1

Character table for $G = S_5$

	1	10	20	30	24	15	20
	e	(12)	(123)	(1234)	(12345)	$(12)(34)$	$(12)(345)$
$\mathbb{1}$	1	1	1	1	1	1	1
sgn	1	−1	1	−1	1	1	−1
χ_3	4	2	1	0	−1	0	−1
$\mathrm{sgn}\cdot\chi_3 = \chi_4$	4	−2	1	0	−1	0	1
χ_5	5	1	−1	−1	0	1	1
$\mathrm{sgn}\cdot\chi_5 = \chi_6$	5	−1	1	1	0	1	−1
χ_7	6	0	0	0	1	−2	0

V = standard permutation repr. of S_5 on $\mathbb{C}[[1,5]]$, $[[1,5]] = \{1,2,3,4,5\}$

$= \mathbb{1} \oplus U \qquad \dim_{\mathbb{C}} U = 4$

$\chi_3 = \chi_U$, character of S_5 on U

$S^2V \underset{S_5}{\cong} V \oplus \underbrace{(W,\eta)}_{\cong S^2U}$, where (W,η) = repr. of S_5 from the perm. rep. on $\left\{\{a,b\} \,\middle|\, \begin{array}{l}1\le a,b\le 5\\ a\ne b\end{array}\right\}$ $\quad \dim_{\mathbb{C}} W = 10$

$\cong S^2U$ ← either by computation or by pure thought

Computation shows:

$$(\chi_\eta \mid \mathbb{1})_{S_5} = 1 = (\chi_\eta \mid \chi_U)_{S_5} \qquad (\chi_\eta \mid \chi_\eta) = 3$$

Let: $\chi_5 = \chi_\eta - \chi_U - 1$, an irreducible character, 5-dim^l of a repr. (Y, ξ_5) s.t. $Y \oplus V \underset{\mathbb{C}[S_5]}{\cong} S^2U \cong (W,\eta)$

$\chi_6 = \mathrm{sgn}\cdot\chi_5$,

Then χ_7 is determined.

One can compute $\chi_{\Lambda^2 U}$ by $\chi_{\Lambda^2 U} = \chi_U^{\,2} - \chi_{S^2U}$

and see: $\quad \chi_7 = \chi_{\Lambda^2 U}$

Note: a) S_5 does not have any irred. 3-dim^l repr., unlike A_5

b) χ_5 is contained in $(W,\eta) \cong S^2U$ ← 10-dim^l

It is induced from either of the two irred. 3-dim^l character of A_5

Character table for $G = A_5$

	1	20	12	12	15
	e	(123)	(12345)	(12354)	$(12)(34)$
$\mathbb{1}$	1	1	1	1	1
χ_2	4	1	-1	-1	0
χ_3	5	-1	0	0	1
χ_4	3	0	a	$c=b$	$e=-1$
χ_5	3	0	b	$d=a$	$f=-1$
ψ	6	0	1	1	-2

$x^{-1} \underset{\text{conj}}{\sim} x \ \forall x \in A_5$
$\Rightarrow$ all entries are tot. real

χ_2 = restriction to A_5 of either of the two 4-dim^l irred char. of S_5
= (restriction to A_5 of) the complement of $\mathbb{1}$ in the standard permutation repr. on $\{1,2,3,4,5\}$

χ_3 = restriction to A_5 of either of the two 5-dim^l irred. character of S_5

[ψ = restriction to A_5 of the 6-dim^l irred. character of S_5

easy computation: $(\psi|\psi)_{A_5} = 2$, $(\psi|\chi_2) = (\psi|\chi_3) = (\psi|\mathbb{1}) = 0$

So: ψ split into two irreducible characters]

Note: 1) The $\begin{smallmatrix} 3 & 0 \\ 3 & 0 \end{smallmatrix}$ part of the last two rows are forced by the orthogonality relation

2) $\deg(\chi_4) = \deg(\chi_5) = 3$, order of $(12)(34) = 2 \Rightarrow e, f \in \{\pm 1, \pm 3\}$

orthogonality relation $\Rightarrow \quad e+f=-2, \ e^2+f^2=2$

Conclude: $e=f=-1$

3) orthogonality relation gives: $\left.\begin{array}{ll} a+b=1, & c+d=1 \\ a+c=1, & b+d=1 \end{array}\right\} \Rightarrow a=d,\ b=c$

$a^2+b^2=3, \quad c^2+d^2=3, \quad ac+bd=-2$

$\Rightarrow \quad ab=-1,$

So $a=d$ and $b=d$ are the two roots of $x^2-x-1=0$

i.e. $\{a,b\} = \{1+\zeta_5+\zeta_5^{-1},\ 1+\zeta_5^2+\zeta_5^{-2}\}$

4) χ_4, χ_5: come from $A_5 \hookrightarrow SO_3(\mathbb{R})$, symmetry group of a regular icosahedron

character table for S_6 , $|S_6| = 720$

	1	15	40	90	144	120	15	45	120	90	40
	e	(12)	(123)	(1234)	(12345)	(123456)	$(12)(34)(56)$	$(12)(34)$	$(123)(45)$	$(1234)(56)$	$(123)(456)$
$\mathbb{1}$	1	1	1	1	1	1	1	1	1	1	1
sgn	1	−1	1	−1	1	−1	−1	1	−1	1	1
α	5	3	2	1	0	−1	−1	1	0	−1	−1
sgn. α	5	−3	2	−1	0	1	1	1	0	−1	−1
β	9	3	0	−1	−1	0	3	1	0	1	0
sgn. β	9	−3	0	1	−1	0	−3	1	0	1	0
γ	10	2	1	0	0	1	−2	−2	−1	0	1
sgn. γ	10	−2	1	0	0	−1	2	−2	1	0	1
δ	5	1	−1	−1	0	0	−3	1	1	−1	2
sgn. δ	5	−1	−1	1	0	0	3	1	−1	−1	2
ϕ	16	0	−2	0	1	0	0	0	0	0	−2
χ_V	6	4	3	2	1	0	0	2	1	0	0
χ_V^2	36	16	9	4	1	0	0	4	1	0	0
χ_{S^2V}	21	11	6	3	1	0	3	5	2	1	0
χ_{Λ^2V}	15	5	3	1	0	0	−3	−1	−1	−1	0
ξ	15	7	3	1	0	0	3	3	1	1	0
ψ	20	8	2	0	0	0	0	4	2	0	2

$V =$ 6-dim^l std permutation repr. of S_6 on $\{1, \dots, 6\}$, $V = \mathbb{1} \oplus U$.

$\underline{\alpha \overset{def}{=} \chi_U} = \chi_V - 1$, $\xi =$ permutation repr. of S_6 on $\big\{ \{a, b\} \,\big|\, \underset{\in \mathbb{Z}}{1 \le a, b \le 6}, \ a \ne b \big\} = \chi_{S^2V} - \chi_V$

$\underline{\beta \overset{def}{=} \xi - \alpha - 1}$ $\quad (\xi | \xi)_{S_6} = 3$, $\underline{\gamma := \chi_{\Lambda^2 U}} = \chi_{\Lambda^2 V} - \chi_U = \chi_{\Lambda^2 V} - \alpha$

$\psi =$ the 20-dim^l permutation repr. of S_6 on $\big\{ \{a, b, c\} \,\big|\, \underset{\in \mathbb{N}}{1 \le a, b, c \le 6}, \ a, b, c \text{ distinct} \big\}$

$(\psi | \psi)_{S_6} = 4$, $(\psi | \alpha) = (\psi | \beta) = (\psi | 1) = 1$, $\quad \underline{\delta :=} \ \psi - \alpha - \beta - 1$

Notes

1) $\phi =$ the 11^{th} irreducible repr. of S_5, $\phi(1) = 16$

To get the representation:

$$(\alpha\cdot\delta \mid \alpha\delta)_{S_6} = 2 \qquad (\alpha\cdot\delta \mid \beta)_{S_6} = (\alpha\cdot\delta \mid \phi) = 1$$

So ϕ appears exactly once in $(W_\alpha, \rho_\alpha) \otimes (W_\delta, \rho_\delta)$

$\chi(\rho_\alpha) = \alpha$, $\chi_{\rho_\delta} = \delta$

$\dim(W_\alpha) = \dim(W_\delta) = 5$

2) $\delta^2 = 1 + sgn\cdot\alpha + \beta + sgn\cdot\beta$. $(\delta^2 \mid \delta^2)_{S_6} = 4$