

Hilbert Nullstellensatz

§1. Hilbert Nullstellensatz

(1.1) Proposition *Let $R \rightarrow S \hookrightarrow T$ be homomorphisms of commutative rings. Assume that R is noetherian, T is a finitely generated R -algebra and T is a finite S -module. Then S is a finitely generated R -algebra.*

SKETCH OF A PROOF. Pick $t_1, \dots, t_n \in T$ such that they generate T as an R -algebra and they also generate T as an S -module. Write $t_i \cdot t_j = \sum_{k=1}^n s_{i,j,k} t_k$ with $s_{i,j,k} \in S$. Let $S' := R[s_{i,j,k}]_{1 \leq i,j,k \leq n}$ be the R -algebra of S generated by the $s_{i,j,k}$'s; S' is noetherian because R is. Consider the finite S' -module $T' := \sum_{i=1}^n S' \cdot t_i \subseteq T$; it is stable under multiplication by the definition of S' , hence $T' = T$ because it is an R -subalgebra of T which contains all the t_i 's. Now S is an S' -submodule of the noetherian S' -module T , so S is a finite S' -module a fortiori finitely generated over S' . ■

(1.2) Corollary (Alternative form of Nullstellensatz) *Let k be a field and let A be a finitely generated k -algebra. The residue field $A/\mathfrak{m}$ of every maximal ideal $\mathfrak{m} \subset A$ is a finite algebraic extension of k .*

PROOF. Apply 1.1 to $R = k$, $T = A/\mathfrak{m}$ and $S = k(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ is a transcendental basis of the extension field $(R/\mathfrak{m})/k$. We conclude that the rational function field $k(x_1, \dots, x_n)$ is a finitely generated ring over k , i.e. there exists a non-zero element $f \in k[x_1, \dots, x_n]$ such that $k(x_1, \dots, x_n) = k[x_1, \dots, x_n, \frac{1}{f}]$. It follows that every non-zero prime ideal of $k[x_1, \dots, x_n]$ contains f , which implies that $n = 0$ in view of the fact 1.3 recalled below. We have proved that the residue field $A/\mathfrak{m}$ of $\mathfrak{m}$ is algebraic over k . ■

(1.3) Lemma *Let K be an infinite field and let $g(u_1, \dots, u_n) \in K[u_1, \dots, u_n]$ be a non-zero polynomial in the variables $u_1, \dots, u_n$ with coefficients in K . There exists $a_1, \dots, a_n \in K$ such that $g(a_1, \dots, a_n) \neq 0$.*

(1.4) Corollary (Hilbert Nullstellensatz) *Let k be a field, and let I be an ideal in a polynomial ring $k[x_1, \dots, x_n]$ over k . Let k^a be an algebraic closure of k . Suppose that $f \in k[x_1, \dots, x_n]$ is a polynomial such that $f(a_1, \dots, a_n) = 0$ for every zero $(a_1, \dots, a_n) \in k^a$ of I in k^a . Then there exists an integer m such that $f^m \in I$.*

PROOF. The assumption implies that the $\bar{f}$ of f in $k[x_1, \dots, x_n]/I$ belongs to every maximal ideal of $k[x_1, \dots, x_n]/I$ according to 1.2. ■