Character tables for some small groups

1. How to guess the character table of S_5.
The symmetric group S_4 has 5 conjugacy classes. It has a normal subgroup consisting of e and the conjugacy class of (12)(34), We have

$$N \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}), \qquad S_4/N \cong S_3\,,$$

where the isomorphism is induced by the conjugation action of S_4 on the 3 non-trivial elements of N. The 3 irreducible characters of S_3 pulls back to three irreducible representations of S_4, of degree $1, 1, 2$ respectively. At this point we get a temporary form of the character table below. The two 0's in the column of 123 are forced by the orthogonality relation.

	1	6	8	6	3
S_4	e	(12)	(123)	(1234)	(12)(34)
1	1	1	1	1	1
sgn	1	-1	1	-1	1
χ_3	2	0	-1	0	2
χ_4	3	a	0	b	e
χ_5	3	c	0	d	f

(a) We have $a, b, c, d, e, f \in \mathbb{R}$ because every element $x \in S_4$ is conjugate to x^{-1}. Moreover $a, c, e, f \in \{\pm 1, \pm 3\}$ because (12) and (12)(34) both have order 2.

(b) Similarly we have

$$b, d \in \{\pm 3, \pm 2 \pm \sqrt{-1}, \pm 1 \pm 2\sqrt{-1}, \pm 1\} \cap \mathbb{R} = \{\pm 3, \pm 1\}$$

because the element $(1234) \in S_4$ has order 4 and $b, d \in \mathbb{R}$. From the orthogonality relations, we get $e + f = -2$, so $e = f = -1$.

(c) From the product of the second and the fourth column with the first column, we get $a + c = 0 = b + d$. From $(\chi_4|\mathbf{1}) = 0 = (\chi_5|\mathbf{1})$ we get $a + b = 0 = c + d$. Therefore $a = d$ and $b = c$.

(d) From $a, c \in \{\pm 1, \pm 3\}$ and the orthogonality relation $a^2 + c^2 = 2$ (length of the second column), we conclude that $a = -c = \pm 1$. We have determined the two characters χ_4 and χ_5. We choose χ_4 by requiring that $a = \chi_4((12)) = 1$.

Remark. The character χ_4 is equal to $\xi - \mathbf{1}$, where ξ is the character of the permutation representation of S_4 on the set $\{1, 2, 3, 4\}$.

S_4	1 e	6 (12)	8 (123)	6 (1234)	3 (12)(34)
1	1	1	1	1	1
sgn	1	-1	1	-1	1
χ_3	2	0	-1	0	2
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	1	-1

Table 1: character table for S_4

2. The character table of the alternating group A_4 is easier. The eight cyclic permutations of order 3 in S_4 is the union of two conjugacy classes in A_4, each with two elements. The quotient group A_4/N is cyclic of order 3; this gives three one-dimensional characters of A_4. The fourth is easily determined by the orthogonality relation. Note that the restriction to A_4 of the two 3-dimensional characters χ_4 and χ_5 of S_4 are both equal to ξ_4.

A_4	1 e	4 (123)	4 (132)	3 (12)(34)
1	1	1	1	1
ψ_2	1	$e^{2\pi\sqrt{-1}/3}$	$e^{-2\pi\sqrt{-1}/3}$	1
ψ_3	1	$e^{-2\pi\sqrt{-1}/3}$	$e^{2\pi\sqrt{-1}/3}$	1
ψ_4	3	0	0	-1

Table 2: character table for A_4

3. Let G be a non-commutative group with 21 elements. The 7-Sylow subgroup H_7 is normal. Let α be an element of order 7 and let β be an element of order 3 such that $\beta \cdot \alpha \cdot \beta^{-1} = \alpha^2$. The quotient group $G/H_7 \cong \mathbb{Z}/3\mathbb{Z}$ gives 3 one-dimensional characters of G. At this point we get a partially-filled character table.

	1 e	3 $\{\alpha, \alpha^2, \alpha^4\}$	3 $\{\alpha^3, \alpha^5, \alpha^6\}$	7 $H_7 \cdot \beta$	7 $H_7 \cdot \beta^2$
1	1	1	1	1	1
χ_2	1	1	1	$e^{2\pi\sqrt{-1}/3}$	$e^{-2\pi\sqrt{-1}/3}$
χ_3	1	1	1	$e^{-2\pi\sqrt{-1}/3}$	$e^{2\pi\sqrt{-1}/3}$
χ_4	3	a	b	0	0
χ_5	3	c	d	0	0

(a) Since the third conjugacy class consists of the inverse of the second conjugacy class, we have $a = \bar{b}$, $c = \bar{d}$. Moreover $a, b, c, d \in \mathbb{Q}(\mu_7)$, the cyclotomic field generated by $\zeta_7 = e^{2\pi\sqrt{-1}/7}$.

(b) The orthogonality relations gives us $a + c = -1 = b + d$, $1 + a + b = 0$, $1 + c + d = 0$. Therefore $b = c = \bar{a}$, $a = d = \bar{c}$.

(c) Pairing the second and the third column of the character table gives us $a^2 + b^2 = -3$, hence $ab = ((a+b)^2 - (a^2+b^2))/2 = 2$. We conclude that a, b are the two roots of the quadratic equation $x^2 + x + 2 = 0$ whose discriminant is -7.

Below is the completed character table for this group with 21 elements; α, β are non-trivial elements satisfying $\alpha^7 = e = \beta^3$, $\beta\cdot\alpha\cdot\beta^{-1} = \alpha^2$.

	1	3	3	7	7
	e	$\{\alpha, \alpha^2, \alpha^4\}$	$\{\alpha^3, \alpha^5, \alpha^6\}$	$H_7\cdot\beta$	$H_7\cdot\beta^2$
1	1	1	1	1	1
χ_2	1	1	1	$e^{2\pi\sqrt{-1}/3}$	$e^{-2\pi\sqrt{-1}/3}$
χ_3	1	1	1	$e^{-2\pi\sqrt{-1}/3}$	$e^{2\pi\sqrt{-1}/3}$
χ_4	3	$(-1+\sqrt{-7})/2$	$(-1-\sqrt{-7})/2$	0	0
χ_5	3	$(-1-\sqrt{-7})/2$	$(-1+\sqrt{-7})/2$	0	0

Table 3: character table for a non-commutative group with 21 elements

4. Let p be a prime number with $p \geq 3$. Recall that $H(\mathbb{F}_p) \subset \mathrm{GL}_3(\mathbb{F}_p)$ is the finite Heisenberg group with p^3 elements, consisting of all 3×3 matrices $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ with $a, b, c \in \mathbb{F}_p$.

(a) The center Z of $H(\mathbb{F}_p)$ consists of all such matrices with $a = b = 0$. Elements of Z will be written as z_c, indexed by the value of $c \in \mathbb{F}_p$. The quotient group $H(\mathbb{F}_p)/Z$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})$.

(b) The complement of Z in $H(\mathbb{F}_p)$ is the disjoint union of $p^2 - 1$ conjugacy classes, namely the $p^2 - 1$ non-trivial Z-cosets in $H(\mathbb{F}_p)$. For any non-zero element $(a, b) \in \mathbb{F}_p^2$, let $s_{a,b}$ be an element in the corresponding conjugacy class, which can be taken to be the matrix $\begin{pmatrix} 1 & a & 1 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$. Note that each $s_{a,b}$ is an element of order p: we have $s_{a,b}^p - 1 = (s_{a,b} - 1)^p = 0$ in the ring $\mathrm{M}_3(\mathbb{F}_p)$ because $p \geq 3$ and $s_{a,b} - 1$ is a nilpoent matrix in $\mathrm{M}_3(\mathbb{F}_p)$.

(c) There are p^2+p-1 irreducible characters. The first p^2 of them, $\chi_1 = \mathbf{1}, \ldots, \chi_{p^2}$, are one-dimensional. The last $p-1$ irreducible characters, $\chi_{p^2+1} =: \psi_1, \ldots, \chi_{p^2+p-1} =: \psi_{p-1}$ have degree at least 2. It follows immediately from the orthogonality relations that $\psi_i(s_{a,b}) = 0$ for all $i = 1, \ldots, p-1$ and all non-zero elements $(a,b) \in \mathbb{F}^p$. (There are already p^2 roots of unity in the column for the conjugacy class of $s_{a,b}$.)

(d) For $j = 1, \ldots, p-1$, let (W_j, ρ_j) be an irreducible representation of $H(\mathbb{F}_p)$ with character ψ_j. Let $\omega_j : Z \to \mathbb{C}^\times$ be the central character of ρ_j, i.e. $\rho_j(z) = \omega_j(z) \cdot \mathrm{Id}_{W_j}$ for all $j = 1, \ldots, p-1$. From (c) above we see that $\psi_j(z) = \psi_j(1)\,\omega(z)$ for all $j = 1, \ldots, p-1$, and $\psi_j(y) = 0$ for all $y \notin Z$. The equality $(\psi_j|\psi_j) = 1$ now becomes $p \cdot \psi_j(1)^2 = p^3$, so $\psi_j(1) = p$ for all $j = 1, \ldots, p-1$.

Conclusion. For every non-trivial character $\omega : Z \to \mathbb{C}^\times$, there exists up to isomorphism exactly one irreducible non-abelian representation (W_ω, ρ_ω) with central character ω. The character ψ_ω of (W_ω, ρ_ω) is

$$\psi_\omega(y) = \begin{cases} p \cdot \omega(y) & \text{if } y \in Z \\ 0 & \text{if } u \notin Z \end{cases} .$$

In particular $\dim_{\mathbb{C}}(W_\omega) = p$, and $\mathrm{Ker}(\rho_\omega) = \mathrm{Ker}(\omega)$. These $p-1$ irreducible characters ψ_ω give all non-abelian characters of the finite Heisenberg group $H(\mathbb{F}_p)$.

5. Let p be a prime number. Let $B(\mathbb{F}_p) \subset \mathrm{GL}_2(\mathbb{F}_p)$ be the subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$ consisting of all matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $a, d \in \mathbb{F}_p^\times$ and $b \in \mathbb{F}_p$. Clearly $\mathrm{card}(B(\mathbb{F}_p)) = (p-1)^2 \cdot p$. We first make some basic observations about the group $B(\mathbb{F}_p)$.

(a) The center Z of $B(\mathbb{F}_p)$ consists of all elements of the form $z_a := \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ with $a \in \mathbb{F}_p^\times$.

(b) For each $a \in \mathbb{F}_p^\times$, the set of all elements of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ for some $b \in \mathbb{F}_p^\times$ form a conjugacy class C_a in $B(\mathbb{F}_p)$ with $p-1$ elements.

(c) For each pair of elements (a,d) in $\mathbb{F}_p^\times$ with $a \neq d$, the set of all elements of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ for some $b \in \mathbb{F}_p$ form a conjugacy class $C_{a,d}$ in $B(\mathbb{F}_p)$ with p elements.

(d) Every conjugacy class of $B(\mathbb{F}_p)$ is of type (a), (b) or (c) above. The number of conjugacy classes of $B(\mathbb{F}_p)$ is $(p-1)+(p-1)+(p-1)\cdot(p-2) = (p-1)\,p = p^2 - p$.

(e) The subgroup N of $B(\mathbb{F}_p)$ consisting of all elements of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for some $b \in \mathbb{F}_p$ is the commutator subgroup of $B(\mathbb{F}_p)$. The quotient group $B(\mathbb{F}_p)/N$ is isomorphic to $\mathbb{F}_p^\times \times \mathbb{F}_p^\times \cong (\mathbb{Z}/(p-1)\mathbb{Z}) \times (\mathbb{Z}/(p-1)\mathbb{Z})$.

(e) The irreducible characters of the quotient group $\mathbb{F}_p^\times \times \mathbb{F}_p^\times$ gives $(p-1)^2 = p^2 - 2p + 1$ one-dimensional characters. The remaining $p-1$ characters $\psi_1, \ldots, \psi_{p-1}$ of $B(\mathbb{F}_p)$ are non-abelian.

(f) For each of the $(p-1)(p-2)$ conjugacy classes $C_{a,d}$ with $a \neq d$, the $(p-1)^2$ abelian characaters gives $(p-1)^2$ roots of 1 in the column of the character table for $C_{a,d}$. So $\psi_i(x) = 0$ for all $x \in C_{a,d}$ and all $j = 1, \ldots, p-1$.

We proceed to construct some non-abelian representations. We have a natural action of $B(\mathbb{F}_p)$ on the space $\mathbb{F}_p^2$ of column vectors with two entries in $\mathbb{F}_p$. Among the $p+1$ lines in $\mathbb{F}_p^2$, the line $\mathbb{F}_p \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is fixed by $B(\mathbb{F}_p)$. The remaining p lines are of the form $\mathbb{F}_p \cdot \begin{pmatrix} z \\ 1 \end{pmatrix}$ with $z \in \mathbb{F}_p$; they form a set S with p elements parametrized $\mathbb{F}_p$. The action of $B(\mathbb{F}_p)$ on S becomes the usual fractional linear transformations

$$\begin{pmatrix} a & 1 \\ 0 & c \end{pmatrix} : z \;\mapsto\; \frac{az+b}{d} \qquad \forall\, z \in \mathbb{F}_p$$

if we identify S with $\mathbb{F}_p$. Let (V, ξ) be the linear permutation representation of $B(\mathbb{F}_p)$ induced from this action. Because the permutation action of $H(\mathbb{F}_p)$ on $\mathbb{F}_p$ is doubly transitive, the one-dimensional trivial subrepresentation in V has a unique complementary $H(\mathbb{F}_p)$-subrepresentation W, and the character η for the subrepresentation W is irreducible. Explicitly

$$\eta(x) = \begin{cases} p-1 & \text{if } x \in Z \\ -1 & \text{if } x \in C_a \text{ for some } a \in \mathbb{F}_p^\times \\ 0 & \text{if } x \in C_{a,d} \text{ for some } a \neq d \in \mathbb{F}_p^\times \,. \end{cases}$$

For each of the one-dimensional characters $\chi : B(\mathbb{F}_p) \to \mathbb{C}^\times$, $\chi \cdot \eta$ is an non-abelian character of $B(\mathbb{F}_p)$ of dimension $p-1$. For any two 1-dimensional characters χ and χ', the equality $\chi \cdot \eta = \chi' \cdot \eta$ holds if and only if $\chi|_Z = \chi'_Z$. This gives us $p-1$ irreducible non-abelian characters.

Conclusion. For each one-dimensional character $\omega : \mathbb{F}_p^\times \to \mathbb{C}^\times$, there exists a unique non-abelian character ψ_ω of $B(\mathbb{F}_p)$ given by

$$\psi_\omega(x) = \begin{cases} (p-1)\cdot\omega(a) & \text{if } \; x = z_a \text{ for some } a \in \mathbb{F}_p^\times \\ -\omega(a) & \text{if } \; x \in C_a \text{ for some } a \in \mathbb{F}_p^\times \\ 0 & \text{if } \; x \in C_{a,d} \text{ for some } \; a \neq d \in \mathbb{F}_p^\times \,. \end{cases}$$