

Complex Algebraic Geometry, Spring 1996

Materials to cover

- (1) Riemann surfaces / Algebraic curves
Riemann-Roch for curves, linear systems, theta and Jacobians
- (2) Surfaces
explain classification, study rational surfaces,
ruled surfaces, K3's and Enriques surfaces
- (3) Chern classes
- (4) Fixed point formulas
- <*> (5) (mixed) Hodge theory

Axiomatic properties of Chern classes:

(I) For every continuous complex vector bundle ξ on X and every integer $i \geq 0$, there is a Chern class

$$c_i(\xi) \in H^{2i}(X, \mathbb{Z}(i))$$

The class $c_0(\xi) \in H^0(X, \mathbb{Z}) \cong \mathbb{Z}^{\pi_0(X)}$ is the unit element.

(II) (naturality) for $f: Y \rightarrow X$, $c_t(\xi) = \sum_{i \geq 0} t^i c_i(\xi)$, the total Chern class
 $c(f^* \xi) = f^* c(\xi)$

(III) (Whitney sum)

$$0 \rightarrow \xi_1 \rightarrow \xi \rightarrow \xi_2 \rightarrow \text{short exact sequence}$$

$$\leadsto c(\xi) = c(\xi_1) \cdot c(\xi_2)$$

(IV) (normalization)

If ξ is a line bundle,

$$\xi \longleftrightarrow \text{a class } \alpha \text{ in } H^1(X, \mathcal{O}_{X, \mathbb{C}}^{\times})$$

$$\text{have: } 1 \rightarrow \mathbb{Z}(1) \rightarrow \mathcal{O}_X^{\mathbb{C}^{\times}} \xrightarrow{\exp} \mathcal{O}_{X, \mathbb{C}}^{\times} \rightarrow 1$$

then: $c_1(\xi)$ = the image of α under the coboundary map $H^1(X, \mathcal{O}_{X, \mathbb{C}}^{\times}) \rightarrow H^2(X, \mathbb{Z}(1))$

$$\text{and } c(\xi) = 1 + c_1(\xi)$$

$$\Rightarrow c_i(\xi) = 0 \text{ for } i > \dim \text{rk}(\xi)$$

(From the splitting principle)

The splitting principle:

X, ξ a vector bundle on X

$\leadsto$ Find $Y \xrightarrow{f} X$ such that

$f^*(\xi) = \xi_r \geq \xi_{r-1} \geq \dots \geq \xi_1 \geq (0)$ $r = \text{rk}(\xi)$
 such that each ξ_k / ξ_{k-1} is a line bundle, flag of subbundles

and $H^*(X) \xrightarrow{f^*} H^*(Y)$ is a split injection

(In fact, one can take Y to be a flag bundle over X)

One uses the splitting principle to verify formal properties of characteristic classes:

(a) (dual bundles): $c_i(E^\vee) = (-1)^i c_i(E)$

(b) (tensor products)

Let $s_k(\alpha_1 + \beta_1, \dots, \alpha_r + \beta_s) =$ the k -th elementary symmetric poly. of $\alpha_i + \beta_j$ ($1 \leq i \leq r$, $1 \leq j \leq s$)

$= f_k(\sigma_1, \dots, \sigma_r, \Sigma_1, \dots, \Sigma_s)$ where $\sigma_i =$ the i -th elementary symmetric poly in the α_i 's

Then $c_k(E \otimes F) = f_k(c_1(E), \dots, c_r(E), c_1(F), \dots, c_s(F))$ $\Sigma_j =$ the j -th elementary symmetric poly in the β_j 's
 $r = \text{rk}(E), s = \text{rk}(F)$

In general the expression is complicated.

But if one is a line bundle, the formula is simple:

exer

$$c_t(E \otimes L) = \sum_{i=0}^r t^i c_t(L)^{r-i} c_i(E) \quad r = \text{rk}(E)$$

$$= \sum_{i=0}^r t^i (1 + t c_1(L))^{r-i} c_i(E)$$

Especially, $c_r(E \otimes L) = \sum_{i=0}^r c_i(L)^{r-i} c_i(E)$

(c) (exterior powers)

$$\text{Let } F_j(\sigma_1, \dots, \sigma_r) = \prod_{i_1 < \dots < i_j} (1 + (\alpha_{i_1} + \dots + \alpha_{i_j})t)$$

$$\text{Then: } C_t(\Lambda^j E) = F_j(C_1(E), \dots, C_r(E)) \quad \sigma_i = \text{the } i\text{-th elementary symm. poly. in } \alpha_1, \dots, \alpha_r$$

Pontrjagin classes.

For a real vector bundle η , consider its complexification

$\eta \otimes_{\mathbb{R}} \mathbb{C}$. Complex conjugation defines an isomorphism (of complex v.b.)

$\eta \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} \overline{\eta \otimes_{\mathbb{R}} \mathbb{C}}$ from $\eta \otimes_{\mathbb{R}} \mathbb{C}$ to its complex conjugate.

Since $c(\overline{\eta \otimes_{\mathbb{R}} \mathbb{C}}) = 1 - c_1(\eta \otimes_{\mathbb{R}} \mathbb{C}) + c_2(\eta \otimes_{\mathbb{R}} \mathbb{C}) - c_3(\eta \otimes_{\mathbb{R}} \mathbb{C}) + \dots$

(as is true for conjugate of complex vector bundles),

we see that

$$c_{2i}(\overline{\eta \otimes_{\mathbb{R}} \mathbb{C}}) = c_{2i}(\eta \otimes_{\mathbb{R}} \mathbb{C}) \quad \forall i, \text{ and}$$

$c_{2i-1}(\eta \otimes_{\mathbb{R}} \mathbb{C})$ are elements killed by 2

Def = The i -th Pontrjagin class

$$p_i(\eta) \in H^{4i}(X, \mathbb{Z}(2i))$$

is defined to be $(-1)^i c_{2i}(\eta \otimes_{\mathbb{R}} \mathbb{C})$.

The total Pontrjagin class is

$$p(\eta) = 1 + p_1(\eta) + \dots + p_{[\frac{r}{2}]}(\eta), \text{ where } r = \text{rk}(\eta)$$

Properties

(a) ξ : complex vector bundle
 $\Rightarrow \xi \otimes_{\mathbb{R}} \mathbb{C} \cong \xi \oplus \overline{\xi}$

exer. $\widehat{\rightarrow} \quad 1 - p_1(\xi) + p_2(\xi) - p_3(\xi) + \dots$

$$= (1 + c_1(\xi) + c_2(\xi) + c_3(\xi) + \dots) (1 - c_1(\xi) + c_2(\xi) - c_3(\xi) + \dots)$$

(b) (naturality) $p(f^* \xi) = f^* p(\xi)$

(c) (Whitney sum)

$p(\xi \oplus \eta) \equiv p(\xi) \cdot p(\eta)$ modulo elements of order 2

Exer

Euler class :

Let ξ be an oriented real vector bundle, $n = \text{rk}(\xi)$,

$E = E(\xi) = \text{total space of } \xi$, $E_0 = E \setminus \{\text{zero section}\}$.

Then: $H^i(E, E_0; \mathbb{Z}) = 0 \quad \forall \quad i < n$

and: $H^n(E, E_0; \mathbb{Z}) \cong \mathbb{Z}$

There exists a unique element $u \in H^n(E, E_0; \mathbb{Z})$ restricting to the orientation class $u_F \in H^n(F, F_0; \mathbb{Z})$ for each fibre. This element u is called the Thom class.

$$\begin{array}{ccc} H^k(E; \mathbb{Z}) & \xrightarrow[\sim]{\cup u} & H^{k+n}(E, E_0; \mathbb{Z}) \\ \uparrow p^* & & \\ H^k(B; \mathbb{Z}) & & \end{array}$$

Def: The Euler class $e(\xi)$ of an oriented rk n bundle ξ

is $e(\xi) = (p^*)^{-1} u|_E \in H^n(B; \mathbb{Z})$

$$\begin{array}{ccccc} H^n(E, E_0; \mathbb{Z}) & \longrightarrow & H^n(E; \mathbb{Z}) & \xleftarrow[\sim]{p^*} & H^n(B; \mathbb{Z}) \\ u & \longmapsto & u|_E & \longmapsto & e(\xi) \end{array}$$

Properties: (a) $e(f^* \xi) \stackrel{(\text{naturality})}{=} f^* e(\xi)$

(b) (orientation) reversing orientation causes Euler class to change sign

(c) When $n \equiv 1 \pmod{2}$, $e(\xi)$ is killed by 2.

$\leadsto$ We shall usually assume that $n \equiv 0 \pmod{2}$ when using Euler classes. ~~Write $n = 2m$, then when ξ is a complex~~

Remark: One can always define the Thom class for every real vector bundle:

$$u \in H^n(E, E_0; \text{orientation sheaf}) \xleftarrow[p^*]{\sim} H^0(B, \underline{H}^n(F, F_0; \mathbb{Z}))$$

(d) (Whitney sum):

$$e(\xi \oplus \xi') = e(\xi) \cup e(\xi')$$

$$(\text{resp. } e(\underline{\xi \times \xi'}) = e(\xi) \times e(\xi'))$$

Cartesian product, i.e. $p_1^* \xi \oplus p_2^* \xi'$

Remark

Now, suppose that ξ is a complex vector bundle, we will modify the Euler class a little bit:

A choice of a square root of -1 determines an orientation of ξ . So one gets a canonical class, the Euler class of a complex vector bundle of rank r

$$e(\xi) \in H^{2r}(X; \mathbb{Z}(r))$$

(e) If ξ has a nowhere zero section, then $e(\xi) = 0$.

(In fact, the Euler class is the primary obstruction class for ξ to have a everywhere non-zero cross section over the $n = \text{rk}(\xi)$ -skeleton of the base B .)

See Milnor - Stasheff, §12, Thm 12.5 on p.147)

(f) For a manifold M , compact oriented,

$$\int_M e(TM) \langle e(TM), [M] \rangle = \chi(M) = \langle e(TM), [M] \rangle$$

the Euler characteristic of M .

More intrinsic formulation:

$$e(TM) \in H^d(M, \Gamma_M^{\text{orientation sheaf}}), \quad M \in H_d(M, \Gamma_M)$$

(g) Suppose that ξ is a complex ~~line~~ vector bundle, $r = \text{rk}(\xi)$.

Its Euler class $e(\xi) \in H^{2r}(X; \mathbb{Z}(r))$ is equal to the top Chern class $c_r(\xi)$.

(h) Suppose that ξ is an oriented $n = 2m$ -bundle.

Then $P_m(\xi) \in H^{4m}(X; \mathbb{Z}(\frac{2m}{2}))$ is equal to the square of the Euler class $e(\xi) \in H^{2m}(X; \mathbb{Z})$.

Hirzebruch Index Theorem

Let X be a compact connected oriented differentiable manifold of dimension $4m$, then the index $\tau(X)$ of X is given by:

$$\tau(X) = \langle L(T_X), [X] \rangle \quad \text{the orientation sheaf}$$

Note (1) Canonically, $[X] \in H_{4m}(X, \Gamma_X)$.

So both sides in the equality depends on the orientation in the same way

(2) Here we use: $p_i(X) \in H^{4i}(X; \mathbb{Z})$
and not $p_i(X) \in H^{4i}(X, \mathbb{Z}(2i))$.

Recall the definition of the L -class:

If E is a real oriented vector bundle, then write

$$p(E) = \prod_{i=1}^{\lfloor \frac{r}{2} \rfloor} (1 + \delta_i) \quad \text{formally.}$$

$$\Rightarrow L(E) = \prod_{i=1}^{\lfloor r/2 \rfloor} \frac{\sqrt{\delta_i}}{\tanh \sqrt{\delta_i}}$$

$$= 1 + \frac{1}{3} p_1(E) + \frac{1}{45} (7 p_2(E) - p_1^2(E)) + \dots$$

exer.

When $m=1$, and X carries an almost complex structure (e.g. if X is a compact complex surface)

$$\text{then } \tau(X) = \frac{1}{3} (c_1^2(X) - 2 c_2(X))$$

$$\text{since } p_1(X) = c_1^2(X) - 2 c_2(X)$$

Exercise

(1) When $n = \dim(X) = 1$, get $\chi(X) = g(X)$, where $g(X)$ is defined by: $\int_X c_1(T_X) = 2 - 2g$

(2) When $n = \dim(X) = 2$, get Noether's formula
 $\chi(X, \mathcal{O}_X) \stackrel{\text{def}}{=} 1 - g(X) + p_g(X) = \frac{1}{12} (c_1^2(X) + c_2(X))$

(3) Riemann-Roch theorem for line bundles on curves:
 $\chi(X, \mathcal{L}) = c_1(\mathcal{L}) - g + 1$

(4) Riemann-Roch for vector bundles on curves
 $\chi(X, \mathcal{E}) = c_1(\mathcal{E}) - \deg(\mathcal{E}) \operatorname{rk}(\mathcal{E}) (g-1)$

(5) Riemann-Roch for line bundles on surfaces:
 $\chi(X, \mathcal{L}) = \chi(X, \mathcal{O}) + \frac{1}{2} (c_1^2(\mathcal{L}) + c_1(\mathcal{L}) c_1(T_X))$

(6) Riemann-Roch for a rank-2 vector bundle on surfaces:

$$\chi(X, \mathcal{E}) = \frac{1}{2} \underbrace{(c_1^2(\mathcal{E}) - 2c_2(\mathcal{E}))}_{p_1(\mathcal{E})} + \frac{1}{2} c_1(\mathcal{E}) \cdot c_1(X) + 2\chi(X, \mathcal{O}_X)$$

Todd class :

E = complex v. bundle of rank r . Write formally

$$c(E) = \prod_{i=1}^r (1 + \alpha_i), \text{ then}$$

$$\text{Todd}(E) = \prod_{i=1}^r \frac{\alpha_i}{1 - e^{-\alpha_i}}$$

$$\stackrel{\text{exer}}{=} 1 + \frac{1}{2} c_1(E) + \frac{1}{12} (c_1^2(E) + c_2(E)) + \frac{1}{24} c_1(E) c_2(E) + \dots$$

of course, the Chern character of E is

$$\text{ch}(E) = \sum_{i=1}^r e^{\alpha_i}$$

Hirzebruch-Riemann-Roch : Let E be a holomorphic vector bundle on a compact connected complex manifold X , then

$$\chi(X, E) = \int_X \text{Todd}(X) \cdot \text{ch}(E)$$

special case :

$$\chi(X, \mathcal{O}_X) = \int_X \text{Todd}(X)$$

Moreover : get the following classical formulas :

Curvature forms and Chern forms for line bundles:
 Let X be a compact complex manifold, $\mathcal{L}$ a line bundle on X

Recall the connection form and curvature form:

Suppose that e is a local section which gives a Riemannian trivialization of $\mathcal{L}$, D is a Riemannian connection on $\mathcal{L}$

$$De = \theta \otimes e$$

$$D^2e = (d\theta - \theta \wedge \theta) \otimes e = d\theta \otimes e$$

$\theta = d\theta$ is the curvature form, indep of the choice

($\theta \mapsto \theta' = \theta + \frac{1}{i} d\psi$ for some local unit u)

Note that if $\|\cdot\|$ is a Hermitian metric, then

$$\theta = \bar{\partial} \log \|e\|^2$$

$$\text{so } \theta = \bar{\partial} \log \|e\|^2 = 2\pi i d^c \log \|e\|^2$$

The real operator d^c is defined by:

$$d^c = \frac{\sqrt{-1}}{2} (\bar{\partial} - \partial) = \frac{1}{2} \frac{\sqrt{-1}}{(\bar{\partial} - \partial)}$$

$$(\Rightarrow \bar{\partial} = \frac{d}{2} + 2\pi i d^c, \quad \partial = \frac{d}{2} - 2\pi i d^c)$$

$$dd^c = \frac{\sqrt{-1}}{2} \partial \bar{\partial} = \frac{-1}{2\pi i} \partial \bar{\partial}$$

The Chern form of $\mathcal{L}$ is:

$$-dd^c \log \|e\|^2 = d^c d \log \|e\|^2 = \frac{\sqrt{-1}}{2\pi} \quad (14)$$

Example: $\mathcal{O}(1)$ on $\mathbb{P}^1$

$$\mathbb{P}^1 = \text{Proj } S(V^*)$$

For $x = [x_0 : x_1] \in \mathbb{P}^1$

the fiber $\mathcal{O}(1)_x = V^* / \text{Ker } x$

$$\downarrow \lambda \quad \langle \lambda_1 | \lambda_2 \rangle = \frac{\langle \lambda_1, x \rangle \langle x, \lambda_2 \rangle}{\langle x | x \rangle}$$

Take the section z_0 ,

$$\text{then } \|z_0\|_x^2 = \frac{x_0 \bar{x}_0}{x_0 \bar{x}_0 + x_1 \bar{x}_1}$$

The Chern form is :

$$\text{let } w = \frac{z_1}{z_0}$$

$$-dd^c \log \frac{1}{1+w\bar{w}} = dd^c \log(1+w\bar{w})$$

Exercise

Prove that

$$\int_{w \in \mathbb{C}} dd^c \log(1+w\bar{w}) = 1$$

Exercise A compact complex surface is called a Hopf surface if its universal covering is biholomorphic to $\mathbb{C}^2 - \{0\}$.

Let $\alpha_1, \alpha_2 \in \mathbb{C}^\times$ be such that $0 < |\alpha_1| \leq |\alpha_2| < 1$.

Let $H = H_{\alpha_1, \alpha_2} = \mathbb{C}^2 - \{0\} / \left((z_1, z_2) \sim (\alpha_1^n z_1, \alpha_2^n z_2) \right)_{n \in \mathbb{Z}}$.

i.e. $H =$ the quotient of $\mathbb{C}^2 - \{0\}$ under the action of $\mathbb{Z}$ sending $(z_1, z_2) \mapsto (\alpha_1^n z_1, \alpha_2^n z_2)$

(a) Show that H is diffeomorphic to $S^1 \times S^3$.

(b) Prove that $\chi_{\text{top}}(H) = e(H) = 0$, $c_1^2(H) = 0$ (Use the Hirzebruch signature theorem)

(c) Show that $p_g(H) = 0$

(If $0 \neq \alpha$ is a holomorphic 2-form,
 $\alpha \neq \partial\varphi$: otherwise $\alpha \wedge \bar{\alpha} = d(\varphi \wedge \bar{\alpha})$
 and $\int_H \alpha \wedge \bar{\alpha} = 0$, and $\alpha = 0$)

(d) Prove that $q(H) = 1$ (Since $\chi(H, \mathcal{O}_H) = 0$)

(e) Prove that $H^0(H, \Omega_H^1) = 0$

(If $\alpha \in H^0(H, \Omega_H^1)$, $\int d\alpha \wedge d\bar{\alpha} = 0$ so $d\alpha = 0$.

~~Write $\alpha = \beta_1 + \beta_2$ with $\beta_1 = \frac{1}{2}(\alpha + \bar{\alpha})$, $\beta_2 = \frac{1}{2}(\alpha - \bar{\alpha})$~~

Thus we have $H^0(H, \Omega_H^1) \oplus \overline{H^0(H, \Omega_H^1)} \rightarrow H^1(H, \mathbb{C})$

if $\alpha_1 + \bar{\alpha}_2 \mapsto 0$, $\exists f$ s.t. $df = \alpha_1 + \bar{\alpha}_2$
 $\Rightarrow \alpha_1 = \partial f \Rightarrow \bar{\partial} \partial f = 0$ i.e. f is harmonic

$\Rightarrow f$ is constant and $\partial f = 0$ ~~$\Rightarrow \bar{\partial} f = 0$~~
 Since $\dim H^1(H, \mathbb{C}) = 1$, $H^0(H, \Omega_H^1) = 0$.

(f) $\text{Pic}(H) \cong \mathbb{C}^\times$

(g) Show that H is not Kähler.

(h) Show that $H^1(H, \Omega^1) = 0$

($\mathbb{C} \hookrightarrow \Omega_H^1 \Rightarrow \chi_{\text{top}}(H) = \sum_{i,j} (-1)^{i+j} \dim H^i(H, \Omega_H^j)$)

The Grassmannians:

Schubert cells: $(=) \quad k\text{-planes in } \mathbb{A}^n$

fix a flag (full flag)

$$(0) \subseteq V_1 \subseteq V_2 \subseteq \dots \subseteq V_n = V$$

Let $W_a(V) = \{A \in G(k, n) \mid \dim(A \cap V_{n-k+a-a_i}) = i\}$

$$a = (a_1, \dots, a_k) \quad n-k \geq a_1 \geq a_2 \geq \dots$$

$$\text{note } \dim(A \cap V_{n-k+a-a_i}) = i \Leftrightarrow \dim(A + V_{n-k+i-a_i}) = n-a_i$$

$$\sigma_a(V) = \overline{W_a(V)} = \{A \in G(k, n) \mid \dim(A \cap V_{n-k+i-a_i}) \geq i \quad \forall i\}$$

$$\text{codim } W_a(V) = \sum a_i$$

In fact, $W_a(V) \cong \mathbb{A}^{k(n-k) - \sum a_i}$

Example: $\sigma_{1,0}(V) = \{A \mid \dim(A \cap V_2) \geq 1\}$

in $G(2, 4)$

$$\sigma_{1,1}(V) = \{A \mid A \subseteq V_3\}$$

$$\sigma_{2,0}(V) = \{A \mid A \supseteq V_1\}$$

$$\sigma_{2,1}(V) = \{A \mid V_1 \subseteq A \subseteq V_3\}$$

Note: $\sigma_a \subseteq \sigma_{a'} \Leftrightarrow a'_i \leq a_i \quad \forall i$

$$\Lambda \mapsto * \Lambda = \text{ann}(\Lambda)$$

$$*: G(k, n) \rightarrow G(n-k, n)$$

Take $V_i^* \text{ s.t. } V_i^* = \text{ann}(V_{n-i})$

Since $\dim(\Lambda \cap V_{n-k+i-a_i}) \geq \dim(*\Lambda \cap V_{k-i+a_i}^*) \geq a_i$

Conclude The image $*\sigma_a(V)$ of $\sigma_a(V)$ under $*$ is

$\sigma_{a'}(V^*)$, where a' is the smallest nondecreasing

sequence s.t. $a'_i \geq i \quad \forall i$

Relation with Chern classes :

Prop: On $G(k, n)$, let S be the $rk = k$ universal subbundle of $\mathcal{O}^{\oplus n}$, and let Q be the universal quotient bundle with $rk(Q) = n - k$.

Then $\forall r \geq 0$, the Chern classes of S and Q are given by

$$c_r(S) = (-1)^r [\sigma_{1, \dots, 1}]^r,$$

$$c_r(Q) = [\sigma_r]$$

Prop: Let $E \rightarrow X$ be a complex vector bundle over a compact connected complex manifold. For $k = rk(E)$ sections $\sigma_1, \dots, \sigma_k$ of E , let the i -th degeneracy set $D_i(\sigma_1, \dots, \sigma_k)$ be

$$D_i(\sigma_1, \dots, \sigma_k) = D_i(\sigma_1, \dots, \sigma_i) = \{x \in X : \sigma_1(x) \wedge \dots \wedge \sigma_i(x) = 0\}$$

Assume that $\sigma_1, \dots, \sigma_k$ are generic sections of E .

Then:
$$c_r(E) = [D_{k-r+1}]$$

Porteous's Formula :

Let $E \rightarrow X$ be as above,

$$D_i^{(j)}(\sigma_1, \dots, \sigma_k) = \{x \in X : \dim \overline{\sigma_1(x), \dots, \sigma_i(x)} \leq i - j\}$$

Assume $\sigma_1, \dots, \sigma_k$ are generic, then

$$[D_i^{(j)}] = \det \begin{bmatrix} c_{k-i+j}(E) & \dots & c_{k-i+2j-1}(E) \\ \vdots & \ddots & \vdots \\ c_{k-i+1}(E) & \dots & c_{k-i+j}(E) \end{bmatrix}$$

Example:

$$*(\sigma_2) = \sigma_{1,1}$$

$$*(\sigma_{2,1,1}) = \sigma_{3,1}$$

Note : $\lambda(a^*) = a_1$
 $\lambda(a) = a_1^*$

In form of Young ~~diagrams~~ tableau : taking the transpose

(Giambelli's formula)

$$\sigma_{a_1, \dots, a_d} = \det \begin{pmatrix} \sigma_{a_1} & \sigma_{a_1+1} & \sigma_{a_1+2} & \dots & \sigma_{a_1+d-1} \\ \sigma_{a_2-1} & \sigma_{a_2} & \sigma_{a_2+1} & \dots & \sigma_{a_2+d-2} \\ \sigma_{a_3-2} & \sigma_{a_3-1} & \sigma_{a_3} & & \\ \vdots & & & \ddots & \\ \sigma_{a_d-d+1} & & & & \sigma_{a_d} \end{pmatrix} = \det (\sigma_{a_i+j-i})_{1 \leq i, j \leq d}$$

(Littlewood-Richardson rule)

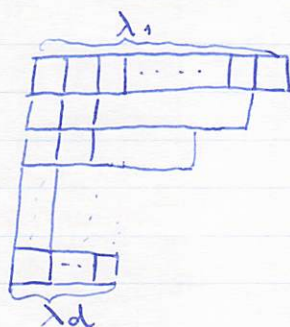
$$\lambda = (\lambda_1, \dots, \lambda_d)$$

$$\mu = (\mu_1, \dots, \mu_e)$$

$$\sigma_\lambda \cdot \sigma_\mu = \sum N_{\lambda, \mu, \rho} \sigma_\rho$$

where: $N_{\lambda, \mu, \rho}$ = the number of ways a Young tableau on the Young diagram of ρ arises by a strict $(\mu_1, \dots, \mu_e)$ -expansion of the diagram for λ .

Young diagram
for λ :



simple expansion of a tableau: obtained from a given tableau by adding m new boxes, to the right of the given rows (including the empty rows), such that no two of the new boxes appear in the same column.

a $(\mu_1, \dots, \mu_e)$ -expansion of a tableau: is the result of a succession of successive simple extensions, first adding μ_1 boxes (with symbol α_1), then μ_2 boxes (with symbol α_2), etc.

The expansion is strict if the following holds:

Given a $(\mu_1, \dots, \mu_e)$ -expansion, write down the added symbols in order, proceeding from right to left in the first row, then right to left in the second row, and so on. We obtain a (non-commutative) monomial

$$\alpha = \alpha_{i_1} \alpha_{i_2} \dots \alpha_{i_t}, \text{ with } t = \mu_1 + \dots + \mu_e$$

The expansion is strict if

$$\forall j \leq t, \forall 1 \leq k < e$$

the number of α_k 's among the first j terms of α $\geq$ the number of α_{k+1} 's among the first j terms of α

Example: $\lambda = (\lambda_1, \dots, \lambda_d)$ $m \geq 1$

$$\sigma_m \cdot \sigma_\lambda = \sigma_\lambda \cdot \sigma_m = \sum_{\mu} \sigma_\mu$$

sum over all $\mu = (\mu_1, \dots, \mu_{d+1})$ s.t.

$$\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \dots \geq \mu_d \geq \lambda_d \geq \mu_{d+1} \geq 0$$

$$\text{and } |\mu| = |\lambda| + m$$

(Pier's formula)

Example:

$$\sigma_{3,1,1} \cdot \sigma_{2,1}$$

$$= \cancel{\sigma_{6,1,1}} + \sigma_{5,1,1} + \sigma_{4,3,1} + \sigma_{4,2,2} + \sigma_{4,2,1} + \sigma_{4,1,1,1} + \sigma_{3,3,2} + \sigma_{3,3,1} + \sigma_{3,2,2,1} + \sigma_{3,2,1,1} + \cancel{\sigma_{3,1,1,1,1}}$$

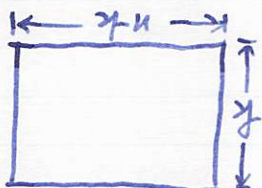
Pairwise diagonally: example

For two Schubert cells in $G(k, n)$ of complementary dimension,

$$\sigma_a \cdot \sigma_b = 0 \text{ unless } b_i + a_{k-i} = n-k \quad \forall i=1, \dots, k$$

- For a description of the complementary Schubert cell:

Take complement in $\square$, and flip the complement upside down.



Example:
$$\sigma_{ij} = \sigma_{i+1} \sigma_j - \sigma_{i+1} \sigma_{j-1} \quad \text{for } i \geq j \geq 0$$

(special case of Giambelli's formula)

Example to work out: $\sigma_{2,1} \cdot \sigma_{2,1}$

Reduction formulas for triple products

Suppose $|a| + |b| + |c| = k \cdot (n-k)$

(I) In the Young tableaux for a, b, c , if for 3 indices $0 \leq \alpha, \beta, \gamma \leq k$ with $\alpha + \beta + \gamma = 2k+1$,

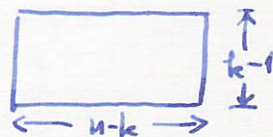
The length of the 3 corresponding rows (some or all may be empty) is $\geq n-k$,

then $\#(\sigma_a \cdot \sigma_b \cdot \sigma_c) = \begin{cases} \#(\sigma_{a'} \cdot \sigma_{b'} \cdot \sigma_{c'}) & \text{if } a_\alpha + b_\beta + c_\gamma = n-k \\ 0 & \text{if } a_\alpha + b_\beta + c_\gamma > n-k \end{cases}$
 $G(n, k) \quad G(k-1, n-1)$

where a', b', c' are obtained from a, b, c

by deleting the α -th row from a , the β -th row from b
and the γ -th row from c

So that the Young tableau becomes restricted to



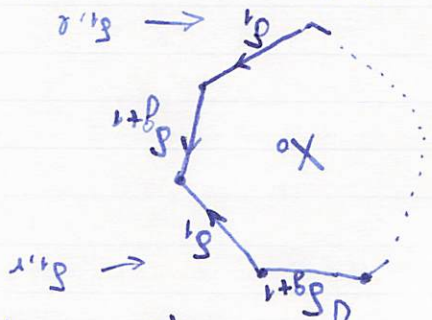
(II) Dual statement to (I), deleting columns (some may be empty) from the Young tableaux

Example: $= \text{coeff. of } \sigma_{4,2,2} \text{ in } \sigma_{3,1,1} \cdot \sigma_{2,1}$
 $\#(\sigma_{3,1,1} \cdot \sigma_{2,1} \cdot \sigma_{2,2}) \text{ in } G(3, 7)$
 $= \#(\sigma_{3,1} \cdot \sigma_2 \cdot \sigma_2) \text{ in } G(2, 6)$
 $= \#(\sigma_2 \cdot \sigma_2 \cdot \sigma_2) \text{ in } G(2, 5)$
 $= \#(\sigma_2 \cdot \sigma_2) \text{ in } G(2, 4)$
 $= 1$

Example that the reduction formulas do not apply:

$$\#(\sigma_{2,1} \cdot \sigma_{2,1} \cdot \sigma_{2,1}) \text{ in } G(3, 6)$$

Reciprocity law (for differentials of the first and the third kind)



ω : holomorphic 1-form
 η : meromorphic, with (at most) simple poles $\{z_\alpha\}_\alpha$

$$\text{Then } \sum_{i=1}^g \left(\int_{\gamma_i} \omega - \int_{\gamma_i} \eta \right) = 2\pi \sqrt{-1} \sum_{\alpha} (\text{Res}_{z_\alpha}(\eta) \cdot \int_{z_\alpha} \omega)$$

$$= 2\pi \sqrt{-1} \sum_{\alpha} (\text{Res}_{z_\alpha}(\eta) \cdot \int_{z_\alpha} \omega)$$

(Note: The right-hand side is independent of the choice of z_α , since $\sum_{\alpha} \text{Res}_{z_\alpha}(\eta) = 0$)

"Hint": Consider $f(z) = \int_{z_0}^z \omega$

$$\text{and } \int_{\gamma_0} f \cdot \eta = 2\pi \sqrt{-1} \sum_{\alpha} \text{Res}_{z_\alpha}(f \cdot \eta)$$

$$\text{For instance } \int_{\gamma_{1,r}} f \cdot \eta - \int_{\gamma_{1,l}} f \cdot \eta = \int_{\gamma_{1,r}} (f|_{\gamma_{1,r}} - f|_{\gamma_{1,l}}) \eta$$

$$= \int_{\gamma_1} (-\int_{\gamma_1} \omega) \cdot \eta = - \int_{\gamma_1} \omega \cdot \int_{\gamma_1} \eta$$

Consequence of the Reciprocity law
(or, the method):

$$(1) \int_{\delta_{g+i}} \omega_j - \int_{\delta_{g+j}} \omega_i = 0$$

$$(2) \int_{\partial X_0} f \cdot \bar{\omega}' = \int_X \omega \wedge \bar{\omega}'$$

$$(f = \int_{z_0}^z \omega) \quad \parallel \quad \sum_{i=1}^g \left(\int_{\delta_i} \omega \cdot \int_{\delta_{g+i}} \bar{\omega}' - \int_{\delta_{g+i}} \omega \cdot \int_{\delta_i} \bar{\omega}' \right)$$

$$\text{Especially: } \sqrt{-1} \cdot \sum_{i=1}^g \left(\int_{\delta_i} \omega \int_{\delta_{g+i}} \overline{\omega} - \int_{\delta_{g+i}} \omega \int_{\delta_i} \overline{\omega} \right) > 0$$

$$\text{Write } \omega = \sum_i a_i \omega_i \quad \int_{\delta_j} \omega_i = \delta_{ij}$$

$$\text{get: } \sqrt{-1} \sum_{i=1}^g \left(a_i \bar{a}_j \int_{\delta_{g+i}} \overline{\omega_j} - \bar{a}_i a_j \int_{\delta_{g+i}} \omega_j \right) \text{ for } j$$

$$= \sqrt{-1} \sum_{i=1}^g \sum_{j=1}^g \left(a_i \bar{a}_j \Omega_{ij} - \bar{a}_i a_j \overline{\Omega_{ij}} \right)$$

$$= \sqrt{-1} \sum_{i,j} -\sqrt{-1} \operatorname{Im}(\Omega_{ij}) a_i \bar{a}_j = \sum_{i,j} \operatorname{Im}(\Omega_{ij}) a_i \bar{a}_j > 0$$

This is the "second Riemann bilinear relation".

Abel's theorem (the first proof)

Assume that $(\int_D \omega_1, \dots, \int_D \omega_g) \equiv 0 \pmod{L_D}$

Want: a meromorphic function f on X s.t. $(f) = D$

Will construct: $\frac{1}{2\pi\sqrt{-1}} d\log f$

First: get a differential of the 3rd kind

η whose residues at the simple poles s_α are integers a_α ,
s.t. $\sum a_\alpha z_\alpha = D$ (as divisors).

May assume:

$$\int_{\delta_i} \eta = 0.$$

Reciprocity law

$$\Rightarrow \int_{\delta_{j+i}} \eta = \int_D \omega_i \quad i=1, \dots, g$$

$$\Rightarrow \exists \text{ cycle } \gamma \text{ s.t. } \gamma \sim \sum_{k=1}^{2g} m_k \delta_k \quad m_k \in \mathbb{Z}$$

$$\text{s.t. } \int_D \omega_i = \int_\gamma \omega_i \quad \forall i$$

$$\text{so: } \int_{\delta_{j+i}} \eta = \int_\gamma \omega_i$$

$$\text{Take: } \eta' = \eta - \sum_{k=1}^g m_{j+k} \omega_k$$

Then: all periods of η' are integers.
(Using Riemann's bilinear relations)

Theta functions

$$\vartheta(\vec{z}, \Omega) = \sum_{\vec{n} \in \mathbb{Z}^g} \mathbb{E} \left(\frac{1}{2} \vec{n} \cdot \Omega \cdot \vec{n} + \vec{n} \cdot \vec{z} \right)$$

(will omit the arrow signs)

converges absolutely and uniformly on each subset of the form

$$\left\{ \begin{array}{l} \max_i |\operatorname{Im} z_i| < \frac{c_1}{2\pi} \quad \text{and} \\ \operatorname{Im} \Omega \geq c_2 I_g \end{array} \right.$$

Theta transformation formula (quasi-periodicity)

$$\left\{ \begin{array}{l} \vartheta(z+m, \Omega) = \vartheta(z, \Omega) \quad \forall m \in \mathbb{Z}^g \\ \vartheta(z+\Omega n, \Omega) = \exp \mathbb{E} \left(-\frac{1}{2} \vec{n} \Omega \vec{n} - \vec{n} \cdot \vec{z} \right) \vartheta(z, \Omega) \end{array} \right.$$

$$\begin{aligned} \theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega) &= \sum_{n \in \mathbb{Z}^g} \mathbb{E} \left(\frac{1}{2} (n+a) \Omega (n+a) + \vec{n} \cdot \vec{a} \right) \vartheta(z+b, \Omega) \\ &= \mathbb{E} \left(\frac{1}{2} \vec{a} \Omega \vec{a} + \vec{a} \cdot \vec{z} \right) \vartheta(z+\Omega \vec{a}+b, \Omega) \end{aligned}$$

$$\left\{ \begin{array}{l} \theta \begin{bmatrix} a \\ b \end{bmatrix} (z+m, \Omega) = \mathbb{E} \left(\vec{a} \cdot m \right) \theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega) \\ \theta \begin{bmatrix} a \\ b \end{bmatrix} (z+\Omega n, \Omega) = \mathbb{E} \left(-\vec{b} \cdot n \right) \cdot \mathbb{E} \left(-\frac{1}{2} \vec{n} \Omega \vec{n} - \vec{n} \cdot \vec{z} \right) \theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega) \end{array} \right.$$

Methods to construct meromorphic functions on $\text{Jac}(X)$

$$(I) \quad f_1(z) = \frac{\prod_{i=1}^n \theta(z+a_i, \Omega)}{\prod_{i=1}^n \theta(z+b_i, \Omega)} \quad \sum_{i=1}^n a_i \equiv \sum_{i=1}^n b_i \pmod{\mathbb{Z}^g}$$

$$(I') \quad f_2(z) = \frac{\prod_{i=1}^n \theta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (Nz, \Omega)}{\prod_{i=1}^n \theta \left[\begin{smallmatrix} a' \\ b' \end{smallmatrix} \right] (Nz, \Omega)}$$

with $\underline{a, b, a', b'} \in \frac{1}{N} \mathbb{Z}^g$

$$(I'') \quad f_3(z) = \frac{\prod_{i=1}^n \theta \left[\begin{smallmatrix} a_i \\ b_i \end{smallmatrix} \right] (z, \Omega)}{\prod_{i=1}^n \theta \left[\begin{smallmatrix} a'_i \\ b'_i \end{smallmatrix} \right] (z, \Omega)}$$

with $\begin{cases} \sum_{i=1}^n a_i \equiv \sum_i a'_i \pmod{\mathbb{Z}^g} \\ \sum_i b_i \equiv \sum_i b'_i \pmod{\mathbb{Z}^g} \end{cases}$

$$(II) \quad \frac{\partial}{\partial z_i} \log \left(\frac{\theta(z+a, \Omega)}{\theta(z+b, \Omega)} \right)$$

$$(III) \quad \frac{\partial^2}{\partial z_i \partial z_j} \log \theta(z, \Omega)$$

$$\omega = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix}$$

Theorem (Riemann)

$\exists \Delta \in \mathbb{C}^g$ s.t. $\forall z \in \mathbb{C}^g$, the function

$$f(z) = f(P) = \theta\left(z + \int_{P_0}^P \omega, \Omega\right) \text{ either vanishes}$$

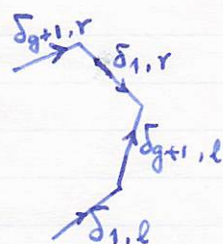
identically on $X \subset \text{Jac}(X)$, or has g zeroes $Q_1, \dots, Q_g$

$$\text{s.t. } z + \sum_{i=1}^g \int_{P_0}^{Q_i} \omega \equiv \Delta \pmod{L_\Omega}$$

D_j : small disc around the zero Q_j

Proof: (1) $0 = \int_{X_0 - \bigcup_j D_j} d\left(\frac{df}{f}\right)$

$$= \int_{\partial X_0} \frac{df}{f} - \sum_j \int_{\partial D_j} \frac{df}{f}$$



$$= - \sum_j \int_{\partial D_j} \frac{df}{f} + \sum_{i=1}^g \int_{\delta_{i,l} - \delta_{i,r}} \frac{df}{f} + \sum_{i=1}^g \int_{(\delta_{g+i,l} - \delta_{g+i,r})} \frac{df}{f}$$

note: f takes the same value on corresponding points of $\delta_{g+i,l}$ and $\delta_{g+i,r}$

But: f on $\delta_{i,l} = \exp\left(-\pi\sqrt{-1} \Omega_{ii} + 2\pi\sqrt{-1} \int_{P_0}^P \omega_i + 2\pi\sqrt{-1} z_i\right) \cdot f$ on $\delta_{i,r}$

$\downarrow$ const. $\downarrow$ on $\delta_{i,r}$ $\downarrow$ const.

by the theta transformation formula

$$\text{so } \frac{df}{f} \text{ on } \delta_{i,l} = 2\pi\sqrt{-1} \omega_i + \frac{df}{f} \text{ on } \delta_{i,r}$$

$$\Rightarrow \sum_j \int_{\partial D_j} \frac{df}{f} = 2\pi\sqrt{-1} \sum_{i=1}^g \int_{\delta_i} \omega_i = 2\pi\sqrt{-1} g$$

So $f(P)$ has g zeroes

(2) Let $g_k(P) = \int_{P_0}^P \omega_k$ on X_0 , repeat the same argument.

$$0 = \int_{X - \bigcup_j D_j} d\left(g_k \frac{df}{f}\right)$$

$$= - \sum_j \int_{\partial D_j} g_k \frac{df}{f} + \sum_{i=1}^g \int_{\delta_{i,l} - \delta_{i,r}} g_k \frac{df}{f} + \sum_{i=1}^g \int_{\delta_{g+i,l} - \delta_{g+i,r}} g_k \frac{df}{f}$$

$$\int_{\partial D_j} g_k \frac{df}{f} = 2\pi\sqrt{-1} g_k(Q_j) = 2\pi\sqrt{-1} \int_{P_0}^{Q_j} \omega_k$$

$$\int_{\delta_{i,l} - \delta_{i,r}} g_k \frac{df}{f} = ?$$

f takes the same value on the corresponding points of $\delta_{g+i,l}$ and $\delta_{g+i,r}$

$$g_k \text{ on } \delta_{i,l} = g_k \text{ on } \delta_{i,r} - \int_{\delta_{g+i}} \omega_k \quad (\text{on the corresponding point})$$

$$\frac{df}{f} \text{ on } \delta_{i,l} = 2\pi\sqrt{-1} \omega_i + \frac{df}{f} \text{ on } \delta_{i,r}$$

$$\underline{\text{So:}} \quad \int_{\delta_{i,l} - \delta_{i,r}} g_k \frac{df}{f} = \int_{\delta_{i,r}} (g_k - \int_{\delta_{g+i}} \omega_k) \cdot \left(\frac{df}{f} + 2\pi\sqrt{-1} \omega_i \right) - \int_{\delta_{i,l}} g_k \frac{df}{f}$$

$$= - \int_{\delta_{i,r}} \frac{df}{f} \cdot \int_{\delta_{g+i}} \omega_k + 2\pi\sqrt{-1} \int_{\delta_{i,r}} g_k \omega_i - 2\pi\sqrt{-1} \int_{\delta_{g+i}} \omega_k \int_{\delta_i} \omega_i$$

$$= \Omega_{ki} \cdot \left(\text{some integer} \right) \cdot 2\pi\sqrt{-1} + 2\pi\sqrt{-1} \int_{\delta_{i,r}} g_k \omega_i - 2\pi\sqrt{-1} \Omega_{ki}$$

$$\circ \circ \quad f|_{\delta_{g+i,l}} = f|_{\delta_{g+i,r}}$$

$$\int_{g_{+i,e} - g_{+i,r}} g_k \frac{df}{f} = \int_{g_{+i,r}} (g_k + \int_{g_{+i,r}}^k \omega_k) \frac{df}{f} - g_k \frac{df}{f}$$

$$= \delta_{k,i} \cdot \left(\int_{g_{+i,r}}^{g_{+i,e}} \frac{df}{f} \right) \cdot \omega_k$$

$$= \delta_{k,i} \cdot \omega_k \cdot (\pi \sqrt{-1} \Omega_{k,i} - 2\pi \sqrt{-1} \int_{P_0}^{P_i} \omega_i - 2\pi \sqrt{-1} z_i) + 2\pi \sqrt{-1} (\text{some integer})_{m_i}$$

Putting everything together, get:

$$\sum_j \int_{P_0}^{g_j} \omega_k = -z_k + \left[\frac{1}{2} \Omega_{k,k} - \int_{P_0}^{P_k} \omega_k + m_k - \sum_i \Omega_{k,i} \right] + 2 \int_{g_{+i,r}}^{g_{+i,e}} g_k \omega_i + \sum_i \Omega_{k,i} \cdot n_i$$

constant modulo periods.

QED.

$$\omega = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix}$$

Corollarys

(1) $\forall P_1, \dots, P_g \in X, \forall z \in \mathbb{C}^g :$

$$\sum_{i=1}^g \int_{P_0}^{P_i} \omega \equiv z \pmod{L_\Omega} \Rightarrow \theta(\Delta - z + \int_{P_0}^{P_i} \omega) = 0 \text{ on } i=1, \dots, g$$

and, if $z \notin E = \{z \mid \theta(\Delta - z + \int_{P_0}^P \omega) = 0 \quad \forall P \in X\}$

then the divisor $\sum_{i=1}^g P_i$ is uniquely determined by z by:

$$\sum_{i=1}^g \int_{P_0}^{P_i} \omega \equiv z \pmod{L_\Omega}$$

$\uparrow$
period lattice

Pf: Consider

$$\begin{array}{ccc} W \subseteq \text{Sym}^g(X) \times \text{Jac}(X) & & \\ \parallel & & \\ \left\{ \left(\sum_i P_i, z \pmod{L_\Omega} \right) \mid \begin{array}{l} \sum_i \int_{P_0}^{P_i} \omega = z \pmod{L_\Omega} \\ \theta(\Delta - z + \int_{P_0}^{P_i} \omega, \Omega) = 0 \quad \forall i=1, \dots, g \end{array} \right\} & & \\ \begin{array}{c} \swarrow P_1 \\ \text{Sym}^g(X) \end{array} & & \begin{array}{c} \searrow P_2 \\ \text{Jac}(X) \end{array} \end{array}$$

(2) Let $E_e(x, y) = \theta(e + \int_x^y \omega, \Omega)$ function on $X \times X$,
depending on $e \in \mathbb{C}^g$
s.t. $\theta(e, \Omega) = 0$ s.t. $\theta(e) = 0$

$$E_e : X \times X \rightarrow \mathbb{C}$$

globally multi-valued,
change by an exponential factor when
 x or y are carried around a δ_{g+i}

There are $2g-2$ points $R_1, \dots, R_{g-1}, S_1, \dots, S_{g-1}$ on X

s.t. the zero divisor of E_e is

$$(\text{diagonal}_X) + \sum_{i=1}^{g-1} \{R_i\} \times X + \sum_{i=1}^{g-1} X \times \{S_i\}$$

Pf: Fix a generic $R \in X$, consider $\theta(e + \int_R^y \omega, \Omega)$.
the zero locus of

this zero divisor is: $R + \sum_{i=1}^{g-1} y_i$

The y_i 's are determined by $\sum_{i=1}^{g-1} \int_{P_0}^{y_i} \omega \equiv \Delta - \left(e + \int_{P_0}^R \omega \right) \pmod{L_\Omega}$

So: ~~the~~ the y_i 's are indep of the choice of R .
call them $s_1, \dots, s_{g-1}$

The same argument provides the $R_1, \dots, R_{g-1}$. q.e.d.

(3) Second proof a Abel's theorem:

Suppose: $I\left(\sum_{i=1}^d P_i - \sum_{i=1}^d Q_i\right) = 0 \pmod{L_\Omega}$

Choose $e \in \mathbb{C}^g$ st. $\theta(e) = 0$, $E_e(P_i, y) \not\equiv 0$ (not identically 0)
 $E_e(Q_i, y) \not\equiv 0$ $\forall i=1, \dots, g$

Consider $f(y) = \frac{\prod_{i=1}^d E_e(Q_i, y)}{\prod_{i=1}^d E_e(P_i, y)} = \frac{\prod_{i=1}^d \theta(e + \int_{Q_i}^y \omega, \Omega)}{\prod_{i=1}^d \theta(e + \int_{P_i}^y \omega, \Omega)}$
the multivalued function on X From the assumption, check:

$f(y)$ is single valued on X

But: From 2) above, the zero locus of f is $\sum_{i=1}^d Q_i$,

while the polar locus of f is $\sum_{i=1}^d P_i$.

So $(f) = \sum_{i=1}^d Q_i - \sum_{i=1}^d P_i$. q.e.d.

$\int_{Q_i}^y \omega$, $\int_{P_j}^y \omega$ are multi-valued expression.

Choose paths σ_i from Q_i to P_0

and τ_j from P_j to P_0

st. $\sum_i \int_{\sigma_i} \omega = \sum_j \int_{\tau_j} \omega$, then evaluate $\int_{Q_i}^y \omega$, $\int_{P_j}^y \omega$
via a path from P_0 to y

4. calculation of the zero locus of θ (in terms of the image of $\text{Sym}^{g-1}(X)$).

for $e \in \mathbb{C}^g$ $\theta(e, \Omega) = 0 \iff \exists \sum_{i=1}^{g-1} P_i \in \text{Sym}^{g-1}(X)$ s.t.

$$e = \Delta - \sum_{i=1}^{g-1} \int_{P_0}^{P_i} \omega$$

Pf $\Leftarrow$: apply Corollary (1) to $z = \sum_{i=1}^g \int_{P_0}^{P_i} \omega$
 for a generic $P_g \in X$

$\Rightarrow$: $\{e \in \mathbb{C}^g \mid \theta(e, \Omega) = 0\}$ has pure codim 1

$\{e \in \mathbb{C}^g \mid \theta(e, \Omega) = 0, \theta(e + \int_{P_0}^y \omega, \Omega) = 0 \forall y \in X\}$
 has pure codim 2.

Suffice to prove $\Rightarrow$ for $e \in \mathbb{C}^g$, $\theta(e, \Omega) = 0$

but $\theta(e + \int_{P_0}^y \omega, \Omega) \neq 0$
 on X

Write e in some way as

$$e = \Delta - \sum_{i=1}^{g-1} \int_{P_0}^{P_i} \omega$$

one of the zeroes is P_0
 since $\theta(e) = 0$.

and by Corollary (1) $\sum_{i=1}^{g-1} P_i$ is exactly the zeroes
 of $\theta(e + \int_{P_0}^y \omega, \Omega)$ on X .

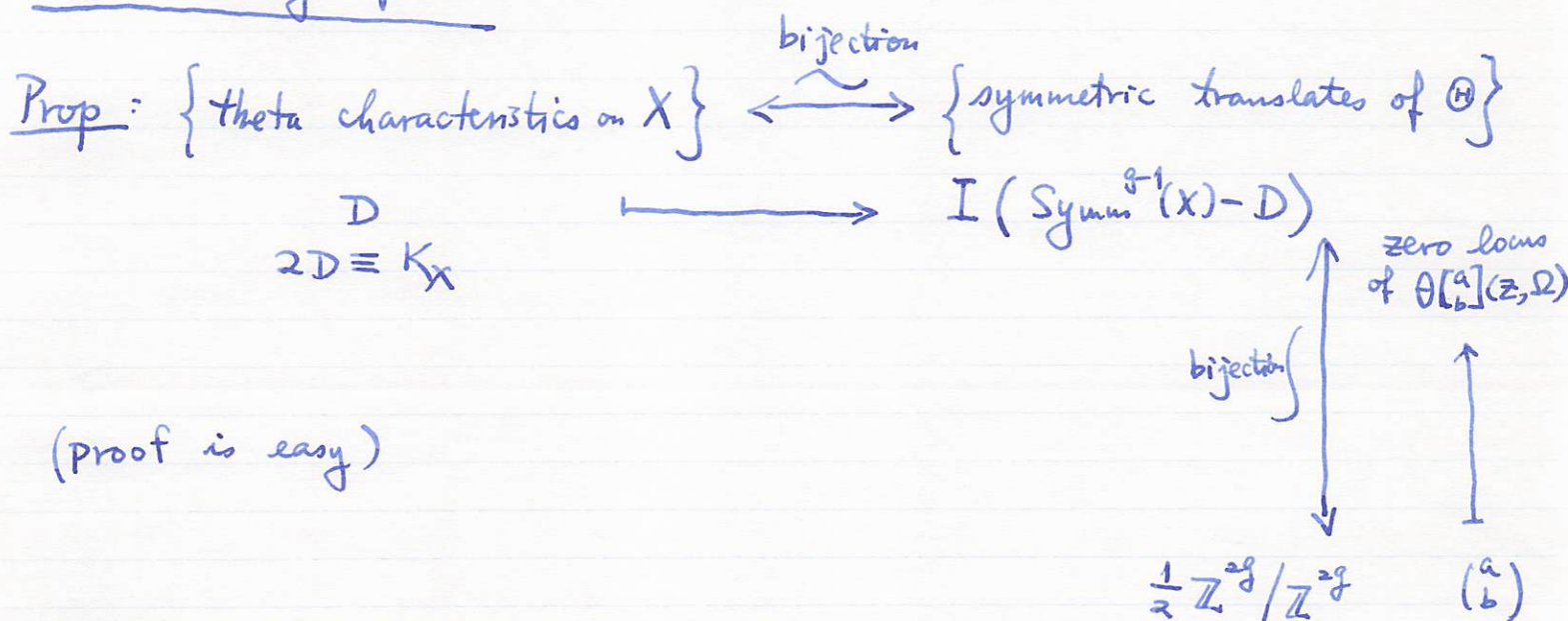
q.e.d.

Note : (H) $\stackrel{\text{def}}{=} \text{zero locus of } \theta(z, \Omega)$

$$= I(\Delta - \text{Sym}^{g-1}(X) + (g-1)P_0)$$

$$= I(\text{Sym}^{g-1}(X) - (g-1)P_0 - \Delta) \quad \text{by the symmetry of (H)}$$

The meaning of $\Delta \in \text{Jac}(X)$



Cor 1 $\Delta = I(D_0 - (g-1)P_0)$ for some theta characteristic D_0

Pf: Let D_0 be the theta characteristic s.t.

$$I(\text{Sym}^{g-1}(X) - D_0) = \Theta \stackrel{\text{as before}}{=} I(\text{Sym}^{g-1}(X) - (g-1)P_0 - \Delta)$$

so $\Delta = I(D_0 - (g-1)P_0)$

($\because$ the only element of $\text{Jac}(X)$ translating Θ to itself is 0)

Cor 2 If a theta characteristic D corresponds to $\begin{bmatrix} a \\ b \end{bmatrix} \in \frac{1}{2} \mathbb{Z}^{2g} / \mathbb{Z}^{2g}$,

then $0 \in \text{the zero locus of } \theta \begin{bmatrix} a \\ b \end{bmatrix} \iff 0 \in I(\text{Sym}^{g-1}(X) - D)$

i.e. $\theta(\Omega a + b, \Omega) = 0 \iff \exists \sum_{i=1}^{g-1} P_i \in \text{Sym}^{g-1}(X)$ s.t.
 $\sum_{i=1}^{g-1} P_i \equiv D$ as divisors

Prop for $\begin{bmatrix} a \\ b \end{bmatrix} \in \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$,

Let $e_*\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = (-1)^{4 \cdot {}^t a \cdot b}$

Note:
$$\frac{e_*\left(\begin{bmatrix} a+a' \\ b+b' \end{bmatrix}\right)}{e_*\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) \cdot e_*\left(\begin{bmatrix} a' \\ b' \end{bmatrix}\right)} = e_2\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a' \\ b' \end{bmatrix}\right) = (-1)^{4({}^t a \cdot b' - {}^t a' \cdot b)}$$

Have: $\theta\left[\begin{bmatrix} a \\ b \end{bmatrix}\right](-z, \Omega) = e_*\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) \cdot \theta\left[\begin{bmatrix} a \\ b \end{bmatrix}\right](z, \Omega)$

Cor For $\begin{bmatrix} a \\ b \end{bmatrix} \in \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$

(by total degree)

$e_*\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = 1 \iff$ either (H) does not contain $\Omega a + b$
or (H) contains $\Omega a + b$ with even multiplicity.

Taylor expansion of ϑ at $\Omega a + b$
has the form $\sum_{|I| \equiv 0 \pmod{2}} a_I z^I$

$e_*\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = -1 \iff$ (H) contains $\Omega a + b$ with odd multiplicity

Thm: For $\sum_{i=1}^{g-1} P_i \in \text{Sym}^{g-1}(X)$,

let $e = \Delta - \sum_{i=1}^{g-1} \int_{P_0}^{P_i} \omega$ (S₀: $e \in (H)$)

Then: $\dim(H^0(\mathcal{O}(\sum_{i=1}^{g-1} P_i))) = \text{multiplicity of } (H) \text{ at } e \xleftarrow{D_0 = \sum_{i=1}^{g-1} P_i}$

Note: From theorem and that $0 \in \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ is the only element of $\frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ s.t. translation by this element leaves e_* invariant, one sees that:

the theta characteristic D_0 (corresponding to (H)) is characterized by:

$\forall E \in \text{Pic}(X_2)$, write $I(E) = \Omega a + b \pmod{L_\Omega}$, $a, b \in \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$

$\dim(H^0(\mathcal{O}(D_0 + E))) \equiv 4 \cdot {}^t a \cdot b \pmod{2}$

Results of maple calculation:

$$\text{Let } \begin{aligned} a &:= \zeta^5 - \zeta^2 \\ b &:= \zeta^3 - \zeta^4 \\ c &:= \zeta^6 - \zeta \end{aligned} \quad \zeta = \zeta_7: \text{ a primitive 7-th root of 1} \quad \left| \begin{array}{l} \text{Note} \\ a^{\sigma_2} = b, \quad a^{\sigma_4} = c \\ \sigma_2 \leftrightarrow [2] \in \mathbb{F}_7^\times \end{array} \right.$$

Then the linear transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} R \\ S \\ T \end{pmatrix} \quad \text{with} \quad \begin{cases} R = \frac{1}{\sqrt{-7}} (ax + by + cz) \\ S = \frac{1}{\sqrt{-7}} (bx + cy + az) \\ T = \frac{1}{\sqrt{-7}} (cx + ay + bz) \end{cases} \quad \text{has order 2.}$$

$$\text{Moreover, } R^3S + S^3T + T^3R = x^3y + y^3z + z^3x$$

Note The linear transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad \text{with} \quad \begin{cases} u = \frac{1}{\sqrt{-7}} (ax + cy + bz) \\ v = \frac{1}{\sqrt{-7}} (cx + by + az) \\ w = \frac{1}{\sqrt{-7}} (bx + ay + cz) \end{cases} \quad \text{has order 2}$$

$$\text{and } uv^3 + vw^3 + wu^3 = x^3y + y^3z + z^3x$$

Let C be the curve in $\mathbb{P}^3$ defined by the equation

$$x^3y + y^3z + z^3x = 0$$

As above, C has an automorphism L of order 2:

$$L: [x:y:z] \mapsto [ax+by+cz : bx+cy+az : cx+ay+bz]$$

Also, C has an automorphism σ of order 7

$$\sigma: [x:y:z] \mapsto [\zeta^4x : \zeta^2y : \zeta z]$$

F. Klein (Math. Annalen 14, 1879; Gesam. Math. Abh. 90-136)
 Über die Transformation siebenter Ordnung der elliptischen Funktionen

$PSL_2(\mathbb{F}_7)$: decomposition into conjugacy classes, $\text{Card}(PSL_2(\mathbb{F}_7)) = 168$

- (1) $\{e\}$: 1
 elements of order 2 : 21, forming one conjugacy class, e.g. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
 elements of order 3 : 56, forming 2 conjugacy classes
 $\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$ and $\begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}$ not conjugate.
 $\exists$ 28 subgroups of order 3, all conjugate
 elements of order 4 : 42, forming 2 conjugacy classes
 $\begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$ not conjugate
 $\exists$ 21 subgroups isomorphic to $\mathbb{Z}/4\mathbb{Z}$, all conjugate
 $\leadsto$ Every element of order 2 belongs to a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z}$
 elements of order 7 : 48, forming 6 conjugacy classes :
 representatives: $\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} : i \in \mathbb{F}_7^*$
 There are 8 subgroups of order 7, all conjugate

(2) The ~~centralizer~~ ^{centralizer} ~~commutator~~ of an element of order 2 is isomorphic to D_8 , the dihedral group of order 8.

e.g. $\mathbb{Z} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}, \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 2 & -3 \end{pmatrix}, \right.$
 $\left. \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$
↖ ↗
order 4

(3) There are 14 subgroups isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$

e.g. $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}, \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} \right\}$

(4) The ~~centralizer~~ ^{centralizer} ~~commutator~~ ^{normalizer} of a ~~subgroup of~~ ^{subgroup of} order 3 is isomorphic to S_3

e.g. $\mathbb{Z} \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix} \right\}$

Wirtinger's inequality:

V : n -dimensional vector space over $\mathbb{C}$

H : pos. definite hermitian form on V ;

$\eta = \operatorname{Re} H$; $\omega = \operatorname{Im} H$

$W \subseteq V$ a real $2k$ -dim^l subspace of V

Then: $\underbrace{|\operatorname{res}_W \omega \wedge \dots \wedge \operatorname{res}_W \omega|}_{k\text{-times}} \leq \|\operatorname{res}_W \eta\|$
 volume form for $\operatorname{res}_W \eta$

and equality holds iff W is a complex subspace

Theorem: Let $X \subseteq \mathbb{P}^n$ be a compact oriented $2k$ -dimensional submanifold of degree d . Then

$$\operatorname{vol}(X) \geq d \cdot \operatorname{vol}(L^{n-k})$$

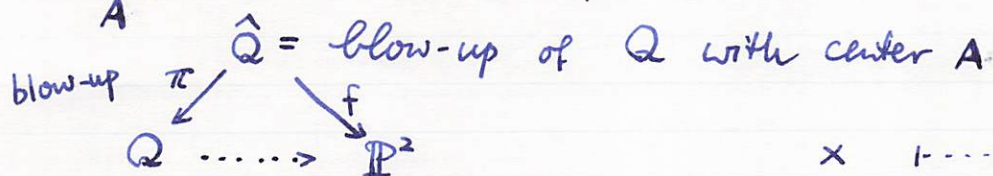
with equality iff X is an algebraic subvariety.
 $\uparrow$ linear subspace of dimension k

$$\begin{aligned} \text{pf: } \operatorname{vol}(X) &\stackrel{\text{def}}{=} \int_X \|\eta_X\| \geq \int_X \underbrace{|\omega \wedge \dots \wedge \omega|}_{k\text{-times}} \\ &\stackrel{\text{Wirtinger's ineq}}{\geq} \left| \int_X \underbrace{\omega \wedge \dots \wedge \omega}_{k\text{-times}} \right| = d \cdot \left| \langle [L^{n-k}], \omega \wedge \dots \wedge \omega \rangle \right| = d \cdot \operatorname{vol}(L^{n-k}) \\ &\quad [X] = d \cdot [L^{n-k}] \quad \uparrow \text{by Wirtinger's ineq applied to } L^{n-k} \end{aligned}$$

If equality holds, then Wirtinger's inequality

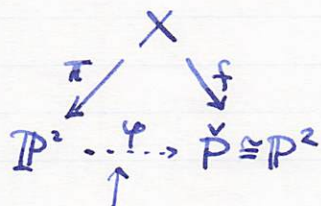
$\Rightarrow$ each $T_x X$ is a complex vector subspace of $T_x \mathbb{P}^n$

so the Cauchy-Riemann equation holds and X is a complex submanifold.

Examples of birational maps(1) $A \in Q \subseteq \mathbb{P}^3$ smooth quadric $x \mapsto \dots$
 $l = \overline{Ax}$
 line in $\mathbb{P}^3$,
 passing through a

There are two lines l_1, l_2 on Q passing through A ; they are separated in $\hat{Q}$.

These two lines map to two points $l_1, l_2 \in \mathbb{P}^2$, and f is the blow-up of these two points

(2) $A, B, C \in \mathbb{P}^2$, $X = \text{blow-up of } A, B, C \in \mathbb{P}^2$
 $P = \text{the linear system of quadrics passing through } A, B, C$ (=conics)


the birational map defined by the linear system P

for $x \in \check{P}$ corresponding to a linear pencil $\lambda C_1 + \mu C_2$, $C_1, C_2 \in P$

— if $C_1 \cap C_2 = A \cup B \cup C \cup X$ $\varphi: X \mapsto x$

— if $C_1 \cap C_2$ contains a line, then this line passes through 2 among the three points A, B, C , and

$C_1 \cap C_2 = \text{this line (say } \overline{AB}) \cup \text{the third point (say } C)$

$\overline{AB}$ defines a linear pencil in P (namely $\overline{AB} \cup (\text{any line passing through } C)$).

hence a point $\check{C} \in \check{P}$;

Similarly, get $\check{A}, \check{B} \in \check{P}$.

$\rightarrow$ The map f is isom. to the blow-up of $\check{A}, \check{B}, \check{C}$ in $\mathbb{P}^2$

Example Kummer K3 surfaces

A: an abelian variety / $\mathbb{C}$ (or, a compact complex surface which is a complex torus)

$\tilde{A}$ = blow-up all 16 2-torsion points of A.
 $X = \tilde{A} / \langle \sigma \rangle$, σ = involution induced by $[-1]_A$

Claim: X is a K3 surface

1) $A \cong \mathbb{C}^2 / \Lambda$ A: lattice
 coord: z, w
 at the origin: $\tilde{A}$ has coordinate $u = \frac{z}{w}$, w for instance

X has coordinate $u, v = w^2$

$$dz \wedge dw = d(uw) \wedge dw = w du \wedge dw = \frac{z}{2} du \wedge dv$$

Conclude: $dz \wedge dw$ gives a global section of K_X with no zero
 $K_X \cong \text{trivial}$

2) Hodge diamonds

$$\begin{array}{c} A: \\ \begin{array}{cccc} & & 1 & \\ & 1 & & \\ 1 & 2 & 2 & \\ & 2 & 2 & \\ & & 1 & \end{array} \end{array}$$

$$\begin{array}{c} \tilde{A}: \\ \begin{array}{cccc} & & 1 & \\ & 1 & & \\ 1 & 2 & 2 & \\ & 2 & 2 & \\ & & 1 & \end{array} \end{array}$$

$$X: \begin{array}{cccc} & & 1 & \\ & 1 & & \\ 1 & 2 & 2 & \\ & 2 & 2 & \\ & & 1 & \end{array}$$

Notice that: $\chi_{\text{top}}(\tilde{A}) = 2 \cdot \chi_{\text{top}}(X) - 16 \chi_{\text{top}}(\mathbb{P}^1)$ as it should hold

(Exercise)

simply connectedness of K3 surfaces:

Suppose that X is a compact complex Kähler surface, smooth,
such that $H^1(X, \mathcal{O}_X) = 0$ and $K_X \cong \text{trivial}$

$\Rightarrow$ By noether's formula, $\chi_{\text{top}}(X) = 24$

Also, $b_1(X) = 0$, and the Hodge diamond is

$$\begin{array}{ccc} & 0 & 0 \\ 1 & 20 & 1 \\ & 0 & 0 \\ & 1 & \end{array}$$

claim: X has no nontrivial finite covers (other than those¹ which are disconnected)

Suppose that $Y \rightarrow X$ is a finite étale cover, Y connected
 $n = \text{degree}(Y/X)$, then again $K_Y \cong \text{trivial}$, $H^1(Y, \mathcal{O}_Y) = 0$

Then by noether's formula, $\chi_{\text{top}}(Y) = 24$,
contradicting that $\chi_{\text{top}}(Y) = n \cdot \chi_{\text{top}}(X)$.

(*) $\uparrow$
since $b_1(Y) = 0$

(*) In fact, we do not even have to know that $H^1(Y, \mathcal{O}_Y) = 0$:

Noether's formula: $\chi_{\text{top}}(Y) = 12 \cdot (1 - h^1(Y, \mathcal{O}_Y) + 1) = 12(2 - h^1(Y, \mathcal{O}_Y))$
 $n = 24$

$$\Rightarrow 2n = 2 - R^1(Y, \mathcal{O}_Y) \Rightarrow R^1(Y, \mathcal{O}_Y) = 0 \text{ and } n = 1$$

This proves: $\pi_1(X)^\wedge$ (profinite completion of X) = $\{1\}$

To formulate this exercise:

Let X be a compact complex nonsingular surface such that $K_X \cong \text{trivial}$ and $H^1(X, \mathcal{O}_X) = \text{trivial}$.

show that every connected finite étale covering $Y \rightarrow X$ is in fact an isomorphism. What does this mean in terms of $\pi_1(X)$?

Godeaux Surface $\mathbb{P}^3 \ni Y$: Fermat quintic surface μ_5 operates on Y :

$$\ni \zeta_5 : [x_0 : x_1 : x_2 : x_3] \mapsto [x_0 : \zeta_5 x_1 : \zeta_5^2 x_2 : \zeta_5^3 x_3]$$

action is without fixed points

$$\leadsto \text{Let } X = \mu_5 \backslash Y$$

numeric invariants:

$$\begin{aligned} \pi_1(Y) = \{e\} \text{ by Lefschetz} &\Rightarrow \pi_1(X) \cong \mathbb{Z}/5\mathbb{Z} \\ \text{so } q(Y) = 0 & \quad b_1(X) = 0 \quad q(X) = 0 \\ b_1(Y) = 0 & \end{aligned}$$

$$K_Y = \mathcal{O}_Y(1)$$

$$c_1^2(Y) = (K_Y \cdot K_Y) = 5$$

$$0 \leftarrow \mathcal{O}_Y(5) \leftarrow T_{\mathbb{P}^3}|_Y \leftarrow T_Y \leftarrow 0$$

$$(1 + c_1(\mathcal{O}_Y(1)))^4 \cdot (1 + c_1(\mathcal{O}_Y(5)))^{-1} = c(T_Y)$$

$$c_2(Y) = c_2(T_Y) = 55$$

$$\Rightarrow c_1^2(X) = 1 = (K_X \cdot K_X)$$

$$c_2(X) = 11$$

$$\chi(X, \mathcal{O}_X) = 1$$

$$\Rightarrow p_g(Y) = h^{0,2}(Y) = 4$$

$$\Rightarrow p_g(X) = h^{0,2}(X) = 0$$

$$h^{1,1}(Y) = 45$$

$$h^{1,1}(X) = 9$$

Kodaira vanishing $\Rightarrow$

$$H^i(Y, K_Y^{\otimes n}) \xleftrightarrow{\text{dual to}} H^{2-i}(Y, K_Y^{\otimes n}) = H^{2-i}(Y, \mathcal{O}_Y(n))$$

$$\Rightarrow H^i(X, K_X^{\otimes n}) = 0 \text{ for } n \geq 2, i = 1, 2$$

$$(H^p(M, \Omega_M^q \otimes \mathcal{L}) = 0$$

if $p+q > \dim(M)$, $\mathcal{L}$ ample, positive

$$= (0) \text{ for } i = 1, 2, n \geq 2$$

$$\text{dually, } H^p(M, \Omega_M^q \otimes \mathcal{L}) = 0 \text{ if } p+q < \dim(M) \text{ } \mathcal{L} \text{ negative}$$

Riemann-Roch $\Rightarrow$ If $n \geq 2$,
$$h^0(X, K_X^{\otimes n}) = 1 + \frac{1}{2} (c_1(K_X^{\otimes n}) \cdot K_X^{\otimes n}) - (K_X^{\otimes n} \cdot K_X)$$

$$= 1 + \frac{1}{2} n(n-1)$$

X is a surface of general type.

Hodge diamond for X :

$$\begin{array}{ccccc} & & 1 & & \\ & \circ & & \circ & \\ & \circ & 9 & \circ & \\ & \circ & & \circ & \\ & & 1 & & \end{array}$$

Cyclic coverings

Y : connected complex manifold

$\mathcal{L}$: ample line bundle on Y

B : effective divisor (or zero)

s.t. $\mathcal{O}_Y(B) = \mathcal{L}^{\otimes n}$ ($n \geq 1$)

$s \in \Gamma(Y, \mathcal{O}_Y(B))$ vanishing exactly along B

$L \supseteq X = \text{zero locus of } p^*s - t^n$
total space of $\mathcal{L}$

$L \xrightarrow{p} Y$, $t \in \Gamma(L, p^*\mathcal{L})$ the tautological section

X in general is an analytic subspace of L

$\pi: X \rightarrow Y$ an n -fold cyclic covering ramified along B

$\leadsto X$ has at most singularities at points over the singular points of B

Especially, if B is reduced and smooth, then X is smooth.

Assume: B is reduced and smooth, then

1) $\pi^*\mathcal{L} \cong \mathcal{O}_X(B_1)$, where $B_1 = \text{the zero locus of } t \text{ on } X$
 $= \pi^{-1}(B)_{\text{red}}$

($p^*\mathcal{L} = \mathcal{O}_L(Y)$: Y is the zero locus of t on L)

$\leadsto \pi^*B = nB_1$

moreover, $K_X \cong \pi^*K_Y \otimes \mathcal{O}_X((n-1)B_1)$

$\cong \pi^*K_Y \otimes \pi^*\mathcal{L}^{\otimes n-1} = \pi^*(K_Y \otimes \mathcal{L}^{\otimes n-1})$

2) $\pi_*\mathcal{O}_X \cong \bigoplus_{j=0}^{n-1} \mathcal{L}^{-j}$

(every $f \in \pi^{-1}(V)$ has a unique power series expansion
 $\leadsto \pi_*\mathcal{O}_X \cong \bigoplus_{j=0}^{n-1} \mathcal{L}^{\otimes -j} \cdot t^j$ $S = \sum_{k \geq 0} a_k t^k$, $a_k \in \Gamma(V, \mathcal{L}^{\otimes -k})$)

invariants for double covers: (Assume B smooth, reduced).

$$\text{from } \pi_* \mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{L}^{-1}, \quad R^i \pi_* \mathcal{O}_X = 0 \quad \forall i \geq 1$$

$$K_X \cong \pi^*(K_Y \otimes \mathcal{L})$$

$$\leadsto \chi(X, \mathcal{O}_X) = 2\chi(Y, \mathcal{O}_Y) + \frac{1}{2}(\mathcal{L}, \mathcal{L}) + \frac{1}{2}(K_Y, \mathcal{L})$$

by Leray spectral sequence for $\pi_* + R.-R.$

$$\begin{aligned} p_g(X) &= \dim H^0(X, K_X) = \dim H^0(Y, \pi_* \pi^*(K_Y \otimes \mathcal{L})) \\ &= \dim H^0(Y, K_Y \otimes \mathcal{L} \oplus K_Y) = p_g(Y) + h^0(Y, K_Y \otimes \mathcal{L}) \end{aligned}$$

$$c_1^2(X) = 2c_1^2(Y) + 4(K_Y, \mathcal{L}) + 2(\mathcal{L}, \mathcal{L})$$

From Noether's formula, deduce that

$$c_2(X) = 2c_2(Y) + 2(K_Y, \mathcal{L}) + 4(\mathcal{L}, \mathcal{L})$$

Enriques surfaces

Def. $K_S \neq \mathcal{O}_S$, $K_S^{\otimes 2} \cong \mathcal{O}_S$, $q(S) = 0$

An example from double covers:

Take $Y = \mathbb{P}^1 \times \mathbb{P}^1 \supseteq B =$ a smooth curve of bidegrees $(4,4)$,
 ι : involution
 $\iota: ([x_0:x_1], [y_0:y_1]) \mapsto ([x_0:-x_1], [y_0:-y_1])$ not passing through any one of the 4 base points of ι ,
 but stable under ι

Let $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{O}_{\mathbb{P}^1}(2)$ on Y

$$K_Y \cong \mathcal{O}_{\mathbb{P}^1}(-2) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-2)$$

The double cover $\pi: X$
 constructed from
 $(Y, B, \mathcal{L})$

$$\rightarrow \chi(X, \mathcal{O}_X) = 2 \chi(Y, \mathcal{O}_Y) + \frac{1}{2} (\mathcal{L}, K_Y) + \frac{1}{2} (\mathcal{L}, \mathcal{L})$$

$$= 2 + \frac{1}{2} \cdot (-8) + \frac{1}{2} \cdot 8 = 2$$

$$p_g(X) = p_g(Y) + h^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}) = 1 \Rightarrow q(X) = 0$$

$$c_1^2(X) = 2 c_1^2(Y) + 4 (K_Y, \mathcal{L}) + 2 (\mathcal{L}, \mathcal{L})$$

$$= 2 \times 8 + 4 \times (-8) + 2 \times 8 = 0$$

$$c_2(X) = 2 c_2(Y) + 2 (K_Y, \mathcal{L}) + 4 (\mathcal{L}, \mathcal{L})$$

$$= 2 \times 4 + 2 \times (-8) + 4 \times (8) = 24$$

Moreover, $K_X \cong \pi^* \mathcal{O}_Y = \mathcal{O}_X$, so X is a K3 surface.

Let $\bar{X} = X/\iota$

$$\rightarrow \chi(\bar{X}, \mathcal{O}_{\bar{X}}) = 1, \quad q(\bar{X}) = 0 \Rightarrow p_g(\bar{X}) = 0 \Rightarrow K_{\bar{X}} \neq \mathcal{O}_{\bar{X}}$$

Hence $\bar{X}$ is an Enriques surface

An Enriques surface:

S = the quartic Fermat surface in $\mathbb{P}^3$, given by equation $x_0^4 + x_1^4 - x_2^4 - x_3^4 = 0$, so S is a K3 surface

$\sigma: S \rightarrow S$ is the automorphism given by
 $(x_0: x_1: x_2: x_3) \mapsto (x_0: \sqrt{-1}x_1: -x_2: -\sqrt{-1}x_3)$

- σ has order 4, no fixed point, but σ^2 has 8 isolated fixed points:
 - 4 are $S \cap (x_0 = x_2 = 0)$
 - 4 are $S \cap (x_1 = x_3 = 0)$

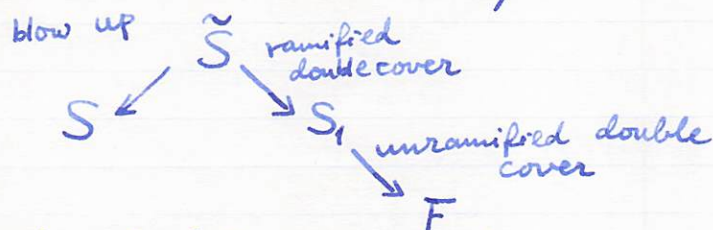
The Enriques surface we get will be a smooth model of $S/\langle \sigma \rangle$.

To do this:

Let $\tilde{S}$ = blow-up S at the 8 fixed points of σ^2

$S_1 = \tilde{S}/\langle \sigma^2 \rangle$: will be a smooth K3 surface;
 σ induces a fix-point-free involution on S_1

Then, let $F = S_1/\langle \sigma \rangle$



local calculation

affine open $x_0 \neq 0$: $1 + u_1^4 - u_2^4 - u_3^4 = 0$
 near $(1:0:1:0)$, i.e. for $(0,1,0)$ in this affine piece

local coordinate: $(u_1, u_3) \xrightarrow{\sigma} (-u_1, -u_3)$

for $\tilde{S}$: $u_1, \frac{u_3}{u_1}$ and $\frac{u_1}{u_3}, u_3$

for S_1 : $u_1^2, \frac{u_3}{u_1}$ and $\frac{u_1}{u_3}, u_3^2$

And, one sees easily that σ induces a fix-point-free involution on S_1

III-9

S is a K3 surface; a generator of $H^0(S, \Omega_S^2)$ is

$$\alpha = \frac{du_1 \wedge du_3}{4 u_2^3}$$

$$\sigma : \alpha \mapsto -\alpha \quad (\because \sigma : \begin{matrix} u_1 \mapsto \sqrt{-1} u_1 \\ u_2 \mapsto -u_2 \\ u_3 \mapsto -\sqrt{-1} u_3 \end{matrix})$$

$$\Rightarrow H^0(F, \Omega_F^2) = 0$$

But, $\sigma^2 : \alpha \mapsto \alpha$, and by local calculation α is a regular 2-form on S_1

For an Enriques surface F :

$$1 \rightarrow \mathbb{Z}(1) \rightarrow \mathcal{O}_F \rightarrow \mathcal{O}_F^* \rightarrow 1$$

$$\Rightarrow H^1(F, \mathcal{O}_F^*) \simeq H^2(F, \mathbb{Z}(1))$$

From Hochschild-Serre : $H^2(F, \mathbb{Z}(1))$ only has 2-torsion.
or existence of trace map $H^2(F, \mathbb{Z}(1))_{\text{tor}} = \text{Ker}([2] : H^2(F, \mathbb{Z}(1)) \rightarrow H^2(F, \mathbb{Z}(1)))$

$$1 \rightarrow \mu_2 \rightarrow \mathcal{O}_F^* \rightarrow \mathcal{O}_F^* \rightarrow 1$$

$$\Rightarrow H^1(F, \mathcal{O}_F^*)[2] \xleftarrow{\sim} H^1(F, \mu_2) = \text{Hom}(\pi_1(F), \mu_2)$$

$$\text{Therefore } H^1(F, \mathcal{O}_F^*)_{\text{tor}} \xrightarrow{\sim} H^2(F, \mathbb{Z}(1))_{\text{tor}} \cong \mathbb{Z}/2\mathbb{Z},$$

the non-trivial element is represented by the canonical bundle Ω_F^2 .

$$\text{Hodge diamond for } S \text{ and } S_1 = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$$\text{Hodge diamond for } F : \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 10 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \quad \left(\text{since } e(F) = \frac{1}{2} e(S) = 12 \right)$$

The 27 lines - one example.

Example: Fermat cubic surface in $\mathbb{P}^3$:

$$\text{Let } S_0 = V(X_0^3 + X_1^3 + X_2^3 + X_3^3) \subseteq \mathbb{P}^3$$

To find all lines on S_0 : by direct calculation, there are 27 lines on S_0 for $1 \leq i, j \leq 2$

$$\left. \begin{aligned} - X_0 + \zeta^i X_1 &= X_2 + \zeta^j X_3 = 0 \\ - X_0 + \zeta^i X_2 &= X_1 + \zeta^j X_3 = 0 \\ - X_0 + \zeta^i X_3 &= X_1 + \zeta^j X_2 = 0 \end{aligned} \right\} \begin{array}{l} 0 \leq i, j \leq 2 \\ i, j \in \mathbb{Z}/3\mathbb{Z} \\ i, j \in \mathbb{Z}/3\mathbb{Z} \end{array} \quad \left. \vphantom{\begin{aligned} - X_0 + \zeta^i X_1 \\ - X_0 + \zeta^i X_2 \\ - X_0 + \zeta^i X_3 \end{aligned}} \right\} \begin{array}{l} 27 \text{ lines} \\ \text{altogether} \end{array}$$

$$\text{Let } \begin{cases} l_1 \text{ be the line } X_0 + X_1 = X_2 + X_3 = 0 \\ l_2 \text{ " " } X_0 + \zeta X_1 = X_2 + \zeta X_3 = 0 \end{cases}$$

There are exactly 5 lines among the 27 which intersect both l_1 and l_2 :

$$\begin{cases} m_1: & X_0 + X_1 = X_2 + \zeta X_3 = 0 \\ m_2: & X_0 + \zeta X_1 = X_2 + X_3 = 0 \\ m_3: & X_0 + X_2 = X_1 + X_3 = 0 \\ m_4: & X_0 + \zeta X_2 = X_1 + \zeta X_3 = 0 \\ m_5: & X_0 + \zeta^2 X_2 = X_1 + \zeta^2 X_3 \end{cases}$$

Consider $f_i: S_0 \setminus l_i \rightarrow \mathbb{P}^1 = \mathbb{P}(\text{planes containing } l_i) \quad i=1, 2$
linear projection

f_i extends to morphism $S_0 \xrightarrow{f_i} \mathbb{P}^1$;

at $x \in l_i$, $f_i(x) \rightarrow$ the tangent plane to S_0 at x
(clearly it contains l_i)

$$\text{Let } \begin{array}{c} \pi_1: S_0 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \\ \parallel \\ (f_1, f_2) \end{array}$$

Clearly π_1 contracts the 5 lines $m_1, \dots, m_5$

and one checks that π_1 is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the 5 ^{distinct} points $P_1, \dots, P_5$

Now, realize $\mathbb{P}^1 \times \mathbb{P}^1$ as a quadric Q in $\mathbb{P}^3$, then linear projection from a point on the quadric Q gives $\hat{Q}$ = blow-up of Q at a point as a blow-up of $\mathbb{P}^2$ at 2 distinct points:

$$\hat{Q} = \text{blow-up of } Q \text{ at a point} \xrightarrow{\sim} \text{blow up of } \mathbb{P}^2 \text{ at } 2 \text{ points.}$$

Therefore: $S_0 \cong$ blow-up of $\mathbb{P}^2$ at 6 distinct points.

Note that:

$$i) K_{S_0} \equiv -H \text{ by adjunction } \Rightarrow h^{2,0} = h^{0,2} = 0, K^2 = 3$$

$$\text{Being a hypersurface, Lefschetz } \Rightarrow \begin{aligned} h^{1,0} &= h^{0,1} = 0 \\ h^{1,2} &= h^{2,1} = 0 \end{aligned}$$

$$\chi(\mathcal{O}_{S_0}) = 1$$

$$\text{Noether's formula: } 12 = \chi_{\text{top}}(S_0) + 3 \Rightarrow b_2 = 9 - 2 = 7$$

$$\text{Hodge diamond is}$$

$$\begin{array}{ccc} & 1 & \\ 0 & & 0 \\ & 7 & \\ 0 & & 0 \\ & 1 & \end{array}$$

$$K : \text{antiample} \Rightarrow q(S_0) = p_2(S_0) = 0 \xRightarrow{\text{Castelnuovo}} S_0 \text{ is rational}$$

$$\text{For each line } l \text{ on } S_0, \quad l \cdot K_{S_0} = -1$$

$$\Rightarrow l \cdot l = -1$$

Thus each of these 27 lines is an exceptional curve of the first kind

The Twenty Seven Lines ; Del Pezzo surfaces :

reference: Chap IV of Manin's book "Cubic forms".
also, see the end of Mumford's book

Thm: S/k smooth cubic surface, k a field

(1) Let $N(S) = \text{Pic}(S \otimes_k \bar{k})$, $\ni [K_S]$, with intersection pairing.

Then $(N(S), [K_S], \langle, \rangle)$ is isomorphic to

$$\left(\mathbb{Z}^7 = \bigoplus_{i=0}^6 \mathbb{Z} e_i, -3e_0 + e_1 + \dots + e_6, \begin{array}{l} \text{the pairing with} \\ \langle e_0, e_0 \rangle = 1, \langle e_i, e_i \rangle = -1 \\ \text{for } i \geq 1 \end{array} \right)$$

(2) $\text{Gal}(k^{\text{sep}}/k)$ operates on $N(S)$ and preserves $[K_S]$ and the intersection pairing

(3) The classes of the 27 lines goes to the subset

$$I_6 = \{ l \in N(S) \mid \langle l, [K_S] \rangle = -1, \langle l, l \rangle = -1 \}$$

Let $R_6 = \{ l \in N(S) \mid \langle l, [K_S] \rangle = 0, \langle l, l \rangle = -2 \}$,

then R_6 forms a root system E_6 (changing the sign of the pairing)

Computing Chern classes of a blow-up:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{j} & \tilde{Y} \\ g \downarrow & & \downarrow f \\ X & \xrightarrow{i} & Y \end{array}$$

embedding of a nonsingular subvariety X into a smooth Y
 $d = \text{codim}(X, Y)$

$N =$ normal bundle $N_{X/Y}$

Use $\begin{cases} 0 \rightarrow \mathcal{O}_X(-1) \rightarrow g^*N \rightarrow Q \rightarrow 0 \\ 0 \rightarrow T_{\tilde{Y}} \rightarrow f^*T_Y \rightarrow j_*Q \rightarrow 0 \end{cases}$

also have $\begin{pmatrix} 0 \rightarrow \mathcal{O}_{\tilde{X}} \rightarrow g^*N \otimes \mathcal{O}(1) \rightarrow T_{\tilde{X}} \rightarrow T_X \rightarrow 0 \\ 0 \rightarrow T_{\tilde{X}} \rightarrow j^*T_{\tilde{Y}} \rightarrow \mathcal{O}_{\tilde{X}}(-1) \rightarrow 0 \end{pmatrix}$

The standard formalism for Chern classes

+ Riemann-Roch theorem (for j_* and Q)

Thm: $c(\tilde{Y}) - f^*c(Y) = j_*(g^*c(X) \cdot \alpha)$,

where
$$\alpha = \frac{1}{\mathfrak{L}} \left[\sum_{i=0}^d g^*c_{d-i}(N) - (1-\mathfrak{L}) \sum_{i=0}^d (1+\mathfrak{L})^i g^*c_{d-i}(N) \right]$$

$$\mathfrak{L} = c_1(\mathcal{O}_{\tilde{X}}(1))$$

(Here α is a polynomial in the Chern classes of the pull-back of N and $\mathfrak{L}$; more precisely it is a polynomial in $\mathfrak{L}$ with coefficients in the subring generated by the Chern classes of the pull-back of N .)

The division by $\mathfrak{L}$ is a formal division, since the difference of the two terms within the square brackets is formally a polynomial in $\mathfrak{L}$ of degree $\leq d+1$ which is divisible by $\mathfrak{L}$.)

Hironaka's example:

nonprojective complete algebraic variety, and an effective cycle, $\neq 0$, which is homologically equivalent to zero.

Ann. of Math., 75, 1962. pp. 190-208

Example: Kodaira dimension of Hopf surfaces $H_{\alpha_1, \alpha_2} : \kappa(H_{\alpha_1, \alpha_2}) = -1$:

$$0 < |\alpha_1| \leq |\alpha_2| < 1$$

$$(z_1, z_2) \mapsto (\alpha_1 z_1, \alpha_2 z_2)$$

$$H_{\alpha_1, \alpha_2} = \mathbb{C}^2 - \{0\} / \mathbb{Z}$$

If $K_{H_{\alpha_1, \alpha_2}}^{\otimes k}$ has a global section ($k \geq 1$)

$\Leftrightarrow$ a function $f(z_1, z_2)$ on $\mathbb{C}^2 - \{0\}$ (extended to $\mathbb{C}^2$)

Consider: $f(z_1, z_2)$. Have:

$$f(\alpha_1 z_1, \alpha_2 z_2) = \alpha_1^{-k} \alpha_2^{-k} f(z_1, z_2)$$

$$\text{Write } f(z_1, z_2) = \sum_{(m,n)} a_{mn} z_1^m z_2^n$$

$$\Rightarrow f(\alpha_1 z_1, \alpha_2 z_2) = \sum_{m,n} \alpha_1^m \alpha_2^n a_{mn} z_1^m z_2^n$$

$$\Rightarrow \alpha_1^{m+k} \alpha_2^{n+k} = 1 \quad \text{if } a_{mn} \neq 0$$

$$\Rightarrow \text{since } |\alpha_1|, |\alpha_2| < 1, \quad k \geq 1, \quad \text{we get: } a_{mn} = 0 \quad \forall m, n$$

Contradiction.

Similarly, if $k \leq -1$,

a section of $K_{H_{\alpha_1, \alpha_2}}^{\otimes k}$ corresponds to an entire function $f(z_1, z_2)$ on $\mathbb{C}^2$ such that

$$f(\alpha_1 z_1, \alpha_2 z_2) = \alpha_1^k \alpha_2^k f(z_1, z_2)$$

$$\Rightarrow \dim H^0(H_{\alpha_1, \alpha_2}, K_{H_{\alpha_1, \alpha_2}}^{\otimes k}) = \#\left\{ (m,n) \in \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 0} \mid \alpha_1^{m+k} \alpha_2^{n+k} = 1 \right\}$$

$\leadsto$ This gives the dimensions of $H^2(H_{\alpha_1, \alpha_2}, K_{H_{\alpha_1, \alpha_2}}^{\otimes k})$ as well, so we also get $\dim H^1(H_{\alpha_1, \alpha_2}, K_{H_{\alpha_1, \alpha_2}}^{\otimes k})$

Example Inverse Surfaces

$$M = (m_{ij}) \in SL_3(\mathbb{Z})$$

$$(Example): \begin{pmatrix} 0 & 1 & 0 \\ n & 0 & 1 \\ 1 & -n & 0 \end{pmatrix}$$

with one real eigenvalue $\alpha > 1$, and 2 complex eigenvalues $\beta, \bar{\beta}$ with $|\beta| \neq 1$.

$$M \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \alpha \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad M \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \beta \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Let $S_M = G \backslash G^2 \times \mathbb{C}$, where

$$G = G_M \cong \mathbb{Z}^3 \rtimes \mathbb{Z} \text{ is generated by } g_0, g_1, g_2, g_3$$

$$g_0 : (w, z) \mapsto (\alpha w, \beta z)$$

$$g_i : (w, z) \mapsto (w + a_i, z + b_i)$$

$$(a_{ij}^k) = \sum_{k=1}^3 m_{jk} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \Rightarrow g_0 \circ g_i \circ g_0^{-1} = g_{m_{i,1}}^1 \circ g_{m_{i,2}}^2 \circ g_{m_{i,3}}^3$$

$$S_M : \pi_1(S_M) \cong G$$

$$H_1(S_M, \mathbb{Z}) \cong G/[G, G] \cong \mathbb{Z} \oplus \bigoplus_{i=1}^3 \mathbb{Z}/e_i \cdot \mathbb{Z}$$

where e_1, e_2, e_3 are the elementary divisors of $M - I_3$.

Claim: S_M does not contain any compact complex curve (nonempty)

$$(1) \quad S_M : \text{an } (S^1)^3\text{-bundle over } S^1$$

$$\Rightarrow c_2(S_M) = e(S_M) = 0 \Rightarrow b_2(S) = 0$$

$$\text{also: } b_1 = b_3 = 1$$

adjunction: if $C \subseteq S_M$ is a curve, $\dim H^1(C, \mathcal{O}_C) = 0$

ie C has to be either a smooth elliptic curve or

Conclude: C is the image of $w \times \mathbb{C}$ for some $w \in \mathbb{H}$

(ii) linear algebra:

$$\begin{cases} w_0 \times \mathbb{C} \subseteq (w_0, 0) + \sum_{i=1}^3 \mathbb{R}(a_i, b_i) \\ (w_0 \times \mathbb{C}) \cap ((w_0, 0) + \sum_{i=1}^3 \mathbb{Q}(a_i, b_i)) \text{ has dimension} \leq 1 \end{cases}$$

In fact, otherwise: The solutions of $\sum_{i=1}^3 a_i x_i$ has dimension ≤ 1
 $\dim_{\mathbb{Q}}(\mathbb{Q}a_1 + \mathbb{Q}a_2 + \mathbb{Q}a_3) \geq 2$
 so that tM has a $\mathbb{Q}$ -rational eigenvector,
 and $\alpha \in \mathbb{Q}$. This is impossible since
 α is integral and $\neq \pm 1$.

$$\left(\begin{array}{l} \text{Riemann-Roch: } \frac{1}{2}(C_1^2 + C_2) \\ X(S_M, \mathcal{L}) = \underbrace{X(S_M, 0)}_0 + \frac{1}{2} \left(C \mathcal{L} \cdot \mathcal{L} - (K_{S_M} \cdot \mathcal{L}) \right) \end{array} \right)$$

Claim $\Rightarrow$ Only the trivial bundle has non-zero sections.

General fact for compact complex surfaces:

$$b_1 = h^{1,0} + h^{0,1}$$

$$b_1 \text{ even} \Rightarrow h^{1,0} = h^{0,1} \text{ and } b^+ = 2p_g + 1$$

$$b_1 \text{ odd} \Rightarrow h^{1,0} = h^{0,1} - 1 \text{ and } b^+ = 2p_g$$

$$b_2 = h^{2,0} + h^{0,2} + h^{1,1}$$

The signature
 of the cup product
 $\Rightarrow$ on $H^{1,1}(X, \mathbb{R})$ is
 $\begin{cases} -h^{1,1} & \text{if } b_1 \text{ odd} \\ 2-h^{1,1} & \text{if } b_1 \text{ even} \end{cases}$

For our Inoue surface S_M :

$$\leadsto p_g = 0, q = h^{0,1} = 1, h^{1,0} = 0, h^{2,0} = h^{1,1} = h^{0,2} = 0$$

$$(b_1 = b_3 = 1) \quad h^{2,1} = 1, h^{1,2} = 0.$$

From the exponential sequence, get $\text{Pic}(S_M) H^1(0, \mathcal{O}_{S_M}^*) \cong (\mathbb{C}/\mathbb{Z}) \times \left(\bigoplus_{i=1}^3 (\mathbb{Z}/e_i \mathbb{Z}) \right)$

e_i : elementary divisors of
 $M - I_3$

notice that

$$H^2(S_M, \mathbb{Z}(1)) \cong \bigoplus_{i=1}^3 (\mathbb{Z}/e_i \mathbb{Z})$$

- Claim: K_{S_M} is not a torsion element in $\text{Pic}(S_M)$. ($\Rightarrow \kappa(S_M) = -1$)
by the previous claim
Moreover, $c_1(K_{S_M}) = 0 \in H^2(S_M, \mathbb{Z}(1))$.

Pf: $H^2(S_M, \mathbb{Z}(1)) = H^2(G_M, \mathbb{Z}(1))$,

$$\begin{array}{ccccccc} H^1(G_M, \mathbb{Z}(1)) & \longrightarrow & H^1(G_M, H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{C}})) & \xrightarrow{\exp} & H^1(G_M, H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{C}}^*)) & \xrightarrow{\delta} & H^2(G_M, \mathbb{Z}(1)) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ H^1(S_M, \mathbb{Z}(1)) & \longrightarrow & H^1(S_M, \mathcal{O}_M) & \longrightarrow & H^1(S_M, \mathcal{O}_M^*) & \longrightarrow & H^2(S_M, \mathbb{Z}(1)) \end{array}$$

and: K_{S_M} comes from an element of $H^1(G_M, \mathbb{C}^*)$,
which is the image of an element of $H^1(G_M, \mathbb{C})$

$$\begin{array}{ccccccc} & & \mathbb{C} & & & & \\ & & \cong & & & & \\ 0 \rightarrow & H^1(G_M, \mathbb{Z}(1)) & \rightarrow & H^1(G_M, \mathbb{C}) & \rightarrow & H^1(G_M, \mathbb{C}^*) & \xrightarrow{\delta} H^2(G_M, \mathbb{Z}(1)) \rightarrow 0 \\ & \downarrow \cong & & \downarrow & & \downarrow & \\ 0 \rightarrow & H^1(G_M, \mathbb{Z}(1)) & \rightarrow & H^1(G_M, H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{C}})) & \rightarrow & H^1(G_M, H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{C}}^*)) & \rightarrow H^2(G_M, \mathbb{Z}(1)) \rightarrow 0 \\ & & & \cong & & & \\ & & & \mathbb{C} & & & \end{array}$$

Hence $H^1(G_M, \mathbb{C}) \cong H^1(G_M, H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{C}}))$

$[K_{S_M}]$ comes from the homomorphism $G_M \rightarrow \mathbb{C}^*$

$$\begin{array}{lcl} g_0 & \mapsto & \log(\alpha\beta) \\ g_1, g_2, g_3 & \mapsto & 0 \end{array}$$

To say that $[K_{S_M}]$ is non-torsion
means that $(\log \alpha\beta) \notin 2\pi i \cdot \mathbb{Q}$.

i.e. $\alpha\beta$ is not a torsion element of $\mathbb{C}^*$
 $\Leftrightarrow \bar{\beta}$ is not a torsion element of $\mathbb{C}^*$.

This is clear, otherwise α is a torsion element, but $\alpha > 1$.

Resolution of surface singularities

reference: Laufer, Normal 2-dim^l singularities.

How to resolve curve and surface singularities:

(I) Embedded curve singularities (in surfaces):

By successive blowing-up

(II) Surface singularities:

- may assume: V irreducible at the given point P .

- V presented as a branched covering of D^2 : $V \xrightarrow{p} D^2$

idea: (1) Perform embedded resolution for the branch locus $B \subseteq D^2$, until the total transform of B is a divisor with normal crossings:

$$\begin{array}{ccc} \tilde{D}^2 & \xrightarrow{\pi} & D^2 \\ \pi^{-1}(B) & \longrightarrow & B \\ \bigcup_i C_i & & \end{array}$$

(of course, $\pi^{-1}(B)$ may not be reduced)

(2) Then locally, the map on $\tilde{D}^2 \supset D^2$

$\tilde{V} \rightarrow \tilde{D}^2$ is a ramified finite

$$\begin{array}{ccccc} V \setminus p^{-1}(B) & \subset & \tilde{V} = V \times_{D^2} \tilde{D}^2 & \rightarrow & V \\ \downarrow & & \tilde{p} \downarrow & & \downarrow p \\ D^2 \setminus B & \subset & \tilde{D}^2 & \xrightarrow{\pi} & D^2 \end{array}$$

abelian covering, since

$$\pi_1(\mathbb{C}^2 - \{z_1 z_2 = 0\}) \cong \mathbb{Z}^2 \text{ is abelian.}$$

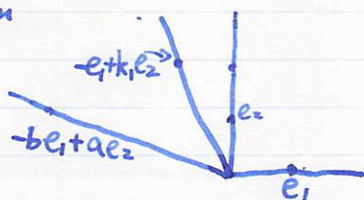
Therefore this is given by a morphism between affine torus embeddings.

Moreover, can use the technique of affine torus embeddings to resolve the singularities

(In the notation of Laufer's book)

given integers $k_1, \dots, k_s \geq 2$

Consider the torus embedding $M(k_1, \dots, k_s)$ with the following cone decomposition



$$\sigma_0 = \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2$$

$$u_1^{(0)} = e_1, \quad u_2^{(0)} = e_2$$

$$\sigma_1 = \mathbb{R}_{\geq 0} u_1^{(1)} + \mathbb{R}_{\geq 0} u_2^{(1)}$$

$$u_1^{(1)} = u_2^{(0)}, \quad u_2^{(1)} = -u_1^{(0)} + k_1 u_2^{(0)}$$

$$\sigma_2 = \mathbb{R}_{\geq 0} u_1^{(2)} + \mathbb{R}_{\geq 0} u_2^{(2)}$$

$$u_1^{(2)} = u_2^{(1)}, \quad u_2^{(2)} = -u_1^{(1)} + k_2 u_2^{(1)}$$

$$\sigma_{i+1} = \mathbb{R}_{\geq 0} u_1^{(i+1)} + \mathbb{R}_{\geq 0} u_2^{(i+1)}$$

$$u_1^{(i+1)} = u_2^{(i)}, \quad u_2^{(i+1)} = -u_1^{(i)} + k_{i+1} u_2^{(i)}$$

$\vdots$

$$\sigma_s = \mathbb{R}_{\geq 0} u_1^{(s)} + \mathbb{R}_{\geq 0} u_2^{(s)}$$

$$u_1^{(s)} = u_2^{(s-1)}, \quad u_2^{(s)} = -u_1^{(s-1)} + k_s u_2^{(s-1)}$$

Write $\underline{u_2^{(i)} = -b_i e_1 + a_i e_2}$

The recursion is:

$$u_2^{(i+1)} = -b_{i+1} e_1 + a_{i+1} e_2$$

$$= -u_1^{(i)} + k_{i+1} u_2^{(i)} = -(-b_i e_1 + a_i e_2) + k_{i+1} (-b_i e_1 + a_i e_2)$$

$$= -(k_{i+1} b_i - b_{i-1}) e_1 + (k_{i+1} a_i - a_{i-1}) e_2$$

i.e.

$$\boxed{\begin{aligned} b_{i+1} &= k_{i+1} b_i - b_{i-1} \\ a_{i+1} &= k_{i+1} a_i - a_{i-1} \end{aligned}}$$

$$b_0 = 0$$

$$b_1 = 1$$

$$b_2 = k_2$$

$$a_0 = 1$$

$$a_1 = k_1$$

$$a_2 = k_2 k_1 - 1$$

(and: $b_s = b, a_s = a$ at the end)

$$a_{i+1} b_i - a_i b_{i+1} = -1 \quad (\text{prove by induction}).$$

Therefore: $\frac{a_{i+1}}{b_{i+1}} < \frac{a_i}{b_i} \quad \forall_i$

• The cones σ_i are smooth w.r.t. $\mathbb{Z}e_1 + \mathbb{Z}e_2$

Also: One proves by induction that

$$\frac{a_i}{b_i} = k_1 - 1\sqrt{k_2} - \dots - 1\sqrt{k_i} \quad \forall i \geq 1$$

(induction step:
 a_i, b_i defined inductively by $\begin{cases} a_{j+1} = k_{j+1} a_j - a_{j-1} \\ b_{j+1} = k_{j+1} b_j - b_{j-1} \end{cases}$
 Show: replacing k_i by $k_i - \frac{1}{k_{i+1}}$, then

$$\begin{aligned} & k_1 - 1\sqrt{k_2} - \dots - 1\sqrt{k_i} - 1\sqrt{k_{i+1}} \\ &= \frac{(k_i - \frac{1}{k_{i+1}}) a_{i-1} - a_{i-2}}{(k_i - \frac{1}{k_{i+1}}) b_{i-1} - b_{i-2}} = \frac{(k_{i+1} k_i - 1) a_{i-1} - k_{i+1} a_{i-2}}{(k_{i+1} k_i - 1) b_{i-1} - k_{i+1} b_{i-2}} \\ &= \frac{k_{i+1} (k_i a_{i-1} - a_{i-2}) - a_{i-1}}{k_{i+1} (k_i b_{i-1} - b_{i-2}) - b_{i-2}} = \frac{k_{i+1} a_i - a_{i-1}}{k_{i+1} b_i - b_{i-1}} \end{aligned}$$

Now: take k_i 's to be integers, ≥ 2 , then

$$a_i < a_{i+1}, \quad b_i < b_{i+1} \quad (\text{since } a_{i+1} - a_i = \underbrace{(k_{i+1} - 1)}_{\text{induction}} a_i - a_{i-1})$$

Consider: The morphism between affine torus embeddings

$$\text{Temp}(\mathbb{R}_{\geq 0} \cdot e_1 + \mathbb{R}_{\geq 0} \cdot (-b e_1 + a e_2); \mathbb{Z}e_1 + \mathbb{Z}e_2)$$

$$\longrightarrow \text{Temp}(\mathbb{R}_{\geq 0} f_1 + \mathbb{R} f_2; \mathbb{Z}f_1 + \mathbb{Z}f_2)$$

(dual to: $f_2^* \mapsto e_2^*$
 $f_1^* \mapsto b e_2^* + a e_1^*$)

Note: e_2^* corresponds to the divisor with self-intersection $-k_1$

is a proper finite morphism. So using torus embedding already includes taking normalization.

Example: of resolution of surface singularities.

$$V: z^5 = x^2 + y^3$$

First: embedded resolution of the branch locus $x^2 + y^3$

$$B: x^2 + y^3 = 0 \subseteq \mathbb{C}^2$$

$$B: \begin{array}{l} \nearrow \\ \searrow \end{array}$$

(A)

(1) blow up the origin:

$$\begin{array}{c} B \\ \nearrow \searrow \\ A_1 \end{array}$$

$$A_1: (\xi = \xi' = 0)$$

$$\text{local equations: } (x, y) = (\xi \xi', \xi) = (\xi', \xi' \xi')$$

$$V': z^5 = \xi'^2 (\xi'^2 + \xi)$$

$$z^5 = \xi'^2 (1 + \xi'^3 \xi')$$

(2) blow up again

$$\begin{array}{c} \nearrow \searrow \\ A_2 \\ \nearrow \searrow \\ A_1 \end{array} \quad B$$

relevant

$$A_2: (\tau = \tau' = 0)$$

$$A_1: (\sigma = 0)$$

$$(\xi, \xi') = (\sigma \tau, \tau) = (\tau', \sigma' \tau')$$

$$V'': z^5 = \sigma^2 \tau^3 (\tau + \sigma) \quad z^5 = \tau'^3 (\sigma' \tau' + 1)$$

relevant

(3) blow up once more:

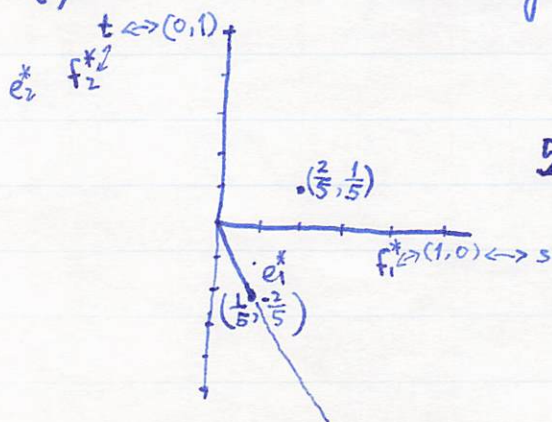
$$(\sigma, \tau) = (st, t) = (t', s't')$$

$$\begin{array}{c} \begin{array}{c} \nearrow \searrow \\ B \\ \nearrow \searrow \\ A_1 \end{array} \quad \begin{array}{c} \nearrow \searrow \\ A_2 \\ \nearrow \searrow \\ A_3 \end{array} \\ \begin{array}{c} (s=-1) \\ (s=0) \\ (s'=0) \end{array} \end{array}$$

$$V''': z^5 = t'^6 s'^3 (s'+1)$$

$$z^5 = s^2 t^6 (s+1)$$

(B) (1) At: $s=0$: The singularity is equivalent to: $z^5 = s^2 t^6$



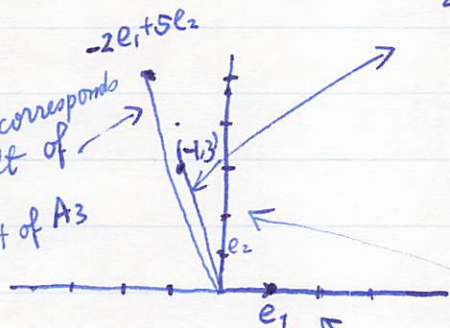
$$e_2^* = f_2^* \leftrightarrow t$$

$$5e_1^* + 2e_2^* = f_1^* \leftrightarrow s$$

$$e_1^* \leftrightarrow s^{\frac{1}{5}} t^{-\frac{2}{5}}$$

dual picture

This edge corresponds to a lift of $t=0$ i.e. a lift of A_3



this edge corresponds to a lift of $s=0$ i.e. A_1 a lift of A_1

$$z \leftrightarrow \frac{2}{5}f_1^* + \frac{6}{5}f_2^*$$

$$= \frac{2}{5}(5e_1^* + 3e_2^*) + \frac{6}{5}e_2^*$$

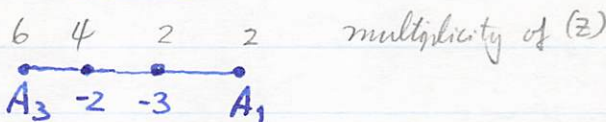
$$= 2e_1^* + 2e_2^*$$

so: multiplicity of $z=0$ has multiplicity 2 at this component the multiplicity of divisor of (z) is 4

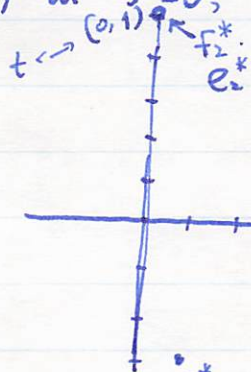
$$\frac{5}{2} = 3 - \frac{1}{2}$$

so: get a part of the dual graph of the resolution:

a_0	a_1	a_2
1	3	5
b_0	b_1	b_2
0	1	2



(2) at $s'=0$, the singularity is equivalent to: $z^5 = t'^6 s'^3$



$$f_1^* \leftrightarrow (1, 0) \leftrightarrow s'$$

$$e_2^* = f_2^* \leftrightarrow t'$$

$$5e_1^* + 3e_2^* = f_1^*$$

$$e_1^* \leftrightarrow s' \frac{1}{5} t' \frac{-3}{5}$$

$$\leftrightarrow (\frac{1}{5}, \frac{-3}{5})$$

here:

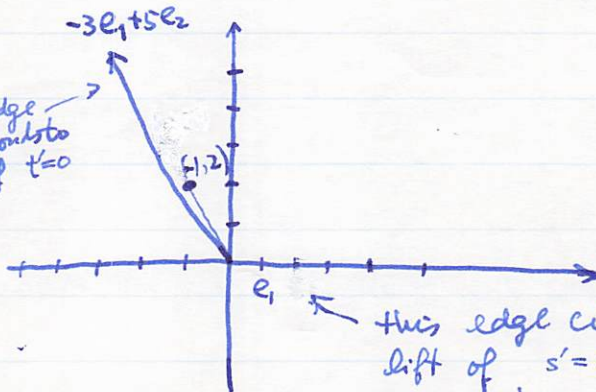
$$z \leftrightarrow \frac{3}{5}f_1^* + \frac{6}{5}f_2^*$$

$$= \frac{3}{5}(5e_1^* + 3e_2^*) + \frac{6}{5}e_2^*$$

$$= 3e_1^* + 3e_2^*$$

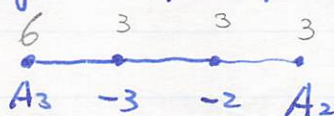
dual picture:

this edge corresponds to a lift of $t'=0$ i.e. A_3



this edge corresponds to a lift of $s'=0$ i.e. a lift of A_2

so: get a part of the dual graph of the resolution:



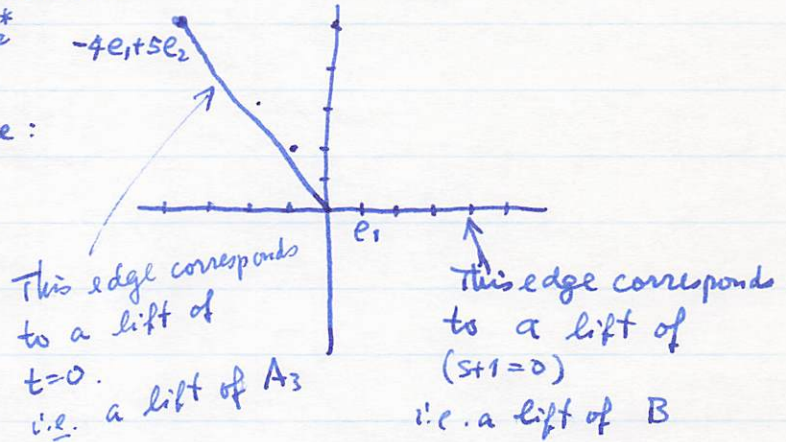
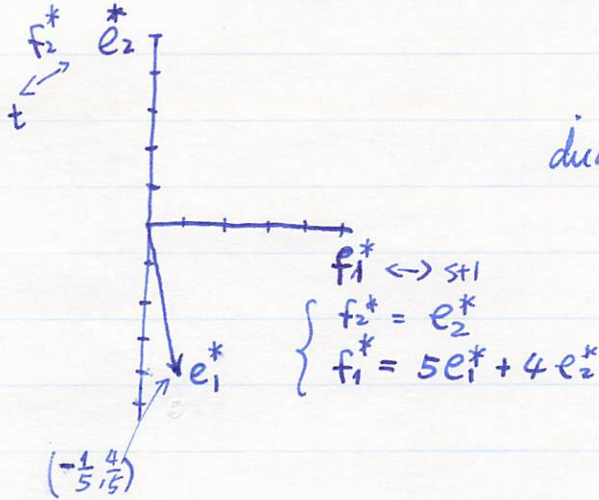
$$\frac{5}{3} = 2 - \frac{1}{3}$$

(3) at $s = -1$, the singularity is ^{equivalent} isomorphic to

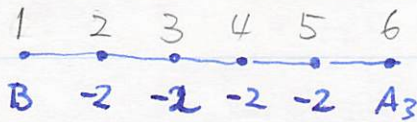
$$z^5 = (s+1)t^6$$

$$\begin{aligned} z &\leftrightarrow \frac{1}{5}f_1^* + \frac{6}{5}f_2^* \\ &= \frac{1}{5}(5e_1^* + 4e_2^*) + \frac{6}{5}e_2^* \\ &= e_1^* + 2e_2^* \end{aligned}$$

dual picture:



$$\frac{5}{4} = 2 - \frac{3}{4} = 2 - \frac{1}{2 - \frac{2}{3}} = 2 - \frac{1}{2 - \frac{2}{3}} = 2 - \frac{1}{2 - 1} = 2 - 1 = 1$$

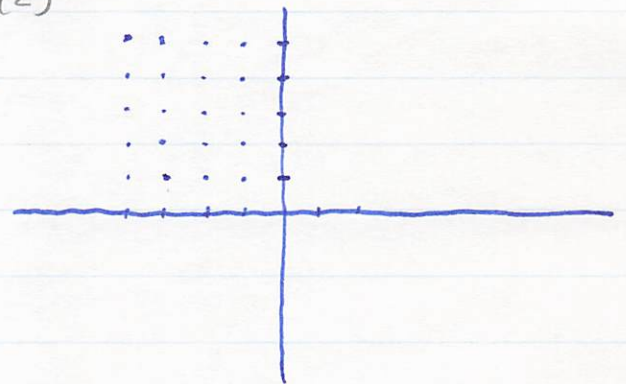


multiplicity for the divisor of (z)

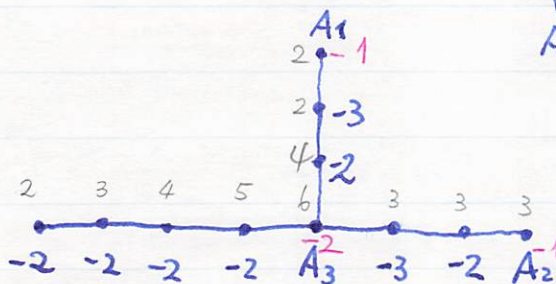
$$(B)^2 = -2$$

$$\begin{array}{cccccc} a_0 & a_1 & a_2 & a_3 & a_4 \\ 1 & 2 & 3 & 4 & 5 \end{array}$$

$$\begin{array}{cccccc} b_0 & b_1 & b_2 & b_3 & b_4 \\ 0 & 1 & 2 & 3 & 4 \end{array}$$



$$\rightarrow (0,1), (-1,2), (-2,3), (-3,4), (-4,5)$$

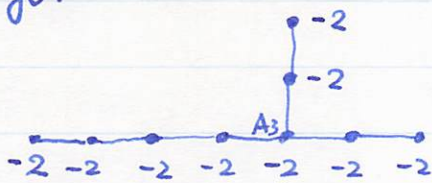


multiplicity for the divisor of (z)

$$0 = 2(A_1 \cdot A_1) + 2 \Rightarrow (A_1 \cdot A_1) = -1$$

$$0 = 6(A_3 \cdot A_3) + 5 + 4 + 3 \Rightarrow (A_3 \cdot A_3) = -2; \quad 0 = 3(A_2 \cdot A_2) + 3 \Rightarrow (A_2 \cdot A_2) = -1$$

contracting the two exceptional divisors A_1 and A_2 with self-intersection -1 , we get:



This is the Dynkin diagram for E_8 !

Grauert's ampleness criterion

X : compact complex space, L : holomorphic line bundle on X

Then L is ample on X iff

$\forall$ irreducible analytic subset $Y \subseteq X$ with $\dim(Y) > 0$,

$\exists n = n(Y) > 0$ s.t. $L^{\otimes n}|_Y$ has a section which is not identically zero on Y but has zeroes on Y .

For surfaces, this means:

(a) For some $m \geq 1$, $L^{\otimes m}$ is effective and non-trivial.

(b) $\forall$ irreducible curve $C \subseteq X$, $\exists n = n(C) \geq 1$ and a section of $L^{\otimes n}|_C$, which is not identically zero on C , but vanishes somewhere on C .

Grauert's criterion for exceptional curves.

A reduced, compact, connected curve on a smooth surface X is exceptional (i.e. $\exists$ a bimeromorphic map $\pi: X \rightarrow Y$, an open nbd U of C in X , a point $y \in Y$, and an open nbd V of y in Y such that $\pi(C) = \{y\}$ and $\pi: U - C \xrightarrow{\text{biholomorphic}} V - \{y\}$.)
iff the intersection matrix $(C_i \cdot C_j)$ of the irreducible components $\{C_i\}$ of C is negative definite.

* Hironaka's example of a nonprojective normal algebraic surface (a normal algebraic space)

$\mathbb{P}^3 \ni C$: a smooth cubic. $P_1, \dots, P_{10} \in C$ ten points on C such that $\nexists$ curve $D \subseteq \mathbb{P}^2$ s.t. $C \cap D \subseteq \{P_1, \dots, P_{10}\}$ set theoretically.

Let $\tilde{X} = \mathbb{P}^2$ with $P_1, \dots, P_{10}$ blown up. $\tilde{X} \rightarrow \mathbb{P}^2$

$\tilde{C} =$ the strict transform of C . $(\tilde{C} \cdot \tilde{C})_{\tilde{X}} = -1$

Let $X =$ blown-down of $(\tilde{X}, \tilde{C})$ $(\tilde{X}, \tilde{C}) \rightarrow (X, \alpha)$

Then every curve in Y has to contain α :

(otherwise $\exists$ a curve $\tilde{D}$ in $\tilde{X}$ s.t. $\tilde{D} \cap \tilde{C} = \emptyset$.
and the image α of $\tilde{D}$ in $\mathbb{P}^2$ is a curve intersecting C in a subset of $\{P_1, \dots, P_{10}\}$.)

hence X is not projective.

Theorem: A compact, connected smooth surface X is projective
iff $\exists$ a line bundle $\mathcal{L}$ on X s.t. $(\mathcal{L} \cdot \mathcal{L}) > 0$.

Cor. 1 Every compact connected smooth surface X with $c_1^2(X) > 0$
is projective

Cor. 2 A line bundle $\mathcal{L}$ on a compact connected smooth
surface X is ample iff $c_1^2(\mathcal{L}) > 0$ and $(\mathcal{L} \cdot D) > 0$
 $\forall$ effective divisor $D > 0$ on X .

Cor. 3 A compact connected smooth surface X is projective
iff $a(X) = 2$

Cor. 4 Every smooth connected complete algebraic surface
is projective.

Castelnuovo's criterion

An algebraic surface X is rational iff $q(X) = 0 = p_2(X)$.

The Enriques-Kodaira Classification

Kodaira dimension = $-\infty$:

(1) minimal rational surfaces

$$b_1 = 0, \quad a(X) = 2 \quad (\text{i.e. must be "algebraic" (i.e. alg. space)})$$

but in fact projective algebraic

$$c_1^2 = 8 \text{ or } 9, \quad c_2 = 4 \text{ or } 3$$

(2) minimal surfaces of type VII

$$\text{i.e. } \text{Kod}(X) = -\infty, \quad b_1(X) = 1$$

(e.g. Hopf surfaces, Inoue surface)

$$a(X) = 0 \text{ or } 1, \quad c_1^2 \leq 0, \quad c_2 \geq 0$$

(3) ruled surfaces (of genus $g \geq 1$)

$$\text{Kod}(X) = -\infty, \quad b_1(X) = 2g, \quad a(X) = 2,$$

$$c_1^2 = 8(1-g), \quad c_2 = 4(1-g)$$

Kodaira dimension = 0

(4) Enriques surfaces

$$\text{order of } K_X = 2 \text{ in } \text{Pic}(X) \quad b_1(X) = 0, \quad a(X) = 2$$

$$c_1^2 = 0, \quad c_2 = 12$$

(5) hyperelliptic surfaces

i.e. $g(X) = 1$ ($b_1(X) = 2$), admitting a holomorphic, locally trivial fibration over an elliptic curve as a typical fibre.

$$a(X) = 2, \quad c_1^2 = 0, \quad c_2 = 0$$

(6) Kodaira surfaces

(a) primary : $b_1(X) = 3$, admitting a holomorphic, locally trivial fibration over an elliptic curve, with an elliptic curve as a typical fiber.

secondary Not a primary Kodaira surface, but have an unramified covering which is a primary Kodaira surface. They have $b_1(X) = 1$, and is an elliptic fibration over a rational curve.

Kodaira surfaces:

(a) primary :

$$b_1(X) = 3, \quad K_X = \text{trivial}, \quad a(X) = 1, \quad c_1^2 = 0, \quad c_2 = 0$$

(b) secondary

$$b_1(X) = 1, \quad \text{order of } K_X \text{ in } \text{Pic}(X) = 2, 3, 4, \text{ or } 6, \quad a(X) = 1, \quad c_1^2 = 0, \quad c_2 = 0$$

(7) K3 surfaces

$$K_X = \text{trivial}, \quad b_1(X) = 0, \quad a(X) = 0, 1, \text{ or } 2, \quad c_1^2 = 0, \quad c_2 = 24$$

(8) tori

$$K_X = \text{trivial}, \quad b_1(X) = 4, \quad a(X) = 0, 1, \text{ or } 2, \quad c_1^2 = 0, \quad c_2 = 0$$

Kodaira dimension = 1

(9) minimal elliptic surfaces with $\text{Kod}(X) = 1$

$$a(X) = 1 \text{ or } 2, \quad c_1^2 = 0, \quad c_2 \geq 0$$

Kodaira dimension = 2

(10) minimal surfaces of general type

$$b_1(X) \equiv 0 \pmod{2}, \quad a(X) = 2, \quad c_1^2 > 0, \quad c_2 > 0$$

Facts about embedded curves

X : a smooth, compact surface

Def: $C \subseteq X$, a curve on X
locally, defined by one equation

$$0 \rightarrow \mathcal{I}_C \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_C \rightarrow 0$$
$$\mathcal{I}_C = \mathcal{O}_X(-C)$$

$\omega_C = K_X \otimes_{\mathcal{O}_X} \mathcal{O}_C(C) =$ the dualizing sheaf for C

$$\leadsto 0 \rightarrow K_X \rightarrow K_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(C) \xrightarrow{r} \omega_C \rightarrow 0$$

— explicit description of r , when C is reduced
 (u, v) : local coordinates on X

if $\varphi = h \frac{du \wedge dv}{f}$

$\nu: \tilde{C} \rightarrow C$
normalization map

then let $r': \varphi \mapsto \left[h \frac{du}{f_v} \right] = \left[-h \frac{dv}{f_u} \right]$

Thus, we can identify ω_C with a subsheaf of $\nu_*(K_{\tilde{C}}(\infty D))$,
or ν_* (meromorphic differentials) on $\tilde{C}$, namely $\text{Im}(r')$.

Riemann-Roch for embedded curves

$\mathcal{F}$: locally free $\mathcal{O}_C$ -module

$$\leadsto \chi(\mathcal{F}) = \chi(\mathcal{O}_C) \cdot \text{rk}(\mathcal{F}) + \deg(\mathcal{F})$$

where $\deg(\mathcal{F}) = \deg(\wedge^{\text{top}} \mathcal{F})$, and degree of line bundles are defined by:

$$H^2(C) \cong \bigoplus_i H^2(\tilde{C}_i)$$
$$\deg(\mathcal{L}) = \sum_{i=1}^r n_i \deg(\mathcal{L}|_{\tilde{C}_i})$$

$C = n_1 C_1 + \dots + n_r C_r$ C_i : reduced

Trace map for embedded curves

$$H_c^1(C, \omega_C) \xrightarrow[\sim]{\text{tr}_D} \mathbb{C}$$

(from $0 \rightarrow K_X \rightarrow K_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(C) \rightarrow \omega_C \rightarrow 0$, get

$$H_c^1(C, \omega_C) \rightarrow H_c^2(X, K_X) \xrightarrow{\text{tr}_X} \mathbb{C};$$

this is compatible with the trace map when C is smooth.)

Serre duality for embedded curves

$\mathcal{F}$: locally free $\mathcal{O}_C$ -module.

$$H^1(\mathcal{F}) \otimes H^0(\mathcal{F}^\vee \otimes \omega_C) \rightarrow H^1(\omega_C) \xrightarrow{\text{tr}} \mathbb{C}$$

defines a perfect pairing.

Intersection theory

intersection multiplicity: $x \in C$, a (non-singular) point of X
 $x \in D$ s.t. x is an isolated point of $C \cap D$.

$f, g \in \mathcal{O}_{X,x}$: local equations for C and D respectively.

$$i_x(C, D) = \dim_{\mathbb{C}} \mathcal{O}_{X,x} / (f, g)$$

$\xrightarrow{\text{C reduced}} \sum_k \text{ord}_{x_k} (v^*(f))$, where $v: \tilde{C} \rightarrow C$ is the normalization map (if C is reduced)

$\uparrow$
summing over all points $x_k \in \tilde{C}$ lying over x .

Properties :

- If $Y \xrightarrow{f} X$ is a proper map between surfaces, $\mathcal{L}, \mathcal{M}$ are line bundles on X , then
$$(f^*\mathcal{L}, f^*\mathcal{M}) = \deg(\pi) \cdot (\mathcal{L}, \mathcal{M})$$
- $(C, \mathcal{M}) = \deg(\mathcal{M}|_C)$, $C \subseteq X$ an embedded curve
- C, D : without common components, then
$$(C \cdot D) = (D \cdot C) = \sum_{x \in C \cap D} i_x(C, D)$$

arithmetic genus

$$g(C) = 1 - \chi(\mathcal{O}_C) \stackrel{\substack{\uparrow \\ \text{Serre duality}}}{=} 1 + \chi(\omega_C)$$

- C reduced, $\nu: \tilde{C} \rightarrow C$ the normalization.

$$\Rightarrow \chi(\mathcal{O}_C) = \chi(\nu_* \mathcal{O}_{\tilde{C}}) - h^0(\nu_* \mathcal{O}_{\tilde{C}} / \mathcal{O}_C)$$

$$\text{So: } g(C) = g(\tilde{C}) + \sum_{x \in C} \dim_{\mathbb{C}} (\nu_* \mathcal{O}_{\tilde{C}} / \mathcal{O}_C|_x)$$

- adjunction formula (genus formula)

$$g(C) = 1 + \frac{1}{2} \deg(K_X \otimes \mathcal{O}(C)|_C)$$

$$\text{pf: } \deg(\omega_C) \stackrel{\substack{\uparrow \\ R \cdot R}}{=} -\chi(\mathcal{O}_C) + \chi(\omega_C) \stackrel{\substack{\uparrow \\ \text{duality}}}{=} 2\chi(\omega_C)$$

- If C is reduced and connected, then $g(C) \geq 0$
If moreover $g(C) = 0$, then C is a tree of nonsingular rational curves.

proof: $g(C) = 1 - h^0(\mathcal{O}_C) + h^1(\mathcal{O}_C) = h^1(\mathcal{O}_C) \geq 0$

since $h^0(\mathcal{O}_C) = 0$

If $g(C) = 0$, use induction: if $C = A+B$, then $(A \cdot B) \geq 1$ ($\because C$ is connected)
so $g(C) \geq g(A) + g(B)$, and $g(C) = 0$ iff $g(A) = g(B) = 0$ and $(A \cdot B) = 1$.

This reduces to the case C is reduced, then use $\nu: \hat{C} \rightarrow C$. $\square$

- If $C = A+B$, then $g(C) = g(A+B) = g(A) + g(B) + (A \cdot B) - 1$
(Immediate from: $g(C) = 1 + \frac{1}{2} \deg(K_X \otimes \mathcal{O}(C)|_C)$)

1-connected curves

Lemma 1 $\mathcal{L}$: a line bundle on C , s.t. $\deg(\mathcal{L}|_{D_i}) = 0$ for each
 $s \in H^0(\mathcal{L})$. irred. component D_i of C

Write $C = C_1 + C_2$

with $C_1 =$ the maximal divisor, $\leq C$, such that
 $s|_{C_1}$ is identically 0.

(in other words, $\min(s, C) = C_1$)

Then $(C_1 \cdot C_2) \leq 0$

Pf: $\mathcal{O}_{C_2} \xrightarrow{h} \mathcal{O}_{C_2}(-C_1) \otimes \mathcal{L}$ is injective with finite
cokernel Q
 $0 \rightarrow \mathcal{O}_{C_2} \rightarrow \mathcal{O}_{C_2}(-C_1) \otimes \mathcal{L} \rightarrow Q \rightarrow 0$ (support of Q is finite)

$$-(C_1 \cdot C_2) = \deg(\mathcal{O}_{C_2}(-C_1) \otimes \mathcal{L}) \quad (\because \deg_{C_2}(\mathcal{L}) = 0)$$

$$\stackrel{\text{R.R.}}{=} \chi(\mathcal{O}_{C_2}(-C_1) \otimes \mathcal{L}) - \chi(\mathcal{O}_{C_2}) = h^0(Q) \geq 0 \quad \square$$

Def: C is 1-connected if $(C_1 \cdot C_2) \geq 1$ for each effective decomposition $C = C_1 + C_2$. Similarly, C is m -connected if $(C_1 \cdot C_2) \geq m \quad \forall$ effective decomposition $C = C_1 + C_2$

Lemma 2: Let C be a 1-connected curve, $\mathcal{L}$ a line bundle on C s.t. $\deg(\mathcal{L}|_{D_i}) = 0 \quad \forall$ irred. component D_i of C .

Then $h^0(\mathcal{L}) \leq 1$, and $h^0(\mathcal{L}) = 1$ iff $\mathcal{L} \cong \mathcal{O}_C$.

Especially, $h^0(\mathcal{O}_C) = 1$.

pf: Let $s \in H^0(\mathcal{L})$, define $C = C_1 + C_2$ as in Lemma 1.

If $s \neq 0$, then $C_2 \neq 0$, and $(C_1 \cdot C_2) \leq 0$ by Lemma 1. C is 1-connected, so $C_1 = 0$, and the exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_C \otimes \mathcal{L} \rightarrow \mathcal{Q} \rightarrow 0$$

and R.-R. shows:

$$\begin{aligned} \chi(\mathcal{O}_C \otimes \mathcal{L}) &= \chi(\mathcal{O}_C) + h^0(\mathcal{Q}) \\ \parallel \text{R.-R.} \\ \chi(\mathcal{O}_C) &\quad \Rightarrow h^0(\mathcal{Q}) = 0 \end{aligned}$$

so $\mathcal{Q} = 0$
and $\mathcal{L} \cong \mathcal{O}_C$. $\square$

Lemma 3 Let C be a 1-connected curve on X .

$E \not\subset \text{supp}(C)$ be an irreducible curve on X with $(C \cdot E) = 1$.

Then either $h^0(\mathcal{O}_C(E)) = 1$

or the component R of C intersecting E is a smooth $\mathbb{P}^1$.

pf: Write $C = D + R$, then $E \cap D = \emptyset$, and we have

$$0 \rightarrow \mathcal{O}_D(E-R) \rightarrow \mathcal{O}_C(E) \rightarrow \mathcal{O}_R(E) \rightarrow 0$$

$$\parallel$$

$$0 \rightarrow \mathcal{O}_D(-R) \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_R \rightarrow 0$$

$$C \text{ is 1-connected} \Rightarrow h^0(\mathcal{O}_C) = 1 \Rightarrow H^0(\mathcal{O}_C) \cong H^0(\mathcal{O}_R) \\ \leadsto H^0(\mathcal{O}_D(-R)) = (0) = H^0(\mathcal{O}_D(E-R)).$$

If $h^0(\mathcal{O}_C(E)) \geq 2$, then $h^0(\mathcal{O}_R(E)) \geq 2$. But $\deg(\mathcal{O}_R(E)) = 1$

(Let $\nu: \tilde{R} \rightarrow R$ be the normalization map for the irred. curve R .
 $\deg(\nu^*(\mathcal{O}_R(E))) = 1$ on the smooth curve $\tilde{R}$, with
 at least 2 independent global sections. This forces
 $\tilde{R} \cong \mathbb{P}^1$. Moreover, $\nu: \tilde{R} \xrightarrow{\sim} R$ since $h^0(\mathcal{O}_R(E)) = 2$
 and $h^0(\nu^*(\mathcal{O}_R(E))) = 2$)

□

Note: $C = A + B$ curves

$$\leadsto \begin{array}{ccccccc} & & \mathcal{O} & & \mathcal{O} & & \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \mathcal{O}_X(-C) & \rightarrow & \mathcal{O}_X(-B) & \rightarrow & \mathcal{O}_A(-B) \rightarrow 0 \end{array}$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \mathcal{O}_X & = & \mathcal{O}_X \end{array}$$

$$\Rightarrow \mathcal{I}_B / \mathcal{I}_C \cong \mathcal{O}_A(-B).$$

$$\begin{array}{ccccccc} & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \mathcal{I}_B / \mathcal{I}_C & \rightarrow & \mathcal{O}_C & \rightarrow & \mathcal{O}_B \rightarrow 0 \end{array}$$

and we have

$$0 \rightarrow \mathcal{O}_A(-B) \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_B \rightarrow 0$$

General fact about smooth connected compact surfaces:

$$b_1 = h^{1,0} + h^{0,1}$$

$$b_1 \text{ even} \Rightarrow h^{1,0} = h^{0,1} \text{ and } b^+ = 2g + 1$$

$$b_1 \text{ odd} \Rightarrow h^{1,0} = h^{0,1} - 1 \text{ and } b^+ = 2g$$

$$b_2 = h^{2,0} + h^{0,2} + h^{1,1}$$

The signature of the cup product on $H^{1,1}(X, \mathbb{R})$ is

$$\begin{cases} -h^{1,1} & \text{if } b_1 \text{ is odd} \\ 2-h^{1,1} & \text{if } b_1 \text{ is even} \end{cases}$$

$$\{K \text{ numerically effective} + (K^2) > 0\}$$

$$\Rightarrow |2K| \neq \emptyset, \text{ and } |nK| \text{ has no base point for } n \gg 0. \\ (\text{or, for some } n > 0)$$

$$\text{pf: Suppose } |2K| = \emptyset. \text{ Then } p_g = 0$$

$$K \text{ nef} + (K^2) > 0 \Rightarrow H^0(\mathcal{O}_X(-K)) = (0)$$

$$\text{R.-R.: } (K^2) + \chi(\mathcal{O}_X) = \chi(\mathcal{O}(2K)) \leq 0.$$

$$\text{Noether's formula: } \Rightarrow 12(1-g) = (K^2) + 2 - 4g + b_2$$

$$\Rightarrow g \leq 1, \text{ hence } \chi(\mathcal{O}_X) \geq 0.$$

Contradiction

Thus we have proved: $|2K| \neq \emptyset.$

Mumford's argument for $|nK|$ to be base-point free when S is a surface of general type:

Assume algebraic
 S : minimal, and not birationally equivalent to a ruled surface

$$(K^2) > 0$$

$$(1) \text{ Riemann-Roch } \Rightarrow h^0(nK) \rightarrow \infty$$

$$(2) \text{ } D \text{ irreducible curve with } (D \cdot K) \leq 0$$

$$\xRightarrow{\text{Hodge index theorem}} (D^2) < 0 \quad (\text{since } (K^2) > 0)$$

$$-2 \leq 2P_a(D) - 2 = (D^2) + (D \cdot K)$$

$$\Rightarrow \overset{D \text{ smooth } \mathbb{P}^1}{(D^2) = -2}, (D \cdot K) = 0 \text{ since } D \text{ is assumed to be not exceptional of the 1st kind}$$

$$(3) \text{ Use an } m \text{ s.t. } \dim |mK| \geq 2 \leadsto \text{exist a pencil } \in |mK| \text{ s.t. gen. member is irred.}$$

$$D: \text{irreducible curve with } (D^2) = -2, (D \cdot K) = 0$$

$$\Rightarrow D \text{ is either a fixed component of } |mK|,$$

or D is contained in a reducible member of the pencil

$$\Rightarrow \text{There are only finitely many such } (-2)\text{-curves}$$

$$(4) \text{ Use Artin:}$$

$$\pi: S \longrightarrow \bar{S} \quad \text{blow down all such } (-2) \text{ curves}$$

$$\text{and: } K_S \text{ descends to a line bundle } K_{\bar{S}}^* \text{ on } \bar{S}$$

"the canonical bundle on $K_{\bar{S}}^*$ ".

$$\text{and: } \chi(\mathcal{O}_{\bar{S}}) = \chi(\mathcal{O}_S)$$

(5) For $n \gg 0$, $|nK_S^*|$ has no fixed components.

Must show: the only possible fixed components of

$|nK|$ are the (-2) -curves D s.t. $(D \cdot K) = 0$

From a theorem of Zariski: If L is numerically effective, then if E_i is a fixed ^{irred} component of $|nL|$ for infinitely many n 's, then $(L, E_i) = 0$

~~But, if $(K \cdot D) > 0$, D an irreducible curve, it is clear that for $n \gg 0$, an irreducible member of $|nK|$ intersects with D , so $H^0(\mathcal{O}(nK)) \rightarrow H^0(\mathcal{O}(nK) \otimes \mathcal{O}_D)$ is not the zero map. So this is clear.~~

(6) By construction, ^{for $n \gg 0$} $|nK_S^*|$ has only a finite number of base points (at most). Then a result of Zariski shows that in fact $|nK_S^*|$ has no base point for $n \gg 0$.

(7) Proof of Zariski's theorem

S^* : a normal ^{projective} surface

$\mathcal{L}$: an invertible sheaf such that $|\mathcal{O}(\mathcal{L})|$ has (at most) a finite number of base point (i.e. non-empty, and) $h^0(S^*, \mathcal{L}) \geq 2$.

Then for $m \gg 0$ $h^0(S^*, \mathcal{L}^{\otimes m})$ has no base point.

Proof: (Use the existence of a projective embedding)

Choose a hyperplane section $H \subseteq S^*$

Then, consider

$$\mathcal{L} \otimes \mathcal{O}_H$$

a line bundle on the 1-dim scheme H .

Then, $\mathcal{L}$ is ample on H , since $\deg_H \mathcal{L} > 0$.
 Replacing $\mathcal{L}$ by a power of $\mathcal{L}$, we may assume that $|H^0(S^*, \mathcal{O}(i) \otimes \mathcal{L}^{\otimes j})|$ has no base point.
 Let $M = \bigoplus_{i,j \geq 0} M_{i,j}$, $M_{i,j} = \text{Image} (H^0(S^*, \mathcal{O}(i) \otimes \mathcal{L}^{\otimes j}) \rightarrow H^0(H, \mathcal{O}(1) \otimes \mathcal{L}^{\otimes j}))$

$$R = \bigoplus_{i,j \geq 0} R_{i,j}, \quad R_{i,j} = \text{Image} (H^0(S^*, \mathcal{O}(i)) \otimes H^0(S^*, \mathcal{L}^{\otimes j}) \rightarrow H^0(H, \mathcal{O}(1) \otimes \mathcal{L}^{\otimes j}))$$

Since $\mathcal{O}(1)$ and $\mathcal{L}$ both have no base point, M is a finitely generated R -module. Hence:

$$(*) \quad \exists N \text{ s.t. } \forall i \geq 1, \forall j \geq N \\ R_{0,1} \otimes M_{i,j-1} \longrightarrow M_{i,j}$$

Now, fix a $j_0 \geq N$, and choose i s.t.

$$|H^0(S^*, \mathcal{L}^{\otimes j_0} \otimes \mathcal{O}(i))| \text{ has no base point.}$$

Consider the long exact sequence induced from

$$0 \rightarrow \mathcal{O}(i-1) \otimes \mathcal{L}^{\otimes j_0} \xrightarrow{s_H} \mathcal{O}(i) \otimes \mathcal{L}^{\otimes j_0} \rightarrow \mathcal{O}(i) \otimes \mathcal{L}^{\otimes j_0} \otimes \mathcal{O}_H \rightarrow 0$$

From (*), we see that

$$H^0(S^*, \mathcal{O}(i) \otimes \mathcal{L}^{\otimes j_0}) = \text{Image} (H^0(S^*, \mathcal{O}(i-1) \otimes \mathcal{L}^{\otimes j_0}) \xrightarrow{s_H} H^0(S^*, \mathcal{O}(i) \otimes \mathcal{L}^{\otimes j_0})) \\ + \text{Image} (H^0(S^*, \mathcal{O}(i) \otimes \mathcal{L}^{\otimes j_0-1}) \otimes H^0(S^*, \mathcal{L}) \rightarrow H^0(S^*, \mathcal{O}(i) \otimes \mathcal{L}^{\otimes j_0}))$$

Thus we deduce:

$$\{\text{base point of } H^0(\mathcal{L})\} \cap \{\text{base point of } H^0(S^*, \mathcal{O}(i-1) \otimes \mathcal{L}^{\otimes j_0})\} = \emptyset$$

Hence $|H^0(S^*, \mathcal{O}(i-1) \otimes \mathcal{L}^{\otimes j_0})|$ has no fixed base point.

Repeating the argument, we get:

$$|H^0(S^*, \mathcal{O}(i-2) \otimes \mathcal{L}^{\otimes j_0})|, \dots, |H^0(S^*, \mathcal{O}(1) \otimes \mathcal{L}^{\otimes j_0})|$$

and then $|H^0(S^*, \mathcal{L}^{\otimes j_0})|$ has no base point. QED

M. Artin $\begin{cases} \text{Amer. J. Math. 84 (1962), 485-496} \\ \text{Amer. J. Math. 88 (1966), 129-136} \end{cases}$

Numerical criteria for contractibility
+ rational singularities of surfaces

Proposition $V \xrightarrow{\pi} \bar{V}$, $\bar{V}$: normal surface, algebraic
(Mumford) $\begin{matrix} \downarrow \\ \pi^{-1}(P) \\ \downarrow \\ E_1 \cup \dots \cup E_n \end{matrix}$ $\begin{matrix} \downarrow \\ P \end{matrix}$ V : smooth surface, algebraic

The intersection matrix $S = ((E_i \cdot E_j))$ is negative definite

Proof (I) follows from the Hodge index theorem.

(Or, combine the Hirzebruch index theorem with Noether's theorem)
(assuming that $b_1 = 2g$, we get: $b_2^+ = \frac{1}{2}(b_2 + \tau) = p_g + 1$)

(II) ^{use} (Mumford)

lemma: (i) If $A = (a_{ij})$ is a symmetric ^{real} matrix such that

$$\begin{cases} a_{ij} \geq 0 & \forall i \neq j \\ \sum_i a_{ij} \leq 0 & \forall j \end{cases}$$

then A is negative semi-definite

(ii) If moreover, $\sum_i a_{ij} < 0$ for some j , and
we cannot split $\{1, \dots, n\}$ into two disjoint parts $I \cup I'$
such that $a_{ij} = 0 \quad \forall i \in I, j \in I'$.

then A is positive semi-definite.

(Apply lemma as follows:

$\left\{ \begin{array}{l} \text{choose hyperplane sections } H_1, H_2 \text{ on } \bar{V} \\ \Rightarrow \pi^* H_2 \sim \widetilde{H}_1 + \sum_i m_i E_i \end{array} \right. \begin{array}{l} H_1 \ni P \\ H_2 \not\ni P \\ m_i > 0 \forall i \end{array} \right\}$

Apply lemma to
 $A = (m_i E_i, m_j E_j)$

Proof of lemma: Use the identity.

$$\begin{aligned} \sum_{i,j} x_i x_j a_{ij} &= \sum_i x_i^2 a_{ii} + 2 \sum_{i < j} x_i x_j a_{ij} \\ (x_i \in \mathbb{R}) \quad &= \sum_j \left(\sum_i a_{ij} \right) x_j^2 - \sum_{i < j} a_{ij} (x_i - x_j)^2 \end{aligned}$$

□

Note: Without using the existence of ample line bundles, can argue as follows:

Pick a function $f \in \mathcal{O}_P$

Write $(f) = \sum_{i=1}^n m_i E_i + D_1 - D_2$

D_1, D_2 effective

May localize near $P \leadsto D_2 = \emptyset$

Again, let $a_{ij} = (m_i E_i \cdot m_j E_j)$

Then from $((f) \cdot m_j E_j) = 0$ follows $\sum_i a_{ij} \leq 0, \forall j$

Moreover, $D_1 > 0$, hence

$$\sum_i a_{ij} < 0 \text{ for some } i.$$

Artin's numerical criterion for contractibility.

1. Fundamental cycle:

$$\pi: V \longrightarrow \bar{V}$$

$\downarrow$
 $\mathbb{P}$

$$\pi^{-1}(\mathbb{P}) = E_1 \cup \dots \cup E_n$$

irreducible components
of the divisor $\pi^{-1}(\mathbb{P})_{\text{red}}$

$\bar{V}$: normal surface, algebraic

V : smooth algebraic surface

Consider: positive cycles $Z = \sum_{i=1}^n r_i E_i$ $r_i \geq 0 \quad \forall i$
some $r_i > 0$
s.t. $(Z \cdot E_i) \leq 0 \quad \forall i$

(This immediately implies that $r_i > 0 \quad \forall i$).

If Z_1, Z_2 both have this property, then so does

$$Z = \min(Z_1, Z_2),$$

Write $Z_1 = \sum r_i^1 E_i$ $Z_2 = \sum r_i^2 E_i$ $Z = \sum r_i E_i$ $r_i = \min\{r_i^1, r_i^2\}$

$$(Z \cdot X_j) = r_j^1 (X_j^2) + \sum_{i \neq j} r_i (X_i \cdot X_j) \leq r_j^1 (X_j^2) + \sum_{i \neq j} r_i^1 (X_i \cdot X_j) = (Z_1 \cdot X_j) < 0.$$

suppose: $r_j^1 \leq r_j^2$

Conclude: $\exists$ a minimal such Z , call the fundamental cycle, characterized by:

$$Z \text{ is a positive cycle, } Z = \sum r_i E_i > 0, \quad r_i \geq 0 \quad \forall i, \quad (Z \cdot X_j) \leq 0 \quad \forall j$$

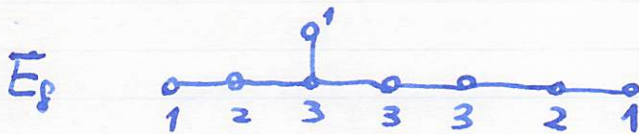
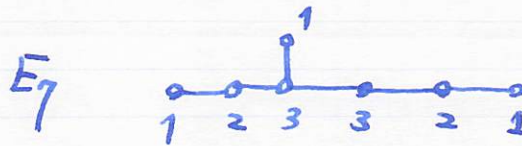
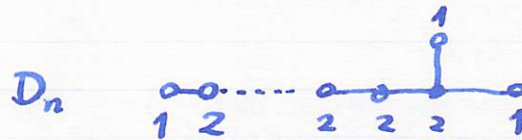
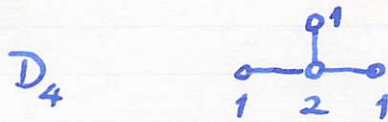
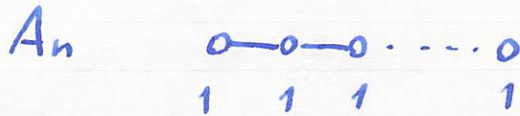
s.t. If Z' is another positive cycle s.t.

$$(Z' \cdot X_j) \leq 0 \quad \forall j, \text{ then } Z \leq Z'.$$

Examples of the fundamental cycles for rational double points:

all self-intersections = -2

(the numbers are the coefficients
 $Z = \sum r_i E_i$ of the fund. cycle)



$(Z \cdot Z) = -2$ (All the above are rational double points)

2. Prop: For a positive cycle $Z = \sum r_i E_i$, the following are equivalent

(a) $H^1(\mathcal{O}_Z) = 0$

(b) $H^1(\mathcal{O}_Z^*) \xrightarrow[\cong]{\deg} \mathbb{Z}^s \quad s = \# \{ i \mid r_i > 0 \}$

(c) $\forall$ positive cycle $Y \leq Z$, $p(Y) \leq 0$

Pf: (a) $\Leftrightarrow$ (b) easy

(a) $\Rightarrow$ (c): Follows from $H^1(\mathcal{O}_Z) \rightarrow H^1(\mathcal{O}_Y)$

$$(h^1(\mathcal{O}_Z) = p(Z) + h^0(\mathcal{O}_Z) - 1)$$

and $h^1(\mathcal{O}_Y) = p(Y) + h^0(\mathcal{O}_Y) - 1$

(c) $\Rightarrow$ (a) For any component $E_i \leq Z$ of Z

if $(Z \cdot E_i) \leq 1 + (E_i)^2$, then from

$$0 \rightarrow \mathcal{O}_{E_i} \otimes \mathcal{O}(-Z+E_i) \rightarrow \mathcal{O}_Z \rightarrow \mathcal{O}_{Z-E_i} \rightarrow 0$$

↑
degree ≥ -1

We deduce $H^1(\mathcal{O}_Z) = 0$.

On the other hand, if $(Z \cdot E_i) \geq 2 + (E_i)^2 \quad \forall i$ with $r_i > 0$,

We get (since $(K \cdot E_i) = -2 - (E_i)^2$ — i.e. $p(E_i) = 0$)

$$((Z+K) \cdot Z) \geq 0 \quad \text{i.e.} \quad p(Z) \geq 1, \text{ contradicting (c)}$$

Theorem: The fundamental cycle Z has the property that $p(Z) \geq 0$. More over, $p(Z) = 0$ iff

$p(Y) \leq 0$ for every cycle Y with support in $\bar{E} = E_1 \cup \dots \cup E_n$.

(In view of the proposition, this means that the singularity of $\bar{V}$ is rational.)

3. Lemma $\pi: V \rightarrow \bar{V}$ as before.

Then $\forall$ positive cycle Z with support $\subseteq E_1 \cup \dots \cup E_n$,

$$\chi(\bar{V}) - \chi(V) \geq p(Z).$$

proof: $\chi(\bar{V}) - \chi(V) = \dim R^1 \pi_* \mathcal{O}_V = \dim R^1 \pi_* (\mathcal{O}_V^{1/X})$
completion along $E_1 \cup \dots \cup E_n$
 $= \dim H^1(\mathcal{O}_{nZ}) \quad n \gg 0.$

(note $Z_1 \geq Z_2 \Rightarrow H^1(\mathcal{O}_{Z_1}) \twoheadrightarrow H^1(\mathcal{O}_{Z_2})$)

Theorem: $\bigcup_{E_1 \cup \dots \cup E_n} E \subseteq V \leftarrow \text{smooth}$

The following are equivalent

- (A) E is contractible, and $\chi(\bar{V}) = \chi(V)$
- (B) The intersection matrix $((E_i \cdot E_j))$ is negative definite,
 and $p(Y) \leq 0$ for every cycle Y with support in E

Remark: (i) We can replace the part $p(Y) \leq 0$ for every Y with support in E by (the apparently weaker, but) $p(Z) = 0$,
(actually equivalent condition)
 where $Z =$ the fundamental cycle

(ii) This is the case when $\bar{V}$ has rational singularities.

(iii) $\dim H^0(Z, \mathcal{O}_Z) = 1$, i.e. $H^0(Z, \mathcal{O}_Z) = k.$

Proof of theorem (B) $\Rightarrow$ (A) :

Let H be a hyperplane section on V . Replacing H by a multiple, we can find (a unique) divisor Z supported in E such that

$$(H \cdot E_i) + (Z \cdot E_i) = 0 \quad \forall i=1, \dots, n.$$

$$\Rightarrow \mathcal{O}(H+Z) \otimes \mathcal{O}_{E_i} \cong \mathcal{O}_{E_i} \quad \forall i=1, \dots, n$$

$$\Rightarrow |H^0(\mathcal{O}(H+Z))| \text{ defines map } V \xrightarrow{f} \mathbb{P}^n(H^0(\mathcal{O}(H+Z))).$$

Clearly : $f(E) = \text{a point}$,

$f|_{V \setminus E}$ is an embedding.

Taking normalization defines $\pi: V \rightarrow \bar{V}$.

4. multiplicity, etc

Theorem: $\pi: V \rightarrow \bar{V}$ as in (3), so that $\bar{V}$ is a rational singularity. Then

(i) $V \times_{\bar{V}} \text{Spec}(\mathcal{O}_{\bar{V}}/\pi_P^n) = nZ$, $Z = \text{the fundamental cycle}$
 esp. $\pi^{-1}(P) = Z$ as a scheme

(ii) $H^0(nZ, \mathcal{O}_{nZ}) = \mathcal{O}_V/\pi_P^n \quad \forall n \geq 1$

(iii) $\dim(\pi_P^n/\pi_P^{n+1}) = -n(Z^2) + 1$.

(so: $\dim(\mathcal{O}_{\bar{V}}/\pi_P^n) = -\frac{1}{2}(Z^2)(n^2 - n) + n$

and the multiplicity of P is equal to $-(Z^2)$.

(When $-(Z^2) = 1$, we recover Castelnuovo's criterion for exceptional curves of the first kind, generalized a little bit.)

Proof:

(i) The whole point is: ^{show} $\pi \cdot \mathcal{O}_V = \mathcal{I}_Z$.

• $\pi \cdot \mathcal{O}_V \subseteq \mathcal{I}_Z$:

for $f \in \pi$, write $(f) = D + Y$ ^{supported in E}
 where D is ^{support not involving} E_i 's $\forall i$

$\Rightarrow (Y \cdot E_i) = -(D \cdot E_i) \leq 0 \quad \forall i$

so $Y \geq Z$

and therefore $f \cdot \mathcal{O}_V \subseteq \mathcal{I}_Z \quad \forall f \in \pi$.

• $\pi \cdot \mathcal{O}_V = \mathcal{I}_Z$: may assume: $\bar{V} = \text{Spec}(A)$, A henselian

Given $x \in Z$. Choose a positive divisor D s.t.

D does not pass through x , and $(D \cdot E_i) = -(Z \cdot X_i) \quad \forall i$

Want: to find an $f \in \pi$ rational function s.t. $(f) = D + Z$

i.e. a nowhere zero section of $\mathcal{I}_{D+Z} = \mathcal{O}(-D-Z)$

Construct a formal function instead:

$$H^0(\hat{A}, \mathcal{O}(-D-Z)^\wedge) = \varprojlim_n H^0(nZ, \mathcal{O}(-D-Z) \otimes \mathcal{O}_{nZ})$$

$\parallel$
 $\mathcal{O}_{nZ}$

Suffices to show:

$$H^0(Z, \mathcal{O}_{(n+1)Z}) \longrightarrow H^0(Z, \mathcal{O}_{nZ})$$

In turn, it suffices to show that

$$H^1(\mathcal{I}_{nZ} \otimes_{\mathcal{O}_V} \mathcal{O}_Z) = (0)$$

$\uparrow$
 degree on E_i is $(-nZ \cdot E_i) \geq 0$, hence this $H^1 = (0)$

$$\left(\begin{array}{l} \text{Moreover,} \\ \dim H^0(\mathcal{I}_{nZ} \otimes \mathcal{O}_Z) = 1 + \sum_i r_i \cdot (-nZ \cdot E_i) \\ \qquad \qquad \qquad = 1 - n(Z \cdot Z) \end{array} \right) \quad Z = \sum r_i E_i$$

(ii) and (iii) Since $\pi_* (\mathcal{O}_V^\wedge) = \mathcal{O}_V^\wedge = A^\wedge$,

and $H^0(Z, \mathcal{O}_{(m+1)Z}) \longrightarrow H^0(Z, \mathcal{O}_{mZ}) \quad \forall m,$

$$A/\mathfrak{m}_P^m \longrightarrow H^0(mZ, \mathcal{O}_{mZ})$$

It suffices to prove that this map is injective.

Suppose: If $g \in A$ s.t. $(g) = mZ + D$ with $D \geq 0$,
we want to show that $g \in \mathfrak{m}_P^m$

Enriques classification of surfaces

(d'après D. Mumford, "Enriques classification of surfaces in char. $p = 1$ " Global Analysis, Papers in honor of Kunihiko Kodaira, p. 325-339, 1969)

Also: II, Bombieri and Mumford, "Complex Analysis and Algebraic Geometry", 1977, p. 23-42

III. Inv. math. 35 (1976) 197-232)

X : a smooth surface without (-1) -smooth rational curves.

Divide into 4 classes

- (a) $\exists$ a curve C on X with $(K_X \cdot C) < 0$ (K_X not nef)
- (b) $\forall$ curve C on X , $(K_X \cdot C) = 0$
(equivalently: $[K_X] = 0$ in $H_{\text{ét}}^2(X, \mathbb{Q}_\ell)$)
- (c) $(K_X \cdot C) \geq 0$ on all curves C ,
 $(K_X^2) = 0$ but $(K_X \cdot H) > 0 \forall$ ample H
- (d) $(K_X \cdot C) \geq 0 \forall$ curve C , and $(K_X^2) > 0$.

Note: Other cases are excluded by Hodge index theorem:

- $(K_X \cdot C) \geq 0 \forall$ curve $C \Rightarrow (K_X^2) \geq 0$
(e.g. $nK_X + H$ is ample $\forall n > 0$, $\forall$ ample H
by Nakai's criterion, so $n^2(K_X^2) + 2n(K_X \cdot H) + (H^2) \geq 0 \forall n \geq 0$)
- $(K_X \cdot H) = 0$ for some ample H and $(K_X^2) = 0$
 $\Rightarrow [K_X] = 0$ in $H_{\text{ét}}^2(X, \mathbb{Q}_\ell)$.

It will turn out that

$$(a) \Leftrightarrow X \text{ is ruled} \Rightarrow \text{Kod}(X) = -1$$

$$(b) \Leftrightarrow \begin{cases} (i) 2K_X \equiv 0, \text{ or} \\ (ii) \exists X \xrightarrow{\pi} D, D \text{ a curve, almost all fibers of } \pi \text{ are} \end{cases}$$

nonsingular elliptic, and for an integer $n \geq 1$ we have $nK_X = \pi^*(\sigma)$, σ a divisor of degree 0 on D

$$\Rightarrow (\text{either } |4K_X| \neq \emptyset \text{ or } |6K_X| \neq \emptyset) \text{ in fact} \\ \text{either } 4K_X \equiv 0 \text{ or } 6K_X \equiv 0$$

$$\Rightarrow \text{Kod}(X) = 0$$

$$(c) \Leftrightarrow \exists X \rightarrow D, \text{ almost all fibres are either nonsingular elliptic or rational with one cusp, hence}$$

$$\exists n \geq 1 \text{ st. } |nK_X| = \pi^*(\sigma), \sigma \text{ a divisor on } D \text{ with } \deg(\sigma) > 0$$

$$\Rightarrow \text{either } 4K_X \text{ or } 6K_X \text{ is represented by a positive divisor}$$

$$\Rightarrow \text{Kod}(X) = 1$$

$$(d) \Rightarrow \text{Kod}(X) = 2$$

(I) $(K \cdot C) < 0$ for some curve C | Will prove: X is ruled.
 ("adjunction terminates")

① $\exists$ ample H s.t. $(K \cdot H) \leq 0$ $(\Rightarrow |nK| = \emptyset \ \forall n \geq 1) \Rightarrow p_g = 0$

Proof $\neg (C^2) < 0 \Rightarrow (K \cdot C) + (C^2) = 2p(C) - 2 \geq -2$
 $\Rightarrow (K \cdot C) = (C^2) = -1 \Rightarrow C$ is an exceptional curve of the first kind, ruled out!

~~$\Rightarrow$~~ if $(C^2) \geq 0$.

$\Rightarrow \forall n \geq 0, nC + H$ is ample, and for $n \gg 0$
 $(K \cdot nC + H) < 0$.

② If $(K^2) > 0$, then X is rational

Proof ~~here~~ Here is a calculation (such calculation appears often in classification)

$g = \dim(\text{Pic}^0(X))$; in char. $p > 0$ g may be smaller than $H^1(X, \mathcal{O}_X)$.

$$12 \underset{11}{(1 - g + p_g)} = (K^2) + (2 - 4g + b_2)$$

$$12 - 12g \quad \text{i.e.} \quad 10 = 8g + (K^2) + b_2$$

$$\Rightarrow g = 0 \text{ or } 1$$

If $g = 1$, then $b_2 = 0, (K^2) = 2$

$b_2 = 0$ is impossible ($\exists$ ample line bundles)

$b_2 = 1$ is impossible - use Albanese

So: $g = 0$ Also: had $|2K| = \emptyset$

$\Rightarrow$
 Calabi-Yau X is rational

③ (Kodaira) If $(K^2) \leq 0$, then $\forall n \geq 0, \exists$ effective D s.t.

$$|K+D| = \emptyset \text{ and } \dim |D| \geq n$$

Proof H ample, $(K_X \cdot H) < 0$.

For any given $n \geq 0$,

$$(nH + mK \cdot H) < 0 \text{ for } m \gg 0.$$

$$\Rightarrow \exists m = m_n \geq 0 \text{ s.t.}$$

$$|nH + mK| \neq \emptyset, \text{ and } |nH + (m+1)K| = \emptyset.$$

$$\text{Let } D = D_n \in |nH + mK|$$

$$\begin{array}{ccc} & \begin{array}{c} \parallel \\ D' + D'' \end{array} & \\ \swarrow & & \searrow \\ \text{all components } C & & \text{all components } C \\ \text{satisfy } (C \cdot K) < 0 & & \text{satisfy } (C \cdot K) \geq 0. \end{array} \Rightarrow (D'' \cdot K) \geq 0$$

$$\forall \text{ all such } C's, (C \cdot K) < 0 \Rightarrow (C^2) \geq 0 \quad | \Rightarrow (D'^2) \geq 0$$

(otherwise C is a (-1) -curve)

$$\text{Also: } |K - D'| = \emptyset \xLeftrightarrow{\text{Serre duality}} H^2(X, \mathcal{O}(D')) = 0$$

(proof: Otherwise, K is effective)
Contradicting $(K \cdot H) < 0$.

$$\begin{aligned} \text{R.-R.: } \dim |D_n| &\geq \frac{1}{2} ((D' \cdot D') - (D' \cdot K_X)) + \chi(\mathcal{O}_X) - 1 \\ &\geq -\frac{1}{2} (D' \cdot K_X) + \chi(\mathcal{O}_X) - 1 \\ &\geq -\frac{1}{2} (D \cdot K_X) + \chi(\mathcal{O}_X) - 1 \end{aligned}$$

$$\stackrel{\text{R.-R.}}{=} -\frac{1}{2} n (H \cdot K_X) - \frac{1}{2} m \cdot (K_X^2) + \chi(\mathcal{O}_X) - 1$$

$$\geq -\frac{1}{2} n + \chi(\mathcal{O}_X) - 1$$

$$(\chi(\mathcal{O}_F) = 1 - g)$$

So, for $D = D_{2(n+1-\chi(\mathcal{O}_X))}$, D satisfies the two properties required.

④ D : effective divisor st. $|K+D| = \emptyset$.
(e.g. the effective divisor D found in ③)

then: $\text{Pic}_F^\circ \rightarrow \text{Pic}_D^\circ$ is surjective

(This is the new idea to take care of $\text{char}(k) = p > 0$)

$$0 \rightarrow \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0$$

$$\Rightarrow H^1(X, \mathcal{O}_X) \twoheadrightarrow H^1(D, \mathcal{O}_D)$$

(note: in char. p , Pic_X° is reduced since $P_g(X) = 0$)

⑤ Suppose that D : effective divisor st. $|K+D| = \emptyset$ as in ④,
Write: $D = \sum n_i E_i$ $n_i > 0$. then

(i) all E_i 's are non-singular

(ii) if $n_i \geq 2$, then

{ either (a) E_i is rational and $(E_i^2) = 0$ or 1 ,
or (b) E_i is elliptic, $(E_i^2) = 0$ with non-trivial normal bundle
or (c) $(E_i^2) < 0$

(iii) the E_i 's are connected and form a tree,
(i.e. without loops)

Proof From ④, Pic_D° is an abelian variety

$\Rightarrow$ all the E_i 's are smooth i.e. (i)

and the E_i 's are connected and without loops i.e. (iii)

For (ii): if $n_i \geq 2$, $\text{Pic}_D^\circ \rightarrow \text{Pic}_{2E_i}^\circ$ is surjective, hence

$\text{Pic}_{2E_i}^\circ$ is abelian, hence $\text{Pic}_{2E_i}^\circ = \text{Pic}_{E_i}^\circ$.

$$\sim H^1(\mathcal{O}_{2E_i}) \xrightarrow{\sim} H^1(\mathcal{O}_{E_i}).$$

$$0 \rightarrow \mathcal{O}_{E_i}(-E_i) \rightarrow \mathcal{O}_{2E_i} \rightarrow \mathcal{O}_{E_i} \rightarrow 0 \quad \text{gives: } H^1(\mathcal{O}_{E_i}(-E_i)) = 0.$$

From $H^1(\mathcal{O}_{E_i}(-E_i)) = (0)$, we see that

if $\deg_{E_i}(\mathcal{O}_{E_i}(-E_i)) = -(E_i^2) \leq 0$, then

either E_i is rational and $\deg_{E_i}(\mathcal{O}_{E_i}(-E_i)) = -1$ or 0
or E_i is elliptic, $(E_i^2) = 0$ and $\mathcal{O}_{E_i}(-E_i) \neq \mathcal{O}_{E_i}$.

□

⑥ There is a nonsingular rational curve on X passing through each given point of X .

Proof: Suppose not. Then: for each degree, there are only
(i) a finite number of ^{smooth} rational curves on X .

Moreover: There are only a finite number of rational
(ii) curves E of each given degree such that
 $(E^2) < 0$

(iii) Also: $\exists$ only a finite number of nonsingular elliptic curves

Curves of type (i), (ii), (iii) gives a countable set of divisors on X .

→ Let $P_1, \dots, P_g$ be distinct points of X not on any of these curves.

Let D be an effective divisor on X s.t.

$|D+K| = \emptyset$ and $\dim |D| \geq 3g$ (exists by ③)

Hence $\exists D' \in |D|$ with ^(at least) double points at $P_1, \dots, P_g$.

$\sum'' n_i E_i$

By ⑤ and our choice of D , ~~all~~ $n_i = 1$ if $n_i \neq 0$, and these E_i 's are nonsingular, forming a connected tree. So each of $P_1, \dots, P_g$ lies on exactly two E_i 's.

Therefore there are at least $g+1$ of the E_i 's in D' .

Also: none of these E_i 's can be rational, by the construction.
(that show up in D')
(and contain a P_j)

Hence: $g = \dim(\text{Pic}_X^0) \geq g+1$, a contradiction. $\square$

Clearly, $\textcircled{6} \Rightarrow X$ is ruled.

This finishes the case (a)

($\exists$ a curve C on X with $(K_X \cdot C) < 0$,)
i.e. K_X not nef.

Classification: continued

$$\text{II. } \left\{ \begin{array}{l} (K^2) = 0 \\ K \text{ numerically effective} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \text{either } 2K \equiv 0, \\ \text{or } \exists \text{ a pencil of curves on } X \\ \text{with } p_a = 1 \end{array} \right\}$$

Def: (i) A curve $D = \sum n_i E_i$ is of canonical type if $(K \cdot E_i) = (D \cdot E_i) = 0$

(ii) A curve $D = \sum n_i E_i$ of canonical type as above is indecomposable if D is connected, and $\text{g.c.d.}(n_i) = 1$.

① Either $2K \equiv 0$, or X contains at least one indecomposable curve of canonical type.

Proof: (a) For any $D \in |2K|$, D is either 0 or of canonical type.

$$\text{Write } D = \sum n_i E_i \quad n_i \geq 0$$

$$0 = (K \cdot D) = \sum n_i (K \cdot E_i) \Rightarrow (K \cdot E_i) = 0 \quad \forall i.$$

$$\Rightarrow (D \cdot E_i) = 2(K \cdot E_i) = 0$$

Conclude: If $|2K| \neq \emptyset$, $\exists$ a curve on X which is indecomposable of canonical type

(b) Now, we may assume that $|2K| = \emptyset$.

Noether's formula:

$$12(1 - g + \frac{p_g}{2}) = (K^2) + c_2 = 2 - 4g + b_2$$

0, since $|2K| = \emptyset$

$$10 = 8g + b_2$$

$\Rightarrow$ either $g = 0$, and hence X is rational by Castelnuovo's criterion, out

or $g = 1$

So: We may assume: $g = 1$, $b_2 = 2$
 $(K^2) = 0$, K numerically effective.

(c) Consider the Albanese morphism $\pi: X \rightarrow E$ $\uparrow$ elliptic curve.

$F = \pi^{-1}(P)$ an irreducible fibre (in char. 0, may take F to be nonsingular)

If $p_a(F) = 1$, then $0 = (K + F \cdot F)$ by adjunction,
 $= (K \cdot F)$

so F is of canonical type, and we are done.

If $p_a(F) > 1$, then $(K \cdot F) = 2p_a(F) - 2 \geq 2$

$\forall Q \in E, Q \neq P$, consider

$$0 \rightarrow \mathcal{O}_X(2K + \pi^{-1}(Q) - F) \rightarrow \mathcal{O}_X(2K + \pi^{-1}(Q)) \rightarrow \mathcal{O}_F \otimes \mathcal{O}_X(2K) \rightarrow 0$$

$$\text{Note: } H^2(\mathcal{O}_X(2K + \pi^{-1}(Q) - F)) \xleftrightarrow{\text{dual}} H^0(F - \pi^{-1}(Q) - K)$$

If $D \in |F - \pi^{-1}(Q) - K|$, then $(K \cdot F) = -(D \cdot F) \leq 0$,
 contradicting $(K \cdot F) \geq 2$. $\uparrow$ since $(F^2) = 0$

$$\text{So: } H^2(\mathcal{O}_X(2K + \pi^{-1}(Q) - F)) = 0$$

$$\chi(\mathcal{O}_X(2K + \pi^{-1}(Q) - F)) \stackrel{R-R}{=} \chi(\mathcal{O}_X) + \frac{1}{2} (K + \pi^{-1}(Q) - F \cdot 2K + \pi^{-1}(Q) - F) \\ = \chi(\mathcal{O}_X) = 0$$

From the long exact sequence, get:

$\forall Q \neq P$ on E ,

either $|2K + \pi^{-1}(Q) - F| \neq \emptyset$

or $H^i(\mathcal{O}_F(2K + \pi^{-1}(Q) - F)) = 0 \quad \forall i$, and hence

$$H^0(\mathcal{O}_F(2K + \pi^{-1}(Q))) \xrightarrow{\sim} H^0(\mathcal{O}(2K) \otimes \mathcal{O}_F)$$

There are two possibilities:

— scenario A —: $|2K + \pi^{-1}(Q) - F| = \emptyset \quad \forall Q \neq P$ on E
i.e. the second situation always hold $\forall Q \neq P$ on E

Since $\dim H^0(\mathcal{O}(2K) \otimes \mathcal{O}_F) = 2p_a(F) - 2 \geq 2$, $H^0(\mathcal{O}(2K) \otimes \mathcal{O}_F) \neq (0)$

choose and fix an $s \in H^0(\mathcal{O}(2K) \otimes \mathcal{O}_F)$

$\forall Q \neq P$ on E , s lifts to a unique $s'_Q \in H^0(\mathcal{O}_X(2K + \pi^*(Q)))$
 Let $D_Q = (s'_Q)$

The D_Q 's form a 1-dimensional algebraic family of divisors
 s.t. $D_Q \cdot F =$ the divisor A of (s) on F , $\forall Q \neq P$ on E .
 Moreover, the D_Q 's are not even linearly equivalent,
 so they are distinct.

As $Q \rightarrow P$, D_Q specialize to a divisor D_P .
 Since $X = \text{closure of } \bigcup_{Q \neq P} D_Q$,

D_P must contain F .

i.e. $D_P = F + D_P^*$, D_P^* effective.
 $D_P \in |2K + F|$

Therefore $D_P^* \in |2K|$ and $|2K| \neq \emptyset$, if scenario (A) holds

Going back: We have shown that
 $|2K + \pi^*(Q) - F| \neq \emptyset$ for some $Q \in E$
 (possibly $Q = P$)

Now: For a divisor $D \in |2K + \pi^*(Q) - F|$,
 $(D \cdot K) = 2(K^2) = 0$ Write $D = \sum_i n_i E_i$ $n_i \geq 0$
 and $(K \cdot E_i) = 0 \quad \forall_i$

So: D is of canonical type, and then we can find
 a divisor/curve on X which is indecomposable
 and of canonical type.

Lemma
Digression : Let $D = \sum n_i E_i$ be an indecomposable divisor of canonical type, and let $\mathcal{L}$ be an invertible sheaf on D .

If $\deg(\mathcal{L} \otimes \mathcal{O}_{E_i}) = 0 \quad \forall i$, then

$$H^0(D, \mathcal{L}) \neq (0) \iff \mathcal{L} \cong \mathcal{O}_D$$

The Hypergeometric function

$$F(a, b, c; z) = 1 + \frac{a \cdot b}{1 \cdot c} z + \frac{a(a+1) b(b+1)}{1 \cdot 2 \cdot c(c+1)} z^2 + \frac{a(a+1)(a+2) b(b+1)(b+2)}{1 \cdot 2 \cdot 3 \cdot c(c+1)(c+2)} z^3 + \dots$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n) \Gamma(b+n)}{n! \Gamma(c+n)} z^n$$

F satisfies the hypergeometric differential equation

$$\left(z(1-z) \frac{d^2}{dz^2} + [c - (a+b+1)z] \frac{d}{dz} - ab \right) F(a, b, c; z) = 0$$

This equation has regular singularity at $0, 1, \infty$; the exponents are

0 :	0, 1-c
1 :	0, c-a-b
∞ :	a, b

λ -family: $y^2 = x(x-1)(x-\lambda)$

$$\left(H(\lambda) = (-1)^{\frac{p-1}{2}} \sum_{j=0}^{\frac{p-1}{2}} \binom{-\frac{1}{2}}{j} \lambda^j \right) \text{ the Hasse invariant in char. } p$$

$$F\left(\frac{1}{2}, \frac{1}{2}, 1, \lambda\right) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n}^2 \lambda^n$$

$$\text{satisfies } \left(4\lambda(1-\lambda) \frac{d^2}{d\lambda^2} + 4(1-2\lambda) \frac{d}{d\lambda} - 1 \right) F\left(\frac{1}{2}, \frac{1}{2}, 1, \lambda\right) = 0$$

Picard-Fuchs equation

$$\left(4\lambda(1-\lambda) \frac{d^2}{d\lambda^2} + 4(1-2\lambda) \frac{d}{d\lambda} - 1 \right) \left(\frac{dx}{y} \right) = -d \left(\frac{y}{(x-\lambda)^2} \right)$$

Note: $\frac{x dx}{y}$ and $\frac{dx}{y}$ form a basis of $H_{DR}^1(E)$

Complex Algebraic Geometry

Home Work Set (I)

1. Suppose that X is a connected compact Kähler manifold of dimension 2 such that $K_X = \text{trivial}$ and $H^1(X, \mathcal{O}_X) = (0)$.

(a) Find the Hodge diamond of X

(b) Show that every connected finite étale cover $Y \rightarrow X$ is in fact an isomorphism. Deduce that $\pi_1(X)$ has no nontrivial finite quotients

2. Let E be a complex vector bundle and $\mathcal{L}$ a complex line bundle. Show that

$$c_t(E \otimes \mathcal{L}) = \sum_{i=0}^r t^i c_t(\mathcal{L})^{r-i} c_i(E), \quad r = \text{rk}(E)$$

Especially
$$c_r(E \otimes \mathcal{L}) = \sum_{i=0}^r c_1(\mathcal{L})^{r-i} c_i(E)$$

3. Let E be a complex vector bundle

(a) Show that $E \otimes_{\mathbb{R}} \mathbb{C} \cong E \oplus \bar{E}$ (as continuous complex vector bundles)

(b) Prove that

$$\begin{aligned} & 1 - p_1(E) + p_2(E) - p_3(E) + p_4(E) - \dots \\ &= (1 + c_1(E) + c_2(E) + c_3(E) + \dots) \cdot (1 - c_1(E) + c_2(E) - c_3(E) + \dots) \end{aligned}$$

4. Verify the initial terms of the L-class of a real oriented bundle E :

$$L(E) = 1 + \frac{1}{3} p_1(E) + \frac{1}{45} (7 p_2(E) - p_1^2(E)) + \dots$$

5. Verify the initial terms of the Todd class of a complex vector bundle E :

$$\begin{aligned} \text{Todd}(E) &= 1 + \frac{1}{2} c_1(E) + \frac{1}{12} (c_1^2(E) + c_2(E)) + \frac{1}{24} c_1(E) c_2(E) \\ &+ \dots \end{aligned}$$

6. Check the following cases of the Hirzebruch-Riemann-Roch formula

(1) When $\dim(X)=1$, $\chi(X) = g(X)$, where the genus $g(X)$ is defined by $c_1(X) = 2-2g = \chi_{\text{top}}(X)$

(2) When $\dim(X)=2$, get Noether's formula

$$\chi(X, \mathcal{O}_X) = \frac{1}{12} (c_1^2(X) + c_2(X))$$

(3) When $\dim(X)=1$ and $\text{rk}(\mathcal{L})=1$, have

$$\chi(X, \mathcal{L}) = c_1(\mathcal{L}) - g + 1$$

(4) When $\dim(X)=1$ and $\mathcal{E}$ is a holomorphic vector bundle on X , we have

$$\chi(X, \mathcal{E}) = c_1(\mathcal{E}) - \text{rk}(\mathcal{E})(g-1)$$

(5) Riemann-Roch for line bundles on surfaces:

$$\chi(X, \mathcal{L}) = \chi(X, \mathcal{O}) + \frac{1}{2} c_1^2(\mathcal{L}) + c_1(\mathcal{L}) c_1(X)$$

(6) Riemann-Roch for rank-two vector bundles on surfaces:

$$\chi(X, \mathcal{E}) = 2 \chi(X, \mathcal{O}) + \frac{1}{2} (c_1^2(\mathcal{E}) - 2 c_2(\mathcal{E})) + \frac{1}{2} c_1(\mathcal{E}) c_1(X).$$

7. Show that $\int_{w \in \mathbb{C}} d d^c \log(1 + w \bar{w}) = 1$

(Hence $\deg \mathcal{O}_{\mathbb{P}^1}(1) = c_1(\mathcal{O}_{\mathbb{P}^1}(1)) = 1$)

8. A connected compact complex surface is called a Hopf surface if its universal covering is analytically isomorphic to $\mathbb{C}^2 - \{0\}$. Given two numbers $\alpha_1, \alpha_2 \in \mathbb{C}^\times$ such

that $0 < |\alpha_1| \leq |\alpha_2| < 1$, define an action of the cyclic group $\mathbb{Z}$ on $\mathbb{C}^2 - \{0\}$ such that the generator $1 \in \mathbb{Z}$ acts on $\mathbb{C}^2 - \{0\}$ sending (z_1, z_2) to $(\alpha_1 z_1, \alpha_2 z_2)$. Let $H = H_{\alpha_1, \alpha_2} = (\mathbb{C}^2 - \{0\}) / \mathbb{Z}$.

(1) Show that H is a Hopf surface, and that H is homeomorphic to $S^3 \times S^1$. Moreover $\chi_{\text{top}}(H) = 0$

(2) Prove that H is not Kähler

(3) Use the Hirzebruch signature theorem to show that $c_1^2(H) = 0$

(4) Show that $H^0(H, \Omega_H^2) = (0) = H^2(H, \mathcal{O}_H)$.

(5) Show that $H^1(H, \mathcal{O}_H) = (0)$. (For instance, use Hirzebruch-Riemann-Roch)

(6) Prove that the natural map

$$H^0(H, \Omega_H^1) \oplus H^0(H, \Omega_H^1) \longrightarrow H^1(H, \mathbb{C})$$

is injective. Deduce that $H^0(H, \Omega_H^1) = (0)$

(7) Show that $\text{Pic}(H) \cong \mathbb{C}^\times$.

(8) Prove that $H^1(H, \Omega_H^1) = 0$.

$$(\text{Hint: } \chi_{\text{top}}(H) = \sum_{i,j \geq 0} (-1)^{i+j} \dim H^i(H, \Omega_H^j))$$

Complex Algebraic Geometry

Homework Set II

A. Weierstrass points

Let X be a compact Riemann surface of genus $g \geq 2$.

- (1) For any point $P \in X$, any natural number $n \geq 1$, show that
 $\nexists$ meromorphic function on X s.t. $(f)_\infty = nP$
 $\Leftrightarrow \exists$ global holomorphic 1-form ω s.t. $\nu_P(\omega) = n-1$,
 (where $\nu_P(\omega)$ = order of vanishing of ω at P).

Def: $G_P = \{n \in \mathbb{N} \mid \text{the equivalent conditions above are satisfied}\}$
 called the gap sequence at P .

- (2) Show that $\text{Card}(G_P) = g$
 (3) Define $w(P) = \left(\sum_{n \in G_P} n \right) - \frac{g(g+1)}{2}$

Prove that

$$w(P) > 0 \Leftrightarrow G_P \neq \{1, 2, \dots, g\}$$

$$\Leftrightarrow \dim H^0(\mathcal{O}(gP)) \geq 2 \Leftrightarrow H^0(K_X \otimes \mathcal{O}(-gP)) \neq (0)$$

Definition: $P \in X$ is a Weierstrass point if the above conditions are satisfied

- (4) Show that if C is hyperelliptic, i.e. there is a branched double cover $\pi: C \rightarrow \mathbb{P}^1$, then

- (a) $P \in C$ is a branch point iff $G_P = \{1, 3, \dots, 2g-1\}$
 (b) $P \in C$ is unramified iff $G_P = \{1, 2, \dots, g\}$.
 Especially, the branch points are exactly the Weierstrass points

- (5) Show that $\forall P \in X$, we have $w(P) \leq \frac{g(g-1)}{2}$.

Moreover, $w(P) = \frac{g(g-1)}{2}$ iff $2 \notin G_P$, i.e. X is a hyperelliptic curve

(2)

(6) Let $\omega_1, \dots, \omega_g$ be a basis of $H^0(X, K_X)$. For a local parameter z , write $\omega_i = f_i(z) dz$. Show that the Wronskian

$$W(\omega_1, \dots, \omega_g) = \det \begin{pmatrix} f_1(z) & \dots & f_g(z) \\ f_1'(z) & & f_g'(z) \\ \vdots & & \vdots \\ f_1^{(g-1)}(z) & & f_g^{(g-1)}(z) \end{pmatrix} (dz)^{\otimes \frac{g(g+1)}{2}}$$

gives a well-defined global section $W(\omega_1, \dots, \omega_g) \in H^0(X, K_X^{\otimes \frac{g(g+1)}{2}})$.

(7) Show that $\nu_P(W(\omega_1, \dots, \omega_g)) = w(P)$.

Conclude that $\sum_{P \in X} w(P) = (g-1)g(g+1)$

Especially, X has at least $2g+2$ Weierstrass points,
and X has exactly $2g+2$ Weierstrass points iff X is hyperelliptic.

B. Let X be the Riemann surface defined by the equation $y^2 = x^3 + 1$. Prove that X is an elliptic curve, and $X \cong \mathbb{C} / \mathbb{Z} + \mathbb{Z}\tau$, where $\tau = e^{\frac{2\pi i}{3}}$.

C. Let X be a Riemann surface of genus $g \geq 2$.

(1) Prove that for any $\phi \in \text{Aut}(X)$, the eigenvalues of $\phi^*: H^1(X, \mathbb{Z}) \rightarrow H^1(X, \mathbb{Z})$ are all roots of 1.

(2) Use the Lefschetz fixed point theorem to conclude that, if $\phi \in \text{Aut}(X)$ fixes all Weierstrass points of X , then $\phi = \text{id}_X$.

(3) Show that $\text{Aut}(X)$ is finite.

(3)

D. Show that all curves of genus 2 are hyperelliptic

E. Show that every nonhyperelliptic curve X of genus 3 is isomorphic to a smooth quartic in $\mathbb{P}^2$.

F. Let E be the elliptic curve : $y^2 = x^3 - 1$,

and let C be the hyperelliptic curve $y^2 = x^6 - 1$.
Show that $\text{Jac}(C)$ is isogenous to $E \times E$ (i.e. there is a surjective holomorphic homomorphism from one to the other.)

G. Let C be a smooth quartic curve in $\mathbb{P}^2$. Consider the dual mapping

$$D: C \longrightarrow \mathbb{P}^{2*} = (\text{the space of lines of } \mathbb{P}^2)$$

$$x \longmapsto \text{tangent line to } C \text{ at } x.$$

(1) Show that the partial derivatives of the defining quartic equation give the projective coordinates of D

(2) Show that there are 24 points of C where D is not étale (counting with multiplicities).

(3) Show that there are 28 bitangents to C (counting with multiplicities).

(4) Show that if L is a bitangent to C , with $(L \cdot C) = 2P_1 + 2P_2$, then $P_1 + P_2$ is an odd theta characteristic.

Conversely, every odd theta characteristic of C arises this way.

H. Suppose X is a hyperelliptic curve of genus 3. Show that every odd theta characteristic of X can be written uniquely as $\mathcal{O}(P_1 + P_2)$, where P_1 and P_2 are distinct branch points of the hyperelliptic double covering.

Complex Algebraic Geometry

Homework set III

This batch of exercises is about the famous Klein's quartic. Intrinsically, this curve C is best described as the modular curve $X(7)$. In other words,

$$C \cong \mathbb{P}^1 \cup \mathbb{Q} \cup \{\infty\} / \Gamma(7),$$

where $\Gamma(7)$ is the principal congruence subgroup $SL_2(\mathbb{Z})(7)$, consisting of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ s.t. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{7}$, and C is given the canonical structure of a smooth Riemann surface.

1. Show the genus $(C) = 3$. (This is quite standard; you can consult Shimura's book for instance. The only point is to use Hurwitz' formula, since $\mathbb{P}^1 \cup \mathbb{Q} \cup \{\infty\} / SL_2(\mathbb{Z})$ is known to be of genus 0, with function field generated by the j -function.)
2. The group $PSL_2(\mathbb{F}_7) = (SL_2(\mathbb{Z})/(\pm 1)) / \Gamma(7)$ operates on C faithfully. And $|PSL_2(\mathbb{F}_7)| = 168 = 84 \cdot (g(C) - 1)$, realizing the bound of the maximal number of automorphisms of a curve of genus 3. Check the following properties of $G = PSL_2(\mathbb{F}_7)$. (Of course, there are many more....)
 - (i) G has 21 elements of order 2; they are all conjugate
 - (ii) G has 56 elements of order 3, forming 2 conjugacy classes. $\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$ and $\begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}$ is a set of representatives of the two conjugacy classes.
 - (iii) G has 42 elements of order 4; the matrices $\begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$ form a set of representatives of the conjugacy classes.

(2)

- (iv) G has 48 elements of order 7, forming 6 conjugacy classes. $\left\{ \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} : i \in \mathbb{F}_7^* \right\}$ is a set of representatives
 - (v) The centralizer of an element of order 2 is isomorphic to D_8 , the dihedral group with 8 elements
 - (vi) The normalizer of a subgroup of order 3 is isomorphic to S_3 , the symmetric group in 3 letters
 - (vii) There are 14 subgroups isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$
3. We shall get a projective equation for C using the geometric properties of C (and its symmetries).
To begin, prove that C is not hyperelliptic.
Conclude that the canonical system embeds C into $\mathbb{P}^2$ as a smooth quartic.
4. The group G does not operate freely on C . Show that the points of C with nontrivial stabilizer subgroups are of the following types
- (i) There are 24 points $\{a_i : 1 \leq i \leq 24\}$ whose stabilizer subgroups are isomorphic to $\mathbb{Z}/7\mathbb{Z}$. These points form one orbit under G .
 - (ii) There are 56 points $\{b_j : 1 \leq j \leq 56\}$ whose stabilizer subgroups are isomorphic to $\mathbb{Z}/3\mathbb{Z}$. The group G operates transitively on $\{b_j\}$.
 - (iii) There are 84 points $\{c_k : 1 \leq k \leq 84\}$ whose stabilizer subgroups are isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The group G operates transitively on $\{c_k\}$.

(3)

5. The points $\{a_i\}$, $\{b_j\}$, $\{c_k\}$ of C can be identified with special points for smooth quartics in $\mathbb{P}^2$ via the canonical embedding of C , as follows.
- (i) The points $\{a_i \mid 1 \leq i \leq 24\}$ are the 24 inflection points of C .
 - (ii) The points $\{b_j \mid 1 \leq j \leq 56\}$ are the points of C given by the 28 bitangents of C .
 - (iii) The points $\{c_k \mid 1 \leq k \leq 84\}$ of C can be described as follows: There are 108 points where C has order of contact ≥ 6 with a conic. These 108 points include the 24 inflection points, where the conics are double lines. The points $\{c_k \mid 1 \leq k \leq 84\}$ are the remaining ones.
6. Starting with an inflection point a_{i_1} , the tangent line at a_{i_1} intersects C with a divisor $3a_{i_1} + b$. Show that b is again an inflection point a_{i_2} . Continue this process to get a_{i_3} and a_{i_4} . Show that $a_{i_4} = a_{i_1}$.
- Thus the 24 inflection points $\{a_i \mid 1 \leq i \leq 24\}$ is the disjoint union of 8 triples as above. For each triple $a_{i_1}, a_{i_2}, a_{i_3}$, the stabilizer subgroups are equal.
7. Use the geometric information obtained in 6. to get a simple projective equation for C as follows. Choose a triple a_1, a_2, a_3 , and then choose projective coordinates of $\mathbb{P}^2$ such that $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$ are a_1, a_2, a_3 respectively. Show that in such a coordinate system, the equation for C takes a particularly simple form:
- $$x_0^3 x_1 + x_1^3 x_2 + x_2^3 x_3 = 0$$

(4)

8. Each element of G induces a linear transformation on $H^0(K_C)$, hence we have a representation $G \longrightarrow GL(H^0(C, K_C)) \cong GL_3(\mathbb{C})$.

Furthermore, G operates on $C \hookrightarrow \mathbb{P}(H^0(C, K_C))$ via the composition $G \longrightarrow GL(H^0(C, K_C)) \twoheadrightarrow PGL(H^0(C, K_C))$. For each element $\sigma \in G$ of order 2, σ operates on $H^0(C, K_C)$ with eigenvalues $-1, -1, +1$. Especially, the eigenspaces give a line L_σ and a point P_σ in $\mathbb{P}(H^0(C, K_C))$. The line L_σ will be called the axis for σ , and the point P_σ will be called the center for σ . So there are 21 axes and 21 centers. The axis L_σ intersects C at 4 points, all fixed under σ . Conclude that the 84 points $\{C_k\}$ form 21 quadruples, each quadruple given by an axis L_σ .

9. Prove that each center P_σ lies on 4 axes, and each axis L_σ contains 4 centers.

10. Show that each double tangent contains 3 centers, and that each center lies on 4 double tangents.

Complex Algebraic Geometry, Homework Set IV

1. Let H_{α_1, α_2} ($0 < |\alpha_1| < |\alpha_2| < 1$) be the Hopf surface

$$H_{\alpha_1, \alpha_2} = \mathbb{C}^2 \setminus \{0\} / (z_1, z_2) \sim (\alpha_1 z_1, \alpha_2 z_2)$$

Prove that the Kodaira dimension of H_{α_1, α_2} is $-\infty$.

(Hint: a section of $K_{\alpha_1, \alpha_2}^{\otimes n}$ corresponds to an entire function on $\mathbb{C}^2$ satisfying certain functional equation.)

2. Let $M = (m_{ij}) \in SL_3(\mathbb{Z})$ such that M has 3 distinct eigenvalues $\alpha, \beta, \bar{\beta}$, with the real eigenvalue $\alpha > 1$.

Let $S_M = G_M \backslash \mathbb{P}^1 \times \mathbb{C}$ be the Inoue surface attached to M ,

where $G_M = \mathbb{Z}^3 \rtimes \mathbb{Z}$ has 4 generators g_1, g_2, g_3, g_0 ; and

G_M operates on $\mathbb{P}^1 \times \mathbb{C}$ by:

$$g_i : (w, z) \mapsto (w + a_i, z + b_i) \quad i = 1, 2, 3$$

$$g_0 : (w, z) \mapsto (\alpha w, \beta z)$$

where $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is a nonzero eigenvector for α , $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is a nonzero eigenvector for β .

(i) Prove that $c_1(K_{S_M}) = 0$. (Note: $H^2(S_M, \mathbb{Z}(1))$ is torsion).

(ii) Show that $[K_{S_M}] \in \text{Pic}(S_M)$ is not a torsion element. Conclude that the Kodaira dimension of S_M is $-\infty$.

(iii) Show that

$$\dim H^0(S_M, K_{S_M}^{\otimes n}) = \begin{cases} 0 & \text{if } n \neq 0 \\ 1 & \text{if } n = 0 \end{cases}$$

$$\dim H^1(S_M, K_{S_M}^{\otimes n}) = \begin{cases} 0 & \text{if } n \neq 0, 1 \\ 1 & \text{if } n = 0 \text{ or } 1 \end{cases}$$

$$\dim H^2(S_M, K_{S_M}^{\otimes n}) = \begin{cases} 0 & \text{if } n \neq 1 \\ 1 & \text{if } n = 1 \end{cases}$$

3. Compute the dimensions of $H^1(H_{\alpha_1, \alpha_2}, K_{H_{\alpha_1, \alpha_2}}^{\otimes n})$ and $H^2(H_{\alpha_1, \alpha_2}, K_{H_{\alpha_1, \alpha_2}}^{\otimes n})$.

Complex Algebraic Geometry

Homework Set 5

1. Prove the following statement on the self-intersection number of a divisor in a 2-dimensional torus embedding: Suppose that $\{\sigma_i\}$ is a 2-dimensional rational polyhedral cone decomposition such that each cone σ_i is generated by part of a $\mathbb{Z}$ -basis of N ($N = \text{Hom}(\mathbb{G}_m, T)$, $\dim(T) = 2$). Suppose that σ_{i_1} and σ_{i_2} are two neighboring cones, σ_{i_1} is generated by $u, v \in N$, σ_{i_2} is generated by $v, w \in N$, $N = \mathbb{Z}u + \mathbb{Z}v = \mathbb{Z}v + \mathbb{Z}w$. Show that there is a linear relation between u, v, w of the form $u + av + w = 0$, $a \in \mathbb{Z}$. Prove that the self-intersection number of the divisor corresponding to v is equal to a . Generalize to higher-dimensional torus embeddings.
 2. Resolve the surface singularity defined by the equation $x^2 + y^2 + z^{n+1} = 0$. Find the weighted diagram for the exceptional divisors. (The weights are the self-intersection numbers.)
 3. Prove that a 2-dimensional torus embedding corresponding to a finite cone decomposition $\{\sigma_i\}$ of a subset of $N \otimes_{\mathbb{Z}} \mathbb{R}$ is compact iff $\bigcup_i \bar{\sigma}_i = N \otimes_{\mathbb{Z}} \mathbb{R}$.
- * There will be an oral at the end of the semester on the homework sets.

Complex Algebraic Geometry

Homework Set 6

1. Let E be an elliptic curve, F be a nontrivial $\mathbb{G}_a$ -bundle over E . Let S be the completion of F over E . In other words, S is a $\mathbb{P}^1$ -bundle over E with a section C , $C \subseteq S$ so that $S - C = F$, and C is "the curve at infinity".
 $\begin{matrix} C \subseteq S \\ \searrow \swarrow \\ E \end{matrix}$ Show that $\mathcal{O}_S(C) \otimes \mathcal{O}_C$ is trivial, while for every integer $n \geq 1$, $\dim |nC| = 0$.

2. The following are the (Dynkin) diagram for the rational double points:

$$A_n: \circ - \dots - \circ \quad D_n: \circ - \dots - \circ \begin{matrix} \uparrow \\ \circ \end{matrix} \quad E_6: \circ - \dots - \circ \begin{matrix} \uparrow \\ \circ \end{matrix}$$

$$E_7: \circ - \dots - \circ \begin{matrix} \uparrow \\ \circ \end{matrix} \quad E_8: \circ - \dots - \circ \begin{matrix} \uparrow \\ \circ \end{matrix} \quad (o^2) = -2$$

- (a) Find the fundamental cycle in each of the above case.
 (b) Show that the above are the only configurations such that each of the exceptional curve E_i is a nonsingular rational curve with $(E_i^2) = -2$.
3. Show that the above configurations are the only possible ones for rational singularities with multiplicity 2, such that there are no exceptional curves of the first kind.
4. Find the multiplicities of the singularities with the following configuration of the exceptional curves:



$$(o^2) = -2, \quad (x^2) = -3$$

5. An embedded curve C is called 1-connected if for every decomposition $C = C_1 + C_2$ with $C_1, C_2 > 0$, we have $(C_1 \cdot C_2) \geq 1$. Prove that if C is 1-connected, and $\mathcal{L}$ is a line bundle on C such that $\deg(\mathcal{L}|_{E_i}) = 0 \quad \forall$ irreducible component E_i of C , then $h^0(\mathcal{L}) \leq 1$. Moreover, if $h^0(\mathcal{L}) = 1$, then $\mathcal{L} \cong \mathcal{O}_C$.

6. Give an example of an embedded curve C with $p_a(C) < 0$.

7. Def An embedded curve $C \subseteq X$ is called of canonical type if $(C \cdot E_i) = (K \cdot E_i) = 0 \quad \forall$ irred. component E_i of C .

Write $C = \sum r_i E_i \quad r_i > 0$. If C is of canonical type, we say it is irreducible of canonical type if C is connected, and $\text{g.c.d.}(r_i) = 1$. Prove the following statement.

Suppose C is irreducible of canonical type, $\mathcal{L}$ an invertible sheaf on C such that $\deg(\mathcal{L}|_{E_i}) = 0 \quad \forall i$.

Then: $H^0(C, \mathcal{L}) \neq (0) \iff \mathcal{L} \cong \mathcal{O}_D$
moreover, $h^0(\mathcal{O}_D) = 1$.

8. (i) Give an example of the diagram of an embedded curve which gives rise to a rational triple point.

(ii) Prove that in the dual graph for a rational triple point, at least one vertex will have self-intersection ≤ -3 .

Complex Algebraic Geometry Homework Set (7)

This batch of problems are about minimal surfaces with $\text{nod} = 0$. We assume that X is a smooth compact algebraic surface such that K_X is numerically effective and $\text{nod}(X) = 0$, and that X does not contain exceptional curves of the first kind.

1. Deduce from Noether's formula that $\chi(\mathcal{O}_X) \geq 0$, using $c_1^2(X) = 0$ and $p_g(X) \leq 1$. Conclude that $g \leq 2$.
2. Suppose that $p_g(X) = 1$ and $g(X) = 1$. Use the Riemann-Roch theorem to show that every line bundle $\mathcal{L}$ on X such that $\mathcal{L}^{\otimes n} \cong \mathcal{O}_X$ for some $n > 1$ must be trivial. Conclude that the case $p_g(X) = 1 = g(X)$ does not occur.
3. Suppose that $p_g(X) = 1$, $g(X) = 0$. Prove that K_X is trivial, therefore X is a K3 surface.
4. Suppose that $p_g(X) = 0$ and $g(X) = 0$. Prove that $K_X^{\otimes 2} \cong \mathcal{O}_X$ and $K_X \not\cong \mathcal{O}_X$, therefore X is an Enriques surface.
5. Suppose that $p_g(X) = 1$ and $g(X) = 2$. Let $f: X \rightarrow \text{Alb}(X)$ be an Albanese morphism for X .
 - (a) Assume that $\dim(f(X)) = 1$. Let X_n be the étale cover of X induced by $[n]: \text{Alb}(X) \rightarrow \text{Alb}(X)$. Show that X_n is connected, and compute $p_g(X_n)$, $g(X_n)$. Also, show that the pull-back C_n of $C = f(X)$ by $[n]_{\text{Alb}(X)}$ is connected. Derive a contradiction from the fact that $\text{genus}(C) = 2$.
 - (b) Assume that f is surjective. Let $D \subseteq X$ be the ramification locus of f . Show that each connected component of D is a ^{multiple of a} smooth elliptic curve, and $\mathcal{O}(D) \cong K_X$.
 - (c) Suppose the ramification divisor $D \neq \emptyset$ in (b). Let E be an irreducible component of D . Show that $f(E) \subseteq \text{Alb}(X)$ is an elliptic curve. Conclude that $\mathcal{O}(mf(E))$ and mK_X have lots of sections. Conclude that $D = \phi^{\text{Alb}(X)}$ and X is an abelian surface.

Complex Algebraic Geometry, Spring 1996

- Study complex algebraic varieties from the point of view of complex analysis + differential geometry + topology.
- Relation with the algebraic approach : GAGA theorems (Serre, Ann. Inst. Fourier, 6 (1956), 1-42)

Will state results needed when they are from
"background materials", i.e. complex analysis, topology
or differential geometry
But will NOT try to give proofs

For instance, this will be our approach to :

Chern classes, Hirzebruch - Riemann - Roch theorems,
(+ Bott Residue formula, Lefschetz formula, holomorphic Lefschetz)
Hodge decomposition (and the theorem on elliptic operators)

Emphasis is on geometry and examples.