

Class field theory: K : number field

$$1. (1a) \mathbb{A}_K^\times / K^\times \xrightarrow{\text{rec}_K} \text{Gal}(K^{ab}/K)$$

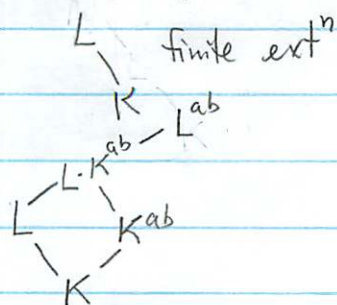
$$\downarrow \quad \nearrow \cong$$

$$\pi_0(\mathbb{A}_K^\times / K^\times)$$

(1b) L/K finite abelian extⁿ

$$\Rightarrow \mathbb{A}_K^\times / K^\times \cdot N_{L/K}(L^\times) \xrightarrow[\cong]{\text{rec}_{L/K}} \text{Gal}(L/K)$$

$$(1c) \text{ (Functoriality 1)} \quad \mathbb{A}_K^\times / K^\times \xrightarrow{\text{rec}_K} \text{Gal}(K^{ab}/K)$$

 $\sim$

$$\mathbb{A}_L^\times / L^\times \xrightarrow{\text{rec}_L} \text{Gal}(L^{ab}/L)$$

$$N_{L/K} \downarrow$$

$$\mathbb{A}_K^\times / K^\times \xrightarrow{\text{rec}_K} \text{Gal}(K^{ab}/K)$$

 $\downarrow$ restriction to K^{ab}

2. Characterization by Frobenii:

 L/K finite abelian, $v \in \Sigma_K$ unramified in L Then: $\text{rec}_{L/K}(\pi_v) =$ the arithmetic Frobenius of v in $\text{Gal}(L/K)$

$$\text{i.e. } \bigcap_{\substack{w|v \\ \sum_L}} \mathbb{A}_w^\times / v$$

 $\text{rec}_{L/K}(\pi_v) =$ the element in $D_{w/v}$ which induces $x \mapsto x^{N_v}$ on $K(w) = \mathbb{Q}_w / w \mathbb{Q}_w$ Thus: The ^{key} point of the existence of rec_K is the following "crude form" of the reciprocity law: L/K finite abelian extⁿ $\exists \varepsilon > 0$ s.t. $\forall a \in K^\times$ with $|a-1|_v < \varepsilon$ for each ramified (in L/K) or arch. place v ,have $\prod_{v \in \Sigma_{K, \text{co}} \cup \{\text{ramified}\}} \text{Frob}_v^{\text{ord}_v(\pi_v)} = 1$ in $\text{Gal}(L/K)$

(1d) (Functoriality 2)

$$\begin{array}{ccc}
 L & & \\
 \downarrow \text{finite ext}^n & & \\
 K & &
 \end{array}
 \quad
 \begin{array}{ccc}
 A_K^x / K^x & \longrightarrow & \text{Gal}(K^{ab}/K) = \text{Gal}(\bar{K}/K)^{ab} \\
 \downarrow \text{induced by } K \hookrightarrow L \text{ and } A_K^x \hookrightarrow A_L^x & & \downarrow \text{transfer } (\text{Gal}(\bar{K}/K), \text{Gal}(\bar{K}/L)) \\
 A_L^x / L^x & \longrightarrow & \text{Gal}(L^{ab}/L) = \text{Gal}(\bar{K}/L)^{ab}
 \end{array}$$

General defⁿ of transfer $H \leq G$ G, H : (topological) groups
 finite index

def $\text{transfer}_{(G,H)} : G^{ab} \longrightarrow H^{ab}$

Choose a section $s : H \backslash G \longrightarrow G$ of $G \twoheadrightarrow H \backslash G$

define $h_{\bar{x}, g}$ by $s(\bar{x}) \cdot g = h_{\bar{x}, g} \cdot s(\bar{x}g)$
 $\bar{x} \in H \backslash G, g \in G$
 $\cap$
 H

Define $\text{transfer}_{(G,H)} : G^{ab} \longrightarrow H^{ab}$ by

$$\text{transfer}_{(G,H)}(g \cdot [G, G]) := \prod_{\bar{x} \in H \backslash G} h_{\bar{x}, g}$$

(Need to check: well-defined)

History of Class field theory

Kronecker 1853, 1877, Weber 1886, 1886 $\mathbb{Q}^{ab} = \mathbb{Q}(\mu)$

Hilbert Zahlbericht 1897 reciprocity law as a product formula

Takagi 1920, 1922

New defⁿ of class fields K/k finite extⁿ of nber fields.

$$C = Cl_{K/k} = \left\{ \text{fractional ideals } I \text{ prime to } m \right\} / \left\langle (a), a \equiv 1 \pmod{m} \right\rangle$$

$N_{K/k}(\text{fractional ideals for } K, \text{ prime to } m)$

 m = modulus, sufficiently large

$$\text{always: } \#C_{K/k} =: h \leq n := [K:k]$$

def: A finite abelian extⁿ K/k is a class field
if $\#C_{K/k} = [K:k]$

Takagi: The class field relation establishes a bijection
between finite abelian extensions of k and
congruence class groups

Artin 1924, 1927

Artin L-functions

Hasse Bericht
1926, 1927 1930

Artin, Furtwängler 1930 principal divisor theorem

Chevalley 1933, 1940

Inequalities in class field theory

Thm 1 ("2nd inequality" = universal norm index inequality)

K/k : finite Galois extⁿ of number fields

$$\text{Then } [A_K^x : \underbrace{k^x \cdot N_{K/k} A_K^x}_{\text{open subgroup of } A_K^x}] \leq [K : k]$$

Thm 2 (norm index for cyclic extⁿ of local fields)

L/K finite cyclic extⁿ of local fields of char. 0

$$\text{Then } \left. \begin{aligned} [O_K^x : N_{L/K} O_L^x] &= e(L/K) \\ [K^x : N_{L/K} L^x] &= [L : K] \end{aligned} \right\} \begin{array}{l} \text{True for archimedean} \\ \text{local fields, trivially} \end{array}$$

Thm 3 ("first inequality")

L/K finite cyclic extension of number fields. Then

$$[A_K^x : K^x \cdot N_{L/K} A_L^x] \geq [L : K]$$

Thm 3' L/K finite cyclic extⁿ of number fields as in Thm 3. Then

$$[A_K^x : K^x \cdot N_{L/K} A_L^x] = [L : K] \cdot [K^x \cap N_{L/K} (A_L^{\tilde{S}})^x : N_{L/K} \underbrace{(L^x \cap A_L^{\tilde{S}})^x}_{\substack{\uparrow \\ \tilde{S}\text{-units in } L}}]$$

Where $\underbrace{S}_{\Sigma_{K,\infty}} =$ a sufficiently large finite subset of Σ_K . $\tilde{S}$ -units in L

$$(A_K^{\tilde{S}})^x \stackrel{\text{def}}{=} \left(\prod_{v \in S} K_v^x \right)^x \times \left(\prod_{v \notin S} O_v^x \right)$$

$\tilde{S} = \{w \in \Sigma_L \mid w \text{ lies above some element of } S\}$,

$$(A_L^{\tilde{S}})^x \stackrel{\text{def}}{=} \left(\prod_{w \in \tilde{S}} L_w^x \right)^x \times \left(\prod_{w \notin \tilde{S}} O_w^x \right)$$

Recall Hilbert's Satz 90: L/K ^{finite} cyclic, $n = [L:K]$, $\text{Gal}(L/K) = \langle \sigma \rangle$
 If $x \in L^\times$ with $N_{L/K}(x) = 1$, then $\exists y \in L^\times$ s.t. $x = y \cdot \sigma(y)^{-1}$

Lemma (Herbrand)

A : abelian group $\partial_0, \partial_1 \in \text{End}(A)$, $\partial_0 \circ \partial_1 = \partial_1 \circ \partial_0 = 0$

$\cup$

B : subgroup, stable under ∂_0 and ∂_1

$H_0(A) = A[\partial_0] / \partial_1 A$, $H_1(A) = A[\partial_1] / \partial_0 A$; similarly for $H_i(B), H_i(C)$
 $i = 0, 1$

(a) We have a cyclic exact sequence

$$\begin{array}{ccccc} & & H_0(A) & \rightarrow & H_0(C) \\ & \nearrow & & & \searrow \partial \\ H_0(B) & & & & H_1(B) \\ & \nwarrow \partial & & & \nearrow \\ & & H_1(C) & \leftarrow & H_1(A) \end{array}$$

(b) Define $Q(A) = \frac{\# H_0(A)}{\# H_1(A)}$ if and when $H_0(A)$ and $H_1(A)$ are both finite

Similarly for $Q(B)$ and $Q(C)$

Then if any two of $Q(A), Q(B), Q(C)$ are defined, so is the third, and

$$Q(A) = Q(B) \cdot Q(C)$$

$$\text{Equivalently: } |H_0(A)| \cdot |H_1(B)| \cdot |H_1(C)| = |H_0(C)| \cdot |H_1(A)| \cdot |H_0(B)|$$

(c) Suppose that $\# C < \infty$, then $Q(C) = 1 \Rightarrow Q(A) = Q(B)$

On the finite group C ,

have $\partial_0, \partial_1 \in \text{End}(C)$ s.t. $\partial_0 \circ \partial_1 = \partial_1 \circ \partial_0 = 0$

(If either is defined)

∂_0, ∂_1 defines a chain complex

$$\cdots \rightarrow C \xrightarrow{\partial_0} C \xrightarrow{\partial_1} C \xrightarrow{\partial_0} C \xrightarrow{\partial_1} C \rightarrow \cdots$$

$$H_0(C) = \frac{\text{Ker}(\partial_0)}{\partial_1 C}$$

Pf of (c)

$$|C| = |\text{Ker } \partial_0| \cdot |\partial_0 C| \stackrel{\downarrow}{=} |\partial_1 C| \cdot |H_0(C)| \cdot |\partial_0 C|$$

Similarly $|C| = |\partial_0 C| \cdot |H_1(C)| \cdot |\partial_1 C|$

$$\Rightarrow |H_0(C)| = |H_1(C)| \quad \text{q.e.d.}$$

Proof of Thm 2

Norm index of finite cyclic extⁿ of local fields

L/K : cyclic extⁿ. L, K : ^{nonarch.} local fields

Let $\partial_0: L^\times \rightarrow L^\times \quad \partial_0: x \mapsto x \cdot \sigma(x)^{-1}$

$\partial_1: N_{L/K} \cdot L^\times \rightarrow L^\times$

(i) For $L^\times \xrightleftharpoons[\partial_1]{\partial_0} L^\times \quad H_1(L^\times) = 0 \quad (\text{Hilbert 90})$

$$H_0(L^\times) = K^\times / N_{L/K}(L^\times)$$

(ii) For $\mathcal{O}_L^\times \xrightleftharpoons[\partial_1]{\partial_0} \mathcal{O}_L^\times \quad H_0(\mathcal{O}_L^\times) = \mathcal{O}_K^\times / N_{L/K}(\mathcal{O}_L^\times)$

$$\text{Ker}(\partial_1) = \partial_0(L^\times) \quad \text{--- Hilbert 90}$$

$$H_1(\mathcal{O}_L^\times) = \frac{\partial_0(L^\times)}{\partial_0(\mathcal{O}_L^\times)}$$

Consider the diagram and the Kernel-cokernel exact sequence,

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{O}_L^\times & \hookrightarrow & L^\times & \xrightarrow{\text{ord}_L} & \mathbb{Z} \rightarrow 0 \\ \downarrow \times & & \downarrow \partial_0 & & \downarrow \partial_0 & & \downarrow 0 \\ x \cdot \sigma(x) & & 0 & \rightarrow & \mathcal{O}_L^\times & \rightarrow & L^\times \rightarrow \mathbb{Z} \rightarrow 0 \end{array}$$

Get:

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{O}_K^\times & \rightarrow & K^\times & \xrightarrow{\text{ord}_K} & \mathbb{Z} \rightarrow \mathcal{O}_L^\times / \partial_0 \mathcal{O}_L^\times \rightarrow L^\times / \partial_0 L^\times \rightarrow \mathbb{Z} \rightarrow 0 \\ \downarrow & & \downarrow \text{ord}_K & & \parallel & & \\ 0 & \rightarrow & \mathbb{Z} & \xrightarrow{x \cdot e_{L/K}} & \mathbb{Z} & & \end{array}$$

$$\Rightarrow 0 \rightarrow \mathbb{Z} / e_{L/K} \mathbb{Z} \rightarrow \mathcal{O}_L^\times / \partial_0 \mathcal{O}_L^\times \rightarrow \mathcal{O}_L^\times / \partial_0 L^\times \rightarrow 0$$

Conclude: $\mathfrak{o}_0(L^x)/\mathfrak{o}_0(\mathcal{O}_L^x) \cong \mathbb{Z}/e_{L/K}\mathbb{Z}$

$$\parallel$$

$$H_1(\mathcal{O}_L^x)$$

On the other hand, $\exists$ $\text{Gal}(L/K)$ -invariant open subgroup

$U \subseteq \mathcal{O}_L^x$ s.t. $\log: U \longrightarrow V \subseteq (\mathcal{O}_K, +)$

is a $\text{Gal}(L/K)$ -equivariant isomorphism, and

$$V \cong \mathcal{O}_K[\text{Gal}(L/K)]$$

$\uparrow$
 $\mathcal{O}_K$ -linear

$\text{Gal}(L/K)$ -equivariant.

$$\Rightarrow H_i(U) = 0 \text{ for } i=1, 2, \text{ and}$$

$$Q(\mathcal{O}_L^x) = Q(U) = 1$$

$$\Rightarrow [\mathcal{O}_K^x : N_{L/K}(\mathcal{O}_L^x)] = |H_0(\mathcal{O}_L^x)| = [H_1(\mathcal{O}_L^x)] = e_{L/K}$$

$$0 \rightarrow \mathcal{O}_L^x \rightarrow L^x \xrightarrow{\text{ord}_L} \mathbb{Z} \rightarrow 0$$

$$\leadsto Q(L^x) = Q(\mathcal{O}_L^x) \cdot Q(\mathbb{Z}) = [L:K]$$

$\mathfrak{o}_0 = 0$, $\mathfrak{o}_1 = \text{multiplication by } [L:K]$

Have proved Thm 2 QED.

Corollary of Thm 2 (and its proof)

1) L/K finite Galois extⁿ of local fields of char. 0.

$$\Rightarrow [O_K^x : N_{L/K}(O_L^x)] < \infty \quad Q(O_K^x) = 1$$

$$[K^x : N_{L/K}(L^x)] = [L : K]$$

2) L/K unramified extⁿ of local fields. (therefore cyclic)
^{locally compact}
 Then $O_K^x = N_{L/K}(O_L^x)$.

Cor: L/K finite Galois extⁿ of number fields.

Then $N_{L/K} : \mathbb{A}_L^x \rightarrow \mathbb{A}_K^x$ sends each open nbd of $1_{\mathbb{A}_L^x}$ to an open nbd of $1_{\mathbb{A}_K^x}$.

In particular, $N_{L/K}(\mathbb{A}_L^x)$ is an open subgroup of $\mathbb{A}_K^x$.

Remark Cannot expect that $[K : N_{L/K}(L^x)] < \infty$.

False in the first example $\mathbb{Q}(\sqrt{-1})/\mathbb{Q}$.
 simplest

Proof of Thm 1 (universal norm index inequality)

$$L/K \text{ finite Galois} \Rightarrow [A_K^x : (K^x \cdot N_{L/K}(\mathbb{A}_L^x))] \leq [L : K]$$

Let $A := \mathbb{A}_K^x / K^x \cdot N_{L/K}(\mathbb{A}_L^x)$ (not $\text{Gal}(L/K)$)

Consider $L(s, \chi)$ and $\log L(s, \chi)$

^{holomorphic on $\mathbb{C} \setminus \{1\}$}

$L(1, \chi) \neq 0$ and $L(s, \chi)$ entire if $\chi \neq \text{trivial}$

simple pole at $s=1$ if $\chi=1$

Previous proof (last semester)
 of non-vanishing of $L(1, \chi)$
 used class field theory.

Let $S = \Sigma_{K, \infty} \cup \{v \in \Sigma_K \mid v \text{ ramifies in } L\}$

$$\log L(s, X) = \sum_{v \in \Sigma_K, S} \chi(\pi_v) \cdot N_v^{-s} + \underbrace{\frac{g(s, X)}{[L:K]}}_{\text{holomorphic in a nbd of } s=1}$$

Consider

$$\sum_{\chi \in \text{Hom}(A, \mathbb{C}^\times)} \log(s, X) = |A| \cdot \sum_{\substack{v \in \Sigma_K, S \\ \pi_v \in (K^\times \cdot N_{L/K}(A_L^\times), \text{ i.e. image of } \pi_v \text{ in } A \text{ is } 1}} N_v^{-s} + g_1(s)$$

Key observation Let $T = \{v \in \Sigma_K \mid v \notin S, v \text{ splits completely in } L\}$

If $v \in T$, then $(1 \dots \pi_v \dots 1) \mapsto \pi_v \in \text{Nm}_{L/K}(A_L^\times)$

Consider $s > 1, s \rightarrow 1^+$

$$\begin{aligned} \Rightarrow \sum_{\chi \in \text{Hom}(A, \mathbb{C}^\times)} \log(s, X) &\geq |A| \cdot \sum_{\substack{v \in \Sigma_K, S \\ v \text{ splits completely in } L/K}} N_v^{-s} + \underbrace{g_2(s)}_{\text{holomorphic in a nbd of } s=1} \\ &\quad - \log(s-1) \\ &= \frac{|A|}{[L:K]} \cdot (-\log(s-1)) + g_3(s) \end{aligned}$$

$$\Rightarrow \frac{|A|}{[L:K]} \leq 1 \quad \text{i.e.} \quad |A| \leq [L:K]$$

$$[A_K^\times : (K^\times \cdot N_{L/K}(A_L^\times))]$$

How to eliminate the nonvanishing of $L(1, X)$:

Observation Do not need to use non-vanishing of $L(1, X)$:
will get

$$\left(1 - \sum_{\chi \neq 1} \text{ord}_s L(s, X)\right) \cdot (-\log(s-1)) \geq \frac{|A|}{[L:K]} (-\log(s-1)) + g_3(s)$$

This shows: $L(1, X) \neq 0$

$\forall \chi \in \text{Hom}(A, \mathbb{C}^\times)$

Thm 1 proved

Have shown: $\forall \chi \in \text{Hom}(\mathbb{A}_K^\times / K^\times \cdot N_{M/L/K} \mathbb{A}_L^\times, \mathbb{C}^\times)$, $L(1, \chi) \neq 0$

and $|\mathbb{A}_K^\times / K^\times \cdot N_{L/K} \mathbb{A}_L^\times| \leq [L:K]$.

Remark: CFT $\Rightarrow \forall$ idele class character $\chi: \mathbb{A}_K^\times / K^\times \rightarrow \mathbb{C}^\times$ of finite order, $L(1, \chi) \neq 0$ holds.

However one still needs Hadamard-de la Vallée-Poussin argument to show that $L(s, \chi) \neq 0$ for all s with $\text{Re}(s) = 1$, even just for characters of finite order.

Prop 4: Let G be a finite group action on a finite set S

Let $A \subseteq \mathbb{R}[S]$ be a cocompact $\mathbb{Z}$ -lattice in $\mathbb{R}[G]$
 $\uparrow$
 $\mathbb{R}[G]$ -module stable under G

Then $\exists$ a $\mathbb{Z}[G]$ -submodule $M \subseteq A$ s.t. $M \cong \mathbb{Z}[S]$
 $\uparrow$
 $\mathbb{Z}[G]$ -linear
 (necessarily: $[A:M] < \infty$).

Pf: Give $\mathbb{R}[S]$ the G -invariant inner product with elements of S as an orthonormal basis.

Pick $C > 0$ s.t. $\forall x \in V = \mathbb{R}[S]$, $\exists \lambda \in A$ with $\|x - \lambda\| < C$. Pick $t \in \mathbb{R}_{>0}$ large.

$\forall v \in G \cdot S$, pick a representative w of v in S ,
 and an element $z_v \in A$ s.t. $\|t \cdot w - z_v\| < C$

(z_v depends on the choice of w above v , and on t , and there may be more than one element of A close to $t \cdot w$ within C)

Use this family $(z_v)_{v \in G \cdot S}$ of elements in A to define a family $(Y_w)_{w \in S}$ of elements of A by:

$$Y_w = \sum_{\substack{\sigma \in G \\ \sigma \cdot s(\pi(w)) = w}} \sigma \cdot z_{\pi(w)} \quad \Rightarrow \quad \tau Y_w = Y_{\tau w} \quad \forall \tau \in G, \forall w \in S$$

$$S \xrightarrow[\pi]{\tau} G \backslash S$$

Must show that the Y_w 's are linearly independent over $\mathbb{R}$.

Suppose $\sum_{w \in S} a_w Y_w = 0$

Have $Z_{\pi(w)} = t X_w + B_{\pi(w)} \quad |B_{\pi(w)}| < c$

$$\leadsto Y_w = \sum_{\substack{\sigma \in G \\ \sigma \cdot S(\pi(w)) = w}} \sigma Z_{\pi(w)} = t \sum_{\substack{\sigma \in G \\ \sigma \cdot S(\pi(w)) = w}} X_w + \sum_{\substack{\sigma \in G \\ \sigma \cdot S(\pi(w)) = w}} B_{\pi(w)}$$

$\parallel$
 B'_w

$$Y_w = t \cdot m_w X_w + B'_w \quad \|B'_w\| \leq |G| \cdot c$$

$$m_w = |\{\sigma \in G \mid \sigma \cdot S(\pi(w)) = w\}|$$

$$= |\text{stab}_G(w)|$$

$$\leadsto 0 = \sum_{w \in G} a_w Y_w = t \cdot \sum_{w \in G} a_w m_w X_w + \sum_{w \in G} a_w B'_w$$

Thus if $t > |G| \cdot c$, we get $a_w = 0 \quad \forall w \in G$

QED

Corollary 1 In the situation of the above proposition.

(i) Have

$$\mathbb{Z}[S] \cong \bigoplus_{v \in G \setminus S} \mathbb{Z}[G] \otimes \mathbb{Z}$$

$\uparrow$ as $\mathbb{Z}[G]$ -module

$\mathbb{Z}[G_{S(v)}]$
"stab_G(S(v)), "decomposition group of S(v)"

$$(ii) \quad Q(A) = Q(\mathbb{Z}[S]) = \prod_{v \in G \setminus S} |G_{S(v)}|$$

Corollary 2 K/k finite cyclic extⁿ of number fields. $G = \text{Gal}(K/k)$

Then:

$$\sum_{k \in S} \leq S \leq \sum_{k \in S} \quad \tilde{S} = S^K$$

$$Q(G, \bigoplus_{k \in S} \mathbb{Q}_{k, \tilde{S}}^{\times}) = \frac{1}{[K:k]} \cdot \prod_{v \in S} |D_v|$$

$$K \cap \prod_{w \notin S} \mathbb{Q}_w^{\times}$$

$\uparrow$
#(decomposition group for v)

proof of Thm 3 (global norm index for cyclic extⁿ of number fields)
"first inequality".

L/K cyclic extⁿ of number fields

Pick $\tilde{S} \subseteq \sum_L$ $\tilde{S}$: stable under $G = \text{Gal}(L/K)$ s.t.
 $\sum_{L, \text{ramified}} \cup \sum_{L, \infty}$ finite

$$A_L^x = L^x \cdot A_{L, \tilde{S}}^x \quad \text{where} \quad A_{L, \tilde{S}}^x = \prod_{w \in \tilde{S}} L_w^x \times \prod_{w \notin \tilde{S}} O_w^x$$

$$\Rightarrow A_L^x / L^x = L^x \cdot A_{L, \tilde{S}}^x / L^x \cong A_{L, \tilde{S}}^x / (L^x \cap A_{L, \tilde{S}}^x)$$

$S = \text{image of } \tilde{S} \text{ in } \sum_K$

$O_{L, \tilde{S}}^x \leftarrow \tilde{S}\text{-units}$

$$Q(G, A_L^x / L^x) = Q(A_{L, \tilde{S}}^x / O_{L, \tilde{S}}^x) = Q(A_{L, \tilde{S}}^x) / Q(O_{L, \tilde{S}}^x)$$

if $Q(A_{L, \tilde{S}}^x)$ and

$Q(O_{L, \tilde{S}}^x)$ are both defined

$$(a) \quad A_{L, \tilde{S}}^x = \prod_{w \in \tilde{S}} L_w^x \times \prod_{w \notin \tilde{S}} O_w^x \Rightarrow H^i(G, A_{L, \tilde{S}}^x) = H^i(G, \prod_{w \in \tilde{S}} L_w^x) \quad i=0, -1$$

$$\text{Shapiro's Lemma} \Rightarrow \left| H^i(G, \prod_{w/v} L_w^x) \right| = \left| H^i(G_w, L_w^x) \right| = \begin{cases} 1 & i=1 \\ |G_w| = [L_w : K_v] & i=0 \end{cases}$$

pick one w/v

Thm 2, local norm index

$$\Rightarrow Q(A_{L, \tilde{S}}^x) = \prod_{v \in S} [L_w : K_v]$$

a place of L above v

$$(b) \quad \text{Cor 2 of Proposition 4} \Rightarrow Q(O_{L, \tilde{S}}^x) = \frac{\prod_{v \in S} [L_w : K_v]}{[L : K]}$$

$$\text{So: } Q(G, A_L^x / L^x) = \frac{Q(G, A_{L, \tilde{S}}^x)}{Q(G, O_{L, \tilde{S}}^x)} = [L : K]$$

$$\frac{|H^0(G, A_L^x / L^x)|}{|H^1(G, A_L^x / L^x)|} = \frac{|A_K^x / K^x \cdot N_{L/K}(A_L^x)|}{|H^1(G, A_L^x / L^x)|}$$

So we have proved a bit more:

Thm 3 Let L/K be a finite cyclic extⁿ of number fields

$$(i) \quad [A_K^x : K^x \cdot N_{L/K} A_L^x] = [L:K] \cdot |H^1(G, A_L^x/L^x)|$$

(ii) (combining (i) with the universal index inequality)

$$[A_K^x : K^x \cdot N_{L/K} A_L^x] = [L:K]$$

$$(iii) \quad H^{-1}(G, A_L^x/L^x) = (0)$$

$$1 \rightarrow L^x \rightarrow A_L^x \rightarrow A_L^x/L^x \rightarrow 1$$

$$\hookrightarrow 0 = H^1(A_L^x/L^x) \rightarrow \frac{K^x}{N_{L/K} L^x} \rightarrow A_K^x / N_{L/K} A_L^x$$

Corollary (rephrasing the injection $\frac{K^x}{N_{L/K} L^x} \hookrightarrow A_K^x / N_{L/K} A_L^x$
= Hasse's theorem)

Let L/K be a finite cyclic extension of number fields.

If an element $\alpha \in K^x$ is a local L/K -norm for every place $v \in \Sigma_K$, then α is a global L/K -norm.

Remark

Explicit expression of $H^1(G, A_L^x/L^x) = H^1(G, A_{L,S}^x/\mathcal{O}_{L,S}^x)$.

L/K finite cyclic

$$1 \rightarrow \mathcal{O}_{L,S}^x \rightarrow A_{L,S}^x \rightarrow A_{L,S}^x/\mathcal{O}_{L,S}^x \rightarrow 1$$

$\hookrightarrow$

$$0 \rightarrow H^1(G, A_{L,S}^x/\mathcal{O}_{L,S}^x) \rightarrow \mathcal{O}_{K,S}^x / N_{L/K}(\mathcal{O}_{L,S}^x) \rightarrow A_{K,S}^x / N_{L/K}(A_{L,S}^x)$$

$$\quad \quad \quad \parallel \quad \quad \quad \parallel$$

$$\quad \quad \quad H^0(G, \mathcal{O}_{L,S}^x) \quad \quad \quad H^0(G, A_{L,S}^x)$$

So we have a natural isomorphism

$$H^1(G, A_{L,S}^x/\mathcal{O}_{L,S}^x) \cong \mathcal{O}_{K,S}^x \cap N_{L/K}(A_{L,S}^x) / N_{L/K}(\mathcal{O}_{L,S}^x)$$

So in this incarnation of $H^1(G, A_L^x/L^x)$, the result that it is 0 means that:

If an S -unit α in K is a local L/K -norm for every place $v \in S$, then it is the L/K -norm of an element β of L^x (which is necessarily an $\tilde{S}$ -unit in L).

Of course this statement is included in Hasse's Theorem.

(Recall that S contains all places of K which are ramified in L/K .)

Reciprocity map

Recall: If L, K = finite ext of $\mathbb{Q}$
 L/K is a finite extension field, and all but finitely many places of K splits completely in L , then $L = K$

(Use Dedekind zeta function, and the fact that $\zeta_K(s)$ and $\zeta_L(s)$ both have simple pole at $s=1$.)

Alternative proof when L/K is finite abelian = May assume that

L/K is cyclic. Then $\forall \alpha \in A_K^x$, choose $b \in K^x$ s.t. $\alpha \cdot b^{-1}$ is a local norm at all places of S .

Let $S = \Sigma_{K, \infty} \cup \{v \in \Sigma_K \mid v \text{ does not split completely in } L\}$

Then $\alpha \cdot b \in N_{L/K}(A_L^x)$. Conclude: $A_K^x = K^x \cdot N_{L/K}(A_L^x)$. So $L = K$.

Consequence: Let L/K be a finite abelian extⁿ of number fields

$$S = \Sigma_{K, \infty} \cup S_{L/K, \text{ramified}}$$

$$\text{rec}_{L/K} \left(\begin{array}{l} \text{fractional ideals} \\ \text{of } \mathcal{O}_K \text{ prime to } S \end{array} \right) \rightarrow \text{Gal}(L/K)$$

$\parallel$

$$A_K^x / A_{K,S}^x$$

$$\left(\prod_{v \in S} \mathcal{O}_{K_v}^x \right) \times \prod_{v \notin S} \mathcal{O}_v^x$$

Remarks about class groups

$$\mathbb{A}_K^\times / \mathbb{A}_{K,S}^\times \xrightarrow{\sim} \text{Gal}(L/K)$$

$$\parallel$$

$$\left(\prod_{v \in S} K_v^\times \right) \times \prod_{v \notin S} \mathcal{O}_v^\times$$

finite abelian
extⁿ

$$S = \sum_{K, \text{as}} \cup S_{L/K, \text{ramified}}$$

$$\tilde{S} = \{w \in \Sigma_L \mid w \text{ lies above } S\}$$

Consider moduli / cycles m of K , supported in S .

$$\text{st. } \{x_v \in K_v^\times \mid x_v \equiv 1 \pmod{m_v}\} \subseteq N_{L_v/K_v}(L_v^\times) \quad \forall v \in S \quad \forall w \in \Sigma_L \text{ above } v$$

For such a "admissible modulus m ", we have.

$$(i) \quad \mathbb{A}_K^\times / \mathbb{A}_{K,S}^\times \xrightarrow{\sim} (\text{fractional } \mathcal{O}_{K,S}\text{-ideals}) =: \mathcal{I}_S$$

$$\mathcal{O}_{K,S} = K \cap \left(\left(\prod_{v \in S} K_v^\times \right) \times \prod_{v \notin S} \mathcal{O}_v^\times \right)$$

(ii) weak approximation

$$\Rightarrow \mathbb{A}_K^\times = K^\times \cdot V_m \cdot \left(\prod_{v \notin S} K_v^\times \right) \quad \text{where } V_m = \prod_{v \in S} \{x_v \equiv 1 \pmod{m_v}\} \times \prod_{v \notin S} \mathcal{O}_v^\times$$

$$\Rightarrow \mathbb{A}_K^\times / K^\times \cong \underbrace{V_m \cdot \left(\prod_{v \notin S} K_v^\times \right)}_{\parallel \mathbb{A}_K^{(S)\times}} / \underbrace{K^\times \cap V_m \cdot \mathbb{A}_K^{(S)\times}}_{\parallel \{x \in K^\times \mid x_v \equiv 1 \pmod{m_v} \forall v \in S\}} \\ \parallel K^\times(m)$$

$$\Rightarrow \mathbb{A}_K^\times / K^\times \cdot N_{L/K}(\mathbb{A}_L^\times) \cong \mathbb{A}_K^{(S)\times} / K^\times(m) \cdot N_{L/K}(\mathbb{A}_L^{(S)\times}) \cong \mathcal{I}_S / \text{principal ideals generated by elements of } K^\times(m) \\ N_{L/K} \mathcal{I}_S$$

$\forall m$ admissible, i.e. m supported in S
for L/K

$$\text{and } \{x_v \in K_v^\times \mid x_v \equiv 1 \pmod{m_v}\} \subseteq N_{L_v/K_v}(L_v^\times) \quad \forall v \in S$$

Theorem 5 L/K abelian $\Rightarrow \exists$ admissible modulus m s.t. $\text{rec}_{L/K}^{\text{under}}: (\text{fractional } \mathcal{O}_S\text{-ideals}) \rightarrow \text{Gal}(L/K)$
 "Existence of conductor" $\text{rec}_{L/K}(a) = 1 \quad \forall a \in K^\times \text{ s.t. } a \equiv 1 \pmod{m_v} \quad \forall v \in S$ VII-16

(weak) reciprocity law: $\tilde{S} = \{w \in \Sigma \mid \pi(w) \in S\}$
 $\uparrow$
 image of w in Σ_K

$$\text{rec}_{L/K}: \mathbb{A}_K^\times / \mathbb{A}_{K,S}^\times \rightarrow \mathbb{A}_K^\times / \mathbb{A}_{K,S}^\times \cdot N_{L/K} \left(\prod_{w \in \tilde{S}} L_w^\times \right) \xrightarrow{\downarrow} \mathcal{O}_{K,S} \rightarrow \mathcal{O}_{K,S} / N_{L/K}(\mathcal{O}_{L,\tilde{S}}) \rightarrow \text{Gal}(L/K)$$

Fractional ideals of $\mathcal{O}_{K,S}$, i.e.

Fractional ideals

of $\mathcal{O}_K$ prime to S

$\exists$ admissible modulus m s.t.

$\text{rec}_{L/K}$ factors through $\mathbb{A}_K^{(S)}/K^\times \cdot N_{L/K} \mathbb{A}_L^{(S)\times} \cong \mathbb{A}_K^\times / K^\times \cdot N_{L/K}(\mathbb{A}_L^\times)$
 and induces a homomorphism

$$\text{rec}_{L/K}: \mathbb{A}_K^\times / K^\times \cdot (N_{L/K} \mathbb{A}_L^\times) \rightarrow \text{Gal}(L/K)$$

"existence of conductor"

Since (universal norm index inequality)

$$|\mathbb{A}_K^\times / K^\times \cdot N_{L/K} \mathbb{A}_L^\times| \leq [L:K]$$

weak reciprocity law

$$\Leftrightarrow \text{rec}_{L/K}: \mathbb{A}_K^\times / K^\times \cdot N_{L/K} \mathbb{A}_L^\times \xrightarrow{\sim} \text{Gal}(L/K)$$

Lemma (a) Weak reciprocity law holds for $\mathbb{Q}(\mu_m)/\mathbb{Q} \quad \forall m \geq 1$

i.e. $\exists M \underset{M \geq 1}{\text{ s.t. }} \forall a \in \mathbb{N}_M^\times$ such that $\gcd(a, m) = 1$ and $a \equiv 1 \pmod{M}$

$$\text{rec}_{\mathbb{Q}(\mu_m)/\mathbb{Q}}((a)) = 1 \text{ in } \text{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/m\mathbb{Z})^\times$$

$\parallel$

$a \pmod{m}$

Suffices to take: $M = m$

(b) Weak reciprocity law holds for every cyclotomic extⁿ of a number field
 (consequence of functoriality)

Idea of Artin's proof: reduce to the cyclotomic case.
of theorem 5
(existence of conductor)

Trivial reduction step. Suffices to prove Theorem 5 when L/K is a cyclic extension with $\text{Gal}(L/K) \cong \mathbb{Z}/p^a\mathbb{Z}$ for some $a \in \mathbb{N}_{\geq 1}$

The proof of the existence of conductor (weak reciprocity law) for cyclic uses:

Artin's Lemma Let L/K be a cyclic extⁿ of number fields. only need this case → extensions whose degree is a power of a prime number.
let S_0 be a finite set of prime numbers, and let $v \in \Sigma_{K, \text{finite}}$ be a finite place of K unramified in L

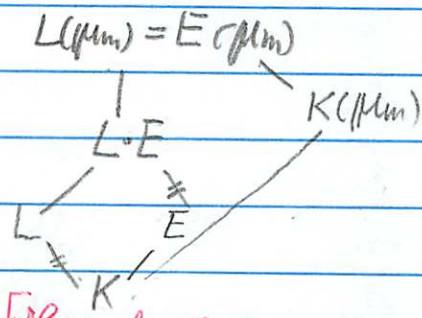
There exists a positive integer $m \in \mathbb{N}_{\geq 1}$, prime to all elements of S_0 and to v , and a finite extⁿ field E of K , $E \subseteq L(\mu_m)$, such that

(1) $L \cap E = K$

(2) $L(\mu_m) = E(\mu_m)$

(3) $L \cap K(\mu_m) = K$

(4) v splits completely in E



Lemma B Let L/K be an abelian extⁿ of nber fields whose degree is a power of a prime number.
let S be a finite set of prime numbers, and let v be a finite place of K unramified in L . There exists an integer $m \in \mathbb{N}_{\geq 1}$, prime to elements of S and to v , such that

(i) The Artin symbol $(v, K(\mu_m)/K) \in \text{Gal}(K(\mu_m)/K)$ has an order $\equiv 0 \pmod{[L:K]}$

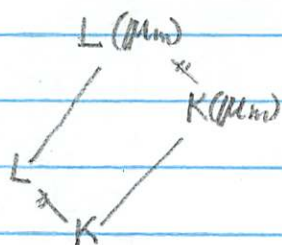
(ii) $K(\mu_m) \cap L = K$

(iii) $\exists \tau \in \text{Gal}(K(\mu_m)/K)$ such that $o(\tau) \equiv 0 \pmod{[L:K]}$
and $\langle \tau \rangle \cap \langle (v, K(\mu_m)/K) \rangle = \{1\}$
 $\uparrow$
 $\mathbb{Z}$

Lemma B $\Rightarrow$ Artin's Lemma:

Let $\sigma \in \text{Gal}(L/K)$ be a generator of $\text{Gal}(L/K) : \langle \sigma \rangle = \text{Gal}(L/K)$

Apply Lemma B:



$$\text{Gal}(L(\mu_m)/K) \cong \text{Gal}(L/K) \times \text{Gal}(K(\mu_m)/K)$$

$\downarrow \quad \uparrow$
 $\sigma \times \tau \quad \text{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q})$

Let H = the subgroup of $\text{Gal}(L(\mu_m)/K)$ generated by $\sigma \times \tau$ and $(\sigma, L/K) \times (\tau, K(\mu_m)/K)$

Let $E = L(\mu_m)^H$. So v splits completely in E by construction/defⁿ of E .

Claim: $H \cap (\text{Gal}(L/K) \times \{1\}) = \{1\}$

Pf of Claim: If $\sigma^i \times 1 = (\sigma^j \times \tau^j) \cdot (\sigma, L/K)^l \cdot (\tau, K(\mu_m)/K)^e$ $i, j, l, e \in \mathbb{Z}$,

$$\text{then } \tau^j = 1 = (\tau, K(\mu_m)/K)^e \quad \because \langle \tau \rangle \cap \langle (\tau, K(\mu_m)/K) \rangle = \{1\}$$

$$\Rightarrow j \equiv 0 \pmod{[L:K]} \quad \because \tau^j = 1$$

$$\Rightarrow \sigma^j = 1 \text{ and } \sigma^i = 1$$

Claim $\Rightarrow E(\mu_m) = E \cdot K(\mu_m) = L(\mu_m)$ by Galois theory

Clearly $\text{Gal}(L(\mu_m)/K)$ is generated by H and $1 \times \text{Gal}(K(\mu_m)/K)$,
so $E \cap L = K$. QED.

$$a = Nr \quad \tau = b \pmod{m} \in \text{Gal}(K(\mu_m)/K) \cong \text{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q}), \quad m \text{ as in Lemma B}$$

Lemma B

See

Lang, p. 201 for a slightly more general version

Lemma A.

Let n be a power of a prime number q . Let S be a finite set of numbers. There exists (i) an integer $m = p \cdot p'$, where $p \neq p'$ are prime numbers, m is prime to elements of S , and (ii) a positive integer b

such that

$$(1) \quad a \pmod{m} \in (\mathbb{Z}/m\mathbb{Z})^\times \text{ has an order } \equiv 0 \pmod{n}$$

$$(2) \quad b \pmod{m} \in (\mathbb{Z}/m\mathbb{Z})^\times \text{ has an order } \equiv 0 \pmod{n}$$

$$(3) \quad \langle a \pmod{m} \rangle \cap \langle b \pmod{m} \rangle = \{1 \pmod{m}\}$$

subgroups of $(\mathbb{Z}/m\mathbb{Z})^\times$

Extended version of Artin's Lemma

L/K : cyclic extⁿ, $[L:K]$ = a power of a prime number $\text{Gal}(L/K) = \langle \sigma \rangle$

$v_1, \dots, v_r$: finite places of K

Apply Artin's lemma to $v_1, \dots, v_r$ in succession, with $S_1 \not\subseteq S_2 \not\subseteq \dots \not\subseteq S_r$,

Get:

- positive integers $m_1, \dots, m_r$, each m_i = product $p_i \cdot p_i'$ of two distinct large prime numbers $m_1, \dots, m_r$ relatively prime to each other

$$\text{Gal}(L(\zeta_{m_1}, \dots, \zeta_{m_r})/K) = \text{Gal}(L/K) \times \prod_{i=1}^r \text{Gal}(\mathbb{Q}(\zeta_{m_i})/\mathbb{Q}) = \Gamma$$

- E_i = the subfield of $L(\zeta_{m_1}, \dots, \zeta_{m_r})$ fixed by the subgroup $\tilde{H}_i \leq \Gamma$ generated by H_i and $\prod_{j \neq i} G_j$

Recall that H_i = the subgroup of $G \times \text{Gal}(\mathbb{Q}(\zeta_{m_i})/\mathbb{Q})$ generated by $\sigma \times \tau_i$

$$\text{Gal}(\mathbb{Q}(\zeta_{m_i})/\mathbb{Q}) \cong \text{Gal}(K(\zeta_{m_i})/K)$$

and $(v, L/K) \times (v, K(\zeta_{m_i})/K) = \text{Fr}_v$ on $K(\zeta_{m_i})/K$

Claim Let $E = E_1 \dots E_r$ = compositum of $E_1, \dots, E_r$.

Then $L \cap E = K$, hence $\text{Gal}(L/K) \hookrightarrow \text{Gal}(KE/E)$

proof : E = the subfield of $L(\zeta_{m_1}, \dots, \zeta_{m_r})$ fixed by $\bigcap_{i=1}^r \text{Gal}(LE_i/E_i) \supseteq \sigma \times \tau_1 \times \dots \times \tau_r$

L is fixed by $1 \times \tau_1 \times \dots \times \tau_r$

$\Rightarrow E \cap L$ is fixed by $\sigma \Rightarrow E \cap L \subseteq L^{\langle \sigma \rangle} = K$. Claim proved.

Proof of the existence of conductor for L/K cyclic

strategy

Let f = the smallest admissible modulus for L/K

$$\{x_v \equiv 1 \pmod{f_v}\} \in N_{L/K}(L_w/K_v) \quad \forall v \in \text{support}(f)$$

$$S = \text{supp}(f) \cup \Sigma_K$$

$$\text{rec}_{L/K} : \mathcal{I}_S \longrightarrow \text{Gal}(L/K)$$

fractional ideals in K prime to f

strategy: Show that $\text{Ker}(\text{rec}_{L/K}) \subseteq \left(\text{principal fractional ideals generated by } x \in K^\times \text{ with } x_v \equiv 1 \pmod{f_v} \right) \cdot N_{L/K}(\mathcal{I}_S^\sim)$

claim

fractional ideals in L prime to f

$$\text{We know: } |\mathcal{I}_S / \text{Ker}(\text{rec}_{L/K})| = [L:K] \quad (\because \text{rec}_{L/K} \text{ is surjective})$$

and

$$\mathcal{I}_S / \left(\text{principal fractional ideals gen. by } x \in K^\times \text{ with } x_v \equiv 1 \pmod{f_v} \right) \cdot N_{L/K}(\mathcal{I}_S^\sim) \cong \underbrace{A_K^\times / K^\times \cdot N_{L/K} A_L^\times}_{\text{cardinality} = [L:K] \text{ norm index of cyclic extensions}}$$

(or, use the "first inequality")

$$\text{get } \text{Ker}(\text{rec}_{L/K}) = N_{L/K}(\mathcal{I}_S^\sim) \cdot \left(\text{principal fractional ideals gen. by } x \in K^\times \text{ with } x_v \equiv 1 \pmod{f_v} \right)$$

!!
 $\mathcal{P}_f$

$$\text{i.e. Have } \text{rec}_{L/K} : A_K^\times / K^\times \cdot N_{L/K} A_L^\times \xrightarrow{\sim} \text{Gal}(L/K)$$

Actual proof Suppose $\alpha = \prod_{i=1}^r \mathfrak{p}_i^{s_i}$ $\mathfrak{p}_i \in$ prime ideals of $\mathcal{O}_K$, prime to f
and $\alpha \in \text{Ker}(\text{rec}_{L/K})$

Apply extended version of Artin's Lemma

$E = E_1 \cdots E_r$ $\mathfrak{p}_i$ splits completely in E_i $i=1, \dots, r$.

point. LE_i/E_i is a cyclotomic extⁿ, so weak reciprocity holds for KE_i/E_i .

Pick $\mathfrak{L}_i =$ fractional $\mathcal{O}_{E_i}$ -ideal, prime to f , s.t. $(\mathfrak{L}_i, LE_i/E_i) = \sigma$

Write $(\mathfrak{p}_i^{s_i}, L/K) = \sigma^{d_i}$ $\text{Gal}(L/K) \xleftarrow{\text{generator of}} \text{Gal}(LE_i/E_i)$

The condition $(\alpha, L/K) = 1$ means: $\sum_{i=1}^r d_i \equiv 0 \pmod{[L:K]}$

$(\mathfrak{L}_i, LE_i/E_i) = \sigma$
 $\Rightarrow (N_{E_i/K} \mathfrak{L}_i, L/K) = \sigma$
 $\Downarrow$
 $\mathfrak{L}_i$

Observe: $\mathfrak{p}_i^{s_i} \cdot \mathfrak{L}_i^{-d_i}$ is an E_i/K norm because both $\mathfrak{p}_i$ and $\mathfrak{L}_i$ are

Write $\mathfrak{p}_i^{s_i} \cdot \mathfrak{L}_i^{-d_i} = N_{E_i/K}(I_i)$ $I_i =$ fractional E_i -ideal prime to f E_i/K splits

Weak reciprocity law for LE_i/E_i

$(I_i, LE_i/E_i) = 1 \xrightarrow{\text{for}} \exists \beta_i \in E_i^\times$ and $\exists J_i$ s.t. $I_i = N_{LE_i/E_i} J_i \cdot (\beta_i)$
 $\beta_i \equiv 1 \pmod{f \cdot \mathfrak{m}_i}$ $\uparrow$ fractional E_i -ideal, prime to $f \cdot \mathfrak{m}_i$
 $\uparrow$ modulus in E_i

$\Rightarrow \mathfrak{p}_i^{s_i} \cdot \mathfrak{L}_i^{-d_i} = (N_{E_i/K} \beta_i) \cdot N_{L/K}(N_{LE_i/L} J_i)$
 $\uparrow$ principal ideal
 $N_{E_i/K} \beta_i \equiv 1 \pmod{f}$

Take product over i :

$$\alpha \cdot \underbrace{\mathfrak{L}_K^{-\sum_{i=1}^r d_i}}_n \in \mathcal{P}_f \cdot N_{L/K}(\mathcal{O}_S^\times)$$

So $\alpha \in \mathcal{P}_f \cdot N_{L/K}(\mathcal{O}_S^\times)$ $\iff -\sum_{i=1}^r d_i \equiv 0 \pmod{[L:K]}$ QED

Existence theorem for global class field theory

$$[k:\mathbb{Q}] < \infty$$

Lemma: Given $H \subseteq \bar{A}_k^x / k^x$ (therefore $[\bar{A}_k^x / k^x : H] < \infty$)

Let F be a cyclic extⁿ field of k . Suppose that $H_F = N_{F/k}^{-1}(H) \subseteq \bar{A}_F^x / F^x$ has a class field, i.e. $\exists$ an abelian extⁿ K/F s.t. $N_{K/F}(\bar{A}_K^x / K^x) = H_F$.

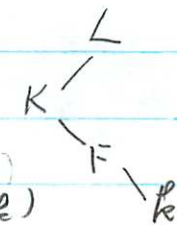
Then

K/k is abelian and $N_{K/k}(\bar{A}_K^x / K^x) \subseteq H$.

In particular H has a class field.

Pf: Let $L/k = \text{Galois closure of } K/k$

Pick $\sigma \in \text{Gal}(L/k)$ s.t. $\sigma|_F$ generates $\text{Gal}(F/k)$



(i) Given any $\tau: K \xrightarrow{k} L$, consider τK
 $\tau K / \tau F$ is the class field of τF
 attached to $\tau H_F = H_F \Rightarrow \tau K = K$, so K/k is Galois, and $L=K$.

(ii) claim $\forall \alpha \in \text{Gal}(K/F)$, have $\sigma \alpha \sigma^{-1} = \alpha$.

Pick $a \in \bar{A}_F^x$ s.t. $\left(\frac{a}{K/F}\right) = \alpha$ Then $\sigma \alpha \sigma^{-1} = \left(\frac{\sigma a}{\sigma K / \sigma F}\right) = \left(\frac{\sigma a}{K/F}\right)$

(iii) K is the class field of k attached to

$$N_{K/k}(\bar{A}_K^x / K^x) = N_{K/F}(H_F) \subseteq H_F.$$

$$= \left(\frac{a}{K/F}\right) \cdot \underbrace{\left(\frac{\sigma a \cdot a^{-1}}{K/F}\right)}_{\substack{\parallel \sigma a \cdot a^{-1} \\ \in \text{Ker}(N_{F/k}) \\ \subseteq H_F}}$$

Cor Given $H \subseteq \bar{A}_k^x / k^x$. Let F/k be a finite Galois

q.e.d.

extⁿ with $\text{Gal}(F/k)$ nilpotent. If $N_{F/k}^{-1}(H) \subseteq \bar{A}_F^x / F^x$ has a class field, so does H .

Thm 7 Let n be a positive integer. Assume $\mu_n \in k^x$

Given $S \subseteq \Sigma_{k,n} \cup \{v \in \Sigma_k \mid v|n\}$, $\#S = s < \infty$

$$A_k^x = k^x \cdot \underbrace{\left(\prod_{v \in S} k_v^x \times \prod_{v \notin S} O_v^x \right)}_{\parallel^x A_{K,S}}$$

Let $H = H_n = \prod_{v \in S} k_v^{x,n} \times \prod_{v \notin S} O_v^x$, Let $U = k^x \cap H_n \leq O_{k,S}^x$.
Let $K = k(u^{\frac{1}{n}} : u \in O_{k,S}^x)$ as in Kummer theory

(i) $N_{K/k}(A_K^x) \cdot k^x = H_n \cdot k^x$ i.e. K = the class field of k^x attached to $H_n \cdot k^x \leq A_k^x$.

(ii) $[K:k] = n^s$, where $s = |S|$

(iii) $U_{S,n} = (O_{k,S}^x)^n$

$\forall$ number field k

Cor Every open subgroup $H_n \leq A_k^x / k^x$ has a class field.

pf of Thm 7 $(O_{k,S}^x \cdot k^{x,n}) / k^{x,n} \xleftarrow{\sim} O_{k,S}^x / O_{k,S}^{x,n}$
 $\uparrow$
 $\circ \circ O_{k,S}^x \cap k^{x,n} = O_{k,S}^{x,n}$

$\hookrightarrow K \stackrel{\uparrow}{=} \text{the Kummer ext}^n \text{ of } k \text{ attached to } O_{k,S}^x$
 $\uparrow$ by def
 $\text{Hom}_{gr}(\text{Gal}(K/k), \mu_n) \cong O_{k,S}^x / O_{k,S}^{x,n} \cong (\mathbb{Z}/n\mathbb{Z})^s$
 $\uparrow$
 $\circ \circ \mu_n \subseteq O_{k,S}^x$
 $\hookrightarrow [K:k] = n^s \quad \nwarrow (ii)$

Must prove: $N_{K/k}(A_K^x) \cdot k^x = H_n \cdot k^x$
 $\geq$ easy $\circ \circ H_n \subseteq N_{K/k}(A_K^x)$

because K/k is unramified

To show $\subseteq$, it suffices to show that $[A_k^x : k^x \cdot H] = n^s$ outside S

$\parallel$
 $[K:k]$
 $\circ \circ [A_k^x : k^x N_{K/k}(A_K^x)] = [K:k]$

A, B abelian groups

$B \subseteq A$ subgroup of finite index

$f: A \rightarrow A, f(B) \subseteq B$

sublemma'

$$\Rightarrow [A:B] = [\text{Ker}(f) \cdot \text{Ker}(f|_B)] \cdot [f(A):f(B)]^{\text{endom.}}$$

pf: $A \supseteq B + A[f] \supseteq B$

$$A/B + A[f] \xrightarrow{\sim} f(A)/f(B)$$

induced by f

$$B + A[f]/B \xleftarrow{\sim} A[f]/(A[f] \cap B)$$

sublemma

$$[k_v^x : \mathcal{O}_v^x] < \infty, \quad n \in \mathbb{N}_{\geq 1}$$

Then $[O_v^x : O_v^{x^n}] = \frac{1}{|n|_v} \cdot |\mu_n(k_v)|$

and $[k_v^x : k_v^{x^n}] = \frac{n}{|n|_v} |\mu_n(k_v)|$ (follows from the first part)

pf Let $U \leq O_v^x$ s.t. $U \xrightarrow[\cong]{\log} \mathbb{Z}_p^{[k_v : \mathbb{Q}_p]}$
finite index

$$1 \rightarrow U \rightarrow O_v^x \rightarrow O_v^x/U \rightarrow 1$$

$$\downarrow [n] \quad \downarrow [n] \quad \downarrow [n]$$

$$1 \rightarrow U \rightarrow O_v^x \rightarrow O_v^x/U \rightarrow 1$$

$$\Rightarrow 1 \rightarrow \mu_n(k_v) \rightarrow \text{Ker}([n]_{O_v^x/U}) \rightarrow U/U^n \rightarrow O_v^x/O_v^{x^n} \rightarrow \text{coker } [n]_{O_v^x/U} \rightarrow 1$$

$$\Rightarrow |\mu_n(k_v)| \cdot \frac{|U/U^n|}{|n|_v^{-1}} = |O_v^x/O_v^{x^n}| \quad \text{q.e.d.}$$

Note $[k_v^x : k_v^{x^n}] = \frac{n}{|n|_v} |\mu_n(k_v)|$ for v archimedean also

Back: Claim $[A_k^x : k^x \cdot H_n] = n^s$

$$\begin{aligned} A_k^x / k^x \cdot H_n &= k^x \cdot A_{k,S}^x / k^x \cdot H \cong A_{k,S}^x / \underbrace{(A_{k,S}^x \cap k^x \cdot H_n)}_{H \cdot O_{k,S}^x} \\ &\cong (A_{k,S}^x / H_n) / (H \cdot O_{k,S}^x / H_n) \end{aligned}$$

$$\Rightarrow [A_k^x : k^x \cdot H_n] = \frac{[A_{k,S}^x : H_n]}{[O_{k,S}^x : O_{k,S}^x \cap H_n]}$$

$$\prod_v [k_v^x : k_v^{x^n}]$$

$$\prod_{v \in S} \frac{n}{|n|_v} |\mu_n(k_v)|$$

$$\frac{n^s \cdot n^s}{1} = n^{2s}$$

Thus: Claim will follow from

$$U_{S,n} = (O_{k,S}^x)^n, \text{ i.e. (iii)}$$

In other words, (iii) $\Rightarrow$ (i)

It remains to show: $\mathcal{O}_{k,S}^x \cap H_n = (\mathcal{O}_{k,S}^x)^n$, $\geq$ obvious

$\Leftarrow$: Given any $\alpha \in \mathcal{O}_{k,S}^x \cap H_n$, Consider $L = k(\alpha^{\frac{1}{n}})$ $\leftarrow$ unramified outside S

Clearly $\mathcal{A}_{k,S}^x \subseteq N_{L/k}(\mathcal{A}_L^x) \because [L:k] | n$

$$\Rightarrow \mathcal{A}_k^x = k^x \cdot \mathcal{A}_{k,S}^x \subseteq k^x \cdot N_{L/k}(\mathcal{A}_L^x) \xrightarrow{\text{CFT}} L = k \Rightarrow \alpha \in (k^x)^n$$

$$\downarrow \\ \alpha \in (k^x)^n \cap \mathcal{O}_{k,S}^x \\ = (\mathcal{O}_{k,S}^x)^n$$

QED.

Further investigation in the set-up of theorem?

Thm 8 In the set-up of theorem 7, let $S = S_1 \sqcup S_2$, $S_1 \neq \emptyset \neq S_2$.

$$\text{Let } H_1 = \left(\prod_{v \in S_1} k_v^x \right) \times \left(\prod_{v \in S_2} k_v^{x^n} \right) \times \prod_{v \notin S} \mathcal{O}_v^x, \quad U_1 = k^x \cap H_1$$

$$H_2 = \left(\prod_{v \in S_1} k_v^{x^n} \right) \times \left(\prod_{v \in S_2} k_v^x \right) \times \prod_{v \notin S} \mathcal{O}_v^x, \quad U_2 = k^x \cap H_2$$

$$\text{Let } L_1 \stackrel{\text{def}}{=} k(U_2^{\frac{1}{n}}), \quad L_2 = k(U_1^{\frac{1}{n}})$$

Then (1) $k^x H_1 = N_{L_1/k}(\mathcal{A}_{L_1}^x)$, $k^x H_2 = N_{L_2/k}(\mathcal{A}_{L_2}^x)$ i.e. L_i = the class field of k attached to H_i

$$[L_1 : k] = \frac{\prod_{v \in S_2} [k_v^x : k_v^{x^n}]}{n^s} [L_2 : k]$$

$$[L_2 : k] = \frac{\prod_{v \in S_1} [k_v^x : k_v^{x^n}]}{n^s} [L_1 : k]$$

(2) L_i split completely above places in S_i , $i=1,2$.
obvious

Cor: Let L/k be an abelian extⁿ of number fields, $\mathcal{M}_{[L:k]} \subseteq k$.

Then a place $v \in \Sigma_k$ splits completely in L

if and only if $k_v^x \subseteq k^x \cdot N_{L/k}(\mathcal{A}_L^x)$

$\uparrow$
regarded as a subgroup of $\mathcal{A}_k^x$

proof of Thm 8. Note first that

$$(\mathcal{O}_{k,S}^x)^n \subseteq U_i \cap (k^x)^n \subseteq H_i \cap (k^x)^n = (\mathcal{O}_{k,S}^x)^n \quad \text{for } i=1,2$$

Since L_i/k is unramified outside S and splits completely $\forall v \in S_i$, we have $H_i \subseteq N_{L_i/k}(\mathcal{A}_{L_i}^x)$ for $i=1,2$.

$$\begin{aligned} [L_i:k] &\leq [\mathcal{A}_k^x : k^x H_i] = [k^x \mathcal{A}_{k,S}^x : k^x H_i] = [\mathcal{A}_{k,S}^x : \underbrace{\mathcal{A}_{k,S}^x \cap k^x H_i}_{H_i'' \cdot \mathcal{O}_{k,S}^x}] \\ &= \frac{[\mathcal{A}_{k,S}^x : H_i]}{[\mathcal{O}_{k,S}^x : \underbrace{\mathcal{O}_{k,S}^x \cap H_i}_{U_i}]} = \frac{\prod_{v \in S_i} [k_v^x : k_v^{x,n}]}{[\mathcal{O}_{k,S}^x : \mathcal{O}_{k,S}^{x,n}]} \cdot [U_i : \mathcal{O}_{k,S}^{x,n}] \\ &\quad \begin{matrix} \nearrow \parallel \circ \circ \\ [L_i:k] \end{matrix} \quad U_i \cdot k^{x,n} / k^{x,n} \cong U_i / U_i \cap \mathcal{O}_{k,S}^{x,n} \end{aligned}$$

Kummer thy for $i=1,2$

Multiply, get:

$$\begin{aligned} [L_1:k] \cdot [L_2:k] &\leq [\mathcal{A}_k^x : k^x H_1] \cdot [\mathcal{A}_k^x : k^x H_2] \\ [L_1:k] \cdot [L_2:k] &= \frac{\prod_{v \in S} [k_v^x : k_v^{x,n}]}{n^s} \cdot [L_1:k] \cdot [L_2:k] = [L_1:k] \cdot [L_2:k] \\ &\quad \begin{matrix} \nearrow \circ \circ \\ [L_1:k] \end{matrix} \quad \begin{matrix} \nearrow \circ \circ \\ [L_2:k] \end{matrix} \quad \begin{matrix} \nearrow \circ \circ \\ [L_1:k] \end{matrix} \quad \begin{matrix} \nearrow \circ \circ \\ [L_2:k] \end{matrix} \end{aligned}$$

Hence: $[L_i:k] = [\mathcal{A}_k^x : k^x H_i] \quad \forall i=1,2$

Since $k^x H_i \subseteq k^x \cdot N_{L_i/k}(\mathcal{A}_{L_i}^x)$ and $[\mathcal{A}_k^x : k^x \cdot N_{L_i/k}(\mathcal{A}_{L_i}^x)] = [L_i:k]$, we see that $k^x H_i = k^x \cdot N_{L_i/k}(\mathcal{A}_{L_i}^x)$ for $i=1,2$.

Have proved (1).

QED.

Cor Let L/k be a class field, $n=[L:k]$, $\mu_n \subseteq k$ $v \in \Sigma_{k, \text{finite}}$ corresponding to $H \subseteq \mathcal{A}_k^x$ k^x open k^x

Then:

v splits completely in $L \iff k_v^x \subseteq H \implies$ "trivial"

Proof of " $\Leftarrow$ ". Pick S as in Thm 8, containing v . Let $S_1 = \{v_0\}$, $S_2 = S - \{v_0\}$

Assume $k_v^x \subseteq H$ the abelian ext L_1/k

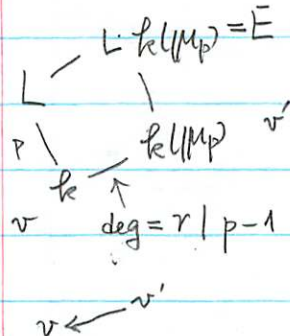
$$L_1 = k(\mathcal{U}_1^{1/n})$$

$$U_2 = k^x \cap H_2$$

S_0 : L_1 splits completely above v_0

$$k_v^{x,n} \times \prod_{v \in S, v \neq v_0} k_v^x \times \prod_{v \notin S} \mathcal{O}_v^x$$

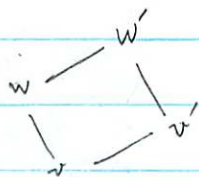
Reduce further to: L/k abelian, $[L:k] = p$ a prime number
 $\text{Gal}(L/k) = G_v$, $\text{ar}_{L/k}(k_v^x) = 1$
 must show: contradiction



factoriality of reciprocity law gives:

$$\text{ar}_{E/k(M_p)}(y) = \text{ar}_{L/k}(N_{k(M_p)/k}(y)) = 1 \quad \forall y \in k(M_p)_{v'}$$

By the result for the Kummer situation,
 the place v' of $k(M_p)$ above v splits completely in E .



Since p and $p-1$ are relatively prime,

$$[E_{w'} : k(M_p)_{v'}] = [L_w : k_v]$$

||
1

So v splits completely in L , hence $G_v = \{1\}$ and $L = k$, contradiction.
 (i) proved.

(ii) follows from (i): (i) says $\text{ar}_{L/k}$ induces a surjection $k_v^x / N_{L_w/k_v} L_w^x \twoheadrightarrow G_v$

On the other hand, the local cyclic norm index ineq. implies

$$|k_v^x / N_{L_w/k_v} L_w^x| \leq [L_w : k_v] = |G_v|. \text{ So } k_v^x / N_{L_w/k_v} L_w^x \cong G_v$$

QED

Remark/Observation

$\forall$ number field k , $\forall v_0 \in \Sigma_k$, $k_{v_0}^x \hookrightarrow A_k^x / k^x$ is a closed embedding
 must show: $\forall \varepsilon > 0$, $\exists$ open nbd U of 1 in A_k^x s.t. $k_{v_0}^x \cap (k^x \cdot U)$

$$\text{Let } U_{S,\delta} = \left(\prod_{v \in S} U_{v,\delta} \right) \times \prod_{v \notin S} \mathcal{O}_v^x \times k_{v_0}^x \quad \cong \{x \in k_{v_0}^x \mid |x-1|_v < \varepsilon\}$$

$\delta < \frac{1}{2}$ $U_{v,\delta} = \{y \in k_v^x \mid \log |y|_v < \delta\}$ $\Sigma_{k,v_0} \{v_0\} \in S \subseteq \Sigma_k \setminus \{v_0\}$, $\tilde{S} = S \cup \{v_0\}$

$$k_{v_0}^x \cap (k^x \cdot U) = \{u_{v_0} \mid u \in \mathcal{O}_{k,\tilde{S}}^x, u_v \in U_{v,\delta} \forall v \in S\} = \{u_{v_0} \mid u \in \Delta\}$$

v_0 -component of u

$$\text{Let } \Delta = \{u \in \mathcal{O}_{k,\tilde{S}}^x \mid u_v \in U_{v,\delta} \forall v \in S\}$$

Dirichlet's thm on units $\Rightarrow \Delta$ is finite. and $\Delta = \mu(k)$ if δ is sufficiently small.

Thm with δ sufficiently small, $S = \sum_{k=0}^{\infty} \{v_0\}$

$$U = \left(\prod_{v \in S} U_{v,\delta} \right) \times U_{v_0,\delta} \times \prod_{v \notin S} \mathcal{O}_v^\times$$

$$\text{Have } k_{v_0}^\times \cap (k^\times \cdot U) \subseteq U_{v_0,2\delta} \quad \text{q.e.d.}$$

$\uparrow$ if $v_0 \in \sum_{k=0}^{\infty}$

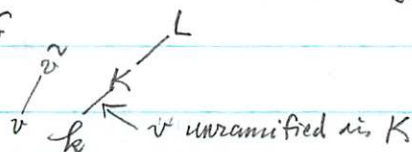
Proposition 10 L/k abelian extⁿ of number fields, $v \in \Sigma_k$

Then $\text{ar}_{L/k}(\mathcal{O}_v^\times) = I_v \subseteq G_v$
inertia subgroup in G_v

pf: Replace k by L^{G_v} (so v splits completely in L^{G_v}), may assume

$$G_v = \text{Gal}(L/k) =: G$$

$$\text{Let } K = L^{I_v}$$



$$\text{Have } I_v = \text{ar}_{L/k}(\mathcal{O}_v^\times) \quad \text{by Prop 9}$$

$$= \text{ar}_{L/k}(N_{K/k}(\mathcal{O}_v^\times)) = \text{ar}_{L/k}(\mathcal{O}_{\tilde{v}}^\times)$$

Done in the archimedean case

The statement of Prop 10 also holds when v is archimedean.

$$I_v = D_v \quad \text{when } v \text{ is archimedean.} \quad \text{q.e.d.}$$

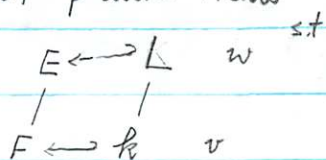
Corollary/restatement L/k abelian extⁿ of number fields

$$k_v^\times / N_{L/k_v} L_w^\times \xrightarrow[\sim]{\text{ar}_{L/k}} G_v$$

$$\begin{array}{ccc} \text{VI} & & \text{VI} \\ \mathcal{O}_v^\times / N_{L/k_v} \mathcal{O}_w^\times & \xrightarrow[\sim]{} & I_v \end{array}$$

Given E/F abelian extⁿ of p-adic fields

Consider



L/k : finite abelian extⁿ of number fields

$v \in \Sigma_k$, $w \in \Sigma_L$ induced by $L \hookrightarrow E$

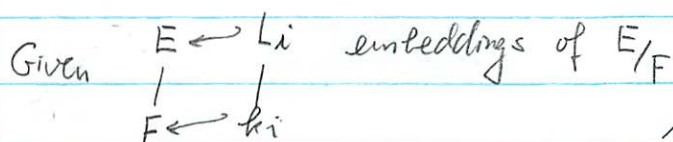
$$E \cong L_w, F \cong k_v$$

induced by $L \hookrightarrow E$ and $k \hookrightarrow F$

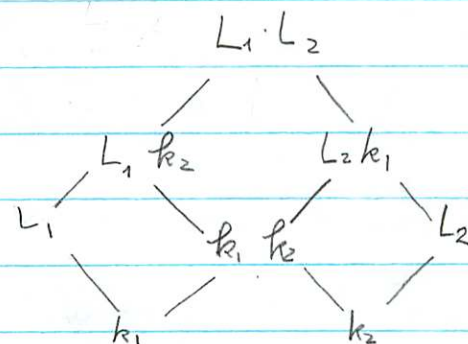
and $L \subseteq E$ is stable under the action of the action of $\text{Gal}(E/F)$ on E

This gives $\text{Gal}(E/F) \hookrightarrow \text{Gal}(L/k)$

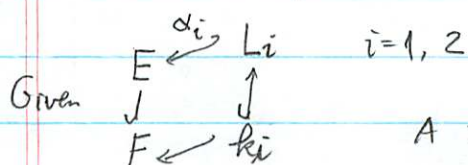
def a) Call such a diagram an "admissible embeddings" of E/F into L/k



$\leadsto$

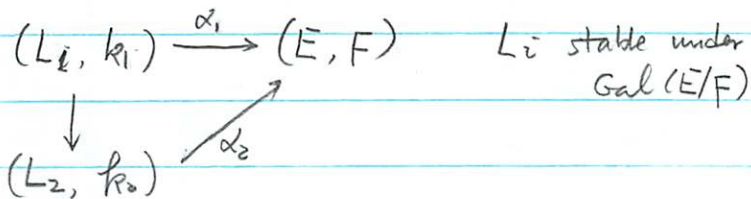
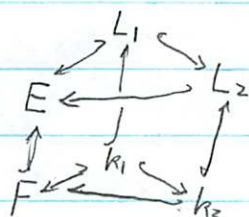


2) notion of morphisms



A morphism from embedding (1) to embedding (2)

is a commutative diagram



Given E/F finite abelian extensions of p-adic fields,

Lemma 1 $\{ \text{admissible embeddings of } E/F \}$ a filtering/directed system
 pf: non-emptiness

Pick a finite extension field k of $\mathbb{Q}$ contained in F st. $k \cdot \mathbb{Q}_p = F$

Then pick subfield $L_1 \subseteq E$ st. $L_1 \supseteq k$, $[L_1:k] < \infty$ and $L_1 \cdot \mathbb{Q}_p = E$

Let $L = \text{compositum of } \{ \sigma L_1 \mid \sigma \in \text{Gal}(E/F) \}$. Let $k = L^{\text{Gal}(E/F)}$ q.e.d.

Remark This construction produced an embedding $(E, F) \hookrightarrow (L, k)$
plus an isom. $\text{Gal}(E/F) \xrightarrow{\sim} \text{Gal}(L/k)$

Lemma 2 Given E/F finite abelian extⁿ of p-adic fields

Global CFT and the family of all admissible embeddings of E/F
define a canonical isom

$$F^{\times}/N_{E/F}(E^{\times}) \xrightarrow{\text{ar}_{E/F}} \text{Gal}(E/F)$$

Given abelian extension E_2/F and a subextension E_1/F of E_2/F
of p-adic fields

$$\begin{array}{ccccc} (E_2, F) & \hookrightarrow & (L_1, k_1) & \hookrightarrow & (L_1 L_2, k_1 k_2) \\ \uparrow & & \text{Gal}(E_2/F)\text{-equivariant} & \rightarrow & \uparrow \\ (E_1, F) & \hookrightarrow & (L_2, k_2) & \hookrightarrow & (L_1 L_2, k_1 k_2) \end{array}$$

$\text{Gal}(E_2/E_1)$

$\Rightarrow$

abelian extⁿ

Lemma 3 Given E_2/F and a subextension E_1/F of p-adic fields,

Have a canonical morphism

$$\left\{ \begin{array}{c} \text{admissible embeddings of } E_1 \\ F \end{array} \right\} \xrightarrow{\uparrow \text{"Gal}(E_2/F)\text{-equivariant"}} \left\{ \begin{array}{c} \text{admissible embeddings of } E_2 \\ F \end{array} \right\}$$

$\Downarrow$

Lemma 4 Given $E_1 \leq E_2$ finite abelian extⁿs of $\mathbb{Q}_p$,

(functoriality in E)
of $\text{ar}_{E/F}$

The diagram

$$\begin{array}{ccc} F^{\times} & \xrightarrow{\text{ar}_{E_2/F}} & \text{Gal}(E_2/F) \\ \downarrow & & \downarrow \\ F^{\times} & \xrightarrow{\text{ar}_{E_1/F}} & \text{Gal}(E_1/F) \end{array} \quad \text{commutes}$$

Cor/Lemma 5 (1) Two finite abelian ext^{ns} E_1/F and E_2/F of p -adic fields are isomorphic iff $N_{E_1/F} E_1^x = N_{E_2/F} E_2^x$

def. A open subgroup $H \leq F^x$ of finite index admits a class field if $\exists$ a finite abelian extⁿ E/F s.t. $H = N_{E/F} E^x$

(2) An open subgroup $H \leq F^x$ of finite index admits a class field if $\exists$ an open subgroup $H_1 \leq H$ of finite index s.t. H_1 admits a class field.

Lemma 6. Let $[F:Q_p] < \infty$, $H \leq F^x$ open, $[F^x:H] < \infty$.

Let F'/F be a ^{finite} cyclic extension, and $H' := N_{F'/F}^{-1}(H) \leq F'$.

- 1) If H' admits a class field E , then E/F is abelian,
- 2) H admits a class field.

(Proof essentially the same as in the global case, p. VII-22)

Cor. Lemma 6 holds if F'/F is a finite nilpotent extension

Theorem 11 Let F be a finite extension field of Q_p , and let $H \leq F^x$ be an open subgroup, $[F^x:H] = n \geq 1$. Then H admits a class field

Pf. May assume: $\mu_n \leq F^x$

Let $E = F(F^{x \frac{1}{n}})$, Kummer theory $\Rightarrow [F^x:F^{x^n}] = [E:F]$

Since E/F is abelian, so $F^x/N_{E/F}(E^x) \xrightarrow{\sim} \text{Gal}(E/F)$

In particular, $[F^x:N_{E/F} E^x] = [E:F]$

Since $F^{x^n} \leq N_{E/F} E^x$, we get $F^{x^n} = N_{E/F}(E^x)$.

i.e. $F(F^{x \frac{1}{n}}) =$ the class field of F attached to F^{x^n}

Since $F^x H \supseteq F^{x^n}$, we conclude that H is also a class field.

Thm 12 (Chebotarev) L/k finite Galois extⁿ of number fields, $G = \text{Gal}(L/k)$
 $\sigma \in G$. $\sigma^G = \{x\sigma x^{-1} : x \in G\}$, $c_\sigma = |\sigma^G|$

Then the subset $\{v \in \Sigma_k \mid v: \text{finite, unramified in } L, \text{ and } \exists w|v \text{ s.t. } \text{Fr}_{w/k} = \sigma\} =: S$
 has Dirichlet density $\frac{c_\sigma}{|G|}$

pf Let $Z = L^\sigma = \text{fixed field of } \sigma$. $\begin{matrix} L \\ \swarrow \langle \sigma \rangle \\ Z \\ \searrow k \end{matrix}$ L/Z cyclic, with generator σ

Let $S_{L,\sigma} = \{w \in \Sigma_{L, \text{finite, ur}} \mid \text{Fr}_{w/k} = \sigma\}$
 bijective $\downarrow$
 $S_{Z,\sigma} = \{v' \in \Sigma_{Z, \text{finite, ur}} \mid \begin{matrix} \text{ar}_{L/Z}(\pi_{v'}) = \sigma, \\ \uparrow \text{CFT} \\ \text{image of } \pi_{v'} \text{ in } A_Z^x / Z^x \cdot N_{L/Z}(A_L^x) \\ = \text{a fixed element in } A_Z^x / Z^x \cdot N_{L/Z}(A_L^x) \end{matrix} \}$
 $Z_{v'} \xleftarrow{\sim} k_{v'}$ v' splits completely

$$\leadsto \sum_{v' \in S_{Z,\sigma}} N_{v'}^{-s} \sim \frac{1}{f} \cdot (-\log(s-1)) \quad \text{This part uses CFT.}$$

$$f = [L:Z] = |\langle \sigma \rangle|$$

Compare $S_{L,\sigma}$ with S $S_{L,\sigma} \xrightarrow{(|G|/|G|f)-1} S$

Given $v \in S$, $w \in S_{L,\sigma}$ above v is $G/G_w \cdot w$
 $\leadsto \# \{w' \mid v : \text{Fr}_{w'/k} = \sigma\} = \frac{\# Z_G(\sigma)}{\# \langle \sigma \rangle} = \frac{|G|/c_\sigma}{f}$

$$\leadsto \text{density of } S = \left(\frac{1}{f}\right) / \left(\frac{|G|}{|G|f}\right) = \frac{|G|}{|G|}$$

QED.

Remark: The key is a counting argument

$$\begin{matrix} \text{density} \\ = \frac{1}{f} \end{matrix} \rightarrow S_{L,\sigma} \xrightarrow{\frac{|Z_G(\sigma)|}{f} - 1} S$$

$$f = [L:Z] = \text{order of } \sigma$$

Thm, p.54, Ch2§3. G ^{Hausdorff} totally disconnected locally compact group.

$V \ni e$ a compact nbd of e . Then V contains a compact open subgroup of G .

Remark: Conversely if G is a ^{Hausdorff} locally compact group such that every compact open neighborhood V of e contains a compact open subgroup of G , then G is totally disconnected.

pf: Let $G^\circ =$ neutral component of G

$\forall$ compact open subgroup H of G , $H \cap G^\circ$ is an open subgroup of G° .

Hence $G^\circ \subseteq H$, so $G^\circ \subseteq \bigcap H = \bigcap V = \{e\}$

$H =$ compact open subgroup of G $V =$ compact open nbd of e

Application to $\mathbb{A}_k^\times / k^\times$ totally disconnected.

$$\begin{array}{ccc} \mathbb{A}_k^\times & \longrightarrow & \mathbb{A}_k^\times / (k_\infty^\times)^\circ \\ \downarrow & & \downarrow \\ \mathbb{A}_k^\times / k^\times & \longrightarrow & \mathbb{A}_k^\times / (k^\times \cdot (k_\infty^\times)^\circ)^{\text{closure}} \end{array}$$

$$\Rightarrow (k^\times \cdot (k_\infty^\times)^\circ)^{\text{closure}} / k^\times \supseteq (\mathbb{A}_k^\times / k^\times)^\circ$$

" $\subseteq$ " obvious

Conclusion:

$$(\mathbb{A}_k^\times / k^\times)^\circ = \text{closure of the image of } (k_\infty^\times)^\circ \text{ in } \mathbb{A}_k^\times / k^\times$$

Structure of $(A_k^x/k^x)^0$, k : number field

VIII-1

From general/elementary properties on Hausdorff locally compact groups, get

Proposition 1 (i) $(A_k^x/k^x)^0$ = the closure of $(k \otimes_{\mathbb{Q}} \mathbb{R})^{\times}$ in A_k^x/k^x

(ii) $(A_k^x/k^x)^0$ is divisible (follows from divisibility of $(k \otimes_{\mathbb{Q}} \mathbb{R})^{\times}$)

(iii) $\text{Ker}(\text{ar}_k: A_k^x/k^x \rightarrow \text{Gal}(\bar{k}/k)^{\text{ab}} = \text{Gal}(k^{\text{ab}}/k)) = (A_k^x/k^x)^0$

$\supseteq$ obvious ($\because \text{Gal}(k^{\text{ab}}/k)$ is totally disconnected)

$\subseteq$ $\because$ every open subgroup of A_k^x/k^x contains $(A_k^x/k^x)^0$

and is of finite index + CFT

(iv) Every divisible subgroup of A_k^x/k^x is contained in $(A_k^x/k^x)^0$
 $(A_k^x/k^x)/(A_k^x/k^x)^0$ is compact and totally disconnected, hence profinite.
 Hausdorff

Theorem 2 $1 \rightarrow \prod_{v \in \Sigma_{k, \text{cpx}}} k_{v,1}^x \xrightarrow{\substack{\uparrow \\ \text{closed embedding}}} (A_k^x/k^x)^0 \rightarrow (A_k^x/k^x)^0 / \prod_{v \in \Sigma_{k, \text{cpx}}} k_{v,1}^x \rightarrow 1$
 $\uparrow \text{ top. isom.}$
 $\mathcal{O}_k^x \otimes_{\mathbb{Z}} (A_{\mathbb{Q}}/\mathbb{Q})$

Theorem 3 k : number field, $m = 2^t m'$ positive integer, m' odd

Let S be a set of places, analytic density $(S) = 0$

Let $\alpha \in k^x$, $\alpha \in k_v^{x,m} \quad \forall v \notin S$

(1) If $k(\mu_{2^t})/k$ is cyclic, then $\alpha \in k^{x,m}$. Note that this condition holds if $t \leq 2$, or if $\sqrt{-1} \in k$.

(2) $\alpha \in (k^x)^{\frac{m}{2}}$ if $t \geq 3$.

Proved by the argument in Chevalley's 1951 paper, Deux théorèmes d'arithmétique

Reduction ① If $\text{gcd}(m_1, m_2) = 1$, $\alpha = \beta_1^{m_1} = \beta_2^{m_2}$, $\beta_1, \beta_2 \in k^x$, Choose $b_1, b_2 \in \mathbb{Z}$ s.t.
 $m_1 b_1 + m_2 b_2 = 1$, then $\alpha = \alpha^{m_1 b_1} \cdot \alpha^{m_2 b_2} = \beta_1^{m_1 m_2 b_1} \cdot \beta_2^{m_1 m_2 b_2} = (\beta_2^{b_1} \cdot \beta_1^{b_2})^{m_1 m_2}$

$\leadsto$ May assume: $m = p^e$, p = a prime number

step ② Let $k' = k(\mu_m)$, and $L = k'(\alpha^{\frac{1}{m}})$. Then L/k' splits at every place ^{the abel. extⁿ} outside S
 $\sim L = k'$ i.e. $\alpha \in (k')^m$

step ③ Consider the polynomial $f(x) = x^m - \alpha$ over k , whose splitting field is k'
 $f(x) = \prod_{i=1}^r f_i(x)$ $f_i(x)$ irreducible polynomial

Let $\beta_i =$ a root of $f_i(x)$. Then $k(\beta_i)$ is an abelian extⁿ of k
 $k_i =$ splitting field of $f_i(x)$ contained in k'

Since $\forall v \notin S$, there exists $\beta_v \in k_v^*$ s.t. $\beta_v^m = \alpha$, β_v is a root of $f_i(x)$ for some i

$\Rightarrow$ One of the subfields k_i splits completely above v

Consequence: If $k' = k(\mu_m)$ is cyclic, subextⁿ of k'/k are linearly ordered, so among the k_i/k 's, there is a smallest one, say k_1 .
 Every $v \notin S$ splits completely in $k_1 \Rightarrow k_1 = k$, and $\alpha \in (k^*)^m$

step ④ Suppose p odd, $m = p^e$, let $k_1 = k(\mu_p) \subseteq k' = k(\mu_{p^e})$

k'/k_1 is cyclic. so by step ③, $\exists \beta \in k_1$ s.t. $\alpha = \beta^m$

But $d = [k_1 : k] \mid p-1$, and $N_{k_1/k} \alpha = N_{k_1/k} \beta^m = (N_{k_1/k} \beta)^m$

$\Rightarrow \alpha^d \in (k^*)^m$

$\alpha^m \in (k^*)^m$ $\gcd(d, m) = 1$

So by reduction step ①, $\alpha \in (k^*)^m$

step ⑤ The remaining case $m = 2^t$

- If $k(\mu_{2^t})/k$ is cyclic: done by step ③

- If $k(\mu_{2^t})/k$ is not cyclic, let $k_2 = k(\sqrt{-1}) = k(\mu_4)$

Then $k(\mu_{2^t})/k_2$ is cyclic $\xRightarrow{\text{step 3}} \exists \beta_2 \in k_2^*$ s.t. $\alpha = \beta_2^m$

$\Rightarrow N_{k_2/k}(\alpha) = (N_{k_2/k} \beta_2)^{2^t}$ so $\alpha = \pm \beta_3^{2^{t-1}}$
 $\parallel \alpha^2$ $\parallel \beta_3$

Claim: $\alpha \neq -\beta_3^{2^{t-1}}$ ^{Known} $t \geq 3$

Suppose $\alpha = -\beta_3^{2^{t-1}}$ Then the assumption on α implies:

$$\text{i.e. } -1 = \frac{\alpha}{\beta_3^{2^{t-1}}}$$

$$\forall v \notin S, -1 \in (\mathbb{P}_r^x)^{2^{t-1}} \subseteq (\mathbb{P}_r^x)^2$$

So $\mathbb{P}(\sqrt{-1})/\mathbb{P}$ splits at every $v \notin S \Rightarrow \sqrt{-1} \in \mathbb{P}$, a contradiction.
QED.

Note on references for Thm 2

Artin-Tate seminar, conducted 1951-52, Chap IX

Artin, Representatives of connected component of the idele class group
1955, Proc. International Symposium on Algebraic Number Theory, Tokyo 1955

Weil, On certain type of characters of idele-class group of an algebraic number-field, 1955, Proc. International Symposium on Algebraic Number Theory, Tokyo 1955.

Weil, Sur la théorie du corps de classes, 1951,

On p.10 of Weil's paper, Weil proved:

$$\begin{aligned} \left((A_k/k^x)^0 \right)^v &\stackrel{\text{def}}{=} \text{Hom}_{\text{cont}} \left((A_k/k^x)^0, \mathbb{C}_1^x \right) = \bigcup_{\mathcal{M} = \text{modulus}} \left(\prod_{v \in \Sigma_{k,\infty}} (A_v^x / \mathcal{O}_h^x(\mathcal{M})) \right)^v \\ &\quad \uparrow \qquad \qquad \qquad \uparrow \\ &\text{Pontryagin dual} \qquad \qquad \qquad \left\{ u \in \mathcal{O}_k^x \mid \begin{array}{l} u_v \in (k_v^x)^0 \quad \forall v \in \Sigma_{k,\infty} \\ u_v \equiv 1 \pmod{m_v} \quad \forall v \in \text{supp}(\mathcal{M}) \end{array} \right\} \\ &\text{of } (A_k/k^x)^0 \end{aligned}$$

$$\cong \mathbb{R} \times \mathbb{Q}^{r_1 r_2 - 1} \times \mathbb{Z}^{r_2}$$

Note: Weil used Chevalley's theorem.

Artin-Tate used Thm 3

Theorem 4 (Chevalley) K : number field. $E \leq K^\times$ a finitely generated subgroup.
 Let $m > 0$ be a positive integer, and let S be a subset of $\{\text{prime numbers}\}$ of Dirichlet density 0.

There exists a non-zero integer b prime to S such that

$$E(b) := \{x \in E \mid x \equiv 1 \pmod{b}\} \subseteq E^m$$

Proof Reduction step ①. May assume $m = p^e$, p : a prime number.

Reduction step ② Let $E' = \{y \in K^\times \mid \exists \neq 0 n \in \mathbb{N} \text{ s.t. } y^n \in E\}$. Then E' is a finitely generated subgroup of $K^\times$, and $E \leq E'$ of finite index.

$\Rightarrow$ $\forall$ prime power m . Suffices to show: (*) $\exists$ a non-zero integer b prime to S s.t. $E(b) \subseteq K^{*m}$

Main step ③ $m = p^e$, suppose $K \not\supseteq \mu_m$, then (*) holds

Let $L = K(E^{\frac{1}{m}})$. Then L/K is a finite abelian extⁿ, $[L:K] \in p^{\mathbb{N}}$

May assume $L \not\supseteq K$. Let $L_1, \dots, L_s$ be the subextⁿs of L/K with $[L_i:K] = p$

$\forall j$, $\exists$ a prime ideal $\mathfrak{q}_j \subseteq \mathcal{O}_K$ s.t. $\mathfrak{q}_j$ is prime to S , and $\mathfrak{q}_j$ stays prime in L_j and prime to m

Let p_j = the prime number below $\mathfrak{q}_j$.

$$b := \text{lcm}(p_1, \dots, p_s)$$

Given any $x \in E_K^{\neq 0}$ s.t. $x \equiv 1 \pmod{b}$. Consider $M := K(x^{\frac{1}{m}})$

Have $x \equiv 1 \pmod{\mathfrak{q}_j} \forall j=1, \dots, s$. because $\mathfrak{q}_j \mid a$

$\Rightarrow x \in (K_{\mathfrak{q}_j}^\times)^m$ $\circ^\circ$ $\mathfrak{q}_j$ is prime to m
 by Hensel's lemma

$\Rightarrow M/K$ is split at $\mathfrak{q}_j \Rightarrow M \not\subseteq L_j \forall j=1, \dots, s$

$$\Rightarrow M = K$$

Sublemma 1 Suppose $k(\sqrt{-1}) \not\subseteq k$, and $\#(k(\sqrt{-1}) \cap \mu_{2^\infty}) = 2^c$

Then $(k(\sqrt{-1})^{x^{2^{e+c}}}) \cap k^x \subseteq (k^x)^{2^e}$

p. 37 of Chevalley

Sublemma 2 In the case $p=2$, assume $\sqrt{-1} \in k$

p. 38 of Chevalley

$$k^x \cap (k(\mu_{p^e})^x)^{p^e} = (k^x)^{p^e}$$

Ref. Chevalley, Deux théorèmes arithmétiques, 1951, pp. 36-44

Left as "exercises"

Proof of sublemma 1. Given $x \in k^x$, $\sqrt{-1} \notin k$, $y \in k(\sqrt{-1})^x$, $x = y^{2^{e+c}}$

Let $f = \min \{f_1 \in \mathbb{N} : y^{2^{f_1}} \in k^x\}$ May assume $f > 0$.

So $y^{2^{f-1}} \notin k$. Consider the conjugate $\bar{y}$ of y relative to k .

$$\bar{y}^{2^f} = y^{2^f} \Rightarrow ((\bar{y}y^{-1})^{2^{f-1}})^2 = 1 \text{ and } (\bar{y}y^{-1})^{2^{f-1}} \neq 1 \text{ so } y^{2^{f-1}} \notin k$$

So: $(\bar{y}y^{-1})^{2^{f-1}} = -1$. So: $\bar{y}y^{-1} \in \mu_{2^f}$ is a primitive 2^f -th root of 1.

$$\text{i.e. } \langle \bar{y}y^{-1} \rangle = \mu_{2^f} \Rightarrow f \leq c \text{ i.e. } y^{2^c} \in k^x$$

$$\text{So } x = (y^{2^c})^{2^e} \in (k^x)^{2^e}$$

Proof of sublemma 2. Prove: $h \leq e-1$
Given $x \in k^x \cap (k(\mu_{p^{h+1}})^x)^{p^e}$, $\sqrt{-1} \in k$ if $p=2$

Then $x \in (k(\mu_{p^h})^x)^{p^e}$

May assume: $k(\mu_{p^h}) \not\subseteq k(\mu_{p^{h+1}})$

$$\text{Case } h=0 \Rightarrow [k(\mu_p) : k] \mid p-1,$$

$$\begin{aligned} \text{Write } x &= y^{p^e} \quad y \in k(\mu_p) \\ \Rightarrow x^d &= \left[N_{k(\mu_p)/k}(y) \right]^{p^e} \\ &\stackrel{||}{=} \end{aligned}$$

$$\begin{aligned} \left\{ \begin{array}{l} x^d \in (k^x)^{p^e} \\ x^{pe} \in (k^x)^{p^e} \end{array} \right. & \quad \gcd(d, p) = 1 \\ \Downarrow & \\ x &\in (k^x)^{p^e} \end{aligned}$$

So: We may assume $h \geq 1$, $k(\mu_{p^{h+1}}) \not\subseteq k(\mu_{p^h})$

Write the given element $x \in k^\times$ as $x = y^{p^e}$, $y \in k(\mu_{p^{h+1}})$

$h \geq 2$ if $p = 2$

$$\frac{k(\mu_{p^{h+1}})}{k(\mu_{p^h})} \text{ Gal}(k(\mu_{p^{h+1}})/k(\mu_{p^h})) = \langle \sigma \rangle$$

Let $\zeta = \zeta_{p^e}$ be a generator of μ_{p^e}

$$\frac{\sigma(y) \cdot y^{-1}}{\zeta^f} \in \mu_{p^e}$$

Pick $\tilde{\sigma} \in \text{Gal}(k(\mu_{p^e})/k)$ s.t. $\tilde{\sigma}|_{k(\mu_{p^{h+1}})} = \sigma$

Will show

$$f \equiv 0 \pmod{p^{e-1}}$$

$$\tilde{\sigma}(\zeta) = \zeta^g, \quad g \in \mathbb{Z}$$

$$\tilde{\sigma}(\zeta^{p^{e-h}}) = \zeta^{p^{e-h}g}$$

$$\leadsto g p^{e-h} \equiv p^{e-h} \pmod{p^e}$$

$$\leadsto g \equiv 1 \pmod{p^h}$$

$$\sigma(y) = \zeta^f \cdot y, \quad \sigma^p(y) = y$$

$$\zeta^f + f g + f g^2 + \dots + f g^{p-1} = \zeta^{f(1+g+\dots+g^{p-1})} \cdot y$$

$$\leadsto f(1+g+\dots+g^{p-1}) \equiv 0 \pmod{p^e}$$

In other words, used

$$\sigma^p - 1 = (\sigma - 1)(\sigma^{p-1} + \dots + \sigma + 1) \text{ in } \mathbb{Z}[\langle \sigma \rangle]$$

$$\text{Know: } g \equiv 1 \pmod{p^h}, \quad h \geq 2 \text{ if } p = 2$$

$$\Rightarrow 1+g+\dots+g^{p-1} = \frac{(g-1)^p}{g-1} \equiv p \pmod{p^2} \text{ by binomial expansion, if } p \neq 2$$

$$\text{if } p = 2, \quad 1+g \equiv 2 \pmod{2^h}$$

$$\because g \equiv 1 \pmod{2^h}$$

$$\text{So: } f \equiv 0 \pmod{p^{e-1}}$$

Note: The proof that $f \equiv 0 \pmod{p^{e-1}}$ follows from

$$\sigma(y) = \zeta^f \cdot y, \quad \sigma^p(y) = y, \text{ by examining } \tilde{\sigma}(\zeta) \in \mu_{p^e}$$

$$\sigma(y) \cdot y^{-1} \in \mu_p \subseteq k(\mu_{p^h})$$

$$\text{Let } \zeta_{p^{h+1}} \stackrel{\text{def}}{=} \zeta^{p^{e-h-1}}$$

$$\zeta_p \stackrel{\text{def}}{=} \zeta^{p^{e-1}}$$

$$\text{So: } \sigma(y) \cdot y^{-1} = (\sigma(\zeta_{p^{h+1}}) \cdot \zeta_{p^{h+1}}^{-1})^r$$

$$\text{assumed } \zeta_{p^{h+1}} \notin k(\mu_{p^h})$$

$\leadsto \sigma(\zeta_{p^{h+1}}) \cdot \zeta_{p^{h+1}}^{-1}$ is a primitive p -th root of 1

for some $r \in (\mathbb{Z}/p\mathbb{Z})^\times$

$$\Rightarrow \sigma(\underbrace{y \cdot \zeta_{p^{h+1}}^{-r}}_{y_1}) = y \cdot \zeta_{p^{h+1}}^{-r}$$

$$y_1 \in k(\mu_{p^h})$$

$$\text{and } x = y^{p^e} = y_1^{p^e} \in (k(\mu_{p^h})^\times)^{p^e} \cap k^\times. \text{ q.e.d. (of sublemma 2)}$$

Lemma (Corollary of Chevalley's theorem)

k : number field. $v_0 \in \Sigma_k$. $H \subseteq k_{v_0}^\times$ an open subgroup of finite index

Then: $\exists$ an open subgroup $U \subseteq A_k^\times / k^\times$ s.t. $U \cap k_{v_0}^\times \subseteq H$.

pf: May assume: v finite

$m = a$ modulus

$$\text{Let } U_0 = (k \otimes \mathbb{R})^{\times 0} \times \prod_{v: \text{finite}} \mathcal{O}_v^\times, \quad U_0(m) = (k \otimes \mathbb{R})^{\times 0} \times \prod_{v: \text{finite}} \mathcal{O}_v^\times(m_v)$$

$$k_{v_0}^\times \cap (k^\times \cdot U_0) = \mathcal{O}_{v_0}^\times \cdot \text{pr}_{v_0}(E_0)$$

$$k_{v_0}^\times \cap (k^\times \cdot U_m) = \mathcal{O}_{v_0}^\times(m_{v_0}) \cdot \text{pr}_{v_0} \left(\underbrace{\left\{ u \in E_0 \mid u_v \equiv 1 \pmod{m_v} \forall v \in \text{supp}(m) \right\}}_{E_m} \right)$$

E_0 finitely generated

$\exists$ modulus m s.t. $E_m \subseteq E_0^n$ by Chevalley's thm.

Take $n = [k_{v_0}^\times : H]$, m_1 s.t. $E_{m_1} \subseteq E_0^{[k_{v_0}^\times : H]}$

Then E and $\mathcal{O}_{k_0}^\times(m_{v_0}) \subseteq H$

$U(m_1) \subseteq H$. q.e.d.

This lemma justifies the proof of the existence theorem for local class field theory in the 1st ed. of Lang's "Alg. Nbr. Thy".

Back to $(A_k^x/k)^\circ$ k : number field $r = r_1 + r_2 - 1$

Proposition 5 Let $\varepsilon_1, \dots, \varepsilon_r$ be units in $\mathcal{O}_k^x$, image of $\varepsilon_1, \dots, \varepsilon_r$ in $\prod_{v \in \Sigma_{k,\infty}} \mathbb{R}_v$ $\mathbb{Z}$ -linearly indep.
Then $\langle \varepsilon_1, \dots, \varepsilon_r \rangle \otimes \hat{\mathbb{Z}} \rightarrow A_{k,f}^x$ is injective

pf: Suppose $x_1, \dots, x_r \in \hat{\mathbb{Z}}$ and $\varepsilon_1^{x_1} \dots \varepsilon_r^{x_r} = 1$ in $A_{k,f}^x$
Must show: $x_1 = \dots = x_r = 0$, or equivalently, $x_i \equiv 0 \pmod{m} \forall m, i=1, \dots, r$
Let $d = [\mathcal{O}_k^x : \langle x_1, \dots, x_r \rangle]$, $\forall m \in \mathbb{N}_{\geq 1}$, pick $n_1, \dots, n_r \in \mathbb{Z}$ $n_i \equiv x_i \pmod{2dm}$
 $i=1, \dots, r$

Consider $\varepsilon = \varepsilon_1^{n_1} \dots \varepsilon_r^{n_r}$, So $\varepsilon_f = \varepsilon_{1,f}^{n_1 - x_1} \dots \varepsilon_{r,f}^{n_r - x_r}$
 $\varepsilon = \varepsilon_{1,\infty}^{n_1} \dots \varepsilon_{r,\infty}^{n_r} \cdot \varepsilon_{1,f}^{n_1 - x_1} \dots \varepsilon_{r,f}^{n_r - x_r}$

Thm 3 (or, use Chevalley's thm at the cost of using a multiple of d) $\Rightarrow \varepsilon \in (\mathcal{O}_k^x)^{dm}$, Write ε as $\varepsilon = \eta^{dm}$ $\eta \in \mathcal{O}_k^x$

$\eta^d \in \langle \varepsilon_1, \dots, \varepsilon_r \rangle \Rightarrow \exists$ integers $b_1, \dots, b_r$ st. $\eta^d = \varepsilon_1^{b_1} \dots \varepsilon_r^{b_r}$
 $\Rightarrow \eta_{\infty}^d = \varepsilon_{1,\infty}^{b_1 m} \dots \varepsilon_{r,\infty}^{b_r m} = \varepsilon_{1,\infty}^{n_1} \dots \varepsilon_{r,\infty}^{n_r}$

$\Rightarrow n_i \equiv 0 \pmod{m} \forall i=1, \dots, r$

$\Rightarrow x_i \equiv 0, \forall m \Rightarrow x_i = 0 \forall i=1, \dots, r$

Assume moreover that each ε_i is totally positive
Define $\langle \varepsilon_1, \dots, \varepsilon_r \rangle \otimes (\mathbb{R} \times \hat{\mathbb{Z}}) \xrightarrow{\psi} A_{k,1}^x / k^x$ ψ not canonical.

Choose $\log \varepsilon_{i,\infty} \in (k \otimes \mathbb{R})$ $i=1, \dots, r$
 $\prod_{v \in \Sigma_{k,\infty}} k_v$

Indeterminately $\prod_{v \in \Sigma_{k,\text{cpx}}} 2\pi\sqrt{-1} \mathbb{Z}$

define $\psi' \left(\sum_{i=1}^r \varepsilon_i \otimes (t_i, x_i) \right) \stackrel{\text{def}}{=} \prod_{i=1}^r \left(\underbrace{e^{t_i \log \varepsilon_{i,\infty}}}_{(k \otimes \mathbb{R})^x} \cdot \underbrace{\varepsilon_{i,f}^{x_i}}_{A_{k,f}^x} \right) \in A_{k,1}^x / k^x$

Clearly ψ' descends to a well-defined continuous homomorphism

$$\psi: (\varepsilon_1, \dots, \varepsilon_r) \otimes ((\mathbb{R}^\times \hat{\mathbb{Z}})/\mathbb{Z}) \longrightarrow A_{k,1}^\times / k^\times$$

Lemma: $\tilde{\psi}: (\varepsilon_1, \dots, \varepsilon_r) \otimes ((\mathbb{R}/\hat{\mathbb{Z}})/\mathbb{Z}) \oplus \prod_{v \in \Sigma_{k,1}^\times} k_{v,1}^\times \longrightarrow A_{k,1}^\times / k^\times$ is injective

pf: Suppose that we have $(t_i, x_i) \in \mathbb{R}^\times \hat{\mathbb{Z}}$, $i=1, \dots, r$, $z_v \in k_{v,1}^\times$, $v \in \Sigma_{k,1}^\times$ and an element $u \in k^\times$ st.

$$\left(\prod_{i=1}^r e^{t_i \log \varepsilon_i, \infty} \right) \cdot \left(\prod_{i=1}^r \varepsilon_{i,f}^{x_i} \right) \cdot \prod_{v \in \Sigma_{k,1}^\times} z_v = u = (u_\infty, u_f)$$

Must show: $z_v = 1 \quad \forall v \in \Sigma_{k,1}^\times$ and $\exists n_i \in \mathbb{Z}$ st $t_i = n_i = x_i \quad \forall i=1, \dots, r$ and $u = \prod_{i=1}^r \varepsilon_i^{n_i}$

First look at the finite component, get

$$\prod_{i=1}^r \varepsilon_{i,f}^{x_i} = u_f \Rightarrow u \in \mathcal{O}_k^\times \leadsto u^d \in \langle \varepsilon_1, \dots, \varepsilon_r \rangle$$

Write u^d as $u^d = \varepsilon_1^{c_1} \dots \varepsilon_r^{c_r}$, $c_1, \dots, c_r \in \mathbb{Z}$

$$\begin{aligned} \leadsto \prod_{i=1}^r \varepsilon_{i,f}^{dx_i - c_i} &= 1 \Rightarrow dx_i = c_i \quad \forall i=1, \dots, r \text{ and } c_i \equiv 0 \pmod{d} \\ &\Rightarrow x_i = \frac{c_i}{d} \in \mathbb{Z} \quad \forall i=1, \dots, r \end{aligned}$$

Look at the image of $\xi_\infty: A_k^\times \longrightarrow \prod_{v \in \Sigma_{k,\infty}^\times} \mathbb{R}_{>0}^\times$

$\xi_\infty(\varepsilon_1), \dots, \xi_\infty(\varepsilon_r)$ are linearly indep. in $\prod_{v \in \Sigma_{k,\infty}^\times} \mathbb{R}_{>0}^\times$

Since $\prod_{i=1}^r \xi_\infty(\varepsilon_i)^{dt_i - c_i} = 1$, we get:

$$t_i = \frac{c_i}{d} = x_i \in \mathbb{Z} \quad \forall i=1, \dots, r$$

$$=: n_i$$

$$\leadsto \prod_{i=1}^r \varepsilon_i^{n_i} \cdot \prod_{v \in \Sigma_{k,1}^\times} z_v = u \in \prod_{k^\times} \leadsto u = \prod_{i=1}^r \varepsilon_i^{n_i} \text{ and } \prod_{v \in \Sigma_{k,1}^\times} z_v \in \langle \varepsilon_1, \dots, \varepsilon_r \rangle \cap \prod_{v \in \Sigma_{k,1}^\times} k_{v,1}^\times = \{1\}.$$

q.e.d.

$$\mathbb{R} \times \hat{\mathbb{Z}} / \mathbb{Z} \xrightarrow{\sim} A_{\mathbb{Q}} / \mathbb{Q}$$

We have:
$$\tilde{\psi} = \underbrace{\left((\varepsilon_1, \dots, \varepsilon_r) \otimes ((\mathbb{R} \times \hat{\mathbb{Z}}) / \mathbb{Z}) \right)}_{\substack{\parallel \\ \mathcal{O}_k^x \otimes_{\mathbb{Z}} (A_{\mathbb{Q}} / \mathbb{Q})}} \oplus \prod_{v \in \text{cpx}} k_{v,1}^x \longrightarrow A_{k,1}^x / k^x$$

(non-canonical)

continuous, injective,

source connected $\because A_{\mathbb{Q}} / \mathbb{Q}$ is connected.

$\Rightarrow \text{Image}(\tilde{\psi})$ is a connected compact closed subgroup of $A_{k,1}^x / k^x$

$$\text{and } \sim \text{Image}(\tilde{\psi}) \subseteq (A_{k,1}^x / k^x)^{\circ}$$

On the other hand, $\text{Image}(\tilde{\psi})$ contains the closure of $\text{Image}(\prod_{v \in \text{arch}} k_v^x \rightarrow A_k^x / k^x)$, i.e. $\text{Image}(\tilde{\psi}) \supseteq (A_k^x / k^x)^{\circ}$.
And clearly the image of $(\varepsilon_1, \dots, \varepsilon_r) \otimes \mathbb{R}$ in $A_{k,1}^x / k^x$ is connected $\Rightarrow \text{Image}(\tilde{\psi}) = (A_{k,1}^x / k^x)^{\circ}$

Have $1 \rightarrow (A_{k,1}^x / k^x)^{\circ} \rightarrow (A_k^x / k^x) \rightarrow \mathbb{R}_{\geq 0} \rightarrow 1$

$$1 \rightarrow \underbrace{\prod_{v \in \text{cpx}} k_{v,1}^x}_{\substack{\uparrow \\ \text{divisible}}} \rightarrow (A_{k,1}^x / k^x)^{\circ} \rightarrow (A_{k,1}^x / k^x) / (\prod_{v \in \text{cpx}} k_{v,1}^x) \rightarrow 1$$

induced
by ψ ,
canonical
isomorphism

$$\begin{array}{c} \xrightarrow{\psi} \uparrow s \\ \mathcal{O}_k^x \otimes_{\mathbb{Z}} (A_{\mathbb{Q}} / \mathbb{Q}) \end{array}$$

uniquely divisible

Remark

canonically,

$$(A_{k,1}^x / k^x)^{\vee} \cong \left(\prod_{v \in \Sigma_k^{\text{cpx}}} \mathbb{Z} \right) \times \text{Hom}_{\mathbb{Z}}(\mathcal{O}_k^x, \mathbb{Q})$$

(via the trace map $\text{tr}: A_{\mathbb{Q}} / \mathbb{Q} \rightarrow \mathbb{C}_1^x$)

Example: $k = \mathbb{Q}$, $s = 2$, $F = \mathbb{Q}(\sqrt{14}) = \mathbb{Q}(\sqrt{14}, \sqrt{2})$ $S_0 = \{2\}$

Special case occurs iff $8 \mid m$

For $m = 8$, $\alpha_0 = (1 + \sqrt{14})^8 = \sqrt{2}^8 = \sqrt{-2}^8 = 16$

$\Rightarrow 16$ is an 8-th power in $\mathbb{Q}(\sqrt{14})$, $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{-2})$

$\mathbb{Q}(\sqrt{14}, \sqrt{2})$ is unramified at every odd prime p .

$\mathbb{Q} \mid \Rightarrow$ One of -1 , 2 and -2 is a square in $\mathbb{Q}_p$

i.e. One of $\mathbb{Q}(\sqrt{14})_{/\mathbb{Q}}$, $\mathbb{Q}(\sqrt{2})_{/\mathbb{Q}}$, $\mathbb{Q}(\sqrt{-2})_{/\mathbb{Q}}$ splits at p .

So: α_0 is an 8-th power in $\mathbb{Q}_p$

(Clearly, α_0 is also an 8-th power in $\mathbb{R}$)

However $\alpha_0 \notin (\mathbb{Q}_2^\times)^8$ (and $\alpha_0 \notin \mathbb{Q}^{\times 8}$ either)

Class Formations

Definition A formation is a triple $(G, \{G_E\}_{E \in X}, A)$ as follows

- (a) G : a group, $\{G_E\}_{E \in X}$ a family of subgroups of G , $[G:G_E] < \infty$
 s.t. $G_E = G_{E'} \Rightarrow E = E'$ and satisfy (i)–(iii) $\forall E \in X$
- (i) $\{G_E\}$ is closed under finite intersection
- (ii) $G_F \leq G', F \in X \Rightarrow \exists! F' \in X$ s.t. $G' = G_{F'}$ notation
- (iii) $F \in X, s \in G \Rightarrow \exists! F'' \in X$ s.t. $G_{F''} = s \cdot G_F \cdot s^{-1}$; $F'' = sF$
 closed under conjugation

- (b) A : G -module s.t. $\forall a \in A, \exists E \in X$ s.t. $\text{stab}_G(a) = G_E$

↑ "formation module"

notation: $A_F = A^F = H^0(F, A)$

terminology: $E \in F \stackrel{\text{def}}{\iff} G_E \geq G_F$

F/E is Galois $\iff G_F \trianglelefteq G_E$

similarly: cyclic extensions $\rightsquigarrow G_{F/E} \stackrel{\text{def}}{=} G_E/G_F$
 abelian extensions
 solvable extensions

Axioms on class formations

axiom I F/E Galois $\Rightarrow H^1(F/E) = (0)$

$\stackrel{\text{def}}{=} H^1(G(E/F), A_F)$

↑↓

axiom I' F/E cyclic of prime order $\Rightarrow H^1(F/E) = (0)$

Notation $H^2(/E) \stackrel{\text{def}}{=} \varinjlim_{F/E} H^2(F/E)$

transition maps: $F'/F/E$

$\text{Inf}_{F'/F/E} = H^2(F'/E) \rightarrow H^2(F/E)$

axiom II

There is a family of homomorphisms $\text{axiom I} \Rightarrow \text{Inf}_{F'/F/E}$ is injective (Hochschild-Serre s.s.) for $q=2$

$\text{inv}_E: H^2(/E) \rightarrow \mathbb{Q}/\mathbb{Z} \quad E \in X$

s.t.

- (a) inv_E is injective and $\text{inv}_E(H^2(F/E)) =$ the unique subgroup of $\mathbb{Q}/\mathbb{Z}$ of order $[F:E]$ $\forall F/E$

- (b) $\forall E'/E, \quad H^2(/E) \xrightarrow{\text{Res}_{E'/E}} H^2(/E') \quad \text{commutes}$
 $\text{inv}_E \downarrow \quad \quad \downarrow \text{inv}_{E'}$
 $\mathbb{Q}/\mathbb{Z} \xrightarrow{[E':E]} \mathbb{Q}/\mathbb{Z}$

Class Formations

Definition A formation is a triple $(G, \{G_E\}_{E \in X}, A)$ as follows

(a) G : a group. $\{G_E\}_{E \in X}$ a family of subgroups of G , $[G:G_E] < \infty$
 s.t. $G_E = G_{E'} \Rightarrow E = E'$ and satisfy (i)–(iii) $\forall E \in X$

(i) $\{G_E\}$ is closed under finite intersection

(ii) $G_F \leq G_{F'} \Rightarrow \exists! F'' \in X$ s.t. $G_{F'} = G_{F''}$ notation

(iii) $F \in X, s \in G \Rightarrow \exists! F'' \in X$ s.t. $G_{F''} = s \cdot G_F \cdot s^{-1}$; $F'' = sF$

closed under conjugation

(b) A : G -module s.t. $\forall a \in A, \exists E \in X$ s.t. $\text{stab}_G(a) = G_E$

↑ "formation module"

notation: $A_F = A^F = H^0(F, A)$

terminology: $E \in F \stackrel{\text{def}}{\iff} G_E \geq G_F$

F/E is Galois $\iff G_F \trianglelefteq G_E$

similarly: cyclic extensions $\rightsquigarrow G_{F/E} \stackrel{\text{def}}{=} G_E/G_F$
 abelian extensions
 solvable extensions

Axioms on class formations

axiom I F/E Galois $\Rightarrow H^1(F/E) = (0)$

↑ $\stackrel{\text{def}}{=} H^1(G(E/F), A_F)$

↑

axiom I' F/E cyclic of prime order $\Rightarrow H^1(F/E) = (0)$

Notation $H^i(_/E) \stackrel{\text{def}}{=} \varinjlim_{F/E} H^i(F/E)$

transition maps: $F'/F/E$

$\text{Inf}_{F/E} = H^1(F/E) \rightarrow H^1(F'/E)$

axiom II

There is a family of homomorphisms $\text{axiom I} \Rightarrow \text{Inf}_{F/E} \text{ is injective (Hochschild-Serre s.s.) for } q=2$

$\text{inv}_E: H^2(_/E) \rightarrow \mathbb{Q}/\mathbb{Z} \quad E \in X$

s.t.

(a) inv_E is injective and $\text{inv}_E(H^2(F/E)) =$ the unique subgroup of $\mathbb{Q}/\mathbb{Z}$ of order $[F:E]$ $\forall F/E$

(b) $\forall E'/E, H^2(_/E) \xrightarrow{\text{Res}_{E'/E}} H^2(_/E')$ commutes
 $\text{inv}_E \downarrow \quad \text{inv}_{E'} \downarrow$
 $\mathbb{Q}/\mathbb{Z} \xrightarrow{[E':E]} \mathbb{Q}/\mathbb{Z}$

Class formations

Definition A formation is a triple $(G, \{G_E\}_{E \in X}, A)$ as follows

(a) G : a group, $\{G_E\}_{E \in X}$ a family of subgroups of G , $[G:G_E] < \infty$
 s.t. $G_E = G_{E'} \Rightarrow E = E'$ and satisfy (i)–(iii) $\forall E \in X$

(i) $\{G_E\}$ is closed under finite intersection

(ii) $G_F \leq G_{F'} \Rightarrow \exists F'' \in X$ s.t. $G_{F'} = G_{F''}$

notation

(iii) $F \in X, s \in G \Rightarrow \exists F' \in X$ s.t. $G_{F'} = s \cdot G_F \cdot s^{-1}$; $F' = sF$

closed under conjugation

(b) A : G -module s.t. $\forall a \in A, \exists E \in X$ s.t. $\text{Stab}_G(a) = G_E$

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Axioms on class formations

axiom I

F/E Galois $\Rightarrow H^1(F/E) = (0)$

↑ $\stackrel{\text{def}}{=} H^1(G(E/F), A_F)$

↕

axiom I'

F/E cyclic of prime order $\Rightarrow H^1(F/E) = (0)$

Notation $H^2(/E) \stackrel{\text{def}}{=} \varinjlim_{F/E} H^2(F/E)$

transition maps: $F'/F/E$

$\text{Inf}_{F/E} = H^2(F/E) \rightarrow H^2(F'/E)$

axiom II

There is a family of homomorphisms $\text{Inf}_{F/E} \Rightarrow \text{Inf}_{F'/E}$ is injective (Hochschild-Serre s.s.) for $q=2$

$\text{inv}_E: H^2(/E) \rightarrow \mathbb{Q}/\mathbb{Z} \quad E \in X$

s.t.

(a) inv_E is injective and $\text{inv}_E(H^2(F/E)) =$ the unique subgroup of $\mathbb{Q}/\mathbb{Z}$ of order $[F:E]$ $\forall F/E$

(b) $\forall E'/E, H^2(/E) \xrightarrow{\text{Res}_{E'/E}} H^2(/E')$ commutes
 $\text{inv}_E \downarrow \quad \text{inv}_{E'} \downarrow$
 $\mathbb{Q}/\mathbb{Z} \xrightarrow{[E:E']} \mathbb{Q}/\mathbb{Z}$

Lemma

$$(1) \quad E'/E \rightsquigarrow H^2(F/E) \xrightarrow{\text{Res}_{E'/E}} H^2(F/E')$$

$$(2) \quad E'/E \rightsquigarrow H^2(F/E') \xrightarrow{\text{Cor}_{E'/E}} H^2(F/E)$$

$$\begin{array}{ccc} \text{inv}_{E'} \searrow & & \swarrow \text{inv}_E \\ & \mathbb{Q}/\mathbb{Z} & \end{array} \quad \text{commutes}$$

$$(3) \quad s \in G. \quad H^2(F/E) \xrightarrow{s^*} H^2(F/sE)$$

$$\begin{array}{ccc} \text{inv}_E \searrow & & \swarrow \text{inv}_{sE} \\ & \mathbb{Q}/\mathbb{Z} & \end{array}$$

Here s^* is defined by

$$\begin{array}{ccc} H^2(F/E) & \xrightarrow{s^*} & H^2(F/sE) \\ \parallel & & \parallel \\ H^2(G_E/G_F, A_F) & \xrightarrow{\quad} & H^2(G_{sE}/G_{sF}, A_{sF}) \end{array}$$

↑
induced by

$$\begin{array}{ccc} G_E/G_F & \xleftarrow{\sim} & G_{sE}/G_{sF} \\ \downarrow \psi & & \downarrow t \\ s.t.s & \xleftarrow{\quad} & t \\ A_F & \xrightarrow{\sim} & A_{sF} \\ a & \xrightarrow{\quad} & s.a \end{array}$$

Fundamental class:

$$u_{F/E} \in H^2(F/E), \quad \text{inv}_E(u_{F/E}) = \frac{1}{h}$$

Properties

$$(i) \quad \text{Inf}_{F'/F/E} : H^2(F/E) \longrightarrow H^2(F'/E) \quad F'/E \text{ and } F/E \text{ Galois}$$

$$u_{F/E} \longmapsto [F':F] \cdot u_{F'/E}$$

$$(ii) \quad \text{Res}_{F/E'/E} : H^2(F/E) \longrightarrow H^2(F/E') \quad F/E \text{ Galois}$$

$$u_{F/E} \longmapsto u_{F/E'}$$

$$(iii) \quad \text{Cor}_{F/E'/E} : H^2(F/E') \longrightarrow H^2(F/E) \quad F/E \text{ Galois}$$

$$u_{F/E'} \longmapsto [E':E] \cdot u_{F/E}$$

$$(iv) \quad s^*: H^2(F/E) \longrightarrow H^2(sF/sE) \quad F/E \text{ Galois}$$

$$u_{F/E} \longmapsto u_{sF/sE}$$

Theorem F/E Galois, $n \in \mathbb{Z}$

$$\hat{H}^n(G_{F/E}, \mathbb{Z}) \xrightarrow{u_{F/E} \cdot} \hat{H}^{n+2}(G_{F/E}, A_F) = \hat{H}^{n+2}(F/E)$$

$$x \longmapsto u_{F/E} \cdot x$$

is bijective.

Special case: (1) $n=1 \leadsto H^1(G_{F/E}, \mathbb{Z}) = 0 = H^3(F/E)$

From $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$, get

$$H^2(G_{F/E}, \mathbb{Z}) \xleftarrow{\sim} \text{Hom}(G_{F/E}, \mathbb{Q}/\mathbb{Z})$$

$$\downarrow \text{Is}$$

$$H^4(F/E)$$

(2) $n=-2 \leadsto \hat{H}^{-2}(G_{F/E}, \mathbb{Z}) = G_{F/E}^{ab}$

$$\theta_{F/E} \downarrow \text{Is}$$

$$\hat{H}^0(G_{F/E}) = A_E / \underbrace{N_{E/F}(A_E)}$$

$$N_{E/F} \longleftrightarrow \sum_{\sigma \in G_E/G_F} \sigma$$

The inverse of the above isomorphism is the reciprocity law map

$$\text{rec}_{F/E} = A_E / N_{E/F} A_E \xrightarrow{\sim} G_{F/E}^{ab}$$

Remark: If the fundamental class $u_{F/E} \in H^2(G_{F/E}, A_F)$

is represented by a 2-cocycle $u_{\sigma, \tau} \in Z^2(G_{F/E}, A_F)$,

then $\theta(s) = \sum_{t \in G} u(t, s)$ sum over "the 1st variable"

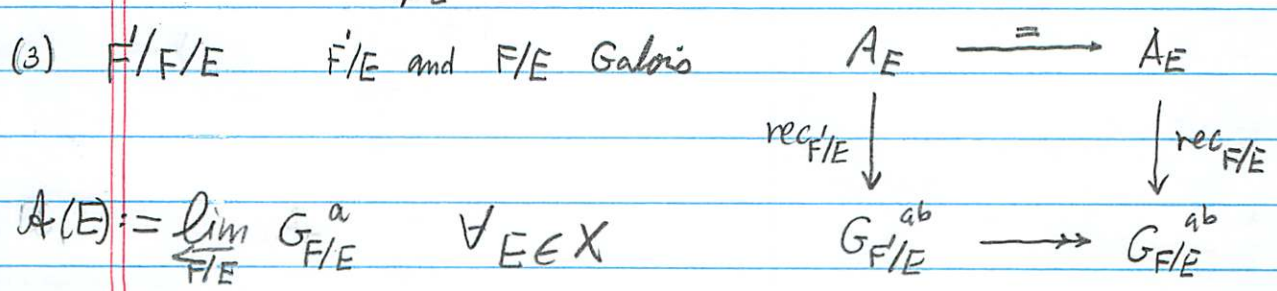
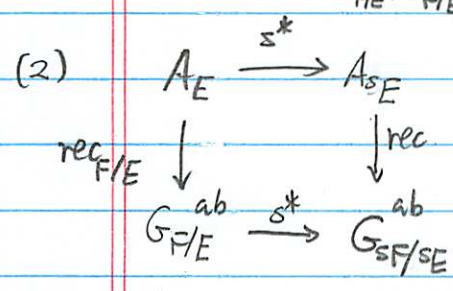
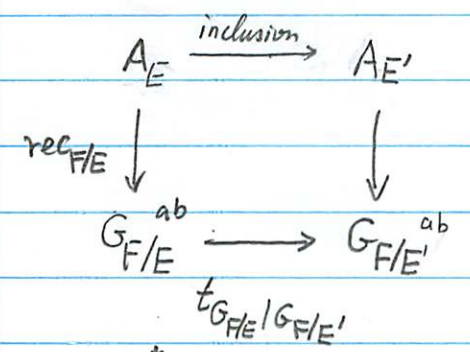
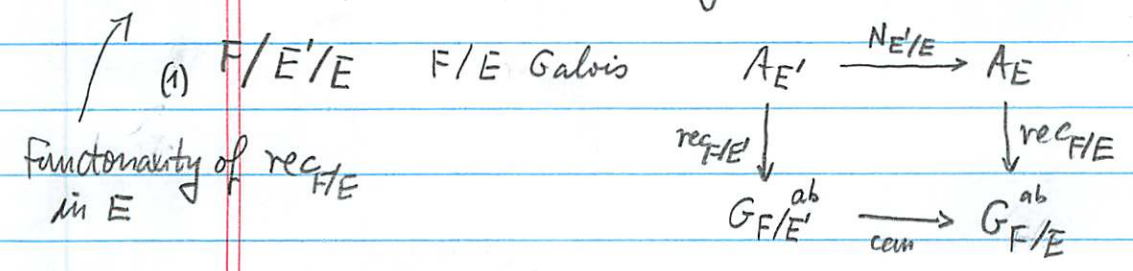
Given $\chi \mapsto d\chi$ from $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$

Proposition $\text{Hom}(G_{F/E}^{\text{an}}, \mathbb{Q}/\mathbb{Z}) \rightarrow H^2(G_{F/E}, \mathbb{Z})$

and $a \in A_E$, $\bar{a} \in A_E/N_{F/E}A_F = \hat{H}^0(G_{F/E}, A_F) = \hat{H}^0(F/E)$, have

$$\langle \chi, \underset{\substack{\uparrow \\ \text{rec}_{F/E}(a)}}{(a, F/E)} \rangle = \text{inv}_E(\bar{a} \cdot d\chi)$$

Functorial properties of reciprocity law:



Def $A(E) := \varprojlim_{F/E} G_{F/E}^a \quad \forall E \in X$

$\leadsto \text{rec}_E = A_E \rightarrow A(E)$

norm subgroups

Def. $E'/E \rightsquigarrow N_{E'/E} : A_{E'} \rightarrow A_E$
 $a \mapsto \sum_{s \in G_E/G_{E'}} s \cdot a$

A subgroup $I \leq A_E$ is a norm subgroup of A_E if $\exists E'/E$
 s.t. $I = N_{E'/E}(A_{E'})$

Prop Given an extⁿ E'/E , let E'' = the largest subextⁿ of E'/E
 s.t. E''/E is abelian.

Then $N_{E'/E}(A_{E'}) = N_{E''/E}(A_{E''})$

pf. $\subseteq$ obvious $\because N_{E'/E} = N_{E''/E} \circ N_{E'/E''}$

$\supseteq$: Given $a \in N_{E'/E}(A_{E'}) \rightsquigarrow (a, E''/E) = 1$ in $G_{E''/E}$
 Pick $F/E'/E$ with F/E Galois $= G_{F/E}/[G_{F/E}, G_{F/E}] \cdot G_{F/E'}$

So the elt $(a, F/E) \in G_{F/E}^{ab}$ is in Image $(G_{F/E'}^{ab} \rightarrow G_{F/E}^{ab})$

$$\begin{array}{ccc} A_{E'} & \xrightarrow{N_{E'/E}} & A_E \\ \downarrow & & \downarrow \\ G_{F/E'}^{ab} & \longrightarrow & G_{F/E}^{ab} \end{array} \rightsquigarrow \exists a' \in A_{E'} \text{ s.t. } (N_{E'/E}(a'), F/E) = (a, F/E)$$

$$\rightsquigarrow \exists b \in A_F \text{ s.t. } N_{E'/E}(a') - a = N_{F/E}(b)$$

$$\rightsquigarrow a = N_{E'/E}(a' - N_{F/E}(b)) \in N_{E'/E}(A_{E'})$$

q.e.d.

Axioms for the existence theorem

Hausdorff

Assume the groups A_E are endowed with topologies s.t.

- $\forall E \subseteq F$, A_E is a topological subgroup of A_F
- $\forall s \in G$, $A_E \xrightarrow{a \mapsto s \cdot a} A_{sE}$ is a topological isomorphism

Axiom III-1 $\forall \text{ ext}^n F/E$, $N_{F/E} : A_F \rightarrow A_E$
 $\text{Ker}(N_{F/E})$ is compact and $\text{Im}(N_{F/E})$ is closed in A_E

$$D_E := \text{universal norms in } E = \bigcap_{F/E} N_{F/E} A_F$$

Axiom III-2 $\forall$ prime number $p \exists$ a "field" E_p s.t.

$$\forall E \text{ s.t. } E \supseteq E_p, [p]_{A_E} : A_E \rightarrow A_E \text{ satisfies}$$

$$\text{Ker}([p]_{A_E}) \text{ is compact and } \text{Im}([p]_{A_E}) \supseteq D_E$$

Axiom III-3 $\exists$ a compact subgroup U_E of A_E s.t. every close subgroup containing U_E and of finite index in U_E is a norm subgroup of A_E

Theorem Axioms III-1, III-2, III-3 implies: a subgroup of A_E is a norm subgroup of A_E if and only if it is a closed subgroup and of finite index in A_E

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(generalized)

IX-7

Principal ideal

Theorem: W : group s.t. $[W, W]$ is finitely generated and is of finite index in W .
 H : subgroup s.t. $W' \subseteq H$ and $[H, H] \subseteq [W', W']$. Let $e = [H : W']$.
 Let $\text{Ver}_{W/H} : W^{ab} \rightarrow H^{ab}$ be the transfer map.

Then $e \cdot \text{Ver}_{W/H}$ is the zero map.

Note The special case when $H = [W, W']$ is the traditional principal ideal theorem

Pf: Replacing (W, H) by $(W/[W', W], H/[W', W])$, we may assume H is abelian

$$1 \rightarrow H \rightarrow W \xrightarrow{\pi} W/H \rightarrow 1$$

$$\text{Let } n = \# G = [W : H]$$

$$e = [H : W']$$

Let $s : W/H \rightarrow W$ be a section of π

Let $(a_{\sigma, \tau}) \in Z^2(G, H)$ be the 2-cocycle with

$$a_{\sigma, \tau} = s_{\sigma} \cdot s_{\tau} \cdot s_{\sigma\tau}^{-1} \quad \text{i.e. } s_{\sigma} \cdot s_{\tau} = a_{\sigma, \tau} s_{\sigma\tau}$$

Let $B =$ the G -module defined by $B = H \oplus \sum_{\tau \neq 1, \tau \in G} \mathbb{Z} \cdot x_{\tau}$

with G -action

$$\sigma \cdot x_{\tau} = x_{\sigma\tau} - x_{\sigma} + a_{\sigma, \tau} \quad \text{where } x_1 = 0 \text{ by convention}$$

Note that $a_{1, \tau} = 1 = a_{\sigma, 1} \quad \forall \sigma, \tau \in G$ (i.e. $\tau^{-1} \cdot x_{\tau} = -x_{\tau^{-1}} + a_{\tau^{-1}, \tau}$)

$$\text{Fact (i)} \quad 0 \rightarrow H \rightarrow B \rightarrow I_G \rightarrow 0$$

$$x_{\tau} \mapsto \tau - 1$$

$$\left(\begin{array}{l} x_{\tau} \mapsto \tau - 1 \\ \sigma \cdot x_{\tau} \mapsto \sigma\tau - \sigma = \sigma \cdot (\tau - 1) \\ x_{\sigma\tau} - x_{\sigma} + a_{\sigma, \tau} \end{array} \right)$$

$$(ii) \quad \{ \gamma \in \mathbb{Z}[G] \mid \gamma \cdot B \subseteq H \} = \mathbb{Z} \cdot N_G$$

By (i), this is equiv to $\{ \gamma \in \mathbb{Z}[G] \mid \gamma \cdot I_G = 0 \} = \mathbb{Z} \cdot N_G$

$$(iii) \quad B/I_G \cdot B \xrightarrow[\log]{\exp} W^{ab} = W/W'$$

$$h + x_{\sigma} \mapsto h \cdot s_{\sigma} \text{ mod } W'$$

$$(iii a) \quad \log(h_1 \cdot s_{\sigma} \cdot h_2 \cdot s_{\tau}) = \log(h_1 \cdot h_2 \cdot s_{\sigma} \cdot s_{\tau})$$

$$\log(h_1 \cdot s_{\sigma}) + \log(h_2 \cdot s_{\tau})$$

$$= h_1 + x_{\sigma} + h_2 + x_{\tau}$$

$$\text{mod } I_G B$$

$$h_1 + h_2 + a_{\sigma, \tau} + x_{\sigma\tau} \text{ mod } I_G B$$

and

$$x_{\sigma} + x_{\tau} - x_{\sigma\tau} + (1 - \sigma) h_2 - a_{\sigma, \tau}$$

$$= (1 - \sigma) x_{\tau} + (1 - \sigma) h_2 \in I_G B$$

$$(iii) \exp: B \longrightarrow W^{ab} = W/W'$$

$$H \ni h \longmapsto h \bmod W'$$

$$x_\tau \longmapsto s_\tau \bmod W$$

$$\text{check } \exp(I_G B) = 0 :$$

$$\exp((\sigma-1)h) = h^{\sigma-1} W' = s_\sigma \cdot h \cdot s_\sigma^{-1} \cdot h^{-1} W' = W' \text{ in } W/W'$$

$$\begin{aligned} \exp((\sigma-1)x_\tau) &= \exp(x_{\sigma\tau} - x_\tau - x_\tau + a_{\sigma,\tau}) \\ &= s_{\sigma\tau} \cdot s_\tau^{-1} \cdot s_\sigma^{-1} \cdot a_{\sigma,\tau} \pmod{W'} = W' \text{ in } W/W' \end{aligned}$$

$$(iv) \begin{array}{ccc} W^{ab} & \xrightarrow{\text{Ver}_{W/H}} & H^{ab} \\ \log \downarrow & & \parallel \\ B/I_G B & \xrightarrow{N_G} & H \end{array}$$

$$\text{Ver}_{W/H}(s_\tau W') = \prod_{\sigma \in G} s_\sigma \cdot s_\tau \cdot s_{\sigma\tau}^{-1} = \prod_{\sigma \in G} a_{\sigma,\tau} \text{ in } H$$

$$\begin{aligned} &= \prod_{\sigma \in G} \sigma x_\tau - x_{\sigma\tau} + x_\tau = N_G \cdot x_\tau \text{ in } H \\ &= N_G \cdot \log(s_\tau W') \end{aligned}$$

$$h \in H$$

$$\text{Ver}_{W/H}(h \cdot W') = N_G \cdot h \cdot \text{Ver}_{W/H}(W')$$

$$= N_G(\log(h \bmod W'))$$

Remark: (ii)-(iv) hold w/o assuming H is a f.gen. abelian gp. q.e.d.

(i)-(iv) Give a convenient model/form of W^{ab} and $\text{Ver}_{W/H}: W^{ab} \rightarrow H$

Back to the proof of Theorem (principal ideal)

Must show: $B/I_G B \xrightarrow{N_G} H$ is killed by $e = [H: W']$

$$\text{Let } n = |G| = [W: H]$$

Note: G is a finite abelian group

H is a fin. gen. abelian group

$B/I_G B$ finite abelian group

Pick

$$b_1, \dots, b_r \in B$$

$$B/I_G B = \langle \bar{b}_1 \rangle \oplus \dots \oplus \langle \bar{b}_r \rangle \quad \bar{b}_i = \text{image of } b_i \text{ in } B/I_G B$$

Let $d_i = \text{order of } \bar{b}_i \text{ in } B/I_G B$

$$\Rightarrow \prod_{i=1}^r d_i = [W: W'] = ne$$

$$d_i \cdot b_i \in I_G B$$

Pick generator $b_{r+1}, \dots, b_m$ of the f.gen. abelian group $I_G B$

$\Rightarrow b_1, \dots, b_m$ generate the $\mathbb{Z}$ -module B . Define $d_{r+1} = \dots = d_m \stackrel{\text{def}}{=} 1$

Pick $\xi_{ij} \in I_G$ s.t.

$$d_i \cdot b_i = \sum_{j=1}^m \xi_{ij} \cdot b_j$$

$$\forall i=1, \dots, m$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad d_i \cdot b_i \in I_G B$$

$$\forall i=1, \dots, m$$

Let $C = (d_i \delta_{ij} - \xi_{ij}) \in M_m(\mathbb{Z}[G])$

$\Rightarrow \gamma \stackrel{\text{def}}{=} \det(C) \cdot B = 0 \quad \gamma \in \mathbb{Z}[G]$

$\Rightarrow \gamma \in \mathbb{Z} \cdot N_G \quad \exists! c \in \mathbb{Z} \text{ s.t. } \gamma = c \cdot N_G$
(ii)

Apply $\varepsilon: \mathbb{Z}[G] \rightarrow \mathbb{Z}$ to γ : $\varepsilon(\gamma) = \det(d_i) = ne$
 $\sigma \mapsto 1$
 $= \varepsilon(c \cdot N_G) = c \cdot n \Rightarrow c = e$

$\Rightarrow \gamma = e \cdot N_G$ i.e.

$e \cdot N_G: B/IB \rightarrow H$ is the zero map. QED

Hausdorff

Def A Weil group $(W, \{W_F\})$ is a topological group W , together with a family of subgroups $\{W_F\}_{F \in X}$ s.t

WG 1) $W_F \subseteq W$ is open of finite index $\forall F$

$\{W_F\}_{F \in X}$ is closed under finite intersection and conjugation, and if a subgroup $H \leq W$ contains some W_F , then $\exists! E$ s.t. $H = W_E$

WG 2) $\forall$ pair $W_K \trianglelefteq W_F$, the transfer map $\text{Ver}_{K/F}$ factors as

$$\text{Ver}_{K/F}: W_F^{ab} \rightarrow W_K^{ab}$$

$$\uparrow \quad \searrow \quad \uparrow$$

$$\quad \quad \quad (W_K^{ab})^{W_F/W_K}$$

topological isom

and $\text{coker}(\text{Ver}_{K/F}) = W_K^{ab} / (W_K^{ab})^{W_F/W_K}$ is compact.

WG 3) The canonical map $W \rightarrow \varprojlim_F W / [W_F, W_F]$ is a topological isomorphism.

Nakayama-Tate theorem

(space version)

Prop 1 G : finite p -group A : $\mathbb{F}_p[G]$ -module. The following are equiv.

(i) $\exists q$ s.t. $\hat{H}^q(G, A) = (0)$

(ii) A is cohomologically trivial(iii) A is an induced $\mathbb{F}_p[G]$ -module (from the trivial subgroup)(iv) A is a free $\mathbb{F}_p[G]$ -module(key step: $H^1(G, A) = (0) \Rightarrow A$ is a free $\mathbb{F}_p[G]$ -module)Prop 1' G : finite p -group A : torsion free G -module. The following are equiv.

(i) $\hat{H}^{q_0}(G, A) = (0) = \hat{H}^{q_0+1}(G, A)$ for some $q_0 \in \mathbb{Z}$

(ii) A is cohomologically trivial(iii) $A \otimes_{\mathbb{Z}} \mathbb{F}_p$ is a free $\mathbb{F}_p[G]$ -module(use $0 \rightarrow A \xrightarrow{[p]} A \rightarrow A/pA \rightarrow 0$)Prop 2 G : finite group A : free $\mathbb{Z}$ -module G -module A is a projective $\mathbb{Z}[G]$ -module $\Leftrightarrow \forall$ prime number p , the G_p -module A satisfies the equiv. cond^{ns} in prop 1' $\uparrow$
 p -Sylow subgroupProp 3 G : finite group A : $\mathbb{Z}[G]$ -module. The following are equiv.

(i) $\forall$ prime number p , $\exists q_p$ s.t. $\hat{H}^{q_p}(G_p, A) = \hat{H}^{q_p+1}(G_p, A) = (0)$

(ii) $\text{proj. dim}_{\mathbb{Z}[G]}(A) < \infty$

(iii) $\text{proj. dim}_{\mathbb{Z}[G]}(A) \leq 1$

 $\uparrow$ free $\mathbb{Z}[G]$ -module(Apply Prop 2 to R , $0 \rightarrow R \rightarrow L \rightarrow A \rightarrow 0$ $\uparrow$ free $\mathbb{Z}$ -module $\Rightarrow R$ is a free $\mathbb{Z}$ -module)

(map version)

Prop. 4 G : finite group $A \xrightarrow{f} A'$ G -equiv. map of G -modulesSuppose that $\forall$ prime number p , $\exists n_p \in \mathbb{Z}$ s.t.

$$\hat{H}^i(f) = \hat{H}^i(G_p, A) \longrightarrow \hat{H}^i(G_p, A')$$

is surjective for $i = n_p$.isom. for $i = n_p + 1$ injective for $i = n_p + 2$.Then $\forall$ G -module B s.t. $\text{Tor}_1^{\mathbb{Z}}(A, B) = (0) = \text{Tor}_1^{\mathbb{Z}}(A', B)$.

the map

$$\hat{H}^i(H, A \otimes_{\mathbb{Z}} B) \xrightarrow{\hat{H}^i(f)} (H, A' \otimes_{\mathbb{Z}} B)$$

is bijective $\forall$ subgroup $H \leq G$ and $\forall i \in \mathbb{Z}$.Theorem 5 G : finite group A, B, C = G -modules $f: A \times B \longrightarrow C$ G -equiv. bilinear map

$$a \in \hat{H}^q(G, A)$$

Assume: $\forall$ prime p , $\exists n_p \in \mathbb{Z}$ s.t.

$$\hat{H}^i(G_p, B) \xrightarrow{a \cup \cdot} \hat{H}^{i+q}(G_p, C)$$

$$\text{Res}_{G_p}^G(a) \cup \cdot$$

is surjective for $i = n_p$,bijective for $i = n_p + 1$ injective for $i = n_p + 2$ Then $\forall$ G -module D s.t. $\text{Tor}_1^{\mathbb{Z}}(B, D) = \text{Tor}_1^{\mathbb{Z}}(C, D) = (0)$ $\forall H \leq G$.

$$\hat{H}^i(H, B \otimes_{\mathbb{Z}} D) \xrightarrow[\text{Res}_H^G(a) \cup \cdot]{\cong} \hat{H}^{i+q}(H, C \otimes_{\mathbb{Z}} D)$$

Cor 6 G : finite group A : G -module $a \in H^2(G, A)$

Assume $H^1(G_p, A) = (0) \quad \forall$ prime number p

$$H^2(G_p, A) = \mathbb{Z} \cdot \text{Res}_{G_p}^G(a), \text{ and } |H^2(G_p, A)| = |G_p|$$

Then $\forall$ G -module D s.t. $\text{Tor}_1^{\mathbb{Z}}(A, D) = 0$,

$$\hat{H}^i(H, D) \xrightarrow[\text{Res}_{H(a)}^G \cup]{\cong} \hat{H}^i(H, A \otimes_{\mathbb{Z}} D) \quad \forall i \in \mathbb{Z}$$

In particular,

Cor 7 (= case $D = \mathbb{Z}$ in Cor. 6)

$$\hat{H}^i(H, \mathbb{Z}) \xrightarrow[\text{Res}_{H(a)}^G \cup]{\cong} \hat{H}^{i+2}(H, A) \quad \forall i \in \mathbb{Z}$$

Back to group extensions

$$1 \rightarrow A \rightarrow W \rightarrow G \rightarrow 1$$

$$u \in Z^2(G, A)$$

a 2-cocycle attached to this extⁿ

Prop 1 $I_G \cdot A \subseteq [W, W] \cap A \subseteq A[N_G]$

and $[W, W] \cap A = \{a \in A[N_G]\}$

$$\exists \xi \in \hat{H}^{-3}(G, \mathbb{Z}) \leq 4$$

$$[u] \cdot \xi = a \pmod{I_G A}$$

$$\text{in } \hat{H}^{-1}(G, A) = A[N_G]/I_G A$$

Prop 2

$$\begin{array}{ccccc} 0 \rightarrow A/[W, W] \cap A & \xrightarrow{W^{ab}} & W/[W, W] & \xrightarrow{\hat{H}^{-2}(G, \mathbb{Z})} & G^{ab} \rightarrow 0 \\ \downarrow N_G & & \downarrow \text{Ver}_{W/A} & & \downarrow [u] \cup \\ 0 \rightarrow NA & \xrightarrow{} & A^G & \xrightarrow{} & \hat{H}^0(G, A) \rightarrow 0 \\ & & \uparrow N_G \cdot A & & \end{array}$$

$\Rightarrow$ Have short exact sequence

$$0 \rightarrow \hat{H}^{-1}(G, A) / ([u] \cdot \hat{H}^{-3}(G, \mathbb{Z})) \rightarrow \text{Ker}(\text{Ver}_{W/A}) \rightarrow \text{Ker}(G^{ab} \xrightarrow{[u]} \hat{H}^0(G, A)) \rightarrow 0$$

and $A^G / \text{Im}(\text{Ver}_{W/A}) \xrightarrow{\cong} \text{Coker}(G^{ab} \xrightarrow{[u]} \hat{H}^0(G, A))$

Prop 1. reformulate:

$$\chi: A \rightarrow \mathbb{Q}/\mathbb{Z}$$

$$\chi(1 \rightarrow A \rightarrow W \rightarrow G \rightarrow 1)$$

$$\begin{array}{ccccc} \downarrow \chi & \downarrow & \downarrow \chi & \downarrow & \downarrow \\ 1 \rightarrow \mathbb{Q}/\mathbb{Z} & \rightarrow & \chi(W) & \rightarrow & G \rightarrow 1 \end{array}$$

$$\chi_W \text{ splits} \Leftrightarrow [\chi_W u] = 0$$

$$\text{in } \hat{H}^{-2}(G, \mathbb{Q}/\mathbb{Z})$$

$$s_\sigma \cdot s_\tau = u_{\sigma, \tau} \cdot s_{\sigma\tau}$$

$[u]$ = the class of $(u_{\sigma\tau})$
in $H^2(G, A)$ IX-15

$$1 \rightarrow A \rightarrow W \xrightarrow{\xi} G \rightarrow 1$$

How to see directly: (i) $I_G \cdot A \subseteq A \cap [W, W]$

$$\because \forall a \in A, \forall \sigma \in G, a^\sigma \cdot a^{-1} = u a u^{-1} \cdot a^{-1} \in [W, W]$$

Pick $u \in W, u \mapsto \sigma$

$$(ii) [W, W] \cap A \subseteq A[N_G]$$

$\because \exists$ a group homomorphism (namely $\text{Ver}_{W/A}$) from W to A
whose restriction to W is $N_G: A \rightarrow A$

Proof that $[W, W] \cap A / I_G A \hookrightarrow A[N_G] / I_G A = \hat{H}^{-1}(G, A)$

$$\begin{array}{c} \uparrow [u] \cdot \\ \hat{H}^{-3}(G, \mathbb{Z}) \end{array}$$

$$\text{Hom}(\hat{H}^{-3}(G, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \cong H^2(G, \mathbb{Q}/\mathbb{Z}), \quad \text{Hom}(\hat{H}^{-1}(G, A), \mathbb{Q}/\mathbb{Z}) \cong \hat{H}^0(G, A^\vee)$$

$\uparrow$
 $\text{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$

perfect pairing $\rightarrow H^2(G, \mathbb{Q}/\mathbb{Z}) \times \hat{H}^{-3}(\mathbb{Z}) \longrightarrow \hat{H}^{-1}(G, \mathbb{Q}/\mathbb{Z}) = I_G \Gamma^1 \mathbb{Z} / \mathbb{Z}$

$\uparrow \chi_* \times 1 \qquad \qquad \qquad \uparrow \chi_*$

$$H^2(G, A) \times \hat{H}^{-3}(\mathbb{Z}) \longrightarrow \hat{H}^{-1}(G, A) = A[N_G] / I_G A$$

$$\forall \chi \in \text{Hom}_G(A, \mathbb{Q}/\mathbb{Z}) \text{ st. } \chi|_{A[N_G]} = 0 \quad (\text{i.e. } \chi \in \hat{H}^0(G, A^\vee))$$

$$\chi \text{ kills } \text{Im} \left(\hat{H}^{-3}(G, \mathbb{Z}) \xrightarrow{[u] \cdot} \hat{H}^{-1}(G, A) \right)$$

$$\iff \chi_*([u]) \cup \hat{H}^{-3}(\mathbb{Z}) = 0$$

$$\iff \chi_*([u]) = 0 \text{ in } H^2(G, \mathbb{Q}/\mathbb{Z})$$

$$\iff \chi: A \rightarrow \mathbb{Q}/\mathbb{Z} \text{ can be extended to a group homomorphism } \hat{\chi}: W \rightarrow \mathbb{Q}/\mathbb{Z}$$

basics about
group extⁿ

$$\iff \chi|_{A \cap [W, W]} = 0$$

To check that

$$\begin{array}{ccc}
 W^{ab} & \xrightarrow{\text{induced by}} & G^{ab} \\
 \downarrow \tau & & \downarrow [u] \\
 A^G & \xrightarrow{\text{can}} & \hat{H}^0(G, A)
 \end{array}$$

commutes,

Will compare (i) an explicit formula of $\text{Ver}_{W/A}$ with (ii) Nakayama's formula for cup product.

(i) Explicit formula for $\text{Ver}_{W/A}$:

Every element of W can be written uniquely as

$$W \ni w = a \cdot s_\sigma \quad a \in A.$$

$$s_\sigma \cdot s_\tau = u_{\sigma, \tau} s_{\sigma\tau} \Rightarrow \text{Ver}_{W/A}(s_\sigma \bmod [W, W]) = \prod_{\sigma \in G} u_{\sigma, \tau} \bmod [A, A]$$

$$\text{Ver}_{W/A}(a \bmod [W, W]) = N_G a \bmod [A, A]$$

$\Rightarrow$ The image in G^{ab} of $\text{Ver}_{W/A}(a \cdot s_\sigma \bmod [W, W])$ is:

$$\underbrace{\prod_{\sigma \in G} u_{\sigma, \tau}}_{\in A^G \pmod{N_G A}}$$

(no cocycle condⁿ for $(u_{\sigma, \tau})$)

$$u_{\sigma, \tau}^\mu \cdot u_{\mu\sigma, \tau}^{-1} \cdot u_{\mu, \sigma\tau}^{-1} \cdot u_{\mu, \sigma}^{-1} = 1$$

product over σ : $\left(\prod_{\sigma} u_{\sigma, \tau}\right)^\mu = \prod_{\sigma} u_{\sigma, \tau} \quad \forall \mu \in G$

(ii) General form of Nakayama's formula:

$$u \in \mathbb{Z}^2(G, B) \quad \bar{z} \in G, \quad \bar{z} = \text{image of } z \text{ in } G^{ab} = G/[G, G]$$

$$\begin{aligned}
 \text{Under } \hat{H}^2(G, B) \times \hat{H}^{-2}(G, \mathbb{Z}) &\longrightarrow \hat{H}^0(G, B) = \hat{H}^{-2}(G, \mathbb{Z}) \\
 [u] \cdot \bar{z} &= \prod_{\sigma \in G} u_{\sigma, \tau} \bmod N_G B = B^G / N_G B
 \end{aligned}$$

ref Serre, Corps Locaux, Chap XI, Annexe, Lemma 4

$$S_\alpha \cdot S_\beta = U_{\alpha\beta} S_\beta$$

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Prop 2 implies $1 \rightarrow A \rightarrow W \xrightarrow{s} G \rightarrow 1$

$$0 \rightarrow \hat{H}^{-1}(G, A) / ([u] \cdot \hat{H}^{-3}(G, \mathbb{Z})) \rightarrow \text{Ker}(\text{Ver}_{W/A}) \rightarrow \text{Ker} \left(\begin{array}{c} G^{ab} \\ \downarrow [u] \\ \hat{H}^0(G, A) \end{array} \right) \rightarrow 0$$

$$\hookrightarrow \left(A^G / \text{Im}(\text{Ver}_{W/A} : W^{ab} \rightarrow A^{ab}) \right) \xrightarrow{\sim} \left(\hat{H}^0(G, A) / [u] \cdot G^{ab} \right)$$

Cor

$$W^{ab} \xrightarrow{\quad V \quad} A^G$$

↑
induced by $\text{Ver}_{W/A}$

$$V \text{ is surjective} \iff \hat{H}^{-2}(G, \mathbb{Z}) \xrightarrow{[u]} \hat{H}^0(G, A) \text{ is surjective}$$

$$V \text{ is injective} \iff \hat{H}^{-3}(G, \mathbb{Z}) \xrightarrow{[u]} \hat{H}^{-1}(G, A) \text{ is surjective}$$

$$\text{and } \hat{H}^{-2}(G, \mathbb{Z}) \xrightarrow{[u]} \hat{H}^0(G, A) \text{ is injective}$$

Note: $\hat{H}^{-1}(G, \mathbb{Z}) = (0)$

Nakayama-Tate says: The map $W^{ab} \xrightarrow{V} A^G$ induced by $\text{Ver}_{W/A}$ is bijective iff

$$\hat{H}^i(G, \mathbb{Z}) \xrightarrow{[u]} \hat{H}^{i+2}(G, A) \text{ is bijective } \forall i \in \mathbb{Z}.$$

G : finite group

$$H \trianglelefteq G$$

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§ The deflation map
(or, Akizuki-Witt)

$$\begin{array}{ccc} \hat{H}^2(G, A) & \xrightarrow{\text{def}_H^G} & \hat{H}^2(G/H, A^H) \\ & \nwarrow \text{def}_H^G & \nearrow [G, H] \\ & \hat{H}^2(G, A) & \end{array}$$

Fact: $\hat{H}^2(G/H, A^H) \xrightarrow{\text{inf}_H^G} \hat{H}^2(G, A)$

Definition/Construction:

Given $[u] = \hat{H}^2(G, A)$, given by $(u_{\sigma, \tau}) \in Z^2(G, A)$
= class of $\text{ext}^n \quad 1 \rightarrow A \rightarrow W \rightarrow G \rightarrow 1$

Look at $1 \rightarrow A \rightarrow W \xrightarrow{\pi} G \rightarrow 1$

$$\begin{array}{ccccccc} & & & \uparrow & & \uparrow j = j_{H \hookrightarrow G} & \\ & & \parallel & & & & \\ 1 & \rightarrow & A & \rightarrow & E & \rightarrow & H \rightarrow 1 \end{array}$$

$$E = \pi^{-1}(H)$$

$$1 \rightarrow E \rightarrow W \rightarrow G/H \rightarrow 1$$

and

$$1 \rightarrow E/[E, E] \xrightarrow{V_{E/A}} W/[E, E] \rightarrow G/H \rightarrow 1$$

$V_{E/A}: E^{ab} \rightarrow A^H$
is induced by
 $V_{E/A}$

$$\begin{array}{ccccccc} & & \downarrow & & \downarrow & & \parallel \\ 1 & \rightarrow & A^H & \rightarrow & D & \rightarrow & G/H \end{array}$$

Define $\text{def}_H^G([u]) :=$ the class given by the ext^n

$$1 \rightarrow A^H \rightarrow D \rightarrow G/H \rightarrow 1$$

$$= (V_{E/A})_* \left(\text{the class given by} \right. \\ \left. 1 \rightarrow E^{ab} \rightarrow W/[E, E] \rightarrow G/H \rightarrow 1 \right)$$

Proposition $\text{def}_H^G([u]) =$ the class of the 2-cocycle

$$\text{def}(u)_{r,s} = N_H(u_{\tilde{r},\tilde{s}} \cdot u_{\gamma(r,s),\tilde{rs}}^{-1}) \cdot \prod_{h \in H} u_{h,\gamma(r,s)}$$

Let $(\tilde{r})_{r \in G/H}$

be a set of repr. of G/H in G

where $\gamma(r,s) = \tilde{r} \cdot \tilde{s} \cdot \tilde{rs}^{-1}$
 $\forall r,s \in G/H$

Pf: $\forall r,s \in G/H$, $s_{\tilde{r}}$ is an elt of W above $\tilde{r}$

$\Rightarrow$ The image of $s_{\tilde{r}}$ in $W/[E,E]$ is an element of $W/[E,E]$ above $\tilde{r}$

$\Rightarrow$ Get a 2-cocycle $(b_{r,s})_{r,s \in G/H} \in Z^2(G/H, E^{ab})$ corresponding to $1 \rightarrow E^{ab} \rightarrow W/[E,E] \rightarrow G/H \rightarrow 1$,

where $b_{r,s}$ is determined by

$$s_{\tilde{r}} \cdot s_{\tilde{s}} \equiv b_{r,s} \cdot s_{\tilde{rs}} \pmod{[E,E]}$$

$$s_{\tilde{r}} \cdot s_{\tilde{s}} = u_{\tilde{r},\tilde{s}} \cdot s_{\tilde{rs}}$$

$$\gamma(r,s) \cdot \tilde{rs} = \tilde{r} \cdot \tilde{s}$$

$$\begin{aligned} &= u_{\tilde{r},\tilde{s}}^{-1} \cdot u_{\gamma(r,s),\tilde{rs}} \cdot s_{\tilde{rs}} \Rightarrow s_{\gamma(r,s)} \cdot s_{\tilde{rs}} = u_{\gamma(r,s),\tilde{rs}} \cdot \underbrace{s_{\gamma(r,s),\tilde{rs}}}_{s_{\tilde{rs}}} \\ &\quad \parallel \\ &\quad s_{\tilde{rs}} \end{aligned}$$

Conclude $b_{r,s} \equiv u_{\tilde{r},\tilde{s}}^{-1} \cdot u_{\gamma(r,s),\tilde{rs}} \cdot s_{\gamma(r,s)} \pmod{[E,E]}$

$$\forall r,s \in G/H$$

where $\gamma(r,s) \stackrel{\text{def}}{=} \tilde{r} \cdot \tilde{s} \cdot \tilde{rs}^{-1}$
 $\forall r,s \in G/H$

Apply $V_{E/A}$: $\text{def}(u)_{rs} = V_{E/A}(b_{r,s})$

$$V_{E/A}(s_{\gamma(r,s)}) = \prod_{h \in H} u_{h,\gamma(r,s)}$$

$$\text{Ver}_{E/A}(u_{\tilde{r},\tilde{s}} \cdot u_{\gamma(r,s),\tilde{rs}}^{-1})$$

$$= N_H(u_{\tilde{r},\tilde{s}} \cdot u_{\gamma(r,s),\tilde{rs}}^{-1})$$

$$= \prod_{h \in H} (u_{\tilde{r},\tilde{s}}^h \cdot u_{\gamma(r,s),\tilde{rs}}^{-h})$$

special case
of explicit formula
for $\text{Ver}_{E/A}$ for

$$1 \rightarrow A \rightarrow E \rightarrow H \rightarrow 1$$

$$\Rightarrow \text{def}(u)_{r,s} = V_{E/A}(b_{r,s}) = N_H(u_{\tilde{r},\tilde{s}} \cdot u_{\gamma(r,s),\tilde{rs}}^{-1}) \cdot \prod_{h \in H} u_{h,\gamma(r,s)} = \prod_{h \in H} (u_{\tilde{r},\tilde{s}}^h \cdot u_{\gamma(r,s),\tilde{rs}}^{-h} \cdot u_{h,\gamma(r,s)})$$

Compare the definition of Weil group on p. IX-10 with Weil's IX-20 construction

Key properties used

(1) K/F normal



$$C_E \xrightarrow{r_E} G_E / [G_E, G_E] = G_E^{ab}$$

(2) $D_K := \text{Ker}(r_E)$ is isomorphic to a direct sum of $G_{K/F}$ -modules induced from cyclic subgroups of $G_{K/F}$ actually, of order 1 or 2

(3) $H^1(G_{K/E}, D_K) = (0)$ and $N_{K/E} D_K = D_E$

What Weil constructed

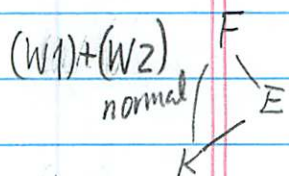
$$W_{K/F} \xrightarrow{g_{K/F}} G_F / [G_K, G_K]$$

K/F normal

$$A_F \xrightarrow{\text{rec}_{K/F}} G_F^{ab}$$

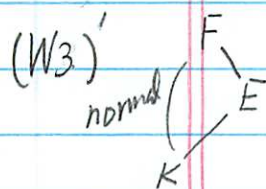
satisfying the usual functoriality properties

(as in field formations)



$$A_E \xrightarrow[r_E]{\cong} W_{F/E} / [W_{F/E}, W_{F/E}] = W_{F/E}^{ab}$$

satisfy standard functoriality properties



$$A_E \xrightarrow[r_E]{\cong} W_{F/E}^{ab} \xrightarrow{g_{K/F}^{ab}} G_F^{ab}$$

$$g_{K/F}^{ab} \circ r_E = \text{rec}_{K/F} \leftarrow \text{reciprocity law map}$$

Defⁿ / Axioms of a Weil group for a topological class formation $(G, \{G_F\}, A)$

data: $(U, g, \{f_F\})$ $U \xrightarrow{g} G$
 Hausdorff topological group $\uparrow$ continuous homom
 with dense image

$A_F \xrightarrow[\sim]{f_E} U_F^{ab}$ topological isom

axioms W1) $\forall$ layer E/F , $A_F \xrightarrow[\sim]{f_F} U_F^{ab}$
 $\downarrow$ $A_E \xrightarrow[\sim]{f_E} U_E^{ab}$ $\downarrow \text{Ver}_{U_F/U_E}$

W2) $\forall u \in U, \sigma = g(u) \in G$ $A_E \xrightarrow[\sim]{f_E} U_E^{ab}$
 $\sigma \downarrow$ $A_{E^\sigma} \xrightarrow[\sim]{f_{E^\sigma}} U_{E^\sigma}^{ab}$ $\downarrow \text{conjugation by } u$

* W3) $\forall$ normal layer K/F , the class of

$$1 \rightarrow A_K \rightarrow U_F/[U_K, U_K] \rightarrow U_F/U_K \cong G_{K/F} \rightarrow 1$$

is the fundamental class of the layer K/F

W4) $U \xrightarrow{\sim} \varprojlim_K U/[U_K, U_K]$

Equivalent description (equivalence needs to be proved!)

$(U, \{U_F\})$ U = Hausdorff topological group $U_F \leq U$

axioms (i) Each U_F is open of finite index in U , $\{U_F\}$ is closed under finite intersection, every subgroup of U containing a U_F is in this family of subgroups

(ii) $U_K \triangleleft U_F \Rightarrow U_F^{ab} \xrightarrow{\text{Ver}_{U_F/U_K}} U_K^{ab}$

(iii) $U \xrightarrow{\sim} \varprojlim_K U/[U_K, U_K] \xrightarrow{\sim} \varprojlim_K (U_K^{ab})^{U_F/U_K}$

Weil's variant

$\forall K/F$ normal. $\leadsto (U_{K/F}, U_{K/F} \xrightarrow{g_{K/F}} G_F/[G_K, G_K], A_F \xrightarrow{f_E} U_F^{ab})$
 satisfy W1), W2) and

W3)': $\begin{array}{c} K \\ \swarrow \searrow \\ E \quad F \\ \text{normal} \end{array} \leadsto \begin{array}{c} A_E \xrightarrow{f_E} U_E^{ab} \xrightarrow{g_E} G_E^{ab} \\ \underbrace{\hspace{10em}} \\ r_E \leftarrow \text{reciprocity law map} \end{array}$

(define $g: U \rightarrow G$ by passing to the limit.)

Key properties used by Weil, with $A_E = A_E^*/E^*$, $G_E = \text{Gal}(\bar{\mathbb{Q}}/E)$

- $r_E: A_E \rightarrow G_E^{ab}$ is surjective
- $\forall$ normal layer K/F , $D_K := \text{Ker}(r_K)$ is isomorphic to a direct sum of modules induced from cyclic subgroups of $G_{K/F}$ of order 1 or 2
- $\begin{array}{c} E \\ \swarrow \searrow \\ F \quad K \\ \text{normal} \end{array} \quad H^1(G_{K/E}, D_K) = 0 \quad \text{and} \quad N_{K/E}(D_K) = D_E$

Weil's idea: Produce $1 \rightarrow A_K \xrightarrow{i} U_{K/F} \xrightarrow{j} G_{K/F} \rightarrow 1$ by climbing up from $1 \rightarrow G_K^{ab} \rightarrow G_F/[G_K, G_K] \rightarrow G_{K/F} \rightarrow 1$

via $A_K \xrightarrow{r_K} G_K^{ab}$, i.e. a "Weil diagram"

(WD)
$$\begin{array}{ccccccc} 1 & \rightarrow & A_K & \xrightarrow{i} & U_{K/F} & \xrightarrow{j} & G_{K/F} \rightarrow 1 \\ & & \downarrow r_K & & \downarrow g_{K/F} & & \downarrow = \\ 1 & \rightarrow & G_K^{ab} & \rightarrow & G_F/[G_K, G_K] & \rightarrow & G_{K/F} \rightarrow 1 \end{array}$$

Note: The notion of Weil diagrams is more rigid than the notion of extensions $1 \rightarrow A_K \xrightarrow{i} U_{K/F} \xrightarrow{j} G_{K/F} \rightarrow 1$

Lemma: The set of isomorphism classes of Weil diagrams for K/F has a natural structure as a torsor for $H^2(G_{K/F}, D_K)$

pf: 1) Existence of a Weil diagram: consider $0 \rightarrow D_K \rightarrow A_K \rightarrow G_K^{ab} \rightarrow 0$
and $\rightarrow H^2(G_{K/F}, A_K) \rightarrow H^2(G_{K/F}, G_K^{ab}) \rightarrow H^3(G_{K/F}, D_K) \rightarrow 0$
 $\downarrow \quad \quad \quad \parallel$
 $[1 \rightarrow G_K^{ab} \rightarrow G_F/[G_K, G_K] \rightarrow G_{K/F} \rightarrow 1] \quad 0$

2) Fix a set theoretic section $\sigma \mapsto \gamma_\sigma$ of $G_F/[G_K, G_K] \rightarrow G_{K/F}$

Let $a_{\sigma, \tau}^{gal} = \gamma_\sigma \gamma_\tau (\gamma_{\sigma\tau})^{-1} \in Z^2(G_{K/F}, G_K^{ab})$

Pick a 2-cocycle $(a_{\sigma, \tau}) \in Z^2(G_{K/F}, A_K)$ which lifts $(\gamma_{\sigma, \tau})$

i.e. $\gamma_K(a_{\sigma, \tau}) = a_{\sigma, \tau}^{gal} \quad \forall \sigma, \tau \in Gal(K/F)$

$\longleftrightarrow (u_\sigma)_{\sigma \in G_{K/F}} \quad \begin{matrix} \uparrow \\ \text{defined for this} \\ \text{Weil diagram} \end{matrix} \quad \begin{matrix} g_{K/F}(u_\sigma) = \gamma_\sigma \\ \forall \sigma \in G_{K/F} \end{matrix}$

s.t. $u_\sigma \cdot u_\tau \cdot u_{\sigma\tau}^{-1} = a_{\sigma, \tau} \quad \forall \sigma, \tau \in G_{K/F}$

Note: Every Weil diagram (for K/F) comes from a 2-cocycle

$(a_{\sigma, \tau}) \in Z^2(G_{K/F}, A_K)$ s.t. $\gamma_K(a_{\sigma, \tau}) = a_{\sigma, \tau}^{gal} \quad \forall \sigma, \tau$

Conversely, every 2-cocycle lifting $(a_{\sigma, \tau}^{gal})$ gives rise to a Weil diagram

3) Suppose $(a_{\sigma, \tau})$ and $(b_{\sigma, \tau})$ are two 2-cocycles in $Z^2(G_{K/F}, A_K)$

lifting $(a_{\sigma, \tau}^{gal})$, i.e. $a_{\sigma, \tau} \cdot b_{\sigma, \tau}^{-1} \in D_K \quad \forall \sigma, \tau$ Need to find

nec. and suff. conditions for the two Weil diagrams to

be isomorphic

$$v_\sigma \cdot v_\tau \cdot v_{\sigma\tau}^{-1} = b_{\sigma, \tau}$$

in Weil diagram for $U_{K/F}$

defined by $(b_{\sigma, \tau})$

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Suppose we have an isom. $\psi: U_{K/F} \rightarrow U'_{K/F}$ inducing an isom of Weil diagrams. Then $\psi(u_\sigma) = d_\sigma v_\sigma$, $d_\sigma \in D_K \forall \sigma$ uniquely determined by ψ

Condition for ψ to preserve group structure:

$$(d_\sigma \cdot v_\sigma) \cdot (d_\tau v_\tau) = a_{\sigma, \tau} d_{\sigma\tau} v_{\sigma\tau}$$

$$\parallel$$

$$d_\sigma \cdot d_\tau^\sigma \cdot b_{\sigma, \tau} v_{\sigma\tau}$$

i.e. ($\because A_K$ is commutative)

$$a_{\sigma, \tau} \cdot b_{\sigma, \tau}^{-1} = d_\sigma^\sigma \cdot d_{\sigma\tau}^{-1} \cdot d_\tau \quad \forall \sigma, \tau \in G_{K/F}$$

This defines a ψ action of $H^2(G_{K/F}, D_K)$ on the set of all isom classes of Weil diagrams for K/F

For a Weil diagram

$$\begin{array}{ccccccc} 1 & \rightarrow & A_K & \xrightarrow{i} & U_{K/F} & \xrightarrow{j} & G_{K/F} \rightarrow 1 \\ & & \eta_K \downarrow & & \eta_{K/F} \downarrow & & \downarrow = \\ 1 & \rightarrow & G_K^{ab} & \rightarrow & G_F/[G_K, G_K] & \rightarrow & G_{K/F} \rightarrow 1 \end{array}$$

Have

$$\begin{array}{ccc} U_{K/E}^{ab} & \xrightarrow{V_{K/E}} & (A_K)^{G_{K/E}} \\ \downarrow \text{Ver}_{U_{K/F}, A_K} & & \downarrow \\ A_K & & \end{array}$$

$\forall$ intermediate field E $\begin{array}{c} K \\ \swarrow \searrow \\ E \\ \swarrow \searrow \\ F \end{array}$

Weil's insight. K/F normal layer.

(1) Given a Weil diagram

$$1 \rightarrow A_K \xrightarrow{i_{K/F}} U_{K/F} \xrightarrow{j_{K/F}} G_{K/F} \rightarrow 1$$

$$1 \rightarrow G_K^{ab} \xrightarrow{\bar{v}_{K/F}^{gal}} G_F/[G_K, G_K] \xrightarrow{\downarrow \theta_{K/F}} G_{K/F} \rightarrow 1, \quad \downarrow =$$

there exists a unique element $[(a_0, z)] \in \hat{H}^2(G_{K/F}, D_K)$ st. the twist by (a_0, z) of the original Weil diagram satisfy the additional condition

$$(*) \quad r_F \circ V_{K/F} = g_{K/F}^{ab}$$

(This is condition (D) on p.8 of Weil's 1951 paper)

(2) Suppose we have a Weil diagram which satisfy the condition (*).

Then $V_{K/F} : U_{K/F}^{ab} \xrightarrow{\sim} A_F$

Step 1 The maps $r_F \circ V_{K/F}$ and $g_{K/F}^{ab} : U_{K/F}^{ab} \longrightarrow G_F/[G_K, G_K]$ coincide on $\text{Im}(A_K \xrightarrow{i_{K/F}} U_{K/F}^{ab})$

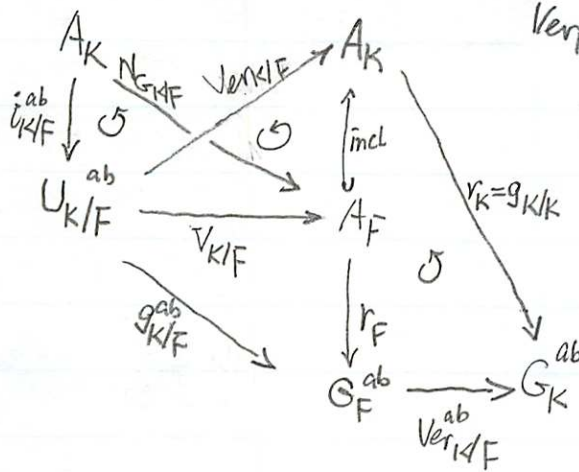
$$\text{and } \bar{v}_{K/F}^{gal} \circ r_F \circ V_{K/F} = \bar{v}_{K/F}^{gal} \circ g_{K/F}^{ab},$$

where $\bar{v}_{K/F}^{gal} : G_F/[G_K, G_K] \longrightarrow G_K^{ab}$ is the transfer map

$\bar{v}_{G_F/[G_K, G_K], G_K^{ab}}$

pf:

$$V_{K/F} \circ i_{K/F}^{ab} = N_{G/K/F}$$



$$Ver_{K/F} = incl \circ V_{K/F}$$

transfer thm

$$\leadsto r_K \circ incl = Ver_{K/F}^{gal} \circ r_F$$

$$Ver_{K/F}^{gal} \circ g_{K/F}^{ab} = r_K \circ Ver_{K/F}$$

functoriality of Ver.

$$\Rightarrow g_{K/F}^{ab} \circ i_{K/F}^{ab} = (i_{K/F}^{gal})^{ab} \circ r_K$$

$$= r_F \circ N_{G/K/F} \quad \text{translation theorem}$$

$$= r_F \circ V_{K/F} \circ i_{K/F}^{ab}$$

first part proved

To show the 2nd part:

$$Ver_{K/F}^{ab} \circ g_{K/F}^{ab} = r_K \circ Ver_{K/F} \quad \text{Functoriality of Ver}$$

$$= r_K \circ incl \circ V_{K/F}$$

$$= Ver_{K/F}^{ab} \circ r_F \circ V_{K/F}$$

transfer theorem

2nd part proved
q.e.d.

Step 1

↓

Step 2

$$\begin{array}{ccc} U_{K/F} & \xrightarrow[\cdot g_F^{ab}]{r_F \circ V_{K/F}} & G_{K/F}^{ab} \\ \downarrow i_{K/F}^{ab} & & \uparrow \\ G_{K/F}^{ab} & \xrightarrow{\psi_{K/F}} & (\mathbb{D}_K \cap A_F) / \mathbb{D}_F \\ \hat{H}^{-2}(G_{K/F}, \mathbb{Z}) & & \hat{H}^0(G_{K/F}, \mathbb{D}_K) \end{array}$$

← ratio

$$\begin{aligned} \text{Ker}(Ver_{K/F}^{ab}) &\xleftarrow{\sim} (\mathbb{D}_K \cap A_F) / \mathbb{D}_F = (\mathbb{D}_K^{G_{K/F}} \cap A_F) / \mathbb{D}_F = (\mathbb{D}_K^{G_{K/F}} \cap N_{K/F} \mathbb{D}_K) / \mathbb{D}_F \\ &\quad \uparrow \text{transfer thm} \\ &= \hat{H}^0(G_{K/F}, \mathbb{D}_K) \end{aligned}$$

Lemma/recall: G = finite cyclic group, M = G -module. $n = |G|$

Then: $\hat{H}^2(G, \mathbb{Z}) \otimes_{\mathbb{Z}/n\mathbb{Z}} \hat{H}^i(G, M) \xrightarrow{\sim} \hat{H}^{2+i}(G, M) \quad \forall i \in \mathbb{Z}$
 (Serre, CL. VIII §4) p.141

$$\hat{H}^{-2}(G, \mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}/n\mathbb{Z}}(\hat{H}^2(G, \mathbb{Z}), \mathbb{Z}/n\mathbb{Z})$$

$$\hat{H}^{2+i}(G, M) = \text{Hom}_{\mathbb{Z}/n\mathbb{Z}}(\hat{H}^{-2}(G, \mathbb{Z}), \hat{H}^i(G, M)) \quad \forall i \in \mathbb{Z}$$

Step 2 $\Rightarrow \exists! \delta_{K/F} \in \hat{H}^2(G_{K/F}, D_K)$ s.t. $\psi_{K/F} = \delta_{K/F}^u$?

$$\hat{H}^{-2}(G_{K/F}, \mathbb{Z}) \xrightarrow[\psi_{K/F}]{\delta u.} \hat{H}^0(G_{K/F}, D_K).$$

We have proved

Functoriality of Galois cohomology:

twisting by a class $[u] \in \hat{H}^2(G_{K/F}, D_K)$

changes $\delta_{K/F} \leftrightarrow \psi_{K/F} = \frac{\tau_F \circ \tau_{K/F}}{g_F^{ab}}$ exactly by $[u]$.

We have proved Weil's insight (1)

Step 3 $\rightarrow$ Now suppose we are given a Weil diagram satisfying

$$\tau_F \circ \tau_{K/F} = g_{K/F}^{ab}$$

Must show: $\tau_{K/F}: U_{K/F} \xrightarrow{ab} A_F$ is an isomorphism

Have proved Weil's insight (2).

More about step 2

Strictly speaking, from step 1, we know: $\forall$ intermediate layer

$\begin{array}{c} K \\ \swarrow \\ E \\ \swarrow \\ F \end{array}$ we have a homomorphism $\psi_E: \hat{H}^{-2}(G_{K/E}, \mathbb{Z}) \rightarrow \hat{H}^0(G_{K/E}, D_K)$
 $\Rightarrow \forall$ intermediate E s.t. $G_{K/E}$ is cyclic, get $\delta_{K/E} \in \hat{H}^2(G_{K/E}, D_K)$

s.t. $\psi_E = \delta_{K/E} \cup ?$

• Functoriality

$$\begin{array}{c} K \\ \swarrow \\ E' \\ | \\ E \\ \swarrow \\ F \end{array} \quad \begin{array}{c} G_{K/E}^{ab} = \hat{H}^{-2}(G_{K/E}, \mathbb{Z}) \\ \downarrow \text{Ver} \\ G_{K/E'}^{ab} = \hat{H}^{-2}(G_{K/E'}, \mathbb{Z}) \end{array} \xrightarrow{\psi_E} \hat{H}^0(G_{K/E}, D_K) = D_K / N_{K/E} D_K \xrightarrow{\text{induced by id}_{D_K}} D_K / N_{K/E'} D_K = D_K / N_{K/E'} D_K$$

$\downarrow \text{Res}_{G_{K/E'}}^{G_{K/E}} \quad \downarrow \text{Res}_{G_{K/E'}}^{G_{K/E}} \quad \downarrow \text{Res}_{G_{K/E'}}^{G_{K/E}}$

$G_{K/E}^{ab} \xrightarrow{\psi_{E'}} \hat{H}^0(G_{K/E'}, D_K) = D_K / N_{K/E'} D_K$

$\text{Res}_{G_{K/E'}}^{G_{K/E}} \delta_{K/E} = \delta_{K/E'}$

Shows $\begin{array}{c} K \\ \swarrow \\ E' \\ | \\ E \\ \swarrow \\ F \end{array}$ s.t. $G_{K/E}$ (hence $G_{K/E'}$) is cyclic

• Also: $\forall \sigma \in G_{K/E}, \sigma \cdot \delta_{K/E} = \delta_{K/E^\sigma}$

$$\begin{array}{c} G_{K/F}^{ab} = \hat{H}^{-2}(G_{K/F}, \mathbb{Z}) \\ \uparrow \text{induced by } G_{K/E} \hookrightarrow G_{K/F} \\ G_{K/E}^{ab} = \hat{H}^{-2}(G_{K/E}, \mathbb{Z}) \end{array} \xrightarrow{\psi_F} \hat{H}^0(G_{K/F}, D_K) = D_K / N_{K/F} D_K \xrightarrow{\text{cores}_{G_{K/E}}^{G_{K/F}}} D_K \cap A_F / D_F$$

$$\begin{array}{c} G_{K/E}^{ab} = \hat{H}^{-2}(G_{K/E}, \mathbb{Z}) \\ \uparrow \text{cores}_{G_{K/E}}^{G_{K/F}} \\ G_{K/E}^{ab} = \hat{H}^{-2}(G_{K/E}, \mathbb{Z}) \end{array} \xrightarrow{\psi_E} \hat{H}^0(G_{K/E}, D_E) = D_K \cap A_E / D_E$$

and the formula $\text{Cores}(a \cdot \text{Res}(b)) = \text{Cores}(a) \cdot b$ Atiyah-Wall p.107 shows

(i) $\psi_F = 0$ if $\psi_E = 0 \quad \forall \begin{array}{c} K \\ \swarrow \\ E' \\ | \\ E \\ \swarrow \\ F \end{array}$ with $G_{K/E}$ cyclic

(ii) If $\exists \delta_{K/F}$ s.t. $\text{Res}_{G_{K/E}}^{G_{K/F}} \delta_{K/F} = \delta_{K/E}$, then $\psi_F = \delta_{K/F} \cup ?$
 $\hat{H}^2(G_{K/F}, D_K) \quad \forall E \text{ with } K/E \text{ cyclic}$

"Gluing Lemma" $G = \text{a finite group}$. $\mathcal{F} = \text{a family of subgroups of } G \text{ s.t.}$

Let $M = \text{a } G\text{-module of the form}$

$$M = \bigoplus_{H \in \mathcal{F}} \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M_H$$

$r \in \mathbb{Z}$

$M_H = H\text{-module}$

Let

$$\text{Coh}(\mathcal{H}^r, \mathcal{F}, M) = \left\{ (\alpha_H) \in \prod_{H \in \mathcal{F}} \hat{H}^r(H, M) \mid \begin{array}{l} \bullet \text{res}_H^H(\alpha_H) = \alpha_{H'} \\ \forall \text{ pair } (H, H') \text{ with } H' \leq H, H \in \mathcal{F} \\ \bullet \forall \text{ pair } (H, \sigma) \text{ with } H \in \mathcal{F}, \sigma \in G \\ \alpha \cdot \alpha_H = \alpha_{H\sigma} \end{array} \right\}$$

Then the canonical map $H^r(G, M) \xrightarrow{\sim} \text{Coh}(\mathcal{H}^r, \mathcal{F}, M)$ is bijective.

$$\downarrow \quad \longmapsto (\text{Res}_H^G(\alpha))_{H \in \mathcal{F}}$$

Down-to-earth version: $G_0 \in \mathcal{F}$, $M_0 = G_0\text{-module}$, the diagram

$$\begin{array}{ccccc} \hat{H}^r(G_0, M_0) & \xrightarrow{\sim} & \hat{H}^r(G, \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_0]} M_0) & \xrightarrow{\pi_G \circ \text{Res}_{G_0}^G} & \hat{H}^r(H, M_\tau) \\ (*) \quad \text{Res}_{\tau H \cap G_0}^{G_0} \downarrow & & \downarrow & \searrow \sim & \text{pr}_\tau \circ \text{Res}_{H \cap G_0 \tau^{-1}}^H \\ \hat{H}^r(\tau H \cap G_0, M_0) & \rightarrow & \hat{H}^r(H \cap \tau G_0 \tau^{-1}, \tau M_0) & & \end{array}$$

$M_\tau := \mathbb{Z}[H] \otimes_{\mathbb{Z}[H \cap G_0 \tau^{-1}]} \tau M_0$

Proof: Recall given a group B , a subgroup C , and a left C -module N ,

Have: left B -module $\text{Hom}_C(B, N) := \left\{ f: B \rightarrow N \mid f(cx) = cf(x) \right\}$

left B -action: $(b \cdot f)(x) = f(xb) \quad \forall b, x \in B$

a C -equivariant map

$$\text{Hom}_C(B, N) \xrightarrow{\text{can}} N$$

and $H^i(B, \text{Hom}_C(B, N)) \xrightarrow[\simeq]{(C \hookrightarrow B, \text{can})} H^i(C, N)$

If $[B:C] < \infty$, then $\mathbb{Z}[B] \otimes_{\mathbb{Z}[C]} N \xrightarrow{\simeq} \text{Hom}_C(B, N)$

$$\forall b \in B, \forall n \in N \quad b \otimes n \mapsto f_{b \otimes n}, \quad f_{b \otimes n}(x) = \delta_{C, b^{-1}x}(x) \cdot x b \cdot n$$

indicator function of Cg
i.e. $\delta_{C, b^{-1}x}(x) = \begin{cases} 1 & \text{if } xb \in C \\ 0 & \text{if } xb \notin C \end{cases}$

Under this isomorphism $\mathbb{Z}[B] \otimes_{\mathbb{Z}[C]} N \xrightarrow{\sim} \text{Hom}_C(B, N)$, the map
can: $\text{Hom}_C(B, N) \rightarrow N$ becomes

$$\text{pr}: \mathbb{Z}[B] \otimes_{\mathbb{Z}[C]} N \rightarrow N$$

$$b \otimes n \mapsto \begin{cases} b \cdot n & \text{if } b \in C \\ 0 & \text{if } b \notin C \end{cases}$$

$b \in B, n \in N$

The proof of the "gluing lemma" comes down to a careful look at Mackey's theorem

$$\mathbb{Z}[G] \otimes_{\mathbb{Z}[G_0]} M_0 = \bigoplus_{\tau \in T} \bigoplus_{\sigma \in R_\tau} \sigma \cdot \tau M_0$$

$$M_\tau := \bigoplus_{\sigma \in R_\tau} \sigma \cdot \tau M_0$$

$\tau G_0 \tau^{-1}$ -module,
restrict to $H \cap \tau G_0 \tau^{-1}$

$1 \in T =$ a set of repr. of $H \backslash G / G_0$

and $\forall \tau \in T$, let $R_\tau =$ a set of repr. of $H / H \cap \tau G_0 \tau^{-1}$

$$\text{So } G = \bigsqcup_{\tau \in T} H \tau G_0 = \bigsqcup_{\tau \in T} \bigsqcup_{\sigma \in R_\tau} \sigma \cdot \tau G_0$$

$$H \tau G_0 \tau^{-1} / \tau G_0 \tau^{-1} \xleftarrow{\sim} H / H \cap \tau G_0 \tau^{-1}$$

$$M_\tau \cong \mathbb{Z}[H] \otimes_{\mathbb{Z}[H \cap \tau G_0 \tau^{-1}]} \tau M_0$$

$\uparrow$
as an H -module

The Shapiro isomorphism for $(B \geq C, C\text{-module } N)$ is the composition

$$H^r(B, \mathbb{Z}[B] \otimes_{\mathbb{Z}[C]} N) \xrightarrow{\text{Res}_C^B} H^r(C, \mathbb{Z}[B] \otimes_{\mathbb{Z}[C]} N) \xrightarrow{\text{pr}} H^r(C, N)$$

$$\text{Given } (\alpha_H)_{H \in \mathcal{F}} \in \text{Coh}(\mathcal{J}^r, \mathcal{F}, M), \alpha_H = \sum_{\tau \in T} \alpha_{H, \tau} \quad \text{pr}_\tau: \mathbb{Z}[H] \otimes_{\mathbb{Z}[H \cap \tau G_0 \tau^{-1}]} M_0 \rightarrow \tau M_0$$

$$\text{Let } \alpha_{H, \tau}^0 = \text{pr}_\tau \circ \text{Res}_{H \cap \tau G_0 \tau^{-1}}^H (\alpha_H) \in H^r(H \cap \tau G_0 \tau^{-1}, \tau M_0)$$

corresponds to $\alpha_{H, \tau} \in H^r(H, M_\tau)$

The commutative diagram (*) (IX-27b) shows:

$$(*) \quad \alpha_{H, \tau}^0 = \tau \cdot \left(\text{res}_{(\tau^{-1} H \tau) \cap G_0}^G \alpha_{G_0, 1} \right) \quad \forall \tau$$

(**) means: $\alpha_H = \text{res}_H^G(\alpha)$

$\alpha :=$ the element of $H^r(G, M)$ corresponding to $\alpha_{G_0, 1}$
 $\uparrow$
 $H^r(G, M)$ $\uparrow$
 $H^r(G_0, M_0)$

$T = \text{sys. of repr of } G_0 \backslash G / G_0$

Gluing Lemma proved.

Proof surjectivity. (Use: elements of D_F are universal norms.)

Given any $a \in A_F$. Since $g_{K/F}^{ab} : U_{K/F}^{ab} \twoheadrightarrow G_F^{ab}$, $\exists x \in U_{K/F}^{ab}$ s.t.
 $g_{K/F}^{ab}(x) = r_F(a) \Rightarrow r_F(V_{K/F}(x) \cdot a^{-1}) = g_{K/F}^{ab}(x) \cdot r_F(a)^{-1} = 1$
 i.e. $V_{K/F}(x) \cdot a^{-1} \in \text{Ker}(r_F) = D_F = N_{K/F}(D_K)$

$$\Rightarrow \exists d \in D_K \text{ s.t. } V_{K/F}(x) \cdot a^{-1} = N_{K/F}(d)$$

$$\Rightarrow a = V_{K/F}(x) \cdot N_{K/F}(d)^{-1} = V_{K/F}(i_{K/F}^{ab}(d)^{-1} \cdot x) \in V_{K/F}(U_{K/F}^{ab})$$

injectivity: (Use: $\hat{H}^1(G_{K/F}, D_K) \overset{\circ\circ}{\cong} \hat{H}^1(G_{K/F}, D_K) = (0)$ $\because D_K$ is induced from cyclic subgroups of $G_{K/F}$)

Suppose that $\bar{u} = \text{image of } u \in U_{K/F} \text{ and } V_{K/F}(\bar{u}) = 0$

$$\Rightarrow g_{K/F}^{ab}(\bar{u}) = r_F(V_{K/F}(\bar{u})) = 0$$

$$\sim u \in i_{K/F}(D_K) \cdot [U_{K/F}, U_{K/F}], \text{ i.e. } \exists d \in D_K \text{ s.t. } u = i_{K/F}(d) \cdot [U_{K/F}, U_{K/F}]$$

$$\text{Know } (0) = \hat{H}^1(G_{K/F}, D_K) = D_K[N_{K/F}] / I_{G_{K/F}} \cdot D_K$$

$$\text{i.e. } D_K[N_{K/F}] = I_{G_{K/F}} \cdot D_K$$

ψ
 $\downarrow$

$$[U_{K/F}, U_{K/F}]$$

$$V(\bar{u}) = 0$$

$$\Leftrightarrow N_{K/F}(d) = 0 \text{ in } A_K$$

$\because$ The action of $G_{K/F}$ on A_K is induced by the conjugation action of $U_{K/F}$ on A_K

Have shown: $V_{K/F}(\bar{u}) = 0 \Rightarrow u \in [U_{K/F}, U_{K/F}]$

i.e. $\bar{u} = 0$. q.e.d.

Notes on Weil's 1951 paper

"Sur la théorie du corps de classes"

k : number field.

K/k : finite Galois extⁿ. $G(K/k) = \text{Gal}(K/k)$

$$A_k = k^{ab}, \quad A_K = K^{ab}, \quad C_k = A_k^{\times}/k^{\times}, \quad C_K = A_K^{\times}/K^{\times}, \quad D_k = (A_k^{\times}/k^{\times})^{\circ}, \quad D_K = (A_K^{\times}/K^{\times})^{\circ}$$

$$C'_k = C_k/D_k, \quad C'_K = C_K/D_K$$

$$G'_{K,k} = \text{Gal}(K^{ab}/k). \quad D'_k = \text{image in } D_k \text{ of } \prod_{v \in \Sigma_{k, \text{cpx}}} k_v^{\times}, \quad \text{Similarly for } D'_K$$

Convention for cocycles and transfer: g acts on the right of A

$$\text{1-cocycles} \quad \xi_{\alpha\beta} = \xi_{\alpha}^{\beta} \cdot \xi_{\beta}, \quad \text{1-coboundary:} \quad \xi_{\alpha} = a^{\alpha} \cdot a^{-1}$$

$$\text{2-cocycles} \quad a_{\alpha\beta, \gamma} \cdot a_{\alpha, \beta}^{\gamma} = a_{\alpha, \beta\gamma} \cdot a_{\beta, \gamma} \quad a_{\alpha, \beta} \in A \quad \forall \alpha, \beta$$

$$\text{Consequence:} \quad \prod_{\beta} a_{\alpha, \beta} \in A^g$$

Example from extensions:

$$1 \rightarrow A \rightarrow E \xrightarrow{\pi} G \rightarrow 1 \quad \text{s.t.} \quad s^{-1} a s = a^{\pi(s)} \quad \forall a \in A, \forall s \in E$$

Let $\alpha \mapsto s_{\alpha}$ be a section of π .

$$\text{Define a 2-cocycle } a_{\alpha, \beta} \text{ by} \quad s_{\alpha} \cdot s_{\beta} = s_{\alpha\beta} \cdot a_{\alpha, \beta}$$

General defⁿ of transfer:

Lubin-Tate:

 k : nonarch. locally compact.

$$\mathcal{O} = \mathcal{O}_k \supseteq \pi_k = \pi \quad \mathfrak{f} = |\mathcal{O}/\pi|$$

 π = generator of π_k

$$P_\pi = \left\{ f \in \mathcal{O}_k[[X]] \mid f(X) \equiv \pi X \pmod{X^2}, \quad f(X) \equiv X^{\mathfrak{f}} \pmod{\pi_k} \right\}$$

Technical

/Elementary Lemma Given $f(X), g(X) \in P_\pi$, and $L(Y_1, \dots, Y_n) = \sum_{i=1}^n a_i Y_i$, $a_i \in \mathcal{O}_k$, $i=1, \dots, n$

linear form

$$\exists! F(Y_1, \dots, Y_n) \in \mathcal{O}[[Y_1, \dots, Y_n]] \text{ s.t.}$$

$$F(Y_1, \dots, Y_n) \equiv L(Y_1, \dots, Y_n) \pmod{(Y_1, \dots, Y_n)^2}$$

and

$$f(F(Y_1, \dots, Y_n)) = F(g(Y_1), \dots, g(Y_n))$$

Only need: $\mathcal{O}_k$ is a DVRDefine (i) $\forall f \in P_\pi$ $G_f(X, Y) \in \mathcal{O}[[X, Y]]$ is the unique solution of

$$\begin{cases} G_f(X, Y) \equiv X + Y \pmod{(X, Y)^2} \\ f(G_f(X, Y)) = G_f(f(X), f(Y)) \end{cases}$$

(ii) $\forall a \in \mathcal{O}_k$, $\forall f, g \in P_\pi$, define $[a]_{f,g}(X) \in \mathcal{O}_k[[X]]$ be the unique solution of

$$\begin{cases} [a]_{f,g}(X) \equiv aX \pmod{X^2} \\ f([a]_{f,g}(X)) = [a]_{f,g}(g(X)) \end{cases}$$

Notation: $[a]_f(X) \stackrel{\text{def}}{=} [a]_{f,f}(X)$

k : non-arch. locally compact local field. $\sigma = \sigma_k, w_k = w_k$
 $q = |\sigma/w|$

π, ω : generators of w_k

$L = k^{ur} = \text{max. unramified ext}^n \text{ of } k$

$f \in P_\pi, g \in P_\omega$

Fact-Lemma $\exists \theta(x) \in \hat{L}[1/x]$ which defines an isom $G_f \xrightarrow{\sim} G_\pi$.

In other words.

$$\begin{cases} \theta(G_f(x, y)) = G_g(\theta(x), \theta(y)) \\ \theta([a]_f(x)) = [a]_g(\theta(y)) \end{cases} \quad \forall a \in \mathcal{O}$$

Admit this.

σ : q -Frob on L and $\mathcal{O}_L$

Have $f(x) \equiv g(x) \equiv x^q \pmod{w_k}$

$$\leadsto g(\theta(x)) \equiv \theta(x)^q \equiv \theta^\sigma(x^q) \equiv \theta^\sigma(f(x)) \pmod{w_k}$$

$$\text{i.e.} \quad \theta^{-1} \circ g \circ \theta \equiv \theta^{-1} \circ \theta^\sigma \circ f \pmod{w_k}$$

$$\begin{array}{ccc} \parallel & & \parallel \\ \theta^{-1} \circ [w]_g \circ \theta & & \theta^{-1} \circ \theta^\sigma \circ [\pi]_f \end{array}$$

$$\parallel$$

rigidity of isomorphisms
of p -divisible groups

$$\Rightarrow \theta^{-1} \circ \theta^\sigma = [\frac{\pi}{w}]_f = [u]_f$$

$$\pi = w u \quad u \in \mathcal{O}_k^\times$$

Compute the reciprocity law on $k^{ur} \cdot G_f[\pi^\infty]$

$$\text{rec}_k(\pi) \Big|_{G_f[\pi^\infty]} = 1 \quad \because \pi \text{ is a norm in } G_f[\pi^m] / k \quad \forall m \geq 1$$

$$\begin{aligned} \text{Given } t \in G_f[\pi^\infty] &\leadsto \theta(t) = \theta(t)^{\text{rec}_k(\pi u)} \\ &= \theta^\sigma(t^{\text{rec}_k(u)}) \end{aligned} \quad \begin{aligned} \because \theta(t) &\in G_g[w^\infty], w = \pi u \\ \because \int \theta^{\text{rec}_k(\pi u)} &= \theta^\sigma \\ \because t^{\text{rec}_k(\pi u)} &= t \end{aligned}$$

$$\leadsto t = \theta^{-1} \circ \theta^\sigma(t^{\text{rec}_k(u)}) = [u]_f(t^{\text{rec}_k(u)}) \quad \forall t \in G_f[\pi^\infty]$$

$$\leadsto t^{\text{rec}_k(u)} = [u^{-1}]_f(t) \quad \forall t \in G_f[\pi^\infty], \forall u \in \mathcal{O}_k^\times$$

Theorem 1) $\begin{cases} t^{\text{reg}_k(u)} = [u^{-1}](t) \\ t^{\text{reg}_k(\pi)} = t \end{cases} \quad \begin{array}{l} \forall t \in G_f[\pi^{-\infty}], \quad \forall u \in \mathcal{O}_k^\times \\ \forall t \in G_f[\pi^\infty] \end{array} \quad f \in P_\pi$

2) $k^{ab} = G_f[\frac{\infty}{\pi}](k^{\text{sep}}) \cdot k^{\text{ur}}$