Burnside's (p, q)-theorem

Theorem. (Burnside) Let G be a finite group such that $|G| = p^a \cdot q^b$, where p, q are distinct prime numbers and $a, b \in \mathbb{N}$. Then G is solvable.

Proof. May assume that $a, b \geq 1$. Let $K = \mathbb{Q}(\zeta_{|G|})$. Let P be a p-Sylow subgroup of G, and let $1 \neq h \in \mathrm{Z}(P)$. So $c := [G : \mathrm{Z}_G(h)] = q^d$ for some $d \in \mathbb{N}$. Let $\mathbf{1} = \chi_1, \ldots, \chi_r$ be the irreducible complex characters of G.

1. (Locate a suitable irreducible representation)
Claim. Let $K = \mathbb{Q}(\mu_{|G|})$. $\exists\, i_0 \geq 2$ (i.e. $\chi_{i_0} \neq \mathbf{1}$) such that $\chi_{i_0}(h) \notin q \cdot \mathcal{O}_K$ and $q \nmid \chi_{i_0}(1)$. In particular $\chi_{i_0}(h) \neq 0$.

This follows immediately from

$$0 = \sum_{i=1}^{r} \chi_i(1)\, \chi_i(h) = 1 + \sum_{i \geq 2} \chi_i(1)\, \chi_i(h)\,,$$

for otherwise $1 \in q\, \mathcal{O}_K$.

2. (Use integrality) We know that $\gcd(c, \chi_{i_0}(1)) = 1$ and

$$c \cdot \chi_{i_0}(h)/\chi_{i_0}(1) \in \mathcal{O}_K\,,$$

which implies that $\chi_{i_0}(h)/\chi_{i_0}(1) \in \mathcal{O}_K$.

3. (Produce a non-trivial proper normal subgroup)

Let (V, ρ) be the irreducible $\mathbb{C}$-representation with character χ_{i_0}. We know that each Galois conjugate of $\chi_{i_0}(h)/\chi_{i_0}(1)$ is non-zero, integral and has complex absolute value at most 1. So the complex absolute value of $\chi_{i_0}(h)/\chi_{i_0}(1)$ is 1. This implies that all eigenvalues of $\rho(h)$ are equal, i.e. $\rho(h) \in \mathbb{C}^\times \cdot \mathrm{Id}_V$.

The subgroup $\rho^{-1}(\mathbb{C}^\times \cdot \mathrm{Id}_V)$ of G is a non-trivial proper normal subgroup of G unless $\dim(V) = 1$. If $\dim(V) = 1$, then $\rho^{-1}(\mathrm{Id}_V)$ is a non-trivial proper normal subgroup of G.

Induction finishes the proof. ∎