Solution of cubic and quartic equations

We work over a base field k of characteristic different from 2 and 3.

§1. Cubic equations

Consider a cubic polynomial

$$f(x) = x^3 + bx + c \in k[x]\,.$$

Its discriminant is given by the formula

$$D = [(\alpha-\beta)\cdot(\beta-\gamma)\cdot(\alpha-\gamma)]^2 = -4b^3 - 27c^2\,.$$

where α, β, γ are the three roots of $f(x)$ in a separable closure k^{sep} of k.

Assume that $f(x)$ is irreducible over k. We first go to $k_1 = k(\sqrt{D})$. The Galois group of the splitting field K of $f(x)$ is isomorphic to S_3 if $k_1 \neq k$, and isomorphic to $\mathbb{Z}/3\mathbb{Z}$ if $k_1 = k$.

To get to K from k_1, one first adjoins μ_3 to k_1, then we can write down the roots using radicals (Cardan's formula). Let ω be a primitive third root of 1, and let

$$\begin{array}{rcl} L(\omega) & = & \alpha + \omega\beta + \omega^2\gamma \\ L(\omega^2) & = & \alpha + \omega^2\beta + \omega\gamma. \end{array}$$

Write $\sqrt{-3} := \omega - \omega^2$, a square root of -3, and $\sqrt{D} := (\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)$. Then

$$\begin{array}{rcl} L(\omega)^3 & = & -\frac{27}{2}c + \frac{3}{2}\sqrt{-3}\sqrt{D} \\ L(\omega^2)^3 & = & -\frac{27}{2}c - \frac{3}{2}\sqrt{-3}\sqrt{D}\,. \end{array}$$

In addition we have

$$L(\omega)\cdot L(\omega^2) = -3b$$

So the Lagrange resolvants $L(\omega)$ and $L(\omega^2)$ are $\sqrt[3]{-\frac{27}{2}c \pm \frac{3}{2}\sqrt{-3D}}$, where $\sqrt{-3D} = \sqrt{-3}\cdot\sqrt{D}$ and the two cubic roots are determined so that their product is $-3b$. The three roots α, β, γ are permuted cyclically by A_3:

$$\begin{array}{rcl} \alpha & = & \frac{1}{3}(L(\omega) + L(\omega^2)) \\ \beta & = & \frac{1}{3}(\omega^2 L(\omega) + \omega L(\omega^2)) \\ \gamma & = & \frac{1}{3}(\omega^L(\omega) + \omega^2 L(\omega^2))\,. \end{array}$$

The set $\{\alpha, \beta, \gamma\}$ is well-defined as long as the cubic roots $L(\omega), L(\omega^2)$ of the two values of $(1/2)(-27\,c + 3\sqrt{-3D})$ satisfy the compatibility relation $L(\omega)\cdot L(\omega^2) = -3b$.

§2. Quartic equations

Let $g(x) = x^4 + bx^2 + cx + d$ be an irreducible quartic polynomial in $k[x]$. The discriminant is

$$D = 16\,b^4 d - 4\,b^3 c^2 - 128\,b^2 d^2 + 144\,bc^2 d - 27\,c^4 + 256\,d^3\,.$$

Let the four roots be $\alpha, \beta, \gamma, \delta$, and write

$$\begin{array}{rcl} \theta_1 & = & (\alpha+\beta)(\gamma+\delta) \\ \theta_2 & = & (\alpha+\gamma)(\beta+\delta) \\ \theta_3 & = & (\alpha+\delta)(\beta+\gamma)\,. \end{array}$$

Clearly every θ_i is fixed by the Klein four subgroup in S_4. Moreover θ_1 is also fixed by (12), (34), (1324), (1423), which together with the Klein form a 2-Sylow subgroup of S_4. The θ_i's are roots of a cubic polynomial

$$h(T) := T^3 - 2bT^2 + (b^2 - 4d)T + c^2 \ \in k[x] \, .$$

(Since the θ_i's are conjugate over S_3, their symmetric polymomials are of course in k.) So we can write down formulas for θ_1, θ_2 and θ_3. Incidentally the discriminant of $h(T)$ is equal to the discriminant of $g(x)$, because

$$\begin{aligned} \theta_1 - \theta_2 &= -(\alpha - \delta)(\beta - \gamma) \\ \theta_2 - \theta_3 &= -(\alpha - \gamma)(\beta - \delta) \\ \theta_2 - \theta_3 &= -(\alpha - \beta)(\gamma - \delta) \, . \end{aligned}$$

Since $\alpha + \beta + \gamma + \delta = 0$, have

$$\begin{aligned} \alpha + \beta &= \sqrt{-\theta_1} = -\gamma - \delta \\ \alpha + \gamma &= \sqrt{-\theta_2} = -\beta - \delta \\ \alpha + \delta &= \sqrt{-\theta_3} = -\beta - \gamma \, . \end{aligned}$$

The compatibilty condition for the three square roots above is

$$\sqrt{-\theta_1} \cdot \sqrt{-\theta_2} \cdot \sqrt{-\theta_3} = (\alpha + \beta)(\alpha + \gamma)(\alpha + \delta) = -c \, .$$

The final formula is

$$\begin{aligned} 2\alpha &= \sqrt{-\theta_1} + \sqrt{-\theta_2} + \sqrt{-\theta_3} \\ 2\beta &= \sqrt{-\theta_1} - \sqrt{-\theta_2} - \sqrt{-\theta_3} \\ 2\gamma &= -\sqrt{-\theta_1} + \sqrt{-\theta_2} - \sqrt{-\theta_3} \\ 2\delta &= -\sqrt{-\theta_1} - \sqrt{-\theta_2} + \sqrt{-\theta_3} \, . \end{aligned}$$

This set $\{\alpha, \beta, \gamma, \delta\}$ is well defined as long as the three square roots of $-\theta_i$'s satisfy the compatibility relation $\sqrt{-\theta_1} \cdot \sqrt{-\theta_2} \cdot \sqrt{-\theta_3} = -c$.