Math 620 Algebraic Number Theory. Fall 2016

What is algebraic number theory
- study arithmetic properties of number fields (and other fields / rings), and systems of polynomial equations over number fields
- zeta and L-functions play an important role (miracle of L-functions / automorphic forms / Langlands program)

What is "basic number theory" / What are covered in a standard graduate course

1. basic properties of number rings, up to finiteness of class numbers. (Including: local fields, global fields, different and discriminant)
2. adeles and idels, idele class group, compactness $\leftrightarrow$ finiteness

(2a) i.e. How to think in terms of adels

(2b) abelian L-functions and function equations via Tate's method

(2c) non-vanishing of L-values + applications

3. local and global class field theory

Will do 1 + 2a+2b this semester. Will state main theorems of class field theory

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Dedekind domains:

- Def DVR

Equivalent characterization of DVR, $(R, \mathfrak{m})$ noetherian local integral domain

(i) $\mathfrak{m}$ is generated by a non-nilpotent element

(ii) R is integrally closed in its fraction field, and R has exactly one non-zero prime ideal

Dedekind domains A = noetherian integral domain

$A_{\mathfrak{p}}$ is a DVR $\forall$ non-zero prime ideal $\mathfrak{p} \subseteq A$

$\iff$ A is integrally closed in $\mathrm{frac}(A)$ and $\dim(A) \leq 1$

i.e. every non-zero prime ideal is maximal

The above are defining properties of Dedekind domains.

Examples. $\mathcal{O}_K$ K = number field

- K = a finitely generated field over $\mathbb{F}_p$, of trans. degree 1.

$\mathcal{O}$ = a finitely generated over K and integrally closed in K subring of K

e.g. the integral closure of $\mathbb{F}_p[t]$ in K, where t = a elt of K, transcendental over $\mathbb{F}_p$

def global fields = finite extensions of $\mathbb{Q}$ or $\mathbb{F}_p[t]$

(strong) Approximation in a Dedekind domain

A: Dedekind domain. $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ distinct maximal ideals in A

$x_1, \dots, x_r \in K := \mathrm{frac}(A)$, $n_1, \dots, n_r \in \mathbb{N}$

Then: $\exists\, x \in K$ s.t. $v_{\mathfrak{p}_i}(x - x_i) \geq n_i$ for $i = 1, \dots, r$

AND $v_{\mathfrak{p}}(x) \geq 0$ $\forall$ maximal ideal $\mathfrak{p}$ of A different from $\mathfrak{p}_1, \dots, \mathfrak{p}_r$

Behavior under finite extensions

set-up L/K: finite extension of fields

$A \subseteq K$ Dedekind domain, $\mathrm{frac}(A) = K$,

B = integral closure of A in B

Prop 1) If L/K is separable, then B is a finite A-module

2) If B is a finite A-module, then B is a Dedekind domain

In particular, if $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ are prime ideals of B s.t. $\mathfrak{p}_1 \cap A = \mathfrak{p}_2 \cap A$, then $\mathfrak{p}_1 = \mathfrak{p}_2$

Note This finiteness assumption is automatically satisfied if K is a global field

Assume B is a finite A-module (finite locally free)

Prop: $\mathfrak{p}$ = maximal ideal of A $\quad \mathfrak{p}B = \prod_{i=1}^{r} \mathfrak{P}_i^{e_i}$

where $\mathfrak{P}_1, \dots, \mathfrak{P}_r$ are maximal ideals of B lying above $\mathfrak{p}$

Then: $[L:K] = \sum_{i=1}^{r} e_i f_i$ $\quad$ where $f_i = f_{\mathfrak{P}_i} = [B/\mathfrak{P}_i : A/\mathfrak{p}]$

In particular $\forall$ maximal ideal $\mathfrak{p}$ of A. $\exists$ a maximal ideal $\mathfrak{P}$ of B s.t. $\mathfrak{P} \cap A = \mathfrak{p}$

Pf: $\underset{\text{define and}}{[L:K]} = [B:A] = \mathrm{length}_A(B \otimes (A/\mathfrak{p})) = \mathrm{length}_A(B/\mathfrak{p}B) = \sum_{i=1}^{r} e_i \cdot \overbrace{\mathrm{length}_A(B/\mathfrak{P}_i)}^{f_i}$

How to think about the norm map

$$\begin{matrix} L & \supseteq & B \\ | & & | \\ K & \supseteq & A \end{matrix}$$

as before

defn $\mathcal{C}_A = \{A\text{-modules of finite length}\}$

$$\mathcal{C}_A \xrightarrow{\text{essentially surjective}} K(\mathcal{C}_A)$$

I_A = the group of fin. gen. A-submodules in K

i.e. the group of fractional ideals of A

$\cup$

I_A^+ = the semi-group of non-zero ideals of A

Have:

$$\begin{tikzcd} K(\mathcal{C}_A) \arrow[r, "\chi_A", "\sim"'] & I_A \\ Ob(\mathcal{C}_A) \arrow[u] \arrow[r] & I_A^+ \arrow[u] \end{tikzcd}$$

I_A = free abelian group with free generators $\{\mathfrak{p} \mid \mathfrak{p}: \text{non-zero max. ideals of } A\}$

Now: back to the set-up

$$\begin{matrix} B & - & L \\ | & & | \\ A & - & K \end{matrix}$$

B: finite A-module

Define norm map $\quad \mathcal{C}_B \xrightarrow{N_{B/A}} \mathcal{C}_A \quad \rightsquigarrow \quad K(\mathcal{C}_B) \xrightarrow{N_{B/A}} K(\mathcal{C}_A)$

regarding a finite length B-module as a finite length A-module

Alternative def^n $I_A = \left\{\begin{array}{l}\text{the } K\text{-group attached to the category of finitely} \\ \text{generated projective } A\text{-modules}\end{array}\right\} / \left\{\begin{array}{l}\text{finite free} \\ A\text{-modules}\end{array}\right\}$

Have
$$\begin{array}{ccc} I_A & \xrightarrow{i_{B/A}} & I_B \\ \wr\| & \xleftarrow[N_{B/A}]{} & \wr\| \\ K(\mathcal{C}_A) & & K(\mathcal{C}_B) \end{array}$$
$i_{B/A} : \underset{\substack{\uparrow \\ \text{fractional} \\ A\text{-ideal in } K}}{[I]} \longmapsto [I \underset{A}{\otimes} B]$

description of $i_{B/A}$ in terms of $K(\mathcal{C}_A) \to K(\mathcal{C}_B)$

$$\mathcal{C}_A \ni M \longmapsto M \underset{A}{\otimes} B$$

Note: B is faithfully flat over A, so the functor $M \mapsto M \underset{A}{\otimes} B$ is exact

Prop: (i) M: finite length B-module

$\Rightarrow \underset{\substack{\uparrow \\ M \text{ regarded} \\ \text{as an } A\text{-module}}}{\chi_A(M)} = N_{B/A}(\chi_B(M))$ restate def^n

(ii) N: finite length A-module restate def^n

$\Rightarrow i_{B/A}(\chi_A(N)) = \chi_B(N \underset{A}{\otimes} B)$

(iii) $N_{B/A}([x \cdot B]) = [N_{L/K}(x) \cdot A]$ in I_B

$\forall x \in L^\times$

Proof of (iii) Reduction (1): suffices to pass to completion: $A = \hat{A}_{\mathfrak{q}}$ $B = B \underset{A}{\otimes} \hat{A}_{\mathfrak{q}}$

Reduction (2) May assume $B = B^\wedge_{\mathfrak{p}}$

$\rightsquigarrow$ May assume $x = \pi_B \underset{\substack{\uparrow \\ \text{must show}}}{\overset{\text{reduce to showing}}{\rightsquigarrow}} N_{L/K}(\pi_B) = \pi_{\mathfrak{p}}^{[L:K]} \cdot (\text{unit in } B)$

By def^n: $\mathrm{ord}_{\mathfrak{q}}(N_{L/K}(\pi_B)) = \mathrm{ord}_{\mathfrak{q}}\left(\det\left(\text{mult. by } \pi_B \text{ on } B\right)\right)$

$$\text{length}_A(B/\pi_B A) = f_{\mathfrak{p}/\mathfrak{q}}$$

basic linear algebra
i.e. using suitable two bases of $\hat{B}$ of B to diagonalize π_B

Remark: 1) The free abelian group $K(\mathcal{C}_A)$ is the group of Weil divisors on Spec A

2) The natural isomorphism is

notation $\widetilde{c}_1$ → $-\chi_A : K(\mathcal{C}_A) \longrightarrow I_A$ = Cartier divisors on Spec(A)

$$[A/\mathfrak{p}] \longmapsto \mathfrak{p}^{-1}$$

which attaches a Weil divisor on Spec A the corresponding Cartier divisor

3) The homomorphism

$$K(\mathcal{C}_A) \xrightarrow{-\chi_A} I_A \to H_A$$

is the first Chern class map

4) Let $K(A)$ = the K-group of all finitely generated A-modules

Have $rk : K(A) \longrightarrow \mathbb{Z}$

(a) $\mathrm{Ker}(rk) =: K(A)^\circ$ is isomorphic to H_A

$$\begin{array}{ccccc} K(\mathcal{C}_A) & \twoheadrightarrow & K(A)^\circ & \xrightarrow{\sim} & H_A \\ \Psi & & \Psi & & \\ [A/\mathfrak{p}] & \mapsto & [A/\mathfrak{p}] = [1] - [\mathfrak{p}] & \mapsto & [\mathfrak{p}^{-1}] \end{array}$$

5) The multiplicative structure on $K(A)$ is quite trivial: the product of any two elements of $K(A)^\circ$ is 0

$$\because \quad ([1]-[\mathfrak{p}])\cdot([1]-[\mathfrak{q}]) = [1]-[\mathfrak{p}]-[\mathfrak{q}]+[\mathfrak{p}\mathfrak{q}]$$

while a dévissage shows the the RHS is 0.

Reset notation

A: Dedekind domain

$C(A)$ = the abelian category of finite length A-modules

$$K(\mathcal{C}_A) = \bigoplus_{\mathfrak{p}:\text{ max ideal of } A} \mathbb{Z}\cdot[A/\mathfrak{p}] \quad ; \quad I_A = \bigoplus_{\mathfrak{p}:\text{max. ideal}} \mathfrak{p}^{\mathbb{Z}}$$

Cartier divisor = associated to a Weil divisor

canonical $\tilde{c}$: lift of 1st Chern class

$$\tilde{c} : K(\mathcal{C}_A) \xrightarrow{\sim} I_A$$

$$[A/\mathfrak{p}] \longmapsto \tilde{c}([A/\mathfrak{p}]) = \mathcal{O}([A/\mathfrak{p}]) = \mathfrak{p}^{-1}$$

$$\underbrace{\bigoplus n_{\mathfrak{p}} [A/\mathfrak{p}]}_{= D} \longmapsto \tilde{c}(D) = \mathcal{O}(D) = \prod_{\mathfrak{p}} \mathfrak{p}^{-n_{\mathfrak{p}}}$$

$$K^\times \ni A \rightsquigarrow \mathrm{div}(f) = \prod_{\mathfrak{p}\subseteq A} \mathrm{ord}_{\mathfrak{p}}(f)\cdot[A/\mathfrak{p}]$$

integral closure of A in L

$$\begin{array}{ccc} B & \subset & L \\ | & & |\ \text{finite} \\ A & \subset & K \end{array}$$

Assume B: finite

<u>Lemma</u> $\tilde{c}(\mathrm{div}(f)) = f^{-1} \cdot A$ $\forall f \in K^\times$

<u>Have</u> 1) $K(A) \underset{N_{B/A}}{\overset{i_{B/A}}{\rightleftarrows}} K(B)$

def: $i_{B/A}: \sum m_{\mathfrak{q}} [A/\mathfrak{q}] \longrightarrow \sum_{\mathfrak{q} \subseteq A \text{ max. ideal}} \sum_{\mathfrak{P}/\mathfrak{q}} m_{\mathfrak{q}} \cdot e(\mathfrak{P}/\mathfrak{q}) \cdot [B/\mathfrak{P}]$

$\mathfrak{q}$: max. ideal of A

$[M] \longmapsto [B \otimes_A M]$

finite length A-module

$[N]_A \longleftarrow [N]_B : N_{B/A}$

N regarded as an A-module; finite length B-module

$\sum_{\mathfrak{q} \subseteq A,\ \mathfrak{P}/\mathfrak{q}} n_{\mathfrak{P}} f(\mathfrak{P}/\mathfrak{q}) \cdot [\mathfrak{q}] \longleftarrow \sum n_{\mathfrak{P}} [B/\mathfrak{P}]$

2) $I_A \underset{N_{B/A}}{\overset{i_{B/A}}{\rightleftarrows}} I_B$

$\mathfrak{P} \cap A \to \mathfrak{q}^{f(\mathfrak{P}/\mathfrak{q})} \longleftarrow \mathfrak{P}$

$\mathfrak{q}^{f(\mathfrak{P}/\mathfrak{q})} \in I_A$, $\mathfrak{P} \in I_B$

$i_{B/A}: \mathfrak{q} \longmapsto \prod_{\mathfrak{P}/\mathfrak{q}} \mathfrak{P}^{e(\mathfrak{P}/\mathfrak{q})}$

$\prod_{\mathfrak{q}} \mathfrak{q}^{n_{\mathfrak{q}}} \longmapsto \prod_{\mathfrak{q}} \prod_{\mathfrak{P}/\mathfrak{q}} \mathfrak{P}^{m_{\mathfrak{q}} e(\mathfrak{P}/\mathfrak{q})}$

<u>Prop</u>: (1) The following diagrams commute

$$\begin{array}{ccc} K(A) & \xrightarrow[\tilde{c}]{\sim} & I_A \\ {\scriptstyle i_{B/A}}\downarrow\uparrow{\scriptstyle N_{BA}} & & {\scriptstyle N_{B/A}}\uparrow\downarrow{\scriptstyle i_{B/A}} \\ K(B) & \xrightarrow[\tilde{c}]{\sim} & I_B \end{array}$$

$\tilde{c} = \tilde{c}_1$

(2) $\forall f \in K^\times$, have $i_{B/A}(\mathrm{div}_A(f)) = \mathrm{div}_B(f)$ <u>and</u> $i_{B/A}(fA) = fB$

(3) $\forall g \in L^\times$, have

$N_{B/A}(\mathrm{div}_B(g)) = \mathrm{div}(N_{L/K}(g))$ in $K(A)$ and $N_{B/A}(gB) = N_{L/K}(g) \cdot A$ in I_A

(4) $$\underbrace{K(A) \xrightarrow{i_{B/A}} K(B) \xrightarrow{N_{B/A}} K(A)}_{[L:K]}$$

$$\underbrace{I_A \xrightarrow{i_{B/A}} I_B \xrightarrow{N_{B/A}} I_A}_{[L:K]}$$

Exercise In general $i_{B/A} \circ N_{B/A} \neq [L:B]$

(5) If L/K is finite Galois, then
$$i_{B/A} \circ N_{B/A}(J) = \prod_{\sigma \in Gal(L/K)} \sigma(J)$$

Exer: (a) $i_{B/A}$ is injective but not surjective unless $A \xrightarrow{\sim} B$

(b) In general $N_{B/A}$ is neither injective nor surjective

Assume L/K finite Galois $\Rightarrow$ B = integral closure of A in L is a finite A-module locally free i.e. proj.

Proposition $\forall \mathfrak{z} \underset{\text{max. ideal}}{\subseteq} A$

(1) $\Gamma = Gal(L/K)$ operates <u>transitively</u> on $\{\mathcal{P} \underset{\text{max. ideal}}{\subseteq} B \mid \mathcal{P} \cap A = \mathfrak{z}\}$

Def $D(\mathcal{P}/\mathfrak{z}) = \{\sigma \in Gal(L/K) \mid \sigma(\mathcal{P}) = \mathcal{P}\}$

Def $I(\mathcal{P}/\mathfrak{z}) = Ker\left(D(\mathcal{P}/\mathfrak{z}) \xrightarrow[can]{} Gal(\overbrace{\kappa(\mathcal{P})}^{B/\mathcal{P}} / \overbrace{\kappa(\mathfrak{z})}^{A/\mathfrak{z}})\right)$

(1) $\Rightarrow$ $D(\sigma(\mathcal{P})/\mathfrak{z}) = \sigma \cdot D(\mathcal{P}/\mathfrak{z}) \cdot \sigma^{-1}$, $I(\sigma(\mathcal{P})/\mathfrak{z}) = \sigma \cdot I(\mathcal{P}/\mathfrak{z}) \cdot \sigma^{-1}$ $\forall \sigma \in \Gamma$

Def

L

$|$

$K^{I(\mathcal{P}/\mathfrak{z})}$ = fixed subfield of $I(\mathcal{P}/\mathfrak{z})$

$|$

$K^{D(\mathcal{P}/\mathfrak{z})}$ = fixed subfield of

$|$

K

$\text{Card}\{Q \underset{\text{max. ideal}}{\subseteq} B \text{ lying above } \mathfrak{z}\} = [K^{D(\mathcal{P}/\mathfrak{z})} : K]$ by (1)

(2) $[K^{I(\mathcal{P}/\mathfrak{z})} : K^{D(\mathcal{P}/\mathfrak{z})}] = [\kappa(\mathcal{P}) : \kappa(\mathfrak{z})]_{sep}$ = separable degree

$Gal(K^{I(\mathcal{P}/\mathfrak{z})}/K^{D(\mathcal{P}/\mathfrak{z})}) \xrightarrow{\sim} Aut(\kappa(\mathcal{P})/\kappa(\mathfrak{z})) = Gal(\underset{\uparrow}{\kappa(\mathcal{P})_{sep}}/\kappa(\mathfrak{z}))$

max. separable subext^n of $\kappa(\mathcal{P})/\kappa(\mathfrak{z})$

(3) $\underset{\text{Card}(I(\mathcal{P}/\mathfrak{z}))}{\underset{\parallel}{[L : K^{I(\mathcal{P}/\mathfrak{z})}]}} = e(\mathcal{P}/\mathfrak{z}) \cdot [\kappa(\mathcal{P}) : \kappa(\mathfrak{z})]_{p.insep}$

Def: $\mathcal{P}/\mathfrak{z}$ is unramified $\Longleftrightarrow$ $e(\mathcal{P}/\mathfrak{z}) = 1$ <u>and</u> $\kappa(\mathcal{P})/\kappa(\mathfrak{z})$ is $\underset{\downarrow}{\overset{\text{automatic}}{\text{finite}}}$ separable

(4) If $\mathcal{P}/\mathfrak{z}$ is unramified, then $can: D(\mathcal{P}/\mathfrak{z}) \xrightarrow{\sim} Gal(\kappa(\mathcal{P})/\kappa(\mathfrak{z}))$

Def Arithmetic Frob. $Fr_{\mathcal{P}/\mathfrak{z}}$ = the element of $D(\mathcal{P}/\mathfrak{z})$ which induces $x \mapsto x^{\text{Card}(\kappa(\mathfrak{z}))}$ $\forall x \in \kappa(\mathcal{P})$

$Fr_{\mathcal{P}/\mathfrak{z}} \longmapsto (x \mapsto x^{\text{Card}(\kappa(\mathfrak{z}))})$ $\forall x \in \kappa(\mathcal{P})$

Suppose $\kappa(\mathfrak{p})$ finite

(5) $\forall \sigma \in \Gamma$, have $\mathrm{Fr}_{\sigma(\mathfrak{P})/\mathfrak{p}} = \sigma \cdot \mathrm{Fr}_{\mathfrak{P}/\mathfrak{p}} \cdot \sigma^{-1}$

(Without the assumption that $\kappa(\mathfrak{p})$ is finite, then one can "only" assert that $D(\sigma(\mathfrak{P})/\mathfrak{p}) = \sigma \cdot D(\mathfrak{P}/\mathfrak{p}) \cdot \sigma^{-1}$

$I(\sigma(\mathfrak{P})/\mathfrak{p}) = \sigma \cdot D(\mathfrak{P}/\mathfrak{p}) \cdot \sigma^{-1}$)

In particular $\mathrm{F}_{\sigma(\mathfrak{P})/\mathfrak{p}}$ and $\mathrm{Fr}_{\mathfrak{P}/\mathfrak{p}}$ are conjugate in Γ

Def Let L/K be a finite Galois ext^n.

$\mathfrak{a} \subseteq K$ fractional ideal in K, i.e. $\mathfrak{a}$ is an invertible $\mathcal{O}_K$-module

(i) $\mathfrak{a}$ is unramified in L/K if $\forall \mathfrak{p}$: max. ideal of $\mathcal{O}_K$ with $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{a}) > 0$ have $\mathfrak{p}$ is unramified in L/K

(ii) Define $\left(\frac{L/K}{\mathfrak{a}}\right) = \prod_{\mathfrak{p}} \mathrm{Fr}_{\mathfrak{p}}^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{a})} \in Gal(L^{ab}/K) = Gal(L/K)^{ab}$

$\forall$ fractional ideal $\mathfrak{a}$ in K unramified in L/K

Consequence of global class field theory (and a major part of it):

$\forall$ finite abelian ext^n L/K of nber fields $\exists$ an ideal $\mathfrak{f}$ of $\mathcal{O}_K$ whose support contains all ramified primes of L/K such that

$$\left(\frac{L/K}{x\mathcal{O}_K}\right) = 1 \text{ in } Gal(L/K)$$

$\forall x \in K^{\times}$ s.t. $x-1 \equiv 0 \mod \mathfrak{f}$

(i.e $v_{\mathfrak{p}}(x-1) \geq v_{\mathfrak{p}}(\mathfrak{f})$ $\forall$ max ideal $\mathfrak{p} \subseteq \mathcal{O}_K$ ramified in $\mathcal{O}_L/\mathcal{O}_K$)

AND $\alpha(x) > 0$ $\forall$ real embedding $\alpha: K \hookrightarrow \mathbb{R}$

Homework assignments A: Dedekind domain

$K^0(A) =$ K-group attached to all fin. gen. proj. A-modules

$K_0(A) =$ K-group attached to all finitely generated A-modules

Fact 1) $\forall$ surjection $A^{\oplus m} \overset{\beta}{\twoheadrightarrow} M$ of A-modules,

Exer $\mathrm{Ker}(\beta)$ is a finitely generated projective A-module

Exer 2) Show that the natural map $K^0(A) \longrightarrow K_0(A)$ is an isomorphism

3) Show that the natural map $K(\mathcal{C}(A)) \longrightarrow K_0(A)$ is a surjection

4) Define $K^0(A)^0 = \mathrm{Ker}\,(K^0(A) \xrightarrow{rk} \mathbb{Z})$

Show that the map

$$\begin{array}{ccc} I_A & \longrightarrow & K^0(A)^0 \\ \Downarrow & & \Downarrow \\ J & \longmapsto & \langle J\rangle - \langle A\rangle \end{array}$$

invertible A-submodule in $\mathrm{frac}(A)$

induces a group isomorphism $H_A \xrightarrow[\sim]{\delta} K^0(A)^0$

Note/example $\mathfrak{p}, \mathfrak{q}$: distinct max ideals ↝ have exact sequence

$$0 \to \mathfrak{p}\cdot\mathfrak{q} \to \mathfrak{p} \oplus \mathfrak{q} \longrightarrow A \to 0$$

↝ $\langle \mathfrak{p}\rangle + \langle \mathfrak{q}\rangle = \langle \mathfrak{p}\cdot\mathfrak{q}\rangle + \langle A\rangle \Leftrightarrow \langle \mathfrak{p}\rangle - \langle A\rangle + \langle \mathfrak{q}\rangle - \langle A\rangle = \langle \mathfrak{p}\cdot\mathfrak{q}\rangle - \langle A\rangle$ in $K^0(A)$

5) Tensor product of fin. gen. projective A-modules induces a natural commutative ring structure on $K^0(A)$.

(a) show that $K^0(A)^0$ is an ideal in $K^0(A)$

and $K^0(A) = K^0(A)^0 \oplus \mathbb{Z}\cdot\langle A\rangle$ as abelian groups

(b) show that the product of any two elements of $K^0(A)^0$ is 0.

Note 6) The equality $\langle \mathfrak{p}\rangle + \langle A/\mathfrak{p}\rangle - \langle A\rangle = 0$ hold in $K(A)$

for every max ideal $\mathfrak{p} \subseteq A$ (and similarly $\forall$ non-zero ideal of I)

Consequently the image of the natural map $K(\mathcal{C}(A)) \longrightarrow K(A)$ lies in $K^0(A)^0$

Or rather directly $K(A) \overset{\wr}{\leftarrow} K^0(A)$

Show that the resulting map $K(\mathcal{C}(A)) \xrightarrow{\gamma} K^0(A)^0$ is a surjection

and the diagram commutes

$$\begin{tikzcd} K(\mathcal{C}(A)) \arrow[r, two heads, "\gamma"] \arrow[d, "\tilde{c}_1", "\wr"'] & K^0(A)^0 \\ I_A \arrow[r, two heads] & H_A \arrow[u, "\delta"', "\wr"] \end{tikzcd}$$

(7) "determinant"

Show that for any two finite projective resolution

$0 \to P_m \to \cdots \to P_1 \to P_0 \twoheadrightarrow M \to 0$ of a finite length A-module M

$0 \to Q_n \to \cdots \to Q_1 \to Q_0 \twoheadrightarrow M \to 0$.

There exists a functorial isomorphism $\underset{\beta(Q_\bullet, P_\bullet)}{\underset{\parallel}{\beta}} : \bigotimes_{i=0}^{m} \Lambda^{top}(P_i)^{(-1)^{i-1}} \xrightarrow{\sim} \bigotimes_{j=0}^{n} \Lambda^{top}(Q_j)^{(-1)^{j-1}}$

(Construct it!)

of rank-one projective A-modules

add remark p.11

Completions and local fields

(non-archimedian)

def/notation

$0 \to \mathcal{O}_K^\times \to K^\times \xrightarrow[\mathrm{ord}_K]{} \mathbb{Z} \to 0$

$\mathfrak{m}_K = \mathrm{ord}_K^{-1}(\mathbb{Z}_{\geq 0}) \cup \{0\}$

local fields = complete discretely valued fields

Lemma / Exer. A local field K is locally compact iff $\mathcal{O}_K/\mathfrak{m}_K$ is finite.

Def K: locally compact local field. The normalized absolute value $\|\cdot\|$ is defined by

$$\mu(xE) = \|x\|\mu(E) \quad \forall \text{ measurable subset } E \subseteq K$$

where μ is a Haar measure on K.

(def) / **Exer** Alternative definition

Let $q = q_K = \mathrm{Card}\, \mathcal{O}_K/\mathfrak{m}_K$

$$\forall x \in K^\times, \quad \|x\| = q_K^{-\mathrm{ord}_K(x)}$$

Prop K: local field, L = finite ext^n of K

(i) $\mathcal{O}_L$ is a free $\mathcal{O}_K$-module of rank $[L:K]$, and $\mathcal{O}_L$ is a complete d.v.r

(ii) $\exists!$ discrete valuation (with values in $\mathbb{Q}$) extending ord_K

Consequently: $\mathrm{ord}_K(x) = \mathrm{ord}_K(\sigma(x))$ $\forall x \in K^{alg}$, $\forall \sigma \in \mathrm{Aut}(K^{alg}/K)$

Remark: Why is local field useful? Or, how to use local fields to study global fields?

"Answer": K = number fields $\mathcal{O}_K \subseteq K$

$\mathfrak{p}$ = max. ideal of $\mathcal{O}_K$

The key fact is that $\mathcal{O}_{K,\mathfrak{p}} \longrightarrow \mathcal{O}^{\wedge}_{K,\mathfrak{p}} := \varprojlim_n \mathcal{O}_K/\mathfrak{p}^n$ is faithfully flat

Thus: get local information about $\mathcal{O}_K$ and modules over $\mathcal{O}_K$ by passing to $\mathcal{O}^{\wedge}_{K,\mathfrak{p}}$.

"Life becomes easier for local fields (in comparison with global fields).

Notation: $v_L(x) = \mathrm{ord}_L(x)$ discrete valuation on $\mathcal{O}_L$ s.t $\mathrm{ord}_L(L^\times) = \mathbb{Z}$

$[L:K] = e_{L/K} \cdot f_{L/K}$ alternatively $e_{L/K} = e(L/K)$, $f_{L/K} = f(L/K)$

Lemma / Exer.

(i) $v_L(x) = \frac{1}{f} v_K(N_{L/K}(x)) \ \forall x \in L^\times$

(ii) Suppose that K is locally compact, then
$$\|x\|_L = \|N_{L/K}(x)\|_L \quad \forall x \in L^\times$$

Note: This is obvious if $x \in \mathcal{O}_L$:

$$\|x\| = \mathrm{Card}(\mathcal{O}_L/x\mathcal{O}_L) = \mathrm{Card}\ N_{L/K}\Big(\text{the class of } \underbrace{\mathcal{O}_L/x\mathcal{O}_L}_{\text{finite length } \mathcal{O}_L\text{-module}} \text{ in } K(\mathcal{C}(\mathcal{O}_L))\Big)$$
$$= \mathrm{Card}(\mathcal{O}_K / N_{L/K}(x)\cdot \mathcal{O}_K)$$

Krasner's Lemma

say $\mathbb{Q}$-valued

K = complete under a valuation (not nec. discrete)

$\alpha, \beta \in K^{alg}$ α separable over $K(\beta)$

If $|\beta - \alpha| < |\sigma(\alpha) - \alpha|$ $\forall$ K-linear embedding $\sigma : K(\alpha) \hookrightarrow K^{alg}$, $\sigma \neq id$

suffices to assume the inequality for all $K(\beta)$-linear embedding $\sigma : K(\beta, \alpha) \to K(\beta)^{alg}$

("β is very close to α")

then $K(\alpha) \subset K(\beta)$ $\leftarrow$ $\forall \beta$ separable over $K(\alpha)$ and close to α

Pf: $\forall \tau : K(\alpha, \beta) \xrightarrow[K(\beta)\text{-linear}]{} K(\beta)^{alg}$

Have: $|\beta - \alpha| \underset{\uparrow}{=} |\beta - \tau(\alpha)|$

uniqueness of valuation extending v_K

So: $|\tau(\alpha) - \alpha| = |\tau(\alpha) - \beta + \beta - \alpha| \leq \max(|\tau(\alpha) - \beta|, |\beta - \alpha|)$

$= |\beta - \alpha| < |\tau(\alpha) - \alpha|$ unless $\tau(\alpha) = \alpha$.

Q.E.D.

Hensel's Lemma: How to solve polynomial equations in complete fields.

One variable version

Given $f(T) \in \mathcal{O}_K[T]$, $\alpha_0 \in \mathcal{O}_K$ s.t. $|f(\alpha_0)| < |f'(\alpha_0)^2|$

("α_0 is an approximate root")

$\exists!$ $\alpha \in \mathcal{O}_K$ s.t. $f(\alpha) = 0$ and $\alpha \equiv \alpha_0 \left(\text{mod } \frac{f(\alpha_0)}{f'(\alpha_0)}\right)$

Pf. Construct a sequence $(\alpha_i)_{i \geq 1}$

$$\alpha_{i+1} = \alpha_i - \frac{f(\alpha_i)}{f'(\alpha_i)}$$

Note $\left|\frac{f(\alpha_0)}{f'(\alpha_0)}\right| < |f'(\alpha_0)| \leq 1$

Inductive assumption:

$$\left|\frac{f(\alpha_i)}{f'(\alpha_i)^2}\right| \leq \left|\frac{f(\alpha_0)}{f'(\alpha_0)^2}\right|^{2^i}$$ obviously holds for $i=0$

$\leadsto$ $$f(\alpha_{i+1}) = f\left(\alpha_i - \frac{f(\alpha_i)}{f'(\alpha_i)}\right) \equiv f(\alpha_i) - \frac{f(\alpha_i)}{f'(\alpha_i)} \cdot f'(\alpha_i) \mod \left(\frac{f(\alpha_i)}{f'(\alpha_i)}\right)^2$$

So $$|f(\alpha_{i+1})| \leq \left|\frac{f(\alpha_i)}{f'(\alpha_i)}\right|^2$$

Must understand $f'(\alpha_{i+1})$

$$f'(\alpha_{i+1}) = f'(\alpha_i) + \text{"}\frac{f''}{2}(\alpha_i)\text{"} \cdot \frac{f(\alpha_i)}{f'(\alpha_i)} \mod \left(\frac{f(\alpha_i)}{f'(\alpha_i)}\right)^2$$

Let $c = \left|\frac{f(\alpha_0)}{f'(\alpha_0)^2}\right| < 1$ $\qquad$ $\left|\frac{f(\alpha_i)}{f'(\alpha_i)}\right| \leq c^{2^i} \cdot |f'(\alpha_i)| < |f'(\alpha_i)|$

$\Longrightarrow$ $f'(\alpha_{i+1})$

Note [induction hypothesis: $\left|\frac{f(\alpha_i)}{f'(\alpha_i)^2}\right| \leq c^{2^i}$

So $$\left|\left(\frac{f(\alpha_i)}{f'(\alpha_i)}\right)^2\right| \leq c^{2^{i+1}} |f'(\alpha_i)|^2$$

Have $$|f'(\alpha_{i+1})| = |f'(\alpha_i)|$$

$$|f(\alpha_{i+1})| \leq \left|\frac{f'(\alpha_i)}{f'(\alpha_i)}\right|^2 \leq c^{2^{i+1}} \cdot |f'(\alpha_i)|^2$$

i.e. $$\left|\frac{f(\alpha_{i+1})}{f'(\alpha_{i+1})^2}\right| \leq c^{2^{i+1}} = \left|\frac{f(\alpha_0)}{f'(\alpha_0)^2}\right|^{2^{i+1}}$$

. Have shown $\left|\frac{f(\alpha_i)}{f(\alpha_i)^2}\right| \leq \left|\frac{f(\alpha_0)}{f(\alpha_0)^2}\right|^{2^i}$ $\forall i \geq 0$

Note also $$\left|\frac{f(\alpha_i)}{f'(\alpha_i)}\right| \leq c^{2^i} \cdot |f'(\alpha_i)| \quad \forall_i$$

$\Longrightarrow$ $\alpha = \lim_{i\to\infty} \alpha_i$ exists in $\mathcal{O}_K$ and $f(\alpha) = 0$ $\qquad$ Q.E.D.

$\alpha \equiv \alpha_0 \left(\mod \frac{f(\alpha_0)}{f'(\alpha_0)}\right)$

Exercise: Formulate and prove a Hensel's Lemma in several variables

$$\alpha_0 \in \mathcal{O}_K^n \qquad F = \begin{bmatrix} f_1(T_1, \dots, T_n) \\ \vdots \\ f_n(T_1, \dots, T_n) \end{bmatrix} \in \mathcal{O}_K[T_1, \dots, T_n]^n$$

$\mathcal{O}_K$: complete d.v.r. $\qquad \partial F = \left[\dfrac{\partial f_i}{\partial T_j} \right]_{1 \le i, j \le}$

Assume: $F(\alpha_0) \equiv 0 \pmod{\pi_K^{2d+r}}$ $\qquad \operatorname{ord}_K(\underbrace{\det(\partial F(\alpha_0))}_{\in K^\times}) = d$

$$r > 0 \quad (\text{i.e. } r \ge 1)$$

Inductively: define $\alpha_i = \alpha_{i-1} - \partial F(\alpha_{i-1})^{-1} \cdot F(\alpha_{i-1}) \quad \forall i \ge 1$

Then $\qquad F(\alpha_i) \equiv 0 \pmod{\pi^{2d + r \cdot 2^i}} \quad \forall i \ge 1$

and $\partial F(\alpha_{i-1})^{-1} \cdot \partial F(\alpha_i) \equiv I_n \pmod{\pi^{r \cdot 2^i}} \quad \forall i \ge 1$

(straight-forward induction)

$\Longrightarrow$

$$\alpha_i \equiv \alpha_{i-1} \mod \pi^{d + r \cdot 2^{i-1}}$$

Let $\alpha := \lim_{i \to \infty} \alpha_i \rightsquigarrow F(\alpha) = 0$ and $\alpha \equiv \alpha_0 \pmod{\pi^{d+r}}$

Conclusion

$\exists! \ \alpha \in \mathcal{O}_K^n$ with $\alpha \equiv \alpha_0 \mod \pi^{d+r}$

s.t. $F(\alpha) = 0$

Different and discriminant

– way to measure ramification

integral closure of A in L $= B \subseteq L$

$$\begin{array}{ccc} B & \subseteq & L \\ | & & |\ \text{finite separable} \\ A & \subset & K = \mathrm{frac}(A) \end{array}$$

Dedekind domain

$\Rightarrow$ $T: L \times L \longrightarrow K$

$(a, b) \mapsto \mathrm{Tr}_{L/K}(a \cdot b)$

is a perfect K-bilinear pairing

<u>Idea</u>: $T: B \times B \longrightarrow A$

$(a, b) \mapsto \mathrm{Tr}_{L/K}(a \cdot b)$

is a perfect A-bilinear pairing if B/A is unramified at $\mathcal{P}$ $\forall \mathcal{P} \subseteq B$ max. ideal $\forall \mathfrak{p} = \mathcal{P} \cap A$

<u>So</u>: the degree of failure of this trace pairing

Def: $\mathcal{D}^{-1}_{B/A} = \mathcal{D}^{-1}(B/A) = \{ x \in L \mid \mathrm{Tr}_{L/K}(xy) \in A \ \ \forall y \in B \}$

inverse different

$\mathcal{D}_{B/A} = \mathcal{D}(B/A) = \mathrm{Hom}_B(\mathcal{D}^{-1}(B/A), B)$

Clearly: $\mathcal{D}(B/A) \subseteq B \subset \mathcal{D}^{-1}(B/A)$

Def $\underset{A\text{-ideal}}{\mathrm{disc}(B/A)} = N_{B/A}(\underset{B\text{-ideal}}{\mathcal{D}(B/A)})$

Note: $\tilde{c}_B([B/\mathcal{D}_{B/A}]) = \mathcal{D}^{-1}_{B/A}$

$\rightsquigarrow$ $\tilde{c}_B([B/\mathcal{D}_{B/A} \text{ as an } A\text{-module}]) = \mathrm{disc}(B/A)^{-1}$

$T \Longleftrightarrow \underset{A\text{-linear map}}{L} : B \longrightarrow \mathrm{Hom}_A(B, A) =: B^\vee$ $\qquad$ B: finite projective A-module

$\det(L): \det(B) \to \det(B^\vee)$

$\rightsquigarrow$ $N_{B/A}([B/\mathcal{D}_{B/A}]) = [\det(B^\vee)/\det(L)(\det B)]$

Key computation

$\alpha \in K^{alg.}$ finite separable, integral over A (Dedekind domain). $f(T) =$ irred. polynomial (min. poly of α/K)

$f(\alpha) = 0 \quad f(T) \in A[T]$

$n = [K(\alpha) : K] = \deg f(T)$

$f(T) = \prod_{i=1}^{n} (T - \alpha_i)$

$\alpha_i =$ K-conjugates of α

$$\frac{1}{f(T)} = \sum_{i=1}^{n} \frac{1}{f'(\alpha_i)(T-\alpha_i)}$$

$$= \sum_{i=1}^{n} \sum_{j\geq 0} \frac{1}{f'(\alpha_i)} \frac{\alpha_i^j}{T^{j+1}}$$

$$= \sum_{j \geq 0} \mathrm{Tr}_{L/K}\left(\frac{\alpha^j}{f'(\alpha)}\right) T^{-(j+1)}$$

$$\Longrightarrow \quad \mathrm{Tr}_{L/K}\left(\frac{\alpha^j}{f'(\alpha)}\right) = \begin{cases} 0 & \text{for } j = 0, \ldots, n-2 \\ 1 & j = n-1 \end{cases}$$

$$(b_{ij}) = \left(\mathrm{Tr}\left(\alpha^i \cdot \frac{\alpha^j}{f'(\alpha)}\right)\right) = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \underbrace{*}_{\in A} \end{pmatrix}$$

So

Prop $\{x \in K(\alpha) \mid \mathrm{Tr}(x \cdot y) \in A \ \forall y \in A[\alpha]\}$

$$= \frac{1}{f'(\alpha)} \cdot A[\alpha]$$

Define $\mathfrak{c}(A[\alpha] \subseteq B) \overset{\text{def}}{=} \{x \in L \mid x \cdot B \subseteq A[\alpha]\}$

$=$ the largest ideal I of B contained in $A[\alpha]$

? $\mathfrak{c}(A[\alpha] \subseteq B) \cdot f'(\alpha)^{-1} \overset{?}{=} \mathcal{D}^{-1}(B/A)$

$\subseteq$: Prop + def^n

$\supseteq$: $\mathrm{Tr}_{L/K}(x \cdot B) \subseteq A \Rightarrow \ldots$ see better way below

Equivalent but easier to show: $\mathfrak{c}(A[\alpha] \subseteq B) = f'(\alpha) \cdot \mathcal{D}^{-1}(B/A)$

$x \in \mathfrak{c}(A[\alpha] \subseteq B) \Leftrightarrow x \cdot B \subseteq A[\alpha] \Leftrightarrow f'(\alpha)^{-1} \cdot x \cdot B \subseteq \{y \in L \mid \mathrm{Tr}(y \cdot A[\alpha]) \subseteq A\} =: A[\alpha]^\vee$

$\Leftrightarrow \mathrm{Tr}(f'(\alpha)^{-1} \cdot x \cdot B \cdot A[\alpha]) \subseteq A \Leftrightarrow f'(\alpha)^{-1} \cdot x \in \mathcal{D}^{-1}(B/A) \Leftrightarrow x \in f'(\alpha) \cdot \mathcal{D}^{-1}(B/A)$

What to say about local fields (= fraction field of complete dvr)
and application to global fields

1. $K = \mathrm{frac}(A)$ A complete dvr

L/K finite ext^n $\rightsquigarrow$ $B :=$ integral closure of A in L is a complete dvr, and a free A-module of rk $[L:K]$

+ consequences

2.
$$\begin{array}{ccc} B & \subset & L \\ | & & | \\ A & \subset & K \end{array}$$
$\uparrow$ Dedekind

Assume B : finite A-module

passing to $A_{\mathfrak{p}}$ and $\hat{A}_{\mathfrak{p}}$

3. structure of complete dvr.

More about Euler's formula

conjugates of α: $\alpha = \alpha_1, \alpha_2, \ldots \alpha_n$

$f(T) =$ separable monic polynomial over a field K, $f(\alpha) = 0$, $L = K(\alpha)$

$n = \deg f(T)$

$0 \le i \le n-1$

$$\frac{T^i}{f(T)} = \sum_{j=1}^{n} \frac{\alpha_j^i}{(T-\alpha_j)\, f'(\alpha_j)}$$

$$f(T) = T^n + b_{n-1} T^{n-1} + \cdots + b_1 T + b_0$$

$$\frac{1}{f(T)} = \sum_{m=0}^{\infty} \sum_{j=1}^{n} \frac{\alpha_j^m}{f'(\alpha_j) \cdot T^{m+1}} = \sum_{m=0}^{\infty} \mathrm{tr}_{L/K} \frac{\alpha^m}{f'(\alpha)}\, T^{-(m+1)}$$

$$1 = f(T) \cdot \sum_{m \geq 0} \mathrm{tr}\left[\frac{\alpha^m}{f'(\alpha)}\right] T^{-(m+1)}$$

$$= \sum_{\substack{m \geq 0 \\ 0 \le r \le n}} \mathrm{tr}\left(\frac{b_r \alpha^m}{f'(\alpha)}\right) T^{r-m-1}$$

Look at terms with $0 \le m \le n-1$

$\rightsquigarrow -n \le r-m-1 \le n-1$

$$\mathrm{tr}_{L/K}(\alpha^m / f'(\alpha)) = \begin{cases} 0 & \text{if } 0 \le m \le n-2 \\ 1 & \text{if } m = n-1 \end{cases}$$

$\uparrow$
The b_r's here are coeff. of $f(T)$

Another way to look at Euler's formula:

$$\frac{1}{f(T)} = \sum_i \frac{1}{(T-\alpha_i)\, f'(\alpha_i)} = \mathrm{Tr}_{L/K} \frac{1}{(T-\alpha)\cdot f'(\alpha)}$$

$$0 \le j \le n-1 \quad \frac{T^j}{f(T)} = \sum_i \frac{\alpha_i^j}{(T-\alpha_i)\, f'(\alpha_i)} = \mathrm{Tr}_{L/K}\left(\frac{\alpha^j}{(T-\alpha)\, f'(\alpha)}\right)$$

$$\rightsquigarrow \boxed{T^j = \mathrm{Tr}_{L/K}\left(\frac{f(T)}{T-\alpha} \cdot \frac{\alpha^j}{f'(\alpha)}\right)}$$

Write $\displaystyle \frac{f(T)}{T-\alpha} = \sum_{r=0}^{n-1} b_r T^r \qquad b_r \in K[\alpha] \subseteq L$

$$\rightsquigarrow T^j = \mathrm{Tr}\left(\left(\sum_{r=0}^{n-1} b_r T^r\right) \cdot \frac{\alpha^j}{f'(\alpha)}\right)$$

$$= \mathrm{Tr}\left(b_{n-1} \frac{\alpha^j}{f'(\alpha)}\right) T^{n-1} + \cdots + \mathrm{Tr}\left(b_r \frac{\alpha^j}{f'(\alpha)}\right) T^r + \cdots + \mathrm{Tr}\left(\frac{b_0 \alpha^j}{f'(\alpha)}\right)$$

$$\Rightarrow \boxed{\mathrm{Tr}\left(b_r \frac{\alpha^j}{f'(\alpha)}\right) = \delta_{rj} \qquad \forall_{0 \le r, j \le n-1}}$$

← The b_r's here are coeff. of $f(T)/(T-\alpha)$

Lemma: The dual basis of ~~$1, \alpha, \ldots, \alpha^{n-1}$~~ $\alpha^{n-1}, \alpha^{n-2}, \cdots \alpha, 1$ w.r.t $\mathrm{Tr}_{L/K}$ is

$$\left(\frac{b_{n-1}}{f'(\alpha)}, \frac{b_{n-2}}{f'(\alpha)}, \ldots, \frac{b_0}{f'(\alpha)}\right)$$

where $b_{n-1}, \ldots, b_0 \in K(\alpha)$ is defined by:

$$\sum_{r=0}^{n-1} b_r T^r = \frac{f(T)}{T-\alpha}$$

Note $\left(\sum_{r=0}^{n-1} b_r T^r\right) \cdot (T-\alpha) = f(T)$

$$\frac{1}{T-\alpha} = \frac{f(T)^{-1} \cdot \sum_{r=0}^{n-1} b_r T^r}{\cancel{f(T) \cdot \frac{1}{\sum_{r=0}^{n-1} b_r T^r}}} \in K[b_1, \dots, b_r][[\tfrac{1}{T}]]$$

$$\overset{\shortparallel}{\sum_{m \geq 0} \frac{\alpha^m}{T^{m+1}}} \qquad \Rightarrow \quad K[\alpha] \subseteq K[b_1, \dots, b_r]$$

("$\supseteq$" is obvious)

Ideles, Adeles

(i) $\| \cdot \| : \mathbb{A}_K \to \mathbb{R}_{\geq 0} = [0, \infty)$

$(x_v)_{v \in \Sigma_K} \mapsto \prod_v \| x_v \|_v$

is lower semi-continuous

i.e. $\{ x \in \mathbb{A}_K \mid \|x\| < a \}$ is open $\forall a \in \mathbb{R}$

and $\mathbb{A}_K^\times = \{ x \in \mathbb{A}_K \mid \|x\| > 0 \}$

(ii) $\mathbb{A}_K^\times$ is a locally closed subset of $\mathbb{A}_K$.

(iii) However the idelic topology on $\mathbb{A}_K^\times$ is strictly stronger than the topology induced from the topology on $\mathbb{A}_K$ (and the embedding $\mathbb{A}_K^\times \hookrightarrow \mathbb{A}_K$)

(iv) Of course $\mathbb{A}_K^\times \hookrightarrow \mathbb{A}_K \times \mathbb{A}_K$ is a closed embedding

$x \mapsto (x, x^{-1})$

(v) $\mathbb{A}_{K,1}^\times \hookrightarrow \mathbb{A}_K$ is a closed embedding

Key: $\forall \varepsilon > 0$ $\exists$ finite subset $\underset{\Sigma_{K,\infty}}{\overset{}{T}} \subset \Sigma_K$ s.t. ($T \supseteq \Sigma_{K,\infty}$)

$\|y\|_v < \varepsilon \quad \forall y \in \mathfrak{p}_v \subset \mathcal{O}_v, \ \forall v \notin T$

Lemma (Minkowsky)

Let $V_1 = \{ \underset{(x_v)}{x} \in \mathbb{A}_K \mid \|x_v\|_v \leq 1 \} \quad \forall v \in \Sigma_K$

$\exists$ constant $C > 0$ (depending only on K) s.t.

$\forall z \in \mathbb{A}_K^\times$ s.t. $\|z\| > C$, have $K^\times \cap (z \cdot V_1) \neq \phi$

proof Let
$$C = 2^{[K:\mathbb{Q}]} \cdot \frac{\mu(\mathbb{A}_K/K)}{\mu(V_1)}$$

Let $V_2 = \left\{ x \in \mathbb{A}_K \;\middle|\; \begin{array}{ll} \|x_v\|_v \leq 1 & \forall v \text{ finite} \\ \|x_v\|_v \leq \frac{1}{2} & \forall v \text{ real} \\ \|x_v\|_v \leq \frac{1}{4} & \forall v \text{ complex} \end{array} \right\}$

$\leadsto \quad \mu(z \cdot V_2) = 2^{-[K:\mathbb{Q}]} \cdot \mu(V_1) \cdot \|z\| > \mu(\mathbb{A}_K/K)$

$$\Rightarrow \quad \exists\, y_1 \neq y_2 \in z V_2 \text{ s.t. } \underbrace{y_1 - y_2}_{\substack{\cap \\ z \cdot V_1}} \in K$$

q.e.d.

Thm: $\mathbb{A}_{K,1}^\times / K^\times$ is compact

Pf: 1^{st} proof: Pick $z_0 \in \mathbb{A}_K^\times$ with $\|z_0\| > C$ (as in Lemma)

Lemma $\Rightarrow \quad (z_0 \cdot V_1) \cap \mathbb{A}_{K,1}^\times \twoheadrightarrow \mathbb{A}_{K,1}^\times / K^\times$

2^{nd} proof: Given any $x \in \mathbb{A}_{K,1}^\times$, with $\|z_0\| > C$ as above.

$\exists\, y_1 \in (z_0 \cdot x \cdot V_1) \cap K^\times$
$\quad y_2 \in (z_0 \cdot x^{-1} \cdot V_1) \cap K^\times \Rightarrow \quad y_1 \cdot y_2 \in \underbrace{(z_0^2 \cdot V_1) \cap K^\times}_{\subset \text{ a finite set!}}$

($\because z_0^2 \cdot V_1$ is compact)

$\Rightarrow \quad x \in y_1^{-1} z_0 \cdot V_1$

$x^{-1} = \underbrace{y_2^{-1} \cdot z_0 \cdot V_1}_{\substack{\parallel \\ (y_1 \cdot y_2)^{-1} \cdot y_1 \cdot z_0 \cdot V_1}}$ i.e. $y_1 x \in (z_0 \cdot V_1)$

$(y_1 \cdot x)^{-1} \in \bigcup_{u \in (z_0^2 \cdot V_1 \cap K^\times)} u^{-1} \cdot (z_0 \cdot V_1)$

i.e $y_1 \cdot x \in$ a fixed compact subset of $\mathbb{A}_{K,1}^\times$

QED

<u>Lemma</u> $\overset{\text{exer.}}{\Longrightarrow}$ strong approximation for $(\mathbb{A}_K, +)$.

<u>Finiteness</u> of class number

S: a finite subset of Σ_K, including all infinite places

$$\begin{array}{ccc} \mathbb{A}_{K,S}^\times = \left(\prod_{v\in S} K_v^\times\right) \times \prod_{v\notin S} \mathcal{O}_v^\times & & \\ \cup\!\shortmid & & \\ \prod_{v\notin S} \mathcal{O}_v^\times & & \end{array}$$

$$\begin{array}{ccc} \mathbb{A}_{K,S}^\times & \longrightarrow & \mathbb{R}_{>0}^{r_1+r_2} \times \mathbb{Z}^{s-r_1-r_2} \\ \cup & & \cup \\ \mathbb{A}_{K,S,1}^\times & \longrightarrow & \left(\mathbb{R}_{>0}^{r_1+r_2} \times \mathbb{Z}^{s-r_1-r_2}\right)_0 \end{array}$$

$\therefore$ $\mathbb{A}_{K,S,1}^\times / \underbrace{(K^\times \cap \mathbb{A}_{K,S,1}^\times)}_{\mathcal{O}_S^\times}$ is compact

Image of $\mathcal{O}_S^\times$ is a lattice in !

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dual measures:

G: locally compact abelian group, $G^\vee = \mathrm{Hom}_{\mathrm{cont}}(G, \mathbb{C}_1^\times)$ unitary dual of G

μ: a Haar measure on G

$f \longmapsto \hat{f}_\mu$ Fourier transform: $\hat{f}_\mu(\hat{x}) = \int_G f(x) \langle \hat{x}, x\rangle \, d\mu(x)$

$\leadsto$ For $\hat{\nu}$ = Haar measure on $G^\vee$, have:

$\exists!$ $C(\mu, \hat{\nu}) \in \mathbb{R}_{>0}^\times$ s.t.

$$\int_{\hat{G}} \hat{f}(\hat{x}) \langle x, \hat{x}\rangle \, d\nu(\hat{x}) = C(\mu, \hat{\nu}) \cdot f(-x)$$

<u>defn</u>: $\hat{\nu}$ is the <u>dual</u> Haar measure to μ if $C(\mu, \hat{\nu}) = 1$

<u>Note</u>: Clearly $C(t\cdot\mu, \hat{\nu}) = C(\mu, t\hat{\nu}) = t\, C(\mu, \hat{\nu})$ $\forall t \in \mathbb{R}_{>0}^\times$

1. For $K = \mathbb{R}$, identify $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{R}, \mathbb{C}_1^\times)$ with $\mathbb{R}$ via
$\mathbb{R}\times\mathbb{R} \to \mathbb{C}_1^\times$, $(x,y) \mapsto e^{2\pi\sqrt{-1}\,xy}$. $\rightsquigarrow$ dx = usual Lebesgue measure is self-dual

2. For $K = \mathbb{C}$: identify $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{C}, \mathbb{C}_1^\times)$ with $\mathbb{C}$ via
$\mathbb{C}\times\mathbb{C} \to \mathbb{C}_1^\times$, $(z,w) \mapsto e^{2\pi\sqrt{-1}\,(zw+\bar{z}\bar{w})}$

Find dual measure of $dz\,d\bar{z} = |dz\wedge d\bar{z}| = 2\,dx\,dy$

Write $z = x+\sqrt{-1}\,y$, $x,y\in\mathbb{R}$ $\rightsquigarrow$ $dz\,d\bar{z} = 2dx\,dy$

$w = r+\sqrt{-1}\,s$

$$(z,w) = e^{2\pi\sqrt{-1}\,\mathrm{tr}_{\mathbb{C}/\mathbb{R}}[(xr-ys)+\sqrt{-1}(xs+yr)]}$$
$$= e^{2\pi\sqrt{-1}\cdot 2(xr-ys)}$$

The factor "2" here and the factor "2" in compensate each other

$\rightsquigarrow$ $|dz\wedge d\bar{z}|$ is the self-dual measure on $\mathbb{C}$ w.r.t. the pairing $(z,w) \mapsto e^{2\pi\sqrt{-1}(zw+\bar{z}\bar{w})} = e^{2\pi\sqrt{-1}\,\mathrm{tr}_{\mathbb{C}/\mathbb{R}}(zw)}$

3. For K = a finite extension field of $\mathbb{Q}_p$, have
$K\times K \to \mathbb{C}_1^\times$, $(x,y)\mapsto e_p(\mathrm{tr}_{K/\mathbb{Q}_p}(xy))$

where $e_p : \mathbb{Q}_p \to \mathbb{C}_1^\times$ is defined by: $e_p|_{\mathbb{Z}[\frac{1}{p}]} : a \mapsto e^{2\pi\sqrt{-1}\,a}$ $\forall a\in\mathbb{Z}[\frac{1}{p}]$

$$\begin{array}{c} t \\ \Uparrow \\ a + \mathbb{Z}_p \\ \Uparrow \\ \mathbb{Z}[\frac{1}{p}] \end{array} \longmapsto e^{2\pi\sqrt{-1}\,a}$$

and $e_p|_{\mathbb{Z}_p}$ = trivial

Find the dual Haar measure on K to μ: $\mu(\mathcal{O}_K)=1$

Let f = characteristic function on $\mathcal{O}_K$

$$\rightsquigarrow \quad \hat{f}(x) = \int_{\mathcal{O}_K} \underbrace{e_p(\mathrm{tr}_{K/\mathbb{Q}_p}(xu))}\,d\mu(u)$$

$\uparrow$ = trivial character on $(\mathcal{O}_K,+)$ iff $x\in\mathcal{D}_{K/\mathbb{Q}_p}^{-1}$

= characteristic function on $\mathcal{D}_{K/\mathbb{Q}_p}^{-1}$

Conclusion: the Haar measure on K dual to μ w.r.t. the pairing $(x,y)\mapsto e_p(\mathrm{tr}_{K/\mathbb{Q}_p}(xy))$ is the Haar measure ν on K s.t. $\nu(\mathcal{D}_{K/\mathbb{Q}_p}^{-1}) = 1$

because $\nu = \frac{1}{[\mathcal{O}_K : \mathcal{D}_{K/\mathbb{Q}_p}]} \mu$

In other words:

the self-dual Haar measure on K w.r.t. the pairing

$(x,y) \mapsto e_p(tr_{K/\mathbb{Q}_p}(xy))$

is the Haar measure on K, denoted by $\mu_{s.d.}$, s.t.

$$\mu_{s.d.}(\mathcal{O}_K) = \#(\mathcal{O}_K / \mathcal{D}_{K/\mathbb{Q}_p})^{-\frac{1}{2}}$$

4. For a number field K, the self-dual Haar measure on $\mathbb{A}_K$ w.r.t. the pairing

$$(x,y) \longmapsto e(tr_{\mathbb{A}_K/\mathbb{A}_\mathbb{Q}}(x \cdot y))$$

is the Haar measure on K, equal to $\prod'_{v \in \Sigma_K} \mu_{v,s.d.}$

Consequence: Since the volume of $\mathbb{A}_K/K$ w.r.t. the self-dual measure is $\underline{1}$

$\uparrow$
Exercise

$\Rightarrow$ Let μ = the product measure on $\mathbb{A}_K$, of:

$|dz \wedge d\bar{z}|$ for cpx places
dx for real places
volume$(\mathcal{O}_{K_v})=1$ for non-arch. places

Then $\mu(\mathbb{A}_K) = |disc_{K/\mathbb{Q}}|^{\frac{1}{2}} = \#\left(\mathbb{Z}/\text{discriminant ideal } disc_{K/\mathbb{Q}}\right)^{\frac{1}{2}}$

3. Supplementary fact about dual Haar measure:

Prop: G: locally compact abelian group

(i) If G is compact, then the dual measure to μ with $\mu(G)=1$ is the counting measure on $\hat{G}$

(ii) If G is discrete, then the dual of the counting measure on G is the Haar measure on $G^\vee$ s.t. $G^\vee$ has volume $=1$

(iii) Similar statements if $\exists\ H \subseteq G$ compact with G/H discrete or $\exists\ H \subseteq G$ s.t. H is discrete and G/H compact

~~Exer~~
Illustration: Use this fact to compute the self-dual Haar measure on $\mathbb{R}$. use: $\mathbb{Z}^{\perp} = \mathbb{Z}$ w.r.t the pairing $(x,y) \mapsto e^{2\pi\sqrt{-1}xy}$

$\mathbb{Z} \subseteq \mathbb{R}$

(What about $\mathbb{C}$: derive it from $\mathbb{R}$)

Remark: If we use the "more traditional product Haar measure" on $\mathbb{A}_K$:

dx for real places
dx dy for cpx places ($z = x + \sqrt{-1}y$) embedding/isom. to $\mathbb{C}$

$\mu_{1,v}(O_{K_v}) = 1$ for non

$\mu_1 = \prod_v \mu_{1,v}$

then $\mu_1(\mathbb{A}_K/K) = 2^{r_2} \cdot \left|disc(K/\mathbb{Q})\right|^{\frac{1}{2}}$

Question: $\mu_v^{\times} =$ the Haar measure on $K_v^{\times}$ s.t. $\begin{cases} \mu_v(O_v^{\times}) = 1 & v \text{ non-arch} \\ \frac{dx}{|x|} & v \text{ real} \\ \frac{|dz\, d\bar{z}|}{z \cdot \bar{z}} & v \text{ cpx} \end{cases}$

$\mu^{\times} = \prod'_{v \in \Sigma_K} \mu_v^{\times}$

Will use the following Haar measures:

on $\mathbb{A}_K$: use the self-dual Haar measure w.r.t. $(x,y) \mapsto e(tr_{K/\mathbb{Q}}(x \cdot y))$

$\mu_{\mathbb{A}_K} = \prod'_v \mu_{K_v}$ $\quad \mu_{K_v}$ = self-dual Haar measure w.r.t. $(x,y) \mapsto e_v(tr_{K_v/\mathbb{Q}_v}(xy))$

Note: In Tate's thesis, he used $2\,dx\,dy$, = our $|dz\,d\bar{z}|$ at archimedian places. Thus the additive measure on $\mathbb{A}_K$ used in Tate's thesis is (our $\mu_{\mathbb{A}_K}$)

Thus

$\mu_{K_v}(O_{K_v}) = \#(O_{K_v}/\mathcal{D}_{K_v/\mathbb{Q}_p})^{-\frac{1}{2}}$ for v non-arch.

$\mu_{K_v}([-1,1]) = 2$ $\quad v$ real

$\mu_{K_v}(\{\|z\|_v \le 1\}) = 4\pi$

On $\mathbb{A}_K^\times$: use $\prod_v' \mu_v^\times =: \mu^\times$

where $\mu_v^\times = \begin{cases} \dfrac{\mu_v}{\|\cdot\|_v} & v \text{ arch.} \\ \dfrac{\mu_v}{1-q_v^{-1}}\Big|_{\mathcal{O}_v^\times} = \dfrac{1}{(1-q_v^{-1})}\mu_v & \mu_v \text{ nonarch} \end{cases}$

Again, this means that the Haar measure Tate used in his thesis is exactly (our $\mu^\times$)

Now, compute the volume of $\mathbb{A}_{K,1}^\times / K^\times$

$$1 \to \mathbb{A}_{K,1}^\times \longrightarrow \mathbb{A}_K^\times \longrightarrow \mathbb{R}_{>0}^\times \to 0$$

$\mu_1^\times \qquad \mu^\times \qquad dt/t$ Use $\frac{dt}{t}$ for Haar measure on $\mathbb{R}_{>0}^\times$

$\Rightarrow$ Specifies a Haar measure $\mu_1^\times = \mu_{\mathbb{A}_{K,1}^\times}^\times$ on $\mathbb{A}_{K,1}^\times$

Want to compute $\mu(\mathbb{A}_{K,1}^\times / K^\times)$

$$\begin{array}{ccccccc} 1 \to & K^\times & \longrightarrow & \mathbb{A}_{K,1}^\times & \longrightarrow & \mathbb{A}_{K,1}^\times / K^\times & \to 1 \\ & \cup & & \cup & & \cup & \\ 1 \to & \mathcal{O}_K^\times & \longrightarrow & \underbrace{\mathbb{A}_{K,\infty,1}^\times}_{\substack{\| \\ (K\otimes_{\mathbb{Q}}\mathbb{R})_1^\times}} \cdot \prod_{v\in\Sigma_{K,f}} \mathcal{O}_v^\times & \longrightarrow & \mathbb{A}_{K,\infty,1}^\times \cdot \prod_{v\in\Sigma_{K,f}} \mathcal{O}_v^\times / \mathcal{O}_K^\times & \to 1 \end{array}$$

On $(K\otimes_{\mathbb{Q}}\mathbb{R})_1^\times$: use $\prod_{v\in\Sigma_{K,\infty}} \mu_v^\times \Big/ \underbrace{\frac{dt}{t}}_{\text{measure on } \mathbb{R}_{>0}^\times} =: \mu_{\infty,1}^\times$

$\Rightarrow$ product measure of $\prod_{v\in\Sigma_{K,f}} \mu_v^\times|_{\mathcal{O}_v^\times}$ and $\frac{dt}{t}$ on

In other words, have $\quad \|\cdot\|: \mathbb{A}_{K,1}^\times \twoheadrightarrow_f \mathbb{R}_{>0}^\times$

$$1 \to (K\otimes_{\mathbb{Q}}\mathbb{R})_1^\times \to (K\otimes_{\mathbb{Q}}\mathbb{R})^\times \xrightarrow{\mathrm{Nm}} \mathbb{R}_{>0}^\times \to 1$$

$$\mu_{\infty,1}^\times \qquad \prod \mu_v^\times \qquad \frac{dt}{t}$$

$$x = (x_v)_{v\in\Sigma_{K,\infty}} \mapsto t = \mathrm{Nm}(x) = \prod_{v\in\Sigma_{K,\infty}} \mathrm{Nm}_{K_v/\mathbb{R}}(x_v)$$

$$(\mathbb{A}_{K,1}^\times / K^\times) \Big/ \Big((K\otimes_{\mathbb{Q}}\mathbb{R})_1^\times \cdot \prod_{v\in\Sigma_{K,f}} \mathcal{O}_v^\times / \mathcal{O}_K^\times\Big) \xrightarrow{\sim} \mathrm{Cl}(\mathcal{O}_K)$$

$$\Rightarrow \mu_{\mathbb{A}_{K,1}}^\times(\mathbb{A}_{K,1}^\times / K^\times) = h_K \cdot \mu_{\mathbb{A}_{K,1}}^\times\Big(\Big[(K\otimes_{\mathbb{Q}}\mathbb{R})_1^\times \times \prod_{v\in\Sigma_{K,f}} \mathcal{O}_v^\times\Big] / \mathcal{O}_K^\times\Big)$$

$$\mu^{\times}_{\mathbb{A}^{\times}_{K,1}}(\mathbb{A}^{\times}_{K,1}/K^{\times}) = h_K \cdot \mu^{\times}_{\mathbb{A}^{\times}_{K,1}}\Big(\big[(K \otimes_{\mathbb{Q}} \mathbb{R})^{\times}_1 \cdot \prod_{v \in \Sigma_{K,f}} O_v^{\times}\big]/O_K^{\times}\Big)$$
$$= h_K \cdot |\mathrm{disc}_{K/\mathbb{Q}}|^{-\frac{1}{2}} \cdot \mu^{\times}_{K_{\infty},1}(K^{\times}_{\infty,1}/O_K^{\times})$$

Remains to compute $\mu^{\times}_{K_\infty,1}(K^{\times}_{\infty,1}/O_K^{\times})$

$$1 \to K^{\times}_{\infty,1} \to K^{\times}_{\infty} \to \mathbb{R}^{\times}_{>0} \to 1$$

$$\begin{array}{ccccccc} 1 \to & U & \to & K^{\times}_{\infty} & \xrightarrow[\prod_{v\in\Sigma_{K,\infty}} N_v]{} & \prod_{v\in\Sigma_{K,\infty}} \mathbb{R}^{\times}_{>0} & \to 1 \\ & \cup & & \cup & & \cup & \\ 1 \to & U \cap K^{\times}_{\infty,1} & \to & K^{\times}_{\infty,1} & \to & V_1 & \to 1 \end{array}$$

$$V_1 = \Big\{ (u_v)_{v\in\Sigma_{K,\infty}} \ \Big|\ u_v \in \mathbb{R}^{\times}_{>0}\ \forall v,\ \prod_{v\in\Sigma_{K,\infty}} u_v = 1 \Big\}$$

Compare two Haar measures on $\mathbb{C}^{\times}$:

(a) $\dfrac{|dz\,d\bar z|}{z.\bar z}$ (b) product measure for $1 \to \mathbb{C}^{\times}_1 \to \mathbb{C}^{\times} \to \mathbb{R}^{\times}_{>0} \to 1$, $z \mapsto z\bar z$

$$\int_{a \le z\bar z \le b} \frac{1}{z\bar z}|dz\,d\bar z| = \iint_{a\le r^2\le b} \frac{1}{r^2} 2r \cdot dr\, d\theta = (2\pi)\int_{a\le t\le b} \frac{dt}{t}$$

$t = r^2$

Conclusion $0 \to \mathbb{C}^{\times}_1 \to \mathbb{C}^{\times} \to \mathbb{R}^{\times}_{>0} \to 1$, $z \mapsto z\bar z$; $\dfrac{dz\,d\bar z}{z\bar z}$, $\dfrac{dt}{t}$

get measure with total volume 2π

i.e. $\mu^{\times}_{\mathbb{C}^{\times}}(\{a \le z\bar z \le b\}) = 2\pi \cdot \int_a^b \frac{dt}{t}$ $\forall\, 0<a<b$

Easier:

$$1 \to \{\pm 1\} \to \mathbb{R}^{\times} \to \mathbb{R}^{\times}_{>0} \to, \quad x \mapsto |x|$$

$\dfrac{dx}{|x|}$, $\dfrac{dt}{t}$

$\mu^{\times}_{\mathbb{R}}\{x \in \mathbb{R}^{\times} \mid a < |x| < b\} = 2\int_a^b \frac{dt}{t}$ $\forall\, 0<a<b$

obvious

give counting measure

$$\begin{array}{ccccc} U_1 := U \cap K^{\times}_{\infty,1} & \rightarrowtail & K^{\times}_{\infty,1} & \twoheadrightarrow & V_1 \\ \cup & & \cup & & \cup \\ U_1 \cap O_K^{\times} & \to & O_K^{\times} & \twoheadrightarrow & \Gamma_1 \\ \| & & & & \\ \mu(K) & & & & \end{array}$$

given measure whose product with $\frac{dt}{t}$ is $\prod_v \frac{du_v}{u_v}$

$t = \|x\| = \prod_{v\in\Sigma_{K,\infty}} \|x_v\|$

$u_v = \|x_v\|$

the product measure on $\prod_{v\in\Sigma_{K,\infty}} \mathbb{R}^{\times}_{>0}$

Conclusion $\mu_{K_\infty,1}^{\times}(K^{\times}_{\infty,1}/O_K^{\times})$
$$= w^{-1} \cdot 2^{r_1} \cdot (2\pi)^{r_2} \cdot \underset{\substack{\| \\ \text{volume}(V_1/\Gamma_1)}}{R}$$

Putting everything together:

$$\boxed{\mu^\times_{\mathbb{A}^\times_{K,1}}\left(\mathbb{A}^\times_{K,1}/K^\times\right) = \frac{2^{r_1}\cdot(2\pi)^{r_2}\cdot h_K\cdot R_K}{w\cdot \operatorname{disc}(K/\mathbb{Q})^{\frac12}}}$$

$$w = \#\mu(K)$$

where $r =$ volume of $V_1/\left(\begin{smallmatrix}\text{image of } \mathcal{O}_K^\times\\ \text{in } V_1\end{smallmatrix}\right)$

$$V_1 = \Big(\prod_{v\in\Sigma_{K,\infty}} \mathbb{R}^\times_{>0}\Big)_1 = \left\{ (u_v)_{v\in\Sigma_{K,\infty}} \;\middle|\; \begin{array}{l} u_v\in\mathbb{R}^\times_{>0}\ \forall v\\ \prod_v u_v = 1\end{array}\right\}$$

with measure μ_{V_1} s.t.

$$1\to V_1\to \prod_{v\in\Sigma_{K,\infty}}\mathbb{R}^\times_{>0}\to \mathbb{R}^\times_{>0}\to 1$$

μ_{V_1}, product measure $\prod_{v\in\Sigma_{K,\infty}}\frac{du_v}{u_v}$, $\frac{dt}{t}$, $t = \prod_{v\in\Sigma_{K,\infty}} u_v$

Recall

How did the factors enter this formula

- $\operatorname{disc}(K/\mathbb{Q}_p)^{-\frac12}$:

$$d\mu_v^\times = \#\big(\mathcal{O}_v/\mathcal{D}(K_v/\mathbb{Q}_v)\big)^{-\frac12}\circ\begin{pmatrix}\text{the Haar measure}\\ \text{on } (K_v,+) \text{ s.t. } \mathcal{O}_{K_v}\\ \text{has volume } 1\end{pmatrix}$$

- 2^{r_1} and $(2\pi)^{r_2}$.

In $1\to K^\times_{v,1}\to K^\times_v \xrightarrow{\|\cdot\|_v} \mathbb{R}^\times_{>0}\to 1$

give $K^\times_{v,1}$ the Haar measure with total volume $\begin{cases} 2 & v \text{ real}\\ 2\pi & v \text{ cpx}\end{cases}$.

and $\mathbb{R}^\times_{>0}$ the Haar measure $\frac{dt}{t}$, the resulting "product measure" on $K_v^\times$

is $\begin{cases}\frac{dx}{|x|} & v \text{ real}\\ |dz\cdot d\bar z|/z\cdot\bar z & v \text{ cpx}\end{cases}$

[These factors appear in the numerator because we are using the measure $\mu^\times_{\mathbb{A}^\times_{K,1}}$ on $\mathbb{A}^\times_{K,1}$]

- The factor h_K appears (in the numerator) because we actually computed only $\mu^\times_{\mathbb{A}^\times_{K,1}}\Big(\big(\prod_{v\in\Sigma_{K,f}}\mathcal{O}_v^\times\big)\cdot(K\otimes_{\mathbb{Q}}\mathbb{R})^\times_1/\mathcal{O}_K^\times\Big)$, and

$$\mathbb{A}^\times_{K,1}/K^\times\cdot\Big(\prod_{v\in\Sigma_{K,f}}\mathcal{O}_v^\times\times(K\otimes\mathbb{R})^\times_1\Big)\cong \mathrm{Cl}(\mathcal{O}_K)$$

- The factor w appears in the denominator because $\mu(K) = \mathcal{O}_K^\times \cap \prod_{v\in\Sigma_{K,\infty}} K^\times_{v,1}$

Exer.

Checking a trivial case: $K=\mathbb{Q}$

fundamental domain for $\mathbb{A}_{\mathbb{Q},1}^{\times}/\mathbb{Q}^{\times}$:

$$1 \to \{\pm 1\} \to \mathbb{A}_{\mathbb{Q},1}^{\times} \longrightarrow \prod_{p=\text{prime nber}}' \mathbb{Q}_p^{\times} \to 1$$

$$\mathbb{A}_{\mathbb{Q},1}^{\times}/\mathbb{Q}^{\times} \longleftarrow \Big(\prod_{p \text{ prime nber}} \mathbb{Z}_p^{\times}\Big) \times \{\pm 1\}/\{\pm 1\}$$

and $\mu^{\times}_{\mathbb{A}^{\times}_{\mathbb{Q},1}}\left(\mathbb{A}_{\mathbb{Q},1}^{\times}/\mathbb{Q}^{\times}\right) = 1$

Poisson summation formula:

$\underset{\substack{\text{closed}\\ \text{discrete}}}{\Gamma \subseteq G}$ $\quad G/\Gamma$: compact $\qquad \mu_G = \underset{\substack{\uparrow\\ \text{counting measure on } \Gamma}}{\mu_\Gamma} \cdot \mu_{G/\Gamma} \quad \mu_{G/\Gamma}(G/\Gamma)=1$

$f \in \mathcal{S}(G)$ schwartz function $\qquad G^{\vee} \supseteq (G/\Gamma)^{\vee} = \Gamma^{\perp}$ discrete

$\qquad G^{\vee}/\Gamma^{\perp} \xrightarrow{\sim} \Gamma^{\vee}$ compact

$\rightsquigarrow g(\bar{x}) = \sum_{\gamma\in\Gamma} f(x+\gamma) \qquad x \mapsto \bar{x}$

$\qquad \Rightarrow \mu_{G^{\vee}} = \underset{\substack{\uparrow\\ \text{counting measure on } \Gamma^{\perp}}}{\mu_{\Gamma^{\perp}}} \cdot \mu_{G^{\vee}/\Gamma^{\perp}} \quad \mu_{G^{\vee}/\Gamma^{\perp}}(G^{\vee}/\Gamma^{\perp}) = 1$

Apply Fourier inversion to G/Γ:

$$\hat{g}_{\mu(G/\Gamma)}(\underset{\substack{\cap\\ \Gamma^{\perp}}}{\alpha}) = \int_{G/\Gamma} g(\bar{x})\, \langle \bar{x}, \alpha\rangle\, d\mu_{G/\Gamma}$$

$$= \sum_{\gamma\in\Gamma} \int_G f(x+\gamma)\, \langle x, \alpha\rangle\, d\mu_G = \sum_{\gamma\in\Gamma} \langle -\gamma, \alpha\rangle \cdot \hat{f}_{\mu_G}(\alpha)$$

$$g(\bar{0}) = \sum_{\alpha\in\Gamma^{\perp}} \hat{f}_{\mu_G}(\alpha)$$

$$\| \qquad$$

$$\sum_{\gamma\in\Gamma} f(\gamma)$$

Conclusion

Poisson summation formula

$$\sum_{\gamma\in\Gamma} f(\gamma) = \sum_{\alpha\in\Gamma^{\perp}} \hat{f}_{\mu_G}(\alpha) \qquad \forall \text{ Schwartz function } f \text{ on } G$$

where

μ_G = product measure on G, of the counting measure on Γ and the Haar measure on G/Γ normalized by: volume $(G/\Gamma)=1$

characters on locally compact fields (the multiplicative group of) general notation

$\|\cdot\|^s = \omega_s \qquad s\in\mathbb{C}$

1. $K \cong \mathbb{R}$

$\mathrm{Hom}_{\mathrm{cont}}(\mathbb{R}^\times, \mathbb{C}^\times) \ni \chi$

cpx mfd

$$\chi(x) = \begin{Bmatrix} \|x\|^t \\ \mathrm{sgn}(x)\cdot\|x\|^t \end{Bmatrix} = x^{-N}\cdot\|x\|^s \qquad N=0 \text{ or } 1$$

terminology $\quad Re(s) = $ exponent of χ

2. $K \underset{z}{\overset{\sim}{\to}} \mathbb{C}$

$\mathrm{Hom}_{\mathrm{cont}}(K^\times, \mathbb{C}^\times) \ni \chi \qquad |z| = (z\bar z)^{\frac12}$ (positive square root)

$$\chi(z) = \left(\frac{z}{|z|}\right)^{-m}\cdot (z\bar z)^t \qquad m\in\mathbb{Z}$$

$$= \begin{cases} z^{-N}\cdot (z\bar z)^s & \text{with } s = t+m/2,\ N = m \text{ if } m\geq 0 \\ \bar z^{-N}\ (z\bar z)^s & \text{with } s = t-m/2,\ N=-m \text{ if } m<0 \end{cases}$$

$$\left(\frac{z}{|z|}\right)^{-m} = \frac{\bar z^m (z\bar z)^{-m}}{(z\bar z)^{-m/2}} = \bar z^m\cdot (z\bar z)^{-m/2}$$

exponent of $\chi = Re(t) = Re(s) - \frac{N}{2}$

(N, s) (and (m,t)) are uniquely determin

$N\in\mathbb{N},\ s\in\mathbb{C}$

3. K non-archimedian $\qquad$ get a splitting once we fix a generator of $\mathfrak{p}_K$ to split $1\to O_K^\times \to K^\times \to \mathbb{Z}\to 0$

$$1\to O_K^\times \to K^\times \xrightarrow{\mathrm{ord}_K} \mathbb{Z}\to 0$$

$$1\to \mathrm{Hom}(K^\times/O_K^\times, \mathbb{C}^\times) \to \underset{\text{cpx mfd}}{\mathrm{Hom}_{\mathrm{cont}}(K^\times, \mathbb{C}^\times)} \to \overset{\text{cpt}}{\mathrm{Hom}_{\mathrm{cont}}(O_K^\times, \mathbb{C}^\times)} \to 1$$

$\mathrm{Hom}_{\mathrm{cont}}(O_K^\times, \mathbb{C}^\times) = \mathrm{Hom}_{\mathrm{cont}}(O_K^\times, \mathbb{C}_1^\times)$ discrete torsion

So $\mathrm{Hom}_{\mathrm{cont}}(K^\times, \mathbb{C}^\times)$

$=$ a $\mathrm{Hom}(K^\times/O_K^\times, \mathbb{C}^\times)$-torsor over a discrete torsion group.

$\mathrm{Hom}(K^\times/O_K^\times, \mathbb{C}^\times) \cong \mathbb{C}^\times$ — unramified characters of $K^\times$

every $\chi \in \mathrm{Hom}_{\mathrm{cont}}(K^\times, \mathbb{C}^\times)$ can be written as $\chi_1\cdot\omega_s$

χ_1: torsion element in $\mathrm{Hom}_{\mathrm{cont}}(O_K^\times, \mathbb{C}^\times)$

terminology

exponent of (a torsion character) $\cdot\ \omega_s$ is $\mathrm{Re}(s)$

<u>Note</u>: Every unramified character has the form (unique) ω_s

$$s \in \mathbb{C}/\mathbb{Z}\cdot\left(\frac{2\pi\sqrt{-1}}{\log q_K}\right)$$

<u>i.e.</u> $\mathrm{Hom}_{\mathrm{cont}}(K^\times/\mathcal{O}_K^\times, \mathbb{C}^\times) \cong \underset{\text{algebraic torus}}{\mathbb{C}^\times} \cong \mathbb{C}/\mathbb{Z}\cdot\left(\frac{2\pi\sqrt{-1}}{\log q_K}\right)$

<u>and</u> $\mathrm{Hom}_{\mathrm{cont}}(K^\times, \mathbb{C}^\times)$ is an extension of a 1-dim^l algebraic torus$/\mathbb{C}$,

a discrete torsion group

$$\cong (\mathbb{Q}_p/\mathbb{Z}_p)^{[K:\mathbb{Q}_p]} \times (\text{a finite group})$$

<u>Exercise</u>: $\mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}_K^\times, \mathbb{C}_1^\times) \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{[K:\mathbb{Q}_p]} \times \left(\text{a finite group}\right)$

<u>Def^n</u> local L-functions

(i) $K \underset{x}{\overset{\sim}{\to}} \mathbb{R}$

$$\int_{\mathbb{R}^\times_{>0}} e^{-\pi x^2}|x|^s \frac{dx}{|x|}$$

$$L(x^{-N}\omega_s) = \Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma\left(\tfrac{s}{2}\right)$$

$$\rightsquigarrow L(\mathrm{sgn}\cdot\omega_s) = L(x^{-1}\cdot\omega_{s+1}) = \underbrace{\pi^{-\frac{s+1}{2}}\Gamma\left(\tfrac{s+1}{2}\right)}_{=\Gamma_{\mathbb{R}}(s+1)}$$

$\mathbb{C}$ | $\mathbb{R}$ $\qquad \mathrm{Ind}_{\mathbb{R}}^{\mathbb{C}}(\mathbb{1}) = \mathbb{1}_{\mathbb{R}} \oplus \mathrm{sgn}$

<u>Classical</u>:

$$\Gamma(z)\cdot\Gamma(z+\tfrac{1}{n})\cdots\Gamma(z+\tfrac{n-1}{n}) = (2\pi)^{\frac{1}{2}(n-1)}\cdot 2^{\frac{1}{2}-nz}\cdot\Gamma(nz)$$

$$\boxed{\begin{aligned}\Gamma_{\mathbb{R}}(s) &= \pi^{-s/2}\Gamma(\tfrac{s}{2})\\ \Gamma_{\mathbb{C}}(s) &= 2\cdot(2\pi)^{-s}\Gamma(s)\end{aligned}}$$

$\underline{n=2}$:

$$\Gamma(z)\cdot\Gamma(z+\tfrac{1}{2}) = (2\pi)^{\frac{1}{2}}\cdot 2^{\frac{1}{2}-2z}\,\Gamma(2z)$$

$$\Gamma(\tfrac{s}{2})\cdot\Gamma(\tfrac{s+1}{2}) = (2\pi)^{\frac{1}{2}}\cdot 2^{\frac{1}{2}-s}\cdot\Gamma(s)$$

$$\Gamma_{\mathbb{R}}(s)\cdot\Gamma_{\mathbb{R}}(s+1) = \pi^{-s-\frac{1}{2}}\Gamma(\tfrac{s}{2})\cdot\Gamma(\tfrac{s+1}{2}) = (2\pi)^{\frac{1}{2}}\cdot 2^{\frac{1}{2}-s}\cdot\pi^{-s-\frac{1}{2}}\Gamma(s)$$

$$= 2^{1-s}\cdot\pi^{-s}\,\Gamma(s) = \Gamma_{\mathbb{C}}(s)$$

$\rightsquigarrow$ $\Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\cdot\Gamma_{\mathbb{R}}(s+1)$

(ii) $\mathbb{C} \xrightarrow[z]{\sim} \mathbb{R}$

$$\underset{N\geq 0}{L(z^{-N}\omega_s)} = L(\bar{z}^{-N}\cdot\omega_s) = \Gamma_{\mathbb{C}}(z^{-N}\cdot\omega_s)$$

(iii) K non-archimedian

$$L(\chi) = \begin{cases} 1 & \text{if } \chi \text{ is ramified} \\ \dfrac{1}{1-\chi(\pi)} & \text{if } \chi \text{ is unramified} \end{cases}$$

i.e. $L(\omega_s) = (1 - q_K^{-s})^{-1}$ standard Euler factor

Note $1 \to \mathbb{A}_{K,1}^\times / K^\times \to \mathbb{A}_K^\times / K^\times \xrightarrow{\|\cdot\|} \mathbb{R}_{>0}^\times \to 1$

$\exists$ splitting

$\rightsquigarrow 1 \to \mathrm{Hom}_{cont}(\mathbb{R}_{>0}^\times, \mathbb{C}^\times) \longrightarrow \mathrm{Hom}_{cont}(\mathbb{A}_K^\times / K^\times, \mathbb{C}^\times) \longrightarrow \mathrm{Hom}_{cont}(\overbrace{\mathbb{A}_{K,1}^\times / K^\times}^{\text{compact}}, \mathbb{C}_1^\times) \to 1$

$\mathrm{Hom}_{cont}(\mathbb{A}_K^\times / \mathbb{A}_{K,1}^\times, \mathbb{C}^\times)$

$\| \{ \omega_s : s \in \mathbb{C} \}$

idele class characters

discrete, but not torsion

$\rightsquigarrow$ $\mathrm{Hom}_{cont}(\mathbb{A}_K^\times / K^\times, \mathbb{C}^\times)$ has a natural structure as a complex manifold

Moreover: Every idele class character χ can be written as
$\chi = \omega_s \cdot \chi_1$ where $\chi_1 : \mathbb{A}_K^\times / K^\times \to \mathbb{C}_1^\times$ is a unitary idele class character

$\rightsquigarrow$ $\mathrm{Re}(s)$ is uniquely determined by χ, called the *exponent* of χ

$$L_v(\chi_1 \cdot \omega_s) = \prod_{v \text{ finite}} (\chi_1 \cdot \omega_s|_{K_v^\times}) \text{ is absolutely convergent}$$

unitary idele class character

for $\mathrm{Re}(s) > 1$

—

Local F.E.

K = local field

dx : a Haar measure on $(K, +)$

$\psi : (K,+) \longrightarrow \mathbb{C}_1^\times$ non-trivial unitary character

$d^\times x$: a Haar measure on $(K^\times, \times)$

$f \rightsquigarrow \hat{f}_\psi(x) = \int_K f(y)\, \psi(xy)\, dy$ Fourier transform attached to ψ and dx

Schwartz function on K

$\forall f \in \mathcal{S}(K)$

$$\boxed{\frac{\int_{K^\times} \hat{f}_\psi(x) \cdot \omega_1 \chi^{-1}(x)\, d^\times x}{L(\omega_1 \cdot \chi^{-1})} = \varepsilon(\chi, \psi, dx) \frac{\int_{K^\times} f(x)\, \chi(x)\, d^\times x}{L(\chi)}}$$

local zeta functions

f: Schwartz function on K

$\psi : (K,+) \to \mathbb{C}_1^\times$ nontrivial
dx : a Haar measure for $(K,+)$
$d^\times x$: a Haar measure for $K^\times$
$\exists C>0$ s.t. $d^\times x = C \cdot \frac{dx}{\|x\|}$

Define 1) $\hat{f}_\psi(y) = \hat{f}_{\psi,dx}(y) = \int f(x)\,\psi(xy)\,dx$

Fourier transform on K

Note: For compatibility with classical formula (esp. Tate's thesis) use $\psi(x) = \mathbb{e}(x)$ defined below

(2) $\zeta_{d^\times x}(f,\chi) = \zeta(f,\chi) \underset{\text{def}}{=} \int_{K^\times} f(x)\,\chi(x)\,d^\times x$

thought of / regarded as a function in $\chi \in \mathrm{Hom}_{cont}(K^\times, \mathbb{C}^\times)$ ← complex Lie group

global:

$\mathbb{A}_\mathbb{Q}$: On $\mathbb{Q}_p$ $\quad \mathbb{e}_p(x) = e^{2\pi\sqrt{-1}\,x_1} \qquad x_1 \in \mathbb{Z}[\frac{1}{p}],\ x \equiv x_1 \pmod{\mathbb{Z}_p}$

On $\mathbb{R}$ $\quad \mathbb{e}_\infty(x) = e^{-2\pi\sqrt{-1}\,x \,(\mathrm{mod}\ \mathbb{Z})}$

On $\mathbb{A}_\mathbb{Q}$: $\quad \underset{(x_v)}{\mathbb{e}(x)} = \prod_v \mathbb{e}_v(x_v)$

On $\mathbb{A}_K$: $\quad \mathbb{e}_K(x) = \mathbb{e}(tr_{K/\mathbb{Q}}(x))$

K local

Lemma 1 $\quad f \in \mathcal{S}(K),\ \mathrm{Re}(\chi) > 0 \Rightarrow$ the zeta integral is absolutely convergent for $\mathrm{Re}(\chi)>0$

(elementary)

Prop: $\quad f, g \in \mathcal{S}(K),\ 0<\mathrm{Re}(\chi)<1 \qquad \hat{\chi} := \omega_1 \cdot \chi^{-1} \qquad d^\times x = c\,\frac{dx}{\|x\|}$

$$\Rightarrow \zeta(f,\chi)\cdot\zeta(\hat{g}_\psi, \hat{\chi}) = \zeta(\hat{f}_\psi, \hat{\chi})\,\zeta(g,\chi)$$

Pf: $\zeta(f,\chi)\cdot\zeta(\hat{g}_\psi,\hat{\chi}) = \int_{K^\times} f(x)\,\chi(x)\,d^\times x \cdot \int_{K^\times} \int g(z)\,\psi(yz)\,\|y\|\,\chi^{-1}(y)\,dz\,d^\times y$

$$\underset{\text{y} \mapsto xy}{=} \iiint f(x)\,g(z)\,\psi(xyz)\,\|xy\|\,\chi(y^{-1})\,dz\,d^\times x\,d^\times y$$

$$= c^2 \iiint f(x)\,g(z)\,\psi(xyz)\,\chi^{-1}(y)\cdot dz\,.\,dx\,dy$$

symmetric in f and g $\qquad$ QED

Corollary / Consequence

assume: denominator is not identically 0 on any component

"$\rho(\chi)$" in Tate's thesis

$$\frac{\zeta_{d^\times x}(f,\chi)}{\zeta_{d^\times x}(\hat{f}_{\psi,dx},\hat{\chi})}$$

, as a meromorphic function on $\{\chi \in \mathrm{Hom}_{cont}(K^\times,\mathbb{C}^\times) \mid 0 < \mathrm{Re}(\chi) < 1\}$

is independent of $f \in \mathcal{S}(K)$ and also independent of the choice of the Haar measure $d^\times x$ on $K^\times$, but it depends on the choice of ψ and the Haar measure dx on $(K,+)$

$$\frac{\zeta(\hat{f}_{\psi,dx},\omega\chi^{-1})}{L(\omega,\chi^{-1})} = \varepsilon(\chi,\psi,dx)\,\frac{\zeta(f,\chi)}{L(\chi)}$$

Clearly $\varepsilon(\chi,\psi,c\,dx) = c\,\varepsilon(\chi,\psi,dx) \quad \forall c \in \mathbb{R}$

$$\forall a \in K^\times, \quad \underbrace{\int_K \psi(axy)f(y)\,dy}_{\hat{f}_{\psi(ax),dx}(y)} = \hat{f}_{\psi,dx}(ay)$$

$$\rightsquigarrow \quad \zeta(\hat{f}_{\psi(ax),dx},\omega\chi^{-1}) = \zeta(\hat{f}_{\psi,dx}(ay),\omega\chi^{-1})$$

$$\Rightarrow \quad \varepsilon(\chi,\psi(ax),dx) = \|a\|^{-1}\chi(a)\,\varepsilon(\chi,\psi,dx) \quad \forall a \in K^\times$$

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Vanishing of integrals related to Gauß sums:

$\psi : (K,+) \to \mathbb{C}_1^\times$ non-trivial unitary additive character

$\chi : K^\times \to \mathbb{C}^\times$ multiplicative character | $d^\times x = \frac{dx}{\|x\|}$

$d = d(\psi)$ = conductor of ψ

i.e. $\psi|_{\pi^d\mathcal{O}_K} = 1$, $\psi|_{\pi^{d-1}\mathcal{O}_K} \neq 1$ (Note Tate used $n(\psi)$ for $-d(\psi)$ in Corvalis)

(a) If χ is unramified, then

$$\int_{x\in\pi^n\mathcal{O}_K^\times} \psi(x)\chi(x)\,d^\times x = \chi(\pi)^n\cdot\Big[\underbrace{\int_{\pi^n\mathcal{O}_K}\psi(x)\,dx}_{=0 \text{ if } n<d(\psi)} - \underbrace{\int_{\pi^{n+1}\mathcal{O}_K}\psi(x)\,dx}_{=0 \text{ if } n+1<d(\psi)}\Big] = 0 \quad \text{if } n \le d(\psi)-2$$

(b) If χ is ramified, $a(\chi) =$ conductor of $\chi > 0$

i.e. $\chi|_{1+\pi^{a(\chi)}\mathcal{O}_K} = 1$, $\chi|_{\underbrace{1+\pi^{a(\chi)-1}}_{\overset{\parallel}{\mathcal{O}_K^\times}}}$ non-trivial

if $a(\chi)=1$ by convention

then $$\int_{\pi^n \mathcal{O}_K^\times} \psi(x)\, \chi(x)\, d^\times x = 0 \quad \text{unless} \quad n = d(\psi) - a(\chi)$$

i.e. $d(\psi) - n = a(\chi)$

$$= \chi(\pi^n) \int_{\mathcal{O}_K^\times} \psi(\pi^n u)\, \chi(u)\, d^\times x$$

conductor of $x \mapsto \psi(\pi^n u)$ is $d(\psi) - n$

Roughly: if $n \gg 0$. $\psi(\pi^n u)|_{u \in \mathcal{O}_K^\times} = 1$ while $\chi|_{\mathcal{O}_K^\times}$ is non-trivial

if $n \ll 0$: look at $u \in$ cosets (additive) mod $\pi^{a(\chi)}\mathcal{O}_K$;

at such a coset, χ is constant but ψ is not constant

Remarks about local zeta functions

(1) If $f \in \mathcal{S}(K^\times)$, then $\chi \mapsto \zeta(f,\chi) := \int_{K^\times} f(x)\, \chi(x)\, d^\times x$

(depending on the Haar measure $d^\times x$ for $K^\times$)

(2) $L(\chi)$: gives all possible poles of $\zeta(f,\chi)$ (in a strong sense), so that $\dfrac{\zeta(f,\chi)}{L(\chi)}$ is entire

i.e. holomorphic on $\mathrm{Hom}_{\mathrm{cont}}(K^\times, \mathbb{C}^\times)$

Note also that $L(\chi)^{-1}$ is entire on $\mathrm{Hom}_{\mathrm{cont}}(K^\times, \mathbb{C})$,
so can think of $L(\chi)$ as the "lcm" of all polar divisors of all $\zeta(f,\chi)$'s (as f varies in $\mathcal{S}(K)$

Back to local zeta functions:

Have
$$\frac{\zeta(f,\chi)}{\zeta(\hat{f}_{\psi,dx},\omega_1\cdot\chi^{-1})} =: \underset{\substack{\uparrow\\ \text{Tate's notation}}}{\rho_{\psi,dx}(\chi)} \overset{\text{will be}}{=\!=} \frac{L(\chi)}{L(\omega_1\cdot\chi^{-1})}\cdot\varepsilon(\chi,\psi,dx)^{-1}$$

(I) $K \underset{x}{\xrightarrow{\sim}} \mathbb{R}$

Will use two Schwartz functions, $f_0(x) := e^{-\pi x^2}$ and $f_1(x) = x\,e^{-\pi x^2}$

2 kinds of characters: $x \mapsto \underset{=\omega_s}{|x|^s}$ and $x \mapsto x^{-1}\cdot|x|^s$

parity: $\zeta(f_0, x^{-1}\omega_s) = 0 \qquad \zeta(f_1,\omega_s)=0$ Use dx = Lebsque measure, $d^\times x = \frac{dx}{|x|}$

$$\zeta(f_0,\omega_s) = 2\int_{\mathbb{R}^\times_{>0}} e^{-\pi x^2} x^s \frac{dx}{x} \underset{\substack{t=\pi x^2\\ \Rightarrow x=\pi^{-\frac12}t^{\frac12}\\ 2\frac{dx}{x}=\frac{dt}{t}}}{=\!=} \pi^{-\frac{s}{2}}\cdot\int_{\mathbb{R}^\times_{>0}} e^{-t}t^{\frac{s}{2}}\frac{dt}{t} = \pi^{-\frac{s}{2}}\cdot\Gamma\left(\tfrac{s}{2}\right)$$
$$= \Gamma_{\mathbb{R}}(s) = L_{\mathbb{R}}(\omega_s)$$

$$\zeta(f_1, x^{-1}\cdot\omega_s) = \int_{\mathbb{R}^\times_{>0}} e^{-\pi x^2}|x|^s\frac{dx}{x} = \Gamma_{\mathbb{R}}(s) = L_{\mathbb{R}}(\omega_s)$$

Find Fourier transform

$$\hat{f}_{0\,\underset{\substack{\|\\ \exp(-2\pi\sqrt{-1}x)}}{e_\infty(x)},\,dx}(y) = \int_{\mathbb{R}} e^{-\pi x^2}\cdot e^{-2\pi\sqrt{-1}xy}\,dx = \int_{\mathbb{R}} e^{-\pi(x+\sqrt{-1}y)^2}\cdot e^{-\pi y^2}\,dx$$
$$= e^{-\pi y^2}$$

$$\left(\int_{\mathbb{R}} e^{-\pi x^2}dx\right)^2 = \iint_{\mathbb{R}^2} e^{-\pi(x^2+y^2)}dx\,dy = \iint_{\substack{r>0\\ 0\le\theta\le 2\pi}} e^{-\pi r^2}r\,dr\,d\theta = 2\pi\cdot\left(\left[-\frac{1}{2\pi}e^{-\pi r^2}\right]_0^\infty\right) = 1$$

Take the previous equality, $\frac{\partial}{\partial y}$ both sides:

$$-2\pi\sqrt{-1}\int_{\mathbb{R}} x e^{-\pi x^2}\cdot e^{-2\pi\sqrt{-1}xy}dx = -2\pi y\, e^{-\pi y^2}$$

i.e.
$$\hat{f}_{1,\,\underset{\substack{\|\\ \exp(-2\pi\sqrt{-1}x)}}{e_\infty(x)},\,dx}(y) = -\sqrt{-1}\cdot y e^{-\pi y^2} = -\sqrt{-1}\,f_1(y)$$

The General formula

$$\varepsilon(\chi, \psi, dx)\,\frac{\zeta(f,\chi)}{L(\chi)} = \frac{\zeta(\hat{f}_{\psi,dx}, \omega\chi^{-1})}{L(\omega\chi^{-1})}$$

$$\zeta(\hat{f}_{1,e_\infty,dx}, \underbrace{\omega(x^{-1}\omega_s)^{-1}}) = \zeta(-\sqrt{-1} f_1, x^{-1}\omega_s) = -\sqrt{-1}\,\Gamma_{\mathbb{R}}(2-s)$$

$$\shortparallel$$
$$x\cdot\omega_{1-s}$$
$$\shortparallel$$
$$x^{-1}\omega_{2-s} \qquad L_{\mathbb{R}}(x^{-1}\omega_s) = \Gamma_{\mathbb{R}}(2-s)$$

Conclusion : (0) $\zeta(f,\chi) = L(\chi) +$ entire function in χ

(1) $\varepsilon\left(\begin{smallmatrix}\omega_s \\ x^{-1}\omega_s\end{smallmatrix}, \exp(-2\pi\sqrt{-1}x), dx\right) = \begin{cases} 1 \\ -\sqrt{-1}\end{cases}$

i.e. $\varepsilon\left(x^{-N}\omega_s, \exp(-2\pi\sqrt{-1}x), dx\right) = -\sqrt{-1}^{\,N}$

$N=0,1$

$\varepsilon\left(x^{-N}\omega_s, \exp(2\pi\sqrt{-1}x), dx\right) = \sqrt{-1}^{\,N}$

Lebesgue measure

Recall that

$f_0 = e^{-\pi x^2}$ $\qquad \zeta(f_0, \omega_s) = L_{\mathbb{R}}(s) = \pi^{\frac{s}{2}}\,\Gamma(\tfrac{s}{2}) = \int_{\mathbb{R}^\times} e^{-\pi x^2} e^{-2\pi\sqrt{-1}x}\,\frac{dx}{x}$

$f_1 = x e^{-\pi x^2}$ $\qquad \zeta(f_1, x^{-1}\omega_s) = L_{\mathbb{R}}(s)$

(II) $K \xrightarrow[z]{\sim} \mathbb{C}$ $\qquad n \in \mathbb{N}_{\geq 0}$ $\quad f_n(z) = z^n e^{-2\pi z\bar{z}}$ $\qquad$ Haar measure for $\mathbb{C}$: $|dz\,d\bar{z}|$

$\psi : z \mapsto \exp(-2\pi\sqrt{-1}(z+\bar{z}))$

$n \geq 0$

$$\zeta(f_n, z^{-n}\cdot\omega_s) = \int_{\mathbb{C}} e^{-2\pi z\bar{z}}\cdot(z\bar{z})^s\,\frac{dz\,d\bar{z}}{z\bar{z}}$$

$$= \iint_{\mathbb{R}^2} e^{-2\pi(x^2+y^2)}(x^2+y^2)^{s-1}\,2\,dx\,dy = \iint_{\substack{r>0 \\ 0\leq\theta\leq 2\pi}} e^{-2\pi r^2} r^{2s-1}\,2\,dr\,d\theta$$

$$\underset{\substack{2\pi r^2 = t \\ r = (2\pi)^{-\frac{1}{2}}\cdot t^{\frac{1}{2}} \\ \frac{2dr}{r} = \frac{dt}{t}}}{=} (2\pi)\cdot\int_{t\in\mathbb{R}_{>0}} e^{-t}\cdot t^s (2\pi)^{s}\cdot\frac{dt}{t} = (2\pi)\cdot(2\pi)^{-s}\cdot\Gamma(s)$$

$$= (2\pi)^{1-s}\cdot\Gamma(s)$$

$$= \pi\cdot\Gamma_{\mathbb{C}}(s)$$

Recall $\Gamma_{\mathbb{C}}(s) = 2\cdot(2\pi)^{-s}\,\Gamma(s)$

$$\zeta(f_n, z^{-n}\cdot\omega_s) = \pi\cdot\Gamma_{\mathbb{C}}(s) = \zeta(g_n, \bar{z}^{-n}\cdot\omega_s) \qquad \forall n \geq 0$$

use $\frac{|dz\,d\bar{z}|}{z\bar{z}}$ on $\mathbb{C}^\times$

where $f_n(z) = z^{-n}\cdot\omega_s$

$g_n(z) = \bar{z}^{-n}\cdot\omega_s$

Of course, $\zeta(f_m, z^{-n}\omega_s) = 0$ $\forall m\geq 0, m\neq n, n\geq 0$ $\qquad \zeta(g_m, \bar{z}^{-n}\omega_s) = 0$ $\forall m,n\geq 0, m\neq n$

$\zeta(f_m, \bar{z}^{-n}\omega_s) = 0$ $\forall m\geq 0\ \forall n>0$ $\qquad \zeta(g_m, z^{-n}\omega_s) = 0$ $\forall m\geq 0\ \forall n>0$

Compute $\hat{f}_{0,\psi,dzd\bar{z}}(w) = \int_{\mathbb{C}} e^{-2\pi z\bar{z}} \cdot e^{-2\pi\sqrt{-1}(zw+\bar{z}\bar{w})} |dz\cdot d\bar{z}|$

$\psi: z \mapsto \exp(-2\pi\sqrt{-1}(z+\bar{z}))$

$z = x+\sqrt{-1}y$

$w = u+\sqrt{-1}v$

$$= \iint e^{-2\pi x^2} \cdot e^{-4\pi\sqrt{-1}xu} \cdot e^{-2\pi y^2} \cdot e^{-4\pi\sqrt{-1}yv} \cdot 2dxdy \qquad zw+\bar{z}\bar{w} = 2(xu-yv)$$

$$= \left(\int e^{-2\pi(x+\sqrt{-1}u)^2} \cdot e^{-2\pi u^2} dx\right) \cdot \left(\int e^{-2\pi(y+\sqrt{-1}v)^2} \cdot e^{-2\pi v^2} dy\right)$$

$$= 2\cdot e^{-2\pi(u^2+v^2)} \cdot \left(\int_{\mathbb{R}} e^{-2\pi x^2} dx\right)^2$$

$$= e^{-2\pi(u^2+v^2)} = e^{-2\pi w\bar{w}}$$

$$\int_{\mathbb{R}} e^{-2\pi x^2} dx \underset{x=\frac{t}{\sqrt{2}}}{=} \frac{1}{\sqrt{2}} \int_{\mathbb{R}} e^{-\pi t^2} dt = \frac{1}{\sqrt{2}}$$

Take $\left(\frac{\sqrt{-1}}{2\pi}\frac{\partial}{\partial w}\right)^n$ of $\int_{\mathbb{C}} e^{-2\pi z\bar{z}} \cdot e^{-2\pi\sqrt{-1}(zw+\bar{z}\bar{w})} |dz\,d\bar{z}|$

$n\geq 0$

$$= e^{-2\pi w\bar{w}}$$

Get $\hat{f}_{n,\psi,dzd\bar{z}} = (-\sqrt{-1})^n g_n$ $\qquad n\geq 0$ $\qquad$ Recall $f_n(z) = z^n e^{-2\pi(z\bar{z})}$

$(\psi(z) = \exp(-2\pi\sqrt{-1}(z+\bar{z})))$ $\qquad g_n(z) = \bar{z}^n e^{-2\pi z\bar{z}}$

Similarly $\hat{g}_{n,\psi,dzd\bar{z}} = (-\sqrt{-1})^n f_n$

$$\frac{\zeta(\hat{f}, \omega\chi^{-1})}{L(\omega\chi^{-1})} = \varepsilon(\chi, \psi, dzd\bar{z}) \frac{\zeta(f,\chi)}{L(\chi)}$$

for $\chi = z^{-n}\cdot\omega_s$

$f = f_n$

$\omega\chi^{-1} = z^n \cdot \omega_{1-s}$

$= \bar{z}^{-n}\omega_{n+1-s}$

$\rightsquigarrow \hat{f}_n = (-\sqrt{-1})^n \cdot g_n$

Conclude

$$\varepsilon(z^{-n}\omega_s, \exp(-2\pi\sqrt{-1}(z+\bar{z})), dzd\bar{z}) = (-\sqrt{-1})^n \qquad \forall n\geq 0$$

$$\varepsilon(\bar{z}^{-n}\omega_s, \exp(-2\pi\sqrt{-1}(z+\bar{z})), dzd\bar{z}) = (-\sqrt{-1})^n$$

III $K/\mathbb{Q}_p$ Use $d^\times x$: $\int_{\mathcal{O}_K^\times} d^\times x = 1$ (Remembering e.g. global situation, want to multiply this Haar measure by $\frac{1}{\mathrm{disc}(K/\mathbb{Q}_p)^{1/2}}$)

and $\int_{\mathcal{O}_K} dx = 1$ also here

(a) χ unramified Consider $\chi \cdot \omega_s$ s.t. $\mathrm{Re}(\chi) + \mathrm{Re}(s) > 0$

Here $d^\times x$: $\mathcal{O}_K^\times$ has volume 1; dx: $\mathcal{O}_K$ has volume 1

f_0 = characteristic function of $\mathcal{O}_K$ $\qquad q = \#(\mathcal{O}_K/\mathfrak{p}_K)$

$$\zeta(f_0, \chi\cdot\omega_s) = \int_{\mathcal{O}_K} (\chi\cdot\omega_s)(x)\, d^\times x = \sum_{n\geq 0} \chi(\pi)^n q^{-sn} = \frac{1}{1-\chi(\pi)\cdot q^{-s}} = L(\chi\cdot\omega_s)$$

$\hat{f}_{0,\psi,dx}$ for general ψ.

[possibility $\psi_{\text{(of)}} e_K(x) = e_p(\mathrm{Tr}_{K/\mathbb{Q}_p}(x))$]

Say $d(\psi)$ s.t. $\psi|_{\pi^n\mathcal{O}_K} = 1\!\!1$, $\psi|_{\pi^{n-1}\mathcal{O}_K} \neq 1\!\!1$

$d(\psi)$ = Conductor of ψ

$$\hat{f}_{0,\psi,dx}(y) = \int_{x\in\mathcal{O}_K} \psi(xy)\, dx = \begin{cases} 0 & \text{if } \mathrm{ord}_K(y) < d(\psi) \\ 1 & \text{if } \mathrm{ord}_K(y) \geq d(\psi) \end{cases}$$

$$\zeta(\hat{f}_{0,\psi,dx}, \chi^{-1}\omega_{1-s}) = \int_{\pi^{d(\psi)}\mathcal{O}_K} (\chi^{-1}\cdot\omega_{1-s})(y)\, d^\times y$$

$$= \frac{\chi(\pi^{d(\psi)})^{-1}\cdot q^{d(\psi)(s-1)}}{1-\chi(\pi)^{-1}\cdot q^{s-1}}$$

Conclusion

$$\varepsilon(\underbrace{\chi\omega_s}_{\text{unramified}}, \psi, dx) = (\chi^{-1}\cdot\omega_{1-s})(\pi^{d(\psi)})$$

$\psi|_{\pi^{d(\psi)}\mathcal{O}_K} = 1\!\!1$, $\psi|_{\pi^{d(\psi)-1}\mathcal{O}_K} \neq 1\!\!1$ $\qquad \int_{\mathcal{O}} dx = 1$

"const" $\cdot\, q^{-ds}$

integer, depending only on ψ

More succinctly:

$$\boxed{\varepsilon(\chi, \psi, dx) = (\omega_1\chi^{-1})(\pi^{d(\psi)})\cdot\int_{\mathcal{O}} dx \quad \text{where } \psi|_{\pi^{d(\psi)}\mathcal{O}_K} = 1\!\!1,\ \psi|_{\pi^{d(\psi)-1}\mathcal{O}_K} \neq 1\!\!1}$$

χ unramified

(b) χ ramified, i.e. $a(\chi) > 0$ $\quad \chi|_{1+\pi^{a(\chi)}\mathcal{O}_K} = \mathbb{1}$

$\chi|_{1+\pi^{a(\chi)-1}\mathcal{O}_K} \neq \mathbb{1}$

$d(\psi)$: $\quad \psi|_{\pi^{d(\psi)}\mathcal{O}_K} = \mathbb{1}, \quad \psi|_{\pi^{d(\psi)-1}\mathcal{O}_K} \neq \mathbb{1}$

Take f = characteristic function of $\mathcal{O}_K$ $\qquad Re(\chi) + Re(s) > 0$

$$\int_{\mathcal{O}_K} (\chi\cdot\omega_s)(x)\, d^\times x = \sum_{n\geq 0} q^{-ns} \cdot \underbrace{\int_{\pi^n\cdot\mathcal{O}_K^\times} \chi(y)\, d^\times y}_{=\,0}$$

no good!

But $\displaystyle\int_{\pi^n\mathcal{O}_K} (\chi\cdot\omega_s)(x)\, d^\times x = 0 \quad \forall n \geq 0.$

Conclude: $\zeta(f,\chi)$ is entire on $\chi \in Hom_{cont}(K^\times, \mathbb{C}^\times)$!

$\rightsquigarrow$ It is reasonable to take $L(\chi) = 1$ for χ unramified!

Thus: $\zeta(\hat{f}_{\psi,dx}, \omega_1\cdot\chi^{-1}) = \varepsilon(\chi,\psi,dx)\cdot\zeta(f,\chi)$

F.E. becomes

Need a function f s.t. $\zeta(f,\chi) \neq 0$

Let:

Let f = characteristic function of $1+\pi^{a(\chi)}\mathcal{O}$

Choose $d^\times x$ s.t. $d^\times x = \frac{dx}{\|x\|}$

$$\Rightarrow \zeta(f,\chi\cdot\omega_s) = \int_{1+\pi^{a(\chi)}\mathcal{O}} \omega_s(x)\, d^\times x = \int_{1+\pi^{a(\chi)}\mathcal{O}} d^\times x$$

What is $\hat{f}_{\psi,dx}$?

$$\hat{f}_{\psi,dx}(y) = \int_{x\in 1+\pi^{a(\chi)}} \psi(xy)$$

Not a good choice.

Use Gauss sums:

Use $f = \chi^{-1}\cdot$(characteristic function of $\mathcal{O}_K^\times$)

Note: As reflected / indicated in the formula $\hat{f}_{\psi,dx} = \chi|_{\pi^{d(\psi)-a(\chi)}O_K^\times} \cdot \int_{\pi^{d(\psi)-a(\chi)}O_K^\times} \psi(z)\cdot\chi^{-1}(z)\, dz$ $\cdot$ (char. function of $\pi^{d(\psi)-a(\chi)}\cdot O_K^\times$) the r.h.s. depends only on $\chi|_{O_K^\times}$

Let $f(x) = \begin{cases} \chi^{-1} & \text{on } O_K^\times \\ 0 & \text{elsewhere} \end{cases}$

$\Rightarrow \quad \zeta(f, \chi\cdot\omega_s) = \int_{O_K^\times} d^\times x$

$$\hat{f}_{\psi,dx}(y) = \int_{O_K^\times} \psi(xy)\,\chi^{-1}(x)\,\frac{dx}{|x|} \underset{x=y^{-1}z}{=} \chi(y)\int_{yO_K^\times} \psi(z)\,\chi^{-1}(z)\,\frac{dz}{|z|}$$

$$= \left[\int_{\pi^{d(\psi)-a(\chi)}O_K^\times} \psi(z)\cdot\chi^{-1}(z)\,\frac{dz}{|z|}\right]\cdot\chi(y) \leftarrow y\in\pi^{d(\psi)-a(\chi)}O_K^\times$$

0 otherwise

$= 0$ unless $ord_K(y) = d(\psi) - a(\chi)$

$$\Rightarrow \zeta(\hat{f}_{\psi,dx}, \omega_{1-s}\chi^{-1}) = \left[\int_{\pi^{d(\psi)-a(\chi)}O_K^\times} \psi(z)\,\chi^{-1}(z)\,\frac{dz}{|z|}\right]\cdot \underbrace{\int_{\pi^{d(\psi)-a(\chi)}O_K^\times} \chi(y)\cdot(\omega_{1-s}\cdot\chi^{-1})(y)\,d^\times y}_{=\ \|\pi\|^{(d(\psi)-a(\chi))(1-s)}\cdot\int_{O_K^\times} d^\times y}$$

$|z| = \|\pi\|^{-(d(\psi)-a(\chi))}$

$$\hat{f}_{\psi,dx} = \left[\int_{\pi^{d(\psi)-a(\chi)}O_K^\times} \chi^{-1}(z)\,\psi(z)\,dz\right]\cdot\chi\cdot\left(\text{char. function of } \pi^{d(\psi)-a(\chi)}\cdot O_K^\times\right)$$

$$= \int_{\pi^{d(\psi)-a(\chi)}O_K^\times} \psi(z)\,(\chi^{-1}\cdot\omega_{-s})(z)\,\underbrace{dz}_{\text{additive Haar measure}}$$

Conclusion:

$$\varepsilon(\chi\cdot\omega_s, \psi, dx) = \int_{\pi^{d(\psi)-a(\chi)}O_K^\times} \psi(x)\cdot(\chi^{-1}\cdot\omega_{-s})(x)\,dx$$

or.

$$\boxed{\varepsilon(\chi, \psi, \underline{dx}) = \int_{\pi^{d(\psi)-a(\chi)}O_K^\times} \psi(x)\,\chi^{-1}(x)\,\underline{dx}}$$ if χ is ramified

+ (of course) $\zeta_{d^\times x}(f, \chi)$ is an entire function in $\chi \in Hom(K^\times, \mathbb{C}^\times)$

Summarize:

(1) $K \xrightarrow[x]{\sim} \mathbb{R}$ $\rightsquigarrow$

$$\varepsilon(x^{-n}\omega_s,\ \exp(-2\pi\sqrt{-1}\,x),\ \underset{\substack{\uparrow \\ \text{Lebesque}}}{dx}) = (-\sqrt{-1})^n \qquad n=0,1$$

$$\zeta_{\frac{dx}{|x|}}(e^{-\pi x^2},\ \omega_s) = \underset{\substack{\uparrow \\ \pi^{-s/2}\Gamma(\frac{s}{2})}}{\Gamma_{\mathbb{R}}(s)} = \zeta_{\frac{dx}{|x|}}(x e^{-\pi x^2},\ x^{-1}\omega_s)$$

$$\mathrm{F.T.}_{\exp(-2\pi\sqrt{-1}x),\,dx}(e^{-\pi x^2}) = e^{-\pi y^2}$$

$$\mathrm{F.T.}_{\exp(-2\pi\sqrt{-1}x),\,dx}(x e^{-\pi x^2}) = -\sqrt{-1}\ y\cdot e^{-\pi y^2}$$

(2) $K \xrightarrow[z]{\sim} \mathbb{C}$

$$\varepsilon(z^{-n}\omega_s,\ \exp(-2\pi\sqrt{-1}(z+\bar z)),\ |dz\,d\bar z|) = (-\sqrt{-1})^n \qquad \forall n\geq 0$$

$$\varepsilon(\bar z\ \omega_s,\ \exp(-2\pi\sqrt{-1}(z+\bar z)),\ |dz\,d\bar z|) = (-\sqrt{-1})^n \qquad \forall n\geq 0$$

$$\zeta_{\frac{dz\,d\bar z}{z\bar z}}(z^n e^{-2\pi z\bar z},\ z^{-n}\omega_s) = \pi\underbrace{\Gamma_{\mathbb{C}}(s)}_{2\cdot(2\pi)^{-s}\Gamma(s)}$$

$$= \zeta_{\frac{dz\,d\bar z}{z\bar z}}(\bar z^n e^{-2\pi z\bar z},\ \bar z^{-n}\omega_s) \qquad \forall n\geq 0 \ (\in \mathbb{N})$$

$$\mathrm{FT}_{\exp(-2\pi\sqrt{-1}(z+\bar z)),\,dz\,d\bar z}(z^n e^{-\pi z\bar z}) = (-\sqrt{-1})^n\ \bar w^n e^{-\pi w\bar w}$$

$$\mathrm{FT}_{\exp(-2\pi\sqrt{-1}(z+\bar z)),\,dz\,d\bar z}(\bar z^n e^{-\pi z\bar z}) = (-\sqrt{-1})^n\ w^n e^{-\pi w\bar w} \qquad \forall n\in\mathbb{N}$$

(3) K / $\mathbb{Q}_p$

(A) χ unramified

$$\Rightarrow \varepsilon(\chi,\psi,dx) = (\omega_1\cdot\chi^{-1})(\pi^{d(\psi)})\cdot\int_{\mathcal{O}_K} dx \qquad \psi|_{\pi^{d(\psi)}\mathcal{O}_K} = \mathbb{1},\quad \psi|_{\pi^{d(\psi)-1}\mathcal{O}_K} \neq \mathbb{1}$$

$$L(\chi) = \frac{1}{1-\chi(\pi)}$$

$$\zeta_{d^\times x}(\text{char. function of } \mathcal{O}_K,\ \chi) = L(\chi)\cdot\int_{\mathcal{O}_K^\times} d^\times x$$

$$\mathrm{F.T.}_{\psi,dx}\begin{pmatrix}\text{char. fnc} \\ \text{of } \mathcal{O}_K\end{pmatrix} = \begin{pmatrix}\text{char. fnc of} \\ \pi^{d(\psi)}\mathcal{O}_K\end{pmatrix}\cdot\int_{\mathcal{O}_K} dx$$

(B) χ ramified $\qquad a(\chi) = \text{conductor of } \chi > 0 \quad \Rightarrow L(\chi) = 1$

$$\varepsilon(\chi,\psi,\underline{dx}) = \int_{\pi^{d(\psi)-a(\chi)}\mathcal{O}_K^\times} \psi(x)\cdot\chi^{-1}(x)\,\underline{dx}$$

$$\zeta_{d^\times x}(\underset{\substack{\uparrow \\ =\chi^{-1} \text{ on } \mathcal{O}_K^\times \\ =0 \text{ elsewhere}}}{\widetilde{\chi^{-1}}},\ \chi) = \int_{\mathcal{O}_K^\times} d^\times x\ ,\quad \mathrm{F.T.}_{\psi,dx}(\widetilde{\chi^{-1}})(y) = \left(\int_{\pi^{d(\psi)-a(\chi)}\mathcal{O}_K^\times}\psi(z)\chi^{-1}(z)\frac{dz}{|z|}\right)\cdot\underset{\substack{\text{on } \pi^{d(\psi)-a(\chi)}\mathcal{O}_K^\times \\ =0 \text{ outside } \pi^{d(\psi)-a(\chi)}\mathcal{O}_K^\times}}{\hat{\tilde{\chi}}(y)}$$

F.E. for
Global zeta function

dx: self-dual Haar measure on $\mathbb{A}_K/K$

$d^\times x = \frac{dx}{|x|}$ $\quad |x|$ = adelic norm of x

$$\mathcal{S}(\mathbb{A}_K) \ni f \rightsquigarrow \zeta_{d^\times x}(f,\chi) := \int_{\mathbb{A}_K^\times} f(x)\chi(x)\, d^\times x = \int_{\mathbb{R}^\times_{>0}} \frac{dt}{t} \overbrace{\int_{u\in\mathbb{A}^\times_{K,1}} f(tu)\chi(tu)\, d^\times u}^{\zeta_t(f,\chi) \text{ notation}}$$

$\chi \in \mathrm{Hom}_{cont}(\mathbb{A}_K^\times/K^\times, \mathbb{C}^\times)$

Write $x = t\cdot u$, $t \in \mathbb{A}_K^\times/\mathbb{A}^\times_{K,1} \xrightarrow[\cong]{Nm} \mathbb{R}^\times_{>0}$, $u \in \mathbb{A}^\times_{K,1}$ $\qquad d^\times x = \frac{dt}{t}\cdot d^\times u$

$$\underset{u=\xi\cdot v}{=} \int_{\mathbb{R}^\times_{>0}} \frac{dt}{t} \cdot \int_{v\in \mathbb{A}^\times_{K,1}/K^\times} \Big(\sum_{\xi\in K^\times} f(t\xi v)\Big)\cdot \chi(tv)\, d^\times v$$

$\xi \in K^\times$, $v \in \mathbb{A}^\times_{K,1}/K^\times$, $d^\times u$ "=" $d^\times v$

i.e. $d^\times u$ induces $d^\times v$ on $\mathbb{A}^\times_{K,1}$

$$\sum_{\xi\in K^\times} f(\xi t v) = \Big(\sum_{\xi\in K} f(\xi t v)\Big) - f(0) \underset{\text{Poisson summation}}{=} \frac{1}{t}\cdot \sum_{\xi\in K^\times} \hat f(\xi\cdot t^{-1} v^{-1}) + \frac{1}{t}\hat f(0) - f(0)$$

Note $\int_{y\in K} f(ytv)\,\psi(yz)\,dy$

$$\underset{y=w\cdot t^{-1}v^{-1}}{=} \Big(\int_{w\in K} f(w)\,\psi\big(w\tfrac{z}{tv}\big)\,dw\Big)\cdot\frac{1}{t} = \frac{1}{t}\cdot \hat f_\psi\big(\tfrac{z}{tv}\big)$$

$$\rightsquigarrow \zeta_{d^\times x}(f,\chi) = \int_{\mathbb{R}^\times_{>0}} \frac{dt}{t}\cdot \zeta_t(f,\chi).$$

$$\zeta_t(f,\chi) = \int_{v\in\mathbb{A}^\times_{K,1}/K^\times} \Big[\sum_{\xi\in K^\times} \hat f(\xi\cdot t^{-1}v^{-1})\Big]\cdot \overbrace{\tfrac{1}{t}\cdot\chi(tv)}^{\hat\chi(w\cdot t^{-1})}\, d^\times v \qquad w^{-1}=v$$
$$+ \frac{1}{t}\hat f(0)\cdot\int_{v\in\mathbb{A}^\times_{K,1}/K^\times} \chi(tv) - f(0)\cdot\int_{v\in\mathbb{A}^\times_{K,1}/K^\times}\chi(tv)\,d^\times v$$
$$\underset{v=\frac{1}{w}}{=} \int_{w\in\mathbb{A}^\times_{K,1}/K^\times}\Big[\sum_{\xi\in K^\times}\hat f(\xi\cdot w\cdot t^{-1})\Big]\cdot\hat\chi(w\cdot t^{-1})\,d^\times w$$
$$+ \hat f(0)\int_{w\in\mathbb{A}^\times_{K,1}/K^\times}\hat\chi\big(\tfrac{w}{t}\big)\,d^\times w - f(0)\int_{v\in\mathbb{A}^\times_{K,1}/K^\times}\chi(tv)\,d^\times v$$
$$= \zeta_{\frac{1}{t}}(\hat f,\hat\chi) + \hat f(0)\int_{\mathbb{A}^\times_{K,1}/K^\times}\hat\chi\big(\tfrac{w}{t}\big)\,d^\times w - f(0)\int_{\mathbb{A}^\times_{K,1}/K^\times}\chi(tv)\,d^\times v$$

More symmetrically:

$$\boxed{\zeta_t(f,\chi) + f(0)\int_{\mathbb{A}^\times_{K,1}/K^\times}\chi(tv)\,d^\times v = \zeta_{t^{-1}}(\hat f,\hat\chi) + \hat f(0)\int_{\mathbb{A}^\times_{K,1}/K^\times}\hat\chi(t^{-1}w)\,d^\times w}$$

Note
$$\int_{\mathbb{A}^\times_{K,1}/K^\times}\chi(tv)\,d^\times v = \begin{cases} 0 & \text{if } \chi \text{ is ramified} \\ \kappa t^s. & \text{if } \chi = \omega_s\end{cases}$$

$$\left[\int_{\mathbb{A}^\times_{K,1}/K^\times}\hat\chi(t^{-1}w)\,d^\times w = \begin{cases} 0 & \text{if } \chi \text{ is ramified} \\ \kappa\cdot t^{s-1} & \text{if } \chi = \omega_s\end{cases}\right]$$

$$\kappa := \int_{\mathbb{A}^\times_{K,1}/K^\times} d^\times u = \frac{2^{r_1}(2\pi)^{r_2}\, h_K\cdot R_K}{\#\mu(K)\cdot|\mathrm{disc}(K/\mathbb{Q})|^{\frac{1}{2}}}$$

Case χ unramified, $\chi = \omega_s$

$$\zeta(f,\chi) = \int_{\mathbb{R}^\times_{>0}} \zeta_t(f,\chi)\frac{dt}{t} = \underset{\substack{\uparrow \\ \text{abs. converg.} \because f \in \mathcal{S}(\mathbb{A}_K)}}{\int_1^\infty} \zeta_t(f,\chi)\frac{dt}{t} + \int_0^1 \underbrace{\zeta_t(f,\chi)}_{\zeta_{t^{-1}}(\hat{f},\hat{\chi}) + \hat{f}(0)\cdot\kappa\cdot t^{s-1} - f(0)\cdot\kappa\cdot t^{s}} \frac{dt}{t}$$

$$= \int_1^\infty \zeta_t(f,\chi)\frac{dt}{t} + \int_1^\infty \zeta_t(\hat{f},\hat{\chi})\frac{dt}{t} - \underbrace{\hat{f}(0)\cdot\kappa\cdot\frac{1}{1-s}}_{} - \underbrace{f(0)\cdot\kappa\frac{1}{s}}_{}$$

abs. converg. $\forall\chi$ need: $Re(s)>1$ for $\int_0^1 t^{s-1}\frac{dt}{t}$ to converge need: $Re(s)>0$ for $\int_0^1 t^s \frac{dt}{t}$ to converge

Case χ ramified: $\zeta(f,\chi) = \int_1^\infty \zeta_t(f,\chi)\frac{dt}{t} + \int_1^\infty \zeta_t(\hat{f},\hat{\chi})$ entire in $\chi \in \mathrm{Hom}_{cont}(\mathbb{A}_K^\times/K^\times, \mathbb{C}^\times)$

Theorem: (1) $\zeta(f,\chi)$ is abs. conv. if $Re(\chi)>1$, and admits an analytic continuation to $\mathrm{Hom}_{cont}(\mathbb{A}_K^\times/K^\times)$ with simple poles at $\chi_1 = \omega_1$ and $\chi_2 = \omega_0$, holomorphic elsewhere

trivial character

(2) $\operatorname{Res}_{\chi=\omega_1} \zeta(f,\chi) = \hat{f}(0)\cdot\kappa$, $\operatorname{Res}_{\chi=\omega_0}\zeta(f,\chi) = -f(0)\cdot\kappa$

(3) $\zeta(f,\chi) = \zeta(\hat{f},\hat{\chi})$ $\left(\hat{f}(0)\cdot\frac{\kappa}{s-1} - f(0)\cdot\kappa\frac{1}{s}\right.$ does not change under $f\mapsto\hat{f}$, $\left.s\mapsto 1-s\right)$

equality of two meromorphic functions on $\mathrm{Hom}_{cont}(\mathbb{A}_K^\times/K^\times, \mathbb{C}^\times)$

(4a) $\zeta(f,\omega_s) + f(0)\cdot\kappa\cdot\frac{1}{s} + \hat{f}(0)\cdot\kappa\cdot\frac{1}{1-s} = \int_1^\infty \zeta_t(\hat{f},\omega_{1-s})\frac{dt}{t} + \int_1^\infty \zeta_t(f,\omega_s)\frac{dt}{t}$

$\parallel$ equality of entire functions

$\zeta(\hat{f},\omega_{1-s}) + \hat{f}(0)\cdot\kappa\cdot\frac{1}{1-s} + f(0)\cdot\kappa\cdot\frac{1}{s}$

Consequence $\prod_{v\in\Sigma_K} L_v(\hat{\chi}) \cdot \prod_{v\in\Sigma_f} \varepsilon_v(\chi_v,\psi_v,dx_v) = \prod_{v\in\Sigma_K} L_v(\chi)$

(4b) $\zeta(f,\chi) = \zeta(\hat{f},\hat{\chi}) = \int_1^\infty \zeta_t(f,\chi)\frac{dt}{t} + \int_1^\infty \zeta_t(\hat{f},\hat{\chi})\frac{dt}{t}$

Integral representation of (the holomorphic part of) the completed L-functions.

Remark: This integral representation is useful for instance in the estimates for the Brauer-Siegel theorem.