Math 6200 – 6210 Fall 2022 – Spring 2023

Fermat	1601–1667	Dedekind	1831–1916
Euler	1707–1783	H. Weber	1841–1913
Lagrange	1736–1813	Hensel	1861–1941
Legendre	1752–1833	Hilbert	1862–1943
Gauss	1777–1855	Minkowski	1864–1909
Dirichlet	1805–1859	Takagi	1875–1960
Kummer	1810–1893	Landau	1877–1938
Hermite	1822–1901	Hecke	1887–1947
Eisenstein	1823–1852	Artin	1898–1962
Kronecker	1823–1891	H. Hasse	1898–1979
Riemann	1826–1866	Weil	1909–1998
		Chevalley	1909–1984
		Iwasawa	1917–1998
		Tate	1925–2019
		Serre	1926–
		Shimura	1930–2019

Hadamard 1865–1963
de la Vallée Poussin 1866–1962

Hardy 1877–1947
Littlewood 1885–1977

Siegel 1896–1981

Gauss Disquisitiones 1801 — quadratic reciprocity
classes of quadratic forms, composition
$\rightsquigarrow$ classes of ideals in algebraic number fields

Dirichlet 1837 mémoir — L-functions (Dirichlet character)

Riemann 1859 — Riemann's zeta function

1896 Hadamard / de la Vallée Poussin — Prime Number Theorem $\Leftarrow$ nonvanishing of $\zeta(s)$

1897 Hilbert Zahlbericht

1920 Takagi — 1918, 1920 Hecke — Hecke L-functions

1927 Artin proof of reciprocity law

1950 Iwasawa (ICM lecture), Tate (Princeton thesis) — adelic approach

1935 Siegel 2 essentially equivalent forms

(i) $\forall \varepsilon > 0$, $\exists$ const. $C_1(\varepsilon) > 0$ s.t.
$\forall$ real primitive Dirichlet character $\chi : (\mathbb{Z}/q\mathbb{Z})^\times \to \{\pm 1\}$,
$L(1, \chi) > C_1(\varepsilon)\, q^{-\varepsilon}$

i.e. $h(d) > C_2(\varepsilon) \cdot |d|^{\frac{1}{2}-\varepsilon}$ for $d<0$
$h(d) \log \eta > C_2(\varepsilon)\, d^{\frac{1}{2}-\varepsilon}$ for $d>0$

(Here $(\pm 1) \times \eta^{\mathbb{Z}} = \{ z \in \frac{1}{2}\mathbb{Z}[\sqrt{d}] : Nm_{\mathbb{Q}(\sqrt{d})/\mathbb{Q}}(z) = 1 \}$
$\eta = \frac{1}{2}(t_0 + u_0\sqrt{d})$, $t_0, u_0 > 0$, $t_0^2 - d u_0^2 = 4$
i.e. η = a fund. solution of Pell's equation)

(ii) $\forall \varepsilon > 0$, $\exists C_3(\varepsilon) > 0$ s.t. $\forall$ real Dirichlet non-trivial Dirichlet character χ, with modulus q, then
$L(s, \chi) \neq 0$ for all $s > 1 - C_3(\varepsilon) \cdot q^{-\varepsilon}$

Dirichlet L-function $\chi : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$

$$L(s, \chi) = \sum_{n \geq 1} \chi(n)\, n^{-s} = \prod_p \left(1 - \chi(p)\, p^{-s}\right)^{-1}$$

sketch Dirichlet's idea Assume q = an odd prime number

$$\log L(s, \chi) = \sum_{p \neq q} \sum_{m \geq 1} \frac{1}{m} \chi(p)^m p^{-ms}$$

given a, $(a,q)=1$

$$\frac{1}{q-1} \sum_{\chi : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times} \chi(a)^{-1} \cdot \log L(s, \chi) = \sum_{p \equiv a \ (\mathrm{mod}\ q)} p^{-s} + O(1)$$

$$\sum_{p \neq q} \sum_{m \geq 2} \frac{1}{m} \chi(a)^{-1} \cdot \chi(p)^m \cdot p^{-ms}$$

Goal: Show that $\frac{1}{q-1} \sum_{\chi:(\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times} \chi(a)^{-1} \cdot \log L(s,\chi) \to \infty$ as $s \to 1^+$

$$\parallel$$

$$\frac{1}{q-1}(1-q^{-s}) \cdot \zeta(s) + \frac{1}{q-1} \sum_{\chi \neq 1} \chi(a)^{-1} \cdot \log L(s,\chi)$$

Suffices to show: $\forall \chi \neq \mathbb{1}$, $\lim_{s \to 1^+} L(s,\chi)$ exists by Dirichlet's test

$$\lim_{s\to 1^+} L(s,\chi) \overset{!!}{=} L(1,\chi) \neq 0$$

Note: $L(1,\chi) \neq 0$ easy if $\bar{\chi} \neq \chi$

The case where $\bar{\chi} = \chi$, i.e. $\chi^2 = \mathbb{1}$, is more difficult and involves explicit evaluation of $L(1,\chi)$ as a finite exponential sum.

For χ real, Dirichlet showed: q : odd prime

(a) If $q \equiv 3 \pmod 4$, then $L(1,\chi) = -\frac{\pi}{q^{3/2}} \underbrace{\sum_{m=1}^{q-1} m\left(\frac{m}{q}\right)}_{\equiv 1 \pmod 2}$

(b) If $q \equiv 1 \pmod 4$, then

$$L(1,\chi) = \frac{\log Q}{q^{1/2}}, \quad Q = \frac{\prod_N \sin(\pi N/q)}{\prod_R \sin(\pi R/q)}$$

N = non-residue mod q

R = residue mod q

$$\prod_R \left(x - e_q(R)\right) = \frac{1}{2}\left(Y(x) - q^{1/2} Z(x)\right)$$

Gauss

$$\prod_N \left(x - e_q(N)\right) = \frac{1}{2}\left(Y(x) + q^{1/2} Z(x)\right)$$

$Y(x), Z(x) \in \mathbb{Z}[x]$

With $Y = Y(1)$, $Z = Z(1)$. $Y^2 - qZ^2 = 4q \rightsquigarrow Z \neq 0$

$$Q = \frac{Y + q^{1/2} Z}{Y - q^{1/2} Z}$$

Class field theory $\Leftrightarrow$ $\left\{ \text{Hecke L-functions} \right\} \underset{\substack{\uparrow \\ \text{Artin's motivation}}}{=} \left\{ \begin{array}{l} \text{Artin L-Function with} \\ \quad \text{abelian characters} \end{array} \right\}$

Global fields = either finite extension fields of $\mathbb{Q}$
or finitely generated extension fields of $\mathbb{F}_p$
of transcendence degree 1

K: global field

def^n: a normalized absolute value $|\cdot|_v : K \to \mathbb{R}$ of K
is an absolute value (i.e. $|\cdot|_v : K \longrightarrow \mathbb{R}$ is multiplicative
$|\ |_v(a) = 0$ iff $a = 0$,
+ triangle inequality)

s.t. $\forall$ Haar measure μ_v on K_v,
$\mu_v(x \cdot U) = |x|_v \cdot \mu_v(U)$ $\quad\forall$ open (or closed) subset $U \subseteq K_v$

e.g. 1) $K_v \overset{\alpha}{\cong} \mathbb{C}$: on $\mathbb{C}$, $\mu(z \cdot U) = z \cdot \bar{z} \cdot \mu(U)$

2) On $\mathbb{Q}_p$, $|a|_p = p^{-\mathrm{ord}_p(a)}$

3) On $\mathbb{F}_q((t))$, $|u| = q^{-\mathrm{ord}(u)}$

Generally: K_v = fraction field of a complete dvr $\mathcal{O}_{K_v}$ with
finite residue field κ_v, $\quad q_v = \# \kappa_v$

$$|x| = q_v^{-\mathrm{ord}_v(x)}$$

product formula: $\forall_{x \in K}$, $\displaystyle\prod_{v \in \Sigma_K} |x|_v = 1$
K: global field

trivial consequence: K = number field.

$$\forall x \in \mathcal{O}_K, \quad \prod_{v:\ \text{arch.}} |x_v| \geq 1$$

Define $\mathbb{A}_K$, and its topology ← (and Hausdorff) commutative locally compact top. ring

$\mathbb{A}_K^\times$, induced topology ← locally compact commutative top. group

Harmonic analysis on locally compact commutative top. group

unitary characters

duality (Pontrygin)

Fourier inversion

$$\hat{f}(\hat{x}) = \int f(x) \langle x, \hat{x} \rangle \, dx$$

$$f(x) = \int \hat{f}(\hat{x}) \langle x, \hat{x} \rangle \, d\hat{x}$$

dual Haar measure

(The Haar measure $d\hat{x}$ on $\hat{G}$ is uniquely determined by this)

Moreover, F.T. determines a unitary $L^2(G) \to L^2(\hat{G})$

<u>Note</u>: Weil's original proof uses the following structural theorem

1940 L'intégration dans les groupes topologiques et ses applications

<u>Thm</u>: Every locally compact abelian group G is topologically isomorphic to $K \times \mathbb{R}^a \times \mathbb{Z}^b$, where K is a compact abelian group, $a, b \in \mathbb{N}$. (Under such an isomorphism, K corresponds to the largest compact subgroup of G.)

<u>Reference</u> Chap 1, §1.1–§1.7 (pp. 1–30) of Rudin, Fourier Analysis on Groups

Poisson summation $G \supseteq \underset{\text{discrete}}{\Gamma}$ G/Γ compact

$$f \in \mathcal{S}(G) \rightsquigarrow \sum_{\gamma \in \Gamma} f(\gamma) = \sum_{\delta \in \Gamma^\perp} \hat{f}(\delta)$$

"product measure"

$$1 \to \Gamma \to G \to G/\Gamma \to 1 \qquad d\mu_\Gamma \cdot d\mu_{G/\Gamma}$$

dual to

$$1 \leftarrow \Gamma^\vee \leftarrow G^\vee \leftarrow \Gamma^\perp \leftarrow 1 \qquad d\mu_{\Gamma^\vee} \cdot d\mu_{\Gamma^\perp}$$

Regulator as example of Fubini theorem

K: nber field

$\mathcal{O}_K^\times$: f.g. abelian group of rank r_1+r_2-1

$$L : \begin{array}{ccc} \mathcal{O}_K^\times & \longrightarrow & \mathbb{R}^{\Sigma_{K,\infty}} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \longmapsto & (\log|x|_v)_{v \in \Sigma_{K,\infty}} \end{array}$$

$|\cdot|_v$: normalized absolute value

$$0 \to \mathbb{R}_0^{\Sigma_{K,\infty}} \longrightarrow \mathbb{R}^{\Sigma_{K,\infty}} \longrightarrow \mathbb{R} \to 0$$

$$(u_v) \longmapsto \sum_{v \in \Sigma_{K,\infty}} u_v$$

$\mu_{\mathbb{R}^{\Sigma_{K,\infty}}}/\mu_{\mathbb{R}}$ product of Lebesgue measures   Lebesgue measure

Fact: $\operatorname{Ker}(L) = \mu(K)$

$\operatorname{Im}(L) =$ a cocompact lattice in $\mathbb{R}_0^{\Sigma_{K,\infty}}$

Define

$$R_K = \int_{\mathbb{R}^{\Sigma_{K,\infty}}/L(\mathcal{O}_K^\times)} dx$$

regulator

dx: the measure $\mu_{\mathbb{R}^{\Sigma_{K,\infty}}}/\mu_{\mathbb{R}}$ on $\mathbb{R}_0^{\Sigma_{K,\infty}}$

another expression $\varepsilon_1, \dots, \varepsilon_{r_1+r_2-1} \in \mathcal{O}_K^\times$: free basis of $\mathcal{O}_K^\times/\mu(K)$

$$R_K = \left| \text{det of any } (r_1+r_2-1)\text{-minor of the matrix } \left(\log|\varepsilon_j|_i\right) \right|$$

$r \times (r_1+r_2-1)$ $\Sigma_{K,\infty} = \{ |\cdot|_1, \dots, |\cdot|_{r_1+r_2} \}$

$$= \frac{1}{[K:\mathbb{Q}]} \left| \det \begin{pmatrix} \log|\varepsilon_1|_1 & \cdots & \log|\varepsilon_{r_1+r_2-1}|_1 & [K_{v_1}:\mathbb{R}] \\ \vdots & & \vdots & \\ \log|\varepsilon_1|_{r_1+r_2} & & \log|\varepsilon_{r_1+r_2-1}|_{r_1+r_2} & [K_{v_{r_1+r_2}}:\mathbb{R}] \end{pmatrix} \right|$$

dual groups of locally compact fields

<u>Lemma</u> K: locally compact nondiscrete field.
$\psi : (K,+) \to \mathbb{C}_1^\times$ non-trivial unitary character.
Then every unitary character of $(K,+)$ is of the form
$x \mapsto \psi(ax)$ for a uniquely determined elt $a \in K$.

1^{st}/direct proof:
Clearly $T : (K,+) \longrightarrow (K,+)^\vee = \mathrm{Hom}_{ct}(K, \mathbb{C}_1^\times)$
$a \longmapsto (x \mapsto \psi(ax))$
is injective with dense image.

Pick $y_0 \in K$ s.t. $\psi(y_0) \neq 1$
Pick ε_0 s.t. $|\psi(y_0)-1| > \varepsilon_0$

Consider open nbd $U_{M,\varepsilon_0} = \{ \chi \in K^\vee \mid |\chi(\underset{\substack{\| \\ \{x\in K \mid |x| \le M\}}}{B_M}) - 1| \le \varepsilon_0 \} \subseteq K^\vee$

$$b \in T^{-1}(U_{M,\varepsilon_0}) \Rightarrow |\psi(bB_M) - 1| \le \varepsilon_0$$
$$\Rightarrow y_0 \notin bB_M \Leftrightarrow |y_0| \ge |b| \cdot M$$
$$\Leftrightarrow |b| \le M^{-1}|y_0|$$

So $K \xrightarrow[\text{homeom}]{\sim} T(K) \Longrightarrow K = TK$.
$\because$ $T(K)$ is dense in $K^\vee$ q.e.d

2^{nd} <u>proof</u>: $K^\vee$: has a natural structure as a K-module
$\rightsquigarrow K^\vee \cong K \oplus Y$ Y: (non-unique) K-submodule of $K^\vee$
$$\Rightarrow K \underset{\substack{\uparrow \\ K\text{-linear}}}{\xrightarrow{\sim}} K^{\vee\vee} \cong K \oplus Y \oplus Y^{\vee\vee}$$
$\Rightarrow Y = (0)$ q.e.d.

<u>Lemma</u> $\mathbb{Q} \times [0,1) \times \prod_p \mathbb{Z}_p \xrightarrow{\text{bij}} \mathbb{A}_{\mathbb{Q}}$

$$(a, \quad b_\infty, \quad (b_p)) \longmapsto a + b_\infty + (a + b_p)_p$$

$\mathbb{R} + \prod_p \mathbb{Z}_p = \mathbb{A}_{\mathbb{Q}}$, so surjective

injective: verify directly

<u>Cor</u>: $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$ is compact $\left(\because [0,1] \times \prod_p \mathbb{Z}_p \twoheadrightarrow \mathbb{A}_{\mathbb{Q}}/\mathbb{Q}\right)$

$\mathbb{Q}$ is discrete in $\mathbb{Q}$

$\Rightarrow$ <u>Prop</u> K: number field

$\mathbb{A}_K/K$ is compact, $\because \mathbb{A}_K/K \underset{\text{homeom}}{\cong} (\mathbb{A}_{\mathbb{Q}}/\mathbb{Q})$

K is discrete in $\mathbb{A}_K$

<u>Note</u> $\mathbb{Q}$ is discrete in $\mathbb{A}_{\mathbb{Q}}$ $\because \mathbb{Q} \cap \left\{ (x_p) \;\middle|\; \begin{array}{l} x_p \in \mathbb{Z}_p \quad \forall p \neq \infty \\ x_\infty \in (-\frac{1}{2}, \frac{1}{2}) \end{array} \right\} = \{0\}$

Self-dual measure on $\mathbb{Q}_p$ and $\mathbb{Q}_\infty$

use
$$\begin{cases} \psi_p: \mathbb{Q}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \xleftarrow{\sim} \mathbb{Z}[\frac{1}{p}]/\mathbb{Z} \longrightarrow \mathbb{C}_1^\times \\ \qquad\qquad\qquad\qquad\qquad\quad a \longmapsto \mathbf{e}(-a) = e^{-2\pi\sqrt{-1}a} \\ \psi_\infty: \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \xrightarrow{\sim} \mathbb{C}_1^\times \\ \qquad\qquad\qquad x \longmapsto \mathbf{e}(x) = e^{2\pi\sqrt{-1}x} \end{cases}$$

to identify $\mathbb{Q}_p^\vee$ with $\mathbb{Q}_p$, $\mathbb{Q}_\infty^\vee$ with $\mathbb{Q}_\infty$

<u>Exer</u> self-dual measure on $\mathbb{R}$: dx

$\mathbb{Q}_p$: $\int_{\mathbb{Z}_p} dx_p = 1$

$p \mid v$

$K/\mathbb{Q}$ $\quad [K:\mathbb{Q}] < \infty$ $\quad$ use $\psi_v : (K_v, +) \to \mathbb{C}_1^*$. $\psi_v(x_v) = \psi_p(\mathrm{Tr}_{K_v/\mathbb{Q}_p}(x_v))$

self-dual measure

$K_v \cong \mathbb{C}$ $\quad |dz \wedge d\bar{z}| = 2dx\,dy$

$K_v \cong \mathbb{R}$ $\quad dx$

v: nonarch $\quad \int_{\mathcal{O}_v} d\mu_v = (\mathbb{N}\mathfrak{D}_v)^{-\frac{1}{2}}$

where $\mathfrak{D}_v^{-1} = \{ x \in K_v \mid \mathrm{Tr}_{K_v/\mathbb{Q}_p}(x \cdot \mathcal{O}_v) \subseteq \mathbb{Z}_p \}$

$\quad \| $

$y_v \cdot \mathcal{O}_v \leadsto \mathbb{N}\mathfrak{D}_v = |y_v|_v^{-1}$ ← normalized abs. value

self-dual Haar measure on $\mathbb{A}_K$ w.r.t.

$\psi_K : \mathbb{A}_K/K \longrightarrow \mathbb{C}_1^*$ $\quad \psi_K((x_v)) = \prod_{v \in \Sigma_K} \psi_{K_v}(x_v)$

$\mu_{\mathbb{A}_K} = \prod_v{}' \mu_{K_v}$ $\quad \mu_{K_v}$ = self-dual Haar measure on K_v w.r.t. $\psi_v = \psi_p \circ \mathrm{Tr}_{K_v/\mathbb{Q}_p}$

global discriminant of K, $[K:\mathbb{Q}] < \infty$

disc_K $\quad \mathrm{disc}_K$ = the disc. of the integral quadratic form

$$\mathcal{O}_K \times \mathcal{O}_K \to \mathbb{Z}$$
$$(x, y) \mapsto \mathrm{Tr}_{K/\mathbb{Q}}(x \cdot y)$$

Let $\mathbb{Z}u_1 + \cdots + \mathbb{Z}u_n = \mathcal{O}_K$, $n = [K:\mathbb{Q}]$ $\quad \sigma_1, \ldots, \sigma_n : K \to \mathbb{C}$

$$\leadsto \mathrm{disc}_K = \det\left(\mathrm{Tr}_{K/\mathbb{Q}}\, u_i \cdot u_j\right)_{1 \le i,j \le n} = \det\left(\sum_{\ell=1}^{n} \sigma_\ell(u_i \cdot u_j)\right)_{1 \le i,j \le n}$$

$$= \det\left(\sigma_\ell(u_i)\right)_{1 \le \ell, i \le n} \cdot \det\left(\sigma_\ell(u_j)\right)_{1 \le \ell, j \le n}$$

$$= \left(\det\left(\sigma_\ell(u_i)\right)_{1 \le \ell, i \le n}\right)^2$$

$$|\mathrm{disc}_K| = \left| \det\left(\sigma_\ell(u_i)\right)_{1\le \ell, i\le n}\right|^2$$

$$\left|\det(\sigma_\ell(u_i))_{1\le \ell, i\le n}\right| = \int_{K_\infty/O_K} \overbrace{\prod_{v\in\Sigma_{K,\infty}} \mu_v}^{|dx_1\wedge\cdots\wedge dx_{r_1}\wedge dz_1\wedge d\bar z_1\wedge\cdots\wedge dz_{r_2}\wedge d\bar z_{r_2}|}$$

self dual measure on K_v

w.r.t. $\psi_v : K_v \to \mathbb{C}^\times_1$

$\psi_v = \psi_p \circ \mathrm{Tr}_{K_v/\mathbb{R}}$

Conclude

$$\int_{K_\infty/K} |dx_1\wedge\cdots\wedge dx_{r_1}\wedge dz_1\wedge d\bar z_1\wedge\cdots\wedge dz_{r_2}\wedge d\bar z_{r_2}| = |\mathrm{disc}_K|^{\frac12}$$

$x_1,\dots,x_{r_1} : K \hookrightarrow \mathbb{R}$

$z_1,\dots,z_{r_2} : K \hookrightarrow \mathbb{C}$

equiv Let $\mu_{\mathbb{A}_K,\text{naive}} = \prod'_{v\in\Sigma_K} \mu_{K_v,\text{naive}}$ where

$\mu_{K_v,\text{naive}}(O_v) = 1$ v: nonarch

$$\mu_{K_v,\text{naive}} = \begin{cases} dx_v & v \text{ real} \\ dz_v\wedge d\bar z_v & v: \text{cpx}\end{cases}$$

Then

$$\int_{\mathbb{A}_K/K} \mu_{\mathbb{A}_K,\text{naive}} = |\mathrm{disc}_K|^{\frac12}$$

equiv

$$\int_{\mathbb{A}_K/K} \mu_{\mathbb{A}_K,\text{sd}} = 1$$

self-dual w.r.t. $\psi_{\mathbb{A}_K} = \psi_{\mathbb{A}_\mathbb{Q}} \circ \mathrm{Tr}_{K/\mathbb{Q}}$

$$\begin{array}{c}\mathbb{A}_K\\ \cup|\end{array}$$

Let $V_1 := \{ x=(x_v) \in \mathbb{A}_K \mid \|x_v\|_v \le 1 \ \forall v \}$

<u>Lemma</u> (Minkowski) $\forall$ idele $z=(z_v) \in \mathbb{A}_K^\times$ with $\|z\| > 2^{r_2}\pi^{-r_2} \cdot |\mathrm{disc}_{K/\mathbb{Q}}|^{\frac{1}{2}}$

have $(z \cdot V_1) \cap K \neq \{0\}$

<u>Proof</u>: $\mu_{\mathbb{A}_{K,sd}}(V_1) = 2^{r_1} \cdot (2\pi)^{r_2} \cdot |\mathrm{disc}_K|^{-\frac{1}{2}} = 2^{[K:\mathbb{Q}]} 2^{-r_2} \pi^{r_2} \cdot |\mathrm{disc}_K|^{-\frac{1}{2}}$

Let $\underset{(y_{0,v})_v}{\overset{\|}{y_0}}$ be the element of $\mathbb{A}_K^\times$ s.t. $y_{0,v} = \begin{cases} 1 & v \text{ non-arch} \\ \frac{1}{2} & v \text{ arch} \end{cases}$

Suppose that $(z \cdot V_1) \cap K = \{0\}$. then:

$\forall\, a \in K_{\neq 0}$, $a + (yzV_1) \cap yzV_1 = \phi$ $\left(\begin{array}{l}\text{Otherwise } \exists\, t_1, t_2 \in V_1 \\ \text{s.t. } z\underbrace{(yt_1 - yt_2)}_{\in V} = a\end{array}\right)$

$\rightsquigarrow \mu_{\mathbb{A}_K}(yzV_1) < 1$

$n = [K:\mathbb{Q}]$ $\qquad \overset{\|}{\left(\tfrac{1}{2}\right)^n \cdot \|z\| \ 2^n 2^{-r_2} \pi^{r_2} |\mathrm{disc}_K|^{-\frac{1}{2}}} \underset{\text{assumption}}{\overset{\uparrow}{>}} 1$

Apply this lemma: Given an ideal $\underset{\|}{\mathfrak{a}} \subseteq \mathcal{O}_K$, let $\mathfrak{a} = \underset{\mathbb{A}_{K,f}^\times}{\overset{}{z_0}} \cdot \mathcal{O}_K$ ($z_0 \in \mathbb{A}_{K,f}^\times$)

Pick $z_1 \in \mathbb{A}_{K,\infty}^\times$ s.t. $\|z_1 \cdot z_0\| > 2^{r_2}\pi^{-r_2} \cdot |\mathrm{disc}_{K/\mathbb{Q}}|^{\frac{1}{2}}$

$\rightsquigarrow \exists\, \alpha \in K^\times \quad \alpha \in \overbrace{z_1 \cdot z_0}^{z} \cdot V_1$

$(\alpha) = \mathfrak{a} \cdot \mathfrak{b}$ $\qquad \mathfrak{b}$: an $\mathcal{O}_K$-ideal

$\underset{\|z_0\|^{-1}}{\underline{\mathbb{N}\mathfrak{a}}} \cdot \mathbb{N}\mathfrak{b} \le \|z_1\| \quad \Rightarrow \quad \mathbb{N}\mathfrak{b} \le 2^{r_2}\pi^{-r_2} \cdot |\mathrm{disc}_{K/\mathbb{Q}}|^{\frac{1}{2}}$

<u>Conclude</u>: slightly weaker form of Minkowski's bound:

$\forall$ ideal class $[\mathfrak{a}]^{-1} \in K(\mathcal{O}_K)$, $\exists$ an ideal $\mathfrak{b}$ s.t

$\mathbb{N}\mathfrak{b} \le 2^{r_2}\pi^{-r_2} \cdot |\mathrm{disc}_K|^{1/2}$

Classical Minkowski bound:

In any ideal class $\in H(\mathcal{O}_K)$, $\exists$ an ideal $\mathfrak{b}$ with

$$\mathbb{N}\mathfrak{b} \leq \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^{r_2}|\mathrm{disc}_K|^{\frac{1}{2}}$$

(Improvement comes from better choice of region in K_∞)

$\psi_{\mathbb{A}_K} = \psi_{\mathbb{A}_\mathbb{Q}} \circ \mathrm{Tr} : (\mathbb{A}_K, +) \longrightarrow \mathbb{C}^\times_1$ identifies $\mathbb{A}_K^\vee$ with $\mathbb{A}_K$

$$\underset{\text{a } K\text{-module}}{(\mathbb{A}_K/K)^\vee} = \{ x \in \mathbb{A}_K \mid \psi_{\mathbb{A}_K}(x \cdot y) = 1 \ \forall y \in K \} \supseteq \underset{\text{discrete}}{K}$$

⇝ If $\underbrace{(\mathbb{A}_K/K)^\vee}_{= \Gamma} \supsetneq K$, then $\mathbb{A}_K$ contain a discrete subgroup $\Gamma \supseteq K$

and $\Gamma/K \underset{\text{discrete}}{\hookrightarrow} \mathbb{A}_K/K$

nontrivial K-module, hence infinite. Contradiction

Have proved: $(\mathbb{A}_K/K)^\vee$ = 1-dimensional K-vector space.

Lemma identified with K via $\psi_{\mathbb{A}_K}$

$$= \{ (y \mapsto \psi_{\mathbb{A}_K}(a y)) \mid a \in K \}$$

Thm $\mathbb{A}^\times_{K,1} / K^\times$ is compact

$$V_1 = \left\{ x = (x_v) \in \mathbb{A}_K \;\middle|\; \begin{array}{l} x_v \in \mathcal{O}_v \\ \|x_v\| \le 1 \end{array} \right\}$$

Pf: $\iota : \mathbb{A}^\times_{K,1} \overset{\text{closed}}{\hookrightarrow} \mathbb{A}_K \times \mathbb{A}_K$

$\qquad x \longmapsto (x, x^{-1})$

Pick $z_0 \in \mathbb{A}_K^\times$ with $\|z_0\| > 2^{r_2} \pi^{-r_2} \cdot |\mathrm{disc}_K|^{\frac{1}{2}}$

Will find compact subsets $B_1, B_2 \subseteq \mathbb{A}_K$ s.t

s.t. $\forall x \in \mathbb{A}^\times_{K,1}$, $\exists c \in K^\times$ s.t $x \cdot c \in B_1$ and $x^{-1} c^{-1} \in B_2$

i.e. $\underbrace{\mathbb{A}^\times_{K,1} \cap \iota^{-1}(B_1 \times B_2)}_{\text{compact}} \longrightarrow\!\!\!\!\to \mathbb{A}^\times_{K,1} / K^\times$

$\forall x \in \mathbb{A}^\times_{K,1}$ ⇝ $\exists\ a \in K^\times \cap x^{-1} z_0 V_1$, $\exists\ b \in K^\times \cap x z_0 V_1$ $\Rightarrow a \cdot b \in (z_0^2 V_1) \cap K =$ a finite subset $C \subseteq K$

$\Rightarrow x a \in z_0 V_1$, $(x \cdot a)^{-1} = x^{-1} a^{-1} = x^{-1} b^{-1} \cdot b^{-1} a^{-1} \in C \cdot z_0 V_1$

Take $B_1 = z_0 V_1$ $\quad B_2 = C \cdot z_0 V_1$ $\qquad$ QED

Cor. Dirichlet-Hasse unit thm

$\Sigma_{K,\infty} \subseteq S$ finite $\quad r = r_1 + r_2 \quad s = \# S$

Consider: $\mathbb{A}_K^\times \supseteq U_S \overset{\text{def}}{=} \{ (x_v) \in \mathbb{A}_K^\times \mid |x_v|_v = 1 \ \ \forall v \notin S \}$

K: nber field

$$\mathbb{A}_K^\times = \mathbb{A}_{K,1}^\times \cdot U_S$$

$$U_S / U_{\Sigma_K} \cong \mathbb{Z}^{s-r} \times \mathbb{R}^r$$

$$\rightsquigarrow (\mathbb{A}_{K,1}^\times \cap U_S) / U_{\Sigma_K} \cong \mathbb{Z}^{s-r} \times \mathbb{R}^{r-1}$$

Consider $\quad (\mathbb{A}_{K,1}^\times \cap U_S) \supseteq \mathcal{O}_{K,S}^\times \cdot U_{\Sigma_K} \supseteq U_{\Sigma_K}$

$\because$ $(\mathbb{A}_{K,1}^\times \cap U_S) / \mathcal{O}_S^\times \cdot U_{\Sigma_K}$ compact

$\mathcal{O}_S^\times \cdot U_{\Sigma_K} / U_{\Sigma_K} \cong \mathcal{O}_{K,S}^\times / \mu(K)$ discrete

$\cong \mathbb{Z}^{s-1}$ $\because$ its image in $\mathbb{Z}^{s-r} \times \mathbb{R}^{r-1}$ is a cocompact lattice

$\Rightarrow \quad \mathcal{O}_{K,S}^\times \cong \mu(K) \times \mathbb{Z}^{s-1}$

II. Examples

Tools, methods $\qquad n = [k : \mathbb{Q}] \quad k$: nber field

Minkowski's bound: Every ideal class of $\mathcal{O}_k$ is represented by an ideal $\mathfrak{b}$ with $N\mathfrak{b} \le \frac{n!}{n^n} \left(\frac{4}{\pi}\right)^{r_2} \cdot |disc_k|^{\frac{1}{2}}$

Cor. $|disc_k| \ge \frac{n^{2n}}{(n!)^2} \left(\frac{\pi}{4}\right)^{2r_2}$

example: $n = 3 \quad r_1 = 1 = r_2 \quad |disc_k| \ge \frac{3^6}{3^2 2^2} \cdot \frac{\pi^2}{2^4} = \frac{3^4}{2^6} \cdot \pi^2 > 12.47$

i.e. $|disc_k| \ge 13$

Fact: $\mathrm{Min}\left(\{|disc_k| : k \text{ : cubic nber field with } r_1 = 1 = r_2\}\right) = -23$

example 0a $\mathbb{Q}[x]/(x^3-x-1)$ is a cubic field

$disc(x^3-x-1) = -(-4+27) = -23$

α = image of x in $\mathbb{Q}[x]/(x^3-x-1) =: k$

$\mathcal{O}_k = \mathbb{Z}[\alpha]$, and $disc_k = -23$

Elementary Fact (0) $sgn(disc_k) \equiv r_2 \pmod 2$

(i) $disc_k \equiv 0, 1 \pmod 4$ (Stickelberger. Artin p. 132)

(ii) $\mathcal{O} \subseteq \mathcal{O}_k$ order $\Rightarrow$ $disc\,\mathcal{O} = \begin{pmatrix}\text{square of}\\ \text{an integer}\end{pmatrix} \cdot disc\,\mathcal{O}_k$

Formula For $f(x) = x^3 + a_1x^2 + a_2x + a_3$

$$disc(f) = a_1^2a_2^2 - 4a_1^3a_3 + 18a_1a_2a_3 - 4a_2^3 - 27a_3^2$$

homog. of wt 6

example 0b. $\mathbb{Q}[x]/(\underbrace{x^3+x+1}_{g(x)}) \ni \alpha$ = image of x

$disc(g) = -(4+27) = -31 \quad \rightsquigarrow \quad \mathcal{O}_k = \mathbb{Z}[\alpha]$ and $disc_k = -31$

Remark: Minkowski bound for nontotally real cubic fields (i.e. $r_1 = 1 = r_2$) is $|\mathrm{disc}_k| \geq 13$

among $-\cancel{13}, -\cancel{14}, -15, -16, -\cancel{17}, -\cancel{18}, -19, -20, -\cancel{21}, -\cancel{22}$ all but $-15, -16, -19, -20$ are ruled out by the congruence condition mod 4 (Stickelberger)

To rule out the existence of cubic fields with disc $= -15, -16, -19, -20$ can use one of two methods

- brute force: such a cubic field admits a generator with bounded height (Minkowski). Then check a finite number of possibilities

- Let k be such a cubic number field. Let N = normal closure of k ← Galois group $= S_3$

  Show that $N/\mathbb{Q}(\sqrt{\Delta_k})$ is everywhere unramified (including arch. places) in case $\mathrm{disc}_k = -15, -16, -19$

  $\Rightarrow$ N is contained in the Hilbert class field of $\mathbb{Q}(\sqrt{\Delta_k})$

  $\Rightarrow h_{\mathbb{Q}(\sqrt{\Delta_k})} \equiv 0 \pmod 3$

  Fact $h_{\mathbb{Q}(\sqrt{-15})} = 2$, $h_{\mathbb{Q}(\sqrt{-19})} = 1$, $h_{\mathbb{Q}(\sqrt{-1})} = 1$

  This rules out $-15, -16, -19$

For the case $\mathrm{disc}_k = -20$: show N is contained in the ray class field $F_{\mathfrak{z}}/\mathbb{Q}(\sqrt{5})$ with conductor $\mathfrak{z}$, where $\mathfrak{z}$ = the unique prime of $\mathbb{Q}(\sqrt{5})$ above 5

and $[F_{\mathfrak{z}} : \mathbb{Q}(\sqrt{5})] = 4$ ← Fact, class field theory + $h_{\mathbb{Q}(\sqrt{5})} = 2$

Example 1 (Dedekind) $k = \mathbb{Q}[x] / (x^3 + x^2 - 2x + 8)$

$f(x)$ $\qquad \alpha = \text{image of } x$

$\mathrm{disc}(f) = -2^2 \cdot 503$

Claim 1 $\mathrm{disc}_k = -503$ ← prime. Note: Stickelberger congruence doesn't help

Will find $\mathcal{O}_k$ $\qquad \alpha^3 + \alpha^2 - 2\alpha + 8 = 0 \qquad (1)$

$$\mathrm{disc}(f) = 4 - \underbrace{4 \cdot 8}_{32} + 18 \cdot (-16) + \underbrace{4 \cdot 2^3}_{32} - 27 \cdot 8^2$$

$$= 4(1 - 72 - 432) = -4 \cdot 503$$

$f(-2) = 8$, $f(1) = 8$, $f(0) = 8$. $f(-3) = -4$

$\rightsquigarrow$ $f(x)$ has a real root in $(-3, -2)$

$\alpha \mid 8$ in $\mathbb{Z}[\alpha] \subseteq \mathcal{O}_k$

$$8 + \frac{8}{\alpha} - \frac{16}{\alpha^2} + \frac{64}{\alpha^3} = 0$$

Let $\beta = \frac{4}{\alpha}$ get a slightly better estimate: $\alpha \mid 4$ in $\mathcal{O}_k$

$$\beta^3 - \beta^2 + 2\beta + 8 = 0 \qquad (2)$$

Conclusion:
$$\begin{cases} \alpha^2 + \alpha - 2 + 2\beta = 0 & \text{(divide (1) by } \alpha) \\ \beta^2 - \beta + 2 - 2\alpha = 0 & \text{(divide (2) by } \beta) \\ \alpha\beta = 4 \end{cases}$$

$\because \dim_{\mathbb{Q}}(\text{L.H.S.}) \le 3$

$$\mathbb{Q}[x,y] / (x^2 + x - 2 - 2y,\ y^2 - y + 2 - 2x,\ xy - 4) \xrightarrow{\sim} k$$

$$\mathbb{Z}[x,y] / (x^2 + x - 2 - 2y,\ y^2 - y + 2 - 2x,\ xy - 4) \cong \mathbb{Z}[\alpha, \beta] \subseteq \mathcal{O}_k$$

$$\rightsquigarrow \mathbb{Z}[\alpha, \beta] \xleftarrow{\sim} \mathbb{Z} + \mathbb{Z}\alpha + \mathbb{Z}\beta$$

Note: $\beta = 1 - \frac{1}{2}\alpha - \frac{1}{2}\alpha^2 \notin \mathbb{Z}[\alpha]$

$\Longrightarrow \mathrm{disc}(\mathbb{Z}[\alpha, \beta]) = -503$ $\qquad$ Claim 1 proved

Claim 2: 2 splits completely in k

$\because$ Have 3 surjective ring homom $\mathcal{O}_k = \mathbb{Z}[\alpha,\beta] \twoheadrightarrow \mathbb{F}_2$

Note $\mathcal{O}_k/2\mathcal{O}_k \cong \mathbb{F}_2[x,y]/(x^2+x,\ y^2-y,\ xy)$

define

$\mathfrak{p}_2 = \operatorname{Ker}(\mathcal{O}_k \overset{h_1}{\twoheadrightarrow} \mathbb{F}_2,\ \alpha \mapsto 0,\ \beta \mapsto 0)$

$\mathfrak{p}_2' = \operatorname{Ker}(\mathcal{O}_k \overset{h_2}{\twoheadrightarrow} \mathbb{F}_2,\ \alpha \mapsto 1,\ \beta \mapsto 0)$

$\mathfrak{p}_2'' = \ker(\mathcal{O} \overset{h_3}{\twoheadrightarrow} \mathbb{F}_2,\ \alpha \mapsto 0,\ \beta \mapsto 1)$

Some Factorizations (involving $\mathfrak{p}_2, \mathfrak{p}_2', \mathfrak{p}_2''$)

Let $f(x) = x^3+x^2-2x+8$

$g(x) = x^3-x^2+2x+8$

Factor $\alpha - a,\ \beta - b \quad a, b \in \mathbb{Z}$

Use $N(\alpha - a) = -f(a) \qquad N(\beta - b) = -g(b)$

Have $N(\alpha) = N(\beta) = -8 = -2^3 \qquad N(\mathfrak{p}_2) = N(\mathfrak{p}_2') = N(\mathfrak{p}_2'') = 2$

From def of $\mathfrak{p}, \mathfrak{p}', \mathfrak{p}''$: $\mathfrak{p}_2 | \alpha,\ \mathfrak{p}_2 | \beta,\ \mathfrak{p}_2' | \beta,\ \mathfrak{p}_2' \nmid \alpha,\ \mathfrak{p}_2'' | \alpha,\ \mathfrak{p}_2'' \nmid \beta$

$(\alpha\beta) = (4) = \mathfrak{p}_2^2 \mathfrak{p}_2'^2 \mathfrak{p}_2''^2$, $\quad \mathfrak{p}_2'$ prime to $\alpha \Rightarrow \mathfrak{p}_2'^2 | \beta$

$\Downarrow \qquad \mathfrak{p}_2''$ prime to $\beta \Rightarrow \mathfrak{p}_2''^2 | \alpha$

$\operatorname{ord}_{\mathfrak{p}_2}(\alpha) = \operatorname{ord}_{\mathfrak{p}_2}(\beta) = 1$

$$\Rightarrow \boxed{\alpha \cdot \mathcal{O}_k = \mathfrak{p}_2 \cdot \mathfrak{p}_2''^2, \quad \beta\mathcal{O}_k = \mathfrak{p}_2 \cdot \mathfrak{p}_2'^2} \quad \text{— (i)}$$

$-f(1) = 8 \quad h_1(\alpha-1) \neq 0$ in $\mathbb{F}_2$, $h_3(\alpha-1) \neq 0$ in $\mathbb{F}_2$

$$\Rightarrow \boxed{(\alpha-1)\cdot\mathcal{O}_k = \mathfrak{p}_2'^3} \quad \text{— (ii)}$$

Similarly, $f(-3) = -4 \Rightarrow h_1(\alpha+3) \neq 0$ in $\mathbb{F}_2$, $h_3(\alpha+3) \neq 0$ in $\mathbb{F}_2$

$$\Rightarrow \boxed{(\alpha+3)\cdot\mathcal{O}_k = \mathfrak{p}_2'^2} \quad \text{— (iii)}$$

$f(2) = 16$, $\rightsquigarrow$ $\mathbb{N}(\alpha-2) = 16$

$\mathrm{ord}_{\mathfrak{p}_2}(\alpha) = 1$, $\mathrm{ord}_{\mathfrak{p}_2}(2) = 1$ $\quad$ $\mathrm{ord}_{\mathfrak{p}_2'}(\alpha) = 0 \Rightarrow \mathrm{ord}_{\mathfrak{p}_2'}(\alpha-2) = 0$

$\mathrm{ord}_{\mathfrak{p}_2''}(\alpha) = 2$, $\mathrm{ord}_{\mathfrak{p}_2''}(2) = 1$ $\Rightarrow \mathrm{ord}_{\mathfrak{p}_2''}(\alpha-2) = 1$

$$\Rightarrow \boxed{(\alpha-2)\cdot\mathcal{O}_k = \mathfrak{p}_2^3\,\mathfrak{p}_2''} \quad \text{— (iv)}$$

$f(-3) = -4$ $\rightsquigarrow$ $\mathbb{N}(\alpha+3) = 2$ $\quad$ $\mathrm{ord}_{\mathfrak{p}_2}(\alpha+3) = 0$, $\mathrm{ord}_{\mathfrak{p}_2'}(\alpha+3) \geq 1$, $\mathrm{ord}_{\mathfrak{p}_2''}(\alpha+3) = 0$

$$\Rightarrow \boxed{(\alpha+3)\cdot\mathcal{O}_k = \mathfrak{p}_2'^{\,2}} \quad \text{— (iii) done already}$$

$f(-2) = 8$ $\rightsquigarrow$ $\mathbb{N}(\alpha+2) = 8$. $\quad$ $\mathrm{ord}_{\mathfrak{p}_2}(\alpha+2) \geq 1$, $\mathrm{ord}_{\mathfrak{p}_2'}(\alpha+2) = 0$, $\mathrm{ord}_{\mathfrak{p}_2''}(\alpha+2) = 1$

$$\Rightarrow \boxed{(\alpha+2)\cdot\mathcal{O}_k = \mathfrak{p}_2^2\cdot\mathfrak{p}_2''} \quad \text{— (v)}$$

$$\left.\begin{array}{l} \text{(iii)+(v)} \Rightarrow \boxed{\dfrac{\alpha-2}{\alpha+2}\,\mathcal{O}_k = \mathfrak{p}_2} \\[2ex] \text{(ii)+(iii)} \Rightarrow \boxed{\dfrac{\alpha-1}{\alpha+3}\cdot\mathcal{O}_k = \mathfrak{p}_2'} \end{array}\right\} \Rightarrow \frac{2\cdot(\alpha+2)}{(\alpha-2)}\,\frac{(\alpha+3)}{(\alpha-1)}\cdot\mathcal{O}_k = \mathfrak{p}_2''$$

In particular, $\mathfrak{p}_2, \mathfrak{p}_2', \mathfrak{p}_2''$ are principal ideals

Application of the fact that $\mathcal{O}_k \otimes \mathbb{F}_2 \cong \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$

$$\Rightarrow \# \mathrm{Hom}_{\text{ring}}(\mathcal{O}_k, \mathbb{F}_2) = 3$$

$\Rightarrow \nexists\, \gamma \in \mathcal{O}_k$ such that $\mathcal{O}_k = \mathbb{Z}[\gamma]$

$\because$ $\mathrm{Hom}_{\text{ring}}(\mathbb{Z}[\gamma], \mathbb{F}_2) \leq 2$ $\quad$ (Cannot use any other prime)

More factorization

Have factored $\alpha, \beta, \alpha-1, \alpha+3, \alpha-2, \alpha+2$

$f(-1)=10 \rightsquigarrow \mathbb{N}(\alpha+1)=10 \qquad \mathrm{ord}_{\mathfrak{p}_2'}(\alpha+1) \geq 1$

$\Rightarrow \exists$ a prime $\mathfrak{p}_5$ above 5, $\mathcal{O}_k/\mathfrak{p}_5 \cong \mathbb{F}_5$, s.t.

$$\boxed{(\alpha+1)\mathcal{O}_k = \mathfrak{p}_2' \cdot \mathfrak{p}_5} \quad -(\text{vi})$$

$f(3)=38 \rightsquigarrow \mathbb{N}(\alpha-3) = 2 \cdot 19 \qquad \mathrm{ord}_{\mathfrak{p}_2'}(\alpha-3) \geq 1$

$$\Rightarrow \boxed{(\alpha-3)\cdot\mathcal{O}_k = \mathfrak{p}_2' \cdot \mathfrak{p}_{19}} \quad -(\text{vii}) \qquad \mathcal{O}_k/\mathfrak{p}_{19} \cong \mathbb{F}_{19}$$

The factorizations (i) — (vii) shows:

None of $\alpha, \alpha+1, \alpha-1, \alpha-2, \alpha+2, \alpha-3, \alpha+3$ are divisible by any prime above 7

$\Rightarrow \quad \mathbb{Z}[\alpha]\otimes\mathbb{F}_7 \cong \mathbb{F}_7[x]/(f(x))$ is a field

i.e. $7\cdot\mathcal{O}_k$ is a prime ideal of $\mathcal{O}_k$

Claim In addition to 7, 11 and 13 also remain prime in $\mathcal{O}_k$

To show $11\cdot\mathcal{O}_k$ and $13\mathcal{O}_k$ are prime ideals, need to factor $\alpha-4, \alpha+4$
$\alpha-5, \alpha-5$
$\alpha-6, \alpha+6$

$f(4) = 64+16-8+8 = 80 \leftarrow$ prime to 11 and 13

$f(-4) = -64+16+8+8 = -32 \leftarrow$ prime to 11 and 13

$f(5) = 125+25-10+8 = 148 = 2^2\cdot 37$

$f(-5) = -125+25+10+8 = -82 = -2\cdot 41$

$f(6) = 216+36-12+8 = 248 = 2^3\cdot 31$

$f(-6) = -216+36+12+8 = -160 = -2^5\cdot 5$ Claim proved.

Also: The above data implies that 3 is inert in $\mathcal{O}_k$

Decomposition of $5 \cdot \mathcal{O}_k$: $\quad ord_{\mathfrak{p}_5}(\alpha+1) = 1$

$\rightsquigarrow$ $\boxed{5 = \mathfrak{p}_5 \cdot \mathfrak{p}_5''}$ $\quad \mathcal{O}_k/\mathfrak{p}_5'' \cong \mathbb{F}_{5^2}$

$(\alpha+1)\cdot\mathcal{O}_k = \mathfrak{p}_2' \cdot \mathfrak{p}_5$

$\rightsquigarrow$ $\boxed{\mathfrak{p}_5 = \frac{(\alpha+1)(\alpha-3)}{(\alpha+1)} \cdot \mathcal{O}_k}$

$\boxed{\mathfrak{p}_5' = \frac{5\ (\alpha+1)}{(\alpha+1)(\alpha-3)} \cdot \mathcal{O}_k}$

$h_k = 1$

Application of the above factorization:
All prime ideals above primes $2, 3, 5, 7$ (and 11, 13) actually are principal

Minkowski's bound: $\quad \frac{n!}{n^n} \cdot \left(\frac{4}{\pi}\right)^{r_2} |disc_k|^{\frac{1}{2}} \quad n=3$

$n=3$
$r_2=1$ $\quad \frac{3!}{3^3} \cdot \left(\frac{4}{\pi}\right) \doteqdot 0.28299 \quad \sqrt{503} \doteqdot 22.4276614$

$\frac{3!}{3^3}\left(\frac{4}{\pi}\right) \cdot \sqrt{503} < 6.35$

Hence $\underline{Cl(\mathcal{O}_k) = (1)}$.

Decomposition of the ramified prime 503

Obviously tame. $\rightsquigarrow$ $503\,\mathcal{O} = \mathfrak{p}_{503}^2 \cdot \mathfrak{p}_{503}'$ $\quad \mathcal{O}/\mathfrak{p}_{503}' \cong \mathbb{F}_{503}$
$\quad \mathcal{O}/\mathfrak{p}_{503} \cong \mathbb{F}_{503}$

$f(x) = x^3 + x^2 - 2x + 8 \pmod{503}$

$\equiv (x-149)^2(x+299) \pmod{503}$ $\quad \mathfrak{p}_{503} = Ker\left(\mathbb{Z}[x]/(f(x)) \twoheadrightarrow \mathbb{F}_{503}\right)$
$\quad x \longmapsto 149$

solve $\begin{cases} x^3 + x^2 - 2x + 8 \equiv 0 \pmod{503} \\ 3x^2 + 2x - 2 \equiv 0 \pmod{503} \end{cases}$

Example 2 $k = \mathbb{Q}[x]/(\underset{\substack{\uparrow \\ f(x)}}{x^3+10x+1}) \ni \alpha = $ image of x

$\mathrm{disc}(f) = -4027 \rightsquigarrow \mathcal{O}_k = \mathbb{Z}[\alpha]$

$r_1 = r_2 = 1$. $\sqrt{4027} \doteqdot 63.4586479$

$$\frac{3!}{3^3}\cdot\left(\frac{4}{\pi}\right)\cdot\sqrt{4027} \doteqdot 0.28299 \times 63.4587 < 18.242$$

mod 2 $x^3+10x+1 \equiv (x-1)(x^2+x+1) \pmod 2$: $2 = \mathfrak{p}_2 \cdot \mathfrak{p}_2'$ $\mathcal{O}_k/\mathfrak{p}_2 \cong \mathbb{F}_2$, $\mathcal{O}_k/\mathfrak{p}_2' \cong \mathbb{F}_{2^2}$

$f(1) = 12 \rightsquigarrow N(\alpha-1) = 12 = 2^2\cdot 3$

$\boxed{(\alpha-1)\cdot\mathcal{O}_k = \mathfrak{p}_2^2\cdot\mathfrak{p}_3}$ $\mathcal{O}_k/\mathfrak{p}_2 \cong \mathbb{F}_2$

$f(-1) = -10 \rightsquigarrow N(\alpha+1) = 10 \rightsquigarrow \boxed{(\alpha+1)\cdot\mathcal{O}_k = \mathfrak{p}_2\cdot\mathfrak{p}_5}$ $\mathcal{O}_k/\mathfrak{p}_5 \cong \mathbb{F}_5$

$f(2) = 29 \rightsquigarrow \boxed{(\alpha-2)\,\mathcal{O}_k = \mathfrak{p}_{29}}$ $\mathcal{O}_k/\mathfrak{p}_{29} \cong \mathbb{F}_{29}$

$f(-2) = -27 \rightsquigarrow N(\alpha+2) = 27 = 3^3 \rightsquigarrow \boxed{(\alpha+2) = \mathfrak{p}_3^3}$ $\mathfrak{p}_3 =$ the unique prime ideal containing $\alpha+2$ = image of $(x+2, 3)$ in $\mathbb{Z}[x]/(3, f(x))$

$f(3) = 58 \rightsquigarrow (\alpha-3)\cdot\mathcal{O}_k = \mathfrak{p}_2\cdot\mathfrak{p}_{29}'$ $\mathcal{O}_k/\mathfrak{p}_{29}' \cong \mathbb{F}_{29}$

Similarly

$(\alpha-6)\,\mathcal{O}_k = \mathfrak{p}_{277}$ $f(6) = 216+60+1 = 277$

$(\alpha+6)\,\mathcal{O}_k = \mathfrak{p}_{11}\cdot\mathfrak{p}_5^2$ $f(-6) = -216-60+1 = -275 = -11\cdot 5^2$

$(\alpha-4)\,\mathcal{O}_k = \mathfrak{p}_3\cdot\mathfrak{p}_5\cdot\mathfrak{p}_7$ $f(4) = 64+40+1 = 105 = 3\cdot 5\cdot 7$

$(\alpha+4)\,\mathcal{O}_k = \mathfrak{p}_{103}$ $f(-4) = -64-40+1 = -103$

$(\alpha-7)\,\mathcal{O}_k = \mathfrak{p}_2\cdot\mathfrak{p}_3^2\cdot\mathfrak{p}_{23}$ $f(7) = 343+70+1 = 414 = 2\times 3^2\times 23$

$(\alpha+7)\,\mathcal{O}_k = \mathfrak{p}_2^2\cdot\mathfrak{p}_{103}'$ $f(-7) = -343-70+1 = -412 = -2^2\times 103$

$(\alpha-5)\,\mathcal{O}_k = \mathfrak{p}_2^4\,\mathfrak{p}_{11}$ $f(5) = 125+50+1 = 176 = 2^4\times 11$

$(\alpha+5)\,\mathcal{O}_k = \mathfrak{p}_2\,\mathfrak{p}_3\,\mathfrak{p}_{29}''$ $f(-5) = -125-50+1 = -174 = -2\times 3\times 29$

$(\alpha-8)\,\mathcal{O}_k = \mathfrak{p}_{593}$

$(\alpha+8)\,\mathcal{O}_k = \mathfrak{p}_3\cdot\mathfrak{p}_{197}$

$$\frac{(\alpha-1)^3}{(\alpha+2)}\cdot\mathcal{O}_k = \mathfrak{p}_2^6$$

In $Cl(\mathcal{O}_k)$, have

$$\mathfrak{p}_2^6 \sim 1, \quad \mathfrak{p}_2^2 \cdot \mathfrak{p}_3 \sim 1, \quad \mathfrak{p}_5 \sim \mathfrak{p}_2^{-1} \sim \mathfrak{p}_2^5, \quad \mathfrak{p}_7 \sim \mathfrak{p}_3^{-1} \cdot \mathfrak{p}_5^{-1} \sim \mathfrak{p}_2^{-2} \cdot \mathfrak{p}_2^{-5} \sim \mathfrak{p}_2^3$$

$$\mathfrak{p}_{11} \sim \mathfrak{p}_2^2$$

For $p = 2, 3, 5, 7, 11$, $\quad p\mathcal{O}_k = \mathfrak{p}_p \cdot \mathfrak{p}_p''$ with $\mathcal{O}_k/\mathfrak{p}_p \cong \mathbb{F}_p$

$\rightsquigarrow \mathfrak{p}_p'' \sim \mathfrak{p}_p^{-1}$ $\qquad \mathcal{O}_k/\mathfrak{p}_p'' \cong \mathbb{F}_{p^2}$

For $p = 13, 17$, $\quad p$ stays prime

Conclude: $Cl(\mathcal{O}_k)$ is generated by $\mathfrak{p}_2$, therefore is a quotient of $\mathbb{Z}/6\mathbb{Z}$. And: it appears that $Cl(\mathcal{O}_k) \cong \mathbb{Z}/6\mathbb{Z}$

To prove this guess, we must show: neither $\mathfrak{p}_2^2$ nor $\mathfrak{p}_2^3$ is principal

Application of the above: produce units

For instance,

$$\frac{(\alpha+1)^2(\alpha+2)(\alpha-5)}{(\alpha-1)^3\cdot(\alpha+6)}\mathcal{O}_k = (\mathfrak{p}_2\cdot\mathfrak{p}_5)^2\cdot\mathfrak{p}_3^3\cdot\mathfrak{p}_2^4\cdot\mathfrak{p}_{11}\cdot\left[(\mathfrak{p}_2^2\cdot\mathfrak{p}_3)^3(\mathfrak{p}_5^2\cdot\mathfrak{p}_{11})\right]^{-1}$$

$$= \mathcal{O}_k$$

So $\dfrac{(\alpha+1)^2\cdot(\alpha+2)(\alpha-5)}{(\alpha-1)^3\cdot(\alpha+6)} \in \mathcal{O}_k^\times$

$\cong (\pm 1) \times \mathbb{Z}$

Note: Take α to be the real root of $f(x)$: $\alpha^3 + 10\alpha + 1 = 0$

$\alpha < 0$

Let $\beta = -\frac{1}{\alpha}$ $\quad \beta^3 - 10\beta^2 - 1 = 0$, $\quad g(x) = y^3 - 10y^2 - 1$

$g(10) = -1$, $g(11) = 120 > 0$

$\Rightarrow 10 < \beta < 11$

$-\frac{1}{10} < \alpha < -\frac{1}{11}$

Artin's inequality (p.170 of Artin's Göttingen Lecture) shows: α generates $\mathcal{O}_k^\times/(\pm 1)$

Idea of proving that neither $\mathfrak{P}_2^3$ nor $\mathfrak{P}_2^2$ is principal.

Since $\mathfrak{P}_2^6 = \frac{(\alpha-1)^3}{\alpha+2}\mathcal{O}_k$

- If $\mathfrak{P}_2^3$ is principal $\Longrightarrow \exists\, \delta\in\{0,1\}$, $\varepsilon\in\{1,-1\}$ s.t. $\varepsilon\cdot\alpha^{\delta}\cdot\frac{\alpha-1}{\alpha+2}$ is a square in $k^\times$

- If $\mathfrak{P}_2^2$ is principal $\Rightarrow \exists\, \nu\in\{0,1,-1\}$ s.t. $\frac{\alpha^{\nu}}{\alpha+2}$ is a cube in $k^\times$

For $\varepsilon\cdot\alpha^{\delta}\frac{\alpha-1}{\alpha+2}$: examine its image in $\mathcal{O}_k/\mathfrak{P}_p$ + use the real place ($\delta=1$) for $p=29$ (ε must be 0), 7 to get a contradiction.

For $\frac{\alpha^{\nu}}{\alpha+2}$: examine its image in $\mathcal{O}_k/\mathfrak{P}_p$ for $p=7$ (ν must be 0), 31 to get a contradiction

mod $\mathfrak{P}_{29}$

$\varepsilon\alpha^{\delta}\frac{\alpha-1}{\alpha+2}$ mod $\mathfrak{P}_{29}$: $\alpha\equiv 2 \pmod{\mathfrak{P}_{29}} \Rightarrow \alpha+2 \equiv 4 \pmod{\mathfrak{P}_{29}}$, $\alpha+2$ mod $\mathfrak{P}_{29}$ is a square

$\left.\begin{array}{l} -1 \text{ is a quadratic residue mod } \mathfrak{P}_{29} \\ \left(\frac{2}{29}\right) = -1 \quad \because 29\equiv 5 \pmod 8 \end{array}\right\} \Rightarrow \varepsilon = 0$

use the real place: $\alpha<0$, $-\frac{1}{10}<\alpha<-\frac{1}{11}$, $\frac{\alpha-1}{\alpha+2}<0 \Rightarrow \delta=1$

Remains to show: $-\frac{\alpha-1}{\alpha+2}$ is not a squre

mod $\mathfrak{P}_7$: $\alpha\equiv 4 \pmod{\mathfrak{P}_7} \leadsto -\frac{\alpha-1}{\alpha+2} \equiv -\frac{3}{6} \equiv 3 \pmod{\mathfrak{P}_7}$
$\equiv -4 \pmod{\mathfrak{P}_7}$

$\left(\frac{-1}{7}\right) = -1$ not a quad. residue

contradiction

$\frac{\alpha^\nu}{\alpha+2}$ is a cube $\quad$ mod $\mathfrak{p}_7$: $\quad \alpha \equiv 4 \pmod{\mathfrak{p}_7}$ $\quad \alpha^\nu \equiv (-3)^\nu \pmod{\mathfrak{p}_7}$

$\alpha+2 \equiv -1 \pmod{\mathfrak{p}_7}$

$\rightsquigarrow$ Need to use primes $\equiv 1 \pmod 3$

$\frac{\alpha^\nu}{\alpha+2}$ is a cube $\Rightarrow$ α^ν is a cube mod $\mathfrak{p}_7$

$\Rightarrow \nu = 0$

In $\mathbb{Z}/7\mathbb{Z}$, only ± 1 are cubes

Want to rule out:

$\Rightarrow \alpha+2$ is a cube

$f(-14) = 1 + (-14)(14^2+10) = 1 + (-14) \cdot 206 = -2883 = -3 \times 31^2$

$\rightsquigarrow \quad (\alpha+14)\cdot \mathcal{O}_k = \mathfrak{p}_3 \cdot \mathfrak{p}_{31}^2$

$\alpha+2$ mod $\mathfrak{p}_{31} \equiv -12$ mod $\mathfrak{p}_{31}$

? Is -12 mod $\mathbb{F}_{31}$ a cube ?

$12^2 \equiv -11 \pmod{31}$, $12^4 \equiv -3 \pmod{31}$, $12^8 \equiv 9 \pmod{31}$

$12^{10} \equiv -99 \equiv -6 \pmod{31}$ $\quad$ So -12 is <u>not</u> a cube mod <u>31</u>.

<u>Artin's inequality</u>: lower bound for the fundamental unit $\varepsilon > 1$ in $\mathcal{O}_k^\times$, k: cubic field, $r_1 = r_2 = 1$:

$$|\mathrm{disc}_k| < 4\varepsilon^3 + 24$$

Application: $\quad \alpha^3 + 10\alpha + 1 = 0 \quad \alpha < 0$ (under the real embedding)

Let $\beta = -\frac{1}{\alpha}$ $\quad 1 + 10\left(\frac{1}{\alpha}\right)^2 + \frac{1}{\alpha^3} = 0$ <u>i.e</u> $\beta^3 - 10\beta^2 - 1 = 0$

Let $g(y) = y^3 - 10y^2 - 1$ $\quad g(10) = -1$, $g(11) = 120$

So $g(y)$ has a root $\in (10, 11)$

and $f(x)$ has a root $\in \left(-\frac{1}{11}, -\frac{1}{10}\right)$

Artin's inequality says in this case:

$4027 < 4\varepsilon^3 + 24 \qquad \varepsilon^3 > \frac{4027}{4} - 24 = 982\frac{3}{4}$

If $\beta = \varepsilon^m$ $m \geq 2$, then $\varepsilon = \beta^{\frac{1}{m}} \leq \beta^{\frac{1}{2}} < \sqrt{11}$,

and $\varepsilon^3 < 11^{3/2}$, contradiction.

Have proved that β is the fundamental unit > 1 in $\mathcal{O}_k^{\times}$

<u>Proof</u> of Artin's inequality: Let $u = \varepsilon^{\frac{1}{2}}$. The two other conjugates of α are $u e^{\sqrt{-1}v}$, $u e^{-\sqrt{-1}v}$ $\quad 0 \leq v \leq \pi$

$$\mathrm{disc}_k^{\frac{1}{2}} = (u^2 - u^{-1}e^{\sqrt{-1}v})(u^2 - u^{-1}e^{-\sqrt{-1}v}) \cdot (u^{-1}e^{\sqrt{-1}v} - u^{-1}e^{-\sqrt{-1}v})$$

$$= 2\sqrt{-1}\,(u^3 + u^{-3} - 2\cos v) \cdot \sin v$$

Let $\xi = \frac{1}{2}(u^3 + u^{-3})$ > 1 $\because u > 1$

$$|\mathrm{disc}_k|^{\frac{1}{2}} = 4\,(\xi - \cos v) \cdot \sin v$$

$$\leq 4 \operatorname*{Max}_v (\xi - \cos v) \cdot \sin v$$

such a maximum happens at v_0

i.e.

$$|\mathrm{disc}_k| \leq 16(\xi - t_0)^2(1 - t_0^2)$$

Have $\xi \cos v_0 - \cos^2 v_0 + \sin^2 v_0 = 0$

$$= 16(\xi^2 - 2\xi t_0 + t_0^2)(1 - t_0^2)$$

Let $t_0 = \cos v_0$

$\rightsquigarrow \xi t_0 - 2t_0^2 + 1 = 0$

$$= 4u^6 + 24 + 4(u^{-6} - 4t_0^2 - 4t_0^4)$$

use $\xi t_0 = 2t_0^2 - 1$

Let $g(t) = 2t^2 - \xi t - 1$

$t_0 =$ a root of $g(t)$ with $|t_0| \leq 1$

$$\begin{cases} g(1) = 1 - \xi < 0 \quad \because \xi > 1 \\ g(-\frac{1}{2u^3}) = \frac{1}{2}u^{-6} + \frac{1}{4}(u^3 + u^{-3})u^{-3} - 1 \\ \qquad = \frac{1}{4}(u^{-6} - 1) < 0 \end{cases}$$

$\Downarrow$

One root of $g(t)$ is > 1, and the other root of $g(t)$ is $< -\frac{1}{2u^3}$; i.e. $t_0 < -\frac{1}{2u^3}$

$\Rightarrow 4t_0^2 > u^{-6} \iff u^{-6} - 4t_0^2 < 0$

$\Rightarrow u^{-6} - 4t_0^2 - 4t_0^4 < 0$

$$|\mathrm{disc}_k| \leq 4u^6 + 24 + 4\underbrace{(u^{-6} - 4t_0^2 - 4t_0^4)}_{<0}$$

$$\leq 4u^6 + 24 \quad \text{q.e.d.}$$

From estimate of t_0

III. Ramification

$$\begin{array}{ccc} \mathcal{O}_L = B & \subseteq & L \\ | & & | \\ \mathcal{O}_K = A & \subseteq & K \end{array}$$

Compute $\mathcal{D}^{-1}_{A[\alpha]/A}$ $\quad$ L/K separable

$\alpha \in L$ $\quad$ $C = A[\alpha]$ $\quad$ $K(\alpha) = L$ $\quad$ $n = [L:K]$ $\quad$ $f(T) =$ min. poly of α/K

<u>Euler's Lemma</u> $\quad \mathrm{Tr}_{L/K}\left(\frac{\alpha^i}{f'(\alpha)}\right) = 0 \quad \forall i = 0, 1, \dots, n-2, \quad \mathrm{Tr}_{L/K}\left(\frac{\alpha^{n-1}}{f'(\alpha)}\right) = 1$

$$f(T) = (T - \underset{\alpha}{\underset{\shortparallel}{\alpha_1}})(T-\alpha_2)\cdots(T-\alpha_n)$$

pf: $\displaystyle \frac{1}{f(T)} = \sum_{i=1}^{n} \frac{1}{f'(\alpha_i)(T-\alpha_i)}$ $\quad$ Lagrange interpolation

[More generally $\displaystyle \frac{T^j}{f(T)} = \sum_{i=1}^{n} \frac{\alpha_i^j}{f'(\alpha_i)(T-\alpha_i)} \quad \forall j = 0, 1, \dots, n-1$

$\displaystyle \rightsquigarrow \quad \mathrm{Tr}_{L/K}\left(\frac{\alpha^j}{f'(\alpha)(T-\alpha)}\right) = \frac{T^j}{f(T)}$]

expand $\displaystyle \frac{1}{f(T)} = \sum_{i=1}^{n} \frac{1}{f'(\alpha_i)(T-\alpha_i)}$ in powers of $\frac{1}{T}$

$$\frac{1}{T^n}\cdot\left(1 + O\left(\frac{1}{T}\right)\right) = \sum_{i=1}^{n}\sum_{j=0}^{n-1} \frac{\alpha_i^j}{T^{j+1}} + O\left(\frac{1}{T^{n+1}}\right) \qquad \text{q.e.d.}$$

Assume $\alpha \in B = \mathcal{O}_L$ in the above

<u>Consequence</u>: Let $C^\vee = \{x \in L \mid \mathrm{Tr}_{L/K}(x \cdot y) \subseteq A \quad \forall y \in C\}$

$x \in C^\vee \iff \mathrm{Tr}(x \cdot \alpha^j) \subseteq A$ for $j = 0, 1, \dots, n-1$

$\iff x \in \frac{1}{f'(\alpha)} C$

by Lemma

<u>i.e</u> $\quad C^\vee = \frac{1}{f'(\alpha)} C$

Let $c(C:B) \overset{\text{def}}{=} \{x \in L \mid xB \subseteq C\}$ — an ideal of B

$x \in c(C:B) \Longleftrightarrow xB \subseteq C \Longleftrightarrow \frac{1}{f'(\alpha)} \cdot xB \subseteq \frac{1}{f'(\alpha)} C = C^\vee$

$\Updownarrow$

$Tr_{L/K}\left(\frac{1}{f'(\alpha)} \cdot x \cdot C \cdot B\right) \subseteq A$

$x \in f'(\alpha) \cdot \mathcal{D}_{B/A}^{-1} \Longleftrightarrow \frac{x}{f'(\alpha)} \in \mathcal{D}_{B/A}^{-1} \Longleftrightarrow$

i.e.

Cor $c(C:B) = f'(\alpha) \cdot \mathcal{D}_{B/A}^{-1}$

In other words, $f'(\alpha) \cdot c(C:B)^{-1} = \mathcal{D}_{B/A}$

K : complete d.v. field L/K = finite ext^n field

π = generator of $\mathcal{O}_L$

totally ramified case : $B = A[\pi]$ $f(T)$: Eisenstein poly — irred. poly. of π

$f(T) = T^e + a_1 T^{e-1} + \cdots + a_{e-1} T + a_e$ $a_1, \ldots, a_e \in \pi \cdot A$

$\operatorname{ord}_A(a_e) = 1$

$w = \operatorname{ord}_\pi$

$v = \operatorname{ord}_A$

$w(f'(T)) = \operatorname*{Inf}_{0 \le i \le e-1} w\left((e-i) \cdot a_i \cdot \pi^{e-i-1}\right) \underset{\text{eg.}}{\le} w(e) + e - 1$

$= e-1$

if tamely ramified, i.e. $w(e) = 0$

Assume L/K finite Galois extension of complete d.v. fields

$\bar{L} = \mathcal{O}_L/\mathfrak{m}_L$ separable over $\bar{K} = \mathcal{O}_K/\mathfrak{m}_K$ $\quad G = Gal(L/K)$

define $i_G : G \longrightarrow \mathbb{N}$ $\quad i_G(s) = \underset{y\in\mathcal{O}_L}{Min}\ v(s(y)-y) \overset{\text{if } \mathcal{O}_L = \mathcal{O}_K(x)}{=} v(s(x)-x)$

$s(y_1y_2) - y_1y_2 = s(y_1)(s(y_2)-y_2) + y_2(s(y_1)-y_2)$

define $G_i = \{ s\in G \mid i_G(s) \geq i-1 \}$

Elementary Properties

(1) $i_G(tst^{-1}) = i_G(s)$ $\quad \forall s, t\ G$

(2) $i_G(st) \geq Min(i_G(s), i_G(t))$

(3) $H \leq G$ $\quad K' \longleftrightarrow H$ $\rightsquigarrow$ $\quad i_H(s) = i_G(s)\ \forall s\in H$

$H_i = G_i \cap H \quad \forall i$

(4) $i_G(s) = \infty \iff s \in 1_G$

$G_0 \longleftrightarrow$ the maximal unramified subfield of L/K

Proposition $H \trianglelefteq G \rightsquigarrow i_{G/H}(\sigma) = \sum\limits_{\substack{s\mapsto\sigma \\ s\in G}} i_G(s) \quad \forall \sigma\in G/H)$

pf: May assume $\sigma \neq 1_{G/H}$ $\quad$ Let $E = L^H$

Pick $x\in\mathcal{O}_L$ s.t. $\mathcal{O}_K[x] = \mathcal{O}_L$

$y\in\mathcal{O}_E$ s.t. $\mathcal{O}_K[y] = \mathcal{O}_K$

Pick $s\in G$ s.t. $s\mapsto\sigma$

Suffices to show that $(s(y)-y)\cdot\mathcal{O}_L = \prod\limits_{t\in H}(st(x)-x)\cdot\mathcal{O}_L$

Let $f(T)\in\mathcal{O}_E[T]$ be the minimal poly of x/E

$({}^sf)(T) = \prod\limits_{t\in H}(T-st(x))$ $\quad f(T) = \prod\limits_{t\in H}(T-t(x))$

Clearly all coeff of ${}^sf - f$ are divisible by $s(y)-y$

$\Rightarrow$ $({}^sf)(x) - f(x)$ is divisible by $s(y)-y$

$\quad\quad ({}^sf)(x) - f(x) = ({}^sf)(x)$

For the converse, $\exists\ g(T)\in\mathcal{O}_K[T]$ s.t. $y = g(x)$

$\Rightarrow$ $g(T) - y = f(T)\cdot h(T)$ for a uniquely determined $h(T)\in\mathcal{O}_K[T]$

$\rightsquigarrow$ $g(T) - {}^sy = ({}^sf)(T)\cdot {}^sh(T)$

$\rightsquigarrow$ $y - {}^sy = ({}^sf)(x)\cdot({}^sh)(x)$ so $({}^sf)(x)$ divides $s(y)-y$ q.e.d.

<u>Prop</u> $v_L(\mathcal{D}_{L/K}) = \sum_{s \neq 1} i_G(s) = \sum_{i \geq 0} \left(\#(G_i) - 1\right)$

$\mathcal{O}_L = \mathcal{O}_K[x] \rightsquigarrow \mathcal{D}_{L/K} = f'(x) \cdot \mathcal{O}_L$

pf: $\sum_{G \ni s \neq 1} i_G(s) = \sum_{s \neq 1} \sum_{\substack{i \geq 0 \\ s \in G_i}} 1 = \sum_{i \geq 0} \sum_{s \in G_i} 1 = \sum_{i \geq 0} \left(\#(G_i) - 1\right)$

$s \in G_i \iff i_G(s) \geq i+1$ q.e.d

<u>Cor</u>

$$\begin{array}{c} L \\ |\ H \\ E \\ | \\ K \end{array} \quad G \qquad v_E(\mathcal{D}_{E/K}) = \frac{1}{e_{L/E}} \sum_{s \in H} i_G(s)$$

pf:

$v_L(\mathcal{D}_{L/K}) = \sum_{s \neq 1_G} i_G(s)$

$v_L(\mathcal{D}_{L/E}) = \sum_{s \neq 1_H} i_H(s)$

$\mathcal{D}_{L/K} = \mathcal{D}_{E/K} \cdot \mathcal{D}_{L/E} \implies v_L(\mathcal{D}_{E/K}) = \sum_{s \notin H} i_G(s)$

$v_L(\mathcal{D}_{E/K}) = e_{L/E} \cdot v_E(\mathcal{D}_{E/K})$.

q.e.d.

<u>Def</u> $\varphi = \varphi_{L/K} : [-1, \infty) \longrightarrow [-1, \infty)$

$$\varphi(u) = \int_0^u \frac{dt}{(G_0 : G_t)}$$

where $G_t \stackrel{\text{def}}{=} \{ s \in G \mid i_G(s) \geq t+1 \}$ $\left(\text{Thus } G_t = \lim_{u \to t-} G_u\right)$

<u>Lemma 1</u>: $\varphi_{L/K}(u) = \left(\frac{1}{\# G_0} \sum_{s \in G} \min\left(i_G(s), u+1\right) \right) - 1$

pf: $\varphi(u) = \frac{1}{\# G_0} \int_0^u \sum_{s \in G} \delta_{G_t}(s) \, dt = \frac{1}{\# G_0} \sum_{s \in G_0} \int_0^u \delta_{G_t}(s) \, dt$

$= \frac{1}{\# G_0} \sum_{s \in G_0} \mathrm{Min}\left(i_G(s) - 1, u\right)$ $\quad \because s \notin G_0 \Rightarrow \delta_{G_0}(s) = 0$

$= \left(\frac{1}{\# G_0} \sum_{s \in G} \mathrm{Min}\left(i_G(s), u+1\right) \right) - 1$

$N \trianglelefteq G$

<u>Def</u>: $\tilde{i}_{G/N}(\sigma) = \underset{s \mapsto \sigma}{\mathrm{Max}}\ i_G(s)$

<u>Note</u> If $\tilde{\sigma} \in G$, $\tilde{\sigma} \mapsto \sigma$, $i_G(\tilde{\sigma}) = \underset{s \mapsto \sigma}{\mathrm{Max}}\ i_G(s)$, then

$$i_G(\tilde{\sigma} t) = \mathrm{Min}\,(i_G(t),\ i_G(\tilde{\sigma}))$$
$$\left(\because\ i_G(s_1 s_2) \geq \mathrm{Min}\,(i_G(s_1),\ i_G(s_2)) \right)$$

$N \leftrightarrow E$

<u>Lemma 2</u> $i_{G/N}(\sigma) - 1 = \varphi_{L/E}(\tilde{i}_{G/N}(\sigma) - 1)$

<u>Pf</u>: $i_{G/N}(\sigma) = \dfrac{1}{e_{L/E}} \displaystyle\sum_{s \mapsto \sigma} i_G(s) = \dfrac{1}{e_{L/E}} \sum_{t \in N} i_G(\tilde{\sigma} t)$ <u>where</u> $\tilde{\sigma} \mapsto \sigma$, $i_G(\tilde{\sigma}) = \underset{t \in N}{\mathrm{Max}}\ i_G(\tilde{\sigma} t)$

$$= \frac{1}{e_{L/E}} \sum_{t \in N} \mathrm{Min}\left(i_G(\tilde{\sigma}),\ i_G(t) \right)$$

$$\underset{\text{by Lemma 1}}{=} 1 + \varphi_{L/E}\big(\underbrace{i_G(\tilde{\sigma})}_{\tilde{i}_{G/N}(\sigma)} - 1\big)$$

<u>Herbrand's theorem</u>

$$G_u N/N = (G/N)_{\varphi_{L/E}(u)}$$

$G/N \ni \sigma$

$\sigma \in G_u N/N \iff \tilde{i}_{G/N}(\sigma) \geq u+1 \iff \varphi_{L/E}(\tilde{i}_{G/N}(\sigma) - 1) \geq \varphi_{L/E}(u)$

$\Updownarrow$

$\sigma \in (G/N)_{\varphi_{L/E}(u)} \iff i_{G/N}(\sigma) - 1 \geq \varphi_{L/E}(u)$

Theorem (Hasse-Arf) G abelian $\Rightarrow$ all breaks ν of the upper numbering filtration are integers

[<u>def</u> ν is a break for the upper-numbering filtration iff $G^\nu \supsetneq G^{\nu+\varepsilon}$ $\forall\, \varepsilon > 0$ $\quad G^\nu = G_{\psi(\nu)}$]

In other words, if $G_i \supsetneq G_{i+1}$, then $\varphi_{L/K}(i) \in \mathbb{N}$

Artin character $a_G(s) \overset{\text{def}}{=} \begin{cases} -f\, i_G(s) & \text{if } s \neq 1_G \\ f \cdot \sum_{s\neq 1} i_G(s) & \text{if } s = 1_G \end{cases}$ Note: $f = \frac{\# G}{\# G_0}$

Swan character $sw_G(s) = a_G(s) - u_G(s) = \begin{cases} -f \cdot i_G(s) + 1 & \text{if } s \neq 1_G \\ f \cdot \sum_{s\neq 1} i_G(s) - \#(G) + 1 & \text{if } s = 1 \end{cases}$

$\forall i \in \mathbb{N}$, let u_i = the augmentation character of G_i

i.e. $u_i(s) = \begin{cases} -1 & \text{if } s \neq 1 \\ \# G_i - 1 & \text{if } s = 1 \end{cases}$

let $u_i^* = \operatorname{ind}_{G_i}^{G} u_i \rightsquigarrow u_i^*(s) = \begin{cases} 0 & \text{if } s \notin G_i \\ -\frac{\# G}{\# G_i} & \text{if } s \in G_i \text{ and } s \neq 1_G \\ \frac{\# G}{\# G_i} \cdot (\# G_i - 1) & \text{if } s = 1_G \end{cases}$

<u>Lemma</u> $a_G = \sum_{i=0}^{\infty} \frac{1}{[G_0 : G_i]} u_i^*$ Let $g = \# G$, $g_i = \# G_i$

Pf: Given $s \in G$, suppose $s \in G_j$, $s \notin G_{j+1}$ i.e. $i_G(s) = j+1$

Then $a_G(s) = -f \cdot (j+1)$

$\sum_{i=0}^{\infty} \frac{1}{[G_0 : G_i]} u_i^*(s) = \sum_{i=0}^{j} \frac{1}{[G_0 : G_i]} \cdot \left(-[G : G_i] \right) = -f \cdot (j+1)$ q.e.d.

<u>Prop</u> $H \leqslant G$, $L^H =: E$

Then $a_G|_H = \operatorname{ord}_K(\operatorname{disc}_{E/K}) \cdot \underset{\substack{\uparrow \\ \text{reg. character}}}{r_H} + f_{E/K} \cdot a_H$

Pf: For $1 \neq s \in H$ $a_G|_H(s) = -f_{L/K}\, i_G(s)$

$r_H(s) = 0$ $a_H(s) = -f_{L/E}\, i_H(s)$ } OK

For $s = 1$, $a_G(1) = f_{L/K} \cdot \sum_{G \ni s \neq 1} i_G(s) = f_{L/K} \cdot v_L(\mathcal{D}_{L/K})$

$a_H(1) = v_E(\mathcal{D}_{L/E})$ The eq^n is: $\underbrace{f_{L/K}\, v_L(\mathcal{D}_{L/K})}_{= v_K(\operatorname{disc}_{L/K})} \overset{?}{=} v_K(\operatorname{disc}_{E/K}) \cdot \# H + f_{E/K} \cdot v_E(\mathcal{D}_{L/E})$

R6

Need to show:

$$v_K(\mathrm{disc}_{E/K}) = [L:E]\cdot v_K(\mathrm{disc}_{E/K}) + v_E(\mathrm{disc}_{L/E})$$

This follows from transitivity of discriminant:

$$\mathrm{disc}_{L/K} = (\mathrm{disc}_{E/K})^{[L:E]} \cdot \mathrm{Nm}_{E/K}(\mathrm{disc}_{L/E})$$

q.e.d.

§1. Dirichlet series

- ordinary Dirichlet series: $\sum a_n n^{-s}$ usual notation: $s = \sigma + it$, $\sigma, t \in \mathbb{R}$

Prop: If $\sum_{n\geq 1} a_n n^{-s}$ converges for $s = s_0$, then it converges uniformly in the angular region $\{s \in \mathbb{C} \mid |\arg(s-s_0)| \leq \frac{\pi}{2} - \theta\}$ $\forall_{0<\theta<\frac{\pi}{2}}$
(Abel summation)

$\rightsquigarrow$ Consequently $\sum_{n\geq 1} a_n n^{-s}$ converges uniformly on compact subsets of $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \mathrm{Re}(s_0)\}$, and defines a holomorphic function on $\{\mathrm{Re}(s) > \mathrm{Re}(s_0)\}$

Given $\sum_{n\geq 1} a_n n^{-s}$ $\rightsquigarrow$ σ_0 = abscissa of convergence

$\bar{\sigma}$ = abscissa of absolute convergence
= abscissa of convergence of $\sum_{n\geq 1} |a_n| n^{-s}$

Ex. $\sum_{n\geq 1} (-1)^{n-1} n^{-s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots$

$= \sum_{n\geq 1} n^{-s} - 2 \cdot \sum_{k\geq 1} \frac{1}{(2k)^s} = (1 - 2^{1-s})\,\zeta(s)$

converges for $\mathrm{Re}(s) > 0$

Prop For any $\sum_{n\geq 1} a_n n^{-s}$, have $\bar{\sigma} - \sigma_0 \leq 1$

Pf: If $\sum_{n\geq 1} a_n n^{-s_0}$ converges, then $\lim_{n\to\infty} |a_n| \cdot n^{-s_0} = 0$,

hence $\sum_{n\geq 1} |a_n| \cdot |n^{-s_0-1-\varepsilon}| \leq C \sum_{n\geq 1} n^{-1-\varepsilon}$ converges q.e.d.
$\forall_{\varepsilon>0}$

Thm (Landau) Suppose $a_n \geq 0$ $\forall n \geq 1$. $\underset{\text{radius of convergence}}{\sigma_0}$ finite.
Then the analytic function $f(s) = \sum_{n\geq 1} a_n n^{-s}$ defined on $\{s \in \mathbb{C} \mid Re(s) > \sigma_0\}$ cannot be analytically continued to any open subset containing $s = \sigma_0$

Pf: May assume $\sigma_0 \overset{\text{obvious}}{=} \bar{\sigma} = 0$. Suppose $f(s)$ can be continued to a holomorphic function on a neighborhood of 0.
$\rightsquigarrow$ $f(s)$ is holomorphic on a circle $\{ s \in \mathbb{C} \mid |s-1| < \rho \}$ for some $\rho > 1$.

$$= \sum_{j\geq 0} f^{(j)}(1) \frac{(s-1)^j}{j!} \qquad f^{(j)}(s) = \sum_{n\geq 1} a_n (-\log n)^j \cdot n^{-s}$$

$$f^{(j)}(1) = \sum_{n\geq 1} a_n (-\log n)^j \cdot \frac{1}{n}$$

$\rightsquigarrow$

$$f(s) = \sum_{j\geq 0} \left(\sum_{n\geq 1} a_n (-\log n)^j \cdot \frac{1}{n} \right) \cdot \frac{(s-1)^j}{j!}$$

$$= \sum_{j\geq 0} \frac{(1-s)^j}{j!} \sum_{n\geq 1} a_n \cdot (\log n)^j \cdot \frac{1}{n}$$

evaluate at $1-\rho < s < 0$, hence $1-s > 0$ and all terms are ≥ 0

$$\underset{1-\rho<s<0}{=} \sum_{n\geq 1} \frac{a_n}{n} \underbrace{\sum_{j\geq 0} \frac{(1-s)^j (\log n)^j}{j!}}_{e^{(\log n)\cdot(1-s)} = n^{-s+1}} = \sum_{n\geq 1} a_n \cdot n^{-s}$$

converges $\forall s > 1-\rho$

So the abscissa of absolute convergence is $\leq 1-\rho$. Contradiction!
QED.

Prop Suppose that $\exists \underset{\sigma_1}{C} > 0$ s.t $|A_n| = |a_1 + \cdots + a_n| \leq C \cdot n^{\sigma_1}$ $\forall n \geq 1$.
Then $\sum_{n\geq 1} a_n n^{-s}$ has abscissa of convergence $\leq \sigma_1$

Pf. $\forall N \geq m$

$$\sum_{n=m+1}^{N} a_n n^{-s} = \sum_{n=m+1}^{N} (A_n - A_{n-1}) \cdot n^{-s} = A_N \cdot N^{-s} + \sum_{j=m+1}^{N-1} A_j \underbrace{\left(j^{-s} - (j+1)^{-s}\right)}\ - A_m (m+1)^{-s}$$

Take $Re(s) \geq \sigma_1 + \delta$, $\delta > 0$

$$s \cdot \int_j^{j+1} \frac{dt}{t^{s+1}}$$

$$\left| \sum_{n=m+1}^{N} A_j \left(j^{-s} - (j+1)^{-s}\right) \right| \leq C \cdot |s| \cdot \int_{m+1}^{N-1} \frac{dt}{t^{\sigma_1 + 1 + \delta}} \longrightarrow 0$$

<u>Cor.</u> Consider $\sum_{n\geq 1} a_n n^{-s}$, $A_n \overset{\text{def}}{=} a_1+\cdots+a_n$. Given $0 \leq \sigma_1 \leq 1$, $\rho \in \mathbb{C}$, $C \in \mathbb{R}_{>0}$

Suppose that $|A_n - n\rho| \leq C \cdot n^{\sigma_1}$ $\forall n \geq 1$

Then $f(s) = \sum_{n\geq 1} a_n n^{-s}$ is absolutely convergent for $Re(s) > 1$.

<u>and</u> has an analytic continuation to $\{ s \in \mathbb{C} \mid Re(s) > \sigma_1 \}$.
and is holomorphic on this half plane except for a simple pole at $s=1$, and $Res_{s=1} f(s) = \rho$

(<u>Consider</u>: $f(s) - \rho \zeta(s)$)

<u>Note</u>: The simple trick that $(1-2^{1-s}) \cdot \zeta(s)$ is holomorphic on $\{Re(s)>0\}$ shows that $\zeta(s)$ has an analytic continuation to $\{Re(s)>0\}$, with a simple pole at $s=1$.

Functional eq^n for $\zeta(s)$

$\zeta(s)$ admits an analytic/meromorphic continuation to $\mathbb{C}$, holomorphic on $\mathbb{C}\setminus\{1\}$, with a simple pole at $s=1$ and $\mathrm{Res}_{s=1}\zeta(s)=1$

It satisfies

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2}\, \Gamma(1-s)\, \zeta(1-s)$$

equiv. formuation: $\xi(s) := \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s).$

$$\xi(s) = \xi(1-s)$$

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}$$

$$\Gamma(s+1) = s\Gamma(s)$$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

$$2^{2s-1}\Gamma(s)\Gamma(s+\tfrac{1}{2}) = \sqrt{\pi}\cdot\Gamma(2s)$$

$$\Leftrightarrow \quad \Gamma_{\mathbb{R}}(s)\cdot\Gamma_{\mathbb{R}}(s+1) = \Gamma_{\mathbb{C}}(s)$$

$$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})$$

$$\Gamma_{\mathbb{C}}(s) = 2\cdot(2\pi)^{-s}\Gamma(s)$$

$$\tilde{\xi}(s) = \Gamma_{\mathbb{R}}(s)\cdot\zeta(s) = \frac{2}{s(s-1)}\xi(s)$$

$$\tilde{\xi}(1-s) = \tilde{\xi}(s)$$

Note $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) = \pi^{-\frac{s}{2}}\int_0^\infty e^{-t} t^{\frac{s}{2}}\frac{dt}{t}$ $\qquad \mathrm{Re}(s)>0$

$$\underset{t=\pi x^2}{=} \pi^{-\frac{s}{2}}\int_0^\infty e^{-\pi x^2}\pi^{\frac{s}{2}}\cdot x^s\cdot 2\frac{dx}{x} = \int_{-\infty}^\infty e^{-\pi x^2}|x|^s\frac{dx}{x}$$

Summation formula:

$f(x)$: C^1 on $[a,b]$,

$$\sum_{a<n\le b} f(n) = \int_a^b f(x)\,dx + \int_a^b (x-\lfloor x\rfloor-\tfrac{1}{2})f'(x)\,dx + (a-\lfloor a\rfloor-\tfrac{1}{2})f(a) - (b-\lfloor b\rfloor-\tfrac{1}{2})f(b)$$

1^{st} proof: Use Euler summation formula

Apply summation formula with $f(x) = x^{-s}$, get

$a,b \in \mathbb{N}_{>0}$

$$\sum_{n=a+1}^{b} \frac{1}{n^s} = \frac{b^{1-s} - a^{1-s}}{1-s} - s\int_a^b \frac{x - \lfloor x \rfloor - \frac{1}{2}}{x^{s+1}} dx + \frac{1}{2}$$

Let $\begin{matrix} a=1 \\ b\to\infty \end{matrix}$, $Re(s)>1$

$$\rightsquigarrow \zeta(s) = s\int_1^\infty \frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{s+1}} dx + \frac{1}{s-1} + \frac{1}{2} \qquad (1)$$

This accomplishes analytic continuation to $Re(s) > 0$

simple pole at $s=1$, with $Res_{s=1}\,\zeta(s) = 1$

For $0 < Re(s) < 1$, have

$$\int_0^1 \frac{\lfloor x \rfloor - x}{x^{s+1}} dx = -\int_0^1 x^{-s} dx = \frac{1}{s-1} \quad , \quad \frac{s}{2}\cdot\int_1^\infty \frac{dx}{x^{s+1}} = \frac{1}{2}$$

<u>So</u>: (1) becomes:

$$\zeta(s) = s\int_0^\infty \frac{\lfloor x \rfloor - x}{x^{s+1}} dx \qquad 0 < Re(s) < 1 \qquad (2)$$

Actually the integral on r.h.s. of (1) converges (conditionally) for $Re(\sigma) > -1$, and

$$\zeta(s) = s\int_0^\infty \frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{s+1}} dx \qquad \text{for } -1 < Re(s) < 0$$

Use Fourier series expansion for $\lfloor x \rfloor - x + \frac{1}{2}$:

$$\lfloor x \rfloor - x + \frac{1}{2} = \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{\pi n} \qquad (3)$$

substitute into (2), integrate term-by-term to get

← need to justify

$$\zeta(s) = \frac{s}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\int_0^\infty \frac{\sin(2\pi n x)}{x^{s+1}} dx$$

$$= \frac{s}{\pi}\sum_{n=1}^{\infty} \frac{(2\pi n)^s}{n}\int_0^\infty \frac{\sin y}{y^{s+1}} dy$$

<u>Use</u> $\int_0^\infty t^{s-1} \sin t \, dt = \Gamma(s)\sin(\frac{\pi s}{2})$ (4)

$$\underset{(4)}{=} \frac{s}{\pi}(2\pi)^s\left(-\Gamma(-s)\right)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s) \qquad (5)$$

This is the FE

2^{nd} proof (Contour integral) – Riemann's

Consider $I(s) = \int_C \frac{z^{s-1}}{e^z - 1}\, dz$

C

where $z^{s-1} = e^{(s-1)\underbrace{\log z}_{0 < \arg z < 2\pi}}$

$$I(s) = -\int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx + \int_0^\infty \frac{x^{s-1}\cdot e^{2\pi\sqrt{-1}(s-1)}}{e^x-1}\,dx$$

$$= \left(e^{2\pi\sqrt{-1}s} - 1\right)\cdot \int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx \qquad -(2)$$

entire !

$$\int_0^\infty x^{s-1} e^{-nx}\,dx = n^{-s}\cdot\int_0^\infty y^{s-1}e^{-y}\,dy = n^{-s}\,\Gamma(s)$$

$$\Gamma(s)\cdot\zeta(s) = \sum_{n=1}^\infty \int_0^\infty x^{s-1}e^{-nx}\,dx = \int_0^\infty x^{s-1}\sum_{n=1}^\infty e^{-nx}\,dx = \int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx$$

$$\zeta(s) = \Gamma(s)^{-1}\int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx \qquad \forall_s \text{ s.t. } Re(s) > 1$$

"starting point"

$$= \frac{1}{2\pi\sqrt{-1}}e^{-\pi\sqrt{-1}s}\,\Gamma(1-s)\int_C \frac{z^{s-1}}{e^z-1}\,dz \qquad (1)$$

Note expand $\frac{z}{e^z-1} = 1 - \frac{z}{2} + B_2\frac{z^2}{2!} + B_4\frac{z^4}{4!} + \cdots$

Bernouli nbers

+ residue at $z=0$

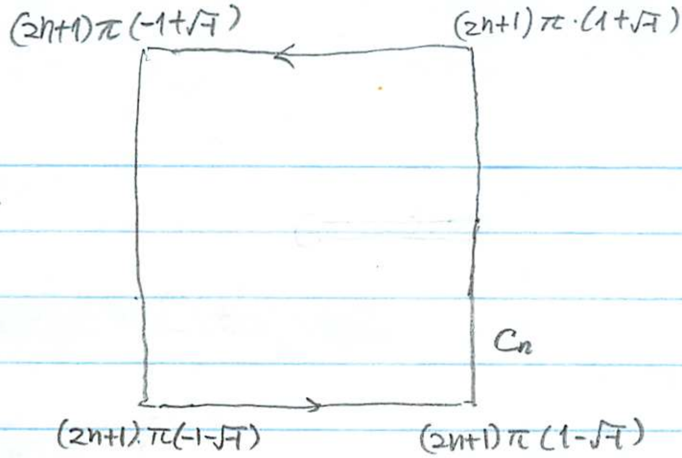
to get: $\zeta(-2m) = 0,\ \zeta(1-2m) = \frac{-B_{2m}}{2m}$

$m = 1, 2, \cdots$

(1)+
(2) $\Rightarrow$ analytic continuation of $\zeta(s)$ to $\mathbb{C}$, with possible/potential poles (simple) at poles of $\Gamma(1-s)$, i.e. $s = 1, 2, 3, 4, \ldots$

Examine $s=1$: $\therefore I(1) = 2\pi\sqrt{-1}$, $\Gamma(1-s) = -\frac{1}{s-1} + \cdots$ $\therefore$ $Res_{s=1}\,\zeta(s) = 1$

modify the contour



$$\operatorname{Res}_{z=2\pi\sqrt{-1}m} \frac{z^{s-1}}{e^z-1} = \begin{cases} 2\pi|m|\, e^{\pi\sqrt{-1}/2} & m>0 \\ 2\pi|m|\, e^{3\pi\sqrt{-1}/2} & m<0 \end{cases}$$

$$\rightsquigarrow \quad I(s) = \int_{C_n} \frac{z^{s-1}}{e^z-1}\,dz + \underbrace{\left(-\sum_{m\geq 1} \operatorname{Res}_{z=m} - \operatorname{Res}_{z=-m}\right)}_{= \, 4\pi\sqrt{-1}\, e^{\pi\sqrt{-1}s} \sin\left(\frac{\pi s}{2}\right) \sum_{m=1}^{n} (2\pi m)^{s-1}}$$

(note: C encircles 0, so the difference does not involve residue at 0)

$n\to\infty$

$$\rightsquigarrow \quad I(s) = 4\pi\sqrt{-1}\, e^{\pi\sqrt{-1}s} \sin\left(\frac{\pi s}{2}\right) \cdot (2\pi)^{s-1} \cdot \zeta(1-s)$$

$$= \frac{2\pi\sqrt{-1}\cdot e^{\pi\sqrt{-1}s}}{\Gamma(1-s)} \zeta(s)$$

Bernoulli numbers and Bernoulli polynomial

def^n by generating functions

$$\frac{t}{e^t-1} = \sum_{r=0}^{\infty} \frac{B_r}{r!} t^r \qquad B_1 = -\frac{1}{2}, \quad B_{2i+1} = 0 \quad \forall i \in \mathbb{N}_{>0}$$

also: $$t\frac{e^t}{e^t-1} = \sum_{r=0}^{\infty} \frac{(-1)^r B_r}{r!} \cdot t^r$$

$$\frac{t e^{tx}}{e^t-1} = \sum_{r=0}^{\infty} \frac{B_r(x)}{r!} t^r \qquad B_r(0) = B_r \quad \forall r$$

$$B_1(x) = x - \frac{1}{2}$$

Euler-MacLaurin summation formula:

$f(x)$: C^N (sufficiently smooth on the given interval)

$a < b$, $a, b \in \mathbb{Z}$, $m \geq 1$, $m \in \mathbb{N}$

$$\boxed{\sum_{n=a+1}^{b} f(n) = \int_a^b f(x)\,dx + \sum_{r=1}^{m} (-1)^r \frac{B_r}{r!} \left[f^{(r-1)}(b) - f^{(r-1)}(a)\right] + \underset{\text{remainder}}{R_m} \quad \text{where } R_m = \frac{(-1)^{m-1}}{m!} \int_a^b B_m(x - \lfloor x \rfloor) \cdot f^{(m)}(x)\,dx}$$

Note: The previous formula on p. IV-4 is the special case $m = 1$.

Reference Rademacher, Topics in Analytic Number Theory, Chap 1

also NIST Handbook of Math. Functions; Bateman Manuscript Project

Andrews-Askey-Roy, Special Functions. Appendix D

Some properties of Bernoulli polynomials and Bernoulli numbers

1) $B_{r+1}'(x) = (r+1)\, B_r(x) \qquad \forall r \geq 1$

2) $$\sum_{n=M}^{N-1} n^r = \frac{1}{r+1}\left[B_{r+1}(N) - B_{r+1}(M)\right] \leftarrow$$ This determines $B_{r+1}(x)$ up to a constant

$B_r(x)$: monic of degree r

$$\rightsquigarrow \quad x^r = \frac{1}{r+1}\left(B_{r+1}(x+1) - B_{r+1}(x)\right)$$

3) $B_r(1-x) = (-1)^r B_r(x) \quad \forall r \geq 1$

$\Rightarrow \begin{cases} B_r(1) = (-1)^r B_r(0), \quad B_{2i+1} = 0 \quad \forall i \geq 1, i \in \mathbb{N} \\ B_{2k+1}(0) = B_{2k+1}(1) = B_{2k+1} = 0 \quad \forall k \geq 1 \quad (\text{i.e. excluding } B_1(x) = x - \frac{1}{2}) \\ B_{2k+1}(\frac{1}{2}) = 0 \quad \forall k \geq 0 \end{cases}$

3b) These are the only zeros of $B_{2k+1}(x)$ in $[0,1]$, and $0, \frac{1}{2}, 1$ are simple zeros of $B_{2k+1}(x)$

3c) $(-1)^{k-1} \cdot B_{2k} > 0 \quad \forall k \geq 1$

4) $B_{2k}(x) - B_{2k}$ does not change sign in $(0,1)$. Moreover

$B_{2k}(x) - B_{2k} \equiv 0 \pmod{(B_2(x) - B_2)^2}$, $\quad B_2(x) - B_2 = x(x-1)$

Special values: $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}$

$B_{10} = \frac{5}{66}$, $B_{12} = -\frac{691}{2730}$, $B_{14} = \frac{7}{6}$, $B_{16} = -\frac{3617}{510}$

Thm (von Staudt-Clausen) $\quad B_{2k} \equiv - \sum_{\substack{p-1 \mid 2k \\ p: \text{prime}}} \frac{1}{p}$

5) (addition formula) $\quad B_r(kx) = k^{r-1} \cdot \sum_{j=0}^{k-1} B_r(x + \frac{j}{k}) \quad \forall r \geq 1$

In particular, with $r = 2m$, $k = 2$, we get

$$B_{2m}(\tfrac{1}{2}) = \left(\frac{1}{2^{2m-1}} - 1\right) B_{2m}$$

0) $0 = \sum_{j=0}^{s-1} \binom{s}{j} B_j \overset{(B_0 = 1)}{=} 1 + \binom{s}{1} B_1 + \binom{s}{2} B_2 + \dots + \binom{s}{s-1} B_{s-1} \quad \forall s \geq 2$

(recursion formula for Bernouli numbers)

This can be taken as the definition of Bernouli numbers with $B_0 = 1$ (instead of by generating function)

Then $B_r(x)$ is defined up to a constant by 2),

and the constant is determined by $B_r(0) = B_r \quad \forall r \geq 1$ (and $B_r(1) = B_r$ $\forall r \geq 2$ follows)

6) $B_r(x) = \sum_{i=0}^{r} \binom{r}{i} B_i\, x^{r-i}$ $\qquad B_0 = 1$ $\qquad$ More generally, $B_r(x+y) = \sum_{i=0}^{r} \binom{r}{i} B_i(x)\, y^{r-i}$

e.g. $B_1(x) = x + B_1 = x - \frac{1}{2}$, $\quad B_2(x) = x^2 + 2\cdot B_1\cdot x + B_2 = x^2 - x + \frac{1}{6}$

$B_2(x+1) - B_2(x) = 2x$ ✓ $\qquad B_2'(x) = 2\cdot(x-\frac{1}{2})$ ✓

Application of Euler-MacLaurin to Gamma fnc:

① $\Gamma(s) = \frac{1}{s} \prod_{\nu=1}^{\infty} \frac{\nu}{\nu+s} \left(\frac{\nu+1}{\nu}\right)^s$ $\quad$ (equivalent to: $\Gamma(s) = \frac{1}{s}\cdot e^{-\gamma s} \prod_{\nu=1}^{\infty} \frac{\nu}{\nu+s} e^{\frac{s}{\nu}}$)

Weistrass' def of $\Gamma(s)$

$\leadsto$ ② $\Gamma(s) = \lim_{N\to\infty} (s-1)\left[\log(N+1) - \sum_{\nu=1}^{N} \log \frac{\nu+s-1}{\nu}\right]$

Apply summation formula with $f(x) = \log(x+s-1) - \log x$; $a=1,\ b=N$, $m=2k$
on p. IV-7

$$\leadsto f^{(r)}(x) = (-1)^{r-1} (r-1)! \left[\frac{1}{(x+s-1)^r} - \frac{1}{x^r}\right]$$

③ $$\sum_{\nu=1}^{N} \log \frac{\nu+s-1}{\nu} = \log s + \sum_{\nu=2}^{N} \left[\log(\nu+s-1) - \log \nu\right]$$

$$\underset{\text{Euler-MacLauren}}{=} \log s + \int_1^N \left[\log(x+s-1) - \log x\right] dx + \frac{1}{2}\left[\log(N+s-1) - \log N - \log s\right]$$

$$+ \sum_{j=1}^{k} \frac{B_{2j}}{(2j-1)\cdot 2j} \left[\frac{1}{(N+s-1)^{2j-1}} - \frac{1}{N^{2j-1}} - \frac{1}{s^{2j-1}} + 1\right]$$

$$+ \frac{1}{(2k)!} \int_1^N B_{2k}(x - \lfloor x \rfloor)\, (2k-1)! \left(\frac{1}{(x+s-1)^{2k}} - \frac{1}{x^{2k}}\right) dx$$

$\leadsto$ (intermediate steps omitted

④ $$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + \sum_{j=1}^{k} \frac{B_{2j}}{(2j-1)2j} \frac{1}{s^{2j-1}}$$

$$- \frac{1}{2k} \int_1^{\infty} B_{2k}(x - \lfloor x \rfloor) \frac{dx}{(x+s-1)^{2k}} + \overbrace{K_k}^{\frac{1}{2}\log(2\pi)}$$

a constant

let $s \to \infty$, get

$$K_k = \lim_{s\to\infty} \left(\log\Gamma(s) - (s - \tfrac{1}{2})\log s + s\right) = \log\sqrt{2\pi}\,!$$

Euler-MacLaurin $\Rightarrow$ meromorphic continuation of $\zeta(s)$ to $\mathbb{C}$:

$$\sum_{n=a+1}^{b} f(n) = \int_a^b f(x)\,dx + \sum_{r=1}^{m} (-1)^r \frac{B_r}{r!}\left[f^{(r-1)}(b) - f^{(r-1)}(a)\right]$$
$$+ \frac{(-1)^{m-1}}{m!}\int_a^b B_m(x-\lfloor x\rfloor)\cdot f^{(m)}(a)$$

with $f(x) = x^{-s}$:

$$\underset{\uparrow\ Re(s)>1}{\zeta(s)} = \lim_{N\to\infty}\left(1+\sum_{n=2}^{n} n^{-s}\right) = \lim_{N\to\infty}\Big[\int_1^N x^{-s}dx + \overset{B_1=-\frac{1}{2}}{\frac{1}{2}}(N^{-s}-1) + 1$$
$$+ \sum_{r=2}^{m} (-1)\cdot\frac{B_r}{r!}\, s(s+1)\cdots(s+r-2)\,(N^{-s-r+1}-1)$$
$$+ (-1)\cdot\frac{1}{m!}\, s(s+1)\cdots(s+m-1)\int_1^N B_m(x-\lfloor x\rfloor)\cdot x^{-s-m}dx\Big]$$

$$\underset{Re(s)>1}{=} \underset{\uparrow\ \int_1^\infty x^{-s}dx}{\frac{1}{s-1}} + \frac{1}{2} + \sum_{r=2}^{m}\frac{B_r}{r!}\, s(s+1)\cdots(s+r-2) - \frac{1}{m!}\, s(s+1)\cdots(s+m-1)\cdot\int_1^\infty B_m(x-\lfloor x\rfloor)\cdot x^{-s-m}dx$$

The r.h.s. is a meromorphic function on $\{Re(s) > 1-m\}$, holomorphic outside $s=1$, with a simple pole and $\mathrm{Res}_{s=1}\zeta(s) = 1$.

Also:
$$\begin{cases} \zeta(0) = -\frac{1}{2} \\ \zeta(-k) = -\frac{1}{k+1} + \frac{1}{2} - \sum_{r=2}^{k+1}\frac{B_r}{r!}\, k(k-1)\cdots(k-r+2) \end{cases}$$
with $m = 1+k$, $k\in\mathbb{N}_{\geq 1}$

$$(k+1)\,\zeta(-k) = \underbrace{\frac{1}{2}}_{\substack{\shortparallel \\ -B_1}}(k+1) - \sum_{r=2}^{k+1}\frac{B_r}{r!}\, k(k-1)\cdots(k-r+2)$$

$$= -\sum_{r=0}^{k+1}\binom{k+1}{r}B_r = -B_{k+1}(1) = -B_{k+1}$$

$$\left(\because\ B_r(x) = \sum_{i=0}^{r}\binom{r}{i}B_i\, x^i\right)$$

$m\in\mathbb{N}_{\geq 1}$
$$\rightsquigarrow \begin{cases} \zeta(-2m) = 0 \\ \zeta(1-2m) = -\frac{1}{2m}B_{2m} \end{cases}$$

Exer

Apply summation formula to $f(x) = \log x$

get:
$$\log N! = (N+\tfrac{1}{2})\log N + \sum_{j=1}^{m} \frac{B_{2j}}{(2j-1)2j} N^{-2j+1}$$
$$- \frac{1}{2m}\int_N^\infty B_{2m}(x-\lfloor x\rfloor)\, x^{-2m}\,dx$$
$$\underbrace{+ \frac{1}{2m}\int_1^\infty B_{2m}(x-\lfloor x\rfloor)\, x^{-2m}\,dx - \sum_{j=1}^{2m}\frac{B_{2j}}{(2j-1)2j}}_{\overset{\text{"}}{K_m}}$$

K_m indep. of N, and also indep of m

$$\therefore\; K_m = \lim_{N\to\infty}\left(\log N! - (N+\tfrac{1}{2})\log N + N\right)$$

Fact. $K_m = \log\sqrt{2\pi}$ Stirling's formula

Follows from "Wallis formula":

$$\frac{\pi}{2} = \lim_{N\to\infty}\prod_{n=1}^{N}\frac{4n^2}{4n^2-1}$$

Application to $\Gamma(s)$

One def^n of $\Gamma(s)$:

$$\Gamma(s) = \frac{1}{s} \cdot \prod_{n=1}^{\infty} \frac{n}{n+s} \left(\frac{n+1}{n}\right)^s \qquad \left(\because \; m! = \underbrace{\left(\prod_{n=1}^{N} \frac{n+1}{n}\right)^m}_{(N+1)^m} \cdot \prod_{\nu=1}^{N} \frac{n}{m+n} \right.$$

$$\underset{\text{exer}}{=} \frac{1}{s} e^{-\gamma s} \prod_{\nu=1}^{\infty} \frac{n}{n+s} e^{\frac{s}{n}} \qquad = \lim_{N\to\infty} \prod_{\nu=1}^{N} \frac{n}{n+m} \left(\frac{n+1}{n}\right)^m$$

$$\text{where } \gamma = \lim_{m\to\infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{m} - \log m\right) \qquad \left. = \prod_{n=1}^{\infty} \frac{n}{n+m} \left(\frac{n+1}{n}\right)^m \right)$$

Euler's const.

Rewrite $\displaystyle \frac{1}{\Gamma(s)} = s \cdot e^{\gamma s} \cdot \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-\frac{s}{n}}$

Apply summation formula (on p I-9) with $f(x) = \log(x+s-1) - \log x$

$$\sum_{n=1}^{N} \log \frac{n+s-1}{n} = \log s + \sum_{n=2}^{N} \left(\log(s+n-1) - \log n\right)$$

$$= \log s + \int_1^N \left[\log(x+s-1) - \log s\right] dx + \frac{1}{2}\left[\log(N+s-1) - \log N - \log s\right]$$

$$+ \sum_{j=1}^{k} \frac{B_{2j}}{(2j-1)\cdot 2j} \left[\frac{1}{(N+s-1)^{2j-1}} - \frac{1}{N^{2j-1}} - \frac{1}{s^{2j-1}} + 1\right]$$

$$+ \frac{1}{(2k)!} \int_1^N B_{2k}(x - \lfloor x \rfloor) \left[\frac{1}{(x+s-1)^{2k}} - \frac{1}{x^{2k}}\right] dx$$

$$= \prod_{n=1}^{\infty} \frac{n}{n+s-1}\left(\frac{n+1}{n}\right)^{s-1}$$

$$\log \Gamma(s) = \log\left((s-1)\cdot\Gamma(s-1)\right)$$

$$= \lim_{N\to\infty}\left[(s-1)\log(N+1) - \sum_{n=1}^{N}\log\left(\frac{n+s-1}{n}\right)\right]$$

Apply summation formula with

$$f(x) = \log(x+s-1) - \log x$$

$$\leadsto \log\Gamma(s) = -\frac{1}{2}\log s + \sum_{j=1}^{m}\frac{B_{2j}}{(2j-1)2j}\left(\frac{1}{s^{2j-1}} - 1\right)$$

$$- \frac{1}{2m}\int_1^{\infty} B_{2m}(x-\lfloor x\rfloor)\left[\frac{1}{(x+s-1)^{2m}} - \frac{1}{x^{2m}}\right]dx$$

$$+ \underbrace{\lim_{N\to\infty}\left[\begin{array}{l}(s-1)\log(N+1) - \int_1^N\left[\log(x+s-1) - \log x\right]dx \\ \quad - \frac{1}{2}\log\frac{N+s-1}{N}\end{array}\right]}_{= s\log s - s - 1}$$

$$\log\Gamma(s) = \left(s-\frac{1}{2}\right)\log s + \sum_{j=1}^{m}\frac{B_{2j}}{(2j-1)\cdot 2j}\frac{1}{s^{2j-1}} - \frac{1}{2m}\int_1^{\infty}B_{2m}(x-\lfloor x\rfloor)\cdot\frac{dx}{(x+s-1)^{2m}}$$

$$+ K_m$$

not involving s

Let $s\to\infty$, get $K_m = \lim_{s\to\infty}\left[\log\Gamma(s) - \left(s-\frac{1}{2}\right)\log(s)\right]$

so K_m is indep. of m

Fact: $K_m = \log\sqrt{2\pi}$ in Stirling's formula

Finally:

$$\boxed{\log\Gamma(s) = \frac{1}{2}\log(2\pi) + \left(s-\frac{1}{2}\right)\log s - s + \sum_{j=1}^{m}\frac{B_{2j}}{(2j-1)2j}\frac{1}{s^{2j-1}} - \frac{1}{2m}\int_0^{\infty}\frac{B_{2m}(x-\lfloor x\rfloor)}{(x+s)^{2m}}dx}$$

FE through Poisson summation

Mellin transform:

f = measurable on $\mathbb{R}^\times_{>0}$ $\quad M(s,f) = \int_{\mathbb{R}^\times_{>0}} f(x) \cdot x^s \frac{dx}{x}$

Inverse Mellin tranform

$$f(x) = \lim_{T\to\infty} \frac{1}{2\pi\sqrt{-1}} \int_{\sigma-\sqrt{-1}T}^{\sigma+\sqrt{-1}T} M(s,f)\, x^{-s}\, ds$$

Prop

Assume: (i) $f(x)\cdot x^{s-1} \in L^1(\mathbb{R}^\times_{>0})$ for $\sigma = \mathrm{Re}(s) \in (\alpha,\beta)$

(def: say $f(x)$ is of type (α,β))

(ii) $f(x)$ is continuous

(iii) $f(x)\cdot x^{s-1}$ is of bounded variation $\forall s$ s.t. $\alpha < \mathrm{Re}(s) < \beta$

Then: $f(x) = \lim_{T\to\infty} \frac{1}{2\pi\sqrt{-1}} \int_{\sigma-\sqrt{-1}T}^{\sigma+\sqrt{-1}T} M(s,f)\, x^{-s}\, ds$

This is a consequence of the Fourier version.

Prop Given $f \in L^1(\mathbb{R})$. Suppose that $\exists$ an open nbd U of x_0 f is of bounded variation in U and f is continuous at x_0

Then $f(x) = \lim_{Y\to\infty} \int_{-Y}^{Y} \hat{f}(y)\, e^{2\pi\sqrt{-1}xy}\, dy$

[Check: For Mellin transformation

$$M(s,f) = \int_0^\infty f(x)\, x^s \frac{dx}{x} \underset{x=e^{2\pi\sqrt{-1}u}}{=} \int_{-\infty}^{\infty} \overbrace{f(e^{2\pi\sqrt{-1}su})\cdot e^{2\pi\sqrt{-1}tu}}^{g(u)} (2\pi\sqrt{-1})\, du$$

$$= 2\pi\sqrt{-1}\cdot \hat{g}(-u)$$

$\rightsquigarrow$ $\frac{1}{2\pi\sqrt{-1}} \int \hat{g}(-u)\, e^{-2\pi\sqrt{-1}ux}\, du$ is the inverse Mellin transform]

Extend meaning of Mellin transform for measurable functions on $\mathbb{R}$ by the same formula: $M(s,f) \overset{\text{def}}{=} \int_{\mathbb{R}^\times} f(x)\cdot |x|^s \frac{dx}{|x|}$

$\mathbb{C}$-valued also OK

$\downarrow$

$\mathbb{R}$-valued

Lemma: Suppose $\eta(x)$, $\eta^*(x)$ are two bounded measurable functions on $\mathbb{R}$

s.t. (i) $\eta(x)+\eta^*(x^{-1}) = 1 \quad \forall x \in \mathbb{R}$

(ii) $\forall A>0$, $\begin{cases} \eta(x) = O(|x|^{-A}) \\ \eta^*(x) = O(|x|^{-A}) \end{cases}$ as $|x| \to \infty$, i.e. $\eta(x)$ decays rapidly as $|x| \to \infty$

Then (1) $\eta(x)$, $\eta^*(x)$ are of type $(0,\infty)$, i.e. $\int_{\mathbb{R}} |\eta(x)| \cdot |x|^{-s} \frac{dx}{|x|}$, $\int_{\mathbb{R}} |\eta^*(x)|\, |x|^{-s} \frac{dx}{|x|}$ are integrable for $\alpha < Re(s) < \beta$

(2) $M(s,\eta) - \frac{2}{s}$ and $M(s,\eta^*)$ have analytic continuation as entire functions on $\mathbb{C}$

(3) $M(s,\eta) + M(-s,\eta^*) = 0$ (after analytic continuation)

Pf. Define $\tilde{\eta}(x) = \begin{cases} 1 & \text{if } |x|<1 \\ \frac{1}{2} & \text{if } |x| = 1. \\ 0 & \text{if } |x|>1 \end{cases}$

Then $\tilde{\eta}(x) + \tilde{\eta}(x^{-1}) = 1 \quad \forall x$, and $\tilde{\eta}(x)$ decays rapidly at ∞ trivially.

$$M(s,\tilde{\eta}) = \underbrace{\int_{-1}^{1} |x|^s \frac{dx}{|x|}}_{\text{Converges for } Re(s)>0} = 2\int_0^1 x^{s-1}\,dx = \frac{2}{s} \quad \text{for } Re(s)>0$$

Thus (1) — (3) holds for $\eta = \tilde{\eta} = \eta^*$ (a special case of Lemma)

General case: Let $f = \eta - \tilde{\eta}$, $f^* = \eta^* - \tilde{\eta}$

Then $\begin{cases} f(x) + f^*(x^{-1}) = 0 \\ f(x) \text{ and } f^*(x) \text{ decay rapidly as } |x| \to \infty \end{cases}$

$\int_{\mathbb{R}^\times} f(x)\, |x|^{s-1}\, dx$: entire on $\mathbb{C}$

$$M(f,s) = \int_{\mathbb{R}^\times} f(x)\, |x|^s \frac{dx}{|x|} = \int_{\mathbb{R}^\times} f(x^{-1})\, |x|^{-s} \frac{dx}{|x|}$$

$$= -M(f^*,s).$$

QED.

"Master Thm"

$\eta(x), \eta^*(x) = \mathbb{R} \to \mathbb{R}$ as before. decay rapidly as $|x| \to \infty$

i.e. $\eta(x) + \eta^*(x^{-1}) = 1$

$\eta(x), \eta^*(x)$ bounded

$\alpha < 1 < \beta$, f: $\mathbb{C}$-valued measurable function on $\mathbb{R}$

<u>Theorem</u> $\int_{\mathbb{R}} f(x) \cdot |x|^s \frac{dx}{|x|}$ integrable for $\alpha < Re(s) < \beta$ $(\Rightarrow f \in L^1(\mathbb{R}))$

<u>Assume</u> (i) $\int_{\mathbb{R}} \hat{f}(y) |y|^s \frac{dy}{|y|}$ integrable for $\hat{\alpha} < Re(s) < \hat{\beta}$, $\hat{\alpha} < 1 < \hat{\beta}$

(ii) f and $\hat{f}$ are continuous and of bounded variation on $\mathbb{R}$

Then $\forall s \in \mathbb{C}$ with $1-\hat{\beta} < Re(s) < \beta$,

$$\underbrace{2\zeta(s) M(s,f)}_{\text{defined for } Re(s)>1} = \underbrace{\int_{\mathbb{R}} \left[\sum_{n\in\mathbb{Z}\setminus\{0\}} f(nx)\right] \cdot \eta(x^{-1}) |x|^s \frac{dx}{|x|} + \int_{\mathbb{R}} \left[\sum_{n\in\mathbb{Z}\setminus\{0\}} \hat{f}(nx)\right] \eta^*(x^{-1}) |x|^{1-s} \frac{dx}{|x|} - f(0) M(s,\eta^*) - \hat{f}(0) M(1-s,\eta)}_{\text{RHS defined, i.e. integrals convergent for } 1-\hat{\beta} < Re(s) < \beta}$$

for $1-\hat{\beta} < Re(s) < \beta$

The proof shows: The 1^{st} integral converges for $\{Re(s) < \beta\}$; the 2^{nd} integral converges for $\{Re(s) > 1-\hat{\beta}\}$

<u>Pf</u>: Let $F(x) = \sum_{n\in\mathbb{Z}\setminus\{0\}} f(nx)$

For $s \in \mathbb{C}$ with $1 < Re(s) < \beta$. Consider

$$\int_{\mathbb{R}} F(x) |x|^s \frac{dx}{|x|}$$

$$\int_{\mathbb{R}^\times} F(x) \cdot |x|^s \frac{dx}{|x|} \underset{\substack{\uparrow \\ \text{need to justify}}}{=} \sum_{m\in\mathbb{Z}\setminus\{0\}} \underbrace{\int_{\mathbb{R}^\times} f(nx) |x|^s \frac{dx}{|x|}}_{\overset{\| \, y=nx}{\sum_{n\in\mathbb{Z}\setminus\{0\}} |n|^{-s} \int_{\mathbb{R}} f(y) |y|^s \frac{dy}{|y|}}} = 2\zeta(s) \cdot M(s,f)$$

$\| \quad \eta(x^{-1}) + \eta^*(x) = 1$

$$\int_{\mathbb{R}^\times} F(x) \eta(x^{-1}) |x|^s \frac{dx}{|x|} + \int_{\mathbb{R}^\times} F(x) \eta^*(x) |x|^s \frac{dx}{|x|}$$

This integral converges for $\alpha < Re(s) < \beta$ $\because$ $\eta(x^{-1})$ is bounded

$\Rightarrow \int_{\mathbb{R}^\times} F(x) \eta(x^{-1}) |x|^s \frac{dx}{|x|}$ converges absolutely for <u>$Re(s) < \beta$</u> $\because$ $|\eta(x^{-1})| = O(x^A) \ \forall A$ as $|x| \to 0$

$\forall x_0 \in \mathbb{R}^\times \quad \int f(x\cdot x_0)\, e^{-2\pi\sqrt{-1}\, x y}\, dx \underset{\substack{\uparrow \\ t = x\cdot x_0}}{=} |x_0|^{-1} \int f(t)\, e^{-2\pi\sqrt{-1}\, t (y\cdot x_0^{-1})}\, dt$

Poisson summation

$$\Rightarrow \quad F(x) = |x|^{-1} \cdot \sum_{n\in\mathbb{Z}\setminus\{0\}} \hat{f}(n\cdot x^{-1}) + \hat{f}(0)\cdot |x|^{-1} - f(0)$$

So The same argument above, with f replaced by $\hat{f}$, s replaced by $1-s$ gives: (Note the $|x|^{-1}$ factor on the r.h.s. of the above formula)

$$\int_{\mathbb{R}^\times} \left[\sum_{n\in\mathbb{Z}\setminus\{0\}} \hat{f}(nx)\right] \cdot \eta^*(x^{-1})\, |x|^{1-s} \frac{dx}{|x|}$$

Converges abs. for $\mathrm{Re}(1-s) < \hat{\beta}$, i.e. $\underline{\mathrm{Re}(s) > 1-\hat{\beta}}$

Apply Poisson summation to get

$$F(x) = |x|^{-1} \cdot \sum_{n\in\mathbb{Z}\setminus\{0\}} \hat{f}(n\cdot x^{-1}) + \hat{f}(0)\cdot |x|^{-1} - f(0)$$

So

$$\int_{\mathbb{R}^\times} F(x)\, \eta^*(x)\, |x|^s \frac{dx}{|x|} = \int_{\mathbb{R}^\times} \sum_{n\in\mathbb{Z}\setminus\{0\}} f(n x^{-1})\, \eta^*(x)\cdot |x|^{s-1} \frac{dx}{|x|}$$

$$+ \underbrace{\int_{\mathbb{R}^\times} \hat{f}(0)\, \eta^*(x)\, |x|^{s-1} \frac{dx}{|x|}} - \underbrace{\int_{\mathbb{R}^\times} f(0)\cdot \eta^*(x) |x|^s \frac{dx}{|x|}}$$

$$= \int_{\mathbb{R}^\times} \sum_{n\in\mathbb{Z}\setminus\{0\}} f(nx)\, \eta^*(x^{-1})\, |x|^{1-s} \frac{dx}{x} \quad + \hat{f}(0)\underbrace{M(s-1, \eta^*)}_{\substack{\| \\ -M(1-s,\, \eta)}} \quad - \quad f(0)\cdot M(s, \eta^*)$$

[Also absolutely convergent for $\mathrm{Re}(s) > 1-\hat{\beta}$]

Putting everything together, get

converges for $\mathrm{Re}(s) < \beta$ ↙ ; converges for $\mathrm{Re}(s) > 1-\hat{\beta}$ ↙

$$2\zeta(s)\, M(s, f) = \int_{\mathbb{R}^\times}\left[\sum_{m\in\mathbb{Z}\setminus\{0\}} f(nx)\right]\eta(x^{-1})\, |x|^s \frac{dx}{x} + \int_{\mathbb{R}^\times}\left[\sum_{m\in\mathbb{Z}\setminus\{0\}} \hat{f}(nx)\right] \eta^*(x^{-1})\, |x|^{1-s} \frac{dx}{|x|}$$

$$- f(0)\, M(s, \eta^*) \quad - \hat{f}(0)\, M(1-s, \eta)$$

QED

Consequences of the Master Thm

1. Take $f(x)$ to be a Schwarz function on $\mathbb{R}^\times$ $\rightsquigarrow$ f and $\hat{f}$ are both of type $(-\infty, \infty)$

$\Rightarrow$ $M(s, f)$ is entire

master FE $\Rightarrow$ $\zeta(s) \cdot M(s, f)$ extends to a meromorphic function on $\mathbb{C} \setminus \{0, 1\}$ with at most simple poles at $0, 1$

So $\zeta(s)$ has a meromorphic continuation to $\mathbb{C}$

Let $\gamma(s) = \zeta(1-s) / \zeta(s)$

2. Use $\eta = \tilde{\eta} = \eta^*$ in Master FE. The RHS of Master Thm does not change under $f \rightsquigarrow \hat{f}$, $s \rightsquigarrow 1-s$

$\Rightarrow$ $\zeta(s) M(s, f) = \zeta(1-s) M(1-s, \hat{f})$ ("local FE" at arch. place)

$\forall f$ satisfying the conditions in Master FE

Know LHS is meromorphic for $\text{Min}(\alpha, 1-\hat{\beta}) < \text{Re}(s) < \beta$

RHS is meromorphic for $\text{Min}(\hat{\alpha}, 1-\beta) < \text{Re}(1-s) < \hat{\beta}$

$\Rightarrow$ Both sides are meromorphic for $\text{Min}(\alpha, 1-\hat{\beta}) < \text{Re}(s) < \text{Max}(1-\hat{\alpha}, \beta)$

Conclude: - $M(s, f)$ has a meromorphic continuation to

$$V := \{ \text{Min}(\hat{\alpha}, 1-\beta) < \text{Re}(s) < \text{Max}(1-\hat{\alpha}, \beta) \}$$

- $M(s, \hat{f})$ has a meromorphic continuation to

$$\hat{V} = \{ \text{Min}(\alpha, 1-\hat{\beta}) < \text{Re}(s) < \text{Max}(1-\alpha, \hat{\beta}) \}$$

- And on the vertical strip V, the "local functional equation"

$$M(s, f) = \gamma(s) M(1-s, \hat{f}) \text{ holds}$$

where $\gamma(s) = \dfrac{\zeta(1-s)}{\zeta(s)}$, a meromorphic function on $\mathbb{C}$.

3. Evaluate $\gamma(s)$ and FE of $\zeta(s)$

Take $f(x) = e^{-\pi x^2}$ $\rightsquigarrow$ $\hat{f}(y) = \int_{\mathbb{R}} e^{-\pi x^2 - 2\pi\sqrt{-1}xy} dx = \int_{\mathbb{R}} e^{-\pi(x+\sqrt{-1}y)^2} \cdot e^{-\pi y^2} dx$
$= C e^{-\pi y^2}$

where $C = \int_{\mathbb{R}} e^{-\pi(x+\sqrt{-1}y)^2} dx \underset{\text{Cauchy}}{=} \int_{\mathbb{R}} e^{-\pi x^2} dx$ $\qquad \hat{f} = C \cdot f \qquad (\hat{\hat{f}})(x) = C \hat{f} = C^2 f$
$= f(-x) = f$

$\rightsquigarrow C^2 = 1 \Rightarrow C = 1$ as $C = \int_{\mathbb{R}} e^{-\pi x^2} dx > 0$

$$M(s,f) = \int_{\mathbb{R}} e^{-\pi x^2} |x|^s \frac{dx}{|x|} \underset{\substack{y=\pi x^2 \\ \frac{dy}{y} = 2\frac{dx}{x}}}{=\!=} \pi^{-\frac{s}{2}} \int_0^\infty e^{-y} \cdot y^{\frac{s}{2}} \frac{dy}{y} = \pi^{-\frac{s}{2}} \cdot \Gamma\left(\tfrac{s}{2}\right)$$

From the local FE $M(s,f) = \gamma(s)\, M(1-s, \hat{f})$, get

$$\gamma(s) = \pi^{-\frac{s}{2}} \Gamma\left(\tfrac{s}{2}\right) \Big/ \pi^{\frac{s-1}{2}} \Gamma\left(\tfrac{1-s}{2}\right) = \pi^{\frac{1}{2}-s} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}$$

And the "local FE" reads:

For $\Lambda(s) \overset{\text{def}}{=} \underbrace{\pi^{-\frac{s}{2}} \cdot \Gamma\left(\tfrac{s}{2}\right)}_{\Gamma_{\mathbb{R}}(s)} \cdot \zeta(s)$, have $\Lambda(s) = \Lambda(1-s)$

4. Determine poles of $\zeta(s)$

With $f(x) = e^{-\pi x^2}$ in Master Thm

$\eta = \tilde{\eta}$

$$2\zeta(s) \cdot \pi^{-\frac{s}{2}} \Gamma(s) = \int_{\mathbb{R}} \Big[\sum_{m \in \mathbb{Z}\setminus\{0\}} f(nx)\Big] \tilde{\eta}(x^{-1})\, |x|^s \frac{dx}{|x|} + \int_{\mathbb{R}} \Big[\sum_{m \in \mathbb{Z}\setminus\{0\}} \hat{f}(nx)\Big] \tilde{\eta}(x^{-1}) \cdot |x|^s \frac{dx}{|x|}$$

$$- f(0) \cdot \frac{2}{s} - \hat{f}(0) \frac{2}{1-s}$$

The two integrals are entire functions in s

and $\frac{1}{\Gamma(s)}$ is entire $\Rightarrow$ $\zeta(s)$ has at most simple poles at $s=1$ and $s=0$

$\Gamma\left(\frac{s}{2}\right) = \frac{2}{s} + O(1)$ near $s=0$ $\Rightarrow$ $\zeta(s) = -\frac{1}{2} + O(s)$ near $s=0$

§5 Tate's thesis

Recall

Have fixed notation

$\psi_p : \mathbb{Q}_p \twoheadrightarrow \mathbb{Q}_p/\mathbb{Z}_p \xleftarrow{\sim} \mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z} \longrightarrow \mathbb{C}^\times$

$a \mapsto e(-a) = e^{-2\pi\sqrt{-1}a}$

$\psi_\infty : \mathbb{R} \to \mathbb{R}/\mathbb{Z} \xrightarrow{\sim} \mathbb{C}_1^\times$

$x \mapsto e(x) = e^{2\pi\sqrt{-1}x}$

$\psi_w = \psi_v \circ Tr_{K_w/\mathbb{Q}_v}$

1. Local FE

K: a locally compact field

General form:

dx: a Haar measure on $(K,+)$

ψ: a non-trivial additive unitary character $\psi : K \to \mathbb{C}_1^\times$

$d^\times x$: a Haar measure on $K^\times$

Fourier transform on K w.r.t. ψ and dx: (Recall: ψ defines $K \xrightarrow{\sim} Hom_{ct}^!(K, \mathbb{C}_1^\times)$)

$$\hat{f}_{\psi,dx}(y) = \int_K f(x)\, \psi(xy)\, dx$$

Local zeta function:

$f \in \mathcal{Z} = \left\{ \begin{array}{l} \text{functions on } K \text{ s.t. } f, \hat{f} \text{ are continuous on } (K,+) \text{ and } \in L^1(K,+) \\ f(x)\cdot |x|^\sigma \text{ and } \hat{f}(x)\,|x|^\sigma \text{ are in } L^1(K^\times) \text{ for all } \sigma > 0 \end{array} \right\}$

{Schwartz functions} $\subseteq$

$$Z(\chi, f) = \int_{K^\times} f(x)\cdot \chi(x)\, d^\times x$$

is defined for exponent of $\chi > 0$ where the exponent of χ is the real number σ s.t. $\chi \cdot |\cdot|^{-\sigma}$ is a unitary character.

Will show:

$\omega_s(x) = |x|^s$

Local F.E.

$$\frac{Z(\omega_1 \cdot \chi^{-1}, \hat{f}_{\psi,dx})}{L(\omega_1 \cdot \chi^{-1})} = \varepsilon(\chi, \psi, dx)\, \frac{Z(\chi, f)}{L(\chi)} \qquad \forall f \in \mathcal{Z}$$

where $L(\chi)$ will be defined below, meromorphic (as a function of χ)

$\Longrightarrow Z(\chi, f)$ extends to a meromorphic function "on $\mathbb{C}$"

and $\varepsilon(\chi\cdot\omega_s, \psi, dx)$ is of the form: "$A \cdot e^{bs}$"

independent of f "local constant"

Definition of $L(\chi)$:

(a) Case when K is non-archimedean $\quad \pi$: a generator of $\mathcal{O}_K$

$$L(\chi) = \begin{cases} (1-\chi(\pi))^{-1} & \text{if } \chi \text{ is unramified} \\ 1 & \text{if } \chi \text{ is ramified} \end{cases}$$

(b) Case when $K \underset{x}{\xrightarrow{\sim}} \mathbb{R}$ $\quad \chi = x^{-N}\omega_s \quad N = 1$ or 1

$$L(x^{-N}\omega_s) = \Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}} \cdot \Gamma(\tfrac{s}{2})$$

e.g. $\chi = sgn = x^{-1}\cdot\omega_1 \Rightarrow L(sgn) = \pi^{-\frac{1}{2}}\Gamma(\frac{1}{2}) = 1$

$$L(sgn\cdot\omega_s) = L(x^{-1}\cdot\omega_{s+1}) = \Gamma_{\mathbb{R}}(s+1)$$

(c) Case when $K \cong \mathbb{C}$ $\quad$ Write χ as $\chi = z^{-N}\omega_s$, where $z : K \xrightarrow{\sim} \mathbb{C}$ $\; N \in \mathbb{N}$

$$L(z^{-N}\omega_s) = \Gamma_{\mathbb{C}}(s) = 2\cdot(2\pi)^{-s}\Gamma(s) = \Gamma_{\mathbb{R}}(s)\cdot\Gamma_{\mathbb{R}}(s+1)$$

Note χ uniquely determines z^{-N} with $N \geq 0$ and s where z is an isomorphism $K \xrightarrow{\sim} \mathbb{C}$

(Unless χ is of the form ω_s for some $s \in \mathbb{C}$, z, N and s are uniquely determined by χ)

Remark $L(\chi)$ can be regarded / thought of as

$$\left[\text{"the" GCD of all } Z(\chi, f) \text{ as } f \text{ varies}\right]^{-1}$$

when K is non-archimedean

Formal properties of $\varepsilon(\chi,\psi,dx)$ / dependence on ψ and dx

(1) $\varepsilon(\chi,\psi, a\,dx) = a\cdot\varepsilon(\chi,\psi,dx) \quad \forall a>0$ — Obvious

(2) $\varepsilon(\chi, \psi(bx), dx) = \chi(b)\cdot|b|^{-1}\cdot\varepsilon(\chi,\psi,dx) \quad \forall b\in K^\times$

Rmk: Clearly, FE holds for $\varepsilon(\chi,\psi,dx)$
$\Rightarrow$ FE holds for $\varepsilon(\chi,\psi(bx), a\,dx)$ $\forall a>0$, $\forall b\in K^\times$

$\because \hat{f}_{\psi(bx),dx}(y) = \int_K f(x)\,\psi(bxy)\,dx = \hat{f}_{\psi,dx}(by)$

Formula for the local ε-factor (by explicit computation)

- Case $K\cong\mathbb{R}$, $\psi(x)=e^{2\pi\sqrt{-1}x}$ $N=0$ or 1. $dx=$ Lebesgue measure on $\mathbb{R}$
$$\varepsilon(x^{-N}\omega_s, dx) = \sqrt{-1}^{\,N} \quad (\text{The same } \sqrt{-1} \text{ in the def of } \psi)$$

- Case $K\cong\mathbb{C}$, $\varepsilon(z^{-N}\cdot\omega_s, e^{2\pi\sqrt{-1}\,\mathrm{Tr}_{K/\mathbb{R}}}, |dz\wedge d\bar z|) = \sqrt{-1}^{\,N}$ $\quad N\in\mathbb{N}$, $z: K \xrightarrow{\sim} \mathbb{C}$

the same $\sqrt{-1}$ in $\psi = e^{2\pi\sqrt{-1}\,\mathrm{Tr}_{K/\mathbb{R}}}$

- Case when K nonarch.

Let $n(\psi)=$ conductor of $\psi=$ the largest integer n s.t. $\psi(\pi^{-n}\mathcal{O}_K)=1$

$a(\chi) =$ conductor of $\chi = \begin{cases} 0 & \text{if } \chi \text{ is unramified} \\ \text{the smallest integer } m\geq 1 & \text{s.t. } \chi(1+\pi^m\mathcal{O}_K)=1 \end{cases}$

Then
$$\varepsilon(\chi,\psi,dx) = \begin{cases} \dfrac{\chi(\pi^{n(\psi)+a(\chi)})}{|\pi^{n(\psi)+a(\chi)}|}\cdot\displaystyle\int_{\mathcal{O}_K} dx \underset{\because a(\chi)=0}{=} \dfrac{\chi(\pi^{n(\psi)})}{|\pi^{n(\psi)}|}\cdot\int_{\mathcal{O}_K} dx & \text{If } \chi \text{ is unramified i.e. } a(\chi)=0 \\ \displaystyle\int_{K^\times}\chi^{-1}(x)\psi(x)\,dx \underset{def}{=} \sum_{n\in\mathbb{Z}}\int_{\pi^n\mathcal{O}^\times}\chi^{-1}(x)\,\psi(x)\,dx & \end{cases}$$
$$\underset{exer.}{=} \int_{\pi^{-n(\psi)-a(\chi)}} \chi^{-1}(x)\,\psi(x)\,dx$$

Note If χ is unramified, $n(\psi)=0$ and $\int_{\mathcal{O}_K} dx = 1$, then $\varepsilon(\chi,\psi,dx)=1$
("generic case")

Key lemma for local FE

Lemma: Let $\hat{\chi} = \omega \cdot \chi^{-1}$ (general notation)

Then $$Z(\chi, f) \cdot Z(\hat{\chi}, \hat{g}) = Z(\hat{\chi}, \hat{f}) \cdot Z(\chi, g)$$

Omit ψ and dx in notation

for all f, g s.t. both sides are defined

pf: $$Z(\chi, f) \cdot Z(\hat{g}, \hat{\chi}) = \int_{K^\times} f(x) \cdot \chi(x)\, d^\times x \cdot \int_{K^\times} \hat{g}(y) \cdot |y| \cdot \chi^{-1}(y)\, d^\times y$$

$$= \iint_{K^\times \times K^\times} f(x)\, \hat{g}(y)\, |y|\, \chi(x \cdot y^{-1})\, d^\times x\, d^\times y$$

Key: change variable $y = xz$

$$= \iint_{K^\times \times K^\times} f(x)\, \hat{g}(xz)\, |xz| \cdot \chi(z^{-1})\, d^\times x\, d^\times z$$

$d^\times x = \frac{dx}{|x|}$, $d^\times z = \frac{dz}{|z|}$

$$= \int_K \chi(z^{-1})\, dz \iint_{K \times K} f(x)\, g(t)\, \psi(xzt)\, dx\, dt$$

symmetric in f and g !

Similarly, $Z(\hat{\chi}, \hat{f})\, Z(\chi, \hat{g})$

$$= \int_K \chi(z^{-1})\, dz \iint_{K \times K} g(t)\, f(x)\, \psi(xzt)\, dx\, dt$$

q.e.d.

In Key Lemma, take f to be a Schwartz function on K

$\Rightarrow$ $Z(\chi, f)$ is holomorphic for $Re(\chi) > 0$

$Z(\hat{\chi}, f)$ is holomorphic for $Re(\hat{\chi}) > 0$

$Re(\hat{\chi}) = 1 - Re(\chi)$, i.e. $Re(\chi) < 1$

case when K is non-archimedean

Take $\hat{g}$ to be a Schwartz function on $K^\times$

s.t. $\int_K g(x)\, dx = 0$

$\Rightarrow \hat{g}(y) = \int g(x)\, \psi(xy)\, dx$ is a Schwartz function on $K^\times$

Then $Z(\chi, g)$ and $Z(\hat{\chi}, \hat{g})$ are holomorphic $\forall \chi \in Hom_{cont}(K^\times, \mathbb{C}^\times)$

$\Rightarrow$ $Z(\chi, f)$ is meromorphic $\forall \chi$

Alternatively, compute $Z(\chi, g)$ for some explicit Schwartz functions

Will do this in the number field case.

1. K non-archimedean.

self-dual Haar measure on K w.r.t. $\psi_p \circ \mathrm{Tr}_{K/\mathbb{Q}_p} = \psi_{can}$

Let $g_m(x) = \begin{cases} 1 & \text{if } x \equiv 1 \pmod{\mathfrak{p}^m} \\ 0 & \text{if } x \not\equiv 1 \pmod{\mathfrak{p}^m} \end{cases}$ $\quad m \in \mathbb{N}$

$$\hat{g}_m(y) = \int_{1+\mathfrak{p}^m} \psi(xy)\, dx \underset{x=1+z}{=} \psi(y)\int_{\mathfrak{p}^m} \psi(yz)\, dz = \begin{cases} \psi(y)\cdot (N\mathfrak{p})^{-m} \int_{\mathcal{O}} dx & \text{if } y \in \mathfrak{p}^{-n(\psi)-m} \\ 0 & \text{if } y \notin \mathfrak{p}^{-n(\psi)-m} \end{cases}$$

general ψ

What is $n(\psi_{can})$, the conductor of the ψ_{can}?

$\psi_{can}(x) = \psi_p(\mathrm{Tr}_{K/\mathbb{Q}_p} x)$

So $\psi_{can}(\mathfrak{p}^{-\mathfrak{D}_{K/\mathbb{Q}_p}}) = 1$ $\quad$ i.e. $n(\psi_{can}) = \mathrm{ord}_K(\mathfrak{D}_{K/\mathbb{Q}_p})$

self-dual measure μ_{sd} : $\int_{\mathcal{O}} \mu_{sd} = \underline{(N\mathfrak{D}_v)^{-\frac{1}{2}}}$

Use general ψ, dx

When $a(\chi) = 0$, let $f_0(x) =$ char. function of $\mathcal{O}_K = \mathcal{O}$

$$Z(\chi, f_0) = \sum_{n\geq 0} \chi(\pi)^n \int_{\pi^n \mathcal{O}^\times} d^\times x = \frac{1}{1-\chi(\pi)} \cdot \int_{\mathcal{O}^\times} d^\times x$$

$$\hat{f}_0(y) = \int_{\mathcal{O}_K} \psi(xy)\, dx = \begin{cases} 0 & \text{if } y \notin \mathfrak{p}^{-n(\psi)} \\ \int_{\mathcal{O}_K} dx & \text{if } y \in \mathfrak{p}^{-n(\psi)} \end{cases}$$

and $$Z(\omega_1\chi^{-1}, \hat{f}_0) = \int_{\mathfrak{p}^{-n(\psi)}} (\omega_1\cdot\chi^{-1})(y)\, d^\times y = \frac{(\omega_1\chi^{-1})(\pi^{-n(\psi)})}{1-(\omega_1\chi^{-1})(\pi)} \cdot \int_{\mathcal{O}_K} dx \cdot \int_{\mathcal{O}_K^\times} d^\times x$$

$$\Rightarrow \varepsilon(\chi, \psi, dx) = (\chi^{-1}\cdot\omega_1)(\pi^{-n(\psi)}) \cdot \int_{\mathcal{O}} dx$$

<u>Next</u>: χ ramified. i.e. $a(\chi) \geq 1$

Elementary computation: What is $Z(\chi, f_0)$?

Assume $\underset{\sigma}{\underset{\shortparallel}{Re(\chi)}} > 0$ $\quad \int_{\mathcal{O}} f_0(x)\, \omega_\sigma(x)\, d^\times x$ converges absolutely.

$$\int_{\substack{\pi^j \mathcal{O}^\times \\ (j \geq 0)}} f_0(x) \cdot \chi(x)\, d^\times x = \int_{\pi^j \mathcal{O}^\times} \chi(x)\, d^\times x = 0 \; !$$

<u>So</u>. $\quad Z(\chi, f_0) = 0$ for all ramified $\chi \in \mathrm{Hom}(K^\times, \mathbb{C})$

$\Longrightarrow \quad Z(\chi \cdot \omega_s, f)$ is holomorphic $\forall s \in \mathbb{C}$, if $\underline{f(0) = 0}$

(Still assume K = nonarch)

Analytic continuation when K is arch.

<u>Key Lemma</u>

$$\frac{Z(\chi, f)}{Z(\chi, g)} = \frac{Z(\chi^{-1}\omega, \hat{f})}{Z(\chi^{-1}\omega, \hat{g})} \qquad f, g \text{ Schwartz function}$$

Take $\psi(x) = e^{2\pi\sqrt{-1}x}$

$$g_0(x) = e^{-\pi x^2} \qquad \hat{g}_0^{\psi, dx}(y) = e^{-\pi y^2} = \int_{\mathbb{R}} e^{-\pi y^2 + 2\pi\sqrt{-1}\, xy}\, dx$$

$$g_\pm(x) = x e^{-\pi x^2} \qquad \hat{g}_\pm^{\psi, dx}(y) = \frac{1}{2\pi\sqrt{-1}} \frac{\partial}{\partial y} \hat{g}_0(y) = \sqrt{-1} \cdot g_\pm(y)$$

$$Z(\omega_s, g_0) = \int_{\mathbb{R}^\times} e^{-\pi x^2} \cdot |x|^s \frac{dx}{|x|} = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) = \Gamma_{\mathbb{R}}(s)$$

$$Z(x^{-1}\omega_s, g_\pm) = \int_{\mathbb{R}^\times} e^{-\pi x^2} |x|^s \frac{dx}{|x|} = \Gamma_{\mathbb{R}}(s)$$

$$\Gamma_{\mathbb{R}}(s) \neq 0 \quad \forall s \in \{Re(s) > 0\}$$

$$\nearrow \frac{Z(\chi,f)}{Z(\chi,g)}$$

$\rightsquigarrow$ When $\chi = \omega_s$ $\rightsquigarrow$ take $g = g_0$ LHS holomorphic on $\{\mathrm{Re}(\chi) > 0\}$
RHS holomorphic on $\{-\mathrm{Re}(\chi)+1 > 0\} \Leftrightarrow \{\mathrm{Re}(\chi) < 1\}$

When $\chi = x^{-1} \cdot \omega_s$ $\rightsquigarrow$ take $g = g_\pm$ LHS holomorphic when $\mathrm{Re}(s) > 1$
RHS holomorphic when $2 - \mathrm{Re}(s) > 0$
i.e. $\mathrm{Re}(s) < 2$

Conclude: $\dfrac{Z(\chi,f)}{L(\chi)}$ is entire (i.e. extends to a holomorphic function on $\mathrm{Hom}_{cont}(K, \mathbb{C}^\times)$)

Case when $K \cong \mathbb{C}$

$$g_0(z) = e^{-2\pi z\bar{z}} \qquad \hat{g}_0(w) = \int_{\mathbb{C}} e^{-2\pi w\bar{w}} e^{2\pi\sqrt{-1}(zw + \bar{z}\bar{w})} |dz \wedge d\bar{z}| = g_0(w)$$

Let $g_N(z) = z^N e^{-2\pi z\bar{z}}$ $\quad N \geq 0$

$$Z(\bar{z}^{-N} \cdot \omega_s, g_N) = \int_{\mathbb{C}^\times} (z\bar{z})^s e^{-2\pi z\bar{z}} \frac{|dz \wedge d\bar{z}|}{z\bar{z}}$$

$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})$
$\Gamma_{\mathbb{C}}(s) = 2 \cdot (2\pi)^{-s} \Gamma(s)$
$= \Gamma_{\mathbb{R}}(s) \cdot \Gamma_{\mathbb{R}}(s+1)$

$$= \iint_{\substack{r > 0 \\ 0 \leq \theta \leq 2\pi}} e^{-2\pi r^2} r^{2s} \frac{r\,dr\,d\theta}{r^2} \underset{t = 2\pi r^2}{=} \int_0^\infty e^{-t} (2\pi)^{-s+1} \frac{dt}{t} = \pi \cdot \Gamma_{\mathbb{C}}(s)$$

Similarly, get $\dfrac{Z(\chi,f)}{L(\chi)}$ is an entire function on $\mathrm{Hom}_{cont}(K, \mathbb{C}^\times)$

Global:

$K=$ a number field.

Let $\psi:(\mathbb{A}_K/K,+)\to\mathbb{C}_1^\times$ be a nontrivial character (necessarily unitary $\because$ $\mathbb{A}_K/K$ is cpt)

dx: a self-dual Haar measure on $(\mathbb{A}_K,+)$ w.r.t. ψ

$\rightsquigarrow\ d^\times x := \prod_v' d^\times x_v$

$d^\times x_v$ normalized by: $\int_{\mathcal{O}_v^\times} d^\times x_v = (N\mathfrak{D}_v)^{-\frac{1}{2}}$

Also: $1\to\mathbb{A}_{K,1}^\times\to\mathbb{A}_K^\times\xrightarrow{1\cdot1}\mathbb{R}_{>0}^\times\to1$

$d^\times u \quad d^\times x \quad d^\times t$ s.t. $d^\times x=d^\times u\cdot d^\times t$

f: Schwartz function on $\mathbb{A}_K$.

$\chi:\mathbb{A}_K^\times/K^\times\to\mathbb{C}^\times$

$\forall t\in\mathbb{A}_K^\times$, define $Z_t(\chi,f)$ by

$$Z_t(\chi,f)\overset{\text{def}}{=}\int_{\mathbb{A}_{K,1}^\times} f(tu)\,\chi(tu)\,d^\times u$$

<u>Lemma</u> $Z_t(\chi,f)+f(0)\cdot\int_{\mathbb{A}_{K,1}^\times/K^\times}\chi(tu)\,d^\times u = Z_{t^{-1}}(\hat{\chi},\hat{f})+\hat{f}(0)\int_{\mathbb{A}_{K,1}^\times}\hat{\chi}(t^{-1}u)\,d^\times u$

where $\hat{\chi} = \chi^{-1}\cdot\omega_1$

<u>pf</u>: Poisson summation:

$$\sum_{\xi\in K} f(t\xi u)=\frac{1}{|tu|}\sum_{\xi\in K}\hat{f}\left(\frac{\xi}{tu}\right)$$

So
$$Z_t(\chi,f)=\int_{\mathbb{A}_{K,1}^\times/K^\times}\left(\sum_{\xi\in K^\times} f(t\xi u)\right)\chi(tu)\,d^\times u$$

$$=\int_{\mathbb{A}_{K,1}^\times/K^\times}\left[\sum_{\xi\in K^\times}\hat{f}\left(\frac{\xi}{tu}\right)\right]\cdot\frac{1}{|tu|}\chi(tu)\,d^\times u+\hat{f}(0)\int_{\mathbb{A}_{K,1}^\times/K^\times}\frac{1}{|tu|}\chi(tu)\,d^\times u - f(0)\int_{\mathbb{A}_{K,1}^\times/K^\times}\chi(tu)\,d^\times u$$

$$=\int_{\mathbb{A}_{K,1}^\times/K^\times}\left[\sum_{\xi\in K^\times}\hat{f}(\xi t^{-1}u^{-1})\right]\cdot(\chi^{-1}\cdot\omega_1)(t^{-1}u^{-1})\,d^\times u + \hat{f}(0)\cdot\int_{\mathbb{A}_{K,1}^\times/K^\times}(\chi^{-1}\cdot\omega_1)(t^{-1}u^{-1})\,d^\times u$$

$$=Z_{t^{-1}}(\hat{\chi},\hat{f})+\hat{f}(0)\int_{\mathbb{A}_K^\times/K^\times}\hat{\chi}(t^{-1}u)\,d^\times u - f(0)\int_{\mathbb{A}_K^\times/K^\times}\chi(tu)\,d^\times u$$

Exer! ↓

Define $Z(f,\chi) := \int_{\mathbb{A}_K^\times} f(x)\,\chi(x)\,d^\times x$ converges absolutely for $Re(\chi) > 1$

<u>Theorem</u>

$$Z(\chi,f) = \begin{cases} \int_1^\infty Z_t(\chi,f)\frac{dt}{t} + \int_1^\infty Z_t(\hat{\chi},\hat{f})\frac{dt}{t} & \text{if } \chi|_{\mathbb{A}_{K,1}^\times} \neq \mathbb{1} \\ \int_1^\infty Z_t(\chi,f)\frac{dt}{t} + \int_1^\infty Z_t(\hat{\chi},\hat{f})\frac{dt}{t} \\ \quad - \frac{\hat{f}(0)}{1-s}\cdot\int_{\mathbb{A}_{K,1}^\times/K^\times} d^\times u - \frac{f(0)}{s}\cdot\int_{\mathbb{A}_{K,1}^\times/K^\times} d^\times u & \text{if } \chi = \omega_s \\ & (\text{i.e. } \chi|_{\mathbb{A}_{K,1}^\times} = \mathbb{1}) \end{cases}$$

<u>Pf</u>: $Z(\chi,f) = \int_1^\infty Z_t(\chi,f)\frac{dt}{t} + \int_0^1 Z_{t^{-1}}(\hat{f},\hat{\chi})\frac{dt}{t} + \hat{f}(0)\int_1^\infty \frac{dt}{|t|}\int_{\mathbb{A}_{K,1}^\times} \hat{\chi}(t^{-1}u)\,d^\times u$

$$- f(0)\int_0^1 \frac{dt}{|t|}\int_{\mathbb{A}_{K,1}^\times} \chi(tu)\,d^\times u$$

If $\chi = \omega_s$, then $\int_1^\infty \frac{dt}{t}\hat{\chi}(t^{-1}) = \int_0^1 \frac{dt}{t}\hat{\chi}(t) = \int_0^1 t^{1-s}\frac{dt}{t} = \frac{1}{1-s}$

$$\int_0^1 \frac{dt}{t} t^s = \frac{1}{s}$$

<u>Cor</u> (i) $Z(\chi\cdot\omega_s, f)$ is entire (as a function of s) if $\chi|_{\mathbb{A}_{K,1}^\times} \neq$ trivial

(ii') $Z(\omega_s, f)$ is holomorphic with at most simple poles at $s=1 \longleftrightarrow \chi = |\cdot|$ and $s=0 \longleftrightarrow \chi = \mathbb{1}$

(iii') $Res_{s=1} Z(\omega_s, f) = \kappa_1 \cdot \hat{f}(0)$

$Res_{s=0} Z(\omega_s, f) = -\kappa\cdot f(0)$, where $\kappa_1 = \int_{\mathbb{A}_{K,1}^\times/K^\times} d^\times u$

$\uparrow \frac{du}{|u|}$ du = self dual Haar meas on $\mathbb{A}_K$ w.r.t. ψ_{can}

Results related to $\int_{\mathbb{A}_{K,1}^\times / K^\times} d^\times u$ ($d^\times u = \frac{du}{|u|}$, du: self-dual Haar measure on $\mathbb{A}_K$ w.r.t. ψ_{can}) (Tamagawa measure for $\mathbb{A}_K^\times$)

Proposition Let dx^{naive} be the "naive Haar measure" on $\mathbb{A}_K$: $\quad w_K = |\mu(K)|$ (roots of 1 in K)

i.e $d^\times x = \prod'_{v \in \Sigma_K} d^\times x_v$ s.t. $\int_{\mathcal{O}_v^\times} d^\times x_v = 1$ and $d^\times x_v = \begin{cases} dx_v/|x_v| & \text{for } v \text{ arch.} \\ |dz d\bar{z}|/z\bar{z} \end{cases}$

$d^\times x = d^\times_{naive} x$

Define $d_1^\times u$ on $\mathbb{A}_{K,1}^\times / K^\times$ by $d^\times_{1,naive} u \cdot \frac{dt}{t} = \frac{dx}{|x|}$ (Haar measure on $\mathbb{R}^\times_{>0}$)

Then

$$\kappa_1 = \int_{\mathbb{A}_{K,1}^\times / K^\times} d^\times_{1,naive} u = 2^{r_1} (2\pi)^{r_2} \cdot h_K \cdot R_K / w_K$$

Note: The factor $|d_K|^{-\frac{1}{2}}$ in $\mathrm{Res}_{s=1} \zeta_K(s)$ comes from Fourier transform

Let $f = \prod_v f_v$, where

$f_v(x_v) =$ characteristic function of $\mathcal{O}_v$ (v-arch)

$$\rightsquigarrow Z(1 \cdot |\ |_v^s, f_v, d^\times x_v) = \sum_{n \geq 0} q_v^{-ns} = \frac{1}{1 - q_v^{-s}} \qquad \int_{\mathcal{O}_v^\times} d^\times x_v = 1$$

$$f_v(x_v) = \begin{cases} e^{-\pi x_v^2} & K_v \cong \mathbb{R} \\ e^{-2\pi z_v \bar{z}_v} & K_v \cong \mathbb{C} \end{cases} \qquad d^\times x_v = \begin{cases} \frac{dx}{|x|} \\ |dz d\bar{z}|/z\bar{z} \end{cases}$$

$$\rightsquigarrow Z(1 \cdot |\ |_v^s, f_v) = \begin{cases} \Gamma_{\mathbb{R}}(s) \\ \Gamma_{\mathbb{C}}(s) \end{cases}$$

v nonarch $\Rightarrow$ $\hat{f}_v = \begin{cases} f_v & v \text{ archimedean} \\ ND_v^{-\frac{1}{2}} \cdot (\text{characteristic function of } \mathcal{D}_v^{-1}) \end{cases}$

inverse different

$$\Longrightarrow \hat{f}(0) = |d_K|^{\frac{1}{2}}$$

$$Z(\omega_s, f) = \Lambda(s) = \Gamma_{\mathbb{R}}(s)^{r_1} \cdot \Gamma_{\mathbb{C}}(s)^{2r_2} \cdot \zeta_K(s)$$

and $\mathrm{Res}_{s=1} \zeta_K(s) = |d_K|^{-\frac{1}{2}} \cdot \kappa_1 = \dfrac{2^{r_1} (2\pi)^{r_2} h_K R_K}{|d_K|^{\frac{1}{2}} \cdot w_K}$

$$\mathrm{Res}_{s=1}\, \zeta_k(s) = \frac{2^{r_1}\cdot(2\pi)^{r_2}\cdot h_k \cdot R_k}{|d_k|^{\frac{1}{2}}\cdot w_k}$$

$$\Lambda_k(s) = \Gamma_{\mathbb{R}}(s)^{r_1}\cdot\Gamma_{\mathbb{C}}(s)^{r_2}\cdot\zeta_k(s) \qquad \Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2}),\ \Gamma_{\mathbb{C}}(s) = 2\cdot(2\pi)^{-s}\Gamma(s)$$

$$\Gamma_{\mathbb{R}}(1) = 1. \qquad \Gamma_{\mathbb{C}}(1) = \pi^{-1}$$

$$\boxed{\mathrm{Res}_{s=1}\, \Lambda_k(s) = \frac{2^{r_1+r_2}\cdot h_k\cdot R_k}{|d_k|^{\frac{1}{2}}\cdot w_k}}$$

With completed zeta functions, one get rid of (powers of) π in the residue at $s=1$

Local FE: $$\frac{Z(\omega_v\cdot\chi_v^{-1}, \hat{f}_{\psi_v, dx_v})}{L(\omega_v\cdot\chi_v^{-1})} = \varepsilon_v(\chi_v, \psi_v, dx_v)\,\frac{Z(\chi_v, f_v)}{L(\chi_v)}$$

$\psi = \prod' \psi_v$, $dx = \prod dx_v$

Take $f = \prod'_{v\in\Sigma_K} f_v$, $\chi = \prod' \chi_v$

$$\hat{f}_{\psi, dx} = \prod'_{v\in\Sigma} \hat{f}_{\psi_v, dx_v}$$

The global FE $Z(\omega\cdot\chi^{-1}, \hat{f}_{\psi, dx}) = Z(\chi, f)$ implies

$$\begin{array}{ccccc} L(\chi) & = & \varepsilon(\chi) & \cdot & L(\chi^{-1}\cdot\omega) \\ \| & & \| & & \| \\ \prod_v L(\chi_v) & & \prod_v \varepsilon_v(\chi_v) & & \prod_v L(\chi_v^{-1}\omega_v) \end{array}$$

$\psi_{can} = \psi_{\mathbb{Q}} \circ \mathrm{Tr}_{k/\mathbb{Q}}$
$$\psi_{\mathbb{Q}_p} : \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \xleftarrow{\sim} \mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z} \longrightarrow \mathbb{C}_1^\times, \quad a \mapsto e^{2\pi\sqrt{-1}a}$$
$$\psi_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{C}_1^\times, \quad x \longmapsto e^{-2\pi\sqrt{-1}x}$$

$dx = \mu_{sd}$ = self-dual Haar measure on $\mathbb{A}_k$ w.r.t. ψ_{can}
$= \prod' \mu_v^{sd}$ $\qquad \mu_v^{sd}(\mathcal{O}_v) = N\mathfrak{D}_{k_v/\mathbb{Q}_p}^{-\frac{1}{2}}$ $\qquad v$ non-arch

Example: k: number field.
$$\varepsilon_v(\omega_s, \psi_{can}, \mu_{sd}) = \begin{cases} N\mathfrak{D}_v^{-s}\cdot N\mathfrak{D}_v\cdot N\mathfrak{D}_v^{-\frac{1}{2}} = N\mathfrak{D}_v^{-s+\frac{1}{2}} & n(\psi_{can\,v}) = \mathrm{ord}(\mathfrak{D}_v) \\ 1 & v \text{ arch.} \end{cases}$$

General formula:
v: nonarch
$$\varepsilon_v(\chi_v, \psi_v, dx_v) = \begin{cases} \chi(\pi_v^{n(\psi)})\cdot|\pi_v|^{-n(\psi)} \int_{\mathcal{O}_v} dx_v & \text{if } \chi_v \text{ is unramified} \\ \int_{\pi_v^{-a(\chi)-n(\psi)}} \chi^{-1}(x)\,\psi(x)\,dx_v & \text{if } \chi \text{ is ramified} \end{cases}$$

ε-factor under
twist by unramified character: v nonarchimedean

(*) $\varepsilon_v(\chi\cdot\omega, \psi, dx) = \omega(\pi^{n(\psi)+a(\chi)})\cdot \varepsilon_v(\chi,\psi,dx)$ $\qquad$ ω unramified

Continue with $\varepsilon_v(\omega_s, \psi_v, dx_v)$:

$$\varepsilon(\omega_s) = \prod_v \varepsilon_v(\omega_s, \psi_v, \mu_{v,can}) = |d_{k/\mathbb{Q}}|^{-s+\frac{1}{2}}$$

$$\Lambda(\omega_s) = \Gamma_{\mathbb{R}}(s)^{r_1}\cdot\Gamma_{\mathbb{C}}(s)^{r_2}\cdot\zeta_k(s) = \left(\pi^{-\frac{s}{2}}\cdot\Gamma(\tfrac{s}{2})\right)^{r_1}\cdot\left(2\cdot(2\pi)^{-s}\right)^{r_2}\cdot\zeta_k(s)$$

$$= 2^{r_2}\cdot\underbrace{2^{-r_2 s}\cdot\pi^{-\frac{ns}{2}}}_{\left(2^{-r_2}\cdot\pi^{-\frac{n}{2}}\right)^s}\cdot\Gamma(\tfrac{s}{2})^{r_1}\cdot\Gamma(s)^{r_2}\cdot\zeta_k(s) \qquad n=[k:\mathbb{Q}]$$

F.E. $\boxed{\Lambda(\omega_s) = \varepsilon(\omega_s)\cdot\Lambda(\omega_{1-s}) = |d_k|^{-s+\frac{1}{2}}\,\Lambda(\omega_{1-s})}$

$$|d_k|^{\frac{s}{2}}\,\Lambda(\omega_s) = |d_k|^{\frac{1-s}{2}}\,\Lambda(\omega_{1-s})$$

$\leadsto$ classical / tradition form.

Let $\xi_k(s) = \left(|d_k|^{\frac{1}{2}}\cdot 2^{-r_2}\cdot\pi^{-\frac{n}{2}}\right)^s\cdot\Gamma(\tfrac{s}{2})^{r_1}\cdot\Gamma(s)^{r_2}\cdot\zeta_k(s)$ $\qquad n=[k:\mathbb{Q}]$

Then $\xi_k(s) = \xi_k(1-s)$

Effect of twisting by an unramified idele class character of finite order

Suppose that $\omega: \mathbb{A}_k^\times/k^\times \longrightarrow \mathbb{C}_1^\times$ s.t. $\omega_\infty(k\otimes_{\mathbb{Q}}\mathbb{R})^\times = 0$ and $\omega_v(\mathcal{O}_v^\times)=1$ $\forall v\in\Sigma_k$

Then $\forall v$ arch: $\varepsilon_v(\omega\cdot\omega_s, \psi_{v,can}, dx_v) = 1$

$\forall v$ nonarch, $\varepsilon_v(\omega\cdot\omega_s, \psi_{v,can}, \overset{\text{self dual}}{dx_v}) = \omega_v(\pi^{ord_v(\mathcal{D}_v)})\cdot\varepsilon_v(\omega_s, \psi_v, dx_v)$

$$\Longrightarrow\ \varepsilon(\omega\cdot\omega_s) = \underset{\prod_v \omega_v(\pi^{ord(\mathcal{D}_{k_v/\mathbb{Q}_p})})}{\underset{\uparrow}{\omega(\mathcal{D}_{k/\mathbb{Q}})}}\cdot\varepsilon(\omega_s) = \omega(\mathcal{D}_{k/\mathbb{Q}})\cdot|d_k|^{-s+\frac{1}{2}}$$

$\omega: \mathbb{A}_k^\times / k^\times \to \mathbb{C}_1^\times$ unramified everywhere

$\Lambda(s,\omega) = \Gamma_{\mathbb{R}}(s)^{r_1} \cdot \Gamma_{\mathbb{C}}(s)^{r_2} \cdot L(s,\omega)$ satisfies FE

$$\Lambda(s,\omega) = \omega(\mathcal{D}_{k/\mathbb{Q}}) \cdot |d_{k/\mathbb{Q}}|^{-s+\frac{1}{2}} \cdot \Lambda(1-s,\omega^{-1})$$

Special case: $\omega^2 = 1$, ω everywhere unramified, so $\Lambda(s,\omega^{-1}) = \Lambda(s,\omega)$

CFT $\Rightarrow$ ω corresponds to a quadratic extension field L/k, everywhere unramified, s.t. (i) L/k splits over archimedean primes

(ii) $\forall$ non-arch $v \in \Sigma_k$, L/k splits at v iff $\omega_v(\pi_v) = -1$

$\Downarrow$

(iii) $L_v(\omega_v; s) \cdot L_v(s) = \prod_{w/v} L_w(s)$, $\forall v \in \Sigma_k$

Which implies:

$$\Lambda_k(s,\omega) \cdot \Lambda_k(s) = \Lambda_L(s)$$

FE for $\Lambda_k(s,\omega)$ and $\Lambda_k(s)$:

$$\Lambda_k(s,\omega) \cdot \Lambda_k(s) = \omega(\mathcal{D}_{k/\mathbb{Q}}) \cdot |d_{k/\mathbb{Q}}|^{-2s+1} \cdot \Lambda_k(1-s,\omega) \cdot \Lambda_k(1-s)$$

$$\Lambda_L(s) = |d_{L/\mathbb{Q}}|^{-s+\frac{1}{2}} \cdot \Lambda_L(1-s) = |d_{k/\mathbb{Q}}|^{-2s+1} \Lambda_L(1-s,\omega)$$

$\because$ $|d_{L/\mathbb{Q}}| = |d_{k/\mathbb{Q}}|^2$

Conclude: $\omega(\mathcal{D}_{k/\mathbb{Q}}) = 1$ $\quad \forall\, \omega: \mathbb{A}_k^\times / k^\times \to \mathbb{C}_1^\times$ everywhere unramified s.t. $\omega^2 = 1$

CFT $\Rightarrow$ $[\mathcal{D}_{k/\mathbb{Q}}]$ = image of $\mathcal{D}_{k/\mathbb{Q}}$ in $\mathbb{A}_k^\times / k^\times \cdot k_\infty^\times \cdot \prod_{v \text{ nonarch}} \mathcal{O}_v^\times$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$

is killed by every element of $H_k^\vee$ $\quad$ H_k ideal class group of k

$H_k^\vee = \mathrm{Hom}_{grp}(H_k, \mu_2)$

i.e. $[\mathcal{D}_{k/\mathbb{Q}}]$ is a square in H_k

<u>Theorem</u> (Hecke; proof via root number due to Serre)

(i) Let k be a number field. The image of $\mathcal{D}_{k/\mathbb{Q}}$ in H_k is a square in H_k

(ii) Let C be a geom. irreducible smooth proper curve over $\mathbb{F}_q$. Then $\exists$ an invertible $\mathcal{O}_C$-module $\mathcal{L}$ such that $\mathcal{L}^{\otimes 2} \cong K_C$

Theorem (Assuming class field theory) K: number field

$\omega : \mathbb{A}_K^\times / K^\times \to \mathbb{C}_1^\times$ unitary idele class character. Then $L(1, \omega) \neq 0$

1^*

(Hadamard–de la Vallée Poussin)

<u>Key Lemma</u>: Let $\varphi(\lambda, t) = (1-t)^3 (1-\lambda t)^4 (1-\lambda^2 t) \in \mathbb{Z}[\lambda, t]$

Then $|\varphi(\lambda, t)| < 1 \quad \forall t \in (0,1) \ \forall \lambda \in \mathbb{C}_1^\times$

Pf:
$$\log |\varphi(\lambda,t)|^2 = \log |\varphi(\lambda,t) \cdot \varphi(\bar\lambda, t)|$$
$$= - \sum_{n=1}^{\infty} \frac{t^n}{n} \left[6 + 4\lambda^n + 4\bar\lambda^n + \lambda^{2n} + \bar\lambda^{2n} \right]$$
$$= - \sum_{n=1}^{\infty} \frac{t^n}{n} \left(2 + \lambda^n + \bar\lambda^n \right)^2 < 0 \quad \text{q.e.d.}$$

Choose $S \subseteq \Sigma_k$, $\#S < \infty$ s.t. ω is unramified at all $v \notin S$
$S \supseteq \Sigma_\infty$

Let $L_S(s, \omega) \overset{\text{def}}{=} \prod_{v \notin S} \left(1 - \omega_v(\pi_v)\, q_v^{-s}\right)^{-1}$ converges absolutely for $\mathrm{Re}(s) > 1$

$$L_S(s, \omega^2) = \prod_{v \notin S} \left(1 - \omega_v(\pi_v)^2 q_v^{-s}\right)^{-1}$$
$$\zeta_S(s) = \prod_{v \notin S} \left(1 - q_v^{-s}\right)^{-1}$$

and consider

$$F(s) \overset{\text{def}}{=} \zeta_S(s)^3 \cdot L_S(s,\omega)^4 \cdot L_S(s, \omega^2) \qquad \mathrm{Re}(s) > 1$$

Key Lemma $\Rightarrow$ $|F(s)| > 1 \quad \forall s$ with $\mathrm{Re}(s) > 1$

Claim 1 If $\omega^2 \neq \mathbb{1}$, then $L(1,\omega) \neq 0$

pf: $\omega^2 \neq \mathbb{1} \Rightarrow$ $L_S(s,\omega^2)$ and $L_S(s,\omega)$ both extends to holomorphic functions near $s=1$

If $L(1,\omega)=0$, i.e. $L_S(1,\omega)=0$, then $F(s)$ extends to a holomorphic function near $s=1$

Key Lemma $\Rightarrow$ $|F(s)|>1$ $\forall s$ with $Re(s)>1$, a contradiction.

Claim 2: If $\omega^2 = 1$, and $\omega \neq \mathbb{1}$, then $L(1,\omega) \neq 1$

Class field theory $\Rightarrow \exists\ L/K$, $[L:K]=2$ corresponding to ω

s.t. $\zeta_L(s) = \zeta_K(s) \cdot L(s,\omega)$

simple pole at $s=1$

$\Rightarrow L(1,\omega) \neq 0$

QED

Siegel's theorem on class numbers of quadratic fields

$K_d = \mathbb{Q}(\sqrt{d})$ a quadratic field $\quad d = \mathrm{disc}(K_d) \quad L_d(s) = \sum_n \left(\frac{d}{n}\right) n^{-s}$

<u>Theorem</u> $\log L_d(1) = o(\log(|d|))$ as $|d| \to \infty$ $\quad$ <u>Recall</u>:

$$L_d(1) = \begin{cases} \pi |d|^{\frac{1}{2}} h(d) & \text{if } d < -4 \\ 2 \cdot |d|^{\frac{1}{2}} h(d) \cdot \log \varepsilon_d & \text{if } d > 0 \end{cases}$$

Logical structure:

<u>Lemma 1</u> Let k be a number field. If $0 < s < 1$ and $\zeta_k(s) \le 0$, then

$$\kappa_k = \mathrm{Res}_{s=1} \zeta_k(s) > s(1-s) \underbrace{2^{-[k:\mathbb{Q}]}}_{\text{can be replaced by } 2^{-[k:\mathbb{Q}]} \cdot \pi^{r_2}} \cdot e^{-2[k:\mathbb{Q}]\pi} \cdot |d_k|^{\frac{s-1}{2}}$$

<u>Note</u>: Proof of Lemma 1 uses the integral representation of $\zeta_k(s)$, or rather $|d_k|^{\frac{s}{2}} \Gamma_{\mathbb{R}}(s)^{r_1} \Gamma_{\mathbb{C}}(s)^{r_2} \zeta_k(s)$

<u>Lemma 2</u>. $L_d(1) < 2 \log |d|$

(Proof by Abel summation)

<u>Note</u>: Lemma 2 $\Rightarrow \limsup_{|d| \to \infty} \frac{\log L_d(1)}{\log |d|} \le 0$

So we must show:

$\forall \varepsilon$ with $0 < \varepsilon < 1$, $\quad L_d(1) \ge |d|^{-\varepsilon}$ if $|d| \gg 0$

$\left(\text{i.e. } \liminf_{|d| \to \infty} \frac{\log L_d(1)}{\log |d|} \ge 0 \right)$

Proof of Theorem (admitting Lemmas 1, 2)

Suppose $\liminf_{|d|\to\infty} \frac{\log L_d(1)}{\log |d|} < 0$

i.e. $\exists\, \varepsilon_1$ with $0<\varepsilon_1<1$ s.t. $L_d(1) < |d|^{-\varepsilon_1}$ for infinitely may fundamental discriminants d

Choose such a fund. discriminant d_1 s.t. $|d_1|^{\frac{\varepsilon_1}{2}} > \frac{4e^{4\pi}}{\varepsilon_1(1-\varepsilon_1)}$

Then $L_{d_1}(1) < \frac{1}{4}\cdot e^{-4\pi}\cdot \varepsilon_1(1-\varepsilon_1)\cdot |d_1|^{-\frac{\varepsilon_1}{2}}$

$\Downarrow$ Lemma 1

$\zeta_{\mathbb{Q}(\sqrt{d_1})}(1-\varepsilon_1) > 0 \implies \exists\, \sigma_1$ with $1-\varepsilon_1<\sigma_1<1$ s.t. $\zeta_{\mathbb{Q}(\sqrt{d_1})}(\sigma_1)=0$

Consider $M = \mathbb{Q}(\sqrt{d_1}, \sqrt{D})$ biquadratic, Let $\mathrm{disc}\, \mathbb{Q}(\sqrt{d_1 D}) = t$. $\exists\, r>0$ s.t. $t\, r^2 = D$

$$\underbrace{\zeta_{\mathbb{Q}(\sqrt{d_1},\sqrt{D})}}_{= M}(s) \underset{\text{C.F.T.}}{=} \zeta_{\mathbb{Q}}(s)\cdot L_{d_1}(s)\cdot L_D(s)\cdot L_t(s)$$

$L_{d_1}(\sigma_1) = 0 \implies \zeta_M(1-\sigma_1) = 0$

$\Downarrow$ Lemma 1

$$L_{d_1}(1)\cdot L_D(1)\cdot L_t(1) > \sigma_1(1-\sigma_1)\cdot 2^{-4}\cdot e^{-8\pi}\, |d\, D\, t|^{\frac{\sigma-1}{2}}$$

$$\xrightarrow{\text{Lemma 2}} L_D(1) > \frac{\sigma_1(1-\sigma_1)\cdot 2^{-4}\cdot e^{-8\pi}}{4\cdot \log|d_1|\, \log|t|}\, |d_1 D t|^{\frac{\sigma-1}{2}} \geq \frac{\sigma_1(1-\sigma_1)\cdot 2^{-6}\cdot e^{-8\pi}}{\log|d_1|\, \log|d_1 D|}\, |d_1 D|^{\sigma-1}$$

$\geq |D|^{-\varepsilon_1}$ for all $|D| \gg 0$ $\because \sigma-1 > -\varepsilon_1$

Have shown: $L_D(1) \geq |D|^{-\varepsilon_1}$ for all fund. discriminant D with $|D|$ sufficiently large. Contradiction!

QED

Outline of strategy

- Key estimate (lemma 1):

  If $0<s<1$ and $\zeta_{\mathbb{Q}(\sqrt{d})}(s)\leq 0$, get lower bound of $\mathrm{Res}_{s=1}\zeta_k(s)$

$$\mathrm{Res}_{s=1}\zeta_k(s) > s(1-s)\, 2^{-[k:\mathbb{Q}]}\cdot e^{-2[k:\mathbb{Q}]\pi}\, |d_k|^{\frac{s-1}{2}}$$

  [Note that the exponent of $|d_k|$ is $\frac{s-1}{2}$]

- Suppose that $L_d(1) < |d|^{-\varepsilon_1}$ for infinitely many fund. disc. d

(1) key estimate $\Rightarrow$ $\exists\, d_1$ with $|d_1|$ large $\zeta_{\mathbb{Q}(\sqrt{d_1})}(1-\varepsilon_1)>0$

(2) Since $\zeta_{\mathbb{Q}(\sqrt{d_1})}(1^-) = -\infty$. $\exists\, 0<\sigma_1<1-\varepsilon_1$ s.t. $\zeta_{\mathbb{Q}(\sqrt{d_1})}(\sigma_1)=0$

equiv. $L_{d_1}(\sigma_1)=0$

$\Rightarrow$ $\zeta_{\mathbb{Q}(\sqrt{d_1},\sqrt{D})}(1-\sigma_1)=0$ $\forall D$. Let $\mathbb{Q}(\sqrt{d_1 D})=\mathbb{Q}(\sqrt{t})$, $t=\mathrm{disc}(\mathbb{Q}(\sqrt{d_1 D}))$

(3) The key estimate gives a lower bound of the product

$$\underbrace{L_{d_1}(1)}_{\leq\, 2\log|d_1|}\cdot L_D(1)\cdot \underbrace{L_t(1)}_{\leq\, 2\log|t|} > \sigma_1(1-\sigma_1)\,2^{-4}e^{-8\pi}\,|dDt|^{\frac{\sigma-1}{2}}$$

$$\geq \sigma_1(1-\sigma_1)\,2^{-4}e^{-8\pi}\,|dD|^{\sigma-1}$$

The exponent of $|dD|$ here is $\sigma-1>-\varepsilon_1$

$\Rightarrow$ $L_D(1) > |D|^{-\varepsilon_1}$ for all D with $|D|$ sufficiently large

More succinctly: The assumption that $L_d(1)<|d|^{-\varepsilon_1}$ for infinitely many fund. disc. d implies (by the key estimate) that $\exists\, d_1$ s.t. $\zeta_{\mathbb{Q}(\sqrt{d_1})}(1-\varepsilon_1)>0$.

This implies that $L_{d_1}(s)$ has a zero between $1-\varepsilon_1$ and 1: call it σ_1

The Key estimate gives a lower bound of $L_{d_1}(1)\cdot L_D(1)\cdot L_t(1)$, the exponent of $|D|$ being $\sigma-1$

While $L_t(1) = O(\log D)$

Hecke's proof of FE for $\zeta_k(s)$ gives: $\qquad n=[k:\mathbb{Q}]$

$$2^{-r_1}\pi^{-\frac{ns}{2}}|d_k|^{\frac{s}{2}}\Gamma(\tfrac{s}{2})^{r_1}\Gamma(s)^{r_2}\cdot\zeta_k(s)$$

$$=-\frac{2^{r_1}h_k\cdot R_k\cdot w_k^{-1}}{s(1-s)}+\sum_{\substack{\mathfrak{a}\subseteq\mathcal{O}_k\\ \text{ideal}}}\int_{Nx\geq 1}\left(Nx^{\frac{s}{2}}+Nx^{\frac{1-s}{2}}\right)\cdot e^{-\pi N\mathfrak{a}^{\frac{2}{n}}\cdot|d_k|^{-\frac{1}{n}}\sigma(x)}\,d^\times x$$

- The integral is taken over the subset $\{x\in\mathbb{R}_{>0}^{r_1+r_2}\mid Nx\geq 1\}$
- For $x=(x_1,\dots,x_{r_1+r_2})\in\mathbb{R}_{>0}^{r_1+r_2}$, $\sigma(x)\overset{\text{def}}{=}x_1+\dots+x_{r_1}+2(x_{r_1+1}+\dots+x_{r_1+r_2})$

  $N(x)=(x_1\cdots x_{r_1})\cdot(x_{r_1+1}^2\cdot\dots\cdot x_{r_1+r_2}^2)$
- $d^\times x=\frac{dx_1}{x_1}\cdots\frac{dx_{r_1+r_2}}{x_{r_1+r_2}}$
- $\sum_{\mathfrak{a}\subseteq\mathcal{O}_k}e^{-\pi N\mathfrak{a}^{\frac{2}{n}}\cdot|d_k|^{-\frac{1}{n}}\sigma(x)}$ $\longleftrightarrow$ F.T. of a translate of the sum over global points of a Schwartz function on $\mathbb{A}_k$

<u>Proof of Lemma 1</u>: Evaluate the above integral repr. at s

- Every term in the sum on the r.h.s is ≥ 0. Take only the term with $\mathfrak{a}=\mathcal{O}_k$
- For the integral $\int_{Nx\geq 1}(Nx^{\frac{s}{2}}+Nx^{\frac{1-s}{2}})\,e^{-\pi\cdot|d_k|^{-\frac{1}{n}}\sigma(x)}\,d^\times x$

  integrate over $\{|d_k|^{\frac{1}{n}}\leq x_i\leq 2|d_k|^{\frac{1}{n}}\ \forall i=1,\dots,r_1+r_2\}$

$$\rightsquigarrow\left.\begin{array}{l}Nx^{\frac{s}{2}}+Nx^{\frac{1-s}{2}}\geq|d_k|^{\frac{s}{2}}\\ \sigma x\leq 2n\,|d_k|^{\frac{1}{n}}\end{array}\right\}\Rightarrow e^{-\pi|d_k|^{-\frac{1}{n}}\cdot\sigma(x)}\geq e^{-2n\pi}$$

$$\rightsquigarrow\int_{Nx\geq 1}\underbrace{\left(Nx^{\frac{s}{2}}+Nx^{\frac{1-s}{2}}\right)}_{\geq|d_k|^{\frac{s}{2}}}e^{-\pi N\mathfrak{a}^{\frac{2}{n}}\cdot|d_k|^{-\frac{1}{n}}\cdot\sigma(x)}\,d^\times x\geq|d_k|^{\frac{s}{2}}e^{-2n\pi}\int_{\{|d_k|^{\frac{1}{n}}\leq x_i\leq 2|d_k|^{\frac{1}{n}}\}}d^\times x=$$

$$=|d_k|^{\frac{s}{2}}e^{-2n\pi}\cdot(\log 2)^{r_1+r_2}$$

$$\geq|d_k|^{\frac{s}{2}}\cdot e^{-2n\pi}\cdot 2^{-n}$$

q.e.d

Proof of Lemma 2

Let $S(m) = \sum_{j=1}^{m} \left(\frac{d}{j}\right) \quad \forall m \in \mathbb{N}$

Clearly $|S(m)| \leq \frac{|d|}{2} \quad \forall m$

$$L_d(1) = \sum_{m=1}^{\infty} \left(S(m) - S(m-1)\right)\frac{1}{m} = \sum_{m=1}^{\infty} S(m)\left(\frac{1}{m} - \frac{1}{m+1}\right)$$

$$\rightsquigarrow \left|L_d(1)\right| \leq \sum_{m=1}^{|d|-1} m\left(\frac{1}{m} - \frac{1}{m+1}\right) + \sum_{m=|d|}^{\infty} \frac{|d|}{2}\left(\frac{1}{m} - \frac{1}{m-1}\right)$$

$$\leq \log|d| + \frac{|d|}{2}\cdot\frac{1}{|d|} = \log|d| + \frac{1}{2} < 2\log|d|$$

$$\left(\because\ |d| \geq 4\right)$$

q.e.d.