)\cdot \tilde{d}_0'(\rho^{-1})\cdot \tilde{t}_0(f^{-})\,\Phi)(y) = f(y)\cdot |\rho|^{\frac{1}{2}}\ (\tilde{t}_0(f^{-})\Phi)^{\wedge}(-y\rho)$$

$$= f(y)\cdot|\rho|^{\frac{1}{2}}\int_V f(-z)\,\Phi(z)\langle z,-y\rho\rangle\, dz.$$

$$\therefore\ (RHS)\Phi(x) = |\rho|\int_V \underline{f(y)}\,\langle y, x\rho\rangle\, dy\int_V \underline{f(-z)}\cdot\Phi(z)\cdot\underline{\langle z,-y\rho\rangle}\, dz.$$

$$= |\rho|\cdot\int_V\langle y, x\rho\rangle\, dy\int_V f(y-z)\cdot\Phi(z)\, dz.$$

$$= |\rho|\cdot(\Phi * f)^{\wedge}(x\rho).$$

"Weil's Thm 2"

Conclusion / Theorem $\quad \exists\ \gamma(f) \in \mathbb{C}_1^\times$ s.t.

$$(\Phi * f)^{\wedge} = \gamma(f)\cdot|\rho|^{-\frac{1}{2}}\cdot g\cdot\hat{\Phi} \quad \text{where } g(x^*) = f(x^*\rho^{-1})^{-1}.$$

shorthand. $\quad \hat{f}(x^*) = \gamma(f)\cdot|\rho|^{-\frac{1}{2}}\cdot f(x^*\rho^{-1})^{-1}$

e.g. $\quad f = e^{2\pi i x^2}$ over $V = \mathbb{R}^1$

$\gamma(f) \rightsquigarrow$ quadratic reciprocity law. Hilbert symbol --

$\rightsquigarrow$ encodes 2-cocycle in metaplectic group.

<u>Exercise</u> From (Δ) on 3/28/97

Suppose $(\sigma_1, f_1)\cdot(\sigma_2, f_2) = (\sigma_3, f_3)$

$\gamma_1, \gamma_2, \gamma_3 : V^* \xrightarrow{\sim} V$

$\rightsquigarrow$ $\tilde{s}_1 \cdot \tilde{s}_2 = \lambda(s_1, s_2)\tilde{s}_3$ where $\tilde{s}_i = \tilde{t}_0(f_{i1})\,\tilde{d}_0'(\gamma_i)\cdot\tilde{t}_0(f_{i2})$.

$\lambda(s_1, s_2)$: 2-cocycle, taking value in $\mathbb{C}_1^\times$.

$\rightsquigarrow$ try to describe λ in terms of $\gamma(f)$'s.

<u>Hint</u>. Figure out: $\tilde{t}_0(f_{11})\,\tilde{d}_0'(\gamma_1)\cdot\tilde{t}_0(f_{12})\cdot\tilde{t}_0(f_{21})\,\tilde{d}_0'(\gamma_2)\cdot\tilde{t}_0(f_{22})$
$= \lambda(s_1, s_2)\cdot\tilde{t}_0(f_{31})\cdot\tilde{d}_0'(\gamma_3)\cdot\tilde{t}_0(f_{32})$

Use relations downstairs, then relate λ to γ.

V vector space $\qquad$ V^* : Pontrjagin dual of V

$\cup$ $\qquad$ $\cup$

Γ : closed subgroup $\qquad$ $\Gamma^\perp = \{\xi^* \in V^* : \langle\xi, \xi^*\rangle = 1\ \forall \xi \in \Gamma$

$\Phi \in L^2(V)$, produce
$\longmapsto \Theta(x, x^*) := \int_\Gamma \Phi(x+\xi)\langle\xi, x^*\rangle\, d\xi$.

Fourier transform of $\Phi_x : \xi \longmapsto \Phi(x+\xi)$, $\xi \in \Gamma$.

- $\Theta(x, x^* + \xi^*) = \Theta(x, x^*)$ $\quad \forall \xi^* \in \Gamma^\perp$.
- $\Theta(x+\xi, x^*) = \langle\xi, -x^*\rangle\ \Theta(x, x^*)$ $\quad \forall \xi \in \Gamma$.

<u>Or</u> On $V \times V^*$ we have $\Gamma\times\Gamma^\perp \subset V\times V^*$

$\Longrightarrow$ $\boxed{\Theta(w+\lambda) = \langle -\xi, x^*\rangle\cdot\Theta(w) = F(\lambda, w)^{-1}\ \Theta(w)}$ (*)

$(w = (x, x^*),\ \lambda = (\xi, \xi^*) \in \Gamma\times\Gamma^\perp)$

Call $Z : L^2(V) \longrightarrow H(V, \Gamma)$.
$\Phi \longmapsto \Theta$

To get back Φ from Θ, we apply the inverse Fourier transform
$$\Phi(x) = \int_{V^*/\Gamma^\perp} \Theta(x, x^*)\, d\bar{x}^* \left(= \Phi_x(-0) = \Phi(x)\right)$$

$\therefore$ $|\Theta|$ depends only on $[w] \in V\times V^*/\Gamma\times\Gamma^\perp$.

$\Longrightarrow$ $H(V, \Gamma) = \{\Theta$ satisfies (*) s.t. $\int_{V\times V^*/\Gamma\times\Gamma^\perp} |\Theta|^2\, d\bar{x}\, d\bar{x}^* < \infty\}$

$\Longrightarrow$ $Z : L^2(V) \xrightarrow{\sim} H(V, \Gamma)$.

Relating with the Heisenberg representation

Goals:

(1) Get a formula for ρ using $H(V, \Gamma)$.

(2) For a "nice subgroup" of $B_0(V)$, can lift by "easy formula"

(3) Lift the Shale-Weil representation over this nice subgroup as an ordinary representation + a homomorphism $L^2(V) \longrightarrow \mathbb{C}$ invariant under the lift of this homomorphism.

(Lifting. $1 \longrightarrow \mathbb{C}_1^\times \longrightarrow Mp(V) \longrightarrow B_0(V) \longrightarrow 1$)

$$\begin{array}{ccc} Mp(V) & & B_0(V) \\ & \overset{\text{lifting}}{\nwarrow} & \cup \\ & & Sp(V, \Gamma) \end{array}$$

Classical example of a homomorphism / linear functional $L^2(V) \longrightarrow \mathbb{C}$:

$V_k \qquad V_{\mathbb{A}} = V_k \otimes \mathbb{A}_k \supseteq V_k$. the quotient is cpt.

so: $\Phi \longmapsto \sum_{\xi \in V_k} \Phi(x+\xi) \langle \xi, x^* \rangle$.

$L^2(V) \longrightarrow \mathbb{C}$ is defined as $\Phi \longmapsto \Theta \longmapsto \Theta(0,0)$

Really classical example: $V = k \quad k = \mathbb{Q}$

$\Phi = \prod_v \Phi_v \quad \in L^2(V_{\mathbb{A}})$

v finite: characteristic function of $\mathbb{Z}_v$. $\Big\} =: \Phi_v$

v infinite: $e^{-\pi x^2}$

take $x = (x_\infty, 0)$ we're looking at:

$$((x_\infty, 0), x^*) = \sum_{n \in \mathbb{Z}} e^{-\pi (x+n)^2} \langle n, x^* \rangle$$

Weil realized: $H(V, \Gamma)$ is the space of adèlic Θ-functions.

our case: $Sp(V, \Gamma) = SL_2(k)$. in general $Sp_{2n}(k)$ if $\dim_k V = n$.

4/2/97

$V \supseteq \Gamma$ closed subgroup. , $L^2(V) \xrightarrow{\sim} H(V,\Gamma)$.

$$\Phi \longmapsto \Theta(x,x^*) = \int_\Gamma \Phi(x+\xi)\ \langle \xi, x^* \rangle\ d\xi$$

$$\Longrightarrow \quad \Phi(x) = \int_{V^*/\Gamma^\perp} \Theta(x,x^*)\ d\bar{x}^*$$; transformation formulae for $\Theta \in H(V,\Gamma)$

<u>Heisenberg representation in the model $H(V,\Gamma)$</u>

say $(u,u^*,t) \in H(V)$, then

$$\begin{aligned}
&(\rho(u,u^*,t)\ \Theta)(x,x^*) \\
&= \int_\Gamma (\rho(u,u^*,t)\ \Phi)(x+\xi)\ \langle \xi, x^* \rangle\ d\xi \\
&= \int_\Gamma \Phi(x+\xi+u)\ t \cdot \langle x+\xi, u^* \rangle \cdot \langle \xi, x^* \rangle\ d\xi \\
&= t \cdot \langle x, u^* \rangle\ \Theta(x+u, x^*+u^*)
\end{aligned}$$

<u>Formula</u>: $$\begin{aligned}(\rho(u,u^*,t)\,\Theta)(x,x^*) &= t \cdot \langle x, u^* \rangle \cdot \Theta(x+u, x^*+u^*) \\ &= t \cdot F((x,x^*),(u,u^*)) \cdot \Theta(x+u, x^*+u^*)\end{aligned}$$

Define a subgroup $B_0(V,\Gamma) = \{ (\sigma, f) \in B_0(V) : f(\xi,\xi^*) = 1$ $\forall\, \xi \in \Gamma,\ \xi^* \in \Gamma^\perp$, and σ preserves $\Gamma \times \Gamma^\perp \}$.

$\rightsquigarrow$ Still have a projective representation of $B_0(V)$ on $H(V,\Gamma)$

$B_0(V) \supseteq B_0(V,\Gamma)$

(Stone - von Neumann)

$\rightsquigarrow$ good thing: a nice formula gives a genuine representation of $B_0(V,\Gamma)$ on $H(V,\Gamma)$ instead of a projective one

i.e. Formula for lifting $B_0(V,\Gamma)$ to a unitary representation of $B_0(V,\Gamma)$ on $H(V,\Gamma)$.

<u>Formula</u>: $s = (\sigma, f) \in B_0(V,\Gamma)$

$$\boxed{(r_\Gamma(s)\ \Theta)(x,x^*) = f(x,x^*) \cdot \Theta((x,x^*)\sigma)}$$

Need to check:

1. $r_\Gamma(s)$ preserves L^2-norm on $H(V,\Gamma)$ (easy to see this)
2. $r_\Gamma(s)$ is a (unitary) representation: $r_\Gamma(s_1)\, r_\Gamma(s_2) = r_\Gamma(s_1 s_2)$ $\forall s_1, s_2 \in B_0(V,\Gamma)$

3. r.h.s $\in H(V,\Gamma)$. i.e. another θ-function

4. right commutator relation for Heisenberg representation:

$$Y_\Gamma(s)^{-1}\ \rho((u,u^*,t))\ Y_\Gamma(s) = \rho((u,u^*)\sigma,\ t\, f(u\,u^*))$$

<u>Check</u>

3 $\quad f(x+\xi, x^*+\xi^*)\cdot\theta((x+u, x^*+u^*)\sigma) \overset{?}{=} \langle \xi, -x^*\rangle \cdot f(x,x^*)\cdot\theta((x,x^*)\sigma)$.

$$f(x+\xi, x^*+\xi^*) = f(x,x^*)\cdot f(\xi,\xi^*)\cdot F((\xi,\xi^*)\sigma,(x,x^*)\sigma)\cdot F((\xi,\xi^*),(x,x^*))^{-1}.$$

$$\therefore \text{l.h.s.} = f(x,x^*)\cdot f(\xi,\xi^*)\ F((\xi,\xi^*)\sigma,(x,x^*)\sigma)\cdot F((\xi,\xi^*),(x,x^*))^{-1}$$
$$F((\xi,\xi^*)\sigma,(x,x^*)\sigma)^{-1}\cdot\theta((x,x^*)\sigma)$$
$$= f(x,x^*)\cdot 1\cdot \langle\xi, x^*\rangle^{-1}\cdot\theta((x,x^*)\sigma) = \text{r.h.s.} \quad \checkmark$$

4 $\quad$ NTS: $\big(\rho(u,u^*,1)\,Y_\Gamma(s)\,\theta\big)(x,x^*) = \big(Y_\Gamma(s)\,\rho((u,u^*)\sigma,\ f(u\,u^*))\theta\big)(x,x^*)$

$$\text{l.h.s.} = F((x,x^*),(u\,u^*))\cdot(Y_\Gamma(s)\theta)(x+u,\ x^*+u^*)$$
$$= \underline{F((x,x^*),(u,u^*))}\cdot\underline{f(x+u,\ x^*+u^*)}\cdot\theta((x+u, x^*+u^*)\sigma)$$
$$\text{r.h.s} = f(x,x^*)\cdot\big(\rho((u,u^*)\sigma, f(u,u^*))\,\theta\big)((x,x^*)\sigma)$$
$$= \underline{f(x,x^*)}\ \theta((x+u,\ x^*+u^*)\sigma)\cdot\underline{F((x,x^*)\sigma,(u,u^*)\sigma)}\cdot\underline{f(u,u^*)}$$

$\checkmark$

2. NTS: $\big(Y_\Gamma(s_1)\,Y_\Gamma(s_2)\,\theta\big)(x,x^*) = \big(Y_\Gamma(s_1 s_2)\,\theta\big)(x,x^*)$.

$$\text{l.h.s.} = f_1(x,x^*)\cdot\big(Y_\Gamma(s_2)\theta\big)((x,x^*)\sigma_1)$$
$$= f_1(x,x^*)\cdot f_2((x,x^*)\sigma)\cdot\theta((x,x^*)\sigma_1\sigma_2)$$
$$= f_3(x,x^*)\ \theta((x,x^*)\sigma_3) = \big(Y_\Gamma(s_1 s_2)\theta\big)(x,x^*) = \text{r.h.s}$$

$\big((\sigma_1, f_1)(\sigma_2, f_2) = (\sigma_3, f_3)$ with $\sigma_3 = \sigma_1\sigma_2,\ f_3 = \cdots\big)$ $\checkmark$.

Now we give a linear functional "theta-null":

$$\mathcal{S}(V) \longrightarrow \mathbb{C}$$
$$\Phi \longmapsto \int_\Gamma \Phi(\xi)\,d\xi = \theta(0,0)$$

extends this to $H(V,\Gamma)$ $\rightsquigarrow$ get a linear functional (in the sense of distribution)

$s \in B_0(V, \Gamma)$. then

$$\int_\Gamma (\Upsilon_\Gamma(s)\Phi)(\xi)\, d\xi = \int_\Gamma \Phi(\xi)\, d\xi.$$

(because $\Phi \longleftrightarrow \Theta$. $\Upsilon_\Gamma(s)\theta(0,0) = f(0,0)\,\theta((0,0)\sigma)$
$= \theta(0,0)$ ✓).

If one thinks about $\Upsilon_\Gamma(s) : L^2(V) \longrightarrow L^2(V)$ then $\Upsilon_\Gamma(s)$ preserves the subspace of Schwartz functions.

<u>REMARK</u>. Weil said this is a generalization of Poisson summuation formula.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(= \begin{pmatrix} 0 & -\gamma^{*-1} \\ \gamma & 0 \end{pmatrix} \right)$$

<u>Example</u> V/k. k : global field. (e.g. # fld.)
$\dim_k V < \infty$. then $V_{\mathbb{A}}^* \xrightarrow{\sim} V_{\mathbb{A}}$. $V_{\mathbb{A}} \supseteq V_k$

PSF : $\displaystyle\sum_{x \in V_k} \Phi(\xi) = \sum_{\xi \in V_k} \hat{\Phi}(\xi)$.

$\Updownarrow$

$$\int_\Gamma \Phi(\xi)\, d\xi = \int_\Gamma (\Upsilon_\Gamma(s)\Phi)(\xi)\, d\xi$$

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Summary. $1 \longrightarrow \mathbb{C}_1^\times \longrightarrow H(V) \longrightarrow V \cdot V^* \longrightarrow 1$

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{C}_1^\times \arrow[r] & Mp(V) \arrow[r] & Bo(V) \arrow[r] & 1 \\ & & \tilde{\gamma}_0 \arrow[u, "\text{lifting}"] & \Omega_0(V) \arrow[ul] \arrow[u, phantom, "\cup"] & Bo(V,\Gamma) \arrow[ul, phantom, "\supset"] \end{tikzcd}$$

Bruhat decomposition (for GL_2) $\bar{N}AN$

$N = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. $\bar{N} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$

$G \geq A$ "maximal torus" $\Phi = \Phi_+ \amalg \Phi_-$

$U_+ = $ "$\prod_{\alpha > 0} U_\alpha$" $\quad \Phi \ni \alpha \to U_\alpha$

$\sim \underbrace{U_- A U_+}_{\text{big cell}} \leq G \quad (\Omega_0(V))$

Γ lattice in V. can define $Bo(V,\Gamma) \subset Bo(V)$

have a genuine lifting $Bo(V,\Gamma) \leq Mp(V)$.

"trivial example" $\Lambda \leq V$. say $\Lambda = (0)$, V.

What are $B_o(V,(0))$. $B_o(V,V)$?

① $B_o(V,V)$. $\Lambda = V$. $\Lambda^\perp = (0)$

$\Rightarrow B_o(V,V) = \{ s = (\sigma, f) \in Bo(V) \cdot \sigma : V \times (0) \xrightarrow{\sim} V \times (0)$.

and $f(x,0) = 1 \quad \forall x \in V \} \ni do(\alpha)$.

$\Rightarrow \sigma = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$.

$f(x,x^*) = f(0,x^*) \; F((x,0)\sigma, (0,x^*)\sigma) \cdot \underbrace{F((x,0),(0,x^*))^{-1}}_{\langle x, x^* \rangle^{-1}}$.

exercise

$\Rightarrow \gamma_{\Gamma=V}(do(\alpha)) = \widetilde{do}'(\alpha)$.

Proposition Let $\Omega_0(V,\Gamma) = \{ (\sigma,f) \in \Omega_0(V) \cap Bo(V,\Gamma) : \begin{matrix} \gamma : V^* \xrightarrow{\sim} V \\ \cup\!\shortmid \qquad \cup\!\shortmid \\ \Gamma^\perp \xrightarrow{\sim} \Gamma \end{matrix} \}$

Then $\tilde{\gamma}_0(s) = \gamma_\Gamma(s) \quad \forall s \in \Omega_0(V,\Gamma)$

and the moduli of $V^*/\Gamma^\perp \longrightarrow V/\Gamma$. $\Gamma^\perp \longrightarrow \Gamma$ are equal

to $|\gamma|^{\frac{1}{2}}$.

Proof. $s = \left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, f \right)$

$\tilde{\gamma}_0(s) = \tilde{t}_0(f_1) \cdot \tilde{d}_0'(\gamma) \cdot \tilde{t}_0(f_2)$, where

$t_0(f_1), t_0(f_2) \in B_0(\bar{V}, \Gamma)$. (clear from formulae)

Also "trivially". $\tilde{t}_0(f_i) = \gamma_\Gamma(t_0(f_i))$ $i = 1, 2$.

(from the def'n of the lifting.)

So the remaining part is to show: $\gamma_\Gamma(d_0'(\gamma)) = \tilde{d}_0'(\gamma)$

(1) $\left(\tilde{d}_0'(\gamma)\,\Phi\right)(x) = |\gamma|^{-\frac{1}{2}}\,\hat{\Phi}(-x\gamma^{*-1})$.

(2) $\left(\gamma_\Gamma(d_0'(\gamma))\,\Phi\right)(x)$ $\qquad d_0'(\gamma) = \left(\begin{pmatrix} 0 & -\gamma^{*-1} \\ \gamma & 0 \end{pmatrix}, \langle -x, x^* \rangle \right)$

$$= \int_{V^*/\Gamma^\perp} \left(\gamma_\Gamma(d_0'(\gamma))\,\theta\right)(x, x^*)\, d\bar{x}^*$$

$$= \int_{V^*/\Gamma^\perp} \langle -x, x^* \rangle\ \theta\left(x^*\gamma, -x\gamma^{*-1}\right)\, d\bar{x}^*$$

$$= \left|\gamma_{V^*/\Gamma^\perp}\right|^{-1} \int_{V/\Gamma} \langle -x, u\gamma^{-1} \rangle\, \theta(u, -x\gamma^{*-1})\, d\bar{u} \quad (u = x^*\gamma)$$

$$= \left|\gamma_{V^*/\Gamma^\perp}\right|^{-1} \int_{V/\Gamma} \langle -x, u\gamma^{-1} \rangle d\bar{u} \int_\Gamma \Phi(u+\xi) \langle \xi, -x\gamma^{*-1} \rangle\, d\xi$$

$$= \left|\gamma_{V^*/\Gamma^\perp}\right|^{-1} \int_V \Phi(y) \cdot \langle y, -x\gamma^{*-1} \rangle\, dy = \left|\gamma_{V^*/\Gamma^\perp}\right|^{-1} \hat{\Phi}(-x\gamma^{*-1})$$

Since both operators are unitary, they are off by a positive number. $\therefore$ (1) = (2) and the proposition is done. $\square$

"Theorem" Let f be a character of 2^d order such that

(1) $f|_\Gamma = 1$.

(2) the symmetric homomorphism $\rho : V \longrightarrow V^*$ attached to f is an isomorphism.

Then $\gamma(f) = 1$

This theorem is followed from the next

Theorem $(\sigma^s, f)(\sigma'^{s'}, f') = (\sigma''^{s''}, f'')$ s.t. $\gamma, \gamma', \gamma'' : V^* \xrightarrow{\sim} V$

Let $f_0(u) = f(0, u\gamma^{-1}) \cdot f'(u, -u\gamma'^{-1})$ (f_2 for s . f_1 for s')

Then f_0 is nondegenerate, giving rise to $\gamma^{-1}\gamma''\gamma'^{-1}$, and we have a cocycle relation: $\boxed{\tilde{\gamma}_0(s)\,\tilde{\gamma}_0(s') = \gamma(f_0) \cdot \tilde{\gamma}_0(s'')}$

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$$(\sigma, f)\cdot(\sigma', f') = (\sigma'', f'')$$
$$s \qquad\qquad s' \qquad = \qquad s'' \qquad\qquad \gamma, \gamma', \gamma'' : V^* \xrightarrow{\sim} V$$

Standard liftings
$$s = t_0(f_1)\, d_0'(\gamma)\cdot t_0(f_2)$$
$$s' = t_0(f_1')\, d_0'(\gamma')\, t_0(f_2')$$
$$s'' = t_0(f_1'')\, d_0'(\gamma'')\, t_0(f_2'')$$

$\Rightarrow$ lift them to $Mp(V)$:
$$\tilde{t}_0(f_1)\cdot \tilde{d}_0'(\gamma)\, \tilde{t}_0(f_2)\, \tilde{t}_0(f_1')\, \tilde{d}_0'(\gamma')\, \tilde{t}_0(f_2')$$
$$= \lambda(s_1, s_2)\, \tilde{t}_0(f_1'')\cdot \tilde{d}_0'(\gamma'')\cdot \tilde{t}_0(f_2'').$$

(Δ) $\Rightarrow$ $\tilde{d}_0'(\gamma)\, \tilde{t}_0(f_0)\cdot \tilde{d}_0'(\gamma') = \lambda(s_1, s_2)\cdot \tilde{t}_0(f_3)\cdot \tilde{d}_0'(\gamma'')\, \tilde{t}_0(f_4)$

where $f_0 = f_2\cdot f_1'$, $f_3 = f_1^{-1}\cdot f_1''$, $f_4 = f_2''\, f_2'^{-1}$.

$f_0(u) = f(0, u\gamma^{-1})\cdot f'(u, -u\alpha'\gamma'^{-1})$

<u>Theorem</u> $\lambda(s_1, s_2) = \gamma(f_0)$

<u>proof</u> Apply both sides of (Δ) to $\Phi \in \mathcal{S}(V)$:

<u>RHS</u> $\left(\lambda(s_1, s_2)\cdot \tilde{t}_0(f_3)\cdot \tilde{d}_0'(\gamma'')\, \tilde{t}_0(f_4)\Phi\right)(x)$
$$= \lambda(s_1, s_2)\cdot f_3(x)\quad \tilde{d}_0'(\gamma'')\left(f_4\,\Phi\right)(x)$$
$$= \lambda(s_1, s_2)\cdot f_3(x)\,|\gamma''|^{-\frac12}\,(f_4\,\Phi)^\wedge(-x\gamma''^{*-1})$$
$$= \underline{\underline{\lambda(s_1, s_2)}}\cdot f_3(x)\cdot|\gamma''|^{-\frac12}\int_V f_4(u)\cdot\Phi(u)\langle u, -x\gamma''^{*-1}\rangle\, du$$

<u>LHS</u> $\left(\tilde{d}_0'(\gamma)\cdot \tilde{t}_0(f_0)\, \tilde{d}_0'(\gamma')\,\Phi\right)(x)$
$$= |\gamma|^{-\frac12}\cdot\left(\tilde{t}_0(f_0)\, \tilde{d}_0'(\gamma')\,\Phi\right)^\wedge(-x\gamma^{*-1})$$
$$= |\gamma|^{-\frac12}\cdot\int_V \left(\tilde{t}_0(f_0)\, \tilde{d}_0'(\gamma')\,\Phi\right)(u)\cdot\langle u, -x\gamma^{*-1}\rangle\, du$$
$$= |\gamma|^{-\frac12}\cdot\int_V f_0(u)\cdot\left(\tilde{d}_0'(\gamma')\,\Phi\right)(u)\,\langle u, -x\gamma^{*-1}\rangle\, du$$
$$= |\gamma|^{-\frac12}\int_V f_0(u)\cdot|\gamma'|^{-\frac12}\cdot\int_V \Phi(v)\cdot\langle v, -u\gamma'^{*-1}\rangle\cdot\langle u, -x\gamma^{*-1}\rangle\, dv\, du$$
$$= |\gamma|^{-\frac12}\cdot|\gamma'|^{-\frac12}\cdot\int_V\int_V \Phi(v)\cdot f_0(u)\cdot\langle u, -v\gamma'^{*-1}\rangle\cdot\langle u, -x\gamma^{*-1}\rangle\, du\, dv$$
$$= |\gamma|^{-\frac12}\,|\gamma'|^{-\frac12}\cdot\int_V \Phi(v)\cdot\underline{\underline{\gamma(f_0)}}\cdot|\rho_0|^{-\frac12}\, f_0\left((-v\gamma'^{*-1} - x\gamma^{*-1})\rho_0^{-1}\right)^{-1}\, dv$$

$\mathcal{F}(f_0)(x^*) = \gamma(f_0)\cdot|\rho_0|^{-\frac12}\cdot f_0(x^*\rho_0^{-1})^{-1}$

<u>Claim</u> $\lambda(s_1, s_2) = \gamma(f_0)$. everything else cancelled

or: $\lambda(s_1, s_2)\cdot\underbrace{|\gamma''|^{-\frac12}} = \underbrace{|\gamma|^{-\frac12}\cdot|\gamma'|^{-\frac12}\cdot|\rho_0|^{-\frac12}}\cdot\gamma(f_0)$ //

"Corollary" f : char. of 2^d order on V s.t.

(1) $f\restriction_\Gamma = 1$

(2) The symmetric homomorphism $\rho : V \longrightarrow V^*$ attached to f induces an isomorphism $(V, \Gamma) \xrightarrow{\sim} (V^*, \Gamma^*)$

Then $\gamma(f) = 1$.

Hint Produce a situation in $B_0(V, \Gamma)$ w/ $f = f_0$. Since we have a genuine representation. $\gamma(f) = 1$.

Application Local reciprocity law.

Properties of the symbol $\gamma(f)$.

k : locally compact field which is not discrete.

$\psi : k \longrightarrow \mathbb{C}_1^\times$ nontrivial additive character.

$$\begin{tikzcd}
1 \arrow[r] & k \arrow[r] \arrow[d, "\psi"] & H'(V) \arrow[r] \arrow[d] & V \times V^* \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & \mathbb{C}_1^\times \arrow[r] & H(V) \arrow[r] & V \times V^* \arrow[r] & 1
\end{tikzcd}$$

$B_0'(V) = \{ (\sigma, f) : \cdots \quad f : \text{quad form} \}$

(ch $k \neq 2 \Rightarrow f$ invariant under $[-1]$)

$\rightsquigarrow \gamma(f)$ for nondegenerate quad. form f on V.

1. $f_1 \sim f_2 \Longrightarrow \gamma(f_1) = \gamma(f_2)$
2. $\gamma(f_1 \oplus f_2) = \gamma(f_1) \cdot \gamma(f_2)$
3. f hyperbolic $\Longrightarrow \gamma(f) = 1$ $\quad (f \sim x_1x_2 + x_3x_4 + \cdots)$

i.e. γ depends only on the image of f in the Witt group $W(k)$.

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k : local field. $\mathrm{char}(k) \neq 2$.
V : finite dimensional vector space over k
$\psi : k \longrightarrow \mathbb{C}_1^\times$ nontrivial additive char.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & k & \longrightarrow & H_\psi(V) & \longrightarrow & V \times V^* & \longrightarrow & 1 \\ & & \downarrow \psi & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathbb{C}_1^\times & \longrightarrow & H(V) & \longrightarrow & V \times V^* & \longrightarrow & 1 \end{array}$$

$B_0'(V) \xrightarrow{\sim} Sp(V)$ k-linear automorphism of $V \times V^*$ which preserves the symplectic form.

$\rightsquigarrow \gamma(f)$ f non-degenerate quad. form on V
depends on ψ.

(Easy) <u>Fact</u> $\gamma(f)$ depends only on the isomorphism classes of the quad form f (by functorality.)

<u>Fact</u>. $\gamma(f_1 \oplus f_2) = \gamma(f_1) \cdot \gamma(f_2)$.

cf. Scharlau <u>Witt groups</u> k: field. $\mathrm{char}(k) \neq 2$ V/k.
q : non-degenerate quad form.

- (Witt) If $(V_1, q_1) \oplus (V_2, q_2) \cong (V_1, q_1) \oplus (V_3, q_3)$
$\Longrightarrow (V_2, q_2) \cong (V_3, q_3)$ (cancellation)

- hyperbolic forms $\cong$ $x_1 x_2 + \cdots + x_{2n-1} x_{2n}$
if a quad form q has the property that $\exists v \neq 0.\ q(v) = 0$, we call q is <u>isotropic</u>.

If it can't be done, then we say q is <u>anisotropic</u>.

<u>Fact</u>. if q ; $\exists v \neq 0.\ q(v) = 0 \Rightarrow \exists w \neq 0.\ q(w) = 0.$
$q(v, w) = 1.$

$\rightsquigarrow$ hyperbolic.

Witt group $\mathrm{Witt}(k) = \left\{ \begin{array}{lll} \text{generators} : & \text{isomorphism classes of non-degenerate quadratic forms.} \\ \text{relations} : & [q_1] + [q_2] = [q_1 \oplus q_2] \\ & \text{hyperbolic} = 0. \end{array} \right\}$

• $\gamma(f)$ depends only on the image of f in $\mathrm{Witt}(k)$. i.e

<u>Lemma</u> f hyperbolic $\Longrightarrow \gamma(f) = 1$.

<u>proof</u> $\dim_k V = 2$. $f = x_1 x_2$. Need to show $\gamma(f) = 1$.

$$\begin{array}{ccc} V & \xrightarrow[\sim]{\beta} & V^* \\ \cup\!\! \mid & & \cup\!\! \mid \\ \Gamma & \xrightarrow{\sim} & \Gamma^\perp \end{array} \qquad V/\Gamma \cong V^*/\Gamma^\perp$$

$f: V \longrightarrow \mathbb{C}_1^\times$ attaches β.

$\exists$ θ lifting. genuine representation $\therefore \gamma(f) = 1$.

Now $V = k \oplus k \xrightarrow{\beta} V^* = k \oplus k$

$\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (attached to $f = x_1 x_2$)

Pick $\Gamma = k \oplus (0) \subset V \xrightarrow[\sim]{\beta} (0) \oplus k = \Gamma^\perp$.

So we have the situation above. done $\square$.

<u>Properties of $\gamma(f)$</u>.

<u>Facts</u>. • $\gamma(f) \in \mu_8$

• $k = \mathbb{C}$, then $\gamma(f) = 1$. (proved.)

($Sp(\mathbb{C}_{2n})$ is 1-conn.)

• $k = \mathbb{R}$ (boils down to Fourier transform of 1 var.)

$\psi: \mathbb{R} \longrightarrow \mathbb{C}_1^\times$ $\psi(x) = e^{2\pi\sqrt{-1}ax}$

$\Longrightarrow \gamma(q_1) = e^{\frac{1}{4}\pi\sqrt{-1}\,\mathrm{sgn}(a)}$ $q_1 = x^2$

• $\gamma(f)^2 = \underbrace{(\mathrm{disc}(f), -1)_k}_{\text{Hilbert norm symbol}} \cdot \gamma(q_1)^{2\dim V}$ $\mathrm{disc}(f) \in k^\times/(k^\times)^2$

• If f = norm form of a quaterion division algebra.

then $\gamma(f) = -1$.

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Recall: properties of $\gamma(f)$.

k: local field f: non-degenerate quad form.

- f = norm form of a quaternion division algebra / k
$\Longrightarrow \gamma(f) = -1$. ($f = x_1^2 - ax_2^2 - bx_3^2 + abx_4^2$).
(if $(a.b)_k = 1 \Rightarrow \gamma(f) = 1$).

- Consequence : $\gamma(f)^2 = (\mathrm{disc}(f), -1)_k \cdot \gamma(q_1)^{2\cdot\dim(V)}$

Recall

$(a.b)_k = 1 \iff (a.b)$ (= quat. alg. gen. by $i.j$, $i^2=a$ $j^2=b$ $ij=-ji$) splits.

$\iff x_1^2 - ax_2^2 - bx_3^2 + abx_4^2$ is isotropic.

$\iff -ax_1^2 - bx_2^2 + abx_3^2$ is isotropic.

$\iff x_1^2 - ax_2^2 - bx_3^2$ is isotropic.

$\iff a$ is a norm in $k(\sqrt{b})$

$\iff b$ is a norm in $k(\sqrt{a})$

exer.

* k: non-arch. local field.

$\gamma(f) = g(\psi.f) / |g(\psi.f)|$.

where $g(\psi.f) = \int_V \psi(f(x))\,dx$ ← "symbolically" " Gaussian sum".

$\left(\int_{K^\times} \chi^{-1}(x)\,\psi(x)\,d^\times x = \varepsilon(\chi.\psi.dx)\right)$

$= \int_M \psi(f(x))\,dx$

$M \subseteq V$. compact open. sufficient large.

$= m(\Gamma) \sum_{x \in M/\Gamma} \psi(f(x))$ Γ: small enough.

Hint. $\int dx \int \Phi(x-y)\,\psi(f(y))\,dy$ Φ: test function.

$= \gamma(f) \cdot |\rho|^{-\frac{1}{2}} \int_V \Phi(x)\,dx$

Then take Φ = char. function for Γ small enough.

The Metaplectic Double Cover

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{C}_1^* & \longrightarrow & Mp(V) & \longrightarrow & Sp(V) & \longrightarrow & 1. \\
 & & \cup\!| & & \cup\!| & & \| & & \\
1 & \longrightarrow & \mu_2 & \longrightarrow & Sp^2(V) & \longrightarrow & Sp(V) & \longrightarrow & 1
\end{array}$$

topological ext.

Weil said algebraic ext

$Sp(V)$ is actually a group scheme.
$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;:\; \begin{array}{l} A^tD - B^tC = I \\ A^tB = B^tA \quad C^tD = D^tC \end{array} \right\}$.

$Sp(V)$ is unirational.
(i.e $\exists$ a rational cover. $\Rightarrow$ rotional points are dense
$\rightsquigarrow$ $Sp(V)$ becomes a linear algebraic group $/k$

But $Sp^2(V)$ is not k-rational points of linear algebraic groups:
it's existence has to do with the topology of k.

V/K K global finite dimensional.
$Sp(V)/K$ is simply connected as a(n) (linear) algebraic group

Adèlization

$$1 \longrightarrow \mathbb{C}_1^\times \longrightarrow Mp(V)_{\mathbb{A}} \longrightarrow Sp(V)_{\mathbb{A}} \longrightarrow 1$$

$$1 \longrightarrow \mathbb{C}_1^\times \longrightarrow Mp(V)_{K_v} \longrightarrow Sp(V)_{K_v} \longrightarrow 1. \qquad \forall v$$

take the restricted product:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mu_2 & \longrightarrow & Sp^2(V)_{K_v} & \longrightarrow & Sp(V)_{K_v} & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \prod_v' \mu_2 & \longrightarrow & \prod_v' Sp^2(V)_{K_v} & \longrightarrow & Sp(V)_{\mathbb{A}} & \longrightarrow & 1. \\
 & & \Big\downarrow \text{multi} & & \nearrow & & \| & & \\
1 & \longrightarrow & \mu_2 & \longrightarrow & Sp^2(V_{\mathbb{A}}) & \longrightarrow & Sp(V)_{\mathbb{A}} & \longrightarrow & 1 \\
 & & & & & \nwarrow^{\theta\text{-lift}} & \cup\!| & & \\
 & & & & & & Sp(V)_K & &
\end{array}$$

restricted product
with respect to $B_0'(V_{K_v}, \Gamma_{K_v})$ use $\Phi_{\Gamma_{K_v}}$ = char. function of Γ_{K_v} to form rest. tensor prod.
being integral element almost everywhere

Strategy to construct $\mathrm{Sp}^2(V)$:

$$1 \longrightarrow \mu_2 \longrightarrow \mathrm{Sp}^2(V) \longrightarrow \mathrm{Sp}(V) \longrightarrow 1.$$

<u>Need</u>. Construct

$$\begin{tikzcd} \mathrm{Mp}(V) \arrow[r, "\chi"] & \mathbb{C}_1^\times . \\ \mathbb{C}_1^\times \arrow[u, phantom, "\cup" sloped] \arrow[ur, "z \mapsto z^2"'] & \end{tikzcd}$$

Then take $\mathrm{Sp}^2(V) = \ker(\chi)$.

Construct such a χ :

$$\tilde{\gamma}_0(s) \cdot \tilde{\gamma}_0(s') = \underset{\overset{\|}{\lambda(s,s')}}{\gamma(f_0)} \cdot \tilde{\gamma}_0(s'')$$

need $\chi : B_0'(V) \longrightarrow \mathbb{C}^\times$ $\qquad B_0'(V) \subseteq \mathrm{Sp}(V)$ dense open

s.t. $\chi(s)\, \chi(s') = \gamma(f_0)^2 \cdot \chi(s \cdot s')$

<u>Def</u>. (Assume char $k \neq 2$)

$$\chi(s) = \left(\det\left(\tfrac{1}{2}\gamma\right),\ -1 \right)_k \cdot \gamma(q_1)^{2\dim(V)} .$$

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To Construct the Metaplectic Double Cover $Sp^2(V)$ of $Sp(V)$

suffices to produce a function $\chi : B_0'(V) \longrightarrow \mathbb{C}_1^\times$.

such that $\chi(s) \cdot \chi(s') = \gamma(f_0)^2 \cdot \chi(s'')$

$s = (\sigma, f)$. $s' = (\sigma', f')$. $s'' = (\sigma'', f'')$

$f_0(u) = f(0, u\gamma^{-1}) \cdot f'(u, -u\alpha'\gamma'^{-1})$

$\Longrightarrow$ get a homomorphism $Mp'(V) \longrightarrow \mathbb{C}_1^\times$

$$\begin{array}{ccc} Mp'(V) & \longrightarrow & \mathbb{C}_1^\times \\ \cup\!\!| & \xrightarrow{z \mapsto z^2} & \\ \mathbb{C}_1^\times & & \end{array}$$

$\sim$ then take $Sp^2(V) = \ker(\chi)$.

Def. / Formula $\chi(s) = (\det(\tfrac{1}{2}\gamma), -1)_k \cdot \gamma(g_1)^{2\cdot \dim V}$

(when char $k \neq 2$.)

from f_0, we can attach a symmetric homomorphism

$\beta_0 = \gamma^{-1}\gamma''\gamma'^{-1} : V \longrightarrow V^*$.

Stated $\gamma(f_0)^2 = (\operatorname{disc}(f_0), -1)_k \cdot \gamma(g_1)^{2\cdot\dim(V)}$

So now only need to show:

$*$. $(\det(\tfrac{1}{2}\gamma), -1)_k \cdot (\det(\tfrac{1}{2}\gamma'), -1)_k = (\operatorname{disc}(f_0), -1)_k \cdot (\det(\tfrac{1}{2}\gamma''), -1$

$\operatorname{disc}(f_0) = (\tfrac{1}{2})^{\dim(V)} \det(\gamma^{-1}\gamma''\gamma'^{-1})$

$= \det((\tfrac{1}{2}\gamma)^{-1}) \cdot \det((\tfrac{1}{2}\gamma'')) \cdot \det((\tfrac{1}{2}\gamma')^{-1})$

then use the multiplicative property of the Hilbert symbol.

and also $(a^2 b, c)_k = (b, c)_k$ for $a \in k^\times$. ✓

"$\det(\tfrac{1}{2}\gamma)$" : pick a basis for V, and look at the dual basis of V^*.

$\Rightarrow$ det is well-defined up to a square.

Have now the metaplectic double covers over local fields

$\rightsquigarrow$ make $\prod'$ (w.r.t. max compact subgroups from lattices.)

$\rightsquigarrow$ push $\prod' /\!\!/ \mu_2$ forward to $/\!\!/ \mu_2$ to get the metaplectic double cover of $Sp(V)_{\mathbb{A}}$.

(classical : modular forms of weight $\frac{1}{2}$.).

<u>Idea of "theta lifting" / "theta correspondence"</u>.

G_1, G_2 : reductive. $G_1 \times G_2 \subset Sp(V)$

& G_1, G_2 are centralizers of each other in $Sp(V)$

("dual reductive pair").

<u>example</u>. $(U, (\cdot|\cdot))$ non-deg. quad. space $\rightsquigarrow G_1 \cong O(n_1)$

$(V, \langle\cdot|\cdot\rangle)$ symplectic form $\rightsquigarrow G_2 \cong Sp(n_2)$

look at $(U \otimes V, (\cdot|\cdot) \otimes \langle\cdot|\cdot\rangle) \rightsquigarrow G_1 \times G_2 \cong Sp(n_1, n_2)$.

restrict the Weil representation to $G_1 \times G_2$:

get $\underset{\text{Weil repr.}}{\omega}|_{G_1 \times G_2} \cong \sum_i \pi_{1,i} \otimes \pi_{2,i}$

where $\pi_{j,i}$ are irred. repr. of G_j. $j = 1, 2$.

this gives a "correspondence" "$\pi_{1,i} \longrightarrow \pi_{2,i}$".

this "usually" is a manifestation of a Langlands lift.

" admissible irreducible automorphic representation of G

$\longleftrightarrow (W \longrightarrow {}^LG)$ ".

$$\begin{array}{c} {}^LG \\ \downarrow \\ {}^LH \end{array} \rightsquigarrow \overset{\text{aia.}}{\text{repr.}} \text{ of } G \rightsquigarrow \overset{\text{aia.}}{\text{repr.}} \text{ of } H.$$

This correspondence "often" can be written as:

$$\int_{G_2} \Theta(g_1, g_2)\, dg_2 \quad \text{or} \quad \int_{G_1} \Theta(g_1, g_2)\, dg_1$$

"Automorphic representations". $\Theta : \mathcal{S}(V) \longrightarrow \mathbb{C}$

then produce $f_v : g \longmapsto \Theta(gv)$. $v \leftrightarrow \Phi \in \mathcal{S}(V)$.

$\rightsquigarrow$ nice function $f_v \in {}_{Sp(V)}\backslash Mp(V)_{\mathbb{A}}$ automorphic forms.

$\Rightarrow$ the integration operators above:

$$\int_{G_2(\mathbb{R})\backslash G_2(\mathbb{A})} \Theta(g_1, g_2)\, dg_2 \quad \text{or} \quad \int_{G_1(\mathbb{R})\backslash G_1(\mathbb{A})} \Theta(g_1, g_2)\, dg_1$$

any non-zero result? $\leftrightsquigarrow$ special value of L-function is non-zero

4/16/97

" Weil representation for $SL_2(F)$ " $\quad F$: local field.

Detour. V/F. $\quad F$: local field.

$\rightsquigarrow$ get a representation of $Mp(V)$ or $Sp^2(V)$ on $\mathcal{S}(V)$.

Unless $V \cong \mathbb{C}$, this is NOT a repr. of $Sp(V)$.

But, $Sp(V)$ has a natural representation on

$\mathcal{S}(V)^* \otimes \mathcal{S}(V) \quad (\cong \mathcal{S}(V \oplus V)$ non-canonically)

Set $K = \begin{cases} \text{quadratic étale extension of } F & \begin{cases} \text{separable quad. field ext.} \\ F \times F \end{cases} \\ \text{central quaternion algebra } /F & \begin{cases} \text{quaternion division alg.} \\ M_2(F) \end{cases} \end{cases}$

$\tau_{K/F}$ = trace of K/F.

ι = standard involution

$\nu_{K/F}$ = norm of K/F

$\forall\, x \in K, \quad x + x^\iota = \tau(x)$

$\qquad\qquad\quad x\, x^\iota = \nu(x)$.

Fix $\psi : F \longrightarrow \mathbb{C}_1^\times$ non-trivial additive char.

$\rightsquigarrow f = \psi \circ \nu$. quadratic form on K

$\rightsquigarrow \gamma = \gamma_{K/F}$ ("basically $\gamma(f)$"). determined by

$$\hat{f}(y) = \gamma \cdot f(y^\iota)^{-1}. \quad (\text{might not be in } \mathbb{C}_1^\times)$$

where $\hat{f}(y) = \displaystyle\int_K f(x) \cdot (\psi \circ \tau)(xy)\, dx$.

Try. the homomorphism attached to $f = \psi \circ \nu$.

$f(x+y) \cdot f(x)^{-1} \cdot f(y)^{-1}$

$= \psi\left((x+y)(x^\iota + y^\iota) - xx^\iota - yy^\iota \right)$

$= \psi\left(x^\iota y + x y^\iota \right) = (\psi \circ \tau)(xy^\iota)$

i.e. $\iota : K \longrightarrow K \xrightarrow{\sim} K^*$ via $\psi \circ \tau$

is the homomorphism attached to f.

<u>Weil representation</u> of <u>$SL_2(F)$</u> <u>on</u> <u>$\mathcal{S}(K)$</u>

generators for $SL_2(F)$. (Steinberg (?) generators)

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad a \in F^\times.$$

$$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \quad z \in F$$

and $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$r = r_{K/F}$.

$$r\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) \Phi(x) = \omega_{K/F}(a) \cdot |a|_K^{\frac{1}{2}} \cdot \Phi(ax).$$

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}.$$

$|a|_F$ or $|a|_F^2$.

$\omega_{K/F} : F^\times \longrightarrow \mu_2$.

quadratic reciprocity (if a = norm then $\nu(a) = 1$ otherwise -1

$$r\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) \Phi(x) = \psi_F(z \cdot \nu(x)) \, \Phi(x).$$

Weil lift

$$r\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \Phi(x) = \gamma \cdot \hat{\Phi}(x^{\iota}).$$

<u>Proof</u>. Complete disgusting!

Now, take K/F : "non-split"

$\rho : K^\times \longrightarrow \mathrm{Aut}(V(\rho))$ irred. finite dim'l repr.

Let $\mathcal{S}(K, \rho) \subseteq \mathcal{S}(K) \otimes V(\rho)$.

$$\mathcal{S}(K, \rho) = \left\{ \Phi \in \underbrace{\mathcal{S}(K, V)}_{\boxed{\text{function on } K \text{ w/ value on } V}} ; \ \Phi(xh) = \rho^{-1}(h)\,\Phi(x) \ \ \forall h \in K^\times \text{ with } \nu(h) = 1 \right\}$$

Prop. (1) $\mathcal{S}(K, \rho)$ is invariant under $SL_2(F)$ (obvious from construction.)

(2) Can extend this to a representation r_ρ of G_+, where $G_+ = \{ g \in GL_2(F) ; \ \det(g) \in \nu(K^\times) \}$ such that the central character $\omega_{r_\rho} = \omega_\rho \cdot \omega_{K/F}$.

Prop. If ρ is unitary, ρ_0 is r_ρ.

If K is separable quad. field ext. $[G : G_+] = 2$
quat div. alg. $[G : G_+] = 1$
($\nu : K^\times \twoheadrightarrow F^\times$ in this case.)

Theorem A. K/F : quaternion division algebra.
then 1) $r_\rho = \pi(\rho)^{\oplus d}$ where $\pi(\rho)$: irred.
$d = \deg(\rho) = \dim V(\rho)$.

2) $d > 1 \Longrightarrow \pi(\rho)$ is supercuspidal.

3) $d = 1 \rightsquigarrow \rho = \chi \circ \nu_K$ then
$\pi(\rho) = \sigma(\chi\, |\cdot|_F^{1/2},\ \chi \cdot |\cdot|_F^{-1/2})$ special repr.

Theorem B Let $\pi(\rho) = \mathrm{Ind}_{G_+}^{G(F)} r_\rho$ (when K is sep. quad. field ext'n)
$\rho : K^\times \longrightarrow \mathbb{C}^\times$ Then $\pi(\rho)$ is supercuspidal
if $\nexists\, \chi : F^\times \longrightarrow \mathbb{C}^\times$ s.t. $\rho = \chi \circ \nu_K$ (i.e. $\rho \neq \rho^\iota$).
While if $\rho = \chi \circ \nu_K$ then $\pi(\rho)$ is a principal series representation $\pi(\rho) = \pi(\chi, \chi\,\omega_{K/F})$.

4/18/97

Relation to Classical Modular Forms

Modular forms $/ GL_2(\mathbb{Q})$. $\quad \Gamma \overset{\text{congruence}}{\subseteq} SL_2(\mathbb{Z})$

a modular form f of weight k is a holomorphic function

$f : \mathbb{H} \longrightarrow \mathbb{C}$ $\quad f|_k \gamma(z) = (\chi(a))(cz+d)^{-k} \cdot \det(\gamma)^{k/2} \cdot f\left(\frac{az+b}{cz+d}\right)$

s.t. $f|_k \gamma = f \quad \forall \gamma \in \Gamma$. $\quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$

and holomorphic at all cusps. $\quad \chi$: nebentypus char.

a cusp form is a modular form vanishing at all cusps.

Peterson inner product

$$(f_1, f_2)_{k\Gamma} = \int_{\Gamma \backslash \mathbb{H}} f_1(z) \cdot \overline{f_2(z)} \cdot y^k \frac{dx\,dy}{y^2}.$$

Detour $\quad M(\Gamma)$ elliptic modular curve.

$\mathcal{E}$ line bundle

$\varepsilon \circlearrowleft \downarrow \pi$ $\quad \varepsilon$: zero section

$M(\Gamma)$ $\quad \mathcal{E}^*(\Omega^1_{\mathcal{E}/M})^{\otimes k} \longleftarrow$ sections of this are meromorphic modular forms.

examples

(1) $\quad q \cdot \prod_{n=1}^{\infty} (1-q^n)^{24} = \Delta(q) = \Delta(\tau)$ $\quad$ abuse of notation Δ.

$q = e^{2\pi\sqrt{-1}\tau}$.

$= \sum_{n=1}^{\infty} \tau(n) \cdot q^n$ $\quad$ Ramanujan's τ-function

Ramanujan's conjecture: $|\tau(n)| = O(n^{11/2+\varepsilon})$ $\forall \varepsilon$ $(\because 11 = 12-$

Deligne proved: $|\tau(p)| \leq 2p^{\frac{11}{2}}$

(2) Eisenstein series:

$E_k = 1 + \frac{(-1)^k \cdot 4k}{B_k} \cdot \sum_{n=1}^{\infty} \sigma_{2k-1}(n) e^{2\pi\sqrt{-1}n\tau}$ $\quad \sigma_{2k-1}(n) = \sum_{d|n} d^{2k-1}$

$= \frac{1}{2\zeta(2k)} \sum_{(c,d)=1}' \frac{1}{(cz+d)^{2k}}$.

Classical Hecke Operators.

$$T_k(p)\, f(\tau) = p^{k-1} \sum_{\substack{a>0 \\ ad=p}} \sum_{b=0}^{d-1} \chi(a)\cdot f\left(\frac{a\tau+b}{d}\right) d^{-k}$$

$(p, N) = 1$

$\gamma \in \Gamma_0(N)$

$$\| \qquad p^{\frac{k}{2}-1} f \Big|_k \Gamma_0(N)\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma_0(N).$$

f: modular form with k, χ

$$\Gamma_0(N)\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma_0(N) = \bigcup_{(a,N)=1}\bigcup_{b=0}^{d-1} \Gamma_0(N)\, \sigma_a \begin{pmatrix}a&b\\0&d\end{pmatrix}$$

$$\sigma_a = \begin{pmatrix}a'&b'\\0&d'\end{pmatrix} \equiv \begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix} \pmod N$$

$\cap$ $SL_2(\mathbb{Z})$

$$\underline{(T(p)\, f, g) = \chi(p)\, (f, T(p)\, g).} \quad \text{Peterson inner product}$$

(Assuming one of f g is a cusp form).

$$D = G(\mathbb{R})/K_\infty \text{ — maximal compact subgroup}$$

$$\rightsquigarrow \quad G(\mathbb{R}) \overset{K_\infty}{\times} V$$

get a vector bundle.
lift back to the group $G(\mathbb{R})$

$f \rightsquigarrow \phi_f(g)$

cusp form

$g = \gamma g_\infty \cdot k_0 \in GL_2(\mathbb{A}_\mathbb{Q})$.

$\gamma \in GL_2(\mathbb{Q})$

$k_0 \in \prod_p K_p^N$.

$K_p^N = \{\begin{pmatrix}a&b\\c&d\end{pmatrix} \in GL_2(\mathbb{Z}_p)\, ;\, c \equiv 0\ (N)$

$g_\infty \in GL_2(\mathbb{R})_+$.

(by strong appro. & $h_\mathbb{Q} = 1$).

where

$$\phi_f(g) = f(g_\infty\cdot\sqrt{-1})\, (c\sqrt{-1}+d)^{-k} \cdot \det(g_\infty)^{\frac{k}{2}} \cdot \chi(k_0)$$

$k_0 = \begin{pmatrix}a&b\\c&d\end{pmatrix} \in K^N$. $\chi(k_0) = \chi(a)$

Then.

(1) ϕ_f is an automorphic form for $GL_2(\mathbb{A}_\mathbb{Q})$.

(2) $\phi_f(g k_0) = \phi_f(g)\cdot \chi(k_0)$ if $k_0 \in K^N$.

(3) $\phi_f\left(g\cdot\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\right) = \phi_f(g)\cdot e^{-\sqrt{-1}k\theta}$

(4) $\Delta\phi_f = -\frac{k}{2}\left(\frac{k}{2}-1\right)\phi_f$. (eigenfunction of Laplacian)

(5) $\phi_f(zg) = \chi(z)\cdot\phi_f(g)$ for $z \in Z(\mathbb{A})$.

(6) ϕ_f is cuspidal.

4/21/97

Adèlic formulation (continued) (won't be precise)

"classical way of thinking"

$\Gamma(1)\backslash\mathbb{H}$. $\Gamma(1) = SL_2(\mathbb{Z})$

projective system

$$\begin{tikzcd}
& \Gamma_1(6)\backslash\mathbb{H} \arrow[dl] \arrow[dr] & \\
\Gamma_0(2)\backslash\mathbb{H} \arrow[dr] & & \Gamma_0(3)\backslash\mathbb{H} \arrow[dl] \\
& \Gamma(1)\backslash\mathbb{H} &
\end{tikzcd}$$

$\Gamma_1(n) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \equiv d \equiv 1 \pmod n,\ c \equiv 0 \pmod n \}$

$\Gamma_0(n) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \pmod n \}$

$\Gamma(n) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \equiv d \equiv 1 \pmod n,\ b \equiv c \equiv 0 \pmod n \}$

$\Gamma(n) \leq \Gamma_1(n) \leq \Gamma_0(n)$.

$S(\Gamma_0(n), \chi)$

cusp forms for $\Gamma_0(n)$ (or $\Gamma_1(n)$, $\Gamma(n)$, etc.)

with nebentypus char. χ etc.

Have . Restrictions to smaller congruence subgroups

$\longrightarrow$ pull-back of cusp forms

- transform by an element $\gamma \in GL_2(\mathbb{Q})_+$ (i.e. $\det(\gamma) > 0$

What does this mean ?

Say $f(z)$: modular form w.r.t. Γ, a congruence subgroup, of weight k.

$\mathbb{H} \xrightarrow{\gamma} \mathbb{H}$.

pull-back : is $f(\gamma z) = \gamma^* f(z)$ a modular form ?

take $\gamma' \in \Gamma$,

is $f(\gamma\gamma' z) = j(\gamma', z)^k \cdot f(\gamma z)$? (an equiv. question)

$\gamma\gamma' z = (\gamma\gamma'\gamma^{-1})\gamma z$. if $\gamma\gamma'\gamma^{-1} \in \Gamma$, then we have transform'n law, then $f(\gamma\gamma' z) = j(\gamma\gamma'\gamma^{-1}, \gamma z)^k f(\gamma z)$. good.

$j(g.z) = (cz+d)\det(g)^{-\frac{1}{2}}$

(

But this is not a good thing to do.

Rather, look at $j(\gamma,z)^{-k}\cdot f(\gamma z)$.

$$\Rightarrow j(\gamma,\gamma' z)^{-k}\cdot f(\gamma\gamma' z)$$
$$= j(\gamma,\gamma' z)^{-k}\cdot j(\gamma\gamma'\gamma^{-1},\gamma z)^{k}\cdot f(\gamma z)$$
$$= j(\gamma,z)^{-k}\cdot j(\gamma,z)^{k}\cdot f(\gamma z) \quad \text{good. if } \gamma\gamma'\gamma^{-1}\in\Gamma.$$
(because j defines a 1-cocycle)

<u>Conclusion</u>. $j(\gamma,z)^{-k} f(\gamma z)$ is a modular form of weight k w.r.t. $\gamma^{-1}\Gamma\gamma$.

(a posteriori, w.r.t. $\Gamma\cap\gamma^{-1}\Gamma\gamma$)

$$\left(\begin{array}{c}\Gamma\backslash\mathbb{H}\\ \downarrow\\ \Gamma(1)\backslash\mathbb{H}\end{array}\right)_{\Gamma}$$

This is a projective system of coverings, with the covering group $SL_2(\hat{\mathbb{Z}})$.

$SL_2(\hat{\mathbb{Z}})$ is not a "big" group since it's compact.

Morphisms between projective systems:

$$\mathrm{Hom}(\varprojlim_I X_i, \varprojlim_I Y_i)$$

$$= \varprojlim_J \mathrm{Hom}(\varprojlim_I X_i, Y_j) = \varprojlim_J \varinjlim_I \mathrm{Hom}(X_i, Y_j)$$

$\therefore$ See that: $(GL_2(\mathbb{Q})_+)^\wedge$ operates on the tower, and also on {modular forms}

$$(GL_2(\mathbb{Q})_+)^\wedge \cong GL_2(\mathbb{A}_f) \qquad \mathbb{A}_f: \text{ finite adèles}$$

<u>Last time</u> Modular form $f \longleftrightarrow \phi$ on $GL_2(\mathbb{A}_\mathbb{Q})$

classic cusp form $f \longmapsto \phi$ adèlic cusp form

wt ≥ 2 satisfies some properties.

$$0=\int_{\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}} \phi\left(g\begin{pmatrix}1&b\\0&1\end{pmatrix}\right) db.$$

(classical Hecke theory) $\longrightarrow$ get a big subspace of cusp forms on $GL_2(\mathbb{A}_\mathbb{Q})$, invariant under $GL_2(\mathbb{A}_f)$) S

$\rightsquigarrow$ "classical Hecke theory" becomes the representation theory for $GL_2(\mathbb{A}_f)$ essentially.

Q. Which cusp forms appear in S ?

S is a representation of $GL_2(\mathbb{A}_\mathbb{Q})$ contained in the space of all cusp forms.

(ignoring the issue about center (normalization problem)).

$$\Longrightarrow \quad S = \widehat{\bigoplus_{i \in I}} \pi_i \qquad \text{(with multiplicity 1)}$$

$$\bigcap\!\!\mid\!\mid \qquad \pi_i = \bigotimes_{p \le \infty}{}' \pi_{i,p}$$

$$\underset{\substack{\shortparallel \\ \text{all cusp forms}}}{C^0} = \bigoplus_j \pi_j \qquad \pi_j = \bigotimes_{p \le \infty}{}' \pi_{j,p}$$

Condition for π_i in S. π_i occurs if and only if $\pi_{i,\infty}$ is in discrete series.

Maaß wave form $\rightsquigarrow$ $C^0 \setminus S$.

if the eigenvalue is $1/4$, $\rightsquigarrow$ very algebraic.

4/23/97

cusp form

$$f \xrightarrow{\text{lift to groups}} \phi_f \quad \text{function on } G(\mathbb{A}_\mathbb{Q})$$

hit f by Hecke operators, blah blah blah --

Canonically. a cusp form f defines an intertwining operator of weight k (≥ 2)

$$D_{k-1} \longrightarrow \{\text{cusp forms on } G(\mathbb{A}_\mathbb{Q})\}$$

$\uparrow$

discrete series for $G(\mathbb{R})$

knowing the image of the lowest weight vector $\rightsquigarrow$ knows everything.

In other words.

$$\left\{\begin{array}{l}\text{holomorphic} \\ \text{cusp forms} \\ \text{of wt } k\end{array}\right\} \xrightarrow{\sim} \mathrm{Hom}_{G(\mathbb{R})}\left(D_{k-1}, \{\text{cusp forms on } G(\mathbb{A}_\mathbb{Q})\}\right)$$

$\circlearrowleft$ $\{T_p\}$ Hecke operators $\qquad$ $\circlearrowleft$ $G(\mathbb{A}_f)$

$\rightsquigarrow$ $\{T_p\}$ & $G(\mathbb{A}_f)$ are essentially the same thing.

$$\rightsquigarrow \mathrm{Hom}_{G(\mathbb{R})}(\text{---}, \text{---}) = \widehat{\bigoplus_i} \pi_i, \quad \pi_i = \bigotimes_{p<\infty} \pi_{i,p}$$

Relations between the two pictures:

Let $\widetilde{T(p)}$ be the action given by the characteristic function of $G(\mathbb{Z}_p)\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}G(\mathbb{Z}_p)$

Then

$$\boxed{p^{\frac{k}{2}-1}\, \widetilde{T(p)}\, \phi_f = \phi_{T(p)\cdot f}}$$

Proposition Over a non-archimedean local field,

$\phi_0 \in \pi(|\cdot|^{s_1}, |\cdot|^{s_2})$ (unramified principal series.) (or spherical representation.)

(Φ_0 : fixed by $G(\mathcal{O}_F)$)

$$\Longrightarrow \quad \phi_0 * \chi_{G(\mathcal{O}_F)\left(\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right)G(\mathcal{O}_F)} \quad (= \tilde{T}(p)\,\Phi_0)$$

$$= q^{\frac{1}{2}} \cdot (q^{s_1} + q^{s_2}) \cdot \Phi_0 \qquad (s_1\ s_2 : \text{purely imaginary})$$

$\left(\left|\frac{a}{d}\right|^{\frac{1}{2}} \text{ in principal series.} \right)$

"Old" and "New" (forms)

F : non-arch. local field

(π, V) : inf dim'l adm. irred. repr. of $G(F) = GL_2(F)$

Theorem Let $\alpha\ \beta : \mathcal{O}_F^\times \longrightarrow \mathbb{C}_1^\times$ be two characters of $\mathcal{O}_F^\times$ s.t. $\alpha\beta = \omega_\pi|_{\mathcal{O}_F^\times}$. Then ① $\exists$ non-zero vector $v \in V$ s.t.

$$\begin{pmatrix} a & b \\ 0 & d\end{pmatrix} v = \alpha(a)\cdot\beta(d)\cdot v \qquad \begin{matrix}\forall\, a, d \in \mathcal{O}_F^\times \\ \forall\, b \in \mathcal{O}_F.\end{matrix}$$

② Let n = the smallest positive integer s.t. $\exists\ 0 \neq v \in V$ as in ① and invariant under $\begin{pmatrix} 1 & 0 \\ \tilde{\omega}^n\mathcal{O}_F & 1\end{pmatrix}$.

Then $n \geq \mathrm{cond}(\alpha\beta^{-1})$ (≥ 0)

③ Let $G_k = \begin{cases} GL_2(\mathcal{O}_F) & \text{if } k=0. \\ \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in G(F) \,\middle|\, a, d \in \mathcal{O}_F^\times,\ b \in \mathcal{O}_F,\ c \in \tilde{\omega}^k\mathcal{O}_F\right\} & \text{if } k \geq 1\end{cases}$

(generalized $\Gamma_0(\tilde{\omega}^k)$)

and let $X_k = \left\{ v \in V \,\middle|\, \pi\begin{pmatrix} a & b \\ c & d\end{pmatrix} v = \alpha\beta^{-1}(a)\,\beta(ad-bc)\,v,\ \forall \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in G_k \right\}$

Then $\dim(X_k) = \max(k-n+1, 0)$

$$X_k = \bigoplus_{r=0}^{k-n} \pi\begin{pmatrix} \tilde{\omega}^{-r} & 0 \\ 0 & 1\end{pmatrix} X_n \quad \text{if } k \geq n$$

($\therefore$ knowing X_k's will enable us to reconstruct (π, V))

(π, V) $\quad V \otimes_{\mathbb{C},\sigma} \mathbb{C}$ $\quad$ with $\sigma \in Aut(\mathbb{C}) \mid \pi^\sigma \cong \pi$.

$H = \{ \sigma \in Aut(\mathbb{C}) \mid \pi^\sigma \cong \pi \}$.
$\mathbb{C}^H =: K =$ " the field of definition of π ".

<u>Corollary</u> π can be realized over K.

(false for representation of finite groups.
true for GL_n as well)

Proof of Thm on 4/23/97:

① $\exists$ Kirillov model ; then
① $\iff \pi\begin{pmatrix} a & b \\ 0 & 1\end{pmatrix} v = \alpha(a)\cdot v$. $\quad (\because \alpha\beta = \omega_\pi|_{O_F^\times})$
choose $\psi : O_F \longrightarrow \mathbb{C}_1^\times$ with $\mathrm{cond}(\psi) = 0$
This means : $v \longleftrightarrow v(x)$, a function on $F^\times$ in $\mathcal{K}(\pi)$.
$v(x)$ has support in O_F.
$\therefore$ we can pick $v(x) \in \mathcal{S}(F^\times)$ with support in O_F.
s.t. $v(ax) = \alpha(a)\cdot v(x)$. ✓.

② the subgroup generated by $\begin{pmatrix} 1 & 0 \\ \varpi O_F & 1\end{pmatrix}$ + $\begin{pmatrix} 1 & O_F \\ 0 & 1\end{pmatrix}$ is

$$\left\langle \begin{pmatrix} 1 & 0 \\ \varpi^n O_F & 1\end{pmatrix}, \begin{pmatrix} 1 & O_F \\ 0 & 1\end{pmatrix} \right\rangle = \begin{cases} \left\{ \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(F) \;\middle|\; \begin{array}{l} c \equiv 0 \pmod{\varpi^n O_F} \\ a \equiv d \equiv 1 \pmod{\varpi O_F} \end{array} \right\} & \text{if } n \geq 0 \\ \text{(this generalizes } \Gamma_1(N)\text{.)} & \\ SL_2(F) & \text{if } n < 0. \end{cases}$$

$\leadsto$ ② follows immediately

③ Observe : $\pi\begin{pmatrix} \varpi^i & 0 \\ 0 & 1\end{pmatrix}$ sends $\begin{pmatrix} 1 & 0 \\ \varpi^k O_F & 1\end{pmatrix}$ - invariants
to $\begin{pmatrix} 1 & 0 \\ \varpi^{k+i} O_F & 1\end{pmatrix}$ - invariants $\quad$ ($\because$ conjugation.)
From the definition of n, we have
$$X_n \cap \pi\begin{pmatrix} \varpi^i & 0 \\ 0 & 1\end{pmatrix} X_{n+i-1} = (0) \qquad \forall\, i \geq 1.$$
$$\Longrightarrow \pi\begin{pmatrix} \varpi^{-i} & 0 \\ 0 & 1\end{pmatrix} X_n \cap X_{n+i-1} = (0) \Longrightarrow \text{q.e.d.} \qquad \#$$

Apply the theorem to $\alpha = $ trivial, $\beta = \omega_\pi|_{O_F^\times}$
then the n that comes out is called the <u>conductor</u> of π.
(if $n = 0$, π is unramified.)
(n measures the ramification of the representation π.)

Pick $v \in X_n \setminus (0)$ then we can regenerate π $\xrightarrow{\text{over}}$

<u>Ramanujan's conjecture</u>: $\pi = \bigotimes \pi_v$ cuspidal unitary automorphic rep.
Then each π_v is tempered, especially, for almost all v, π_v is a principal series induced from unitary characters. (False for general group)

Classical Case . $GL_2(\mathbb{Q})$:

$\pi = \pi_\infty \times \pi_f$. $\qquad \pi_\infty = D_{k-1}$.

$\qquad \pi_f = \bigotimes_p{}' \pi_p$.

Only finite numbers of π_p's are ramified.

Pick v_p to be a generator of X_{n_p}, $\forall p$.

$\Longrightarrow$ take $\bigotimes_p{}' v_p$ "new form"

gives an automorphic representation.