Class functions

§1. Action of the center of the group ring

(1.1) Notation Let G be a finite group and let k be an algebraically closed field of characteristic 0.

(1.1.1) Let $C_1 = \{1\}, C_2, \ldots, C_r$ be the conjugacy classes of G. Let $c_i := \#(C_i)$ for $i = 1, \ldots, r$. Pick elements $z_i \in C_i$ for $1 \le i \le r$. Let $\sigma_i := \sum_{y \in C_i} [y] \in k[G]$, $i = 1, \ldots, r$.

(1.1.2) Let (V_α, ρ_α), $\alpha = 1, \ldots, s$ be the non-isomorphic irreducible k-linear representation of G. Let χ_α be the character of ρ_α.

(1.2) From Schur's lemma we get the orthogonality relations

$$\langle \chi_\alpha, \chi_\beta \rangle := \frac{1}{|G|} \sum_{y \in G} \chi_\alpha(y) \cdot \chi_\beta(y^{-1}) = \delta_{\alpha,\beta} \quad \forall\, \alpha, \beta.$$

In particular the irreducible characters χ_α generate an s-dimensional subspace of $\mathrm{Z}(k[G])$, so $s \le r$.

(1.3) LEMMA (1) *Suppose that $u = \sum_{y \in G} a_y \cdot [y]$ is an element of the center $\mathrm{Z}(k[G])$ of the group ring $k[G]$ and let $f : G \to k$ be the corresponding class function with $f(y) = a_y$ for all $y \in G$. Then u operates on any irreducible representation (V_β, ρ_β) by*

$$\frac{\sum_{y \in G} a_y \cdot \chi_\beta(y)}{\chi_\beta(1)} \cdot \mathrm{Id}_{V_\beta} = \frac{|G|}{\chi_\beta(1)} \cdot \langle f, \chi_{\beta^\vee} \rangle \cdot \mathrm{Id}_{V_\beta},$$

where $\chi_{\beta^\vee}$ is the character of the contragradient representation of ρ_β, i.e. $\chi_{\beta^\vee}(x) = \chi_\beta(x^{-1})$ for all $x \in G$.

(2) *The element $\sigma_i = \sum_{y \in C_i} [y] \in \mathrm{Z}(k[G])$ operates on the irreducible representation V_β as*

$$\frac{c_i \cdot \chi_\beta(z_i)}{\chi_\beta(1)} \cdot \mathrm{Id}_{V_\beta}.$$

(e) *The element $\sum_{y \in G} \chi_\alpha(y) \cdot [y] \in \mathrm{Z}(k[G])$ operates on the irreducible representation V_β as*

$$\frac{|G|}{\chi_\alpha(1)} \cdot \langle \chi_\alpha, \chi_{\beta^\vee} \rangle \cdot \mathrm{Id}_{V_\beta} = \frac{|G|}{\chi_\alpha(1)} \cdot \delta_{\alpha,\beta^\vee} \cdot \mathrm{Id}_{V_\beta},$$

(1.4) COROLLARY (a) $\frac{c_i \cdot \chi_\beta(z_i)}{\chi_\beta(1)}$ *is an algebraic integer for every conjugacy class C_i and every irreducible character χ_α.*

(b) *$\chi_\alpha(1)$ divides $|G|$ for every irreducible character χ_α of G.*

§2. Class function

(2.1) PROPOSITION *The irreducible functions form an orthonormal basis of the class functions.*

PROOF. The orthogonality relation implies that $s \le r$. Suppose that $s < r$, then there exists a non-zero class function $f(x)$ on G such that $\langle f, \chi_\alpha \rangle = 0$ for all $\alpha = 1, \ldots, r$. Lemma 1.3(1) tells us that the element $u := \sum_{y \in G} f(y) \cdot [y]$ operates as 0 on all irreducible representations of G, hence it operates as 0 on every finite representations of G. But the action of u on the regular representation of G is clearly nonzero, which is a contradiction. □

(2.2) For each $i = 1, \ldots, r$, let Δ_i be the class function with value 1 on C_i and value 0 on all other conjugacy classes. The Δ_i's form an orthogonal basis of the space of class functions, while the irreducible characters χ_α's form another.

- Clearly $\chi_\alpha = \sum_{i=1}^{r} \chi_\alpha(z_i) \cdot \Delta_i$
- Write $\Delta_i = \sum_{\alpha=1}^{r} b_{i,\alpha} \chi_\alpha$. From the orthogonality relation we see that $b_{i,\alpha} = \langle \Delta_i, \chi_\alpha \rangle = \frac{c_i \cdot \chi_\alpha(z_i)}{|G|}$, i.e.

  $$\Delta_i = \frac{c_i}{|G|} \cdot \sum_{\alpha=1}^{r} \chi_\alpha(z_i^{-1}) \cdot \chi_\alpha$$

  Equating the values of both sides at z_j we see that

  $$\frac{c_i}{|G|} \cdot \sum_{\alpha=1}^{r} \chi_\alpha(z_i^{-1}) \cdot \chi_\alpha(z_j) = \delta_{i,j} \quad \forall i, j,$$

  which is the other set of orthogonality relations for the character table.