Math 7210, Spring 2024

Siegel's Lemma, classical form

$A = (a_{ij}) \in M_{m\times n}(\mathbb{Z})$, $n > m$

$|a_{ij}| \le B \quad \forall i,j$

Then $\exists\ \vec{x} = (x_1, \dots, x_n)^t \in \mathbb{Z}^n$ with $|x_j| \le \lfloor (nB)^{\frac{m}{n-m}} \rfloor \quad \forall j$

$\ne 0$ s.t. $A \cdot \vec{x} = \vec{0}$

Proof. Let c be a parameter in $\mathbb{N}_{\ge 1}$. Consider $\vec{x} \in \mathbb{Z}^n \cap [0, c]^n$

$\Rightarrow$ Every coord of $\vec{x}$ lies in an interval $[c s_i^-, c s_i^+]$, where $s_i^- = \sum_{a_{ij}<0} a_{ij}$, $s_i^+ = \sum_{a_{ij}\ge 0} a_{ij}$

Under $A : \vec{x} \mapsto A\cdot\vec{x}$, these $(c+1)^n$ points maps to a subset with $\le (nBc+1)^m$ elements

$s_i^+ - s_i^- \le nB$

Suffices if $(nBc+1)^m < (c+1)^n$

$$(nBc+1)^m < (nB(c+1))^m \le (c+1)^{n-m} \cdot (c+1)^m$$

So if $c = \lfloor (nB)^{\frac{m}{n-m}} \rfloor$, then $nB^{\frac{m}{n-m}} < c+1 \Rightarrow nB^m < (c+1)^{n-m}$ QED

Minkowski's lemma (a special case of)

def^n: $V, \langle\,\rangle$: finite dim^l inner product space $/\mathbb{R}$ $\qquad b = \dim_{\mathbb{R}}(V)$

M: $\mathbb{Z}$-lattice in V

$$\mathrm{vol}(V, \langle\cdot\rangle, M) \overset{\text{def}}{=} \frac{\mu_V(B_V(1))}{\mu_V(V/M)}$$

unit ball in V

shorthand: $\mathrm{vol}(V, M)$

μ_V: any Haar measure on V

$$\lambda_1(V,M) \le \dots \le \lambda_b(V,M)$$

$$\lambda_i(V,M) = \min\left\{\lambda \in \mathbb{R}_{>0} \,\middle|\, \begin{array}{l}\exists\ v_1, \dots, v_i \in M \text{ s.t. } \|v_1\|, \dots, \|v_i\| \le \lambda \\ \text{and } v_1, \dots, v_i \text{ linearly indep}\end{array}\right\}$$

Minkowski: $$\frac{2^b}{b!} \le \left(\prod_{i=1}^{b} \lambda_i(V,M)\right) \cdot \mathrm{vol}(V,M) \le 2^b$$

Proposition

Faltings's version of Siegel's lemma:

$$\begin{array}{ccc} \alpha: V & \longrightarrow & W \\ \cup\!\!\text{I} & & \cup\!\!\text{I} \\ M & \longrightarrow & N \end{array}$$

$(V, \|\cdot\|_V)$, $(W, \|\cdot\|_W)$ finite dim$^{\ell}$ inner product spaces $/\mathbb{R}$

$\mathbb{Z}$-lattices in V, W $\quad b = \dim_{\mathbb{R}}(V)$ $\quad a = \dim_{\mathbb{R}}(\operatorname{Ker}\alpha)$

$U = \operatorname{Ker}(\alpha)$, with norm given by $\|\cdot\|_V$

Assume

$C \geq 2$, $\quad |\alpha| \leq C$,

M is generated by elements of norm $\leq C$

Every non-zero element of M or N has norm $\geq C^{-1}$

Then $\quad \lambda_{i+1}(U, \|\cdot\|_V|_U, M) \leq \left(C^{3b-a} \cdot b!\right)^{\frac{1}{a-i}}$ $\quad$ for $i = 0, 1, \cdots, a-1$

Lemma: Give $\alpha(V)$ the quotient norm of $V \overset{\alpha}{\twoheadrightarrow} \alpha(V)$.

<The factor 2^a is from Faltings's paper, but unnecessary>

Then $\quad vol(V, \|\ \|_V, M) \leq 2^a\, vol(U, \|\cdot\|_{sub}, M \cap U) \cdot vol(\alpha(V), \|\cdot\|_{quot}, \alpha(M))$

Pf: The assertion is equivalent to

$$\mu_{V, \text{Euclidean}}(B_V(1)) \leq 2^a \cdot \mu_{U, \text{Euclidean}}(B_U(1)) \cdot \mu_{\alpha(V), \text{Eucl.}}(B_{\alpha(V)}, 1)$$

Easy exercise using Fubini $\qquad$ volume of the unit ball defined by the quotient norm.

Note: $\quad \underbrace{\mu_{\mathbb{R}^n, \text{Euclidean}}(B_{\mathbb{R}^n}(1))}^{vol(B_{\mathbb{R}^n}(1))} = \dfrac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$

It seems very likely that

$$vol(B_{\mathbb{R}^{n+m}}(1)) \leq vol(B_{\mathbb{R}^n}(1)) \cdot vol(B_{\mathbb{R}^m}(1)).$$

i.e. The factor 2^a in the Lemma can be dropped.

This is indeed the case: follows from an easy application of Fubini's thm

Proof of Proposition:

(1) $\lambda_1(\alpha(V), \|\ \|_{quot}, \alpha(M)) \geq C^{-2}$

$\because$ $\forall w \in \alpha(M)$, $w \neq 0$, pick $m \in M$ s.t. $w = \alpha(m)$,

$$\|w\|_{quot} = \inf_{u \in U} \|m+u\| \geq \frac{\|\alpha(m)\|_W}{\|\alpha\|} \geq C^{-2}$$

$$\underset{\text{Minkowski}}{\Longrightarrow} \quad \lambda_1(\alpha(V), \alpha(M))^{b-a} \cdot vol(\alpha(V)) \leq 2^{b-a}$$

$$\rightsquigarrow \quad vol(\alpha(V)) \leq (2C^2)^{b-a}$$

(2) Minkowski applied to (V, M) + $\lambda_b(V, M) \leq C$ ← assumption

$$\Rightarrow \quad C^b \, vol(V, M) \geq \frac{2^b}{b!}$$

$$\rightsquigarrow \quad vol(V, M) \geq \frac{2^b \cdot C^{-b}}{b!}$$

(3) $\lambda_1(U, U \cap M) \geq C^{-1}$ by assumption

Minkowski for $(U, U \cap M)$

$\Rightarrow$ $\forall i = 0, \dots, a-1$, have

$$\lambda_1(U, U\cap M)^i \cdot \lambda_{i+1}(U, U\cap M)^{a-i} \cdot vol(U, U \cap M) \leq 2^a$$

$$\rightsquigarrow \quad \lambda_{i+1}(U, U\cap M) \leq \left(2^a \cdot vol(U, U\cap M)^{-1} \cdot C^i\right)^{\frac{1}{a-i}} \qquad \text{by (3)}$$

$$\leq \left(2^{2a} \cdot \frac{vol(\alpha(V), \alpha(M))}{vol(V, M)} \cdot C^i\right)^{\frac{1}{a-i}} \qquad \text{by Lemma}$$

$$\leq 2^{2a} \cdot \left((2C^2)^{b-a} \cdot C^b 2^{-b} \cdot b! \cdot C^i\right)^{\frac{1}{a-i}}$$

$$= \left(2^{\cancel{a}} \cdot C^{3b-2a+i} \cdot b!\right)^{\frac{1}{a-i}}$$

$$\leq \left(C^{3b-a} b!\right)^{\frac{1}{a-i}} \qquad \because C \geq 2,\ i \leq a-1$$

QED

Corollary of Classical Siegel's Lemma

K: number field. $d=[K:\mathbb{Q}]$. $\exists$ constants $C_1, C_2>0$ s.t. the following holds
$\forall\, A=(a_{ij}) \in M_{m\times n}(\mathcal{O}_K)$, $0<m<n$

$$B=\sup_{\sigma:K\hookrightarrow\mathbb{C}} |\sigma(a_{ij})|$$

$\exists\, \vec{x}\in \mathcal{O}_K^n \setminus \{\vec{0}\}$ s.t. $A\cdot\vec{x}=0$ and

$$H(\vec{x}) \le C_1\cdot (C_2 nB)^{\frac{m}{n-m}}$$

Bombieri-Vaaler's version

K: number field $\quad A\in M_{m\times n}(K) \quad rk(A)=m \quad d=[K:\mathbb{Q}]$

Then $\exists$ a K-basis $\vec{x}_1,\dots,\vec{x}_{n-m}$ of $Ker(A)\cap\mathcal{O}_K^n$

s.t. $$\prod_{l=1}^{n-m} H(\vec{x}_l) \le |disc_{K/\mathbb{Q}}|^{\frac{n-m}{2d}}\cdot H_{Ar}(A)$$

$H([\vec{x}_l])$

$\cap$

$\mathbb{P}^{n-1}(K)$

Def^n Arakelov height of a matrix $A\in M_{m\times n}(K)$, $rk(A)=m$

$\overset{def}{=}$ Arakelov height of the row span of A, an m-dim$^{\underline{l}}$ vector subspace of K^n_{row} $\quad K^m_{row}\cdot A$

$\overset{def}{=} H\left(\Lambda^m(K^m_{row}\cdot A)\right)$

regarded as a point of $\mathbb{P}^{\binom{n}{m}-1}(K)$

Adelic version of Minkowski's theorem

Thm K: number field $\quad d=[K:\mathbb{Q}]$
V: finite dim v.sp over K $\quad \Lambda \subseteq V \otimes_K \mathbb{A}_K$ lattice $\quad n=\dim_K(V)$

Assume. $\Lambda = \prod_v \Lambda_v$ s.t. $\Lambda_v \subseteq V_{K_v}$ fin. gen. O_v-submodul
Λ_v symmetric convex bounded open

$$\mathrm{vol}(\Lambda, V) = \frac{\mu(\Lambda)}{\mu(V_{\mathbb{A}_K}/V)} \quad \mu: \text{any Haar measure on } V_{\mathbb{A}_K}$$

$\lambda_1(\Lambda,V) \leq \cdots \leq \lambda_n(\Lambda,V)$ successive minimum

Then
$$\left(\prod_{i=1}^{n} \lambda_i(\Lambda,V)\right)^d \cdot \mathrm{vol}(\Lambda,V) \leq 2^{dn}$$

Moreover if $\forall$ complex v. Λ_v is $K_{v,1}^\times$-symmetric, then

$$\frac{2^{dn} \cdot \pi^{r_2 n}}{|\mathrm{disc}_{K/\mathbb{Q}}|^{n/2} (n!)^{r_1} \cdot ((2n)!)^{r_2}} \leq \left(\prod_{i=1}^{n} \lambda_i(\Lambda,V)\right)^d \cdot \mathrm{vol}(\Lambda,V)$$

$r_1 = \#$ real places of K
$r_2 = \#$ cpx places of K

ref. Bombieri – Vaaler, On Siegel's lemma, Inv Math 73 (1983) 11-32

relative version of Siegel's Lemma:

Thm K: number field $\quad F/K$ finite ext^n field $\quad r=[F:K]$, $\quad d=[K:\mathbb{Q}]$

$A \in M_{m\times n}(F)$, $\quad rm<n$

Then $\exists$ $\vec{x}_\ell \in O_K^N$, $\ell=1,\dots,n-rm$, K-linearly s.t. $A\cdot\vec{x}_\ell = 0$ $\forall \ell$
and
$$\prod_{\ell=1}^{n-rm} H(\vec{x}_\ell) \leq |\mathrm{disc}_{K/\mathbb{Q}}|^{\frac{n-rm}{2d}} \prod_{i=1}^{m} H_{Ar}(A_{i\text{-th row}})^r$$
$$\leq |\mathrm{disc}_{K/\mathbb{Q}}|^{\frac{n-rm}{2d}} \prod_{i=1}^{m} \left(\sqrt{n}\cdot H(A_{i\text{-th row}})\right)^r$$

<u>Roth's Lemma</u>

$0 \neq P(x_1,\dots,x_m) \in \overline{\mathbb{Q}}[x_1,\dots,x_m]$, $\quad \underset{\underline{\xi}}{(\xi_1,\dots,\xi_m)} \in \overline{\mathbb{Q}}^m$, $\quad 0<\sigma\leq \frac{1}{2}$

<u>Assume</u>

(1) $d_{j+1}/d_j \leq \sigma \quad \forall j=1,\dots,m-1 \qquad \underline{d}=(d_1,\dots,d_m)$

(2) $\underset{j}{\mathrm{Min}}\ d_j \cdot h(\xi_j) \geq \sigma^{-1}(h(P)+4md_1)$

Then $\mathrm{ind}(P;\underline{d};\underline{\xi}) \leq 2m\cdot\sigma^{\frac{1}{2^{m-1}}}$

<u>Roth's Theorem</u> K: number field $\quad S \subseteq M_K$ finite, $\quad \kappa>2$

F/K finite ext^n field, $\quad (\alpha_v)_{v\in S} \quad \alpha_v \in F \ \forall v$

Extend $|\cdot|_v$ to an absolute value $|\cdot|_{v,K}$ on F $\forall v \in S$

$$\#\{\beta \in K \mid \prod_{v\in S} \mathrm{Min}(1,|\beta-\alpha_v|_{v,K}) < H(\beta)^{-\kappa}\} < \infty$$

<u>Definition</u> (i) $\forall \beta\in K$, define $\Lambda(\beta) := \prod_{v\in S} \mathrm{Min}(1,|\beta-\alpha_v|_{v,K})$

and $a(\beta) := \left(\frac{1}{\log\Lambda(\beta)} \log \min(1,|\beta-\alpha_v|_{v,K})\right)_{v\in S} \in [0,1]^S$

For $N \in \mathbb{N}_{>0}$

(ii) Say $\beta, \beta' \in K$ are <u>in the same approximation class</u> of size $\frac{1}{N}$ if $\forall v\in S$, the v-coordinates $a(\beta)_v$ and $a(\beta')_v$ satisfy:

$$\lfloor N\,a(\beta)_v\rfloor = \lfloor a(\beta')_v \rfloor$$

(i.e. $a(\beta)$ and $a(\beta')_v$ lie in the same sub-hypercube of $[0,1]^S$ of size $\frac{1}{N}$)

Given $L, M \geq 2$

(iii) A sequence $\beta_1,\dots,\beta_m \in K$ is (L,M) independent if $h(\beta_1)\geq L$ and $h(\beta_{j+1}) \geq M\,h(\beta_j) \quad \forall j=1,\dots,m-1$

Notation: $r=[F:K]$, $\quad d=[K:\mathbb{Q}]$

nber fields F/K $r = [F:K]$

Lemma 1. Given $\underline{\alpha}_i = (\alpha_{i1}, \dots, \alpha_{im}) \in F^m$, $i = 1, \dots, n$

$\underline{t} = (t_1, \dots, t_n) \in \mathbb{R}^n_{\geq 0}$ $\quad V_m(\underline{t}) \overset{\text{def}}{=} \sum_{i=1}^n V_m(t_i)$

Assume. $r \cdot V_m(\underline{t}) < 1$ $\qquad \overset{\text{def}}{=} \text{volume}\left(\{ \underline{u} \in [0,1]^m \mid \sum_{i=1}^m u_i \leq 1 \}\right)$

Then for all $d_1, \dots, d_m \in \mathbb{N}$, all sufficiently large.

$\exists\, P(X_1, \dots, X_m) \in K[X_1, \dots, X_m]$ with $\deg_{X_i} P \leq d_i$ for $i = 1, \dots, m$

$\neq 0$ $\qquad (d_1, \dots, d_m)$

s.t. (a) $\text{ind}(P; \underline{d}; \underline{\alpha}_i) \geq t_i$ for $i = 1, \dots, n$

(b) $h(P) \leq \dfrac{r}{1 - r V_m(\underline{t})} \sum_{i=1}^n \sum_{j=1}^m V_m(t_i) \left[h(\alpha_{ij}) + \log 2 + o(1) \right] d_j$

as $d_j \to \infty$ for $j = 1, \dots, m$

pf: Use the relative version of Siegel's lemma. p.5

Lemma 2 For any $0 \leq \varepsilon \leq \frac{1}{2}$, $\quad V_m((\frac{1}{2} - \varepsilon)m) \leq e^{-6m\varepsilon^2}$

Pf of Lemma 2 $\quad$ Let $\chi(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$. Have $\chi(x) < e^{-\lambda x}$ $\forall \lambda > 0$

$$V_m((\tfrac{1}{2} - \varepsilon)m) \underset{x_i = u_i - \frac{1}{2}}{=} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} \chi(x_1 + \dots + x_m + m\varepsilon)\, dx_1 \cdots dx_m$$

$$\leq \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\lambda(m\varepsilon + x_1 + \dots + x_m)}\, dx_1 \cdots dx_m$$

$$= \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\lambda(\varepsilon + x)}\, dx \right)^m = \exp\left(-m\, U(\lambda)\right)$$

$$U(\lambda) \overset{\text{def}}{=} \varepsilon\lambda - \log\left(\frac{\sinh(\frac{\lambda}{2})}{(\frac{\lambda}{2})} \right)$$

$$\frac{\sinh(u)}{u} = 1 + \frac{u^2}{3!} + \frac{u^4}{5!} + \dots \underset{\frac{1}{(2k+1)!} \leq \frac{1}{6^k \cdot k!}}{\leq} 1 + \frac{u^2}{6} + \frac{(u^2/6)^2}{2!} + \dots = e^{u^2/6}$$

$$\rightsquigarrow \log \frac{\sinh(u)}{u} \leq \frac{u^2}{6}$$

Choose $\lambda = 12\varepsilon$ $\rightsquigarrow$ $U(12) = 12\varepsilon^2 - \log \dfrac{\sinh(6\varepsilon)}{6\varepsilon} \geq 6\varepsilon^2$

q.e.d.

Remark: Lemma 2 is a crude form of the central limit theorem.

Central limit theorem: $X_1, X_2, \dots$ a sequence of independent identically distributed random variables with mean 0 and standard deviation σ, then

the distribution of $\frac{1}{\sqrt{m}} \sum_{i=1}^{m} X_i \longrightarrow$ the normal distribution with density $\frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{x^2}{2\sigma^2}}$

Here the standard deviation is given by

$$\sigma^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 dx = \frac{1}{12}.$$

and $V_m((\frac{1}{2}-\varepsilon)m)$ corresponds to probability $\left(\frac{S_m}{\sqrt{m}} \geq \varepsilon\sqrt{m}\right)$

Thus $e^{-\frac{(\varepsilon\sqrt{m})^2}{2\cdot\sigma^2}} = e^{-6\varepsilon^2 m}$, exactly as in Lemma 2

Proof of Lemma 1   Use the relative version of Siegel's Lemma,

$E \overset{\text{def}}{=} \#$ of linear equations (for coeff. of P) with entries in F

$$V_m(\underline{t}) \leq E \leq V_m(\underline{t}) \cdot \left[1 + \operatorname{Max}(1, t_1^{-1}, \dots, t_n^{-1})\left(\frac{1}{d_1} + \dots + \frac{1}{d_m}\right)\right]^m \cdot \prod_{j=1}^{m} d_j$$

The matrix a of the linear equations is

$$a = \left(\binom{J}{I} \cdot (\underline{\alpha}_k)^{J-I}\right) \qquad \text{rows: indexed by } (J, k)$$

$$\underline{\alpha}_k = (\alpha_{k1}, \dots, \alpha_{km}) \in F^m \qquad J \in \mathbb{N}^m \cap \prod_{j=1}^{m} [0, d_j], \quad k = 1, \dots, n$$

$$\# \text{ of variables} = \prod_{j=1}^{m} (d_j + 1) =: B$$

Siegel's Lemma (relative version)

$\rightsquigarrow$ $\exists$ non-zero sol^n $\underline{P}(\underline{x})$ with

$$H(P) \leq \left|\operatorname{disc}_{K/\mathbb{Q}}\right|^{\frac{1}{2d}} \cdot \left[\left(\prod_{j=1}^{m}(d_j+1)\right)^{\frac{1}{2}} \cdot \prod_{(I,k)} H\left(a_{(I,k)\text{-th row}}\right)\right]^{\frac{r}{B-rE}}$$

$$H\left(a_{(I,k)\text{-th row}}\right) \leq \prod_{j=1}^{m} \left(2 H(\alpha_{kj})\right)^{d_j}$$

Constraints on the parameters and estimates

0) $h(P) \leq 2 C_1 D/L$ where $C_1 := |S| \cdot \left(\max_{v \in S} h(\alpha_v) + \log 2\right)$

$$\because\ h(P) \leq \sum_{v \in S}\left[\sum_{j=1}^{m} (h(\alpha_v) + \log 2)/h(\beta_j)\right] \cdot D + o(D)$$

used: $h(\beta_j) \geq L$

1) $m > \log(2r|S|)/(6\varepsilon^2) \rightsquigarrow r V_m(\underline{t}) \leq \frac{1}{2}$ $t_v = (\frac{1}{2} - \varepsilon, \dots, \frac{1}{2} - \varepsilon)\ \forall v \in S$

$$\Updownarrow\quad \frac{r V_m(\underline{t})}{1 - r V_m(\underline{t})} \leq 1$$

2) $L > (2C_1 + 5m)\, \varepsilon^{-2^{m-1}}$

$\Rightarrow d_j h(\beta_j) \geq \sigma^{-1}(h(P) + 4m d_1)$

$\left(\because \text{Need } D \geq \sigma^{-1}\left(\frac{2C_1 + 5m}{L}\right) D\right)$

3) $M \geq 2\varepsilon^{-2^{m-1}} \rightsquigarrow d_{j+1}/d_j \leq \sigma$ (The factor 2 is slightly wasteful.)

Overview of proof of Roth's theorem

parameters $N, L, M, \varepsilon, \sigma$ m, D $\sigma = \varepsilon^{2^{m-1}}$

$0 < \varepsilon < \frac{1}{2}$ $m > \log(2r|S|)/6\varepsilon^2$

$L \geq 2(C_1 + 5m)$, $M \geq 2\varepsilon^{-2^{m-1}}$

Assume $\{\beta \in K \mid \Lambda(\beta) < H(\beta)^{-\kappa}\}$ is infinite

$\rightsquigarrow$ For any (large) m, N, L, M

$\exists\ \beta_1, \dots, \beta_m \in K$, $\Lambda(\beta_j) < H(\beta)^{-\kappa}$ $\forall j$

(L,M)-independent, $\beta_1, \dots, \beta_m$ in the same approximation class of size $\frac{1}{N}$

Define $d_j = \lfloor D/h(\beta_j) \rfloor$ for $j = 1, \dots, m$

$\underline{t} = (t_v)_{v \in S} \in \mathbb{R}_{>0}^S$ $t_v := (\frac{1}{2} - \varepsilon) m$

$$m > \log(2r|S|)/6\varepsilon^2 \underset{\text{Lemma 2}}{\Longrightarrow} r V_m(\underline{t}) \leq \frac{1}{2}$$

Step 1

By Lemma 1, get $\exists\, P(x_1, \dots, x_m) \in K[x_1, \dots, x_m]$

s.t. $\deg_{x_j} P \leq d_j$ $\forall j = 1, \dots, m$

$\mathrm{ind}(P; \underline{d}; \underset{(\alpha_v, \dots, \alpha_v)}{\underline{\alpha_v}}) \geq (\frac{1}{2} - \varepsilon) m$ $\forall v \in S$

and

$$h(P) \leq \left[\sum_{v \in S} \sum_{j=1}^{m} (h(\alpha_v) + \log 2)/h(\beta_j)\right] \cdot D + o(D) \quad \text{as } D \to \infty$$

$$\leq 2 C_1 D / L$$

with $C_1 := |S| \cdot (\max_{v \in S} h(\alpha_v) + \log 2)$

Step 2 Assume $\beta_1, \dots, \beta_m$ are (L, M)-indep.

$m > \log(2r|S|)/6\varepsilon^2$,

$L \geq (2C_1 + 5m)\varepsilon^{-2^{m-1}}$, $M \geq 2\varepsilon^{-2^{m-1}}$

where $C_1 := |S| \left(\max_{v \in S} \{h(\alpha_v)\} + \log 2 \right)$

$D >> 0$, $d_j = \lfloor \frac{D}{h(\beta_j)} \rfloor$

Then $\exists_{\neq 0} Q \in K[x_1, \dots, x_m]$ s.t

(a) $\mathrm{ind}(Q; \underline{d}; \underset{(\alpha_v, \dots, \alpha_v) \in F^m}{\underline{\alpha_v}}) \geq (\frac{1}{2} - \varepsilon) m \quad \forall v \in S$

(b) $Q(\underline{\beta}) \neq 0 \qquad \underline{\beta} = (\beta_1, \dots, \beta_m) \in K^m$

(c) $h(Q) \leq 4C_1 D/L$

$Q = \partial_{\underline{\mu}} P$, $P(\underline{x})$ in Step 1, $\underline{\mu}$: given by Roth's Lemma

Apply Roth's Lemma with $\sigma = \varepsilon^{2^{m-1}}$

$\rightsquigarrow \exists \underline{\mu}$ with $\sum_{j=1}^m \frac{\mu_j}{d_j} \leq 2m\varepsilon$ and $\partial_{\underline{\mu}} P(\underline{\beta}) \neq 0$

Step 3 Upper bound of $|Q(\underline{\beta})|_v$, $v \in M_K$

- Use the assumption that $\beta_1, \dots, \beta_m$ are in the same approximation class of size $\frac{1}{N}$

- trivial estimate for $v \notin S$
- use $\Lambda(\beta_j) < H(\beta_j)^{-\kappa}$ $j = 1, \dots, m$ for $v \in S$

$\rightsquigarrow$ $$\boxed{\sum_{v \in M_K} \log |Q(\underline{\beta})|_v \leq -\kappa\left(1 - \frac{|S|}{N}\right)\left(\frac{1}{2} - 3\varepsilon\right) mD + \left(m + \frac{C_2}{L}\right) D + o(D)}$$ as $D \to \infty$

where $C_2 \overset{\text{def}}{=} 4C_1 + 2\log 4 + 2|S| \cdot \mathrm{Max}\{\log^+ |\alpha_v|_{v,K} : v \in S\}$

Step 4 $\sum_{v \in M_K} \log |Q(\underline{\beta})|_v = 0$ (lower bound of $\sum_{v \in M_K} \log |Q(\underline{\beta})|_v$)

Step 5 Compare the upper bound and the lower bound

$0 \leq -\kappa\left(1 - \frac{|S|}{N}\right)\left(\frac{1}{2} - 3\varepsilon\right) + \left(1 + \frac{C_2}{mL}\right)$ (let $D \to \infty$)

$\rightsquigarrow 1 - \frac{\kappa}{2} \geq 0$ (let $\varepsilon \to 0^+$, $N \to \infty$, $L \to \infty$)

contradicting the assumption that $\kappa > 2$!

Subspace Theorem (Schmidt, 1970, Simultaneous approximation to alg. nbers
For $\mathbb{Q}$ and $S=\{\infty\}\subseteq M_{\mathbb{Q}}$ 1971, 1972 Acta Math. 125 (1970) 189-201)
Norm form equations. Ann. of Math 96 (1972) 526–551
Linear Forms with algebraic coefficients
J. Number Thy 3 (1971) 253–277

$\underline{x}=(x_1,\dots,x_n)$

$L_1(\underline{x}),\dots,L_n(\underline{x})\in\bar{\mathbb{Q}}[x_1,\dots,x_m]$, $\bar{\mathbb{Q}}$-linearly indep. linear forms

Given $\delta>0$, $\exists$ finitely many proper $\mathbb{Q}$-linear subspaces of $\mathbb{Q}^n$ s.t. every integer point $\underline{x}\in\mathbb{Z}^n$, $\underline{x}\neq 0$ with

$$\left|\prod_{i=1}^n L_i(\underline{x})\right| < H(\underline{x})^{-\delta}$$

lies in one of these linear subspaces

Consequences of the subspace Thm

Thm Let $\alpha_1,\dots,\alpha_u$ be real algebraic numbers such that $1,\alpha_1,\dots,\alpha_u$ are $\mathbb{Q}$-linearly independent. Given $\delta>0$

1) Then $\#\left\{q\in\mathbb{N}_{>0}:\ q^{1+\delta}\cdot\prod_{i=1}^u \|\alpha_i q\|_{\mathbb{R}/\mathbb{Z}}<1\right\}<\infty$.

where $\|y\|_{\mathbb{R}/\mathbb{Z}} := \min_{n\in\mathbb{Z}} |y-n|$

2) $\#\left\{(q_1,\dots,q_n)\in(\mathbb{Z}\setminus(0))^n:\ \left(\prod_{i=1}^u |q_i|\right)^{1+\delta}\cdot\|q_1\alpha_1+\dots+q_u\alpha_u\|_{\mathbb{R}/\mathbb{Z}}<1\right\}<\infty$

Pf 1): Use $n=u+1$
$$\begin{cases} L_i(\underline{x}) = \alpha_n x_n - x_i & \text{for } i=1,\dots,u \\ L_n(\underline{x}) = x_n \end{cases} \qquad \underline{x}=(p_1,\dots,p_u,q)$$

2) Use: $n=u+1$
$$\begin{cases} L_i(\underline{x}) = x_i & i=1,\dots,u \\ L_n(\underline{x}) = \alpha_1 x_1+\dots+\alpha_u x_u - x_n \end{cases}$$

Strong subspace Theorem

Given $L_1, \dots, L_n \in (\overline{\mathbb{Q}} \cap \mathbb{R})[X_1, \dots, X_n]$ $\overline{\mathbb{Q}}$-linearly indep linear forms
↑ real alg. numbers

$c_1, \dots, c_n \in \mathbb{R}$ with $c_1 + \dots + c_n = 0$,

def^n $\forall Q > 0$, define $\Pi(Q) = \Pi(Q; L_1, \dots, L_n; c_1, \dots, c_n)$
$$\stackrel{def}{=} \{ \underline{x} \in \mathbb{R}^n : |L_j(\underline{x})| \leq Q^{c_j} \quad \forall j = 1, \dots, n \}$$
Let $\lambda_1 = \lambda_1(Q) = \lambda_1(Q; L_1, \dots, L_n, c_1, \dots, c_n) \leq \dots \leq \lambda_n = \lambda_n(Q)$ be the successive minima of $\Pi(Q)$

Suppose $\exists \delta > 0$, an integer d with $1 \leq d \leq n-1$ and an unbounded set $\mathcal{N}$ of positive real numbers such that
$$\lambda_d < \lambda_{d+1} Q^{-\delta} \qquad \forall Q \in \mathcal{N}$$

Then $\exists$ a $\mathbb{Q}$-vector subspace S^d with $d = \dim_{\mathbb{Q}} S^d$ and an unbounded subset $\mathcal{N}' \subseteq \mathcal{N}$ s.t. $\forall Q \in \mathcal{N}'$, the first d successive minima of $\Pi(Q)$ are assumed by elements $\underline{g}_1, \dots, \underline{g}_d \in S^d$ i.e. $\underline{g}_j \in \lambda_j(Q) \cdot \Pi(Q)$ for $j = 1, \dots, d$

$$\left(\Leftrightarrow \ |L_i(\underline{g}_j)| \leq \lambda_j(Q) \cdot Q^{c_i} \quad \text{for all } i = 1, \dots, n \text{ and } j = 1, \dots, d \right)$$

Exer. Strong subspace Thm
$\Downarrow$
Lemma: Let $L_1, \dots, L_n; c_1, \dots, c_n$ be as above (in strong subspace Thm)
$\exists$ $\mathbb{Q}$-vector subspaces $T_1, \dots, T_w \subseteq \mathbb{Q}^n$ s.t.
$$Z := \{ \underline{x} \in \mathbb{Z}^n : |L_i(\underline{x})| \leq H(\underline{x})^{c_i - \delta} \quad \forall i = 1, \dots, n \} \subseteq T_1 \cup \dots \cup T_w$$

Exer.
Lemma $\Longrightarrow$ subspace Theorem (Hint: a compactness argument)

Hint for Lemma: If Lemma is false, then $\exists (\underline{x}_j)_{j=1,2,\dots}$ infinite sequence of integer points in Z with $H(\underline{x}_j) \uparrow \infty$ s.t. every n elements in this sequence is $\mathbb{Q}$-linearly indep.

$$|L_i(\underline{x}_j)| \leq H(\underline{x}_j)^{c_i - \delta} \quad \forall i = 1, \dots, n \quad \forall j$$
$$\rightsquigarrow \lambda_1(H(\underline{x}_j)) \leq Q^{-\delta}$$
$$\stackrel{\forall j}{\rightsquigarrow} \exists\, d, \text{ with } 1 \leq d \leq n-1 \text{ s.t. } \lambda_d(H(\underline{x}_j)) < \lambda_{d+1}(H(\underline{x}_j)) \cdot Q^{-\delta/n}$$

Def $\underset{\mathbb{R}}{\overset{\|}{R_{n,m}}} = \mathbb{R}[x_{11},\dots,x_{1n};x_{21},\dots,x_{2n};\dots;x_{m1},\dots,x_{mn}]$

$\underline{r}=(r_1,\dots,r_m)$, $r_h\in\mathbb{N}_{\geq 1}$ $\forall h=1,\dots,m$ $L_1,\dots,L_m$ nonzero linear forms, $L_h\in\mathbb{R}[x_{h1},\dots,x_{hn}]$ $\forall h=1,\dots,m$

(i) $\forall c>0$, define $I(c)=I_{n,m,\underline{L},\underline{r}}(c)$

$$= \sum_{\substack{(i_1,\dots,i_m)\in\mathbb{N}^m \\ \sum_{1\leq h\leq m} i_h/r_h\geq c}} \left(\prod_{h=1}^m L_h^{i_h}\right)\cdot R_{n,m}$$

$\forall P\in R_{n,m}$, $P \neq 0$, define

$$\mathrm{Ind}(P;L_1,\dots,L_m;r_1,\dots,r_m) = \underset{c\geq 0}{\mathrm{Max}}\ P\in I_{n,m,\underline{L},\underline{r}}(c)$$

Equivalently: Suppose that $\forall h=1,\dots,m$, have

$$L_h = \alpha_{h1}x_{h1}+\dots+\alpha_{hn}x_{hn},\quad \alpha_{h1}\neq 0.$$

$\forall P \neq 0$, write P as $P=\sum_{\underline{J}} b(\underline{J}) \prod_{h=1}^m L_h^{j_h} x_{h2}^{a_{h2}}\cdots x_{hn}^{a_{hn}}$

$\underline{J}=(j_1,a_{12},\dots,a_{1n};j_2,a_{22},\dots,j_{2n};\dots;j_m,a_{m2},\dots,a_{mn}) \in\mathbb{Z}^{mn}$

Then

$$\mathrm{Ind}(P;L_1,\dots,L_m;r_1,\dots,r_m) = \underset{b(J)\neq 0}{\mathrm{Min}}\left(\frac{j_1}{r_1}+\dots+\frac{j_m}{r_m}\right)$$

(ii) For $\underline{I}=(i_{11},\dots,i_{1n};i_{21},\dots,i_{2n};\dots;i_{m1},\dots,i_{mn})\in\mathbb{N}^{mn}$

Let $\partial^{\underline{I}} \overset{\text{def}}{=} \prod_{\substack{1\leq h\leq m\\1\leq j\leq n}} \frac{1}{i_{hj}!}\left(\frac{\partial}{\partial x_{hj}}\right)^{i_{hj}}$, $(\underline{I}/\underline{r}) = \sum_{h=1}^m \frac{i_{h1}+i_{h2}+\dots+i_{hn}}{r_h}$

$$(\underline{I}/\underline{r}) = \sum_{h=1}^m \frac{i_{h1}+\dots+i_{hn}}{r_h}$$

The polynomial theorem / Construction of approximating polynomials

<u>Theorem:</u> Given $L^{(1)}(\underline{X}), \dots, L^{(n)}(\underline{X}) \in \overline{\mathbb{Q}}[x_1, \dots, x_n]$ n linear forms,
$L^{(i)}(x) = \alpha_{i1} x_1 + \dots + \alpha_{in} x_n$ linearly indep. over $\overline{\mathbb{Q}}$

Replicate them in the h-th block:

Let $L_h^{(i)} = \alpha_{i1} x_{h1} + \dots + \alpha_{in} x_{hn}$ for $h = 1, \dots, m$

Given $\varepsilon > 0$, and assume
$$m > 4\varepsilon^{-2} \log(2n\Delta) \qquad \Delta \overset{\text{def}}{=} \max_{1 \le i \le n} \left([\mathbb{Q}(\alpha_{i1}, \dots, \alpha_{in}) : \mathbb{Q}] \right)$$

Given $(r_1, \dots, r_m)$, $r_h \in \mathbb{N}_{>0}$ $\forall h = 1, \dots, m$
$\exists D > 0$ depending only on $L^{(1)}, \dots, L^{(n)}$

Then: $\exists_{\neq 0} P \in \mathbb{Z}[x_{hj} : 1 \le h \le m, 1 \le j \le n]$ with the following properties

(i) P is homogeneous in $(x_{h1}, \dots, x_{hn})$ of degree r_h, $\forall h = 1, \dots, m$

(ii) $H(P) \le D^{r_1 + \dots + r_m}$

$\forall \underline{I} \in \mathbb{N}^{m,n}$, write
$$\partial^{\underline{I}} P = \sum_{\substack{\underline{J} \in \mathbb{N}^{mn} \\ \underline{J} = (j_{11}, \dots, j_{1,n}; j_{21}, \dots, j_{2n}, \dots; j_{m1}, \dots, j_{mn})}} d^{\underline{I}}(\underline{J}) \cdot \prod_{h=1}^{m} \prod_{i=1}^{n} \left(L_h^{(i)}\right)^{j_{hi}}$$

(iii) If $(\underline{I}/\underline{r}) \le 2\varepsilon m$, then $d^{\underline{I}}(\underline{J}) = 0$ unless
$$\left| \left(\sum_{h=1}^{m} \frac{j_{hi}}{r_h} \right) - \frac{m}{n} \right| \le 3nm\varepsilon$$

In particular, $\forall i = 1, \dots, n$,
$$\operatorname{Ind}(P^{\underline{I}}; \underbrace{L^{(i)}, \dots, L^{(i)}}_{m}, r_1, \dots, r_m) \ge \frac{m}{n} - 3nm\varepsilon$$

<u>Cor:</u> $\exists E > 0$ depending only on $L^{(1)}, \dots, L^{(n)}$ s.t.
$$|d^{\underline{I}}(\underline{J})| \le E^{r_1 + \dots + r_m} \qquad \forall \underline{I}, \underline{J} \in \mathbb{N}^{m,n}$$

Davenport's Lemma (1937):

Let $L_1(\underline{x}), \dots, L_n(\underline{x})$ be n $\mathbb{R}$-linearly indep. linear forms on $\mathbb{R}^n$ with determinant 1 (i.e. ± 1). Let $\lambda_1, \dots, \lambda_n$ be the successive minima of the parallelepiped

$$\Pi = \Pi(L_1, \dots, L_n) = \{ \underline{x} \in \mathbb{R}^n : |L_i(\underline{x})| \le 1 \quad \forall i = 1, \dots, n \}$$

Let $\rho_1, \dots, \rho_n > 0$ be positive real numbers s.t

$$\rho_1 \ge \rho_2 \ge \cdots \ge \rho_n > 0 \quad , \prod_{i=1}^n \rho_i = 1$$

$$\rho_1 \lambda_1 \le \rho_2 \lambda_2 \le \cdots \le \rho_n \lambda_n$$

Then $\exists$ permutation $\pi \in S_n$ s.t.

$$\Pi' = \Pi'(\rho_i L_{\pi(i)}) = \{ \underline{y} \in \mathbb{R}^n : |\rho_i L_{\pi(i)}(\underline{y})| \le 1 \quad \forall i \}$$

have successive minima $\lambda_1', \dots, \lambda_n'$ with

$$2^{1-n} \lambda_i \rho_i \le \lambda_i' \le 2^{n-1} \cdot n! \cdot \lambda_i \rho_i \qquad \forall i = 1, \dots, n$$

Moreover suppose $\underline{g}_1, \dots, \underline{g}_n \in \mathbb{Z}^n$ are linearly indep. integer points with $\underline{g}_i \in \lambda_i \Pi$, i.e. $|L_j(\underline{g}_i)| \le \lambda_i \quad \forall j, i$. Then $\forall i = 1, \dots, n$

$\forall \underline{x} \in \mathbb{Z}^n$ s.t. $\underline{x} \notin \mathbb{Q}\underline{g}_1 + \cdots + \underline{g}_{i-1}$, we have

$$\mathrm{Max}\left(|\rho_1 L_1(\underline{x})|, \dots, |\rho_n L_n(\underline{x})| \right) \ge 2^{-n} \rho_i \lambda_n$$

Reciprocal parallelepipeds

(Mahler) $L_i(\underline{x}) = \underline{a}_i \cdot \underline{x} \quad i = 1, \dots, n$ linearly indep

$L_i^\vee(\underline{y}) = \underline{a}_i^\vee \cdot \underline{y} \qquad \underline{a}_i \cdot \underline{a}_j^\vee = \delta_{ij} \quad \forall i, j = 1, \dots, n$

$\Pi = \{ \underline{x} \in \mathbb{R}^n \mid |L_i(\underline{x})| \le 1 \} \qquad \lambda_1, \dots, \lambda_n$ successive minima of Π

$\Pi^\vee = \{ \underline{y} \in \mathbb{R}^n \mid |L_i^\vee(\underline{y})| \le 1 \} \qquad \lambda_1^\vee, \dots, \lambda_n^\vee$ successive minima of $\Pi^\vee$

Then $\lambda_i^\vee \ll \lambda_{n+1-i}^{-1} \ll \lambda_i^\vee$ implied constants depending only on n

Moreover if $\underline{g}_1, \dots, \underline{g}_n \in \mathbb{Z}^n$ are linearly indep with $\underline{g}_j \in \lambda_j \cdot \Pi$ (i.e. $|\underline{a}_i \cdot \underline{g}_j| \le \lambda_j \ \forall i$)

and $\underline{g}_1^\vee, \dots, \underline{g}_n^\vee \in \mathbb{R}^n$ satisfy $\underline{g}_i \cdot \underline{g}_j^\vee = \delta_{ij} \quad \forall i, j$.

then $|\underline{a}_i^\vee \cdot \underline{g}_j^\vee| \ll \lambda_j^{-1}$

Explicit constants: $|\underline{a}_i^\vee \cdot \underline{g}_j^\vee| \le n! \cdot (n-1)! \cdot \lambda_j^{-1}$

$$4^{-n} (n!)^{-n-1} ((n-1)!)^{-n+1} \le \lambda_i^\vee \cdot \lambda_{n+1-i} \le n! \cdot (n-1)!$$

Proof of Mahler's Lemma on reciprocal parallelepipeds.

$|\underline{a}_i \cdot \underline{g}_j| \leq \lambda_j \quad \forall i,j$ — (1)

Minkowski: $1 \ll |\det(\underline{g}_1, \dots, \underline{g}_n)| \ll 1$ — (2)

duality $\rightsquigarrow$ $\sum_{i=1}^{n} (\underline{a}_i \cdot \underline{x})(\underline{a}_i^\vee \cdot \underline{y}) = \underline{x} \cdot \underline{y} \quad \forall \underline{x}, \underline{y} \in \mathbb{R}^n$

$\rightsquigarrow \sum_{i=1}^{n} (\underline{a}_i \cdot \underline{g}_k) \cdot (\underline{a}_i^\vee \cdot \underline{g}_j^\vee) = \delta_{kj} \quad \forall k, j$

$$\Rightarrow \left(\underline{a}_i^\vee \cdot \underline{g}_j^\vee\right)_{1 \leq i,j \leq n} = \left(\underline{a}_k \cdot \underline{g}_i\right)^{-1}_{1 \leq k,i \leq n} \text{ in } M_{n \times n}(\mathbb{R})$$

Let $A = \det(\underline{a}_k \cdot \underline{g}_i)$, $D = \det(\underline{a}_1, \dots, \underline{a}_n)$, $E = \det(\underline{g}_1, \dots, \underline{g}_n)$ $\quad 1 \ll E \ll 1$

$= DE$

$\underline{g}_i = \sum (\underline{a}_k \cdot \underline{g}_i) \cdot \underline{a}_k^\vee$

$\rightsquigarrow E = A \cdot \det(\underline{a}_1^\vee, \dots, \underline{a}_n^\vee) = A \cdot D^{-1}$

Consider the matrix $(\underline{a}_k \cdot \underline{g}_i)_{1 \leq k,i \leq n}$

$$\underline{a}_i^\vee \cdot \underline{g}_j^\vee = A^{-1} \cdot \begin{pmatrix} \text{cofactor of } \underline{a}_i \cdot \underline{g}_j \\ \text{in the matrix } (\underline{a}_i \cdot \underline{g}_j)_{1 \leq i,j \leq n} \end{pmatrix}$$

$$\uparrow$$

$$\ll \lambda_1 \cdots \lambda_{j-1} \lambda_{j+1} \cdots \lambda_n \cdot D^{-1} = \lambda_1 \cdots \lambda_n \cdot D^{-1} \cdot \lambda_j^{-1} \underset{\text{Minkowski}}{\ll} \lambda_j^{-1}$$

Have shown: The last assertion

$\underline{g}_1^\vee, \dots, \underline{g}_k^\vee \in \frac{1}{E} \mathbb{Z}^n$, so

What we just showed $\rightarrow$ $|\underline{a}_i^\vee \cdot E \underline{g}_j^\vee| \ll \lambda_j^{-1} \implies$

$\lambda_1^\vee \ll \lambda_n^{-1}$

$\lambda_2^\vee \ll \lambda_{n-1}^{-1}$

$\lambda_i^\vee \ll \lambda_{n-i+1}^{-1} \quad \forall i = 1, \dots, n$

$\therefore \prod_i \lambda_i \cdot \prod_i \lambda_i^\vee \gg \mathrm{vol}(\Pi)\, \mathrm{vol}(\Pi^\vee) = 1$

$$\implies \lambda_i^\vee \ll \lambda_{n+1-i}^{-1} \ll \lambda_i^{-1} \quad \forall i$$

q.e.d.

Explicit constants:

$$|\underline{a}_i^\vee \cdot \underline{g}_j^\vee| \leq n!\,(n-1)!\,\lambda_j^{-1}$$

$$4^{-n} (n!)^{-n+1} ((n-1)!)^{-n+1} \leq \lambda_i^\vee \cdot \lambda_{n+1-i} \leq n!\,(n-1)!$$

Mahler's Theory of compound sets

$\Pi = \{ \underline{x} \in \mathbb{R}^n \mid |\underline{a}_i \cdot \underline{x}| \le 1 \ \ \forall i \} = \Pi(\underline{a}_1, \dots, \underline{a}_n)$ $\quad 0 < \lambda_1 \le \cdots \le \lambda_n$

successive minima of Π

$\underline{a}_1, \dots, \underline{a}_n$: $\mathbb{R}$-basis of $\mathbb{R}^n$

Define: $\Pi^{(p)} = \Pi^{(p)}(\underline{a}_1, \dots, \underline{a}_n)$ $\quad 1 \le p \le n$

$$= \{ \underline{y} \in \Lambda^p \mathbb{R}^n \mid |\underline{A}_\sigma \cdot \underline{y}| \le 1 \ \ \forall \sigma \in C(n,p) \}$$

$\underline{A}_\sigma = \bigwedge_{i \in \sigma} \underline{a}_i$ $\qquad C(n,p) = \{ (i_1, \dots, i_p) \mid 1 \le i_1 < \cdots < i_p \le n, \ i_j \in \mathbb{N} \ \forall j \}$

Let $\{\tau_1, \dots, \tau_\ell\} = C(n,p)$, $\ell = \binom{n}{p}$
enumerated so that

$0 < \lambda_{\tau_1} \le \cdots \le \lambda_{\tau_\ell}$, where $\lambda_\tau \overset{\text{def}}{=} \prod_{i \in \tau} \lambda_i$

Let $\nu_1 \le \cdots \le \nu_\ell$ be the successive minima for $\Pi^{(p)}$.

then (i) $\quad \lambda_{\tau_i} \ll \nu_i \le \lambda_{\tau_i}$

(ii) Let $\underline{g}_1, \dots, \underline{g}_n \in \mathbb{Z}^n$, linearly indep. s.t.

$\underline{g}_j \in \lambda_j \Pi \ \ \forall j$, i.e. $\ |\underline{a}_i \cdot \underline{g}_j| \le \lambda_j \ \ \forall j$

Then $|\underline{A}_\sigma \cdot \underline{G}_\tau| \le p! \, \lambda_\tau$, $\quad \forall \sigma, \tau \in C(n,p)$

where $\quad G_\tau = \bigwedge_i \underline{g}_i$

Lemma (a manifestation of the central limit theorem) Schmidt. p.122

$r_1, \dots, r_m \in \mathbb{N}_{\geq 1}$. $0 < \varepsilon < 1$ $n \in \mathbb{N}$, $n \geq 2$

Then
$$\#\left\{ I = (i_{hj})_{\substack{1 \le h \le m \\ 1 \le j \le n}} \in \mathbb{N}^{m,n} \;\middle|\; \sum_{j=1}^{n} i_{hj} = r_h \;\; \forall h = 1, \dots, m, \; \left| \sum_{h=1}^{m} \frac{i_{hj}}{r_h} - \frac{m}{n} \right| \geq \varepsilon m \;\; \forall j = 1, \dots, n \right\}$$
$$\leq 2 e^{-\varepsilon^2 m/4} \cdot \prod_{h=1}^{m} \binom{r_h + n - 1}{r_h}$$

The proof shows:

$$M_+ \stackrel{\text{def}}{=} \#\left\{ I = (i_{hj}) \in \mathbb{N}^{m,n} \;\middle|\; \sum_{j=1}^{n} i_{hj} = r_h \;\; \forall h, \; \left(\sum_h \frac{i_{hj}}{r_h} \right) - \frac{m}{n} \geq \varepsilon m \right\} \leq e^{-\frac{\varepsilon^2 m}{4}} \cdot \prod_{h=1}^{m} \binom{r_h + n - 1}{r_h}$$

$$M_- \stackrel{\text{def}}{=} \#\left\{ I = (i_{hj}) \in \mathbb{N}^{m,n} \;\middle|\; \sum_{j=1}^{n} i_{hj} = r_h \;\; \forall h, \; \left(\sum_h \frac{i_{hj}}{r_h} \right) - \frac{m}{n} \leq -\varepsilon m \right\} \leq e^{-\frac{\varepsilon^2 m}{4}} \prod_{h=1}^{m} \binom{r_h + n - 1}{r_h}$$

Used: $e^x \leq 1 + x + x^2 \;\; \forall\; x \in \mathbb{R}$ with $|x| \leq 1$

Remark on $\mathrm{Ind}(P; L_1,\dots,L_m; r_1,\dots,r_m)$

$n \geq 2$

$P \in \mathbb{R}[x_{hj} : 1\leq h\leq m, 1\leq j\leq n] = R_{m,n}$

homog. of deg r_h in $x_{h1},\dots,x_{hn}$ $\forall_{1\leq h\leq m}$

$T = \{ \underline{x} \in \mathbb{R}^{m,n} \mid \underset{=0}{L_h(x_{h1},\dots,x_{hn})} \ \forall_{h=1,\dots,m} \}$

an $\mathbb{R}$-vector subspace of $\dim(T) = m(n-1)$

$\mathrm{Ind}(P; L_1,\dots,L_m; r_1,\dots,r_m) = e$

$\Leftrightarrow$ (i) $\forall\, \underline{I} \in \mathbb{N}^{m,n}$, $\partial^{\underline{I}} P\big|_T = 0$

with $(\underline{I}/\underline{r}) = \sum_{h=1}^{m} \frac{i_{h1}+\dots+i_{hn}}{r_h} < e$

$\partial^I P\big|_T$ homogeneous of degree $r_h - \sum_{j=1}^{n} i_{hj}$ $\forall_{h=1,\dots,m}$ on for T_h

$T = T_1 \times\dots\times T_m$

$T_h = \{(x_{h1},\dots,x_{hn}) \in \mathbb{R}^n \mid L_h(x_{h1},\dots,x_{hn}) = 0\}$

i.e. $P \in \bigotimes_{h=1}^{m} \mathbb{R}[x_{h1},\dots,x_{hm}]_{(r_m)}$ ← homog. of degree r_m

$\partial^{\underline{I}} P\big|_T \in \bigotimes_{h=1}^{m} \mathbb{R}[T_h]_{(r_h - \sum_{j=1}^{n} i_{hj})}$

(ii) $\exists\, \underline{I}_0 \in \mathbb{N}^{m,n}$ with $(\underline{I}_0/\underline{r}) = e$ and

$\partial^{I_0} P\big|_T \neq 0$ on T

Grids: one way to show that a polynomial is identically 0

Def^n Let $w_1, \dots, w_{n-1}$ be $n-1$ linearly indep vectors in $\mathbb{R}^n$,

$V = \mathbb{R}w_1 + \dots + \mathbb{R}w_{n-1} \subseteq \mathbb{R}^n$, $\quad s \in \mathbb{N}_{\geq 1}$

$$\Gamma(s; w_1, \dots, w_{n-1}) = \left\{ \sum_{i=1}^{n-1} a_i w_i \;\middle|\; a_i \in [1, s] \cap \mathbb{N} \;\; \forall i=1,\dots,n-1 \right\}$$

= the grid of size s spanned by $w_1, \dots, w_{n-1}$ in $\mathbb{R}^n$

Lemma 1: $P \in \mathbb{R}[x_1, \dots, x_n]$ $\deg P \leq r$. $w_1, \dots, w_{n-1} \in \mathbb{R}^n$, linearly indep.
$s \in \mathbb{N}_{\geq 1}$, $\Gamma = \Gamma(s; w_1, \dots, w_{n-1})$, $t \in \mathbb{N}$, $\underline{s \cdot (t+1) > r}$

If $\partial^I P$ vanishes at every point of Γ for all $I = (i_1, \dots, i_n) \in \mathbb{N}^n$
with $i_1 + \dots + i_n \leq t$.
then $P|_T = 0$ identically.

$\Uparrow$

Lemma 2 Let $Q \in \mathbb{R}[y_1, \dots, y_\ell]$, $\deg Q \leq r$, $\quad s \in \mathbb{N}_{\geq 1}$

If $\partial^J Q(k_1, \dots, k_\ell) = 0$ $\quad \forall\, k_1, \dots, k_\ell \in \mathbb{N}$, $1 \leq k_i \leq s$
$\forall J = (j_1, \dots, j_\ell) \in \mathbb{N}^\ell$ with $j_1 + \dots + j_\ell \leq t$
then $Q = 0$

pf: Induction on ℓ
$\underline{\ell=1}$: $Q(X) \in \mathbb{R}[X]$, $\left(\frac{d}{dX}\right)^j Q\Big|_{X=k} = 0 \quad \forall\, 0 \leq j \leq t$. $\forall k = 1, \dots, s$

$$\Rightarrow Q(X) \equiv 0 \left(\mathrm{mod} \prod_{k=1}^{s} (X-k)^{t+1}\right)$$

q.e.d.

Remark: 1) Can replace $\mathbb{R}$ by any field of characteristic $> \max(t, s-1)$
2) Can replace $([1, s] \cap \mathbb{N})^\ell$ by a translation of $([\ell, s] \cap \mathbb{N})^\ell$

Corollary Let $V_h \subseteq \mathbb{R}^n$ be codim=1 vector subspaces of $\mathbb{R}^n$ $\forall h = 1, \dots, m$
$P \in R_{m,n}$: polynomial on $\underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{m}$ $\quad \deg_{(x_{h1}, \dots, x_{hn})} P \leq r_h \quad \forall h$
Let $\Gamma_h \subseteq V_h$ be a grid of size s $\quad \forall h = 1, \dots, m$
$t_1, \dots, t_m \in \mathbb{N}$ s.t $s_h (t_h + 1) > r_h \quad \forall h$
Suppose that P^I vanishes on $\prod_{h=1}^{m} \Gamma_h$ $\quad \forall\, \underset{(i_{hj})}{\overset{\parallel}{I}} \in \mathbb{N}^{m.n}$ s.t $\sum_{j=1}^{n} i_{hj} \leq t_h$

Then $P|_{V_1 \times \dots \times V_m}$

Consequence of trivial upper bound

Theorem [Lower bound of $\mathrm{Ind}(P; M_1, \dots, M_m; r_1, \dots, r_m)$, $M_h = M\{g_{h1}, \dots g_{h,n-1}\}$

Given $c_1, \dots, c_n \in \mathbb{R}$ $|c_i| \le 1$ $\forall i$, $\sum_i c_i = 0$

Given $0 < \varepsilon, \delta < 1$ with $16 n^2 \varepsilon < \delta$

Given $L^{(1)}, \dots, L^{(n)}$ linearly indep linear forms in $(\overline{\mathbb{Q}} \cap \mathbb{R})[X_1, \dots, X_n]$

Given $m \in \mathbb{N}_{>0}$ s.t. $m > 4 \varepsilon^{-2} \log(2n\Delta)$, where

$$\Delta = \max_{1 \le i \le n} [\mathbb{Q}(L^{(i)}) : \mathbb{Q}]$$

Given $r_1, \dots, r_m \in \mathbb{N}_{\ge 1}$

Let $P \in R^{m,n}$ satisfies the conditions in the polynomial theorem:

(i) P is homog in $X_{h1}, \dots, X_{hn}$ of degree r_h, $\forall h = 1, \dots, m$

(ii) $H(P) \le D^{r_1 + \dots + r_m}$ $D > 0$: constant depending only on $L^{(1)}, \dots L^{(n)}$

$\forall \underline{J} = (j_{hk})_{\substack{1 \le h \le m \\ 1 \le k \le n}} \in \mathbb{N}^{m,n}$, with $\partial^{\underline{J}} P$ written as

$$\partial^{\underline{J}} P = \sum_{\substack{\underline{I} \in \mathbb{N}^{m,n} \\ \| \\ (i_{h,\ell})}} d^{\underline{J}}(\underline{I}) \prod_{h=1}^{m} \prod_{\ell=1}^{n} \left(L_h^{(\ell)} \right)^{i_{h\ell}}$$

$$L_h^{(\ell)} = L^{(\ell)}(X_{h1}, \dots, X_{hn})$$

(iii) $|d^{\underline{J}}(\underline{I})| \le E^{r_1 + \dots + r_m}$ $\forall \underline{I}$, i.e. $H(\partial^{\underline{J}} P) \le E^{r_1 + \dots + r_m}$

(iv) If $(\underline{J}/\underline{r}) \le 2\varepsilon m$, then

$$d^{\underline{J}}(\underline{I}) = 0 \quad \text{unless} \quad \left| \sum_{h=1}^{m} \frac{i_{h\ell}}{r_h} - \frac{m}{n} \right| \le 3 n m \varepsilon, \ \forall \ell = 1, \dots, n$$

Let $Q_1, \dots, Q_m > 0$ be real numbers s.t.

$$\begin{cases} Q_h^{\varepsilon} > 2^n E, \quad Q_h^{\varepsilon} > n(\varepsilon^{-1} + 1) & \forall h = 1, \dots, m \\ r_1 \log Q_1 \le r_h Q_h \le (1+\varepsilon) r_1 \log Q_1 & \forall h = 1, \dots m \end{cases}$$

Assume moreover that $\forall h = 1, \dots, m$, $\exists\, q \overset{\text{def}}{=} n-1$ $\mathbb{R}$-linearly indep. integer points $\underline{g}_{h1}, \dots, \underline{g}_{hq} \in \mathbb{Z}^n$ s.t.

$$|L^{(k)}(\underline{g}_{ht})| \le Q_h^{c_k - \delta} \quad \forall k = 1, \dots, n, \quad \forall h, 1 \le h < m, \ \forall t = 1, \dots, n-1$$

Then $\mathrm{Ind}(P; M_1, \dots, M_m; r_1, \dots, r_m) \ge m\varepsilon$,

where $M_h = M_h(\underline{g}_{h1}, \dots, \underline{g}_{hq})$ is "the" linear form in $\mathbb{Z}[X_1, \dots X_n]$ with coprime coefficients s.t. $M(\underline{g}_{h,t}) = 0$ for $t = 1, \dots, q$ ($q = n-1$)

M_h is uniquely determined up to ± 1 by $\underline{g}_{h1}, \dots, \underline{g}_{hq}$

Pf. We must show that $\partial^{\underline{J}} P\big|_T = 0 \quad \forall \underline{J} \in \mathbb{N}^{m,n}$ with $(\underline{J}/\underline{r}) < m\varepsilon$

$$\sum_{h=1}^{m} \frac{j_{h1}+\dots+j_{hm}}{r_m}$$

notation $\left[\partial^{\underline{J}} P = \sum_{\substack{\underline{I} \in \mathbb{N}^{m,n} \\ (i_{hl})}} d^{\underline{J}}(\underline{I}) \cdot \prod_{h=1}^{m} \prod_{k=1}^{n} L_h^{(k)\, i_{hk}} \right]$

where $T = T_1 \otimes \dots \otimes T_m$

$T_h = \{ \underline{y} \in \mathbb{R}^n \mid M_h(\underline{y}) = 0 \}$
$= \mathbb{R}\underline{g}_{h1} + \dots + \mathbb{R}\underline{g}_{hq}$

Let $s_h = \lfloor \varepsilon^{-1} \rfloor + 1, \quad t_h = \lfloor r_h \varepsilon \rfloor \quad \forall h = 1, \dots, m \rightsquigarrow s_h \cdot t_h > r_h$

$\Gamma_h = \Gamma(s_h, \underline{g}_{h1}, \dots, \underline{g}_{hq})$ = grid of size s_h spanned by $\underline{g}_{h1}, \dots, \underline{g}_{hq}$

suffices to show: $\quad \left(\partial^{\underline{J}+\underline{I}} P\right)\Big|_{\Gamma_1 \times \dots \times \Gamma_h} = 0$

$\forall \underline{J}, \underline{I} \in \mathbb{N}^{m,n}$ with $(\underline{J}/\underline{r}) < m\varepsilon, \quad t_{h,1} + \dots + t_{h,n} \le \lfloor r_h \varepsilon \rfloor$

For such $\underline{J}, \underline{I}$, $\quad \left(\underbrace{(\underline{J}+\underline{I})}_{\underline{J}'} / \underline{r} \right) < 2m\varepsilon$ $\quad \underline{J}' \longleftarrow$ renamed $\underline{J}$

Thus: It suffices to show that

$$\partial^{\underline{J}} P(\underline{v}_1, \dots, \underline{v}_m) = 0 \qquad \forall \underline{J} \in \mathbb{N}^{m,n} \text{ with } (\underline{J}/\underline{r}) < 2m\varepsilon$$
$$\underline{v}_h \in \Gamma_h$$

- Trivial estimate of $d^{\underline{J}}(\underline{I})$: $\quad \left| d^{\underline{J}}(\underline{I}) \right| \le E^{r_1 + \dots + r_m}$
- estimate $\left| L_h^{(\ell)}(\underline{v}_h) \right|$:

$$\left| L_h^{(\ell)}(\underline{v}_h) \right| \le n \cdot (\varepsilon + 1) \cdot Q_h^{c_k - \delta} \qquad \left(\because \left| L^{(k)}(\underline{g}_{hl}) \right| \le Q_h^{c_k-\delta} \quad \forall k = 1, \dots, n \right)$$

sum of n terms — size of grid.

$$\le Q_h^{c_h - \delta + \varepsilon} \qquad \left(\because Q_h^{\varepsilon} > n(\varepsilon^{-1} + 1) \right)$$
$$\le Q_h^{c_k - 15n^2\varepsilon} \qquad \left(\because 16n^2\varepsilon < \delta \right)$$

$$\rightsquigarrow \log \left| \prod_{h=1}^{m} L_h^{(k)\, i_{hk}}(\underline{v}_h) \right| \le (c_k - 15n^2\varepsilon) \underline{\sum_{h=1}^{m} i_{hk} \cdot \log Q_h}$$

Estimate $\displaystyle\sum_{h=1}^{m} i_{hk} \log Q_h \quad \left(= \sum_{h=1}^{m} \frac{i_{hk}}{r_h} r_h \log Q_h \right)$

lower bound: $\displaystyle\sum_{h=1}^{m} i_{hk} \log Q_h \ge r_1 \log Q_1 \cdot \sum_{h=1}^{m} \frac{i_{hk}}{r_h} \qquad (\because r_1 \log Q_1 \le r_h \log Q_h)$

(otherwise $d^{\underline{J}}(\underline{I}) = 0$) $\displaystyle\ge r_1 \log Q_1 \cdot \left(\frac{1}{n} - 3n\varepsilon\right) m \qquad$ (assumption (iv))

upper bound. $\displaystyle\sum_{h=1}^{m} i_{hk} \log Q_h \le (1+\varepsilon) r_1 \log Q_1 \cdot \sum_{h=1}^{m} \frac{i_{hk}}{r_h} \qquad (\because r_h \log Q_h \le (1+\varepsilon) r_1 \log Q_1)$

$$\le r_1 \log Q_1 \cdot (1+\varepsilon)\left(\frac{1}{n} + 3n\varepsilon\right) m \qquad \text{(assumption (iv))}$$
$$\le r_1 \log Q_1 \cdot \left(\frac{1}{n} + 7n\varepsilon\right) m$$

Combining the two estimates, get

- $$\left|\sum_{h=1}^{m} i_{hk}\log Q_h - (r_1\log Q_1)\frac{m}{n}\right| \le (r_1\log Q)\cdot 7n\varepsilon m$$ whenever the coeff $d^{I}(\underline{I}) \neq 0$

$\leadsto$ • $$\left|\prod_{h=1}^{m} L_m^{(k)\,i_{hk}}(\underline{v}_m)\right| \le Q_1^{\,r_1\frac{m}{n}(c_k-15n^2\varepsilon)+2r_1\cdot 7n\varepsilon m} = Q_1^{\,r_1\left(\frac{m}{n}c_k-nm\varepsilon\right)}$$

($\because$ $|c_k-15n^2\varepsilon|\le 2$, which follows from $|c_k|\le 1$, $16n^2\varepsilon<\delta<1$)

Combining all estimates above, get

$$\left|\partial^{\underline{I}}P(\underline{v},\dots,\underline{v}_m)\right| \le (2^n\cdot E)^{(r_1+\dots+r_m)}\cdot Q_1^{\,r_1\frac{m}{n}\left(\sum_{k=1}^{n}c_k\right)-r_1n^2m\varepsilon} = E^{(r_1+\dots+r_m)}\cdot Q_1^{-r_1n^2m\varepsilon}$$

$\le 2^{n(r_1+\dots+r_m)}$ summands

$$\le (2^n\cdot E)^{\sum_{h=1}^{m}r_h}\cdot\left(Q_1^{-r_1\varepsilon}\cdots Q_m^{-r_m\varepsilon}\right)^{\frac{n^2}{1+\varepsilon}}$$ ($\because$ $r_h\log Q_h<(1+\varepsilon)r_1\log Q_1$)

$$\le \prod_{h=1}^{m}(2^nE)^{r_h}\cdot Q_1^{-r_1\varepsilon}\cdots Q_m^{-r_m\varepsilon}$$

$$= \prod_{h=1}^{m}\left(2^n E\cdot Q_h^{-\varepsilon}\right)^{r_h}$$ ($\because$ $Q_h^{\varepsilon}>2^nE$ $\forall h$)

$\Longrightarrow$ $\partial^{\underline{I}}P(\underline{v}_1,\dots,\underline{v}_m)=0$ $\quad\forall(\underline{v}_1,\dots,\underline{v}_m)\in\Gamma_1\times\dots\times\Gamma_m\subseteq\mathbb{Z}^{m,n}$

$\forall\underline{I}\in\mathbb{N}^{m,n}$ with $(\underline{I}/\underline{r})<3m\varepsilon$

As explained already, this implies

$$\operatorname{Ind}(P;\,M_1\{\underline{g}_{11},\dots,\underline{g}_{1q}\},\dots,M_m\{\underline{g}_{m1},\dots,\underline{g}_{mq}\};\,r_1,\dots,r_m)\ge m\varepsilon.$$

QED

A version of Roth's Lemma

<u>Thm</u> (Roth, 1955) $m \in \mathbb{N}_{\geq 1}$, $0 < \varepsilon < \frac{1}{12}$, $\omega = 24 \cdot 2^{-m} \cdot \left(\frac{\varepsilon}{12}\right)^{2^{m-1}}$

$r_1, \dots, r_m \in \mathbb{N}_{\geq 1}$, $\omega r_h \geq r_{h+1}$ $\forall h = 1, \dots, m$

$0 < \gamma \leq 1$. $(p_1, q_1), \dots, (p_m, q_m)$ m pairs of coprime integers, $q_h > 0$ $\forall h$

$q_h^{r_h} \geq q_1^{\gamma r_1}$ $\forall h = 1, \dots, m$

$q_h^{\gamma \omega} \geq 2^{3m}$ $\forall h$

$0 \neq P(X_1, \dots, X_m) \in \mathbb{Z}[X_1, \dots, X_m]$. $\deg_{X_h}(P_h) \leq r_h$ $\forall h = 1, \dots, m$

$H(P) \leq q_1^{\omega \gamma r_1}$

Then $\operatorname{Ind}(P; \frac{p_1}{q_1}, \dots, \frac{p_m}{q_m}; r_1, \dots, r_m) \leq \varepsilon$

generalized Roth's Lemma, obtained by specialization.

<u>Thm</u>: $n \geq 2$, $m \in \mathbb{N}_{\geq 1}$, $0 < \varepsilon < \frac{1}{12}$, $\omega = 24 \cdot 2^{-m} \left(\frac{\varepsilon}{12}\right)^{2^{m-1}}$

$0 \neq M_1, \dots, M_m \neq 0 \in \mathbb{Z}[X_1, \dots, X_n]$, coeff. of M_h coprime $\forall h = 1, \dots, m$

$M_h = \sum_{j=1}^{n} m_{hj} X_{hj}$

$0 < \Gamma \leq q = n-1$,

$H(M_h)^{r_h} \geq H(M_1)^{\Gamma r_1}$

$H(M_h)^{\Gamma \omega} \geq 2^{3mq^2}$ $q = (n-1)$

$0 \neq P \in \mathcal{R}_{\mathbb{Z}}^{m,n}$, homog. of degree r_h in $(X_{h1}, \dots, X_{hn})$ $\forall h = 1, \dots, m$

$H(P) \leq H(M_1)^{\omega \Gamma r_1}$

Then $\operatorname{Ind}(P; M_1, \dots, M_m; r_1, \dots, r_m) \leq \varepsilon$

<u>Remark</u>: Need easy lemma: $m_1, \dots, m_n \in \mathbb{Z}$, $\gcd(m_1, \dots, m_n) = 1$. $m_1 \neq 0$

Then $\exists j$ with $2 \leq j \leq n$ s.t. $\gcd(m_1, m_j) \leq |m_1|^{\frac{n-2}{n-1}}$

<u>Pf</u>: May assume $H(M_h) = |m_{h1}|$ $\forall h = 1, \dots, m$

Use the remark above, may assume $\gcd(m_{h1}, m_{h2}) \leq |m_{h,1}|^{\frac{n-2}{n-1}} = |m_{h1}|^{\frac{q-1}{q}}$ $\forall h = 1, \dots, m$

Write $P(\underline{X}) = \left(\prod_{h=1}^{m} \prod_{j=3}^{n} X_{hj}^{u_{hj}}\right) \cdot \bar{P}(\underline{X})$, with $0 \neq \bar{P}(X_{11}, X_{12}, 0, \dots, 0; X_{21}, X_{22}, 0 \dots 0, \dots, X_{m1}, X_{m2}, 0 \dots 0)$

$u_{hj} \in \mathbb{N}_{\geq 1}$ $\forall j \geq 3$

$\overset{!!}{\hat{P}}(X_{11}, X_{12}; X_{21}, X_{22}, \dots, X_{m1}, X_{m2})$

homog. of degree $\leq r_h$ in X_{h1}, X_{h2} $\forall h$

Let $\eta = \operatorname{Ind}(P; M_1, \dots, M_m; r_1, \dots, r_m)$

$H(\hat{P}) \leq H(P)$

$\rightsquigarrow \operatorname{Ind}(\hat{P}, \hat{M}_1, \dots, \hat{M}_m; r_1, \dots, r_m) \geq \eta$,

where $\hat{M}_h = m_{h1} X_{h1} + m_{h2} X_{h2} = \gcd(m_{h1}, m_{h2}) \cdot q_h \cdot (X - \frac{p_h}{q_h})$

where $p_h \overset{\text{def}}{=} -\frac{m_{h2}}{\gcd(m_{h1}, m_{h2})} \cdot \frac{m_{h1}}{|m_{h1}|}$, $q_h \overset{\text{def}}{=} \frac{|m_{h1}|}{\gcd(m_{h1}, m_{h2})}$

Let $\gamma = \frac{\Gamma}{q} = \frac{\Gamma}{n-1}$ (so $0 < \gamma \leq 1$)

$$H(M_h)^{\frac{1}{q}} = |m_{h1}|^{\frac{1}{q}} \leq \frac{|m_{h1}|}{\gcd(m_{h1}, m_{h2})} = q_h \leq H(M_h)$$

$$\therefore \gcd(m_{h1}, m_{h2}) \leq |m_{h1}|^{\frac{q-1}{q}}$$

Let $\tilde{P} = \hat{P}(x_1, 1; x_2, 1, \ldots, x_m, 1)$

Check:

$$\begin{array}{ccc} q_h^{r_h} & \geq & q_1^{\gamma r_1} \quad \checkmark \\ \text{IV} & & \| \\ H(M_h)^{r_h/q} & \geq & H(M_1)^{\Gamma r_1/q} \end{array}$$

$\uparrow$ assumption

$$\begin{array}{ccc} q_h^{\gamma\omega} & \geq & 2^{3m} \quad \checkmark \\ \text{IV} & & \| \\ H(M_h)^{\frac{\Gamma\omega}{q^2}} & \geq & 2^{3mq^2/q^2} \end{array}$$

$\uparrow$ assumption

Conditions in Roth's Lemma hold

$$H(\tilde{P}) \leq H(\hat{P}) \leq H(P) \leq H(M_1)^{\omega\gamma q r_1} \leq q_1^{\omega\gamma r_1} \quad \checkmark$$

$\uparrow$ assumption

Thus: $\operatorname{Ind}(\tilde{P}; \frac{p_1}{q_1}, \ldots, \frac{p_m}{q_m}; r_1, \ldots, r_m) \leq \varepsilon$

$\|$

$$\operatorname{Ind}(\hat{P}; \hat{M}_1, \ldots, \hat{M}_m; r_1, \ldots, r_m) \geq \operatorname{Ind}(P; M_1, \ldots, M_m; r_1, \ldots, r_m)$$

QED

Estimate of the size of $\underline{g}_n^\vee$

Given $L_i(\underline{x}) = \underline{a}_i \cdot \underline{x} \in (\overline{\mathbb{Q}} \cap \mathbb{R})[X_1, \dots, X_n]$, $i = 1, \dots, n$. linearly indep $/\mathbb{R}$ (or, equiv. $/(\overline{\mathbb{Q}} \cap \mathbb{R})$)

$\underline{a}_1^\vee, \dots, \underline{a}_n^\vee \in \mathbb{R}^n$, $\underline{a}_i^\vee \cdot \underline{a}_j = \delta_{ij}$

Given $c_1, \dots, c_n \in \mathbb{R}$, $c_1 + \dots + c_n = 0$, $|c_i| \le 1$ $\forall_i$

Given $\delta > 0$

Let $G = \{ i \in [1, n] \cap \mathbb{N} : c_i + \frac{\delta}{2} \ge 0 \}$

<u>Lemma</u>: $n \ge 2$ Given $Q > 1$. Define $\Pi = \Pi(Q; L_1, \dots, L_n; c_1, \dots, c_n)$

$$= \{ \underline{x} \in \mathbb{R}^n : |L_i(\underline{x})| = |\underline{a}_i \cdot \underline{x}| \le Q^{c_i} \ \forall_i \}$$

Let $\lambda_i = \lambda_i(Q; c_1, \dots, c_n; L_1, \dots, L_n)$ be the successive minima of Π

Let $\underline{g}_i = \underline{g}_i(Q; c_1, \dots, c_n; L_1, \dots, L_n)$ be integer points realizing $\lambda_1, \dots, \lambda_n$, so that $\underline{g}_1, \dots, \underline{g}_n$ are $\mathbb{Q}$-linearly independent and $\underline{g}_j \in \lambda_j \cdot \Pi$ $\forall j = 1, \dots, n$

Let $\underline{g}_1^\vee, \dots, \underline{g}_n^\vee$ be reciprocal to $\underline{g}_1, \dots, \underline{g}_n$, i.e. $\underline{g}_i^\vee \cdot \underline{g}_j = \delta_{ij}$ $\forall_{i,j}$

Assume: $\lambda_{n-1} < Q^{-\delta}$

and suppose that $\exists\, i_0 \in G$, i.e. $c_{i_0} + \frac{\delta}{2} \ge 0$.

s.t. $\underline{a}_{i_0}^\vee \cdot \underline{g}_n^\vee \neq 0$

Then $\exists$ constants $C_i = C_i(\delta; L_1, \dots, L_n; c_1, \dots, c_n)$, $i = 1, 2, 3$, s.t.

$$Q^{C_1} \le H(\underline{g}_n^\vee(Q)) \le Q^{C_2} \qquad \forall Q \ge C_3$$

<u>Pf</u>: Apply Mahler's theory of reciprocal parallelepipeds to $(Q^{-c_i} \underline{a}_i)_{1 \le i \le n}$ and $(Q^{c_i} \underline{a}_i^\vee)_{1 \le i \le n}$

$\rightsquigarrow$ $$|Q^{c_i} \underline{a}_i^\vee \cdot \underline{g}_n^\vee| \ll \lambda_n^{-1},$$

$\rightsquigarrow$ $$|\underline{a}_i^\vee \cdot \underline{g}_n^\vee| \ll Q^{-c_i} \cdot \lambda_n^{-1} \ll Q^{-\delta(n-1) - c_i} \qquad \forall i = 1, \dots, n$$

$\uparrow$ $\because$ $\lambda_n^{-1} \ll \lambda_1 \cdots \lambda_{n-1} < Q^{-(n-1)\delta}$

$$\Rightarrow H(\underline{g}_n^\vee) \ll Q^{-\delta(n-1) - c_i} \le Q^{1 - \delta(n-1)} \qquad \because |c_i| \le 1$$

$(\underline{a}_i^\vee)$ is a basis

$\rightsquigarrow$ $\exists\, C_2 = C_2(\delta; L_1, \dots, L_n; c_1, \dots, c_n)$ s.t.

$H(\underline{g}_n) \le Q^{C_2}$ if Q is sufficiently large, proved the 2nd $\le$

lower bound: $$0 \neq |\underline{a}_{i_0}^\vee \cdot \underline{g}_n^\vee| \ll Q^{-\delta(n-1) + \frac{\delta}{2}} \le Q^{-\delta/2}$$

Let $K_{i_0} = \mathbb{Q}(\underline{a}_{i_0}^\vee)$ $\rightsquigarrow$ $|N_{K/\mathbb{Q}}(\underline{a}_{i_0}^\vee \cdot \underline{g})| \ge 1$, while $|\sigma(\underline{a}_{i_0}^\vee \cdot \underline{g}_n^\vee)| \ll H(\underline{g}_n)$ $\forall \sigma : K_{i_0} \hookrightarrow \mathbb{C}$

$\rightsquigarrow$ $$|\underline{a}_{i_0}^\vee \cdot \underline{g}_n^\vee| \gg H(\underline{g}_n^\vee)^{1 - [K_{i_0} : \mathbb{Q}]} \ge H(\underline{g}_n^\vee)^{1 - [K : \mathbb{Q}]}, \quad K = \mathbb{Q}(\underline{a}_1^\vee, \dots, \underline{a}_n^\vee)$$

$Q^{-\delta/2} \gg$

$$\Longrightarrow [K : \mathbb{Q}] > 1, \text{ and } H(\underline{g}_n^\vee) \gg Q^{\delta/2 \cdot ([K:\mathbb{Q}] - 1)}$$

q.e.d.

Consequence: $\exists$ constants $C_4(\delta; L_1,\dots,L_n; c_1,\dots,c_n)$, $C_5(\delta; L_1,\dots,L_n; c_1,\dots,c_n)$, $C_6(\delta; L_1,\dots,L_n; c_1,\dots,c_n)$ s.t. for $M = M(Q) = M\{\underline{g}_1(Q),\dots,\underline{g}_{n-1}(Q)\} = \underline{m}(Q)\cdot\underline{x}$

$$Q^{C_4} \leq H(M) \leq Q^{C_5} \qquad \forall Q \geq C_6$$

$\because$ $1 \leq E := |\det(\underline{g}_1,\dots,\underline{g}_n)| \leq n!$

and $\underline{g}_n^\vee \in \frac{1}{E}\mathbb{Z}^n$ $\qquad \underline{g}_n^\vee = \frac{t}{E}\,\underline{m}$ $\qquad 1 = \underline{g}_n^\vee \cdot \underline{g}_n \in \frac{t}{E}\cdot\mathbb{Z}$

with coprime integer coeff $\qquad$ so: $t \mid E$

i.e. $\underline{g}_n^\vee = \frac{1}{F}\,\underline{m}$ $\qquad F \in \mathbb{Z}$, $|F| \leq n!$

Theorem on the next-last-minimum

$L_i(x) = \underline{a}_i \cdot x$

Given $L_1, \dots, L_n$: l. indep. linear forms on $\mathbb{R}^n$ with real algebraic coeff.

$c_1, \dots, c_n \in \mathbb{R}$, $|c_i| \le 1$ $\forall i$ $c_1 + \dots + c_n = 0$; $\delta > 0$

$$\Pi = \Pi(Q; c_1, \dots, c_n; L_1, \dots, L_n) = \{ \underline{x} \in \mathbb{R}^n \mid |L_i(\underline{x})| \le Q^{c_i} \ \forall i \}$$

$\lambda_i(Q; c_1, \dots, c_n; L_1, \dots, L_n)$ $1 \le i \le n$ successive minimum of $\Pi(Q)$

$\underline{g}_1, \dots, \underline{g}_n \in \mathbb{Z}^n$, $\underline{g}_j \in \lambda_j \Pi$ $\forall j$

$\underline{g}_j(Q; c_1, \dots, c_n; L_1, \dots, L_n)$, linearly indep

$\underline{a}_i^\vee \cdot \underline{a}_j = \delta_{ij}$

$\underline{g}_1^\vee, \dots, \underline{g}_n^\vee$: $\underline{g}_i^\vee \cdot \underline{g}_j = \delta_{ij}$

dual/reciprocal of the $\underline{a}_j$'s

Assume: $\lambda_{n-1}(Q) < Q^{-\delta}$

Then $\exists$ $Q_1(\delta; L_1, \dots, L_n; c_1, \dots, c_n)$ s.t.

$\underline{a}_i^\vee \cdot \underline{g}_n^\vee(Q) = 0$ $\forall Q > Q_1$ and $\forall i \in \mathfrak{S} = \{ i \mid c_i + \frac{\delta}{2} \ge 0 \}$

i.e. $c_i + \frac{\delta}{2} \ge 0$

$\lambda_{n-1}(Q) < Q^{-\delta}$ and

Note: If $\exists i \in \mathfrak{S}$ s.t. $\underline{a}_i^\vee \cdot \underline{g}_n^\vee(Q) \ne 0$,

then $Q^{C_1} \le H(\underline{g}_n^\vee) \le Q^{C_2}$ C_1, C_2 depends only on $\delta; L_1, \dots, L_n; c_1, \dots, c_n$

Lemma 11A, p. 195 of Schmidt

Pf. Let $\mathcal{Q} = \{ Q > 1 : \lambda_{n-1}(Q) < Q^{-\delta}$ and $\exists i \in \mathfrak{S}$ with $\underline{a}_i^\vee \cdot \underline{g}_n^\vee(Q) \ne 0 \}$

i.e. $c_i + \frac{\delta}{2} \ge 0$

Suppose that $\mathcal{Q}$ is unbounded.

Will derive a contradiction

May assume: $\underline{a}_j \in (\mathcal{O}_{\bar{\mathbb{Q}}} \cap \mathbb{R})^n$ $\forall j = 1, \dots, n$

Let $q = n-1$

(i) Pick δ_1 with $0 < \delta_1 < 1$, $\delta_1 \le \delta$

(ii) Pick ε small enough so that $16 n^2 \varepsilon < \delta_1$

(iii) Let $\Delta = \max_{1 \le j \le n} ([\mathbb{Q}(\underline{a}_j^\vee) : \mathbb{Q}])$, Choose m large so that $m \ge 4 \varepsilon^{-2} \log(2n\Delta)$ (cond^n in polynomial thm)

Let $\omega = \omega(m, \varepsilon) = 24 \cdot 2^{-m} \left(\frac{\varepsilon}{2}\right)^{2^{m-1}}$

$\Gamma = \mathrm{Min}\left(\frac{C_4}{C_5}, q\right)$

(iv) Choose $Q_1 \in \mathcal{Q}$ s.t. $Q_1^h > 2^n E$ (polynomial Thm), $Q_1^\varepsilon > n(\varepsilon^{-1} + 1)$, $Q_1 > C_6$ (p. 27), $Q_1^{\omega C_4 \Gamma} \ge 2^{3mq^2}$

$Q_1^{\omega C_4 \Gamma} \ge D^{mq^2}$ (poly thm)

(v) Choose $Q_2, \dots, Q_m \in \mathcal{Q}$ with

$\omega \log Q_{h+1} \ge 2 \log Q_h$ $\forall h = 1, \dots, m-1$ $\rightsquigarrow$ $Q_1 < Q_2 < \dots < Q_m$

(vi) Choose $r_1 \in \mathbb{N}$ large so that $\varepsilon r_1 \log Q_2 \ge \log Q_m$

(vii) Define $r_h = \left\lceil \frac{r_1 \log Q_1}{\log Q_h} \right\rceil$ $\forall h = 2, \dots, m$

$\rightsquigarrow r_1 \log Q_1 \leq r_h \log Q_h \leq r_1 \log Q_1 + \log Q_h \leq (1+\varepsilon) r_1 \log Q_1$

$\because \ \varepsilon r_1 \log Q_1 \geq \log Q_m \geq \log Q_h$

$\Rightarrow$ Have checked one of the conditions in the estimate with trivial upper bound

- The condition $m \geq 4\varepsilon^2 \log(2^n \Delta)$ in the polynomial thm holds by the choice of m

Let $\overset{*}{0}$ $P = P(x_{11}, \dots, x_{1n}; \dots; x_{m1}, \dots, x_{mh})$ be as in the polynomial thm (p 14)

The result on lower bound of $\mathrm{Ind}(P; M_1, \dots, M_m; r_1, \dots, r_m)$ using trivial upper bound estimate $M_h = M\{\underline{g}_{h1}, \dots, \underline{g}_{hq}\}$ $\quad q = n-1$

$\Longrightarrow \quad \mathrm{Ind}(P; M_1, \dots, M_m; r_1, \dots, r_m) \geq m\varepsilon$

Now: verify conditions in generalized Roth's Lemma

$$\omega r_h \geq r_{h+1} \quad \forall h = 1, \dots, m-1 \quad ?$$

$$\omega \frac{r_{h+1} \log Q_{h+1}}{(1+\varepsilon) \log Q_h} \geq \frac{2}{1+\varepsilon} r_{h+1} \quad \because \ \omega \log Q_{h+1} \geq 2 \log Q_h \text{ in (v)}$$

$$\geq r_{h+1} \qquad \text{so indeed } \omega r_h \geq r_{h+1} \quad \checkmark$$

Since $\exists i_0$ with $c_{i_0} + \frac{\delta}{2} \geq 0$ s.t. $\check{\underline{a}}_{i_0} \cdot \check{\underline{g}}_n(Q) \neq 0$, the lemma on the size of $\check{g}_n(Q)$ gives:

$$Q_h^{c_4} \leq H(M_h) \leq Q_h^{c_5} \qquad \forall h = 1, \dots, m$$

$$\Rightarrow \quad H(M_h)^{r_h} \geq Q_h^{r_h c_4} \geq Q_1^{r_1 c_4} \geq H(M_1)^{r_1 c_4 / c_5} \geq H(M_1)^{r_1 \Gamma}$$

where $\Gamma = \min(\frac{c_4}{c_5}, q)$

Check: $H(M_h)^{r_h} \geq H(M_1)^{r_1 \Gamma}$ $\quad$ just shown

$H(M_h)^{\omega \Gamma} \geq 2^{3mq^2}$ $\quad \checkmark$

$\rotatebox{90}{$\leq$}\ Q_h^{\omega \Gamma c_4} \overset{(v)}{\geq}$ $\quad \checkmark$

Check $\quad H(P)^{q^2} \leq H(M_1)^{\omega r_1 \Gamma}$

$$H(P) \leq D^{r_1 + \dots + r_m} \leq D^{m r_1}$$

$$\rightsquigarrow \quad H(P)^{q^2} \leq D^{m q^2 r_1} \underset{(iv)}{\leq} Q_1^{\omega c_4 \Gamma r_1} \leq H(M_1)^{\omega r_1 \Gamma} \quad \checkmark$$

$\xRightarrow{\text{generalized Roth's Lemma}}$ $\quad \mathrm{Ind}(P; M_1(Q), \dots, M_m(Q), r_1, \dots, r_m) \leq \varepsilon$

Contradiction! QED

<u>Corollary</u>: constancy / boundedness of $\underline{g}_n^\vee(Q)$ for $Q >> 0$

Given $L_1, \dots, L_n \in (\mathbb{R} \cap \overline{\mathbb{Q}})[X_1, \dots, X_n]$ linearly forms, linearly indep, $c_1, \dots, c_n \in \mathbb{R}$, $c_1 + \dots + c_n = 0$, δ as in the previous theorem on next-to-last minimum

$$\Pi(Q) = \Pi(Q; c_1, \dots, c_n; L_1, \dots, L_n) = \left\{ \underline{x} \in \mathbb{R} : \left| Q^{-c_i} L_i(\underline{x}) \right| \leq 1 \ \forall_i \right\}$$

$\underline{g}_i = \underline{g}_i(Q) = \underline{g}_i(Q; c_1, \dots, c_n; L_1, \dots, L_n)$: integral points realizing the successive minima

$\underline{g}_i^\vee = \underline{g}_i^\vee(Q)$ = reciprocal / dual to $(\underline{g}_1, \dots, \underline{g}_n)$ $\lambda_1(Q) \leq \dots \leq \lambda_n(Q)$

Assume that $\lambda_{n-1}(Q) < Q^{-\delta}$ for an unbounded subset $\mathcal{Q}$ of $\mathbb{R}_{>0}$

Then $\exists$ an unbounded subset $\mathcal{Q}' \subseteq \mathcal{Q}$ such that

$$\underline{g}_n^\vee(Q) = \underline{g}_n^\vee(Q') \quad \forall Q, Q' \in \mathcal{Q}'$$

<u>Variant</u> /consequence Instead of fixing $c_1, \dots, c_n$ with $c_1 + \dots + c_n = 0$,
suppose that for each $Q \in \mathcal{Q}$, $\mathcal{Q}$ an unbounded subset of $\mathbb{R}_{>0}$,
we are given positive numbers $A_i = A_i(Q)$ s.t.

$$\prod_{i=1}^n A_n(Q) = 1 \quad \text{and} \quad \mathrm{Max}\left\{ A_1(Q), \dots, A_n(Q), A_1(Q)^{-1}, \dots, A_n(Q)^{-1} \right\} \leq Q,$$

and $\Pi(Q) = \Pi(A_1(Q), \dots, A_n(Q); L_1, \dots, L_n) = \left\{ \underline{x} \in \mathbb{R}^n : \left| L_i(\underline{x}) \right| \leq A_1(Q) \ \forall_i \right\}$

Let $\lambda_1(Q), \dots, \lambda_n(Q)$ be the successive minima of $\Pi(Q)$.
Let $\underline{g}_1(Q), \dots, \underline{g}_n(Q)$ be integer points realizing $\lambda_1(Q), \dots, \lambda_n(Q)$
and let $\underline{g}_1^\vee(Q), \dots, \underline{g}_n^\vee(Q)$ be the reciprocals of $\underline{g}_1(Q), \dots, \underline{g}_n(Q)$.
Suppose that $\lambda_{n-1}(Q) < Q^{-\delta} \quad \forall Q \in \mathcal{Q}$.
Then $\exists$ an unbounded subset $\mathcal{Q}' \subseteq \mathcal{Q}$ such that

$$\underline{g}_n(Q) = \underline{g}_n(Q') \quad \forall Q, Q' \in \mathcal{Q}'$$

— An easy compactness argument —

<u>Pf</u> of Corollary: May assume: $Q > Q_1(\delta; L_1, \dots, L_n; c_1, \dots, c_n)$ in the thm on next-to-last minimum

$\rightsquigarrow$ $\underline{a}_i^\vee \cdot \underline{g}_n^\vee(Q) = 0 \quad \forall_i$ s.t. $c_i + \frac{\delta}{2} \geq 0$, $\forall Q \in \mathcal{Q}$

Pick $Q_0 \in \mathcal{Q}$ $\rightsquigarrow$ $\left| \underline{a}_i^\vee \cdot \underline{g}_n(Q_0)^\vee \right| \ll 1$ Let $\underline{h} \in \mathbb{Q}\underline{g}_n(Q_0)^\vee \cap \mathbb{Z}^n$ primitive

For any c_j with $c_j + \delta/2 < 0$, then

$$\left| \underline{a}_j^\vee \cdot \underline{h} \right| \ll 1 \leq Q^{-c_j - \delta/2} \quad \text{for all sufficiently large } Q \in \mathcal{Q}'$$

Conclude: $\underline{h} \in \Pi(Q)^\vee$ for all $Q \in \mathcal{Q}$ and sufficiently large

On the other hand, $\lambda_2^{\vee}(Q) \gg\ll \lambda_{n-1}(Q)^{-1} > Q^{\delta}$

$\leadsto$ For $Q \in \mathcal{Q}$ sufficietly large, $\lambda_2^{\vee}(Q) > 1 \Rightarrow \mathbb{Z}^n \cap \Pi(Q)^{\vee} \subseteq \mathbb{Q}\cdot \underline{\underline{h}}$

$\Rightarrow g_n^{\vee}(Q) \in \mathbb{Q}\cdot h$

On the other hand, $|\det(\underbrace{g_1(Q), \dots, g_n(Q)}_{E(Q)})| \ll 1$

and $g_n^{\vee}(Q) = \frac{1}{E(Q)} \cdot \underline{m}$ ← components relatively prime

Have shown, $\exists\, C > 0$ s.t. $\{g_n^{\vee}(Q) \mid Q \in \mathcal{Q},\ Q \geq C\}$ is a finite set

Have proved the corollary on constancy/boundedness of $g_n^{\vee}(Q)$ for $Q \gg 0$

q.e.d.

<u>Proof</u> of the strong subspace theorem $\quad 1 \leq d \leq n-1$, $d \in \mathbb{N}$ $\quad$ <u>assume</u> $|c_i| \leq \frac{1}{n}\ \forall i$

Given $L_1, \dots, L_n$; $c_1, \dots, c_n$ with $\sum c_i = 0$, $\delta > 0$, s.t. $\exists$ an unbound subset $\mathcal{Q} \subseteq \mathbb{R}_{>0}$ s.t. $\forall Q \in \mathcal{Q}$, the d-th and $d+1^{\text{st}}$ among the successive minimum $\lambda_1(Q), \dots, \lambda_n(Q)$ of

$\Pi(Q) = \{x \in \mathbb{R}^n : |L_i(\underline{x})| \leq Q^{c_i}\ \forall i\}$ satisfies

$$\lambda_d < \lambda_{d+1} Q^{-\delta} \qquad \forall Q \in \mathcal{Q}$$

Must show: $\exists$ a fixed d-dim$^{\underline{l}}$ vector subspace $S^d \subseteq \mathbb{Q}^n$ and an unbound subset $\mathcal{Q}' \subseteq \mathcal{Q}$ s.t. the first d successive minima of $\Pi(Q)$ are assumed by elements of $S^d \cap \mathbb{Z}^n$

Let $p = n-d$. $\forall \sigma \in C(n,p)$, let $\underline{A_\sigma} \stackrel{\text{def}}{=} \bigwedge_{i \in \sigma} \underline{a}_\sigma$, $c_\sigma := \sum_{i \in \sigma} c_i$

$C(n,p) = \{\tau_1, \dots, \tau_{\ell-1}, \tau_\ell\}$ $\quad \leadsto \sum_\sigma c_\sigma = 0$, $|c_\sigma| \leq 1\ \forall \sigma$

$\tau_{\ell-1} = \{d, d+2, \dots, n\}$, $\tau_\ell = \{d+1, \dots, n\}$, $\ell = \binom{n}{p}$

Let $\nu_i(Q)$, $1 \leq i \leq \binom{n}{p}$ be the successive minima of $\Pi^{(p)}(Q) \subseteq \Lambda^p \mathbb{R}^n$

$\Pi^{(p)}(Q)$: p-th pseudo compound of $\Pi(Q)$

Mahler $\Rightarrow$

$$\nu_\ell \ll \lambda_{d+1} \cdots \lambda_n \ll \nu_\ell$$

$$\nu_{\ell-1} \ll \lambda_d \lambda_{d+2} \cdots \lambda_n \ll \nu_{\ell-1}$$

$\leadsto \quad \nu_{\ell-1} \ll \nu_\ell \cdot Q^{-\delta} \Rightarrow \nu_{\ell-1} < \nu_\ell Q^{-\delta/2}$ for Q large in $\mathcal{Q}$

Lemma on the last two minima (using Davenport's Lemma) $\Rightarrow$

$\exists$ an unbounded subset $\mathcal{Q}' \subseteq \mathcal{Q}$ s.t.

$$\underline{V}_\ell^{\vee}(Q) = \underline{V}_\ell^{\vee}(Q') = \underline{H} \qquad \forall Q, Q' \in \mathcal{Q}',$$

where $\underline{V}_1(Q), \dots, \underline{V}_\ell(Q)$ are the successive minima of $\Pi^{(p)}(Q)$, and $\underline{V}_1^{\vee}(Q), \dots, \underline{V}_\ell^{\vee}(Q)$ are reciprocal to $\underline{V}_1(Q), \dots, \underline{V}_\ell(Q)$, $\ell = \binom{n}{p}$

Let $\underline{g}_1(Q), \dots, \underline{g}_n(Q) \in \mathbb{Z}^n$ be linearly independent integer points which realize the successive minima of $\Pi(Q)$.

$$\underline{G}_\tau(Q) = \bigwedge_{i \in \tau} \underline{g}_i(Q) \quad \text{for } \tau \in C(n,p)$$

We know $\quad |\underline{A}_\sigma \cdot \underline{G}_\tau(Q)| \ll \nu_{\ell-1}(Q) \cdot Q^{c_\sigma} \quad \forall \sigma \in C(n,p)$

$\forall \tau \in C(n,p)$

$\tau_\ell = \{d+1, \dots, n\}$

Since $\nu_{\ell-1}(Q) \ll \nu_\ell(Q)^{-\delta}$,

$$|\underline{A}_\sigma \, G_\tau(Q)| < \nu_\ell(Q) \, Q^{-\delta/3} \, Q^{c_\sigma} \quad \forall Q \in q' \text{ sufficiently large}$$

So $\sum_{\tau \neq \tau_\ell} \mathbb{Q}\underline{G}_\tau \subseteq \sum_{j=1}^{\ell-1} \mathbb{Q}\underline{V}_j(Q)$, for otherwise $\exists\, \tau_0 \neq \tau$

s.t. $\underline{V}_1(Q), \dots, \underline{V}_{\ell-1}(Q), \underline{G}_{\tau_0}(Q)$ are ℓ linearly indep. integer points in $\nu_\ell(Q) \cdot Q^{-\delta/3} \cdot \Pi^{(p)}(Q)$, a contradiction.

<u>Conclude</u>: After changing q' to $q' \cap [N, \infty)$ for N sufficiently large, <u>we have</u>: $\mathbb{Q} \cdot \underline{g}^\vee_{d+1}(Q) \wedge \dots \wedge \underline{g}^\vee_n(Q)$ is constant for all $Q \in q'$.

In other words, $\exists$ a fixed d-dimensional subspace $S^d \subseteq \mathbb{Q}^n$ s.t. $\underline{g}_1(Q), \dots, \underline{g}_d(Q) \in S^d \quad \forall Q \in q'$.

<u>QED</u>.

$\underline{n} = (n_1, \dots, n_m)$ $\quad \mathbb{P}^{\underline{n}} = \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_m}$ $\quad M = n_1 + \dots + n_m = \dim \mathbb{P}^{\underline{n}}$

$\Gamma_R^{\underline{n}} = \Gamma(\mathbb{P}^{\underline{n}}_{/R}, \bigoplus_{\underline{d} \in \mathbb{N}^m} \mathcal{O}_{\mathbb{P}^{\underline{n}}}(\underline{d}))$ $\quad \forall$ commutative ring R

$= \bigoplus_{\underline{d} \in \mathbb{N}^m} \Gamma_R^{\underline{n}}(\underline{d})$

Def For $\sigma \geq 0$, $0 \neq F \in \Gamma_R^{\underline{n}}$ multi-homogeneous, $\quad \underline{d} = (d_1, \dots, d_m) \in \mathbb{N}_{>0}^m$

$$Z_\sigma(F, \underline{d}) = V\left(\partial_{\underline{i}} F \;\middle|\; \underline{i} = (i_{hj})_{\substack{1 \leq h \leq m \\ 0 \leq j \leq n_h}} \in \mathbb{N}^{n_1 + \dots + n_m}, \ \underbrace{(\underline{i}/\underline{d})}_{\sum_{h=1}^m \frac{i_{h0} + \dots + i_{hn_h}}{d_h}} \leq \sigma \right)$$

Theorem 1 $k \supseteq \mathbb{Q}$: alg. closed field, $0 \neq F \in \Gamma_k^{\underline{n}}$, $\sigma \geq 0$, $0 < \varepsilon \leq 1$

$\underline{d} = (d_1, \dots, d_m)$

Suppose: 1) $\dfrac{d_h}{d_{h+1}} \geq \left(\dfrac{mM}{\varepsilon}\right)^M$ for $h = 1, \dots, m-1$

2) Z_σ and $Z_{\sigma+\varepsilon}$ have a common irreducible component Z

Then (a) Z is a product variety: $Z = Z_1 \times \dots \times Z_m$ $\quad Z_h \subseteq \mathbb{P}^{n_h}$ reduced irreducible $\forall h$

(b) If $F \in \Gamma_{k_0}^{\underline{n}}$ $\quad k_0 \leq k$ subfield,

then $Z_1, \dots, Z_m$ are defined over an extension field k_1 of k_0

with $[k_1 : k_0] \cdot \prod_{h=1}^m \deg(Z_h) \leq \left(\dfrac{ms}{\varepsilon}\right)^s$, where $s = \sum_{h=1}^m \operatorname{codim}(Z_h \subseteq \mathbb{P}^{n_h})$

Theorem 2 With $k = \overline{\mathbb{Q}}$ in Thm 1, $F \in \Gamma_{k_0}^{\underline{n}}$ $\quad k_0$: a number field

$$[k_1 : k_0] \cdot \left(\prod_{h=1}^m \deg(Z_h) \right) \cdot \left(\sum_{h=1}^m \frac{1}{\deg(Z_h)} d_h \cdot h(Z_h) \right)$$

$$\leq 2 \cdot \left(\frac{s}{\varepsilon}\right)^s m^M M^2 (d_1 + \dots + d_m + \underset{\uparrow}{h(F)})$$

height of polynomial, using L^2-norm at arch. places

i.e. $h_{Ar}(F)$

Cor $F \in \Gamma_{k_0}^{\underline{n}}$, k_0 : number field. $\quad 0 < \varepsilon \leq 1$

Assume $d_h / d_{h+1} \geq \left(\dfrac{mM(M+1)}{\varepsilon}\right)^M$

Then $\forall$ every irreducible component Z' of Z_ε, $\exists k_1$ $\exists Z_h \subseteq \mathbb{P}^{n_h}_{/k_1}$ abs. irred. and reduced $\forall h = 1, \dots m$

s.t. $Z'_{/k_1} \subseteq Z_1 \times \dots \times Z_m$,

$[k_1 : k_0] \prod_h \deg(Z_h) \leq \left(\dfrac{m(M+1)s}{\varepsilon}\right)^s$

and $[k_1 : k_0] \cdot \left(\prod_h \deg(Z_h)\right) \cdot \left(\sum_{h=1}^m \dfrac{1}{\deg(Z_h)} d_h \cdot h(Z_h)\right) \leq 2 \left(\dfrac{(M+1)s}{\varepsilon}\right)^s m^M M^2 (d_1 + \dots + d_m + h(F))$

Thm 3 (improved Roth Lemma)

$0 \neq F \in \overline{\mathbb{Q}}[X_{10}, X_{11}; \cdots; X_{m0}, X_{m1}]$, $0 < \varepsilon < m+1$, homog. of deg. d_h in X_{h0}, X_{h1}

$\underline{d} = (d_1, \ldots, d_m)$, $d_h / d_{h+1} \geqslant 2m^3/\varepsilon$ for $h = 1, \ldots, m-1$

$\underline{P} = (P_1, \ldots, P_m) \in (\underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{m})(\overline{\mathbb{Q}})$

$d_h \cdot h(P_h) > \left(\frac{3m^3}{\varepsilon}\right)^m \cdot \left(d_1 + \cdots + d_m + h(F)\right)$

Then $i_{\underline{d}}(F, \underline{P}) < \varepsilon$

Green's function on Riemann surfaces

$$d^c = \frac{1}{2\pi\sqrt{-1}}\cdot\frac{1}{2}(\partial-\bar\partial) = \frac{1}{4\pi\sqrt{-1}}(\partial-\bar\partial)$$

on $U \subseteq \mathbb{C}$, $d^c f = \frac{1}{4\pi}\left(-\frac{\partial f}{\partial y}dx + \frac{\partial f}{\partial x}dy\right)$

polar coord: $d^c f = \frac{1}{4\pi}\left(\frac{\partial f}{\partial r} r\,d\theta - \frac{1}{r}\frac{\partial f}{\partial \theta}dr\right)$

$$dd^c = \frac{-1}{2\pi\sqrt{-1}}\partial\bar\partial = \frac{1}{2\pi\sqrt{-1}}\bar\partial\partial$$

<u>Lemma</u>

$dd^c \log(|z|^2) = \delta_0$ as currents

<u>pf</u>: Must show $\forall f \in C_0^\infty(\mathbb{C})$,

$$\int \log|z|^2 \cdot dd^c f = f(0)$$

$$= \lim_{\varepsilon\to 0^+}\int_{|z|\geq\varepsilon} \log|z|^2\, dd^c f = -\lim_{\varepsilon\to 0^+}\int_{|z|\geq\varepsilon} d\log|z|^2\cdot d^c f - \underbrace{\lim_{\varepsilon\to 0^+}\int_{|z|=\varepsilon} 2\log\varepsilon\, d^c f}_{\downarrow\ 0}$$

$$= -\lim_{\varepsilon\to 0^+}\int_{|z|\geq\varepsilon} d^c\log|z|^2\cdot df$$

Note: $\left(\frac{dz}{z}+\frac{d\bar z}{\bar z}\right)\wedge\left(\frac{\partial f}{\partial z}dz - \frac{\partial f}{\partial \bar z}d\bar z\right)$
$= -\left(\frac{dz}{z}-\frac{d\bar z}{\bar z}\right)\wedge\left(\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar z}d\bar z\right)$

i.e. $d\log|z|^2\wedge d^c f = -d^c\log|z|^2\wedge df$

$$= \lim_{\varepsilon\to 0^+}\int_{|z|=\varepsilon} \underbrace{d^c\log|z|^2}_{\parallel\ 2d\theta} f + \int_{|z|\geq\varepsilon} \underbrace{dd^c(\log|z|^2)}_{\parallel\ 0}\cdot f$$

$= f(0)$ <u>q.e.d.</u>

<u>Poincare-LeLong formula</u>

Let $\mathcal{L}$ be a holomorphic line bundle on a compact complex manifold X and let $\|\cdot\|$ be a hermitian metric on $\mathcal{L}$. Let s be a meromorphic section on X.

<u>Then</u> $$\underbrace{dd^c\left[-\log\|s\|^2\right]}_{\uparrow} + \underbrace{\delta_{div(s)}}_{\uparrow} = \left[\underbrace{c_1(L;\|\cdot\|)}_{\uparrow}\right]$$

The $(0,0)$-current on X represented by $-\log\|s\|^2$

The C^∞ $(1,1)$-form on X which coincides with $dd^c(-\log\|s\|^2)$ outside support of $div(s)$

The current in $D^{1,1}$
$$\underset{\substack{\uparrow\\ \text{smooth } (n-1,n-1) \text{ form on } X}}{\omega} \longmapsto \int_{div(s)}\omega$$

Note

$$(4\pi\sqrt{-1})\, d\alpha\wedge d^c\beta = (\partial\alpha+\bar\partial\alpha)(\partial\beta-\bar\partial\beta) = (\bar\partial\alpha\wedge\partial\beta - \partial\alpha\wedge\bar\partial\beta) + \partial\alpha\wedge\partial\beta - \bar\partial\alpha\wedge\bar\partial\beta$$

$$(4\pi\sqrt{-1})\, d^c\alpha\wedge d\beta = (\partial\alpha-\bar\partial\alpha)(\partial\beta+\bar\partial\beta) = (-\bar\partial\alpha\wedge\partial\beta + \partial\alpha\wedge\bar\partial\beta) + \partial\alpha\wedge\partial\beta - \bar\partial\alpha\wedge\bar\partial\beta$$

$\leadsto$ If α of type (p,p), β of type (q,q), $p+q=\dim(X)-1$, then $\partial\alpha\wedge\partial\beta=0$ $\because$ type $(\dim X+1, \dim X-1)$

$\bar\partial\alpha\wedge\bar\partial\beta=0$ $\because$ type $(\dim X-1, \dim X+1)$

and $d\alpha\wedge d^c\beta = d^c\alpha\wedge d\beta$

Def^n: X: connected complex manifold, $Z\subseteq X$ closed reduced, irred subvariety, $\operatorname{codim}(Z,X)=c$

$\delta_Z\in D^{c,c}$ fundamental class of Z. $\quad n=\dim X$

i.e. $\delta_Z : \mathcal{A}^{2n-2c}\longrightarrow \mathbb{C}$

$\alpha \longmapsto \int_Z \alpha$

A Green form for $Z\hookrightarrow X$ is a smooth $(c-1,c-1)$-form g_Z of logarithmic type on $X\setminus Z$ such that $\exists$ a smooth (c,c)-form ω on X with

$$dd^c[g_Z]+\delta_Z=[\omega]$$

After resolution of sing, $\tilde X\setminus\tilde Z \xrightarrow{\sim} X\setminus Z$, $\tilde Z$: divisor with normal crossing $\prod_{i=1}^{k} z_i=0$

Note: g_Z is of logarithmic type means: locally $g_Z=\sum_{i=1}^{r}\alpha_i\log|z_i|^2+\gamma$

$\gamma,\alpha_i : C^\infty$.

functoriality: $X\xrightarrow{f} Y$

(a) $D^c_{p,q}(X)\xrightarrow{f_*} D^c_{p+\dim(Y)-\dim(X),\, q+\dim(Y)-\dim(X)}(Y)$ dual to pull-back of smooth forms

(b) f is proper and surjective,

$\mathcal{D}^{p,q}(X)\xleftarrow{f^*}\mathcal{D}^{p,q}(Y)$ dual to $\int_f$

Poincaré-LeLong: $(X,\mathcal{L},\|\cdot\|)$ X: connected compact Riemann surface

s: meromorphic section of $\mathcal{L}$

$\leadsto$ $-\log\|s\|^2$ is a Green form of $\operatorname{div}(s)$,

and $\omega = dd^c(-\log\|s\|^2)$ is the curvature form of $\mathcal{L}$
$=\omega_{\mathcal{L}}$

$\int_X \omega = \deg(\mathcal{L})$, and $-\log\|s\|^2$ is a Green current

Green's functions on a compact Riemann surface w.r.t. to a volume form

φ: C^∞ positive real (1,1)-form on X

locally: $\varphi(z) = f(z)\sqrt{-1}\,dz\wedge d\bar{z}$, $f(z)$: strictly positive, real valued

normalized i.e. $\int_X \varphi = 1$

<u>Def^n</u>: A Green's function for a divisor D on X is a function $g_D : X \setminus \mathrm{supp}(D) \to \mathbb{R}$ s.t.

1) Locally, with $D = \mathrm{div}(f)$ ← merom. function on an open subset $U \subseteq X$

$$g_D = -\log \|f\|^2 + (\text{a } C^\infty \text{ function})$$

2) $dd^c g_D = (\deg D)\cdot\mu$ on $X \setminus \mathrm{supp}(D)$

$$\left(\rightsquigarrow \quad dd^c[g_D] + \delta_D = (\deg D)\cdot[\varphi] \right)$$

3) $\int_X g_D \cdot \varphi = 0$ ← normalization condition : (1) + (2) determine g_D up to additive const.)

<u>Lemma</u> $P \neq Q \in X$. g_P, g_Q : (normalized) Green's function for P and Q.

Then $g_P(Q) = g_Q(P) \iff \int_X g_P\, dd^c g_Q = \int_X g_Q\, dd^c g_P$

<u>Pf</u>: Will show: If g_P, g_Q satisfy 1), 2) above, then

$$\int_X g_P\, dd^c g_Q - g_Q\, dd^c g_P = g_P(Q) - g_Q(P)$$

$$\| $$

$$\int_X g_P\cdot\varphi - \int_X g_Q\,\varphi$$

$U_\varepsilon(P)$, $U_\varepsilon(Q)$ small nbds of P, Q

$$g_P\, dd^c g_Q - g_Q\, dd^c g_P = d(g_P\, d^c g_Q - g_Q\, d^c g_P) + \overbrace{(-dg_P \wedge d^c g_Q + dg_Q\, d^c g_P)}^{0}$$

$(4\pi\sqrt{-1})\, dg_P \wedge d^c g_Q = (\partial g_P + \bar\partial g_P)\wedge(\partial g_Q - \bar\partial g_Q)$

$(4\pi\sqrt{-1})\, dg_Q \wedge d^c g_P = (\partial g_Q + \bar\partial g_Q)\wedge(\partial g_P - \bar\partial g_P)$

$= \bar\partial g_Q \wedge \partial g_P + \bar\partial g_P \wedge \partial g_Q$

$$\rightsquigarrow \int_X g_P\, dd^c g_Q - \int_X g_Q\, dd^c g_P$$

$$= -\lim_{\varepsilon\to 0^+} \int_{\partial U_\varepsilon(P)} (\underbrace{g_P\, d^c g_Q}_{\int \to 0} - \underbrace{g_Q\, d^c g_P}_{\int \to g_Q(P)}) - \lim_{\varepsilon\to 0^+} \int_{\partial U_\varepsilon(Q)} \underbrace{g_P\, d^c g_Q}_{\int \to g_P(Q)} - \underbrace{g_Q\, d^c g_P}_{\int \to 0}$$

compact Riemann surface

Def: A function $g: X \times X \setminus \Delta_X \to \mathbb{R}$ is a Green's function for X if $\forall P \in X$, $Q \mapsto g(P,Q)$ satisfies condition 1), 2) of Green's function for the divisor P and a smooth volume form φ, and $g(P,Q) = g(Q,P)$.

1) Then $\exists$ a constant c s.t. $g+c$ is normalized, i.e. satisfies condition 3)

Lemma/Exer If X has a Green's function for a smooth volume for φ, then X has a Green's function for every smooth volume form φ'.

Example $X = \mathbb{P}^1(\mathbb{C})$

$$g(z,w) = -\log \frac{|z-w|^2}{(1+z\bar{z})(1+w\bar{w})}$$

$$\varphi(z) = \frac{\sqrt{-1}}{2\pi} \frac{dz \wedge d\bar{z}}{(1+z\bar{z})^2} \qquad \text{Exer: } \int_{\mathbb{P}^1(\mathbb{C})} \varphi(z) = 1$$

Remarks (a) $\dfrac{|z-w|}{(1+z\bar{z})^{\frac{1}{2}}(1+w\bar{w})^{\frac{1}{2}}}$ is the global chordal metric for S^2

and also $|1|_z$ ← section of $\mathcal{O}_{\mathbb{P}^1}(w)$

(b) $g(z,w)$ satisfies conditions 1), 2) but is not normalized:

$$0 < \int_{\mathbb{P}^1(\mathbb{C})} -\log\left(\frac{|z|^2}{1+z\bar{z}}\right) \varphi(z) < 1$$

The canonical Green's function

$g = g(X) \geq 1$

$\omega_1, \dots, \omega_g$ orthonormal:

$$\frac{\sqrt{-1}}{2} \int_X \omega_i \wedge \bar{\omega}_j = \delta_{ij}$$

$$\mu = \frac{\sqrt{-1}}{2g} \left(\omega_1 \wedge \bar{\omega}_1 + \dots + \omega_g \wedge \bar{\omega}_g\right)$$

is the canonical volume form

Define Laplacian w.r.t. a volume form φ by:

$$(\Delta_\varphi f) \cdot \varphi = -dd^c f$$

in local coord z,

$$-dd^c f = \frac{+1}{2\pi\sqrt{-1}} \cdot \frac{\partial^2 f}{\partial z \partial \bar{z}} dz \wedge d\bar{z}$$

$$= \frac{+1}{2\pi\sqrt{-1}} \frac{1}{4} \left(\frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right) f \cdot 2(-\sqrt{-1}\, dx \wedge dy)$$

$$= -\frac{1}{4\pi}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f \cdot dx \wedge dy$$

So: $\Delta f = -\frac{1}{4\pi}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$

Conclusion Δ_φ is positive

$$\Phi = pr_1^*\mu + pr_2^*\mu - \frac{\sqrt{-1}}{2}\sum_{j=1}^{g}\left(pr_1^*\omega_j \wedge pr_2^*\overline{\omega_j} + pr_2^*\omega_j \wedge pr_1^*\overline{\omega_j}\right)$$

Claim: $[\Phi] = \delta_{\Delta X}$, i.e. $\forall$ closed harmonic 2-form α on $X \times X$, have

$$\int_{X\times X} \Phi \wedge \alpha = \int_{\Delta X} \alpha$$

Check: For $\alpha = \frac{\sqrt{-1}}{2} pr_1^*\omega_i \wedge pr_2^*\omega_k$

$$\int_{X\times X} \Phi \wedge (pr_1^*\omega_i \wedge pr_2^*\overline{\omega_k}) = 0 \quad \text{if } i \neq k$$

$$\frac{\sqrt{-1}}{2}\int_{X\times X} \Phi \wedge \left(pr_1^*\omega_i \wedge pr_2^*\overline{\omega_i}\right)$$

$$= -\left(\frac{\sqrt{-1}}{2}\right)^2 \int_{X\times X} pr_2^*\omega_i \wedge pr_1^*\overline{\omega_i} \wedge pr_1^*\omega_i \wedge pr_2^*\overline{\omega_i}$$

$$= +\left(\frac{\sqrt{-1}}{2}\right)^2 \int_{X\times X} pr_1^*\omega_i \wedge pr_1^*\overline{\omega_i} \wedge pr_2^*\omega_i \wedge pr_2^*\overline{\omega_i} = 1$$

Thm: Let ρ be a hermitian metric on $\mathcal{O}_{X\times X}(\Delta_X)$ s.t. $c_1(\rho) = \Phi$ above

Let s be the global section of $\mathcal{O}_{X\times X}(\Delta_X)$ corresponding to the constant function 1 on $X\times X$, so that $\operatorname{div}(s) = \Delta_X$

Let $g = -\log|s|_\rho^2$. Then g satisfies conditions 1) and 2) of Green's function on $(X\times X)\setminus\Delta_X$.

(To see that g is symmetric: Let ι be the transposition on $X\times X$
$\rightsquigarrow$ $\iota^* s = \lambda s$ for some $\lambda \in \mathbb{C}^\times$ $\rightsquigarrow$ $\lambda^2 = 1$ i.e. $\lambda = \pm 1$
$\rightsquigarrow$ $\iota^* g = g$)

Def: A hermitian metric $\|\cdot\|$ on a line bundle $\mathcal{L}$ on a compact Riemann surface X ($g = g(X) \geq 1$) is admissible if $c(\mathcal{L}, \|\cdot\|) = \deg(\mathcal{L})\cdot\mu$ ← canonical volume form

Construction

$\exists!$ homomorphism

$$\Phi: \operatorname{Div}(X) \longrightarrow \left\{(\mathcal{L}, \|\cdot\|) \;\middle|\; \begin{array}{l}\mathcal{L}: \text{holomorphic line bundle on } X\\ \|\cdot\|: \text{admissible herm metric on } \mathcal{L}\end{array}\right\}$$

s.t. $X \ni Q \longmapsto (\mathcal{O}(Q), \|\cdot\|)$

s.t. $\|1\|(P) = \exp(-\frac{1}{2}g(P,Q))$ ← global section of $\mathcal{O}(Q)$

<u>def</u> Green current for $\overset{\text{codim}=p}{Y} \subseteq X$ ← Kähler

$$dd^c g + \delta_Y = [\omega]$$ ← smooth, in $A^{p-1,p-1}(X)$

<u>def</u> $*$-product of Green currents of log type along Y ← after resolution of sing.

g_Y : Green form for Y, of log type along Y

g_Z : arbitrary Green current for Z, not necessarily represented by a Green form of log type along Z

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\pi} & X \\ \cup & & \cup \\ E & \to & Y \end{array}$$

$\prod_{j=1}^{k} z_j = 0$ local eq^n for E

$g = \pi_*$ (a current β on $\tilde{X} \setminus E$)

s.t. $\beta = \sum_{j=1}^{k} \alpha_i \log |z_i|^2 + \gamma$

γ smooth

α_i: $\partial \alpha_i = 0 = \bar\partial \alpha_i \ \forall_i$ ← smooth

<u>Define</u>

$$[g_Y] * g_Z := [g_Y] \wedge \delta_Z + [\omega_Y] \wedge g_Z$$

<u>Thm</u>: 1) If Y and Z intersect properly, then g_Y is of log type along $X \setminus Y$

$$dd^c [g_Y] * g_Z = [\omega_Y \wedge \omega_Z] - \sum_i \mu_i \delta_{S_i}$$

$(Y \cdot Z) = \sum_i \mu_i S_i$

$\mu_i = \mu_{S_i}(Y, Z)$ Serre's intersection multiplicity

<u>i.e.</u> $[g_Y] * g_Z$ is a Green current of $Y \cap Z$

2) If g_Y is a Green form of Y of log type along Y and g_Z is a Green form of Z of log type along Z then $g_Y * g_Z \equiv g_Z * g_Y$ (mod $\operatorname{im}(\partial) + \operatorname{im}(\bar\partial)$)

Similarly, associativity holds modulo $\operatorname{im}(\partial) + \operatorname{im}(\bar\partial)$

intersection pairing

X/O_K K: nber field

X regular $X \to \operatorname{Spec}(O_K)$ proper

$$\widehat{CH}^p(X) = \hat{Z}^p(X) / \hat{R}^p(X)$$

<u>where</u>: $\hat{Z}^p(X) = \langle (Z, g_Z) : Z \in Z^p(X), g_Z : \text{a Green current for } Z \rangle$

$g_Z \in D^{p-1,p-1}(X)$ current

$$\subseteq Z^p(X) \oplus \bigoplus_{\sigma : K \hookrightarrow \mathbb{C}} \{T \in D^{p,p}(X_\sigma) : T \text{ real}, F_\infty^* T = (-1)^p T\}$$

$$\hat{R}^p(X) = \langle (\operatorname{div}(f), -[\log |f|^2]) : f \in \kappa(y)^\times, y \in X^{(p-1)} \rangle + \langle (0, \partial u + \bar\partial v) \rangle$$

u: type $(p-2, p-1)$ v: type $(p-1, p-2)$

intersection pairing:

$$\widehat{CH}^p(X) \times \widehat{CH}^q(X) \longrightarrow \widehat{CH}^{p+q}(X)_{\mathbb{Q}}$$

$[(Y, g_Y)] \cdot [(Z, g_Z)] := [([Y \cap Z]), g_Y * g_Z]$ if Y, Z intersect properly

If Y, Z do not intersect properly, use moving lemma:

$\exists \; y_i \in X_{\mathbb{Q}}^{(p-1)}, \; f_{y_i} \in \kappa(y_i)^\times, \; i = 1, \dots, m$ s.t.

$Y + \sum_{i=1}^{m} \mathrm{div}(y_i)$ intersects Z properly

$$\rightsquigarrow [(Y, g_Y)] \cdot [(Z, g_Z)] = \Big[(Y, g_Y) + \sum_{i=1}^{m} \overbrace{\widehat{\mathrm{div}}(y_i)}^{(\mathrm{div}(y_i), -[\log|f_i|^2])}\Big] \cdot \big[(Z, g_Z)\big]$$

Properties:

(a) $\left(\bigoplus_{p \geq 0} \widehat{CH}^p(X)_{\mathbb{Q}}, \text{ intersection product} \right)$ is a commutative graded $\mathbb{Q}$-algebra

(b)
$$\begin{aligned} \bigoplus_{p \geq 0} \widehat{CH}^p(X)_{\mathbb{Q}} &\longrightarrow \bigoplus_{p \geq 0} \left(CH^p(X) \oplus Z^{p,p}(X) \right) \\ (Y, g_Y) &\longmapsto [Y] + \underbrace{\omega_Y}_{= dd^c g_Y + \delta_Z} \end{aligned}$$

is a homomorphism of commutative graded $\mathbb{Q}$-algebras

functoriality.

$f: Y \longrightarrow X \rightsquigarrow f^*: \widehat{CH}^p(X)_{\mathbb{Q}} \longrightarrow \widehat{CH}^p(Y)_{\mathbb{Q}}$, ring homom

If $f_{\mathbb{Q}}: Y_{\mathbb{Q}} \longrightarrow X_{\mathbb{Q}}$ is smooth and f equidimensional

$\delta = \dim Y_{\mathbb{Q}} - \dim X_{\mathbb{Q}}$

$$\begin{aligned} \rightsquigarrow f_*: \widehat{CH}^p(X) &\longrightarrow \widehat{CH}^{p-\delta}(Y) \\ [(Z, g_Z)] &\longmapsto [(f_*(Z), f_* g_Z)] \end{aligned}$$

$(f \circ g)^* = g^* \circ f^*$

$(f \circ g)_* = f_* \circ g_*$

$f^*(\alpha \cdot \beta) = f^*(\alpha) \cdot f^*(\beta)$

$f_*(f^*(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta)$ projection formula

f: meromorphic function on X

Note: $\Phi((f)) \longmapsto \mathcal{O}((f))$

$\uparrow$ meromorphic section 1

$-\log \|1\|^2 = g(P,(f)) = \sum_{Q\in X}' \mathrm{ord}_Q(f)\cdot g(P,Q)$

$\mathcal{O}((f)) \xrightarrow{\sim} \mathcal{O}$

$1 \longmapsto f^{-1}\cdot 1$

$\|s\|^2 = \|f^{-1}\cdot 1\|^2_{\mathcal{O}}(P) = \|f(P)\|^{-2}$

$\int_X -\log\|s\|(P) = \int_X \log\|f(P)\|\,d\mu$

Assign, $\forall r\in\mathbb{R}$, $(0, r\cdot X)$ (formal product); multiply the usual metric on $\mathcal{O}$ by e^{-r}

Proposition $X/\mathcal{O}_K$ K: number field X: proper over $\mathcal{O}_K$, regular X_K = geom. irred curve

1) $\Phi: \mathrm{Div}(X) \oplus \sum_{v\in M_{K,\infty}} \mathbb{R}\cdot X_v \longrightarrow \{(\mathcal{L}, \|\cdot\|) \mid \mathcal{L}$: admissibly metrized line bundle on $X\}$

s.t. (i) $\forall$ horizonal divisor $D \longmapsto \mathcal{O}(D)$, s.t. $\forall v\in\Sigma_{K_\infty}$, the metric on $\mathcal{O}(D)_v$ is specified by the normalized Green's function as above, i.e.

$\|1\|_{\mathcal{O}(D),v}(P_v) = \exp\left(-\frac{1}{2} g_v(P_v, D_v)\right)$

(ii) $\forall v\in M_{K,\infty}$,

$r_v X_v \longmapsto \mathcal{O}_X$, and the metric at v is changed to (standard metric) $\cdot \exp(-r_v)$

2) $\forall$ meromorphic function $f\in K(X)_K^*$, define

$\widehat{(f)} = (f) + \sum_{v\in M_{K,\infty}} \int_{X(K_v)} -\log|f|_v \, d\mu_v$

($|f|_v$ defined by $K_v \cong \mathbb{C}$, standard abs. value on $\mathbb{C}$)

3) Φ induces an isomorphism

$\left(\mathrm{Div}(X) \oplus \sum_{v\in M_{K,\infty}} \mathbb{R}\cdot X_v\right) \Big/ \left(\widehat{f}\right)_{f\in K(X)^*} \xrightarrow{\sim} \{\text{admissibly metrized line bundles on } X\} / (\text{isom})$

Arakelov variety: $X_{/O_K}$ $\overset{X}{\text{regular}}$ $X \to \operatorname{Spec} O_K$ projective $+\ \omega_0$: Kähler metric $F_\infty^* \omega_0 = -\omega_0$

$\bar{X} = (X, \omega_0)$

$CH^p(X) = \{[(Z, g_Z)] :\ dd^c g_Z + \delta_Z \equiv$ a harmonic (p,p)-form (mod $\operatorname{Im}\partial + \operatorname{Im}\bar\partial$)$\}$ w.r.t. ω_0

mod $\operatorname{Im}\partial + \operatorname{Im}\bar\partial$
$+ \langle \widehat{\operatorname{div}}(f_y) \rangle$

$$= \langle Z^p(X) \oplus H^{p-1,p-1}(X) \rangle \Big/ \langle (\operatorname{div} f, -H[\log|f|^2]) \rangle$$

$\uparrow$ harmonic $(p-1, p-1)$ forms w.r.t. ω_0 on X

$f \in \kappa(y)^\times$, $y \in X^{(p-1)}$

H: $\underset{\text{on } D^{p,p}}{\operatorname{Ker}(dd^c)} \longrightarrow H^{p-1,p-1}(X)$

$\hat{c}_1 : \widehat{\operatorname{Pic}}(X) \xrightarrow{\sim} \widehat{CH}^1(X)$, $\hat{c}_1(\mathcal{L}, \|\cdot\|_{\mathcal{L}}) \longmapsto [(\operatorname{div}(s), -[\log\|s\|^2])]$

$\widehat{\operatorname{Pic}}(X)$ $\|$ isom. classes of $(\mathcal{L}, \|\cdot\|_{\mathcal{L}})$

s: meromorphic section of $\mathcal{L}$

Case $X = \mathbb{P}^n$, with $\omega_0 =$ Fubini-Study metric normalized so that $\frac{1}{n!}\int_{\mathbb{P}^n(\mathbb{C})} \omega_0^n = 1$

define: $\forall\ Z \subseteq \mathbb{P}^n_{/\bar{\mathbb{Q}}}$, $\operatorname{codim}(Z, \mathbb{P}^n) = p$

$$h(Z) \underset{\text{def}}{=} \left([(X, g_Z)] \cdot \hat{c}_1(\mathcal{O}(1), \|\cdot\|)^{p+1} \right) \quad \text{height of } Z$$

$\uparrow$ $dd^c g_Z + \delta_Z$ is harmonic $\quad$ $\uparrow$ Fubini-Study

Inductive def^n of height of subvarieties of $\mathbb{P}^n$

$Z \subseteq \mathbb{P}^n_{/K}$ closed subvariety. $f \in \Gamma^{\underline{n}}$, not vanishing identically on Z; $\mathcal{M}_1, \dots, \mathcal{M}_t \in \mathrm{Pic}(\underline{n})$, $t = \dim(Z)$

$\forall \sigma : K \hookrightarrow \mathbb{C}$, define $\kappa_\sigma = \kappa_\sigma(Z, f, \mathcal{M}_1, \dots, \mathcal{M}_t)$ by

$$\kappa_\sigma \stackrel{\text{def}}{=} -[K:\mathbb{Q}]^{-1} \int_{Z_\sigma(\mathbb{C})} \log|\sigma(f)| \; c_1(\mathcal{M}_1) \wedge \cdots \wedge c_1(\mathcal{M}_t)$$

$\forall v \in M_{K,\text{fin.}}$, let $Z_{/O_{\overline{\mathbb{Q}}}}$ = Zariski closure of Z in $\mathbb{P}^n$

$\rightsquigarrow \mathrm{div}_v(f|Z)$ and define $\kappa_v = \kappa_v(Z, f, \mathcal{M}_1, \dots, \mathcal{M}_t)$ by

$$\kappa_v = [K:\mathbb{Q}]^{-1} \log N_v \cdot (\mathrm{div}_v(f) \cdot \mathcal{M}_1 \cdots \mathcal{M}_t) \leftarrow \text{intersection product}$$

$\forall \mathcal{M}_0, \dots, \mathcal{M}_t \in \mathrm{Pic}(\underline{n})$, define

$$h(Z, \mathcal{M}_0, \dots, \mathcal{M}_t) = h(\mathrm{div}(f|_Z), \mathcal{M}_1, \dots, \mathcal{M}_t) + \sum_{v \in M_{K,\text{fin}}} \kappa_v(Z, f, \mathcal{M}_1, \dots, \mathcal{M}_t) + \sum_{\sigma : K \hookrightarrow \mathbb{C}} \kappa_\sigma(Z, f, \mathcal{M}_1, \dots, \mathcal{M}_t)$$

$\rightsquigarrow$ Get a well-defined function

$$h : \underset{\substack{\uparrow \\ \text{cycles of dim} = t \\ \text{in } \mathbb{P}^n_{/\overline{\mathbb{Q}}}}}{Z_t(\mathbb{P}^n_{/\overline{\mathbb{Q}}})} \times \mathrm{Pic}(\underline{n})^{t+1} \longrightarrow \mathbb{R}$$

, additive in $Z_t(\mathbb{P}^n_{/\overline{\mathbb{Q}}})$, and in $\mathcal{M}_0, \dots, \mathcal{M}_t$, symmetric in $\mathcal{M}_0, \dots, \mathcal{M}_t$

Define $h(Z) \stackrel{\text{def}}{=} h(Z, \mathcal{O}(1)^{t+1})$

Examples

(a) $\forall P \in \mathbb{P}^n(\overline{\mathbb{Q}})$, $h(P) = \log \underset{\substack{\uparrow \\ \text{standard ht, } L^2\text{-norm at } \infty}}{H.(P)}$

(b) $h(\mathbb{P}^n) = \frac{1}{2} \sum_{j=1}^{n} \frac{1}{j}$

(c) Z effective, and $\mathcal{M}_0, \dots, \mathcal{M}_t \in \mathrm{Pic}^+(\underline{n})$, then

$$h(Z, \mathcal{M}_0, \dots, \mathcal{M}_t) \geq 0$$

(d) $f \in \Gamma(\mathcal{M}_0)$, not vanishing identically on Z,

then $h(\mathrm{div}(f|_Z), \mathcal{M}_1, \dots, \mathcal{M}_t) \leq h(Z, \mathcal{M}_0, \dots, \mathcal{M}_t) + \log H(f) \cdot \underbrace{(Z \cdot \mathcal{M}_1 \cdots \mathcal{M}_t)}_{\text{usual intersection}}$

$\Phi: \widehat{CH}^1(X) \xrightarrow{\sim} \{(\mathcal{L}, \|\cdot\|): \mathcal{L}: \text{invertible } \mathcal{O}_X\text{-module}\}/(\text{isom}) =: \widehat{Pic}(X)$

The inverse of Φ is given by:

Given $(\mathcal{L}, \|\cdot\|)$, pick a meromorphic section s of $\mathcal{L}$,

$(\mathcal{L}, \|\cdot\|) \longleftrightarrow [(\mathrm{div}(s), -[\log\|s\|^2])]$

$n = \dim(X_K)$ $\qquad \pi: X \to \mathrm{Spec}(\mathcal{O}_K)$

$$\widehat{CH}^{n+1}(X) \xrightarrow{\pi_*} \widehat{CH}^1(X) \xrightarrow{\sim} \widehat{Pic}^1(\mathrm{Spec}(\mathcal{O}_K)) \xrightarrow{\deg} \mathbb{R}$$

$$(Z, g) \longmapsto (\pi_* Z, (f_\sigma))_{\sigma: K \hookrightarrow \mathbb{C}} \longmapsto [K:\mathbb{Q}]^{-1}\left(\sum_v n_v \log(q_v) + \frac{1}{2}\sum_\sigma f_\sigma\right)$$

Z: cycle, vertical, dim $= 0$; g: (n,n)-current on $X(\mathbb{C})$; $\pi_* Z = \sum_{v \in M_{K,fin}} a_v \cdot v$

Def

Faltings Case $X = \mathbb{P}^n$ (over $\mathcal{O}_K$, say) $\qquad$ No problem for vertical cycles

Let $Z_K \subseteq \mathbb{P}^n_K$ be a geometrically irred. subvariety, $\mathrm{codim}(Z, \mathbb{P}^n) = p$

$\rightsquigarrow$ Define $\hat{Z} = (Z, g_Z)$ where g_Z is determined by: $\qquad h = c_1(\mathbb{P}^n, \mathcal{O}(1), \text{std metric})$

Z = Zar. closure of Z_K in $\mathbb{P}^n$; $g_Z = g$

(i) $dd^c g_Z + \delta_Z = \deg(Z_K) \cdot h^p$

(ii) $\int_{\mathbb{P}^n(\mathbb{C})} g_Z\, \omega = 0 \quad \forall$ harmonic form ω

<u>So</u> g is uniquely determined (mod $\mathrm{Im}(\partial) + \mathrm{Im}(\bar\partial)$)

Lemma: If $s \in \Gamma(\mathbb{P}^n, \mathcal{O}(d))$ and $s|_Z$ is not identically 0,

then
$$\hat{c}_1(\mathcal{O}(d), \text{std metric}) \cdot \hat{Z} = \mathrm{div}(s|_Z) - \left(0, h^p \cdot \int_{Z(\mathbb{C})} \log\|s\|^2 h^{n-p}\right)$$
$p = \mathrm{codim}(Z, X)$

<u>Note</u>: Falting's convention

$$\widehat{CH}^p(X) \ni (Z, g)$$

$$\delta_X + dd^c(2g) = [\omega] \quad C^\infty \qquad d^c = \frac{1}{4\pi\sqrt{-1}}(\partial - \bar\partial)$$

$$dd^c(2g) = -\frac{1}{\pi\sqrt{-1}}\partial\bar\partial g, \text{ i.e. } \delta_X + dd^c(2g) = \delta_X - \frac{1}{\pi\sqrt{-1}}\partial\bar\partial g$$

(sign error in Faltings' paper)

Falting's def^n of height continued

Def: Let Z be an effective cycle in $\mathbb{P}^n$ purely of codim p.

Define $h(Z) = \deg\left(\hat{Z} \cdot c_1(\mathcal{O}(1)_{\mathbb{P}^n}, \|\cdot\|)^{n-p+1}\right) \in \mathbb{R}$

where $\deg : \widehat{CH}^{n+1}(\mathbb{P}^n) \longrightarrow \mathbb{R}$

$$\underset{\substack{\uparrow \\ \text{effective}}}{(Z,\ g)} \longmapsto \underbrace{\log(\#\Gamma(Z,\mathcal{O}_Z))}_{\substack{\uparrow \\ \deg(Z)}} + \int_{\mathbb{P}^n(\mathbb{C})} g$$

Falting's convention

Similarly, define $h(X)$ for X = effective cycles on $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$

Inductive definition of heights of effective cycles on $\mathbb{P}^n$

property

$$h : \underset{\substack{\uparrow \\ \text{cycles } \dim = t}}{Z_t(\mathbb{P}^n_{\overline{\mathbb{Q}}})} \times \mathrm{Pic}(\Omega)^{t+1} \longrightarrow \mathbb{R}$$

mult-additive $h(Z, \mathcal{M}_0, \ldots, \mathcal{M}_t)$

symmetric in $\mathcal{M}_0, \ldots, \mathcal{M}_t$

$\mathcal{M}_i = \mathcal{L}(\underline{d}_i)$ $i = 0, 1, \ldots, t$

Z : irreducible$/K$ $\quad s \in \Gamma(\mathcal{M}_0)$, $s|_Z \neq 0$

Then :

$$h(Z, \mathcal{M}_0, \ldots, \mathcal{M}_t) = h(\mathrm{div}(s|_Z), \mathcal{M}_1, \ldots, \mathcal{M}_t) \qquad \forall t \geq 0$$

$$+ \sum_{v \in M_{K,\mathrm{fin}}} \kappa_v(Z, s, \mathcal{M}_1, \ldots, \mathcal{M}_t)$$

$$+ \sum_{\sigma : K \hookrightarrow \mathbb{C}} \kappa_\sigma(Z, s, \mathcal{M}_1, \ldots, \mathcal{M}_t),$$

where $\kappa_v(Z, s, \mathcal{M}_1, \ldots, \mathcal{M}_t) = [K:\mathbb{Q}]^{-1} \log(q_v) \cdot (\underset{\substack{\uparrow \\ \text{vertical cycle for } v}}{\mathrm{div}_v(s|Z)} \cdot \mathcal{M}_1 \cdots \mathcal{M}_t)$

usual intersection number

and

$$\kappa_\sigma(Z, s, \mathcal{M}_1, \ldots, \mathcal{M}_t)$$

$$= -[K:\mathbb{Q}]^{-1} \int_{Z_\sigma(\mathbb{C})} \log\|{}^\sigma s\| \cdot c_1(\mathcal{M}_1) \wedge \cdots \wedge c_t(\mathcal{M}_t)$$

Remark: When $t = 0$, $Q \in \mathbb{P}^n(\overline{\mathbb{Q}})$

$$h(Q, \mathcal{M}_0) = \deg\left(Q^*(\mathcal{M}_0, \|\cdot\|)\right)$$

Lemma (1) $Q \in \mathbb{P}^n(\overline{\mathbb{Q}}) \rightsquigarrow h(Q, \mathcal{O}(1)_{\mathbb{P}^n}) = \log H(P)$ ← L^2-norm

(2) $h(\mathbb{P}^n) = \frac{1}{2} \sum_{j=1}^{n} \sum_{\ell=1}^{j} \frac{1}{\ell}$

Key computation

$$-\int_{\mathbb{P}^n(\mathbb{C})} \log \frac{|z_0|}{(|z_0|^2 + \cdots + |z_n|^2)^{1/2}} \, c_1(\mathcal{O}(1))^n = \frac{1}{2} \sum_{\ell=1}^{n} \frac{1}{\ell}$$

where $c_1(\mathcal{O}_1) = \frac{\sqrt{-1}}{2\pi} \left[\frac{dw_1 \wedge d\bar{w}_1 + \cdots + dw_n \wedge d\bar{w}_n}{1 + w_1\bar{w}_1 + \cdots + w_n\bar{w}_n} - \frac{(\sum_i \bar{w}_i dw_i) \wedge (\sum_i w_i d\bar{w}_i)}{(1 + w_1\bar{w}_1 + \cdots + w_n\bar{w}_n)^2} \right]$

(3) $Z \in Z_t(\mathbb{P}^n)$ effective, and $\mathcal{M}_0, \ldots, \mathcal{M}_t \in \mathrm{Pic}^+(\underline{n})$, then

$h(Z, \mathcal{M}_0, \ldots, \mathcal{M}_t) \geq 0$

(4) $Z \in Z_t(\mathbb{P}^n)$ effective, geom. irred. $\mathcal{M}_0, \ldots, \mathcal{M}_t \in \mathrm{Pic}^+(\underline{n})$, $s \in \Gamma(\mathcal{M}_0)$, $s \longleftrightarrow$ multi-homog. poly f not vanishing on any irred. component of Z

Then $h(\mathrm{div}(s|_Z), \mathcal{M}_1, \ldots, \mathcal{M}_t)$

$$\leq h(Z, \mathcal{M}_0, \ldots, \mathcal{M}_t) + h(f) \, (Z \cdot \mathcal{M}_1 \cdots \mathcal{M}_t)$$

↑ usual intersection number

Pf (4) by the inductive definition

(4) ⇒ (3): Apply (4) with s attached to a monomial f.

(hence $h(f) = 0$

Lemma 1 $Z = Z_1 \times \cdots \times Z_m \subseteq \mathbb{P}^{n_1}_{\overline{\mathbb{Q}}} \times \cdots \times \mathbb{P}^{n_m}_{\overline{\mathbb{Q}}}$ $\quad Z_h \in Z_{\delta_h}(\mathbb{P}^{n_h}_{\overline{\mathbb{Q}}})$ $\forall_{h=1,\ldots,m}$

(i) $\hat{Z} \cdot \hat{c}_1(\hat{\mathcal{L}}_1^{e_1}) \cdots \hat{c}_m(\hat{\mathcal{L}}_m^{e_m}) = \begin{cases} 0 & \text{if } (e_1, \ldots, e_m) \neq (\delta_1, \ldots, \delta_{h-1}, \delta_h + 1, \delta_{h+1}, \ldots, \delta_m) \ \forall_h \\ h(Z_h) \cdot \prod_{j \neq h} \deg(Z_j) \end{cases}$

$\mathcal{L}_i = \mathrm{pr}_i^* \mathcal{O}_{\mathbb{P}^{n_i}}(1)$ $\quad e_1 + \cdots + e_m = \delta_1 + \cdots + \delta_m + 1$

(ii) $\hat{Z} \cdot c_1(\hat{\mathcal{L}}_1(d_1) \otimes \cdots \otimes \hat{\mathcal{L}}_m(d_m))^{\delta_1 + \cdots + \delta_m + 1}$

$$= \left(\prod_{j=1}^{m} d_j^{\delta_j} \right) \cdot \sum_{h=1}^{m} \left[\frac{(\delta_1 + \cdots + \delta_m + 1)!}{\delta_1! \cdots \delta_{h-1}! \, (\delta_h + 1)! \, \delta_{h+1}! \cdots \delta_m!} \cdot d_h \cdot h(Z_h) \cdot \prod_{j \neq h} \deg(Z_j) \right]$$

(ii) follows from (i)

Z_h: reduced, irred $/\overline{\mathbb{Q}}$

Lemma 2 $Z = Z_1 \times \cdots \times Z_m \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ $\quad \mathcal{L}_h = pr_h^* \mathcal{O}_{\mathbb{P}^{n_h}}(1)$ $\quad \dim(Z_h) = \delta_h$ $\quad \forall h=1,\dots,m$

(i) $$(Z \cdot \mathcal{L}_1^{e_1} \cdots \mathcal{L}_m^{e_m}) = \begin{cases} \deg(Z_1)\cdots\deg(Z_h) & \text{if } \delta_h = e_h \ \forall h \\ 0 & \text{otherwise} \end{cases}$$
$e_1 + \cdots + e_m = \delta_1 + \cdots + \delta_h$

(ii) $\mathcal{L}(\underline{d}) \overset{\text{def}}{=} \mathcal{O}_{\mathbb{P}^{\underline{n}}}(\underline{d}) = \mathcal{L}_1^{\otimes d_1} \otimes \cdots \otimes \mathcal{L}_m^{\otimes d_m}$ then
$$\left(Z \cdot \mathcal{L}(\underline{d})^{\delta_1 + \cdots + \delta_h}\right) = \frac{(\delta_1 + \cdots + \delta_h)!}{\delta_1! \cdots \delta_h!} d_1^{\delta_1} \cdots d_m^{\delta_m}$$

In particular, for $Z = \mathbb{P}^{\underline{n}}$, we get
$$\mathcal{L}(\underline{d})^{n_1 + \cdots + n_m} = \frac{(n_1 + \cdots + n_m)!}{n_1! \cdots n_m!} d_1^{n_1} \cdots d_m^{n_m}$$

Lemma 3 $Z \subseteq \mathbb{P}^{\underline{n}}_{\overline{\mathbb{Q}}}$, reduced irred. Assume Z is not a product (of the form $Z = Z_1 \times \cdots \times Z_m$)

Then $\exists\ (e_1, \dots, e_m) \neq (e_1', \dots, e_m') \in \mathbb{N}^m$ s.t. $e_1 + \cdots + e_m = \dim(Z) = e_1' + \cdots + e_m'$

$$(Z \cdot \mathcal{L}_1^{e_1} \cdots \mathcal{L}_m^{e_m}) > 0, \quad (Z \cdot \mathcal{L}_1^{e_1'} \cdots \mathcal{L}_m^{e_m'}) > 0$$

Pf: Let $pr_h : \mathbb{P}^{\underline{n}} \to \mathbb{P}^{n_h}$ $\quad Z_h := pr_h(Z) \subseteq \mathbb{P}^{n_h}$, $\delta_h = \dim(Z_h)$

$\rightsquigarrow \quad Z \subsetneqq Z_1 \times \cdots \times Z_h$

Will prove by induction on m the following stronger statement (than Lemma 2)

(*) $\forall h = 1, \dots, m \quad \exists\ (e_1, \dots, e_m) \in \mathbb{N}^m$ with $e_h = \delta_h$ and $e_1 + \cdots + e_m = \dim(Z)$
s.t. $(Z \cdot \mathcal{L}_1^{e_1} \cdots \mathcal{L}_m^{e_m}) > 0$

May assume: $h = 1$

Assume: (*) holds for $m = r-1$, $r \geq 2$. Will show (*) holds for $m = r$

(double) Induction on δ_1. Case $\delta_1 = 0$ holds because it exactly the induction hypo. for $m = r-1$.

So: $\delta_1 \geq 1$. Choose $s_1 \in \Gamma(\mathbb{P}^{n_1}, \mathcal{O}_{\mathbb{P}^{n_1}}(1))$ s.t. $s_1|_{pr_1(Z)} \neq 0$.

Let $Z_1 = Z \cdot \operatorname{div}(pr_1^*(s_1))$

Induction: $\exists\ e_2, \dots, e_m \in \mathbb{N}$ with $\delta_1 + e_2 + \cdots + e_m = \dim(Z)$ s.t.

$$\left(Z_1 \cdot \mathcal{L}_1^{\delta_1 - 1} \mathcal{L}_2^{e_2} \cdots \mathcal{L}_m^{e_m}\right) > 0$$
$$\|$$
$$\left(Z \cdot \mathcal{L}_1^{\delta_1} \cdot \mathcal{L}_2^{e_2} \cdots \mathcal{L}_m^{e_m}\right).$$

This shows that (*) holds for $m = r$.

QED

<u>Lemma 4</u> Let $A \subseteq \Gamma_{\overline{\mathbb{Q}}}^{\underline{n}}(\underline{d})\setminus\{0\}$ be a finite subset of nonzero multi-homog. poly.
Let $X \subseteq \mathbb{P}^{\underline{n}}$ be the closed subscheme $\mathrm{Proj}\left(\Gamma_{\overline{\mathbb{Q}}}^{\underline{n}} / A\Gamma_{\overline{\mathbb{Q}}}^{\underline{n}}\right)$ defined by A.
Fix $t \in \mathbb{N}$. $Z_1, \dots, Z_r$ = irred components of X of codim $= t$
(but t may not be equal to $\underbrace{\sum_{h=1}^{m} n_h}_{= M} - \dim(X)$)
Let m_{Z_i} = length$_{\mathcal{O}_{X,\eta_{Z_i}}} \mathcal{O}_{X,\eta_{Z_i}}$ = multiplicity of Z_i in X

Artinian local ring

Note: None of $Z_1, \dots, Z_r$ is an embedded component of X

Then
$$\sum_{i=1}^{r} m_{Z_i} \left(Z_i \cdot \mathcal{L}_1^{e_1} \cdots \mathcal{L}_m^{e_m} \right) \leq \left(\mathcal{L}_1^{e_1} \cdots \mathcal{L}_m^{e_m} \cdot \mathcal{L}(\underline{d})^t \right)$$
$$\forall (e_1, \dots, e_m) \in \mathbb{N}^m \text{ with } e_1 + \dots + e_m = M - t = n_1 + \dots + n_m - t$$

<u>proof</u>: Pick $f_1, \dots, f_t \in \langle A \rangle_{\overline{\mathbb{Q}}} = \overline{\mathbb{Q}}$-linear span of A
such that each Z_i is an irreducible component of $V(f_1, \dots, f_t) = X'$.
$\rightsquigarrow$ May assume $X = X'$. Prove this statement by induction on $t = \mathrm{codim}(Z_i, \mathbb{P}^{\underline{n}})$
Let $Y = V(f_1, \dots, f_{t-1})$. $Y_1, \dots, Y_s$ = irreducible components of Y of codim $t-1$
$n_j :=$ multiplicity of Y_j in Y, $j = 1, \dots, s$

Let b_{ji} = multiplicity of Z_i in $Y_j \cap V(f_t)$

$\rightsquigarrow m_{Z_i} = \sum_{j=1}^{s} n_j b_{ji}$ ($\because$ $f_1, \dots, f_t$ gives a regular sequence in $\mathcal{O}_{X,\eta_{Z_i}}$).

<u>Now</u>:
$$\sum_{i=1}^{r} m_{Z_i} \left(\mathcal{L}_1^{e_1} \cdots \mathcal{L}_m^{e_m} \cdot Z_i \right) = \sum_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}} n_j b_{ji} \left(\mathcal{L}_1^{e_1} \cdots \mathcal{L}_m^{e_m} Z_i \right)$$
$$= \sum_{j=1}^{s} b_{ji} \left(\mathcal{L}_1^{e_1} \cdots \mathcal{L}_m^{e_m} V(f_t) \cdot Y_j \right) = \sum_{j=1}^{s} b_{ji} \left(\mathcal{L}_1^{e_1} \cdots \mathcal{L}_m^{e_m} \cdot \mathcal{L}(\underline{d}) \cdot \underset{\text{codim}=t-1}{Y_j} \right)$$
$$\underset{\text{induction on } t}{\leq} \left(\mathcal{L}_1^{e_1} \cdots \mathcal{L}_m^{e_m} \mathcal{L}(\underline{d}) \cdot \mathcal{L}(\underline{d})^{t-1} \right)$$

<u>QED</u>

Key refinement in the construction of $f_0, \dots, f_{t-1}$

- Choose f_0 in A. Let $C_1 = \mathrm{div}^X(f_0) = \sum_{\substack{W:\text{ irred div. of } \mathbb{P}^n \\ W \not\subseteq X}} \mathrm{ord}_W(f)\, W$

Notation: For an irred. sub. V in $\mathbb{P}^n$ and $f \in \Gamma^n_{\overline{\mathbb{Q}}}$, $f|_V \neq 0$

define $$\mathrm{div}^X(f|_V) = \sum_{\substack{W:\text{ irred. component of } \mathrm{div}(f|_V) \\ \text{s.t. } W \not\subseteq X}} \mathrm{ord}_W(f|_V)\cdot W$$

By linearity, for a cycle $V = \sum_i a_i V_i$ of a fixed dimension, and $f \in \Gamma^n_{\overline{\mathbb{Q}}}$ st $f|_{V_i} \neq 0$ $\forall i$

Define $$\mathrm{div}^X(f|_V) = \sum_i a_i\, \mathrm{div}^X(f|_{V_i})$$

- Define $C_2, \dots, C_{t-1}$ inductively, s.t.

$$C_i = \mathrm{div}^X(f_{i-1}|_{C_{i-1}}),$$ where $f_{i-1} \in \langle A \rangle_{\overline{\mathbb{Q}}}$ s.t. $f_{i-1}|_{\text{any irred. component of } C_{i-1}} \neq 0$

- $C_t = \mathrm{div}(f_{t-1}|_{C_{t-1}})$, where $f_{t-1} \in \langle A \rangle_{\overline{\mathbb{Q}}}$ s.t. $f_{t-1}|_{\text{any irred. component of } C_{t-1}} \not\equiv 0$
not identically zero

Then: $$\sum_{i=1}^{r} m_{Z_i}\left(Z_i \cdot \mathcal{L}_1^{e_1} \cdots \mathcal{L}_m^{e_m}\right)$$
$$\leq \left(C_t \cdot \mathcal{L}_1^{e_1} \cdots \mathcal{L}_m^{e_m}\right) \leq \left(C_{t-1} \cdot \mathcal{L}(\underline{d}) \cdot \mathcal{L}_1^{e_1} \cdots \mathcal{L}_m^{e_m}\right)$$
$$\leq \left(C_t \cdot \mathcal{L}(\underline{d})^2 \mathcal{L}_1^{e_1} \cdots \mathcal{L}_m^{e_m}\right) \leq \cdots \leq \left(\mathcal{L}(\underline{d})^t \cdot \mathcal{L}_1^{e_1} \cdots \mathcal{L}_m^{e_m}\right)$$

Note

In the construction above,

$$C_{i-1} \cdot \mathcal{L}(\underline{d}) = \mathrm{div}^X(f_i|_{C_{i-1}}) + (\text{effective cycle})$$

Benefit of this explicit construction:

<u>Lemma 5</u> May choose each of $f_0, f_1, \ldots, f_{t-1}$ to be a $\mathbb{Z}$-linear combination of elements of A with ~~bounded~~ coefficients and bounded number of terms

$$f_i = \sum_{j=1}^{C} b_{ij}\, g_{ij} \quad , \quad g_{ij} \in A$$

(*) $\qquad b_{ij} \in \mathbb{Z}$ with $|b_{ij}| \leq C \;\; \forall i, j$

where $C = \dfrac{(n_1 + \cdots + n_m)!}{n_1! \cdots n_m!}\, d_1^{n_1} \cdots d_m^{n_m}$

Pf: Will choose $f_0, f_1, \ldots, f_{t-1}$ and define $\qquad M = n_1 + \cdots + n_m$

$C_i = \operatorname{div}^X(f_{i-1}|_{C_{i-1}})$ for $i = 1, \ldots, t-2$

and $C_t = \operatorname{div}^X(f_{t-1}|_{C_{t-1}})$

The requirement is: f_i does not vanish identically on each irred. component of the effective cycle C_i for $i = 0, 1, \ldots, t-1$

Have $\left(C_i \cdot \mathcal{L}(\underline{d})^{M-i}\right) \leq \left(C_{i-1} \cdot \mathcal{L}(\underline{d})^{M-i+1}\right) \leq \cdots \leq \mathcal{L}(\underline{d})^M = C = \dfrac{(n_1 + \cdots + n_m)!}{n_1! \cdots n_m!} \prod_{h=1}^{m} d_h^{n_h}$ (of dim $M-i$)

$\rightsquigarrow$ Each C_i has $\leq C$ irred. components

Need to construct $f_0, \ldots, f_{t-1}$ such that each f_j satisfies the additional property (*)

Do this inductively. Suppose we have constructed $f_0, \ldots, f_s$ $\quad s \leq t \leq 2$ satisfying (*). Need to construct f_{s+1} not vanishing identically on each irred. component of C_s and f_{s+1} satisfies (*)

Let $V_1, \ldots, V_r$ be the irreducible components of C_s, so that $r \leq C$

Pick $g_1, \ldots, g_r \in A$ s.t. $g_i|_{V_i} \not\equiv 0 \;\; \forall i = 1, \ldots, r$. Pick $x_i \in V_i(\overline{\mathbb{Q}})$ s.t. $g_i(x_i) \neq 0$

Will construct $h_1, \ldots, h_r \in \mathbb{Z}$-span of A s.t. $h_i|_{V_1} \not\equiv 0, \ldots, h_i|_{V_i} \not\equiv 0$.

and $\quad h_j = \sum_{i=1}^{j} k_i\, g_i \qquad k_i \in [-\lceil \tfrac{i}{2} \rceil, \lceil \tfrac{i}{2} \rceil] \cap \mathbb{Z} \qquad g_i \in A$

and $\quad h_j$ does not vanish identically on $V_1, \ldots, V_j$

Let $h_1 = g_1$. Suppose have constructed $h_1, \ldots, h_{j-1}$.

Will pick $a \in [-\lceil \tfrac{j}{2} \rceil, \lceil \tfrac{j}{2} \rceil] \cap \mathbb{Z}$ s.t. $(h_{j-1} + a\, g_j)(x_i) \neq 0$

(need to avoid at most j values of a) for $i = 1, \ldots, j$

Then let $h_j = h_{j-1} + a\, g_j$ $\qquad$ QED.

Theorem 1. $k \supseteq \mathbb{Q}$, alg. closed. $0 \neq F \in \Gamma_k^{\underline{n}}$ $\sigma \geq 0$, $0 < \varepsilon \leq 1$
$\underline{d} = (d_1, \dots, d_m)$ $M := n_1 + \cdots + n_m$

Suppose (i) $\frac{d_h}{d_{h+1}} \geq \left(\frac{mM}{\varepsilon}\right)^M$ $\forall h = 1, \dots, m$

(ii) Z_σ and $Z_{\sigma+\varepsilon}$ have a common irreducible component Z

Then (a) $Z = Z_1 \times \cdots \times Z_m$ $Z_h \subseteq \mathbb{P}^{n_h}$ $Z_\sigma = Z_\sigma(F, \underline{d})$
$= V(\partial_{\underline{i}} F \mid (\underline{i}/\underline{d}) \leq \sigma)$

(b) If $F \in \Gamma_{k_0}^{\underline{n}}$, $k_0 \leq k$ subfield.
then $Z_1, \dots, Z_m$ are defined over an ext^n field k_1 of k_0
with $[k_1 : k_0] \cdot \prod_{k=1}^{m} \deg(Z_k) \leq \left(\frac{ms}{\varepsilon}\right)^s$, where $s = \sum_{h=1}^{m} \operatorname{codim}(Z_h \subseteq \mathbb{P}^{n_h})$
$= \operatorname{codim}(Z, \mathbb{P}^{\underline{n}})$

Key Lemma 6: Z_σ and $Z_{\sigma+\varepsilon}$ have a common irred. component Z,
$s = \operatorname{codim}(Z, \mathbb{P}^{\underline{n}})$

Then $m_Z \overset{\text{def}}{=}$ multiplicity of Z in $Z_\sigma \geq \left(\frac{\varepsilon}{s}\right)^s d_1^{n_1 - \delta_1} \cdots d_m^{n_m - \delta_m} - 1$

where $\delta_h = \dim(p_h(Z)) - \dim(p_{h+1}(Z))$, $p_h : \mathbb{P}^{\underline{n}} \longrightarrow \mathbb{P}^{n_h} \times \mathbb{P}^{n_{h+1}} \times \cdots \times \mathbb{P}^{n_m}$

$$\left(\begin{array}{ll} \triangle\ \underset{\substack{\| \\ (n_1 + \cdots + n_m) - s}}{M - s} = \delta_1 + \cdots + \delta_m & \text{i.e. } \sum_{h=1}^{m} (n_h - \delta_h) = s \end{array} \right)$$

Admit Lemma 6.

Proof of Thm 1:

Claim For any $(e_1, \dots, e_m) \in \mathbb{N}^m$ with $e_1 + \cdots + e_m = M - s$
and $(Z \cdot \mathcal{L}_1^{e_1} \cdots \mathcal{L}_m^{e_m}) > 0$, have

$\eta_i - \eta_{i+1} = \delta_i - e_i$

$\eta_i \overset{\text{def}}{=} (\delta_i + \delta_{i+1} + \cdots + \delta_m) - (e_i + e_{i+1} + \cdots + e_m) > 0$, $\forall i = 2, \dots, m$

$\because$ $\delta_i + \cdots + \delta_m = \dim(p_i(Z))$, apply the projection formula

$$[k_1 : k] \cdot (Z \cdot \mathcal{L}_1^{e_1} \cdots \mathcal{L}_m^{e_m}) \leq m_Z^{-1} \cdot (\mathcal{L}(\underline{d})^s \cdot \mathcal{L}_1^{e_1} \cdots \mathcal{L}_m^{e_m}) \quad \text{by Lemma 4}$$
$$= m_Z^{-1} \cdot \left((d_1 \mathcal{L}_1 + \cdots + d_m \mathcal{L}_m)^s \cdot \mathcal{L}_1^{e_1} \cdots \mathcal{L}_m^{e_m}\right)$$
$$= m_Z^{-1} \cdot \frac{s!}{(n_1 - e_1)! \cdots (n_m - e_m)!} d_1^{n_1 - e_1} \cdots d_m^{n_m - e_m} \qquad (n_1 - e_1) + \cdots + (n_m - e_m) = s$$
$$\leq m_Z^{-1} \cdot m^s \cdot d_1^{n_1 - e_1} \cdots d_m^{n_m - e_m}$$
$$\leq \left(\frac{s}{\varepsilon}\right)^s \cdot m^s \cdot d_1^{\delta_1 - n_1} \cdots d_m^{\delta_m - n_m} \cdot d_1^{n_1 - e_1} \cdots d_m^{n_m - e_m} \qquad \text{Lemma 6}$$
$$= \left(\frac{ms}{\varepsilon}\right)^s \cdot \left(\frac{d_2}{d_1}\right)^{\eta_2} \cdots \left(\frac{d_m}{d_{m+1}}\right)^{\eta_m}$$

(proof of Thm 1 continued)

Suppose that Z is not a product.

$\overset{\text{Lemma 3}}{\rightsquigarrow}$ $\exists\, (e_1,\dots,e_m) \neq (e_1',\dots,e_m')$ with $e_1+\dots+e_m = M-s = e_1'+\dots+e_m'$

s.t. $(Z\cdot \mathcal{L}_1^{e_1}\cdots\mathcal{L}_m^{e_m})>0$ and $(Z\cdot \mathcal{L}_1^{e_1'}\cdots\mathcal{L}_m^{e_m'})>0$

So at least one of them, say $(e_1,\dots,e_m)$, is not $(\delta_1,\dots,\delta_m)$

So: $\exists h$ s.t. $\eta_h := (\delta_h+\dots+\delta_m)-(e_h+\dots+e_m)>0$

The inequality established before gives:

$$[k_1:k]\cdot(Z\cdot\mathcal{L}_1^{e_1}\cdots\mathcal{L}_1^{e_m}) \leq \left(\frac{ms}{\varepsilon}\right)^s \left(\frac{mM}{\varepsilon}\right)^{-M(\eta_2+\dots+\eta_m)} \leq \left(\frac{ms}{\varepsilon}\right)^s\left(\frac{mM}{\varepsilon}\right)^{-M}$$

$$< 1 \quad (\because s<M)$$

Contradiction.

Have shown: $Z = Z_1\times\dots\times Z_m$

$\rightsquigarrow$ $[k_1:k]\cdot(Z\cdot\mathcal{L}_1^{e_1}\cdots\mathcal{L}_m^{e_m}) = [k_1:k]\prod_{h=1}^{m}\deg(Z_h) \leq \left(\frac{ms}{\varepsilon}\right)^s$

inequality on the previous page

QED

(Have proved Thm 1, but still need to prove Lemma 6. lower bound of the multiplicity m_Z of Z in Z_σ)

<u>Lemma 7</u> Given $\underline{d} = (d_1, \dots, d_m) \in \mathbb{N}^m$ and a finite subset $A \subseteq \Gamma^{\underline{n}}_{O_K}(\underline{d})$ multi-homog. deg. $\underline{d}$

$H_\sigma = \operatorname{Max}_{f \in A} \{H(f)_\sigma\}$ naive L^2-height $\quad \forall \sigma : K \hookrightarrow \mathbb{C}$. $H = \left(\prod_{\sigma : K \hookrightarrow \mathbb{C}} H_\sigma\right)^{[K:\mathbb{Q}]^{-1}}$

$t \in \mathbb{N}$ $\qquad$ $K/\mathbb{Q}$ finite

$Z_1, \dots, Z_r$: irreducible components of $\bigcap_{f \in A} \operatorname{div}(f) =: X \subseteq \mathbb{P}^{\underline{n}}$ of codim $\underline{t}$

as a scheme

m_{Z_i} = multiplicity of Z_i in X

Then
$$\sum_{i=1}^{r} m_{Z_i} \cdot h(Z_i, \mathcal{L}(\underline{d})^{n_1 + \dots + n_m - t + 1}) \qquad M = n_1 + \dots + n_m = \dim \mathbb{P}^{\underline{n}}$$
$$\leqslant \frac{(n_1 + \dots + n_m)!}{n_1! \cdots n_m!} (d_1^{n_1} \cdots d_m^{n_m}) \cdot \left[M^2 (d_1 + \dots + d_m) + t \log H\right]$$

(Recall)
<u>Geo</u>metric analog:

<u>Lemma 4</u> $\quad X \subseteq \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_m}$, scheme-theoretic zero locus of a finite subset $A \subseteq \Gamma^{\underline{n}}_{\overline{\mathbb{Q}}}(\underline{d}) - (0)$

$\{(Z_i, m_{Z_i})\}_{1 \le i \le r}$ as in Prop 1.

$e_1, \dots, e_m \in \mathbb{N}$, $\qquad e_1 + \dots + e_m = t$

Then
$$\sum_{i=1}^{r} m_{Z_i} (\mathcal{L}_1^{e_1} \cdots \mathcal{L}_m^{e_m} \cdot X_i) \le \left(\mathcal{L}_1^{e_1} \cdots \mathcal{L}_m^{e_m} \cdot (\mathcal{L}_1^{\otimes d_1} \otimes \dots \otimes \mathcal{L}_1^{d_m})^t\right)$$

Admit Lemmas 6, 7, prove Theorem 2

Theorem 2 $\underline{n}, \underline{d}, \sigma, \varepsilon$ given $\sigma \geq 0, 0<\varepsilon \leq 1$, $\frac{d_h}{d_{h+1}} \geqslant \left(\frac{mM}{\varepsilon}\right)^M$ $M = \sum_{h=1}^m n_h = \dim \mathbb{P}^{\underline{n}}$

$[k_0:\mathbb{Q}]<\infty$ $0 \neq F \in \Gamma^{\underline{n}}_{k_0}(\underline{d})$, Z = a common irred componenet of both $Z_\sigma(F, \underline{d})$ and $Z_{\sigma+\varepsilon}(F, \underline{d})$

Know by theorem 1 that $Z = Z_1 \times \cdots \times Z_m$ $Z_h \subseteq \mathbb{P}^{n_h}$ irred.

and the field of definition k_1 of Z satisfies

$[k_1 : k_0] \cdot \prod_{h=1}^m \deg(Z_h) \leq \left(\frac{ms}{\varepsilon}\right)^s$, where $s = \operatorname{codim}(Z/\mathbb{P}^{\underline{n}}) = \sum_{h=1}^m \operatorname{codim}(Z_h/\mathbb{P}^{n_h})$

Then $[k_1 : k_0] \cdot \prod_{h=1}^m \deg(Z_h) \cdot \sum_{h=1}^m \frac{1}{\deg(Z_h)} d_h \cdot h(Z_h) \leqslant 2 \cdot \left(\frac{s}{\varepsilon}\right)^s m^M M^2 \left(h(F) + \sum_{h=1}^m d_h\right)$

proof: Replacing k_0 by a finite ext^n if necessary, we may assume that $F \in \Gamma_{\mathcal{O}_{k_0}}(\underline{d})$ and the coefficients of F generate the ideal $\mathcal{O}_{k_0}$.

Note: The factor $[k_1 : k_0]$ is recovered from the coefficient/multiplicity of $[Z]$ in $CH^s(\mathbb{P}^{\underline{n}}_{/k_0}) \subseteq CH^s(\mathbb{P}^{\underline{n}}_{/\overline{\mathbb{Q}}})$

cycle

(a) By Lemma 7,

$\rightsquigarrow$ $Z_\sigma(F, \underline{d})$ is defined by $A = \left\{\frac{1}{\underline{I}!} \partial^{\underline{I}} F\right\}$, and $H_\sigma\left(\frac{1}{\underline{I}!}\partial^{\underline{I}} F\right) \leqslant 2^{d_1 + \cdots + d_m} H(F)$

$$
\begin{aligned}
&[k_1 : k]\, m_Z \cdot h(Z, \mathcal{L}(\underline{d})^{M-s+1}) \\
&\leqslant \frac{M!}{n_1! \cdots n_m!} \left(\prod_{h=1}^m d_h^{n_h}\right) \cdot \left[M^2 \cdot (d_1 + \cdots + d_m) + s\left(\log H(F) + (d_1 + \cdots + d_m) \log 2\right)\right] \\
&= \frac{M!}{n_1! \cdots n_m!} \left(\prod_{h=1}^m d_h^{n_h}\right) \cdot \left[(d_1 + \cdots + d_m) \cdot (M^2 + s \log 2) + s \cdot h(F)\right] \\
&\leqslant 2 m^M M^2 \left(\prod_{h=1}^m d_h^{n_h}\right) \cdot \left(h(F) + \sum_{h=1}^m d_h\right)
\end{aligned}
$$

(b) By Lemma 6 (lower bound of m_Z) and Lemma 1 (height of product subvarieties)

$$
\begin{aligned}
&[k_1 : k]\, m_Z \cdot h(Z, \mathcal{L}(\underline{d})^{M-s+1}) \\
&\geqslant [k_1 : k] \left(\frac{\varepsilon}{s}\right)^s d_1^{n_1 - \delta_1} \cdots d_m^{n_m - \delta_m} \cdot \left(d_1^{\delta_1} \cdots d_m^{\delta_m}\right) \left(\prod_{h=1}^m \deg(Z_h)\right) \cdot \sum_{h=1}^m \frac{d_h \cdot h(Z_h)}{\deg(Z_h)} \\
&= [k_1 : k] \cdot \left(\frac{\varepsilon}{s}\right)^s \left(\prod_{h=1}^m \deg(Z_h)\right) \cdot \left(\prod_{h=1}^m d_h^{n_h}\right) \cdot \left(\sum_{h=1}^m \frac{d_h \cdot h(Z_h)}{\deg(Z_h)}\right)
\end{aligned}
$$

Combine (a) with (b), get

$$
[k_1 : k] \left(\prod_{h=1}^m \deg(Z_h)\right) \cdot \left(\sum_{h=1}^m \frac{d_h \cdot h(Z_h)}{\deg(Z_h)}\right) \leqslant 2 \left(\frac{s}{\varepsilon}\right)^s m^M \cdot M^2 \left(d_1 + \cdots + d_m + h(F)\right)
$$

QED.

Lemma 6 Z: a common irred. component of Z_σ and $Z_{\sigma+\varepsilon}$

$p_i : \mathbb{P}^n \to \mathbb{P}^{n_i} \times \cdots \times \mathbb{P}^{n_m}$ $\quad s = \mathrm{codim}(Z, \mathbb{P}^n)$

$\delta_i = \dim(p_i(Z)) - \dim(p_{i+1}(Z))$

Then $m_Z \overset{\text{def}}{=} \ell_{\mathcal{O}_{\eta_Z}}(\mathcal{O}_{Z_\sigma, \eta_Z}) \geq \left(\frac{\varepsilon}{s}\right)^s d_1^{n_1-\delta_1} \cdots d_m^{n_m-\delta_m} - 1$

Pf. Consider

$$\begin{tikzcd} \mathbb{P}^n \arrow[r, "pr_{\geq h}"] \arrow[dr, "pr_{\geq h+1}"'] & \mathbb{P}^{n_h} \times \cdots \times \mathbb{P}^{n_m} \supseteq pr_{\geq h}(Z) \ni \eta_h = \eta_{pr_{\geq h}(Z)} \arrow[d, "pr^h_{h+1}"] \\ & \mathbb{P}^{n_{h+1}} \times \cdots \times \mathbb{P}^{n_m} \supseteq pr_{\geq h+1}(Z) \ni \eta_{h+1} = \eta_{pr_{\geq h+1}(Z)} \end{tikzcd}$$

$\delta_h = \dim(pr_{\geq h}(Z)) - \dim(pr_{\geq h+1}(Z)) \rightsquigarrow \delta_1 + \cdots + \delta_m = \dim(Z)$

Pick elements $t_{h,j} \in \mathcal{O}_{\mathbb{P}^n, \eta_Z}$ $1 \leq h \leq m$, $1 \leq j \leq n_h$

such that

(i) $\{dt_{h,j} : 1 \leq h \leq m, 1 \leq j \leq n_h\}$ form a free $\mathcal{O}_{\mathbb{P}^n, \eta_Z}$-basis of $\Omega^1_{\mathbb{P}^n/\overline{\mathbb{Q}}, \eta_Z}$

(ii) $\{t_{h,j} \mid j \geq \delta_h + 1, 1 \leq h \leq m\}$ form a regular system of parameters of the regular local ring $\mathcal{O}_{\mathbb{P}^n, \eta_Z}$

(iii) $\{dt_{h,j} \mid 1 \leq j \leq \delta_h, 1 \leq h \leq m\}$ form a basis of the $\kappa(\eta_Z)$-vector space $\Omega^1_{\kappa(\eta_Z)/\overline{\mathbb{Q}}}$

(iv) $\exists$ elements $(s_{hj})_{\substack{1 \leq h \leq m \\ 1 \leq j \leq n_h}}$ s.t. $s_{hj} \in \mathcal{O}_{\mathbb{P}^{n_h} \times \cdots \times \mathbb{P}^{n_m}, \eta_h}$ $\forall h, j$

The s_{hj}'s are constructed inductively, starting with $h = m$

Construction:

- $s_{m,\delta_m+1}, \ldots, s_{m,n_m}$ is an $\mathcal{O}_{\mathbb{P}^{n_m}, \eta_m}$-basis of $\Omega^1_{\mathbb{P}^{n_m}, \eta_m}$

  $ds_{m,1}, \ldots, ds_{m,\delta_m}$ is a free basis of $\Omega^1_{\eta_m/\overline{\mathbb{Q}}}$

- Inductively, suppose s_{ij} have been constructed for all $i \geq h+1$ and all $1 \leq j \leq n_i$

  Pick $s_{h,\delta_h+1}, \ldots, s_{h,n_m} \in \mathcal{O}_{\mathbb{P}^{n_h} \times \cdots \times \mathbb{P}^{n_m}, \eta_h}$

  such that the images of $s_{h,\delta_h+1}, \ldots, s_{h,n_m}$ in $\mathcal{O}_{\mathbb{P}^{n_h}_{/\kappa(\eta_{h+1})}}$ form a regular system of parameters of the regular local ring $\mathcal{O}_{\mathbb{P}^{n_h}_{/\kappa(\eta_{h+1})}, \eta_h}$

$$\begin{tikzcd} \mathbb{P}^{n_h} \times \cdots \times \mathbb{P}^{n_m} \arrow[d, "pr^h_{h+1}"] \arrow[dr, phantom, "\square"] & \supseteq \mathbb{P}^{n_h}_{/\kappa(\eta_{h+1})} \arrow[d] \\ \mathbb{P}^{n_{h+1}} \times \cdots \times \mathbb{P}^{n_m} & \ni \eta_{h+1} \end{tikzcd}$$

  then pick $s_{h,1}, \ldots, s_{h,\delta_h}$ s.t. $ds_{h,1}, \ldots, ds_{h,\delta_h}$ form an $\underbrace{\mathcal{O}_{pr_{\geq h}(Z), \eta_h}}_{= \kappa(\eta_h)}$-basis of $\Omega^1_{pr_{\geq h}(Z), \eta_h/\overline{\mathbb{Q}}} = \Omega^1_{\kappa(\eta_h)/\overline{\mathbb{Q}}}$

Let $I_\sigma \subseteq \mathcal{O}_{\mathbb{P}^n}$ and $I_{\sigma+\varepsilon} \subseteq \mathcal{O}_{\mathbb{P}^n}$ be the sheaf of ideals of Z_σ and $Z_{\sigma+\varepsilon}$ respectively. Define I_Z similarly.

Key: $D\, I_\sigma \subseteq I_{\sigma+\varepsilon} \subseteq I_Z$ $\forall$ linear differential operator D of weigted order $\leq \varepsilon$

(*) $\forall \underline{J} \in \mathbb{N}^{n_1-\delta_1} \times \cdots \times \mathbb{N}^{n_m-\delta_m}$, $\underline{J} = (j_{1,\delta_1+1}, \cdots j_{1,n_1}; \cdots; j_{m,\delta_m+1}, \cdots, j_{m,n_m})$

$$\prod_{\substack{1\leq h\leq m \\ \delta_h+1\leq j\leq n_h}} \left(\frac{\partial}{\partial t_{h,j}}\right)^{j_{h,j}} =: \partial_{\underline{t}}^{\underline{J}}$$

Have $\partial_{\underline{t}}^{\underline{J}} I_{\sigma,\eta_Z} \in \mathfrak{m}_{\mathbb{P}^n,\eta_Z}$ s.t. $(\underline{J}/\underline{d}) = \sum_{h=1}^{m} \frac{(j_{h,\delta_h+1} + \cdots + j_{h,n_h})}{d_h} \leq \varepsilon$

Consider the set $\mathcal{J} := \{ \underline{J} = (j_{1,\delta_1+1}, \cdots, j_{1,n_1}; \cdots, j_{m,\delta_m+1}, \cdots, j_{m,n_m}) \in \mathbb{N}^{n_1-\delta_1} \times \cdots \times \mathbb{N}^{n_m-\delta_m} \mid (\underline{J}/\underline{d}) \leq \varepsilon \}$

with lexicographic order

Let $\underline{J}_1 > \underline{J}_2 > \cdots > \underline{J}_b$ be the elements of $\mathcal{J}$

Consider the following ascending chain of ideals of $\mathcal{O}_{\mathbb{P}^n,\eta_Z}$

$$q_0 \subseteq q_1 \subseteq \cdots \subseteq q_b$$

$$q_0 := I_\sigma \mathcal{O}_{\mathbb{P}^n,\eta} + \sum_{(\underline{J}/\underline{d})>\varepsilon} \mathcal{O}_{\mathbb{P}^n,\eta_Z} \underline{t}^{\underline{J}} \qquad \underline{t}^{\underline{J}} \overset{def}{=} \prod_{\substack{1\leq h\leq m \\ \delta_h+1\leq j\leq n_h}} t_{h,j}^{j_{h,j}}$$

$$q_i = q_0 + (\underline{t}^{\underline{J}_1}, \ldots, \underline{t}^{\underline{J}_i}) \quad \text{for } i = 1, \cdots b$$

Claim $q_i \subsetneq q_{i+1}$ $\forall i = 0, \ldots, b-1$

Pf of claim: Suppose that $q_{i+1} = q_i$

i.e. $\exists\, g_1, \ldots, g_i \in \mathcal{O}_{\mathbb{P}^n,\eta}$, $f_1 \in I_\sigma \mathcal{O}_{\mathbb{P}^n,\eta}$ and $f_2 \in \sum_{(\underline{J}/\underline{d})>\varepsilon} \mathcal{O}_{\mathbb{P}^n,\eta_Z} \underline{t}^{\underline{J}}$

s.t. $\underline{t}^{\underline{J}_{i+1}} = \sum_{1\leq \ell\leq i} g_\ell \underline{t}^{\underline{J}_\ell} + f_1 + f_2$

Have: $\partial_{\underline{t}}^{\underline{J}_{i+1}} \underline{t}^{\underline{J}_{i+1}} \in \mathbb{Q}^\times$, $\partial_{\underline{t}}^{\underline{J}_{i+1}} (\underline{t}^{\underline{J}_1}, \ldots \underline{t}^{\underline{J}_i}) \subseteq \mathfrak{m}_{\mathbb{P}^n,\eta_Z}$, $\because$ $\underline{J}_{i+1} < \underline{J}_i < \underline{J}_{i-1} < \cdots < \underline{J}_1$

$\partial_{\underline{t}}^{\underline{J}_{i+1}} f_1 \in \mathfrak{m}_{\mathbb{P}^n,\eta_Z}$ by (*)

and $\partial^{\underline{J}_{i+1}} f_2 \in \mathfrak{m}_{\mathbb{P}^n,\eta_Z}$. Claim proved

$$\#\mathcal{J} \geq \left(\left\lfloor\frac{\varepsilon d_1}{s}\right\rfloor + 1\right)^{n_1-\delta_1} \cdots \left(\left\lfloor\frac{\varepsilon d_m}{s}\right\rfloor + 1\right)^{n_m-\delta_m} - 1$$

$$\rightsquigarrow \operatorname{length}_{\mathcal{O}_{\mathbb{P}^n,\eta_Z}}(\mathcal{O}_{Z,\eta}) \geq \#\mathcal{J} \geq \left(\frac{\varepsilon d_1}{s}\right)^{n_1-\delta_1} \cdots \left(\frac{\varepsilon d_m}{s}\right)^{n_m-\delta_m}$$

$$= \left(\frac{\varepsilon}{s}\right)^s d_1^{n_1-\delta_1} \cdots d_m^{n_m-\delta_m}$$

QED.

Roth's Lemma improved

$\underline{d} = (d_1, \cdots, d_m)$, $\quad F \in \bar{\mathbb{Q}}[X_{10}, X_{11}; \cdots; X_{m0} X_{m1}]$ multi-form of degree $\underline{d}$

$0 < \varepsilon \leq m+1$, $\quad d_h / d_{h+1} \geq 2m^3/\varepsilon =: \sigma^{-1} \quad (1)$

$\underline{P} = (P_1, \cdots, P_m)$, $P_h \in \mathbb{P}^1(\bar{\mathbb{Q}})$

$d_h \cdot h(P_h) > (\frac{3}{2}\sigma^{-1})^m \cdot [h(F) + (d_1 + \cdots + d_m)] \quad (2) \qquad (\frac{3}{2}\sigma^{-1})^m = (\frac{3m^3}{\varepsilon})^m$

Then $\operatorname{index}_{\underline{d}}(F; \underline{P}) < \varepsilon$

In comparison, in the previous version, the assumptions were

(1)': with $\sigma' = \varepsilon^{2^{m-1}}$

(2)': with $(\frac{3}{2}\sigma^{-1})^m$ replaced by $\frac{1}{\sigma'} = \frac{1}{\varepsilon^{2^{m-1}}}$

and the conclusion was

$\operatorname{index}_{\underline{d}}(F, P) < 2m\varepsilon$

So: (1) strictly weaker than (1)'

(2) strictly weaker than (2)'

and with strictly stronger conclusion

Geometric setup

k: alg. closed field, $A_{/k}$: abelian variety

$X \subseteq A$ reduced, irred. DOES NOT contain any translate of a positive dimensional abelian subvariety

Lemma 1 $X^m \xrightarrow{\alpha_m} A^{m-1}$ is finite for m sufficiently large

$$(x_1, \dots, x_m) \longmapsto (2x_1 - x_2, 2x_2 - x_3, \dots, 2x_{m-1} - x_m)$$

proof: For any k-point $\underline{a}_m = (a_1, \dots, a_m) \in A^m$, $\alpha_{m+1}^{-1}(\underline{a}_m) \subseteq \alpha_{m-1}(\underset{(a_1,\dots,a_{m-1})}{\underline{a}_{m-1}})$

$\rightsquigarrow$ $\exists\, m_0$ s.t. $\operatorname{Max}\{\dim(\alpha_m^{-1}(\underline{a}_m)) : \underline{a}_m \in A^m(k)\}$

$= \operatorname{Max}\{\dim(\alpha_{m_0}^{-1}(\underline{a}_{m_0})) : \underline{a}_{m_0} \in A^{m_0}(k)\}$ $\forall m \geq m_0$

$=: d$. Must show: $d = 0$

$$\alpha_m^{-1}(\underline{a}_m) = \left\{ \underset{(x_1,\dots,x_m)}{\underline{x}} \in X^m \,\middle|\, \begin{matrix} x_2 = 2x_1 - a_1,\ x_3 = 2x_2 - a_2, \dots, \\ x_m = 2x_{m-1} - a_m \end{matrix} \right\} \hookrightarrow X \quad \forall \underline{a}_m$$

$$\underline{x} = (x_1, \dots, x_m) \longmapsto x_1$$

Fix an ample (symmetric) investible $\mathcal{O}_A$-module $\mathcal{L}$

$\forall$ irred. subvariety $Z \subseteq \alpha^{-1}(\underline{a}_m)$ for some $\underline{a}_m$,

define $\deg(Z) :=$ degree of Z w.r.t. $pr_1^* \mathcal{L}$

Clearly, $\exists$ constant $C > 0$ s.t. $\forall d$-dimensional irred. component Z of $\alpha_m^{-1}(\underline{a}_m)$, $\deg(Z) \leq C$, $\forall m \geq m_0$, $\forall \underline{a}_m \in A^m$

def: $\forall m \geq m_0$, Let $Y_m = \{\underline{a}_m \in A^m \mid \dim(\alpha_{m+1}^{-1}(\underline{a}_m)) = d\}$

Clearly: $pr_{\leq m} : Y_{m+1} \to Y_m$

$\rightsquigarrow Y := \bigcap_{m \geq m_0} pr_{\leq m_0}(Y) \neq \phi$

Pick $y \in Y$ $\rightsquigarrow$ $\alpha_{m_0}^{-1}(y)$ has a d-dim$^{\underline{l}}$ irred. component Z which occurs in fibers of α_m for all $m \geq m_0$

Note: Let $Z_1 = pr_1(Z)$.

Then $\forall r > 0$, $\exists\, b_r \in A$ s.t. $2^r Z_1 + b_r \subseteq X$

$\rightsquigarrow \deg([2^r](Z_1)) \leq C$ $\forall r \geq 0$

grows like $4^{rd}\cdot$ ($\mathcal{L}$-degree of Z) $\rightarrow$ $\left(\text{degree of } [2^r](Z_1) \text{ w.r.t } [2^r]^* \mathcal{L}\right) = \left(\deg([2^r] : Z \to [2^r](Z))\right) \cdot \left(\overset{\text{bounded}}{\deg \text{ of } [2^r](Z) \text{ w.r.t. } \mathcal{L}}\right)$

$\rightsquigarrow \dim(B) = r \rightsquigarrow r = 0$ $\qquad = \#(B[2^r])$

QED. where $B = \{w \in A \mid w + Z \subseteq Z\}$

$\mathcal{L}$: very ample invertible $\mathcal{O}_A$-module

<u>Lemma 2</u> $Y = Y_1 \times \cdots \times Y_m \subseteq A^m$ a product subvariety

$$\mathcal{L}(-\varepsilon, s_1, \dots, s_m) \overset{\text{def}}{=\!=} -\varepsilon \sum_{i=1}^{m} s_i^2 \cdot pr_i^* \mathcal{L} + \sum_{i=1}^{m-1} (s_i x_i - s_{i+1} x_{i+1})^* \mathcal{L} \in \mathrm{Pic}(A) \otimes \mathbb{Q}$$

$\varepsilon, s_1, \dots, s_m \in \mathbb{Q}_{>0}$

The intersection number $\left(\mathcal{L}(-\varepsilon, s_1, \dots, s_m)^{\dim(Y)} \cdot Y\right) = c_{Y,\varepsilon} \cdot \prod_{i=1}^{m} s_i^{2\dim(Y_i)}$

a constant

<u>Remark</u>. Recall $X \subseteq A$ does not contain any translate of any pos. dim^l abelian subvariety

m: sufficiently large s.t. $\alpha_m : X^m \longrightarrow A^{m-1}$

$(x_1, \dots, x_m) \mapsto (2x_1 - x_2, \dots, 2x_{m-1} - x_m)$

is finite.

$$\rightsquigarrow \mathcal{O}_{X^m} \underset{\mathcal{O}_{A^m}}{\otimes} \mathcal{L}(0, 2^{m-1}, 2^{m-2}, \dots, 1) = (2^{m-1}x_1 - 2^{m-2}x_2)^* \mathcal{L} + (2^{m-2}x_2 - 2^{m-3}x_3)^* \mathcal{L} + \cdots + (2x_{m-1} - x_m)^* \mathcal{L}$$

$$= \alpha_m^* \left([2^{m-2}]^* \mathcal{L}_1 + \cdots + [2]^* \mathcal{L}_{m-2} + \mathcal{L}_{m-1}\right)$$

$$\cong \alpha_m^* \left(4^{m-2} \mathcal{L}_1 + \cdots + 4 \mathcal{L}_{m-2} + \mathcal{L}_{m-1}\right) \text{ is ample on } X^m$$

<u>Corollary</u>: $\exists\, \varepsilon_0 > 0$ s.t. $\mathcal{L}(-\varepsilon, 2^{m-1}, 2^{m-2}, \dots, 1)$ is ample $\forall \varepsilon \leq \varepsilon_0$.

Hence, $\forall$ product subvariety $Y_1 \times \cdots \times Y_m \subseteq X^m$, the constant $c_{Y,\varepsilon} > 0$ $\forall \varepsilon \leq \varepsilon_0$

In particular $\left(\mathcal{L}(-\varepsilon, s_1, \dots, s_m)^{\dim(Y)} \cdot Y\right) > 0$ $\forall \varepsilon \leq \varepsilon_0$.

<u>proof of Lemma 2</u>

Theorem of ▢

$$\mathcal{L}(-\varepsilon, s_1, \dots, s_m) = (1-\varepsilon) s_1^2 \mathcal{L}_1 + \sum_{i=2}^{m-1} (2-\varepsilon) s_i^2 \mathcal{L}_i + (1-\varepsilon) s_m^2 \mathcal{L}_m - \sum_{i=1}^{m-1} s_i \cdot s_{i+1} \underline{P_{i,i+1}}$$

$pr_{i,j}^* P$

P: $(x_1+x_2)^* \mathcal{L} - x_1^* \mathcal{L} - x_2^* \mathcal{L}$

$\sum_i e_i + \sum e_{ij} = \dim(Y)$

$$\left(\prod_i \mathcal{L}_i^{e_i} \cdot \prod_{i<j} P_{i,j}^{e_{i,j}} \cdot Y\right) = \deg\left(\prod_i c_1(\mathcal{L}_i)^{e_i} \cdot \prod_{i<j} c_1(P_{i,j})^{e_{i,j}} \cdot [Y]\right)$$

$$\neq 0 \implies 2e_i + \sum_{\ell < i} e_{\ell,i} + \sum_{j > i} e_{i,j} \leq \dim(Y_i) \quad \forall i$$

$$\rightsquigarrow 2e_i + \sum_{\ell < i} e_{\ell,i} + \sum_{j > i} e_{i,j} = \dim(Y_i) \quad \forall i$$

q.e.d.

X given, m: satisfies Lemma 1

ε_0: as in Lemma 2

Theorem1 (geometric part)

$\forall \varepsilon < \varepsilon_0$, $\exists$ $s > 0$ s.t. $\mathcal{L}(-\varepsilon, s_1, \dots, s_m) \otimes_{\mathcal{O}_{A^m}} \mathcal{O}_{X^m}$ is ample if $s_h/s_{h+1} \geq s$ $\forall h = 1, \dots, m-1$

Lemma 3 $\forall W$ projective variety. D: ample divisor on W

Fact related to positivity

$\mathcal{M}$: invertible $\mathcal{O}_W$-module s.t. $\mathcal{M} \otimes_{\mathcal{O}_W} \mathcal{O}_D$ is ample on D

Then $H^i(W, \mathcal{M}^{\otimes d}) = (0)$ $\forall d >> 0$

Sublemma V: projective variety. $\mathcal{F}$: coherent $\mathcal{O}_V$-module

$\mathcal{L}, \mathcal{M}$: ample $\mathcal{O}_V$-module

Then the set

$$S_{\mathcal{F}} = \{ (a,b) \in \mathbb{N}^2 \mid H^i(V, \mathcal{F} \otimes \mathcal{L}^{\otimes a} \otimes \mathcal{M}^{\otimes b}) \neq 0 \text{ for some } i \geq 1 \}$$

is finite

Pf of sublemma:

May assume $\mathcal{L}, \mathcal{M}$ are both very ample: $V \xrightarrow{\varphi_{\mathcal{L}}} \mathbb{P}^n$, $V \xrightarrow{\varphi_{\mathcal{M}}} \mathbb{P}^m$

$$\rightsquigarrow V \underset{\varphi_{\mathcal{L}} \times \varphi_{\mathcal{M}}}{\hookrightarrow} \mathbb{P}^n \times \mathbb{P}^m$$

Use descending induction, that every coherent $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^m}$-module $\mathcal{G}$ admits a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_{\mathbb{P}^n}(a) \otimes \mathcal{O}_{\mathbb{P}^m}(b) \longrightarrow \mathcal{G} \to 0 \qquad (a,b) \in \mathbb{N}^2$$

and. $H^i(\mathbb{P}^n \times \mathbb{P}^m, \mathcal{O}_{\mathbb{P}^n}(a) \otimes \mathcal{O}_{\mathbb{P}^m}(b)) = (0)$ $\forall i \geq 1$ if $a + b \geq 1$ sublemma proved

Pf of Lemma:

$$0 \to \mathcal{M}^{\otimes d} \to \mathcal{M}^{\otimes d}(nD) \longrightarrow \mathcal{M}^{\otimes d}(nD) \otimes_{\mathcal{O}_W} (\mathcal{O}_W / \mathcal{I}_D^n) \to 0 \qquad d \geq d_0$$

Choose d_0 s.t.

$H^i(D, \mathcal{M}^{\otimes d}(nD) \otimes_{\mathcal{O}_W} \mathcal{O}_D) = (0)$

$\forall i \geq 1$, $\forall d \geq d_0$

$\forall n \geq 0$

(By sublemma)

$$\rightsquigarrow H^i(W, \mathcal{M}^{\otimes d}) \hookrightarrow H^i(W, \mathcal{M}^{\otimes d}(nD))$$

$\forall d \geq d_0$, $\forall i \geq 2$, $\forall n \geq 0$

$= (0)$ for $n >> 0$

$$\Longrightarrow H^i(W, \mathcal{M}^{\otimes d}) = (0) \quad \forall d \geq d_0, \ \forall i \geq 2$$

QED

Proof of theorem 1 (geometric part)

Claim: $\forall N, r \in \mathbb{N}$ $\exists s_0 = s_0(N,r) \in \mathbb{N}_{>0}$ s.t.
$\forall$ product subvariety $Y = Y_1 \times \cdots \times Y_m \subseteq X^m$ with $\dim(Y) = r$ and $\deg(Y_i) \le N$ $\forall i$.
$\mathcal{L}(-\varepsilon, s_1, \dots, s_m) \otimes_{\mathcal{O}_{A^m}} \mathcal{O}_Y$ is ample if $s_h/s_{h+1} \geq s_0$ $\forall h = 1, \dots, m-1$ $\forall \varepsilon < \varepsilon_0$

Prove the Claim by induction on r (case $r=0$ is trivial).
Suppose Claim is true for all $r < r_0$. Let N be given

Note: For a product subvariety $Y = Y_1 \times \cdots \times Y_m \subseteq X^m$, let $H_i \subseteq Y_i$ be a hyperplane section
then $D = \sum_{i=1}^{m} Y_1 \times \cdots \times Y_{i-1} \times H_i \times Y_{i+1} \times \cdots \times Y_m \subseteq Y$ is an ample divisor on Y. $\rightsquigarrow$ By induction (for $(N, r-1)$), $\mathcal{L}(-\varepsilon, s_1, \dots, s_m) \otimes_{\mathcal{O}_{A^m}} \mathcal{O}_D$ is an ample invertible $\mathcal{O}_D$-module if $s_h/s_{h+1} \geq s_0(N, r-1)$ $\forall h$ $\forall \varepsilon < \varepsilon_0$

Subclaim: $\exists s_0'(N,r)$ s.t. $\forall$ irreducible curve $C \subseteq Y$.
$\deg \mathcal{L}(\underbrace{-\tfrac{\varepsilon+\varepsilon_0}{2}}_{=-\varepsilon'}, s_1, \dots, s_m) \geq 0$ if $s_h/s_{h+1} \geq s_0(N,r)$ $\forall h = 1, \dots, m-1$ and $\varepsilon < \varepsilon_0$

Clearly subclaim $\Rightarrow$ $\mathcal{L}(-\varepsilon, s_1, \dots, s_m) \otimes_{\mathcal{O}_{A^m}} \mathcal{O}_Y$ is an ample invertible $\mathcal{O}_Y$-module

Idea: For d sufficiently large and divisible, produce a global section $0 \neq f \in \Gamma(\mathcal{L}(-\varepsilon', s_1, \dots, s_m)^{\otimes d} \otimes_{\mathcal{O}_{A^m}} \mathcal{O}_Y)$. Consider an irreducible curve $C \subseteq Y$

Clearly, if $f|_{\mathcal{O}_C} \neq 0$, then $\deg(-\varepsilon', s_1, \dots, s_m) \geq 0$

So assume that $f|_{\mathcal{O}_C} = 0$, i.e. f vanishes on C

- If $\operatorname{index}(f, \eta_C, \underline{d})$ is large, apply the product theorem and induction
  $\underline{d} = (d_1, \dots, d_m)$, $d_i = 4ds_i^2$
- If $\operatorname{index}(f, \eta_C, \underline{d})$ is small, use a suitable "$\partial^I f$" to produce a global section g of $\mathcal{L}(-\varepsilon', s_1, \dots, s_m) \otimes_{\mathcal{O}_{A^m}} \mathcal{O}_C$ which does not vanish on η_C and has "suitably bounded poles"

Will apply the product theorem to $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$,
$\pi_i : Y_i \longrightarrow \mathbb{P}^{n_i}$ generically finite, $\deg(\pi_i) = \deg(Y_i)$
étale outside a hypersurface $Z_i \subseteq \mathbb{P}^{n_i}$, s.t. $\mathcal{J}_{Z_i} \cdot \Omega^1_{Y_i/\mathbb{P}^{n_i}} = 0$
and $\deg(Z_i) \leq \underbrace{n}_{} \cdot N^2$.

Admit the existence of such good projections π_i

$$A \overset{\varphi_{\mathcal{L}}}{\hookrightarrow} \mathbb{P}^n$$

Pick $\sigma > 0$ s.t. $\varepsilon' := 4nN^2 \deg(\pi)\sigma + \varepsilon < \varepsilon_0$, $\pi = \pi_1 \times \cdots \times \pi_m : \underbrace{Y_1 \times \cdots \times Y_m}_{= Y} \to \underbrace{\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}}_{= \mathbb{P}^n}$ finite.

Remark/sublemma:

$$0 \to \pi_i^* \Omega^1_{\mathbb{P}^{n_i}/k} \longrightarrow \Omega^1_{Y_i/k} \longrightarrow \Omega^1_{Y_i/\mathbb{P}^{n_i}} \to 0$$

$G_i \in \Gamma(\mathbb{P}^{n_i}, \mathcal{O}(\deg Z_i))$

defining eq^n of $Z_i \subseteq \mathbb{P}^{n_i}$ s.t. $\mathcal{I}_{Z_i} \cdot \Omega^1_{Y_i/\mathbb{P}^{n_i}} = (0)$. $\deg(Z_i) \leq nN^2$

$\rightsquigarrow$ Every holom. vector field on $\mathbb{P}^{n_i}$ extends to a meromorphic vector field on Y_i with at most a simple pole along $\pi_i^{-1}(Z_i)$

$$(s_i x_i - s_{i+1} x_{i+1})^* \mathcal{L} + (s_i x_i + s_{i+1} x_i)^* \mathcal{L} \cong 2 s_i^2 \mathcal{L}_i + 2 s_{i+1}^2 \mathcal{L}_{i+1}$$

$\rightsquigarrow$ d sufficiently large and divisible.

Have "injections without common zeros" on A^m

$$d \sum_{i=1}^{m-1} (s_i x_i - s_{i+1} x_{i+1})^* \mathcal{L} \longrightarrow d\,(2 s_1^2 \mathcal{L}_1 + 4 s_1^2 \mathcal{L}_2 + \cdots + 4 s_{m-1}^2 \mathcal{L}_{m-1} + 2 s_m^2 \mathcal{L}_m)$$

Since $\mathcal{L}$ is very ample, have injections without common zeros

$$d \cdot \mathcal{L}(-\varepsilon, s_1, \ldots, s_m) \longrightarrow d\big((2-\varepsilon) s_1^2 \mathcal{L}_1 + (4-\varepsilon) s_2^2 \mathcal{L}_2 + \cdots + (4-\varepsilon) s_{m-1}^2 \mathcal{L}_{m-1} + (2-\varepsilon) s_m^2 \mathcal{L}_m\big)$$

$$\downarrow$$

$$d\,(4 s_1^2 \mathcal{L}_1 + 4 s_2^2 \mathcal{L}_2 + \cdots + 4 s_m^2 \mathcal{L}_m)$$

Let $d_i :\stackrel{\text{def}}{=} 4 d s_i^2$. Will use only those d which are squares

Have produced a global section $f \in \Gamma(Y, \mathcal{L}(-\varepsilon', s_1, \ldots, s_m)^{\otimes d} \otimes_{\mathcal{O}_{A^m}} \mathcal{O}_Y)$ s.t. $f|_C \neq 0$

Choose an injection $\rho : \mathcal{L}(-\varepsilon, s_1, \ldots, s_m)^{\otimes d} \hookrightarrow \bigotimes_{i=1}^m \mathcal{L}_i^{\otimes d_i}$

$\uparrow$ exists because there are injections without common zeros

Let $g = Nm_{Y/\mathbb{P}^n}(\rho(f)) \in \Gamma(\mathbb{P}^n, \mathcal{O}(d_1, \ldots, d_m))$

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & \mathbb{P}^n \\ \cup & & \cup \\ C & \longrightarrow & \pi(C) \end{array}$$

π étale at η_C

Pick $\sigma > 0$ s.t. $\varepsilon' := 4nN^2 \deg(\pi)\cdot\sigma + \varepsilon < \varepsilon_0$ (so σ is "quite small")
$n = \dim H^0(A, \mathcal{L}) - 1$

If $i(g, \pi(\eta_C), \deg(\pi)\cdot \underline{d}) \geq \sigma \rightsquigarrow$ Apply product theorem
$\rightsquigarrow$ Claim follows by induction

So: Remains to treat the case when

$$i(g, \pi(\eta_C), \deg(\pi)\cdot\underline{d}) < \sigma$$
$$\vee\!/$$
$$i(f, \eta_C, \underline{d})/\deg(\pi)$$

Situation: $i(f, \eta_C, \underline{d}) < \deg(\pi)\cdot\sigma$
$f \in \Gamma(Y, \mathcal{L}(-\varepsilon', s_1, \ldots, s_m) \otimes_{\mathcal{O}_{A^m}} \mathcal{O}_Y)$

$\exists \{\partial_{ij}\}$
vector fields on $\mathbb{P}^{n_i}$, $\{\partial_{ij} \mid j = 1, \ldots, n_i\}$ is a basis of vector fields near $\pi_i(\eta_C) \in \mathbb{P}^{n_i}$.

$$\partial_{ij} = g_i^{-1}\hat{\partial}_{ij}$$

$\hat{\partial}_{ij}$ ← holomorphic on Y_i

g_i: equation of $\pi_i^{-1}(Z_i)$ $\quad \deg(Z_i) \leq nN^2$

$\rightsquigarrow \prod_{i,j}(g_i\cdot\partial_{ij})$ "is" a global section of

$$\left(\bigotimes_{i=1}^m \mathcal{L}_i^{\otimes e_i \deg(Z_i)}\right) \otimes \mathcal{L}(-\varepsilon', s_1, \ldots, s_m)^{\otimes d}$$
$$\cong \bigotimes_{i=1}^m \mathcal{L}_i^{\otimes(e_i\deg(Z_i) - \varepsilon' d s_i^2)} \otimes \bigotimes_{i=1}^{m-1}(s_i x_i - s_{i+1}x_{i+1})^*\mathcal{L}^{\otimes d}$$

whose restriction to C is not zero $\qquad$ where $e_i = \sum_j e_{ij}$.

So $\deg_C\left(\bigotimes_{i=1}^m \mathcal{L}_i^{\otimes(e_i\deg(Z_i) - \varepsilon' d s_i^2)} \otimes \bigotimes_{i=1}^{m-1}(s_i x_i - s_{i+1}x_{i+1})^*\mathcal{L}^{\otimes d}\right) \geq 0$

$\sum_{i,j}(e_{ij}/d_i) \leq \deg(\pi)\cdot\sigma \rightsquigarrow e_i = \sum_j e_{ij} \leq i(f, \eta_C) < \deg(\pi)\cdot\sigma \quad \forall i$
$d_i = 4ds_i^2$

$$\rightsquigarrow \deg_C \mathcal{L}(\underbrace{4nN^2\deg(\pi)\cdot\sigma - \varepsilon'}_{\overset{\shortparallel}{-\varepsilon}}, s_1, \ldots, s_m) \geq 0$$

This completes the induction. Have proved the Claim
$\Downarrow$
Thm 1

QED

Arithmetic Part

$A_{/k}$: abelian variety. k = a number field. $X_{/k} \subseteq A_{/k}$ closed irred subvar. does not contain translate of any pos. dim abel. subvariety

$\mathcal{L}_{/k}$: very ample invertible $\mathcal{O}_A$-module, symmetric

m, $\varepsilon>0$, s satisfy the conditions/properties in the geometric part.

(i) $X^m \longrightarrow A^{m-1}$

$(x_1,\dots,x_m) \mapsto (2x_1-x_2, 2x_2-x_3, \dots, 2x_{m-1}-x_m)$ is finite

(ii) $\mathcal{L}_k(-\varepsilon, s_1,\dots,s_m) \otimes_{\mathcal{O}_{A^m}} \mathcal{O}_{X^m}$ is ample if $s_h/s_{h+1} \geq s \quad \forall h=1,\dots,m-1$,

where $\mathcal{L}_k(-\varepsilon, s_1,\dots,s_m) = -\varepsilon \sum_{i=1}^m s_i^2 \, pr_i^* \mathcal{L}_k + \sum_{i=1}^{m-1} (s_i\, pr_i - s_{i+1}\, pr_{i+1})^* \mathcal{L}_k \in \mathrm{Pic}(A^m_{/k}) \otimes_{\mathbb{Z}} \mathbb{Q}$

m, ε, s are <u>fixed</u>

Overview of the arithmetic part

<u>Step 0</u> Construct suitable proper normal integral models $A_{\mathcal{O}_k}$ (with $A_k^{\text{Neron}} \supseteq A_{\mathcal{O}_k}$) of A and Y of $X^m_{/k}$, together with integral models $\mathcal{L}$ of $\mathcal{L}_k$ on A.

$\mathcal{L}_i$ of $pr_i^* \mathcal{L}_{/k}$ on Y and $\mathcal{P}_{ij}$ of $(pr_i + pr_j)^* \mathcal{L}_k \otimes pr_i^* \mathcal{L}_{/k}^{\otimes -1} \otimes pr_j^* \mathcal{L}_{/k}^{\otimes -1}$ on Y

$Y^0 \overset{\text{def}}{=} Y \cap A^{m,\text{Neron}}_{/\mathcal{O}_K}$ ← Néron model of $A^m_{/k}$

May assume: $\mathcal{L}_{\mathcal{O}_k}$ is very ample on $A_{/\mathcal{O}_K}$

<u>Notation</u> Let $\underline{d_i = 4d\, s_i^2}$, d: suff. large, sufficiently divisible <u>square</u> (s_i will vary)

Choose and <u>fix</u> $(f_\alpha)_{\alpha \in I}$, a finite set of $\mathcal{O}_k$-generators of $\Gamma(A_{\mathcal{O}_K}, \mathcal{L}_{\mathcal{O}_K})$

Notation: $\forall a,b \in \mathbb{Z}$, $\forall 1 \leq i,j \leq m$

$\mathcal{P}_{/k} \overset{\text{def}}{=} (pr_1+pr_2)^* \mathcal{L}_k \otimes pr_1^* \mathcal{L}_k^{\otimes -1} \otimes pr_2^* \mathcal{L}_k^{\otimes -1}$ on A_k^2

Have over A_k^m.

$$(pr_i + pr_j, pr_\ell)^* \mathcal{P}_k \xrightarrow{\sim} (pr_i, pr_\ell)^* \mathcal{P}_{/k} \otimes (pr_j, pr_\ell)^* \mathcal{P}_{/k}$$

and

$$(a\,pr_i + b\,pr_j)^* \mathcal{L}_k \underset{\phi_{a,b,i,j}}{\xrightarrow{\sim}} (a\,pr_i)^* \mathcal{L} \otimes (b\,pr_j)^* \mathcal{L} \otimes (a\,pr_i, b\,pr_j)^* \mathcal{P} \quad \text{on } A_k^m$$

$\forall\, 0 < \varepsilon' \leq \varepsilon$, have

$$\rho : \mathcal{L}(-\varepsilon', s_1,\dots,s_m)_k^{\otimes d} \longrightarrow \bigoplus_{\substack{(\alpha_1,\dots,\alpha_{m-1},\beta_1,\dots,\beta_m) \\ \in I^{2m-1}}} \bigotimes \mathcal{L}_{i/k}^{\otimes d_i}$$

↑ the component $\rho_{(\alpha_1,\dots,\alpha_{m-1},\beta_1,\dots,\beta_m)}$ is : multiplication with

$$\left(\prod_{i=1}^m f_{\beta_i}^{\varepsilon' d s_i^2}\right) \cdot pr_1^* f_{\beta_1}^{2ds_1^2} \cdot pr_m^* f_{\beta_m}^{2ds_m^2} \cdot \prod_{i=1}^{m-1} r_i\, \phi_{s_i\sqrt{d},\, s_{i+1}\sqrt{d},\, i,\, i+1}\, (s_i\sqrt{d}\, pr_i + s_{i+1}\sqrt{d}\, pr_j)^* f_{\alpha_i}$$

Used:

Lemma 4 $\forall a,b \in \mathbb{Z}$, $\forall_{1\le i,j\le m}$. Let $\mathcal{F}_{a,b,i,j} \overset{\text{def}}{=}$ the subsheaf of $\mathcal{L}_{i/k}^{\otimes a^2} \otimes \mathcal{L}_{j/k}^{\otimes b^2} \otimes P_{ij/k}^{\otimes ab}$ generated by $\left(\phi_{a,b,i,j}\left((a\,pr_i + b\,pr_j)^* f_\alpha\right) \mid \alpha \in I\right)$

$\exists\, c_1 \in \mathbb{R}$ s.t. $\forall a,b,i,j$ $\exists\, r \in \mathbb{Z}$ with $0 < r < \exp\left(c_1(1+a^2+b^2)\right)$ s.t.

(i) $r \cdot \mathcal{F}_{a,b,i,j} \subseteq \mathcal{L}_{\mathcal{O}_k}^{\otimes a^2} \otimes \mathcal{L}_{\mathcal{O}_k}^{\otimes b^2} \otimes P_{i,j/\mathcal{O}_k}^{\otimes ab}$ on Y

(ii) $\mathcal{L}_{i/\mathcal{O}_k}^{\otimes a^2} \otimes \mathcal{L}_j^{\otimes b^2} \otimes P_{i,j/\mathcal{O}_k}^{\otimes ab} \subseteq r^{-1} \cdot \mathcal{F}_{a,b,i,j}$ on $A^{m,\text{Neron}}$

Remark: (i): Bounds the poles of $\phi_{a,b,i,j}\left((a\,pr_i + b\,pr_j)^* f_\alpha\right)$ on Y

(ii): Bounds the zeros of $\phi_{a,b,i,j}\left((a\,pr_i + b\,pr_j)^* f_\alpha\right)$ on the $(\text{Néron model of } A_k)^m$ = Néron model of A_k^m

Step 1 Use the beginning of Koszul complex to get a complex

$$0 \longrightarrow \mathcal{L}(-\varepsilon', s_1, \ldots, s_m)^{\otimes d} \longrightarrow \bigoplus_{J \in I^{2m-1}} \bigotimes_{i=1}^{m} \mathcal{L}_i^{\otimes d_i} \longrightarrow \bigoplus_{J' \in (I^{2m-1})^3} \bigotimes_{i=1}^{m} \mathcal{L}_i^{\otimes 3d_i}$$

of coherent $\mathcal{O}_Y$-modules such that the restriction to Y° of its homology sheaves are killed by some $r \in \mathbb{Z}$ with $0 < r < \exp\left(c_1 \sum_{i=1}^{m} d_i\right)$.

Also, examine metrics at arch. places (Proposition 5 precise statement)

Taking global sections gives an exact sequence

$$0 \to \Gamma\left(X_k^m, \mathcal{L}(-\varepsilon', s_1, \ldots, s_m)^{\otimes d}\right) \to \Gamma\left(X_k^m, \bigoplus_{J \in I^{2m-1}} \bigotimes_{i=1}^{m} \mathcal{L}_i^{\otimes d_i}\right) \to \Gamma\left(X_k^m, \bigoplus_{J \in (I^{2m-1})^3} \bigotimes_{i=1}^{m} \mathcal{L}_i^{\otimes 3d_i}\right)$$

Use Faltings' version of Siegel's lemma to get:

$\forall\, 0 < \sigma < \varepsilon$, $\exists\, c(\sigma) \in \mathbb{R}$, depending only on σ (ε fixed) s.t.

$\forall$ smooth k-rational point $x \in X^m(k)$, any sequence $s_1, \ldots, s_m \in \mathbb{Q}_{>0}$ with $s_h / s_{h+1} \ge s$ $\forall h = 1, \ldots, m-1$, and any square integer d sufficiently large and sufficiently divisible,

$\exists\, f \in \Gamma\left(Y^\circ, \mathcal{L}(\sigma - \varepsilon, s_1, \ldots, s_m)^{\otimes d}\right)$ with index$(f, x, \underline{d}) < \sigma$ $\quad d_i = 4 d s_i^2$

and $\|f\|_{L^\infty, Y_k(k \otimes_{\mathbb{Q}} \mathbb{C})} \le \exp\left(c(\sigma) \cdot \sum_{i=1}^{m} d_i\right)$

Recapitulate:

Step 1. Apply Faltings' version of Siegel's lemma to produce an integral global section $f \in \Gamma(Y^\circ, \mathcal{L}(\sigma-\varepsilon, s_1, \dots, s_m)^{\otimes d})$ with $\mathrm{index}(f, \underset{\uparrow}{x}, \underline{d}) < \sigma$ and $\|f\|_{L^\infty, Y_k(k\otimes\mathbb{C})} \leq \exp\left(c(\sigma)\sum_{i=1}^m d_i\right)$. $\underset{\uparrow}{s_h}/s_{h+1} \geq s \quad \forall h=1,\dots,m-1$ given smooth k-pt

Step 2. Assume $\# X(k) = \infty$.

Use two "independent parameters" $\underset{\vee\atop 0}{\overset{\text{small}}{\sigma}}$ and $\underset{\vee\atop 0}{\overset{\text{large}}{b}}$

Find $x = (x_1, \dots, x_m) \in X^m(k)$ s.t.

(i) $h(x_1) = h_{\mathcal{L}}(x_1) \geq b$

(ii) $h(x_2)/h(x_1) \geq 2s^2, \dots, h(x_m)/h(x_{m-1}) \geq 2s^2$

(iii) $\langle x_i, x_{i+1}\rangle = \langle x_i, x_{i+1}\rangle_{NT,\mathcal{L}} \geq \left(1-\frac{\varepsilon}{4}\right)\|x_i\|\cdot\|x_{i+1}\|^2$

$\|x_i\| \overset{\text{def}}{=} \langle x, x\rangle$

Note: $\hat{h}(x_i) =$ Néron-Tate height of x_i w.r.t. $\mathcal{L}$
$= \frac{1}{2}\langle x_i, x_i\rangle$

$\hat{h}(x) \sim h_{\mathcal{L}}(x)$ (same quadratic term)

(iv) $x = (x_1, \dots, x_m)$; $x_i \in U(k) \ \forall i$

$\widetilde{X}_{/\mathcal{O}_k} \xrightarrow{pr} \mathbb{P}^n_{/\mathcal{O}}$

open $\underset{U}{\cup}$ $\quad n = \dim(X_k)$

pr: étale over U

$\Omega^1_{\widetilde{X}/\mathbb{P}^n}$ killed by $0 \neq G \in \Gamma(\mathbb{P}^n_{/\mathcal{O}_k}, \mathcal{O}(g))$

Note: (iii) means: The points $x_1, \dots, x_m$ are "almost aligned" in $A(k)\otimes_{\mathbb{Z}}\mathbb{R}$

Choose $s_i \in \mathbb{Q}_{>0}$, $\underline{|s_i^2 h(x_i) - 1| < b^{-1}}$

Step 3 $\quad X_k^m \longrightarrow \mathbb{P}^n_k \times \dots \times \mathbb{P}^n_k$

$\partial_{1,1}, \dots \partial_{1,n} \qquad \partial_{m,1}, \dots, \partial_{m,n}$

Say $\left(\prod_{i,j} \partial_{i_j}^{e_{i,j}}\right) f \neq 0$, $\quad \sum_{i=1}^m \underbrace{\sum_{j=1}^n e_{ij}}_{= e_i} / d_i = \mathrm{index}(f, x, \underline{d}) < \sigma$

$$f' \overset{\text{def}}{=} \left(\prod_{i,j} \frac{(NG_i)^{e_{ij}}}{e_{ij}!}\right) \cdot \left(\prod_{i,j} \partial_{i_j}^{e_{ij}} f\right)(x) \in \Gamma(\mathrm{Spec}(\mathcal{O}_k), \mathcal{M})$$

$N \in \Gamma(Y, \mathcal{N})$ kills $\Omega^1_{Y/X^m_{/\mathcal{O}_K}}$

$$\mathcal{M} = x^*\left(\mathcal{L}(\sigma-\varepsilon, s_1, \dots, s_m)^{\otimes d} \otimes \bigotimes_{i=1}^m \mathcal{L}_i^{e_{ig}}\right)$$

(a) Upper bound (trivial estimate)

Lemma 5 $\exists c_2, c_3$ not depending on b, σ and the x_i's s.t.

$$\deg \mathcal{M} \leq md\left(c_3\sigma - \frac{\varepsilon}{2}\right) + c_2 d b^{-1}$$

normalized by $[k:\mathbb{Q}]^{-1}$

from f' is an integral section over $\mathrm{Spec}(\mathcal{O}_k)$, + estimate at arch. places

(b) Lower bound $\exists c_8(\sigma)$ not depending on b and the x_i's, s.t

Lemma 6 $\deg \mathcal{M} \geq -c_8(\sigma) d b^{-1}$

Lemma 9

Take $\sigma < \frac{\varepsilon}{2c_3}$ (so that $c_3\sigma - \frac{\varepsilon}{2} < 0$), and b large enough so that

$m\left(c_3\sigma - \frac{\varepsilon}{2}\right) < \left(-c_8(\sigma)d - c_2\right)b^{-1}$, to get a contradiction

Lemma 7. $\exists$ constant $c_4 \in \mathbb{R}$ s.t. $\forall d_i \geq 0$ and all $0 \neq f \in \Gamma(Y_{/\mathcal{O}_k}, \otimes_i \mathcal{L}_i^{\otimes d_i})$ have $\|f\|_{L^\infty, Y(k\otimes_{\mathbb{Q}}\mathbb{C})} \geq \exp(-c_4 \sum d_i)$

Pf: Explain the special case when $\exists$ morphism $\pi : X_k \longrightarrow \mathbb{P}^{\dim(X_k)}$ s.t. $\mathcal{L} \cong \pi^* \mathcal{O}_{\mathbb{P}^{\dim X_k}}(1)$.

$\forall P = (P_1, \dots, P_m) \in Y(\mathcal{O}_L)$ $\quad L \,/\, k$ finite

s.t. $P^* f \neq 0$, have

$$h_{\otimes_i \mathcal{L}_i^{d_i}}(P) = \deg_{\mathcal{O}_L}(P^* \otimes_i \mathcal{L}_i^{\otimes d_i}) = \sum_{i=1}^m d_i h_{\mathcal{L}}(P_i)$$

$$\| \quad [L:k]^{-1}\left(\log \#\left[(P^* \otimes_i \mathcal{L}_i^{d_i})/\mathcal{O}_L \cdot P^* f\right] - \sum_{v \in M_{L,\infty}} \varepsilon(v) \log \|f(P)\|_v\right)$$

$$\geq - \sum_{v \in M_{L,\infty}} \varepsilon(v) \log \|f(P)\|_v$$

Take $P = (P_1, \dots, P_m)$ s.t. all projective coordinates of P_i $\in \mu(\bar{\mathbb{Q}})$ roots, $\forall i$

These points are dense in $\mathbb{P}^{\dim(X_k)}(k \otimes \mathbb{C})$. QED.

ε "already fixed"

Theorem 1. Given $0 < \sigma < \varepsilon$, $\exists\, c(\sigma) > 0$, depending only on σ (and ε) with the following property:

$\forall$ smooth k-point $x \in X_k^m(k)$, any $s_1, \dots, s_m \in \mathbb{Q}_{>0}$ with $s_h/s_{h+1} \geq s$ $\forall h = 1, \dots, m-1$

$\forall$ square d sufficiently large and divisible,

$\exists\, f \in \Gamma(Y^0, \mathcal{L}(\sigma - \varepsilon, s_1, \dots, s_m)^{\otimes d})$ with

$$\text{index}(f, x, \underline{d}) < \sigma \qquad \underline{d} = (d_1, \dots, d_m). \quad d_i = 4 d s_i^2$$

and $\quad \|f\|_{L^\infty, Y(k\otimes_{\mathbb{Q}}\mathbb{C})} \leq \exp\left(c(\sigma) \sum_{i=1}^m d_i\right)$

Note about the term $d b^{-1}$:

$$\sum_i d_i = 4d \sum_i s_i^2 \leq 4d\, b^{-1}(1+b^{-1}) \qquad \text{as } |s_i^2 h_i - 1| \leq b^{-1} \ \forall_i \text{ and } h_i \geq b \ \forall_i$$

Lemma 8. $\exists\, c_2 \in \mathbb{R}$, not depending on b, σ, and the x_i's, s.t.
$\deg x^* \mathcal{L}(\sigma-\varepsilon, s_1, \dots, s_m)^{\otimes d} \leq m d (\sigma - \frac{\varepsilon}{2}) + c_2\, d b^{-1}$

Lemma 9. $\exists\, C_4(\sigma) \in \mathbb{R}$, not depending on b and the x_i's, s.t.

Lemma 6

$\log \| f' \|_v \leq C_4(\sigma)\, d b^{-1} \qquad \forall v \in M_{k,\infty}$

integral section at x