Math 7020 Fall 2024

(Foundations of) synthetic geometry Given • a set of points
• a set of lines

<u>Def^n</u> For any two lines ℓ, m, say $\ell \parallel m$
if either $\ell = m$
or no point P lies on both ℓ and m

<u>Axiom 1</u> Given two distinct points P and Q, there exists a unique line ℓ s.t. P and Q both lie on ℓ
Notation: $\ell = P+Q$.

<u>Axiom 2</u> Given a point P and a line ℓ, $\exists!$ line m s.t. P lies on m and $m \parallel \ell$

<u>Axiom 3</u> $\exists$ distinct points A, B, C s.t. C does not lie on $A+B$

<u>Def^n</u>. $\sigma: \{\text{points}\} \to \{\text{points}\}$ is a <u>dilatation</u> if
$\forall$ points $P \neq Q$, then $\sigma(Q)$ lies on the line ℓ
s.t. ℓ passes through P and $\ell \parallel P+Q$

<u>Exer</u> (a) A dilatation σ is uniquely determined by the images $\sigma(P)$ and $\sigma(Q)$ for two distinct points P, Q
(b) If $\sigma(P) = \sigma(Q)$, $P \neq Q$, then σ is degenerate in the sense that $\sigma(R) = \sigma(P)$ $\forall$ point R

$\Big(\forall R, \forall P \neq Q \quad \sigma(R) = (\text{the line through } \sigma(P) \text{ and } \parallel R+P) \cap (\text{the line through } \sigma(Q) \text{ and } \parallel R+Q) \Big)$
$\underset{P,Q}{\not\!\!\times}$

<u>Exer</u>. Every nondegen dilatation is a bijection

<u>Def^n</u> σ: a non-degenerate dilatation
P: a point
Any line containing P and $\sigma(P)$ is a <u>trace</u> of σ

<u>Def^n</u> A <u>translation</u> is a nondegen. dilatation τ such that either $\tau = id$ or τ has no fixed point

Exer. Let σ be a nondegen dilatation

(a) If $\sigma = id$, then every line is a trace

(b) If $\sigma(P) = P$, $\sigma \neq id$. then $\{\text{traces of } \sigma\} = \{\text{lines through } P\}$

(c) If σ has no fixed point, then
$\{\text{traces of } \sigma\} =$ a pencil of parallel lines

Exer. A translation τ is determined by the image of one point

Hint: Given a point P and $\tau(P)$
Let ℓ = a trace of τ containing P $\Rightarrow$ $\ell' \parallel \ell$, $Q \in \ell'$
$\tau(Q) =$ (the line ℓ' parallel to traces of ℓ and contains Q) $\cap$ (the line parall to $P+Q$ and pass through P)

Exer. $D = \{\overset{\text{nondegen}}{\text{dilatations}}\}$ is a subgroup of Perm(points)
$\cup\!|$
$T = \{\text{translations}\}$ $\quad T \trianglelefteq D$

Exer. If $\exists$ translation $\underset{\neq 1}{\tau_1}, \underset{\neq 1}{\tau_2}$ with different directions (traces), then T is a comm. group.

Axiom 4a $\quad \forall$ points P, Q, $\exists$ a translation τ with $\tau(P) = Q$

Axiom 4b $\quad$ If $\underset{\neq id}{\tau_1} \neq \underset{\neq id}{\tau_2}$ are translations with the same trace.
then $\exists$ a trace preserving endom. $\underset{k}{\alpha}$ of T s.t. $\tau_2 = \tau_1^\alpha$

Def^n of $k = \{\alpha \in End_{gp}(T, T) \mid \tau \parallel \tau^\alpha \quad \forall \tau \text{ s.t. } \tau^\alpha \neq id\}$
i.e. $\{\text{traces of } \tau\} \subseteq \{\text{traces of } \tau^\alpha\}$

Exer k is a division ring

Variant
Axiom 4b P (a point) $\quad$ Given P, Q, R three distinct points on a line.
$\exists$ a dilatation σ s.t. $\tau(P) = P$ and $\tau(Q) = R$

Exer. Axiom 4b $\Longleftrightarrow$ Axiom 4b P for one point P
$\Longleftrightarrow$ Axiom 4b P for all points P

Theorem. Let $\tau_1 \neq id$, $\tau_2 \neq id$ be two translations with different directions
Then $\forall$ translation $\tau \in T$, $\exists\, \alpha, \beta \in k$ uniquely determined by τ
s.t. $\tau = \tau_2^\beta \tau_1^\alpha$

Note
Axiom 4a $\Rightarrow$ $\{\text{points}\} \xleftrightarrow{\text{bijection}} T$

Thm: $T \underset{\text{bijection}}{\longleftrightarrow} k^2$

Introduce coordinates on the "plane" = {points}

Pick a point O ("the origin")

$\rightsquigarrow$ $\forall$ point P, let τ_{OP} = the translation st. $\tau_{OP}(O) = P$

Pick $\underset{\substack{\ncong \\ 1}}{\tau_1}, \underset{\substack{\ncong \\ 1}}{\tau_2}$ dilatations with different directions

$$\rightsquigarrow \quad \underset{\substack{\cap \\ k^2}}{(\alpha, \beta)} \longmapsto \underset{\substack{\cap \\ T}}{\tau_2^{\beta} \tau_1^{\alpha}}$$

$$k^2 \xrightarrow{\sim} T \xrightarrow{\sim} \{\text{points}\}$$

Conversely: In this coordinate system,

$$(\text{lines}) = \left(\text{subsets of } k^2 \text{ of the form } \{P + tA \mid t \in k\} \quad A \in k^2 \setminus (0,0),\ P \in k^2 \right)$$

dilatation on $V \leftarrow$ 2-dim$^{\ell}$ left v.s. $/k$

$$\sigma : \underset{\substack{\cap \\ V}}{x} \longmapsto \alpha_\sigma x + \underset{\substack{\cap \\ V}}{v_0} \qquad \alpha_\sigma \in k$$

In case $v_0 = 0$, so that σ fixed 0, then

σ is k-linear iff $\alpha_\sigma \in Z(k)$.

"Desargues' theorem"

Let l_1, l_2, l_3 be three distinct lines satisfying one of the two conditions below.

(a) l_1, l_2, l_3 are parallel

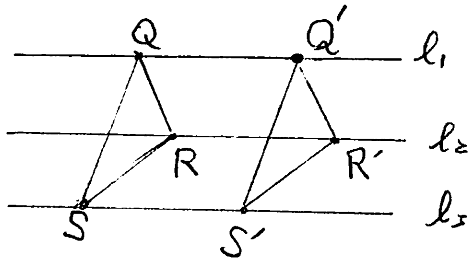
(b) l_1, l_2, l_3 meet at a point P

Let $Q, Q' \in l_1$, $R, R' \in l_2$, $S, S' \in l_3$

Suppose that $Q+R \parallel Q'+R'$, $Q+S \parallel Q'+S'$

then $R+S \parallel R'+S'$

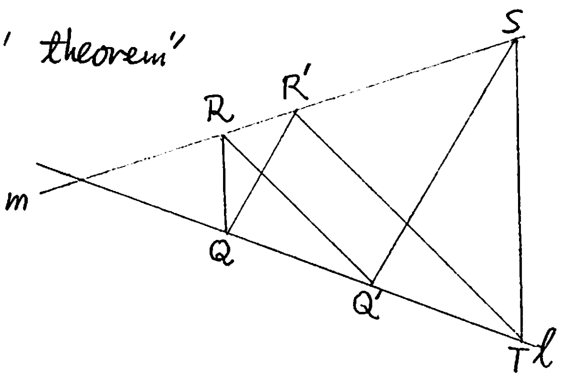
picture of case (a)



<u>Theorem</u> Desargues (a) $\Longleftrightarrow$ axiom 4a
Desargues (b) $\Longleftrightarrow$ axiom 4b

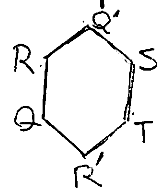
axioms 1, 2, 3, 4a, 4b : Desarguian geometry

"Pappus' theorem"



Consider the "hexagon" with vertices R, Q, R', T, S, Q' "inscribed" in the pair l, m of lines.
and the 3 pairs of "opposite sides" $R+Q$, $T+S$; $Q+R'$, $S+Q'$; $R'+T$, $Q'+R$

If two of the three pairs of lines are parallel, then so is the third



<u>Theorem</u> Pappus' theorem $\Longleftrightarrow$ k is commutative

"Fundamental theorem of projective geometry":

V, V': finite dim^e left vector spaces of dimension ≥ 3 over a division rings k and k'

Let $\sigma : \mathbb{P}(V) \longrightarrow \mathbb{P}(V')$ be a bijection s.t.

$\forall$ points $[L_1], [L_2], [L_3] \in \mathbb{P}(V)$ s.t. $L_1 \subseteq L_2 + L_3$

have $\sigma L_1 \subseteq \sigma L_2 + \sigma L_3$

<u>Then</u>: $\exists$ an isomorphism $\mu : k \xrightarrow{\sim} k'$ and a μ-semilinear isomorphism $\phi : V_1 \xrightarrow{\sim} V_2$ which induces σ

Moreover if $\phi' : V_1 \xrightarrow{\sim} V_2$ is a μ'-semilinear isomorphism which induces σ,
then $\exists\, a \in k^\times$ s.t. $\lambda'(v) = \lambda(av)$ $\forall v \in V$,

<u>and</u> $\mu'(x) = \mu(axa^{-1})$ $\forall x \in k$

[<u>Check</u>: $\lambda'(xv) = \mu'(x)\,\lambda'(v) = \mu'(x)\,\lambda(av) = \mu'(x)\,\mu(a)\,\lambda(v)$
$\quad\quad \lambda'(xv) = \lambda(axv) = \mu(a)\,\mu(x)\,\lambda(v)$

All algebras /F are finite dimensional

<u>Thm</u> (Skolem-Noether) B, A central simple algebras /F $\quad B \subseteq A$

Every F-alg homom $\rho : B \to A$ extends to an inner automorphism of A

Convention: (a) M is a right R-module $\rightsquigarrow$ $\mathrm{End}_R(M)$ operates on M on the left

(b) M is a left R-module $\rightsquigarrow$ $\mathrm{End}_R(M)$ operates on M on the <u>right</u>

<u>ie</u> the composition law on $\mathrm{End}_R(M)$ is:

$$f, g \in \mathrm{End}_R(M), \quad x \in M, \quad x^{fg} = (x^f)^g$$

Consequence: (i) $R \longrightarrow \mathrm{End}_{\mathrm{End}_R(M)}(M)$ is a ring homom. for both left and right R-mod.

(ii) $R = \mathrm{End}_R(M), \quad S \subseteq \mathrm{End}_R(M)$

$$\rightsquigarrow \mathrm{End}_S(M) = \left(C_{\mathrm{End}_R(M)}(S)\right)^{op}$$

<u>Double centralizer</u>: $\quad B \subseteq A \quad$ A: central simple /F, B: simple subalg.

Then: $C_A(C_A B) = B$, $\quad \dim_F(A) = \dim_F(B) \cdot \dim_F C_A(B) \qquad C(C_A B) = C_A(B) =: K$

If $K := C(B) = F$, then $B \otimes_F C_A B \xrightarrow{\sim} A$

In general, $[B \otimes_K C_A(B)] = [A \otimes_F K]$ in $Br(K)$

<u>Def</u> A: central simple /F $\quad M$: left A-module

$$\mathrm{rdim}_A(M) = \frac{\dim_F(M)}{\deg(A)} \qquad M \cong M_{r,s}(D) \qquad A = M_r(D)$$

$$\deg(A)^2 = \dim_F(A) \qquad \mathrm{rdim}(M) = s \cdot \mathrm{ind}(A)$$

$\mathrm{ind}(A)$ = Schur index $(A) = \deg D$

Morita

<u>Prop.</u> 1) A: central simple /F

M: non-zero left A-module $\quad E = \mathrm{End}_A(M)$

<u>Then</u> M is an (A, E)-bimodule (by our convention)

E: central simple, $\quad [E] \sim [A]$

$A = \mathrm{End}_E(M)$

2) Conversely, if A, E are Brauer-equiv central simple alg /F then $\exists$ an (A, E) bimodule M

s.t. $A = \mathrm{End}_E(M)$, $E = \mathrm{End}_A(M)$

$$\mathrm{rdim}_A(M) = \deg(E), \quad \mathrm{rdim}_E(M) = \deg(A)$$

left-right ideals in $\mathrm{End}_D(V)$, V: f. dim$^{\ell}$ right v.s. $/D$

Prop. 1) $\{U : \text{right v. subspace of } V \text{ over } D\} \xrightarrow[\text{bij}]{\sim} \{\text{right ideals of } \underbrace{\mathrm{End}_D(V)}_{\overset{\shortparallel}{A}}\}$

Morita again

$$U \longmapsto \mathrm{Hom}_D(V, U) \ni \alpha$$

$$\alpha \cdot f = \alpha \circ f, \quad f \in \mathrm{End}_D(V)$$

$$\mathrm{rdim}_A \mathrm{Hom}_D(V, U) = \dim_D(U) \cdot \deg D$$

2) $\{U : \text{right v. subsp. of } V \text{ over } D\} \xrightarrow[\text{bij}]{\sim} \{\text{left ideals of } \mathrm{End}_D(V)\}$

$$U \longmapsto \mathrm{Hom}_D(V/U, V)$$

$$\mathrm{rdim}_A(\mathrm{Hom}_D(V/U, V)) = \dim_D(V/U) \cdot \deg D$$

3) $\mathrm{End}_A(\mathrm{Hom}_D(V, U)) \cong \mathrm{End}_D(U)$

$\mathrm{End}_A(\mathrm{Hom}(V/U, U)) \cong \mathrm{End}_D(V/U)$

<u>Corollary</u> $\forall$ left ideal $I \subseteq A = \mathrm{End}_D(V)$, $\exists$ idempotent $e \in A$ s.t. $I = Ae$

Similarly for right ideals of $A = \mathrm{End}_D(V)$

$$I \overset{\rho}{\twoheadrightarrow} \mathrm{End}_A(I) \quad \text{induces} \quad eAe \xrightarrow{\sim} \mathrm{End}_A(I)$$

$$x \longmapsto (\alpha \mapsto (y \mapsto \alpha(yx)))$$

left/right Annihilator

I: left ideal in A $\leadsto$ $I^0 = \{x \in A \mid Ix = (0)\}$ is a right ideal

J: right ideal $\quad I^0 = \{y \in A \mid xI = (0)\}$ is a left ideal

A: central simple

For $A = \mathrm{End}_D(V)$ and $I = \mathrm{Hom}_D(V/U, V) \leftarrow$ left A-ideal

right v.sp. $/D$

$$I^0 = \{\alpha \in \mathrm{End}_D(V) \mid \underbrace{V \xrightarrow{\alpha} V \twoheadrightarrow V/U \xrightarrow{f} V}_{=(0)} \ \forall f\}$$

$$= \mathrm{Hom}_D(V, U)$$

$\underline{(I^0)^0 = I}$ $\quad \forall$ left ideal I of $A = \mathrm{End}_D(V)$

Severi-Brauer variety A: central simple $/F$. $n = \deg(A)$

$\forall r$, with $1 \le r \le n$,

$\forall L/F$ ext^n field.

$SB_r(A)(L) = \{$ nr-dim^l subspaces of $A \otimes_F L$ which are right $(A\otimes_F L)$-ideals$\}$

(Similarly for commutative F-algebras R)

use: local direct summands $A \otimes_F R$

$$SB_r(A) \xrightarrow{\sim} SB_{n-r}(A^{opp})$$
$$I \mapsto I^0$$

<u>Involutions</u> V: f.d^l v.sp $/F$

$b: V \times V \to F \quad \longleftrightarrow \quad \hat{b}: V \to V^\vee$

$\hat{b}(x)(y) = b(x,y)$ Assume: b is non-singular

σ_b: antiautomorphism on $\mathrm{End}_F(V)$

$$\sigma_b(T) = \hat{b}^{-1} \cdot {}^tT \cdot \hat{b}$$

$$b(\sigma_b(T)x, y) = \hat{b}(\sigma_b(T)(x))(y) = \hat{b}(\hat{b}^{-1}\,{}^tT \cdot \hat{b}(x))(y)$$
$$= \hat{T}(\hat{b}(x))y = \hat{b}(x)(Ty) = b(x, T(y))$$

$$b(\sigma_b(T)x, y) = b(x, T(y)) \quad \forall x, y \in V, \ \forall T \in \mathrm{End}(V)$$

<u>Note</u>: For $b'(x,y) \overset{\text{def}}{=} b(y,x)$, $\sigma_{b'} = \sigma_b^{-1}$

$$b'(\sigma_{b'}^{-1}(T)(x), y) = b(y, \sigma_b^{-1}(T)(x)) = b(\sigma_b(\sigma_b^{-1}(T))x, y)$$
$$\| ?$$
$$b'(x, T(y)) = b(T(y), x) \quad // \quad \checkmark$$

$\leadsto$ If $\sigma_b^2 = id$, then $\sigma_b = \sigma_{b'}$

$\Rightarrow \exists c \in F^\times$ s.t. $b' = c \cdot b \Rightarrow c^2 = 1 \Rightarrow c = \pm 1$

i.e. b is either symmetric or skew-symmetric

i.e.

Lemma. σ_b is an involution $\iff$ b is either symmetric or skew-symmetric

$\leadsto$ By Skolem-Noether

For every involution ~~involution~~ σ on $\mathrm{End}_F(V)$ of the first kind,

$\exists$ a regular bilinear form $b: V \times V \to F$, either symmetric or skew-symm.

s.t. $\sigma_b = \sigma$.

Notation: σ: involution of the first kind on a central simple alg. A/F

$$\begin{aligned} \mathrm{Sym}(A,\sigma) &\supseteq \mathrm{Symd}(A,\sigma) = \{a+\sigma(a) \mid a\in A\} \\ \mathrm{Skew}(A,\sigma) &\supseteq \mathrm{Alt}(A,\sigma) = \{a-\sigma(a) \mid a\in A\} \end{aligned}$$

$$\begin{aligned} \mathrm{Symd}(A,\sigma) &= \mathrm{Skew}(A,\sigma)^{\perp} \\ \mathrm{Sym}(A,\sigma) &= \mathrm{Alt}(A,\sigma)^{\perp} \end{aligned}$$ w.r.t Trd_A (reduced trace)

Case $\sigma=\sigma_b$ = adjoint involution attached to a non-sing $b: V\times V\to F$, $A=\mathrm{End}(V)$

(1) If b is symmetric, then $\dim_F \mathrm{Sym}(A,\sigma) = \frac{n(n+1)}{2}$ $\quad A=\mathrm{End}(V)$

(2) If b is skew-symmetric, then $\dim_F \mathrm{Skew}(A,\sigma)=\frac{n(n+1)}{2}$

(3) If $\mathrm{Char}(F)=2$, then b is alternating iff $\mathrm{tr}(T)=0$ $\forall T\in \mathrm{Sym}(A,\sigma)$

Note: b is alternating and regular $\Rightarrow n\equiv 0 \pmod 2$

Pf of (3): If b is not alternating $\Rightarrow \exists v\in V$ s.t. $b(v,v)\neq 0$

Consider the element $T\in A=\mathrm{End}(V)$

$$T(x) = v\cdot b(v,x)\cdot b(v,v)^{-1} \quad \forall x\in V$$

necessarily symmetric

$$b(Tx,y) = \underbrace{b(v,y)\cdot b(v,x)\, b(v,v)^{-1}}_{\text{symmetric in } x \text{ and } y} = b(Ty,x) = b(x,Ty)$$

$\forall x,y\in V$ i.e. T is symmetric

$\Rightarrow \sigma(T)=T$

$$\begin{aligned} T^2(x) &= b(v,x)\, b(v,v)^{-1}\cdot T(v) \\ &= v\cdot b(v,x)\ b(v,v)^{-1}\, b(v,v)\cdot b(v,v^{-1}) = T(x) \end{aligned}$$

Note $T(v)=v$, $\quad T((Fv)^{\perp})=0$

$\Rightarrow \mathrm{Trd}(T)=1 \qquad T$ symmetric

Conversely, suppose that b is alternating.
Pick a symplectic ~~basis~~ for b.

$e_1, e_2, \dots, e_{2n-1}, e_{2n}$ $\qquad b(e_{2i-1},e_{2i}) = 1 = b(e_{2i}, e_{2i-1})$

other entries of the Gram matrix are 0

$T(e_j)=\sum e_i\cdot a_{ij}$

$$T\in \mathrm{Sym}(A,\sigma) \Rightarrow \underset{\substack{\| \\ b(T(e_{2i-1}),\,e_{2i})}}{a_{2i-1,2i-1}} = \underset{\substack{\searrow \\ b(e_{2i-1},\,T(e_{2i}))}}{a_{2i,2i}}$$

$$\Rightarrow \mathrm{Trd}(T) = 2\sum_{i=1}^{\frac{n}{2}} a_{2i,2i} = 0 \qquad n=\dim(V)$$

q.e.d.

σ: involution of the first kind on $A \leftarrow$ central simple$/F$ $\deg(A)=n$

(1) Assume $\mathrm{Char}(F) \neq 2$.

Proposition (1a) If σ is of orthogonal type, then

$$\dim_F \mathrm{Sym}(A,\sigma) = \frac{n(n+1)}{2}, \quad \dim_F \mathrm{Skew}(A,\sigma) = \frac{n(n-1)}{2}$$

(1b) If σ is of symplectic type, then

$$\dim_F \mathrm{Skew}(A,\sigma) = \frac{n(n+1)}{2}, \quad \dim_F \mathrm{Sym}(A,\sigma) = \frac{n(n-1)}{2}$$

(2) Assume $\mathrm{Char}(F) = 2$.

$\leadsto$ $\dim_F \mathrm{Sym}(A,\sigma) = \dim_F \mathrm{Skew}(A,\sigma) = \frac{n(n+1)}{2}$

$\dim_F \mathrm{Alt}(A,\sigma) = \dim \mathrm{Symd}(A,\sigma) = \frac{n(n-1)}{2}$

σ is of symplectic type $\iff$ $\mathrm{Trd}_A(\underbrace{\mathrm{Sym}(A,\sigma)}_{\mathrm{Alt}(A,\sigma)^{\perp}}) = (0)$ $\leftarrow$ w.r.t. Trd_A

$\iff$ $1 \in \mathrm{Alt}(A,\sigma)$

Existence of involution

A/F : central simple

Thm (Albert) $\to$ (1) A has an involution of the 1st kind

$\iff$ $A \otimes A$ splits

(2) (Albert - Riehm - Scharlau) F/E separable quadratic

A admits an involution σ of the 2nd kind s.t. $\langle \underbrace{\sigma|_F}_{\iota} \rangle = \mathrm{Gal}(F/E)$

$\iff$ $N_{F/E}(A)$ splits

$= \{ {}^{\iota}b \in {}^{\iota}A \otimes_F A \mid s(b) = b \}$

where ${}^{\iota}A \leftarrow$ "conjugate algebra of A"

${}^{\iota}A = \{ {}^{\iota}a \mid a \in A \}$ ${}^{\iota}a + {}^{\iota}b = {}^{\iota}(a+b)$ ${}^{\iota}a \cdot {}^{\iota}b = {}^{\iota}(a \cdot b)$

${}^{\iota}(\underset{\in F}{\lambda} \cdot a) = \iota(\lambda) \cdot {}^{\iota}a$

$= F \otimes_{(\iota, F)} A$

$N_{F/E}(A)$ $\leftarrow$ "corestriction"

Clifford algebra

Operations on tensor algebras $\quad$ E: A-module $\quad$ A: commutative ring

$f \in E^\vee \rightsquigarrow \quad i_f : T^\cdot E \to T^\cdot E$ A-linear $\quad i_f(T^n E) \subseteq T^{n-1}(E)$

$$i_f \circ e_x + e_x \circ i_f = f(x) \cdot id \quad \forall x \in E, \quad i_f(1) = 0$$

$$i_f \circ i_g + i_g \circ i_f = 0 \quad \forall f, g \in E^\vee$$

Given a bilinear form $\quad F : E \times E \to A$

$\rightsquigarrow \quad \lambda_F : T^\cdot E \to T^\cdot E$, A-linear, s.t.

inductive def^n
$$\begin{cases} \lambda_F(1) = 1, \\ \lambda_F \circ e_x = (e_x + i_x^F) \circ \lambda_F & \forall x \in E, \text{ where } i_x^F = i_{f_x} \end{cases} \qquad f_x(y) = F(x,y)$$

We have $\quad \lambda_F \circ i_f = i_f \circ \lambda_F \quad \forall f \in E^\vee$

Lemma: $\quad F, G : E \times E \to A$ bilinear forms

(a) $\lambda_{F+G} = \lambda_F \circ \lambda_G$

(b) $\lambda_F : T^\cdot E \to T^\cdot E$ is bijective

Given (E, Q) $\quad Q : E \to A$ quadratic form $\quad$ and $\quad F : E \times E \to A$ bilinear, $Q'(x) = Q(x) + F(x,x)$

$$\lambda_F \circ e_x^2 = (e_x + i_x^F)^2 \lambda_F = (e_x^2 + e_x \circ i_x^F + i_x^F \circ e_x) \circ \lambda_F = (e_x^2 + F(x,x)) \circ \lambda_F$$

$$\rightsquigarrow \lambda_F(x \otimes x \otimes u - Q'(x)u) = (e_x^2 + F(x,x) - Q'(x)) \circ \lambda_F = (e_x^2 - Q(x)) \circ \lambda_F$$
$$= (x \otimes x - Q(x)) \otimes \lambda_F(u)$$

i.e. $\quad \lambda_F(x \otimes x \otimes u - Q'(x) u) = (x \otimes x - Q(x)) \otimes \lambda_F(u)$

So: $\quad \lambda_F$ descends to $\quad \bar{\lambda}_F : Cl(Q') \xrightarrow{\sim} Cl(Q)$
A-linear bijection

Prop: Suppose that E is a free A-module with basis $(x_i)_{i \in I}$ $\quad$ I: totally ordered
$\rho : E \to Cl(E, Q)$ $\quad Q : E \to A$ quadratic form
Then
$Cl(E, Q)$ is a free A-module with basis
$$\{ \rho(x_{i_1}) \cdots \rho(x_{i_\ell}) \mid i_1 < \cdots < i_\ell \}$$

Cor (i) In particular, $E \xrightarrow{\rho} Cl(E, Q)$ is injective

(ii) If E is free of rank n over A, then $Cl(E, Q)$ is free of rank 2^n over A

Cor: $Cl((E_1,Q_1)\overset{\perp}{\oplus}(E_2,Q_2)) \xrightarrow{\sim} Cl(E_1,Q_1)\,\hat{\otimes}\, Cl(E_2,Q_2)$

isom of $(\mathbb{Z}/2\mathbb{Z})$-graded alg $/A$ — tensor product of $(\mathbb{Z}/2\mathbb{Z})$-graded alg $/A$

<u>Thm 1</u> K: field. E: vector space$/K$, $\dim(E)=2r$, $Q: E \to K$ quadratic

Suppose Q is non-degen (i.e. $\Phi(x,y) := Q(x+y)-Q(x)-Q(y)$ induces $E \xrightarrow{\sim} E^\vee$)
and neutral (i.e. $E \cong N \oplus P$, N, P both totally isotropic)

(1) $C(E,Q) \cong M_{2^r}(K)$

(2) $C^+(Q) \cong M_{2^{r-1}}(K) \times M_{2^{r-1}}(K)$, $Z(C^+(Q)) \cong K \times K$

Pf: Pick bases $(n_1,\dots,n_r)$ of N, $(p_1,\dots,p_r)$ of P s.t. $\Phi(n_i,p_j)=\delta_{ij}$

$\forall p\in P$, let $i_p = i_{\Phi(\cdot,p)}: T^\bullet N \to T^\bullet N$

Consider $e_n, i_p: T^\bullet N \to T^\bullet N$

For $x = n+p$ $\quad x^2 = Q(x) = \Phi(n,p)$

define $s(x) = e_n|_{C(N)} + i_p|_{C(N,Q|_N)} \in End_K(C(N,Q|_N))$

$s(x)^2 = (e_n+i_p)^2|_{C(N)} = \Phi(n,p)\, id_{C(N)} = Q(x)\cdot id_{C(N)}$

$\rightsquigarrow$ Get a homomorphism $C(Q) \xrightarrow{s} End_K(C^\bullet N)$

Check this homom. s is surjective:

Let $x_{H,K} \overset{def}{=} \underbrace{n_H}_{\prod_{i\in H} n_i}\, p_{\{1,\dots,r\}}\, \underbrace{n_K}_{\prod_{i\in K} n_i}$ $\qquad H, K \in 2^{\{1,\dots,r\}}$

$\rightsquigarrow s(x_{H,K})(n_L) = \begin{cases} 0 & \text{if } K\neq L \\ \pm n_H & \text{if } K=L \end{cases} \qquad \forall L \in 2^{\{1,\dots,r\}}$

So: $s: C(Q) \xrightarrow{\sim} End_K(C^\bullet(N,Q|_N))$

$C^+(Q) \longrightarrow End_K(C^+(Q|_N)) \times End_K(C^-(Q|_N))$

injective because s is

dimension equal $\rightsquigarrow$ $C^+(Q) \xrightarrow{\sim} End_K(C^+(Q|_N)) \times End_K(C^-(Q|_N))$

QED

Theorem: (E,Q) nondegen$_{/K}$, $\dim_K(E) = 2r+1$ ($\rightsquigarrow$ char$(k) \neq 2$)

(1) $C^+(Q)$ is central simple over K

(1') If index$(Q) = r$, then $C^+(Q) \cong M_{2^r}(K)$

(2) $C(Q)$ is separable $/K$. $\dim_K Z(C(Q)) = 2$.

$C(Q) \cong Z(C(Q)) \otimes_K C^+(Q)$

Pf Take an element $x_0 \in E$ s.t. $Q(x_0) \neq 0$. Let $F = K x_0^{\perp}$

Consider $C(F, \underbrace{Q(x_0)\cdot Q|_F}_{= Q_1}) \longrightarrow C^+(E,Q)$

$y \longmapsto x_0 y \quad \forall y \in F$

$\dim(F) \equiv 0 \pmod 2$ $\rightsquigarrow$ $C(F, Q_1)$ is central simple

$\overset{\rightsquigarrow}{\text{dim. reason}}$ $C(F, Q_1) \xrightarrow{\sim} C^+(E,Q)$

Note: If index$(Q) = r$, choose x_0 s.t. (F, Q_1) is neutral, then we get $C^+(Q) \cong C(F, Q_1) \cong M_{2^r}(K)$

$Z = K\cdot 1 + K\cdot x_0 x_1 \cdots x_{2r}$ $\qquad$ $x_0, \dots, x_{2r}$ orthogonal basis of E

$Z \otimes_K C^+(Q) \xrightarrow{h} C(Q)$ $\qquad$ $h(Z \otimes_K C^+(Q)) = C^+(Q) + (x_0 \cdots x_{2r})\cdot C^+(Q)$

$= C^+(Q) + C^-(Q) = C(Q)$.

$\rightsquigarrow$ h is an isom

QED.

char$(K) \neq 2$

(twisted) Clifford group $\qquad$ $\alpha: C(Q) \to C(Q)$ canonical principal automorm (attached to the orthogonal transf. $-Id_E$)

$t: C(Q) \to C(Q)$ transpose/principal involution

$\Gamma(Q) = \{ x \in C(Q)^{\times} \mid \underbrace{\alpha(x) E x^{-1}}_{\text{twisted adjoint}} \doteq E \}$ $\rightsquigarrow$ $\rho: \Gamma(Q) \to \mathrm{Aut}_K(E)$

$x \mapsto (y \mapsto \alpha(x) y x^{-1})$

Lemma $\mathrm{Ker}(\rho) = K^{\times}\cdot Id_E$

Pf: Given $x \in \mathrm{Ker}(\rho)$, write $x = x_0 + x_1$ $\qquad$ $x_0 \in C^+(Q)$, $x_1 \in C^-(Q)$

Have $\alpha(x) y = y x \quad \forall y \in E$

i.e. $\begin{cases} x_0 y = y x_0 \\ x_1 y = -y x_1 \end{cases}$ $\forall y \in E$ $\qquad$ $a_0 \in C^+(F)$, $b_1 \in C^-(F)$

$\forall e \in E$ with $Q(e) \neq 0$ $\rightsquigarrow$ $E = Ke + \underbrace{e^{\perp}}_{= F}$ $\qquad$ $x_0 = a_0 + e b_1$

$x_0 e = e x_0$ $\rightsquigarrow$ $\underset{= a_0 + e b_1}{x_0} = e x_0 e^{-1} = e(a_0 + e b_1)e^{-1} = e a_0 - e b_1$ $\rightsquigarrow$ $x_0 = a_0 \in C^+(Ke^{\perp})$

$\forall e$ s.t. $Q(e) \neq 0$

Now write x_1 as $x_1 = a_1 + e b_0$ $\quad$ $a_1 \in C^-(F)$, $b_0 \in C^+(F)$ $\qquad$ $\Rightarrow x_0 \in K\cdot 1$

$a_1 + e b_0 = x_1 = -e x_1 e^{-1} = -e(a_1 + e b_0)e^{-1} = -e a_1 e^{-1} - e^2 b_0 e^{-1} = a_1 - e b_0$

$\Rightarrow x_1 \in C^-(F)$ $\quad \forall e$ s.t. $Q(e) \neq 0$

$\Rightarrow x_1 = 0$

QED

(twisted) spinor norm $\quad N: C(Q) \longrightarrow C(Q)$

$$x \longmapsto x\cdot \alpha({}^t x)$$

Lemma $\quad N(x) \in k^\times \cdot 1 \quad \forall x \in \Gamma(Q)$

pf: Will show $\quad N(x) \in \mathrm{Ker}(\rho) \quad \forall x \in \Gamma(Q)$

Given any $x \in \Gamma(Q)$, any $y \in E$ have $\alpha(x)\, y\, x^{-1} = z \in E$

$$ {}^t x^{-1} \cdot y\, \alpha({}^t x) = z = \alpha(x)\, y\, x^{-1}$$

$$\Rightarrow \quad y\, \alpha({}^t x)\cdot x = {}^t x \cdot \alpha(x)\, y$$

$$y = \alpha(\alpha({}^t x)\cdot x)\; y \cdot (\alpha({}^t x)\cdot x)^{-1} \qquad \forall y \in E$$

i.e. $\alpha({}^t x)\cdot x \in \mathrm{Ker}(\rho) = K^\times\cdot 1 \Rightarrow x\cdot \alpha({}^t x) = \alpha({}^t x)\cdot x \in K^\times \cdot 1 \qquad \forall x \in \Gamma(Q)$.

QED

Def $\quad \mathrm{Pin}(K) \overset{\text{def}}{=} \mathrm{Ker}(N: \Gamma(Q) \longrightarrow K^\times) \qquad \mathrm{Char}(K) \neq 2$

Proposition $\quad 1 \to \mu_2 \longrightarrow \mathrm{Pin}(K) \longrightarrow O(Q)(K) \to 1$

Lemma: $\quad \mathrm{Pin}(K) = \left(\mathrm{Pin}(K) \cap C(Q)^+\right) \cup \left(\mathrm{Pin}(K) \cap C(Q)^-\right)$

pf: Image of r.h.s. under the vector representation is equal to $O(Q)(K)$

Cartan-Dieudonné $\quad$ (M,Q) nonsingular $O(Q)(K)$ is generated by symmetries w.r.t hyperplanes orthogonal to non-singular vectors in M, except when

$\quad K = \mathbb{F}_2$, $\dim(M) = 4$, (Q, M) neutral (i.e. index 2)

"exceptional case"

More study of (M, Q) neutral, $\dim(M) = 2r$

N, P maximal totally singular, $M = N \oplus P$

$K\cdot f = \Lambda^r P \leq C(Q) \geq \Lambda^\bullet N$

$\rightsquigarrow \quad C(Q)\cdot f =$ a minimal left ideal of $C(Q)$

$\quad f\cdot C(Q) =$ a minimal right ideal of $C(Q)$

$\quad C(Q)\cdot f = \Lambda^\bullet N \cdot f$

$\quad\quad\quad u\cdot f \in$

$B =$ polar form of Q

$\quad P \ni y$ acts on $\Lambda^\bullet N$ as a $(\mathbb{Z}/2\mathbb{Z})$-graded derivation δ_y s.t $\delta_y|_N = (\underset{N}{x} \mapsto B(x,y))$

Remark on $Z(C(Q))$ when $\mathrm{char}(K) = 2$

use a basis of $M \rightsquigarrow$ get a matrix A representing Q

$B = A + {}^t A$ represent the polar of Q

Let $\quad \mathrm{disc}_\pm(Q) = (-1)^r + s_2(B^{-1} A) \in K/\wp(K) \qquad \wp: K \to K,\ t \mapsto t + t^2$

Then $Z(C(Q)) \cong K[X]/(X^2 + X + \mathrm{disc}_\pm(Q))$

$\dim(N) = 2r \qquad \mathrm{char}(K) \neq 2$

Prop: $C((N, Q_N) \overset{\perp}{\oplus} (P, Q_P)) \cong C(N, Q_N) \otimes_K C((-1)^r \mathrm{disc}(Q_N) \cdot Q_P)$

pf: $(N, Q_N) \cong \langle x_1 \rangle \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} \langle x_{2r} \rangle \qquad \mathrm{disc}(N, Q_N) = \underset{\text{usual, unsigned}}{Q_N(x_1) \cdots Q_N(x_{2r})} = z \qquad z^2 = (-1)^r \mathrm{disc}(Q_N)$

$Z(C_+(N, Q_N)) = K \cdot z$

z anti-commutes with $C_-(N, Q_N)$

$C(N, Q_N)$ commutes with $C_+(P, Q_P) + z \cdot C_-(P, Q_P)$

Get a homom of K-linear alg. $C((-1)^r \mathrm{disc}(Q_N) \cdot Q_P) \xrightarrow{\theta} C(Q_N \overset{\perp}{\oplus} Q_P)$

$$\begin{matrix} \cup\!\mid & & \\ P \ni p & \longmapsto & zp \end{matrix}$$

$\leadsto$ get a K-alg homom

$$C(N, Q_N) \otimes C((-1)^r \mathrm{disc}(Q_N) \cdot Q_P) \longrightarrow C(Q_N \overset{\perp}{\oplus} Q_P)$$

To get the inverse map, consider the map

$$\begin{matrix} N \oplus P & \longrightarrow & C(Q_N) \otimes_K C((-1)^r \mathrm{disc}(Q_N) \cdot Q_P) \\ (n, p) & \longmapsto & n + z^{-1} p \end{matrix}$$

which extends to a K-alg. homom

$$C(Q_N \overset{\perp}{\oplus} Q_P) \longrightarrow C(Q_N) \otimes_K C((-1)^r \mathrm{disc}(Q_N) \cdot Q_P)$$

QED

Prop $C_+((Q, M) \overset{\perp}{\oplus} \langle a \rangle) \cong C(-a \cdot Q)$

$\langle a \rangle$: $K \cdot x_1$, $Q(x_1) = a$

pf: Consider

$$\begin{matrix} M & \longrightarrow & C_+((Q, M) \overset{\perp}{\oplus} \langle a \rangle) \\ y & \longmapsto & y x_1 \end{matrix} \qquad y x_1 \cdot y x_1 = -Q(y) \cdot Q(x_1) \quad \forall y \in M$$

$\leadsto$ get a K-alg. homom

$$C(-a \cdot Q, M) \xrightarrow{\theta} C_+(Q, M), \quad \text{surjective}$$

QED.

spinors: split case $\dim M \equiv 0\ (2)$

$\dim(M) = 2r$, N, P totally singular. $M = N \oplus P$

$N = Kx_1 + \cdots + Kx_r$

$P = Ky_1 + \cdots + y_r \qquad B(x_i \cdot y_j) = \delta_{ij}$

$f = y_1 \cdots y_r$

$e = x_r \cdots x_1 = (-1)^{\frac{r(r-1)}{2}} x_1 \cdots x_r$

$C^N \cdot f = C(Q) f$ minimal left ideal

$e \cdot f$ = an idempotent $\because$ $fef = y_1 \cdots y_r x_r \cdots x_1 f = f$, $(ef)^2 = ef$

$\leadsto f \in C(Q) ef$, $\leadsto C(Q) f = C(Q) ef$

$\forall v \in C(Q)\ \forall u \in C^N = S = \Lambda^{\cdot} N$

$(\rho(v) \cdot u) f = v \cdot u f$

$\rho(v) \cdot u \in C^N$

$S_{even} \overset{\text{def}}{=} \Lambda^{even} N$

$S_{odd} = \Lambda^{odd} N$

Clifford group

$$\Gamma(Q) = \{ s \in C(Q)^\times : \; sMs^{-1} = M \} \leq C(Q)^\times$$

spinorial norm $\quad \lambda : \underset{\Gamma(Q) \cap C^+(Q)}{\overset{\Gamma(Q)}{\shortparallel}} \longrightarrow K^\times \qquad \lambda(s) = {}^t s \cdot s \quad \forall s \in \Gamma(Q)^+$

<u>Lemma</u>: If $\; s \in C(Q)$, $x \in M$, and $\exists y \in M$ s.t. $sx = ys$

then $\quad {}^t s \cdot s \cdot x = x \cdot {}^t s \cdot s$

Pf: $\quad sx = ys \rightsquigarrow x\,{}^t s = {}^t s\, y \rightsquigarrow {}^t s s x = {}^t s y s = x\, {}^t s s$.

$\rightsquigarrow$ Get $\quad \Gamma(Q) \xrightarrow{\lambda} K^\times$

$$\begin{array}{l} \quad \cup \\ \Gamma(Q)^+ \longrightarrow \left(Z(C(Q)) \cap C^+(Q) \right)^\times = K^\times \end{array}$$

$$\Gamma_0(Q) = \mathrm{Ker}(\lambda) \leq \Gamma(Q) \geq \Gamma^+(Q) \overset{\text{def}}{=} \Gamma(Q) \cap C^+(Q)^\times$$

$$\begin{array}{l} \quad \cup \\ \Gamma_0^+(Q) = \Gamma_0(Q) \cap \Gamma^+(Q) \end{array}$$

$\Gamma^+(Q)$: special Clifford group

$\Gamma_0^+(Q)$: reduced Clifford group

$$\chi : \Gamma(Q) \longrightarrow O(Q) \quad \text{vector repr.}$$

denoted by G in Chevalley

$$\begin{array}{l} \qquad \cup \\ \chi(\Gamma^+(Q)) = SO(Q)(K) \end{array}$$

$\Gamma^+(Q)(K)$

denoted by G^+ in Chevalley

$$\chi(\Gamma_0^+(Q)) = SO(Q)(K)_0$$

reduced orthogonal group,

$$SO(Q)(K) / SO(Q)(K)_0 \xrightarrow{\sim} H/(K^\times)^2$$

$$H = \langle Q(x)Q(y) \mid x, y \in M,\; Q(x) \neq 0 \neq Q(y) \rangle$$

Hurwitz's Thm: $\quad Q(\varphi(x,y)) = Q(x)\,Q(y) \qquad Q = \text{nondeg.}$

$\Rightarrow m = \dim(M) = 1, 2, 4, 8$

Example: $M = Ke + Kf$, $Q(e)=0=Q(f)$, $B(e,f)=1$ $\qquad ef+fe=1$ in $C(Q)$ (orthogonal idempotents)

$C^+(Q) = K + Kef \qquad (ef)^2 = e(1-ef)f = ef\ ,\quad (fe)^2 = fe$

What is $\Gamma^+(Q)$? $\qquad$ (Note:

$$N(u+vef) = (u+vef)\cdot(u+vfe) = u^2+uv = u(u+v) = 1$$

change variables: $\quad u+v=t,\ u=t^{-1} \quad v = t-t^{-1}$

$$u+vef = t^{-1}+(t-t^{-1})ef = t\,ef + t^{-1}fe$$

So: $N(tef+t^{-1}fe) = 1$

vector representation:

$$(t\cdot ef+t^{-1}fe)\cdot e\cdot(tfe+t^{-1}ef) = t\,efe\cdot(tfe+t^{-1}ef) = t^2 efe = t^2 e$$

$$(tef+t^{-1}fe)\cdot f\cdot(tfe+t^{-1}ef) = t^{-1}fef\cdot(tfe+t^{-1}ef) = t^{-2}fef = t^{-2}(1-ef)f = t^{-2}f$$

weights: $t^{\pm 2}$

spin representation on $C(Q)\cdot f = Kf + Kef$

$$(tef+t^{-1}fe)\cdot f = t^{-1}fef = t^{-1}(1-fe)f = t^{-1}f$$

$$(tef+t^{-1}fe)\cdot ef = tef\,ef = t\,ef$$

weights: $t^{\pm 1}$

[Note $(ef-fe)^2 = (ef)^2+(fe)^2 = ef+fe = 1$]

Representations of $O(Q)$ on $\Lambda^\bullet M \qquad \dim M = m$

Lemma: $\forall\, 0\le h\le m$. $\quad \Lambda^h M \xrightarrow{\sim} \Lambda^{m-h} M$ as $SO(Q)$-modules

Thm $\mathrm{Char}(K)\neq 2 \quad m=\dim(M)$ $\qquad$ Exceptional situation:

Not the exceptional situation $\quad K=\mathbb{F}_3,\ m=2,\ h=1,\ \mathrm{index}(Q)=1$

(1) The representations of $O(Q)(K)$ on $\Lambda^h M$ is irreducible $\quad \forall\, 0\le h\le 2m$

(2) If $2h\neq m$, then the representation of $SO(Q)(K)$ on $\Lambda^h M$ is irreducible

(3) If $2h=m$, then $\Lambda^h M \underset{SO(Q)(K)}{\cong}$ either irred or $U_1\oplus U_2 \quad U_1, U_2$ irred $SO(Q)(K)$ repr.

Moreover if $\Lambda^h M \cong U_1\oplus U_2$, U_1, U_2 irred. $SO(Q)(K)$-repr

then U_1, U_2 are non-isom as $SO(Q)(K)$ repr, and not isomorphic to any $\Lambda^h M$ with $2h\neq m$

Ex $M = Ke_1 + \cdots + Ke_n + Ke_{-1} + \cdots + Ke_{-n}$ split quadratic form Type D_n

$0 = Q(e_1) = \cdots = Q(e_n)$ $B(e_i, e_j) = \delta_{i,-j}$

$0 = Q(e_{-1}) = \cdots = Q(e_{-n})$

$\leadsto (e_i e_{-i})^2 = e_i e_{-i}, \ (e_{-i} e_i)^2 = e_{-i} e_i$

$(t e_i e_{-i} + t^{-1} e_{-i} e_i) \cdot (t e_{-i} e_i + t^{-1} e_i e_{-i}) = 1$

Consider $\left(\prod_{i=1}^n t_i e_i e_{-i} + t_i^{-1} e_{-i} e_i\right) =: h(\underline{t}) \in \Gamma_0^+(M, Q)$

Let $f = e_{-1} e_{-2} \cdots e_{-n} \in \Lambda^n(Ke_{-1} + \cdots + Ke_{-n})$, consider the minimal left ideal

$C(Q, M) \cdot f = \Lambda^\bullet(Ke_1 + \cdots + Ke_n) \cdot f$ and the left action of $C(Q)$ on $C(Q) f$

$= \bigoplus_{J \subseteq \{1, \dots, n\}} K \cdot e_J \cdot f$ $\quad e_J = e_{i_1} \wedge \cdots \wedge e_{i_r}$, $J = \{j_1, \dots, j_r\}$

Consider

$(t_i e_i e_{-i} + t_i^{-1} e_{-i} e_i) \cdot e_J f$

If $i \in J$, then $e_i \cdot e_J = 0$, and $t_i e_i e_{-i} \cdot e_i = t_i e_i (1 - e_i e_{-i}) = t_i e_i$,

so $(t_i e_i e_{-i} + t_i^{-1} e_{-i} e_i) \cdot e_J \cdot f = t_i \cdot e_J f$

If $i \notin J$, then $(t_i e_i e_{-i} + t_i^{-1} e_{-i} e_i)\, e_J e_{-i} = e_J (t_i e_i e_{-i} + t_i^{-1} e_{-i} e_i)\, e_{-i}$

$= e_J\, t_i^{-1} (1 - e_i e_{-i})\, e_{-i} = t_i^{-1} e_J e_{-i}$

Conclude: $(t_i e_i e_{-i} + t_i^{-1} e_{-i} e_i)\, e_J f = \begin{cases} t_i\, e_J f & \text{if } i \in J \\ t_i^{-1}\, e_J f & \text{if } i \in \{1, \dots, n\} - J = J^c \end{cases}$

$\leadsto h(\underline{t}) \cdot e_J f = \underbrace{\prod_i t_i^{\chi_J(i) - \chi_{J^c}(i)}}_{\varepsilon_J(\underline{t})} \cdot e_J f$ $\quad \chi_J$ = indicator function of J, χ_{J^c} = indicator function of J^c

$\{h(\underline{t}) : \underline{t} \in \mathbb{G}_m^n\} \subseteq \Gamma^+(Q, M)$ is a maximal torus.

The $\Gamma^+(Q)$-module $\Lambda^\bullet(Ke_1 + \cdots + Ke_n) \cdot f$ decomposes into a direct sum

$\bigoplus_{r \equiv n (2)} \Lambda^r(Ke_1 + \cdots + Ke_n) f \ \oplus \ \bigoplus_{r \equiv n-1 (2)} \Lambda^r(Ke_1 + \cdots + Ke_n) f$

heighest weights $\varepsilon_{\{1, \dots, n\}}$ $\qquad \varepsilon_{\{1, \dots, n-1\}}$

usual notation as in Bourbaki.

$\varpi_n = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n)$ $\qquad \varpi_{n-1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n)$

Type B_n $\quad M = Ke_{-n} + \cdots + Ke_{-1} + Ke_0 + Ke_1 + \cdots + Ke_n \qquad N, P$ ~~totally~~ singular

char(K)$\neq 2$ $\quad P = Ke_{-n} + \cdots + Ke_{-1}, \quad N = Ke_1 + \cdots + Ke_n \qquad B(e_i, e_j) = \delta_{i+j,0} \quad \forall i \neq j$

$Q(e_0) = -1 \rightsquigarrow (M,Q) = (N \oplus P, Q|_{N+P}) \oplus \langle -1 \rangle$ in particular $B(e_i, e_0) = 0 \quad \forall i \neq 0$.

Have: $\xi : C(N+P, Q|_{N+P}) \xrightarrow{\sim} C^+(M,Q)$

$$n + p \longmapsto e_0 n + e_0 p \qquad \rightsquigarrow \quad \xi(e_i e_{-i}) = e_0 e_i e_0 e_{-i}$$
$$= -e_0^2 e_i e_{-i} = e_i e_{-i} \text{ in } C(M,Q)$$

$$t_i e_i e_{-i} + t_i^{-1} e_{-i} e_i \longmapsto t_i e_0 e_i e_0 e_{-i} + t_i^{-1} e_0 e_{-i} e_0 e_i$$
$$= t_i e_i e_{-i} + t_i^{-1} e_{-i} e_i \qquad \forall i = 1, \dots, n$$

$\rightsquigarrow \quad C^+(M,Q) \underset{\cong}{\xrightarrow{\xi^{-1}}} C(N+P, Q|_{N+P})$ operates on the minimal left ideal $C(N+P, Q|_{N+P}) \cdot f = \Lambda^\bullet N \cdot f$

$f = e_1 \cdots e_n$

$$= \bigoplus_{J \subseteq \{1,\dots,n\}} K \cdot e_J \cdot f$$

Let $\underline{t} = (t_1, \dots, t_n) \in \mathbb{G}_m^n$

$h(\underline{t}) = \prod_{i=1}^n (t_i e_i e_{-i} + t_i^{-1} e_{-i} e_i) \in C^+(M,Q)^\times \qquad e_J = e_{j_1} \cdots e_{j_r}$ if $J = \{j_1, \dots, j_r\} \quad j_1 < \dots < j_r$

$h(\underline{t})$ operates on $\Lambda^\bullet N \cdot f = \bigoplus_{J \subseteq \{1,\dots,n\}} K e_J \cdot f$

$$h(\underline{t}) \cdot e_J \cdot f = \varepsilon_J(\underline{t}) \cdot e_J \cdot f, \text{ where } \varepsilon_J(\underline{t}) = \prod_{i=1}^n t_i^{\chi_J(i) - \chi_{J^c}(i)}$$

where $\chi_J, \chi_{J^c} : \{1, \dots, n\} \to \{0,1\}$ are the indicator functions of J and J^c $\left(= \{1,\dots,n\} \setminus J\right)$ respectively.

$C^+(M,Q) \triangleq \mathrm{End}_K(\Lambda^\bullet N \cdot f)$, giving a simple $C^+(M,Q)$-module

Note

In Bourbaki's notation, the highest weight of this represent(at)ion is $\tilde{\omega}_n = \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_n)$

B_n. basic roots $\underset{\alpha_1}{\varepsilon_1 - \varepsilon_2}, \dots, \underset{\alpha_{n-1}}{\varepsilon_{n-1} - \varepsilon_n}, \underset{\alpha_n}{\varepsilon_n} \qquad$ roots: $\pm \varepsilon_i, \ \pm \varepsilon_i \pm \varepsilon_j \quad 1 \leq i, j \leq n$

fundamental weights

$$\tilde{\omega}_1 = \varepsilon_1, \quad \tilde{\omega}_2 = \varepsilon_1 + \varepsilon_2, \dots, \tilde{\omega}_{n-1} = \varepsilon_1 + \dots + \varepsilon_{n-1}, \quad \tilde{\omega}_n = \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_n)$$

Exer: $\dim(M) = m = 2r \quad$ char(K)$\neq 2$. Determine whether the main involution on $C(M,Q)$ is orthogonal or symplectic

Direct computation shows:

For $m = 0, 1, 4, 5$, orthogonal

$m = 2, 3$ symplectic

Guess: $(-1)^{r(r-1)/2}$

Proposition: $\mathrm{char}(K)=2$. Not exceptional
Exceptional: $K=\mathbb{F}_2$, $\dim M=2$, $\mathrm{index}(Q)=1$

(1) M is irred / $O(Q)(K)$

(2) Exclude additional exceptional case: $K=\mathbb{F}_2$, $\dim M=4$, $\mathrm{index}(Q)=2$

M is irreducible as $[SO(Q)(K), SO(Q)(K)]_{grp}$ -module

Lemma $\overset{m>2}{\ }$ $\chi(\Gamma_0^+(Q)(K)) = [SO(Q)(K), SO(Q)(K)]_{grp}$ except

when $m=4$, $K=\mathbb{F}_2$, $\mathrm{index}(Q)=2 \leftarrow$ maximal

$SO(Q)(K) = \chi(\Gamma^+(Q)(K))$

- Sesquilinear form

E, F: right A-modules A: ring with an anti-aut $J: A \to A$

$$\Phi(xa, yb) = a^J \Phi(x,y)\, b \quad \forall x \in E,\ \forall y \in F,\ \forall a, b \in A$$

- ε-hermitian form (often $\varepsilon = \pm 1$, or $\varepsilon \cdot \varepsilon^\iota = 1$) $\iota: A \to A$ involution

$$\Phi: E \times E \to A \quad \begin{cases} \Phi(xa, yb) = a^\iota \Phi(x,y)\, b & \forall x, y \in E,\ \forall a, b \in A \\ \Phi(x,y) = \varepsilon\, \Phi(y,x)^\iota \end{cases}$$

- quadratic form : A commutative

$Q: E \to A$. polar form $\underbrace{\Phi(x,y) = Q(x+y) - Q(x) - Q(y)}_{\text{bilinear}}$

Witt decomposition + cancellation

$A =$ a division ring $\Rightarrow \varepsilon \cdot \varepsilon^\iota = 1$

Φ: non-deg. ε-hermitian s.t. $\forall x \in E$, $\exists \alpha \in A$ s.t. $\alpha + \varepsilon \alpha^\iota = \Phi(x,x)$

(1) Any two max. totally isotropic subspace have the same dimension (the index of (E, Φ))

$\leadsto \quad E = \underset{\text{neutral}}{E_1} \overset{\perp}{\oplus} \underset{\text{anisotropic}}{E_2}$

i.e. $E_1 = N \oplus P$ N, P totally isotropic, dual to each other via $\Phi|_{E_1}$

(2) Witt cancellation

$(E, \Phi) \cong (E', \Phi')$ both nondegen

$F \subseteq E$

$\alpha: F \hookrightarrow E'$ respects metrics

$\Rightarrow \alpha$ extends to an isom $(E, \Phi) \cong (E', \Phi')$

Hurwitz's Theorem (M, Q) nondegen quadratic form, $\dim(M) \geq 3$

$\varphi: M \times M \to M$ bilinear s.t $Q(\varphi(x,y)) = Q(x)\,Q(y)$ $\forall x, y \in M$

Then $m = 4$ or 8

Pf: linearize : (1) $B(\varphi(x,y), \varphi(x,z)) = Q(x) \cdot B(y,z)$

(2) $B(\varphi(x,z), \varphi(y,z)) = B(x,y) \cdot Q(z)$ $\forall y, z$

linearize again

(3) $B(\varphi(x,y), \varphi(w,z)) + B(\varphi(w,y), \varphi(x,z)) = B(x,w) \cdot B(y,z)$ $\forall w,x,y,z \in M$

Can check (Exercise): If $M = K$-linear span of a subset S and (1),(2),(3) hold for all elements in S, then $Q(\varphi(x,y)) = Q(x) \cdot Q(y)$ $\forall x, y \in M$

Do not need for the proof

Reduction steps
- May assume : Q neutral
- May assume : $\exists\, x_1 \in M$ s.t. $Q(x_1) = 1$ and $\varphi(x_1, y) = y$ $\forall y \in M$

Pick $x_1 \in M$ s.t. $Q(x_1) = 1 \rightsquigarrow Q(\varphi(x_1,y)) = Q(x_1)\,Q(y) = Q(y)$

i.e. $\sigma = (y \mapsto \varphi(x_1, y)) \in O(Q)$

Define $\psi: M \times M \to M$ by

$$\varphi(x_1, \psi(x,y)) = \varphi(x,y) \quad \text{i.e. } \psi(x,y) = \sigma^{-1}(\varphi(x,y))$$

Then
$$\underset{\substack{\| \\ Q(x)\,Q(y)}}{Q(\varphi(x,y))} = Q(\varphi(x_1, \psi(x,y))) = Q(x_1) \cdot Q(\psi(x,y)) = Q(\psi(x,y)) \quad \forall x, y$$

$$\varphi(x_1, \psi(x_1,y)) = \varphi(x_1, y) \ \forall y \Rightarrow \psi(x_1, y) = y \ \forall y$$

So $\psi: M \times M \to M$ has the same property as φ, and $\psi(x_1 y) = y$ $\forall y \in M$

So we assume: $\varphi(x_1, y) = y$ $\forall y \in M$.

Pick $x_2 \in M$, $0 \neq x_2$, $Q(x_2) = 0$, let $L_{x_2} = (y \mapsto \varphi(x_2, y)) \in \mathrm{End}_K(M)$

(i) $L_{x_2}(M)$ is totally singular

$\because Q(\varphi(x_2, y)) = Q(x_2) \cdot Q(y) = 0$ $\forall y \in M$

(ii) $\mathrm{Ker}(L_{x_2})$ is totally singular

Suppose $\exists\, z \in \mathrm{Ker}(L_{x_2})$, $Q(z) \neq 0$, i.e. $\varphi(x_2, z) = 0$, we get from formula (2) that

$\rightsquigarrow 0 = B(\varphi(x_2, z), \varphi(w z)) = B(x_2, w)\, \underset{\neq 0}{Q(z)}$ $\forall w \in M$

$\Rightarrow x_2 = 0$, contradiction

Conclude: $\dim L_{x_2}(M) \leq \lfloor \frac{m}{2} \rfloor$, $\dim \mathrm{Ker}(L_{x_2}) \leq \lfloor \frac{m}{2} \rfloor$

$\Rightarrow m \equiv 0 \pmod 2$ and $\dim L_{x_2}(M) = \dim \mathrm{Ker}(L_{x_2}) = \frac{m}{2}$.

In particular, $m = 2r$, $r \in \mathbb{N}$

$\forall x \in x_1^\perp$, $\forall y \in M$ have
$$\begin{aligned} Q(\varphi(x + x_1, y)) &= Q(x + x_1) \cdot Q(y) = (1 + Q(x)) \cdot Q(y) \\ \| \quad & \\ Q(y + \varphi(x,y)) &= Q(y) + Q(x)\,Q(y) + B(y, \varphi(x,y)) \end{aligned}$$

Conclude : $B(y, \varphi(x,y)) = 0$ $\forall x \in x_1^\perp$, $\forall y \in M$

Have shown: $B(y, \varphi(x,y)) = 0 \quad \forall x \in x_1^\perp, \ \forall y \in M$

linearize: $B(y, \underset{L_x(z)}{\underset{\parallel}{\varphi(x,z)}}) + B(z, \underset{L_x(y)}{\underset{\parallel}{\varphi(x,y)}}) = 0 \ \forall x \in x_1^\perp, \ \forall y \in M$

Substitute z by $L_x(z)$:

$$B(y, L_x^2(z)) + \underbrace{B(L_x(z), L_x(y))}_{\substack{\parallel (1) \\ Q(x)\cdot B(y,z)}} = 0 \quad \forall x \in x_1^\perp, \ \forall y, z \in M$$

$\Rightarrow \quad L_x^2 = -Q(x)\cdot Id_M \quad \forall x \in x_1^\perp \quad (*) \longleftarrow$ This is the key info.

Exer/easy: $\exists N \subseteq x_1^\perp$, $\dim(N) = m-2$, $B|_{N\times N}$ nonsingular

$\rightsquigarrow$ Get a K-alg. homom $C(N, -Q|_N) \longrightarrow End_K(M)$

$\cong M_{2^{r-1}}(K)$

$\rightsquigarrow \quad 2r = \dim(M) \equiv 0 \pmod{2^{r-1}}$

$\Rightarrow r$ is a power of $2 \Rightarrow r = 2$ or 4. Q.E.D.

Chevalley's Lemma:

Recall: $\forall u \in M^\vee$, $\exists!\, i_u : T^\cdot M \to T^\cdot M$ s.t. $i_u(1) = 0$

$i_u \circ e_x + e_x \circ i_u = u(x)\cdot 1 \quad \forall x \in M$

(Recursive def^n)

Have $\begin{cases} i_{u_1+u_2} = i_{u_1} + i_{u_2} \\ (i_u)^2 = 0 \\ i_{u_1}\circ i_{u_2} + i_{u_2} i_{u_1} = 0 \quad \forall u_1, u_2 \in M^\vee \end{cases}$ i.e. The graded derivations i_{u_1}, i_{u_2} graded commute

Lemma

$F: M\times M \to K$ K-bilinear

$\exists!\ \eta_F : T^\cdot M \to T^\cdot M$ K-linear s.t. $\eta_F(1) = 1$

$\eta_F \circ e_x = e_x \circ \eta_F + i_x^F \circ \eta_F$

attached to $(y \mapsto F(x,y))$

Property: $\begin{cases} \eta_F \circ i_u = i_u \circ \eta_F \quad \forall u \in M^\vee \\ \eta_{F_1}\circ \eta_{F_2} = \eta_{F_1+F_2} \end{cases}$

For $F = 0$, $\eta_0 = Id_{T^\cdot M} \rightsquigarrow \eta_F$ is a K-linear bijection

Key property for us: $Q: M \to K$ quadratic form

$$\eta_F \circ e_x^2 = (e_x + i_x^F)\,\eta_F \circ e_x = (e_x + i_x^F)^2 \circ \eta_F = (e_x^2 + F(x,x)\cdot Id_{T(M)}) \circ \eta_F$$

$\Rightarrow \forall u \in T^\cdot M$

$$\eta_F(x\otimes x \otimes u - Q(x)\,u) = [x\otimes x - (Q(x) - F(x,x))] \otimes u$$

In particular, if $F(x,x) = Q(x)$ $\forall x \in M$, then η_F induces a K-linear isom

$\eta_F : C(Q) \xrightarrow{\sim} \Lambda^\cdot(M)$ of K-vector spaces

Choices of bilinear forms $F: M\times M \to K$ s.t. $F(x,x) = Q(x)$ $\forall x\in M$

(1) Works in all characteristics. Need Q neutral

N, P, tot. singular, $M = N\oplus P$ so $\dim(M) = 2r$ even

Choose B_0: $B_0|_{N\times N} = 0$, $B_0|_{P\times P} = 0$ $B_0|_{N\times P} = 0$, $B_0|_{P\times N} = B|_{P\times N}$

$\leadsto$ For $x = n+p \in M$,

$B_0(x,x) = B_0(p,n) = B(p,n) = Q(x)$ $\forall x\in M$

(2) In case $\mathrm{char}(K)\neq 2$, choose $B_{\frac12} = \frac12 B$ $\leadsto$ $B_{\frac12}(x,x) = \frac12 B(x,x) = Q(x)$ $\forall x\in M$

(M,Q) splits $Kf = \Lambda^{top}P = \Lambda^r P$

$m = \dim M = 2r$

Ex $\eta_{B_0}(f_1\cdots f_r) = f_1\wedge\cdots\wedge f_r$, $\eta_{B_0}(e_1 f_1\cdots f_r) = e_1\wedge f_1\wedge\cdots\wedge f_r$ $\because B_0(e_1, P) = 0$

$\leadsto \eta_{B_0}(e_{i_1}\cdots e_{i_h} f_1\cdots f_r) = e_{i_1}\wedge\cdots\wedge e_{i_h}\wedge f_1\wedge\cdots\wedge f_r$

$\eta_{\frac12 B}(f_1\cdots f_r) = f_1\wedge\cdots\wedge f_r$, $\eta_{\frac12 B}(e_1 f_1\cdots f_r) = e_1\wedge f_1\wedge\cdots\wedge f_r + \frac12 f_2\wedge\cdots\wedge f_r$

$\eta_{\frac12 B}(e_2 e_1 f_1\cdots f_r) = e_2\wedge e_1\wedge f_1\wedge\cdots\wedge f_r + \frac12 e_2\wedge f_2\wedge\cdots\wedge f_r + \frac12 e_1\wedge f_1\wedge f_3\wedge\cdots\wedge f_r$
$+\frac14 f_3\wedge\cdots\wedge f_r$

$(e_1+f_1)\cdot(e_1-f_1) = 0$, $Q(e_1+f_1) = 2$. $Q(e_1-f_1) = -2$

$\eta_{B_0}(\underbrace{(e_1+f_1)\cdot(e_1-f_1)}_{-e_1f_1+f_1e_1}) = -e_1\wedge f_1 + \eta_{B_0}(f_1e_1) = -e_1\wedge f_1 + f_1\wedge e_1 + 1$

$\eta_{\frac12 B}((e_1+f_1)\cdot(e_1-f_1)) = (e_1+f_1)\wedge(e_1-f_1)$

$x_1,\dots,x_h$ mutually orthogonal $\Rightarrow \eta_{\frac12 B}(x_1\cdots x_h) = x_1\wedge\cdots\wedge x_h$

$\eta_{B_0}(C^N\cdot f) = \Lambda^\cdot N\wedge f$, $\eta_{\frac12 B}(C^N\cdot f)\neq \Lambda^\cdot N\wedge f$

$\mathrm{Lie}(\Gamma) = Z(C) + Fil_{\le 2} C^+$

check $\forall x,y,z\in M$ $\quad \underbrace{xyz - zxy}_{ad(xy)(z)} = x(-zy + B(y,z)) + (xz - B(x,z))y = B(y,z)x - B(x,z)y \in M$

$ad(xy)(M)\subseteq M$

$\forall w\in M$ $\quad B(ad(xy)(z), w) + B(z, ad(xy)(w)) = B(y,z)B(x,w) - B(x,z)B(y,w)$
$+B(z,x)B(y,w) - B(z,y)B(x,w) = 0$

i.e. $ad(xy)$ is skew symmetric w.r.t B.

Pure spinors $\quad m = \dim(Q) = 2r$ even $\quad N, P$ tot. singular, $M = N\oplus P$

$Z\subseteq M$ totally singular, define $K\cdot u_Z$ by

$Ku_Z\cdot f = f_Z C\cap Cf$, where $Kf_Z = \Lambda^r Z\subseteq C^Z\subseteq C$

$u_Z\in C^N = \Lambda^\cdot N = S$

Examples of pure spinors

(a) $x_1, \dots, x_h \in N$, linearly indep. $A = Kx_1 + \dots + K_h$, $A' = P \cap A^{\perp} = Ky_{h+1} + \dots + Ky_r$

$Z = A + A'$ Take $f_Z = x_1 \cdots x_h\, y_{h+1} \cdots y_r$

Claim/Exer $f_Z \cdot e_r e_{r-1} \cdots e_{h+1} \cdot f = (-1)^{r-h} x_1 \cdots x_h \cdot f$

$\rightsquigarrow$ $u_Z = x_1 \cdots x_h$ is a representative spinor for Z

In particular

$K \cdot u_{Ky_1 + \dots + Ky_r} = K \cdot 1$. $K u_{Kx_1 + \dots + Kx_r} = K \cdot x_1 \cdots x_r = Ke$

(b) Action by $\Gamma = \Gamma(Q)$

Z: totally singular, $\dim Z = r$ $\quad u_Z f \in f_Z \cdot C \cap C \cdot f \quad u_Z f = f_Z \cdot w \quad w \in C$

$$(\rho(s) \cdot u_Z) \cdot f = s \cdot u_Z \cdot f = s f_Z w = \underbrace{s f_Z s^{-1}}_{= f_{\chi(s)(Z)}} \cdot s w \in f_{\chi(s)(Z)} \cdot C \cap C \cdot f$$

i.e. $\rho(s) \cdot u_Z$ is a representative of $\chi(s)(Z) = s Z s^{-1}$

(c) Combine (a) with (b)

$x_1, \dots, x_h \in N$ linearly indep, $v \in C_2^N$, $Ky_{h+1} + \dots + Ky_r = P \cap (Kx_1 + \dots + Kx_h)^{\perp}$

$\rho(\exp(v)) \cdot x_1 \cdots x_h$ is a representative of $\underbrace{\chi(\exp(v))(\underbrace{Kx_1 + \dots + Kx_h + Ky_{h+1} + \dots + Ky_r}_{Z})}_{Z'}$

$= \underbrace{\exp(v)}_{\uparrow \text{ fixes elements of } N} \cdot x_1 \cdots x_r$

$Z' \cap N = \chi(\exp(v))(Z \cap N) = Kx_1 + \dots + x_h$

Properties of pure spinors $\subseteq \mathbb{P}(C^N) = \mathbb{P}(S)$ $\quad C^N$ left $C(Q)$-module under the repr. ρ

1. They form an orbit under Γ
2. pure spinors $\subseteq \Lambda^{even} C^N \cup \Lambda^{odd} C^N$. $\rightsquigarrow$ {pure spinors} = union of two orbits of Γ^+
3. Z maximal totally singular subspace of M

   $\Rightarrow Z = \{x \in M \mid \rho(x) \cdot u_Z = 0\}$

   Conversely, if $0 \neq u \in S$ and $\rho(x) \cdot u = 0$ $\forall x \in Z$, then $Ku = Ku_Z$
4. Z, Z': max tot. sing. subspaces

   Z and Z are of the same parity $\Leftrightarrow$ $\dim(Z \cap Z') \equiv r \pmod 2$
5. $0 \neq u \in C^N$ is a pure spinor

   $\Leftrightarrow$ $\exists v \in C_2^N$ and $\exists x_1, \dots, x_h \in N$, linearly indep s.t. $K \exp(v) \cdot x_1 \cdots x_h \ni u$

   If so, i.e. $u = u_Z$ for a max. tot. sing subspace Z, then

   $Z \cap N = Kx_1 + \dots + Kx_h$

6. $\forall$ tot. sing subspace $U \subseteq M$ with $\dim U = r-1 = \frac{1}{2}\dim M - 1$
There are exactly two maximal totally singular subspaces containing U, one even and one odd

7. Z, Z' max. totally sing subspaces
(a) $\exists\, a, b \in K^\times$ s.t. $a u_Z + b u_{Z'}$ is pure $\Longleftrightarrow \dim(Z \cap Z') = r-2$
(b) Suppose that $\dim(Z \cap Z') = r-2$. Then
$$\{ Ku \in Ku_Z + Ku_{Z'} \}_{u \neq 0} = \left\{ \begin{array}{l} \text{pure spinors representing max tot. sing} \\ \text{subspaces } Z'' \text{ s.t. } Z'' \cap Z = Z'' \cap Z' \text{ or } Z'' = Z \text{ or } Z'' = Z' \end{array} \right\}$$

Def $\quad S = C^N$
$\beta : S \times S \to K$ bilinear pairing (up to $K^\times$)
$\beta(u,v)$ = homog deg $= r$ component of $\alpha(u)v \in C^N = \Lambda^\bullet N$
$\beta(u,v) f = \alpha(uf)\, v f \qquad$ with a suitable choice of f

Properties
(1) $\beta(\rho(s)\cdot u, \rho(s)\cdot v) = \lambda(s)\, \beta(u,v) \quad \forall s \in \Gamma, \ \forall u, v \in S$
$\therefore\ \alpha(suf)\cdot svf = \alpha(uf)\cdot \underbrace{\alpha(s)\cdot s}_{\lambda(s)}\, vf$

(2) $\beta(\rho(x)\cdot u, \rho(x)\cdot v) = Q(x)\ \beta(u,v) \qquad \forall u, v \in S \ \ \forall x \in M$ (same computation as in (1))
$\beta(\rho(x)\cdot u, v) = \beta(u, \rho(x) v) \qquad \forall u, v \in S, \ \ \forall x \in M$
$\therefore\ \alpha(xuf)\, vf = \alpha(uf)\cdot xvf$

(3) β is non-degen. $\therefore\ \forall u \in C^N, \ \exists\, v \in C^N$ s.t. $\alpha(u)\cdot v = e$

(4) $\beta(v,u) = (-1)^{r(r-1)/2}\ \beta(u,v) \quad \forall u, v \in S$
$\therefore\ \alpha(\alpha(v)u) = \alpha(u) v \rightsquigarrow$ (Taking homog. components of degree r on both sides)
$\beta(v,u)\cdot \alpha(e) = \beta(u,v)\, e \qquad \alpha(e) = (-1)^{r(r-1)/2} e$

(5) Z, Z' max. tot. sing.
$Z \cap Z' \neq (0) \Longleftrightarrow \beta(u_Z, u_{Z'}) = 0$

(6) Except: $m = 2$, $K = \mathbb{F}_2$ or $\mathbb{F}_3$
$$\mathrm{Hom}_{\Gamma_0^+}\big((S, \rho), (S^\vee, \rho^\vee)\big) = K\beta + K\cdot\tilde{\beta},$$
where $\tilde{\beta}(u,v) = \beta(C[-1](u), v) \quad \forall u, v \in S$
Note: $\tilde{\beta}(\rho(s)u, \rho(s)v) = \varepsilon(s)\cdot\lambda(s)\cdot\tilde{\beta}(u,v) \quad \forall u, v \in S \ \forall s \in \Gamma$
$\varepsilon : \Gamma \to \{\pm 1\} = \mu_2$ the parity

Proposition: $\left\{\begin{matrix}\text{max. tot. isotropic} \\ \text{subspaces of } (M,Q)\end{matrix}\right\}$ forms one orbit under $\underset{SO(Q)}{\overset{}{\chi(\Gamma_0)}}$

$\chi(\Gamma_0) = SO(Q)$

Proposition: Assume either $\mathrm{char}(K) \neq 2$ and $r(r-1) \equiv 0 \ (4)$ or $\mathrm{char}(K) = 2$ and $r \geq 3$

$\exists$ a neutral quadratic form γ on S s.t.

$$\begin{cases} \beta(u,v) = \gamma(u+v) - \gamma(u) - \gamma(v) & \forall u, v \in M \\ \gamma(\rho(s)\cdot u) = \lambda(s)\, \gamma(u) & \forall u \in S, \ \forall s \in \Gamma \\ \gamma(\rho(x)\cdot u) = Q(x)\, \gamma(u) & \forall x \in M, \forall u \in S \end{cases}$$

(If $\mathrm{char}(K) \neq 2$, take $\gamma(u) = \frac{1}{2}\beta(u,u)$.
The case $\mathrm{char}(K) = 2$ uses "reduced squaring" in C^N)

Construction of γ when $\mathrm{Char}(K) = 2$, $r > 2$

- Define a "reduced squaring operation" on S (which depends on a choice of a basis $x_1, \dots, x_r$ of N)

Let $\xi_0, \dots, \xi_{2^r-1}$ be the monomials in $x_1, \dots x_r$ in $S = C^N$

$S \ni u = \sum_{a_i \in K} a_i \xi_i$, define $u^{[2]} := \sum_{0 \le i < j \le 2^r-1} a_i a_j \xi_i \xi_j + a_0^2$

define $\gamma(u)$ by:

$$\gamma(u) e = \text{homog degree } r \text{ component of } u^{[2]}$$

(Check: independent of choice of basis of N)

Remark: If $\mathrm{char}(K) = 2$ and $r = 2$, No such quad. form γ exists

$\because$ If such a γ exists, then $\gamma(x) = \gamma(x\cdot 1) = Q(x)\cdot\gamma(1) = 0 \ \forall x \in N$

$\rightsquigarrow \beta(x,y) = 0 \ \forall x, y \in M$

But $r=2$, for a basis x_1, x_2 of N, with $e = x_1 x_2$, we have $\beta(x_1,x_2) = 1$

Prop. $(S, \rho\otimes\rho) \underset{\Gamma}{\simeq} (w \mapsto \lambda(s)\, s w s^{-1})$ via $S\otimes S \xrightarrow[\varphi]{\sim} C(Q)$

$u\otimes v \longmapsto u f \alpha(v)$

Pf 1) $\varphi: S\otimes S \to C$ Check $\mathrm{Im}(\varphi)$ is an ideal of $C(Q)$
$u\otimes v \to C$ $\rightsquigarrow \varphi$ is surjective. hence an K-linear isom by dim count

2) $\forall s \in \Gamma$

$$\varphi(\rho(s) u \otimes \rho(s) v) = \underbrace{\rho(s)(u) f}_{= suf}\, \alpha(\rho(s) v)) = suf\, \alpha(\rho(s)\cdot v)$$

$$= suf \alpha(v) \lambda(s) s^{-1} = \lambda(s)\, \varphi(u\otimes v)\cdot s^{-1}$$

$(\rho(s)\cdot v) f = svf$
$\rightsquigarrow \alpha(f)\cdot\alpha(\rho(s)\cdot v) = \alpha(f)\,\alpha(v)\cdot\alpha(s)$
$\rightsquigarrow f\cdot\alpha(\rho(s)\cdot v) = f\cdot\alpha(v)\,\alpha(s)$

char(K) $\neq 2$ With $C(Q) \xrightarrow{\theta = \eta_{\frac{1}{2}B}} \Lambda^{\bullet} M$

$\rightsquigarrow$ $O(Q)$-equivariant

Def. char(K) $\neq 2$
Q neutral, $\dim M = 2r$

$$\eta_{\frac{1}{2}B}(u f \alpha(v)) = \sum_{h=0}^{m} \beta_h(u,v)$$

defines $m+1$ bilinear maps
$\beta_h : S \times S \longrightarrow \Lambda^h M$

s.t. $\beta_h(\rho(s)\cdot u, \rho(s)\cdot v) = \lambda(s)\, \underset{\uparrow}{\zeta_h}(\chi(s))(\beta_h(u,v)) \quad \forall u,v \in S, \ \forall s \in \Gamma$

repr. of $O(Q)$ on $\Lambda^h M$

char(K) arbitrary

Prop Criterion for $u \in S$ to be pure
(a) $u \in S^{even} \cup S^{odd}$
(b) $u f \alpha(v) \in \mathrm{Fil}_{\leq r} C(Q)$
(The exceptional case $K = \mathbb{F}_2$, $\dim M = 2$, Q splits is checked separately.)

Prop Z, Z' max. tot. sing subspace
$h = \dim(Z \cap Z')$
$\Rightarrow u_Z f u_{Z'} \in \mathrm{Fil}_{m-h} C \setminus \mathrm{Fil}_{m-h-1} C$

Lemma $\eta_{\frac{1}{2}B}(\alpha(\xi)) = \underset{\uparrow}{\Lambda^{\bullet}([-1]_M)}(\eta_{\frac{1}{2}B}(\xi)) \quad \forall \xi \in C(Q)$

$(-1)^{h(h-1)/2}$ on $\Lambda^h M$

(Exer) $\beta_h(v,u) = (-1)^{r(r-1)/2 + h(h-1)/2} \beta(u,v) \quad \forall u,v \in S$

Properties of β_h

1. $h \equiv r \ (2) \Rightarrow \beta_h$ vanishes on $S_{even} \times S_{odd}$ and $S_{odd} \times S_{even}$
   $h \not\equiv r \ (1) \Rightarrow \beta_h$ vanishes on $S_{even} \times S_{even}$ and $S_{odd} \times S_{odd}$
   (Consider the $\mathbb{Z}/2$-graded structure of $u f \alpha(v)$)

2. Pick $\begin{matrix} x_1, \dots, x_r \\ y_1, \dots, y_r \end{matrix}$ basis of $\begin{matrix} N \\ P \end{matrix}$ $\quad B(x_i, y_j) = \delta_{ij} \quad z = \prod_{k=1}^{r} (x_k - y_k)(x_k + y_k)$

   $z^2 = 1$

   $f = y_1 \cdots y_r$

   $f \cdot z = \delta(y_1) \cdots \delta(y_r) z = f, \quad z f = (-1)^r f$

   $\rightsquigarrow \forall v \in S_{even} \cup S_{odd}, \ \forall u \in S$

   $\beta_{m-h}(u,v) = \varepsilon\, \beta_h(u,v) z \qquad \varepsilon = \begin{cases} 1 & v \text{ even} \\ -1 & v \text{ odd} \end{cases}$

Note: $\forall z_1, \dots, z_h \in M$

$(z_1 \wedge \dots \wedge z_h)\, z = \delta(z_1) \cdots \delta(z_h)\, z$

1. Root system $R \subseteq V \setminus \{0\}$ finite
   - $\mathbb{R}$-span of $R = V$
   - $\forall \alpha \in R, \; s_\alpha(R) = R$
   - $\check{\alpha}(R) \subseteq \mathbb{Z}$

   R is <u>reduced</u> if $2\alpha \notin R \;\; \forall \alpha \in R$

   $\rightsquigarrow W(R), \; \check{R}$: dual root system
   $P(R) \supseteq Q(R)$

2. $n(\alpha,\beta) = \langle \alpha, \check{\beta} \rangle \in [-3,3] \cap \mathbb{Z}$ reduced

$\{n(\alpha,\beta), n(\beta,\alpha)\} =$

$0 \longleftrightarrow (\widehat{\alpha,\beta}) = \frac{\pi}{2}$		$s_\alpha s_\beta$ order 2	
$\pm 1 \longleftrightarrow (\widehat{\alpha,\beta}) = \frac{\pi}{3}$ or $\frac{2\pi}{3}$	$\lVert\alpha\rVert = \lVert\beta\rVert$	$s_\alpha s_\beta$ order 3	o—o
$\pm\{1,2\} \longleftrightarrow (\widehat{\alpha,\beta}) = \frac{\pi}{4}$ or $\frac{3\pi}{4}$	$\frac{\lVert\alpha\rVert}{\lVert\beta\rVert} = \sqrt{2}^{\pm 1}$	$s_\alpha s_\beta$ order 4	o═o
$\pm\{1,3\} \longleftrightarrow (\widehat{\alpha,\beta}) = \frac{\pi}{6}$ or $\frac{5\pi}{6}$	$\lVert\alpha\rVert / \lVert\beta\rVert = \sqrt{3}^{\pm 1}$	$s_\alpha s_\beta$ order 6	o≡o

3. C : positive Weyl chamber = a chosen connected component of $V \setminus R$

   $\rightsquigarrow (x|y) > 0 \;\; \forall x,y \in C$ (Choose a $W(R)$-invariant inner product on V)

   $L_1, \dots, L_\ell$ walls of C $L_i = L_{\alpha_i}$

   $B = \{\alpha_1, \dots, \alpha_\ell\}$ basis corresponding to C • $B(\check{C}) = \{\check{\alpha}_1, \dots, \check{\alpha}_\ell\}$ ($\check{C}$ identified with C via $(\cdot|\cdot)$)

   - $\alpha_1, \dots, \alpha_\ell$ is an $\mathbb{R}$-basis of V
   - $(\alpha_i, \alpha_j) \le 0 \;\; \forall i \neq j$
   - Every root $\beta \in R$ is a $\mathbb{Z}$-linear combination of $\alpha_1, \dots, \alpha_\ell$ s.t. all coeff. have the same sign
   - $W(R)$ is generated by $s_{\alpha_1}, \dots, s_{\alpha_\ell}$

   (partial) Order relation on V: positive elements = $\mathbb{R}_+$-linear combinations of $\alpha_1, \dots, \alpha_\ell$

4. $\tilde{\alpha}$ = highest root = $n_1\alpha_1 + \dots + n_\ell\alpha_\ell$

   $\rightsquigarrow n_1 + \dots + n_\ell = h - 1$ h = Coxeter number

   $\# W(R) = \ell! \left(\prod_{i=1}^{\ell} n_i\right) \cdot f$ ($f = \# \frac{P(R)}{Q(R)}$)

   $1 \le m_1 \le \dots \le m_\ell \le h-1$ exponents of R $m_1 = 1, \; m_\ell = h-1$, $m_j + m_{\ell+1-j} = h$

   $m_1+1, \dots, m_\ell+1$ = degrees of min. generators of $S^\bullet(V)^W$

   $m_1 + \dots + m_\ell = \frac{1}{2}\ell \cdot h = \frac{1}{2} \# R$

B_ℓ $\quad e_1, \dots, e_\ell, e_0, e_{-\ell}, \dots, e_{-1}$ basis

$$Q\Big(\sum_{i=-\ell}^{\ell} x_i e_i\Big) = -x_0^2 + \sum_{i=1}^{\ell} x_i x_{-i}, \quad B = \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & -2 & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix} \qquad S = \begin{pmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{pmatrix}, \ \ell\times\ell$$

$\mathfrak{g} = so(Q)$

$$\Bigg\| \left\{ \begin{pmatrix} A & 2S\,{}^t x & B \\ y & 0 & x \\ C & 2S\,{}^t y & D \end{pmatrix} : \begin{matrix} x, y \in M_{1,\ell}(k) \\ A, B, C, D \in M_{\ell,\ell}(k) \end{matrix} \quad \begin{matrix} B = -S\,{}^tB\,S \\ C = -S\,{}^tC\,S \end{matrix} \quad D = -S\,{}^tA\,S \right\}$$

Note: $A \mapsto S\,{}^tA\,S$ is "Flipping w.r.t. the antidiagonal".

$$\dim so(Q) = \underset{A,D}{\ell^2} + 2\cdot\underset{B,C}{\frac{\ell(\ell-1)}{2}} + \underset{x,y}{2\ell} = \ell^2 + \ell^2 - \ell + 2\ell = 2\ell^2 + \ell = \ell(2\ell+1)$$

$\#R = \ell \cdot \underline{2\ell}$ $\leftarrow h$, Coxeter nber

Cartan $\mathfrak{h}$ = diagonal matrices in $\mathfrak{g}$

$\| \quad k\cdot H_1 + \dots + k\cdot H_\ell. \qquad H_i = E_{i,i} - E_{-i,-i} \qquad i = 1, \dots, \ell$

$E_{i,j}$: std basis elts of $M_{[-\ell,\ell]\times[-\ell,\ell]}$

$\{\varepsilon_i\}$ = dual basis to $\{H_i\}$ $\subseteq \mathfrak{h}^*$

generators of $\mathfrak{g}_\alpha$, $\alpha \in R$ $\qquad R = \{\pm(\varepsilon_i \pm \varepsilon_j) : 1 \le i < j \le \ell\} \cup \{\pm\varepsilon_i \mid 1 \le i \le \ell\}$

$X_{\varepsilon_i} = 2E_{i,0} + E_{0,-i}$ $\quad 1 \le i \le \ell$ $\quad$ corresponding to "x"

$X_{-\varepsilon_i} = -2E_{-i,0} - E_{0,i}$ $\quad$ corresponding to "y"

$X_{\varepsilon_i - \varepsilon_j} = E_{i,j} - E_{-j,-i}$

$X_{\varepsilon_j - \varepsilon_i} = -E_{j,i} + E_{-i,-j}$ $\quad 1 \le i < j \le \ell$ $\quad$ off-diagonal parts of A, D

$X_{\varepsilon_i + \varepsilon_j} = E_{i,-j} - E_{j,-i}$ $\quad$ corresponding to B

$X_{-\varepsilon_i - \varepsilon_j} = -E_{-j,i} + E_{-i,j}$ $\quad$ corresponding to C

$[h, X_\alpha] = \alpha(h) X_\alpha \quad \forall \alpha \in R$

$\leadsto [X_\alpha, X_{-\alpha}] = -H_\alpha \quad \forall \alpha \in R$

General Lemma: $\mathrm{Char}(k) \neq 2$

(V, Q) quadratic form, non-degen $\quad B$ = polar form of Q

define $f_0 : o(Q) \to C^+(V, Q)$ as the composition

$$f_0 : o(Q) \hookrightarrow gl(V) \underset{can}{\xrightarrow{\sim}} V \otimes V^* \underset{\text{defined by } B}{\xleftarrow{\sim}} V \otimes V \underset{\text{multiplication in } C(V,Q)}{\longrightarrow} C^+(V, Q)$$

$$f = \tfrac{1}{2} f_0 : o(Q) \longrightarrow C^+(V, Q)$$

Lemma: (1) Let (e_r), (e_r') be dual bases of V w.r.t. B (i.e. $B(e_r, e_s) = \delta_{rs}$)

Then $f_0(a) = \sum_r a(e_r) \cdot e_r'$ $\forall a \in o(V, Q)$

multiplication in $C(V, Q)$

(2) $\forall a, b \in o(Q)$, have

$$\sum_r \underset{\text{product in } C(Q)}{a(e_r) \cdot b(e_r')} = -\sum_r (ab)(e_r) \cdot e_r'$$

(3) $[f(a), v] = a(v)$ $\forall a \in o(Q)$, $\forall v \in V$ (vector representation of $C(Q)$)

(4) $[f(a), f(b)] = f([a, b])$ $\forall a, b \in o(Q)$

(5) $f(o(Q))$ generates $C^+(V, Q)$ (as an associative alg. $/k$)

proof: (1) obvious: $V \otimes V^* \xleftarrow{\sim} V \otimes V$

$a \longmapsto \sum_r a(e_r) \otimes e_r'$

(2) $$\sum_r \underbrace{a(e_r)} \cdot b(e_r') = \sum_{r,s,t} B(a(e_r), e_s') \, B(b(e_r'), e_t) \, e_s \cdot e_t'$$

$\sum_s B(a(e_r), e_s') e_s$

$$= \sum_{r,s,t} B(e_r, a(e_s')) \, B(e_r', b(e_t)) \, e_s \cdot e_t'$$

$$= \sum_{s,t} B(a(e_s'), b(e_t)) \, e_s \cdot e_t'$$

$$= -\sum_{s,t} B(e_s', ab(e_t)) \, e_s \, e_t' = -\sum_t (ab)(e_t) \cdot e_t'$$

(3) $$[f(a), v] = \tfrac{1}{2}\Big[\sum_r a(e_r) e_r', v\Big] = \tfrac{1}{2}\Big(\sum_r a(e_r) \cdot e_r' \cdot v - \sum_r v \cdot a(e_r) \cdot e_r'\Big)$$

$$= \tfrac{1}{2} \sum_r \Big(\underbrace{a(e_r) \cdot e_r' \cdot v + a(e_r) \cdot v \cdot e_r'}_{= a(e_r) \cdot B(e_r', v)} - \underbrace{a(e_r) \cdot v \cdot e_r' - v \cdot a(e_r) \cdot e_r'}_{= B(a(e_r), v) \cdot e_r'}\Big)$$

$$= \tfrac{1}{2} a\Big(\underbrace{\sum_r B(e_r', v) \cdot e_r}_{= v}\Big) + \tfrac{1}{2} \sum_r B(e_r, a(v)) e_r'$$

$$= \tfrac{1}{2} a(v) + \tfrac{1}{2} a(v) = a(v)$$

(4) $$[f(a), f(b)] = [f(a), \tfrac{1}{2} \sum_r b(e_r) e_r'] \quad \text{by (1)}$$

$$= \tfrac{1}{2} \sum_r \Big([f(a), b(e_r)] e_r' + \tfrac{1}{2} \cdot b(e_r) \cdot [f(a), e_r']\Big) \quad \text{ad}(f(a)) \text{ is a derivation}$$

$$= \tfrac{1}{2} \sum_r a(b(e_r)) \cdot e_r' + \tfrac{1}{2} \sum_r b(e_r) \cdot a(e_r') \quad \text{by (3)}$$

$$= \tfrac{1}{2} \sum_r a(b(e_r)) e_r' + \tfrac{1}{2} \sum_r b(a(e_r)) e_r' \quad \text{by (2)}$$

$$= f([a, b]) \quad \text{by (1)}$$

(5) Suffice to prove it when k is alg. closed.

$\leadsto$ May assume $B(e_r, e_s) = \delta_{rs}$ $\forall r, s$ i.e. $e_r' = e_r$ $\forall r$

$\leadsto$ $f(E_{ij} - E_{ji}) = \frac{1}{2}(e_i e_j - e_j e_i)$ by (4)

$= e_i e_j$ $\forall i j$. QED.

Alternative proof (without extending base field)

For $a \in \mathrm{End}(V)$, write $a(e_s) = \sum_r u_{rs} e_r'$ $a \longleftrightarrow (u_{rs})$

$a \in o(Q) \Longleftrightarrow B(a(v), w) + B(v, aw) = 0$

$\Longleftrightarrow B(a(e_m), e_n) + B(e_m, a(e_n)) = 0$ $\forall m, n$

$B(a(e_m), e_n) = B(\sum_r u_{rm} e_r', e_n) = u_{nm}$, $B(e_m, a(e_n)) = u_{mn}$

$\Longleftrightarrow u_{mn} + u_{nm} = 0$ $\forall m, n$

And if so, $f_o(a) = \frac{1}{2}\sum_{r,s} a_{sr}\, e_s' \cdot e_r'$ Such elements form an algebra generator of $C(Q)$.

QED

Back to the general situation/setup of B_ℓ, $\dim V = 2\ell+1$

Let $N = ke_{-\ell} + \cdots + ke_{-1}$ $\quad V \cong \underbrace{(N \oplus P)}_{W} \overset{\perp}{\oplus} k e_0$ $\quad Q(e_0) = -1$

$P = ke_1 + \cdots + ke_\ell$

$C(Q|_W, W) \xrightarrow{\sim} C^+(Q, V)$

$W \ni x \longmapsto e_0 x$

$C(Q|_W) \cdot \underset{\text{spinor module}}{\overset{e_1 \wedge \cdots \wedge e_\ell}{f}} \xleftarrow{\sim} \Lambda^\cdot N \circlearrowleft C^+(Q, V)$

$x_1 \cdots x_r \cdot f \longleftarrow x_1 \wedge \cdots \wedge x_r$

Recall: $H_i = E_{ii} - E_{-i,-i}$ $(1 \le i \le \ell) \longleftrightarrow f(H_i) = \frac{1}{2}(H(e_i)\cdot e_{-i} + H(e_{-i}) e_i)$

$= \frac{1}{2}(e_i \cdot e_{-i} - e_{-i} \cdot e_i)$

$= \frac{1}{2} - e_{-i} e_i = -\frac{1}{2} + e_i e_{-i}$

also $= \frac{1}{2}(e_0 e_i \cdot e_0 e_{-i} - e_0 e_{-i} \cdot e_0 e_i)$

$\leadsto \underset{\text{spinor repr}}{\rho}(H_i)(e_{-i_1} \wedge \cdots \wedge e_{-i_r})$ $\quad 1 \le i_1 < \cdots < i_r \le \ell$

$= \begin{cases} -\frac{1}{2} e_{-i_1} \wedge \cdots \wedge e_{-i_r} & \text{if } i \in \{i_1, \ldots, i_r\} \\ \frac{1}{2} e_{-i_1} \wedge \cdots \wedge e_{-i_r} & \text{if } i \notin \{i_1, \ldots, i_r\} \end{cases}$

Conclude

$$\rho(h)(e_{i_1}\wedge\cdots\wedge e_{i_r}) = \left(\tfrac{1}{2}(\varepsilon_1+\cdots+\varepsilon_\ell)-(\varepsilon_{i_1}+\cdots+\varepsilon_{i_r})\right)(h)\cdot(e_{-i_1}\wedge\cdots\wedge e_{-i_r})$$

$\forall h\in\mathfrak{h}$

Lemma: The $\mathfrak{h}$-weights of the spinor module (for B_ℓ) are:

$$\underbrace{\tfrac{1}{2}(\varepsilon_1+\cdots+\varepsilon_\ell)}_{\overset{\shortparallel}{\omega_\ell}} - \sum_{j\in J}\varepsilon_j \qquad (J\subseteq\{1,\dots,\ell\}),$$

each with multiplicity $=1$

Move on to D_ℓ

$$Q\Big(\sum_{i=1}^{\ell}x_ie_i+\sum_{i=-\ell}^{-1}x_{-i}e_{-i}\Big)=\sum_{i=1}^{\ell}x_i\,x_{-i}$$

$$so(Q)=\left\{\begin{pmatrix}A&B\\C&D\end{pmatrix},\ A,B,C,D\in M_\ell(k)\ \begin{matrix}B=-S\,{}^tB\,S\\ C=-S\,{}^tC\,S\end{matrix}\ \ D=-S\,{}^tA\,S\right\}$$

$$\dim\mathfrak{g}=\dim so(Q)=2\cdot\frac{\ell(\ell-1)}{2}+\ell^2=2\ell^2-\ell=\ell(2\ell-1)$$

$\# R = 2\ell^2-2\ell=\ell(2\ell-2)$, Coxeter nber $=2\ell-2$

$$R=\{\pm(\varepsilon_i\pm\varepsilon_j)\mid 1\le i<j\le\ell\}$$

$$\mathfrak{g}_{\varepsilon_i-\varepsilon_j}=k\,X_{\varepsilon_i-\varepsilon_j}=k\cdot(E_{i,j}-E_{-j,-i})$$
$$\mathfrak{g}_{\varepsilon_i+\varepsilon_j}=k\,X_{\varepsilon_i+\varepsilon_j}=k\cdot(E_{i,-j}-E_{j,-i})$$
$$\mathfrak{g}_{\varepsilon_j-\varepsilon_i}=k\,X_{\varepsilon_j-\varepsilon_i}=k\cdot(-E_{j,i}+E_{-i,-j})$$
$$\mathfrak{g}_{-\varepsilon_i-\varepsilon_j}=k\,X_{-\varepsilon_i-\varepsilon_j}=k\cdot(-E_{-j,i}+E_{-i,j})$$

$i<j$

$H_i=E_{i,i}-E_{-i,-i}$, $1\le i\le\ell$

$\mathfrak{h}=\bigoplus_i k\cdot H_i$

Fundamental weights: $\omega_1,\dots,\omega_\ell$

$\{\varepsilon_1,\dots,\varepsilon_\ell\}$: dual basis to $\{H_1,\dots,H_\ell\}$

$\omega_i=\varepsilon_1+\cdots+\varepsilon_\ell \qquad 1\le i\le\ell-2$

$\omega_\ell=\frac{1}{2}(\varepsilon_1+\cdots+\varepsilon_\ell)$

$\omega_{\ell-1}=\frac{1}{2}(\varepsilon_1+\cdots+\varepsilon_{\ell-2}+\varepsilon_{\ell-1}-\varepsilon_\ell)$

$\alpha_i=\varepsilon_i-\varepsilon_{i+1} \quad 1\le i\le\ell-1$

$\alpha_\ell=\varepsilon_{\ell-1}+\varepsilon_\ell$

$$P=\sum_{i=1}^{\ell}k\cdot e_i \qquad N=\sum_{i=1}^{\ell}k\cdot e_{-i}$$

$f=e_1\wedge\cdots\wedge e_\ell$

$\frac{1}{2}$-spinor repr. $\rho_+\longleftrightarrow\Big(\bigoplus_{\bullet\equiv0(2)}\Lambda^\bullet N\Big)\cdot f$

$\rho_-\longleftrightarrow\Big(\bigoplus_{\bullet\equiv1(2)}\Lambda^\bullet N\Big)\cdot f$,

$J=\{i_1,\dots,i_r\}$ $i_1<\cdots<i_r$

$e_{-i_1}\wedge\cdots\wedge e_{-i_r}\wedge f$

Weights of ρ_+: $\left\{\frac{1}{2}(\varepsilon_1+\cdots+\varepsilon_\ell)-\sum_{\substack{J\subseteq\{1,\dots,\ell\}\\ \#J\text{ even}}}\varepsilon_j\right\}$

ρ_-: $\left\{\frac{1}{2}(\varepsilon_1+\cdots+\varepsilon_\ell)-\sum_{j\in J}\varepsilon_j \mid \#J\text{ odd}\right\}$

continue with D_ℓ

$R = \{ \pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \le i < j \le \ell \}$ $\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{\ell-2} = \varepsilon_{\ell-2} - \varepsilon_{\ell-1}, \alpha_{\ell-1} = \varepsilon_{\ell-1} - \varepsilon_\ell$

$\alpha_\ell = \varepsilon_{\ell-1} + \varepsilon_\ell$

$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i \quad 1 \le i \le \ell-2$

$\omega_{\ell-1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{\ell-2} + \varepsilon_{\ell-1} - \varepsilon_\ell)$

$\omega_\ell = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_\ell)$

$$\tilde{\alpha} = \varepsilon_1 + \varepsilon_2 = (\varepsilon_1 - \varepsilon_2) + 2(\varepsilon_2 - \varepsilon_3) + \cdots + 2(\varepsilon_{\ell-2} - \varepsilon_{\ell-1}) + (\varepsilon_{\ell-1} - \varepsilon_\ell) + (\varepsilon_{\ell-1} + \varepsilon_\ell)$$
$$= \alpha_1 + 2\alpha_2 + \cdots + 2\varepsilon_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell$$ highest root

3 minuscule representations: weights

$\omega_1 \longleftrightarrow$ std

$\omega_{\ell-1} \longleftrightarrow \rho-$ $\Big\}$ $\frac{1}{2}$-spinor

$\omega_\ell \longleftrightarrow \rho+$

$$Q(R) = \left\{ \sum_{i=1}^{\ell} n_i \varepsilon_i \;\middle|\; n_i \in \mathbb{Z},\ \sum_i n_i \equiv 0\ (2) \right\}$$

$$P(R) = \sum_{i=1}^{\ell} \mathbb{Z}\cdot\varepsilon_i + \mathbb{Z}\cdot\frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_\ell)$$

representatives of $P/Q = \{0, \omega_1, \omega_{\ell-1}, \omega_\ell\}$

$\ell \equiv 0\ (2) \iff P/Q$ is killed by 2, i.e. $P/Q \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$

$\ell \equiv 1\ (2) \iff P/Q$ is not killed by 2, i.e. $P/Q \cong \mathbb{Z}/4\mathbb{Z}$

$\leadsto$ image of $\omega_{\ell-1}, \omega_\ell$ are both generators of P/Q

$\omega_1 \equiv 2\omega_{\ell-1} \equiv 2\omega_\ell \pmod{Q}$

$$W \cong \underset{\substack{\uparrow \\ \text{change even \# of signs}}}{(\mathbb{Z}/2\mathbb{Z})^{\ell-1}} \rtimes S_\ell$$

A Coxeter element: $s_{\alpha_1} \cdots s_{\alpha_\ell}$: $\begin{cases} \varepsilon_i \longmapsto \varepsilon_{i+1} & \text{for } i = 1, \ldots, \ell-2 \\ \varepsilon_{\ell-1} \longmapsto -\varepsilon_1 \\ \varepsilon_\ell \longmapsto -\varepsilon_\ell \end{cases}$

an $(\ell-1)$-cycle, plus two -1 signs. $\Longrightarrow$ again $h = 2(\ell-1)$

Back to C_ℓ

$\begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix} \longleftrightarrow \Psi$, $S = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$

$$sp(\Psi) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;\middle|\; \begin{array}{l} A, B, C, D \in M_\ell(K) \\ B = S\,{}^tB\,S,\ C = S\,{}^tC\,S,\ S\,{}^tA\,S + D = 0 \end{array} \right\}$$

$\# R = 2\ell^2$. $h = 2\ell$ $\qquad$ $\dim sp(\Psi) = \ell^2 + (\ell^2 + \ell) = \ell(2\ell+1)$

$H = \sum_{i=1}^{\ell} K \cdot H_{ii}$ $\quad H_{ii} = E_{ii} - E_{-i,-i}$ $\quad (\varepsilon_1, \dots, \varepsilon_\ell)$ dual to $(H_1, \dots H_\ell)$

$$\begin{cases} X_{2\varepsilon_i} = E_{i,-i} \\ X_{-2\varepsilon_i} = -E_{-i,i} & 1 \le i < j \le \ell \\ X_{\varepsilon_i - \varepsilon_j} = E_{ij} - E_{-j,-i} \\ X_{-\varepsilon_i + \varepsilon_j} = -E_{j,i} + E_{-i,-j} \\ X_{\varepsilon_i + \varepsilon_j} = E_{i,-j} + E_{j,-i} \\ X_{-\varepsilon_i, -\varepsilon_j} = -E_{-i,j} - E_{-j,i} \end{cases} \qquad \mathfrak{g}_\alpha = K \cdot X_\alpha$$

Roots: $\{ \pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le \ell \} \cup \{ 2\varepsilon_i \mid 1 \le i \le \ell \} = R$

$\Psi \longleftrightarrow -\sum_{i=1}^{\ell} e_i^* \wedge e_{-i}^* =: \Gamma^*$ (Bourbaki's sign convention)

$\Gamma := \sum_{i=1}^{\ell} e_i \wedge e_{-i}$

Define operators X_-, X_+, H on $\Lambda^\bullet V$ by

$X_- u = \Gamma \wedge u$

$X_+ u = \Gamma^* \wedge u$ (interior product)

$H|_{\Lambda^r V} = (\ell - r) \cdot \mathrm{id}_{\Lambda^r V}$ $\qquad \forall\, 0 \le r \le 2\ell$

standard basis of $\Lambda^\bullet V$: parametrized by disjoint subsets of $\{1, \dots, \ell\}$ $\quad (A, B, C) \subseteq \{1, \dots, \ell\}$

$e_{A,B,C} = e_{a_1} \wedge \dots \wedge e_{a_m} \wedge e_{-b_1} \wedge \dots \wedge e_{-b_n} \wedge e_{c_1} \wedge e_{-c_1} \wedge \dots \wedge e_{c_t} \wedge e_{-c_t}$ $\qquad A = \{a_1 < a_2 < \dots < a_m\}$, $B = \{b_1 < \dots < b_n\}$, $C = \{c_1 < \dots < c_t\}$

$\rightsquigarrow$ $X_- \cdot e_{A,B,C} = \sum_{j \notin A \cup B \cup C} e_{A,B,C \cup \{j\}}$

$X_+ \cdot e_{A,B,C} = -\sum_{j \in C} e_{A,B,C \setminus \{j\}}$

$\Rightarrow$ $[X_+, X_-] = -H$, $[H, X_+] = 2X_+$, $[H, X_-] = -2X_-$ — standard sl_2-triple

sl_2: $X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $X_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\rightsquigarrow$ $\Lambda^r V =$ wt $= \ell - r$ subspace of $\Lambda^\bullet V$ for this sl_2-triple

$\cup$

E_r primitive subspace of $\Lambda^r V$ for this sl_2-triple

$\forall r<\ell \quad X_- : \Lambda^r V \rightarrowtail \Lambda^{r+2} V$

$\Lambda^r V = E_r \oplus X_-(E_{r-2}) \oplus X_-^2(E_{r-4}) \oplus \cdots = E_r \oplus X_- \Lambda^{r-2} V$

X_+, X_-, H commutes with the action of $sp(\Psi, V)$ on $\Lambda^\cdot V$
($\because$ Γ and Γ^* are invariant under $sp(\Psi, V)$)

$\leadsto$ E_r is stable under $sp(\Psi, V)$ $\forall r<\ell$

$\cup\!\!|$

F_r = K- linear span of decomposable elements in $\Lambda^r V$ which correspond to totally isotropic subspaces

Lemma $\quad 1 \le r \le \ell$

$$\Lambda^r V = F_r + X_-(\Lambda^{r-2} V) \quad (= F_r + \underbrace{(\sum_{i=1}^{\ell} e_i \wedge e_{-i})}_{\Gamma} \wedge \Lambda^{r-2} V)$$

$\Uparrow$ Induction on r

Sublemma: $\quad F_{r-1} \wedge V \subseteq F_r + (\sum_{i=1}^{\ell} e_i \wedge e_{-i}) \wedge \Lambda^{r-2} V$

Key $\forall$ totally isotropic subspace $Kf_1 + \cdots + Kf_{r-1} + Kf_r + \cdots + Kf_\ell$

$\Gamma = f_1 \wedge f_{-1} + \cdots + f_\ell \wedge f_{-\ell}$

$$\left\{\begin{array}{l} f_1 \wedge \cdots \wedge f_{r-2} \wedge (f_{r-1} + f_j) \wedge (f_{1-r} - f_{-j}) \\ f_1 \wedge \cdots \wedge f_{r-2} \wedge f_j \wedge f_{1-r} \\ f_1 \wedge \cdots \wedge f_{r-2} \wedge f_{r-1} \wedge f_j \end{array}\right.$$

are decomposable and isotropic $\qquad \forall j = r, r+1, \ldots, \ell$

Obvious

Linear algebra lemma underlying Borel-Tits 6.13 (ii), p.

$V \xrightarrow{\pi} W$ V: Euclidean vector space $/\mathbb{R}$ π: orthogonal projection

$v_1, \dots, v_\ell$: basis of V $\pi : \{v_1, \dots, v_\ell\} \twoheadrightarrow \{w_1, \dots, w_r\}$

$w_1, \dots, w_r$ basis of W

$\rightsquigarrow \mathrm{Ker}(\pi) = \mathbb{R}$-span of $\{v_i \mid \pi(v_i) = 0 \quad i = 1 \dots \ell\} \cup \{v_i - v_j \mid \begin{smallmatrix} i \neq j \\ \pi(v_i) = \pi(v_j)\end{smallmatrix}\}$

Let $A = (a_{ij})_{1 \le i, j \le \ell}$, $a_{ij} = (v_i \mid v_j)$ $\quad C = (c_{ij}) = A^{-1}$

$B = (b_{st})_{1 \le s, t \le r}$, $b_{st} = (w_s \mid w_t)$ $\quad D = (d_{st}) = B^{-1}$

Then $\displaystyle d_{st} = \sum_{\substack{\pi(v_i) = w_s \\ \pi(v_j) = w_t}} c_{ij}$

Pf: Let $v_1^\vee, \dots, v_\ell^\vee \in V$ be the dual basis of V w.r.t $(\cdot \mid \cdot)$

i.e. $(v_i \mid v_j^\vee) = \delta_{ij}$.

$w_1^\vee, \dots, w_r^\vee \in W$ dual basis of W

$\rightsquigarrow \displaystyle v_i^\vee = \sum_j c_{ij} v_j, \quad w_s^\vee = \sum_t d_{st} w_t$

Key: $\pi({}^t\pi(w_s^\vee)) = w_s^\vee \quad \forall s \qquad {}^t\pi : W \to V$ transpose of π

Compute ${}^t\pi(w_s^\vee)$:

$$({}^t\pi(w_s^\vee) \mid v_i) = \begin{cases} 1 & \pi(v_i) = w_s \\ 0 & \pi(v_i) \neq w_s \end{cases} \quad \text{i.e. } {}^t\pi(w_s^\vee) = \sum_{\pi(e_i) = w_s} v_i^\vee$$

$$w_s^\vee = \pi({}^t\pi(w_s^\vee)) = \pi\Big(\sum_{\pi(e_i) = w_s} v_i^\vee\Big) = \pi\Big(\sum_{\pi(e_i) = w_s} \sum_j c_{ij} v_j\Big)$$

$$\| $$

$$\sum_t d_{st} w_t \quad = \sum_t \sum_{\pi(e_i) = w_s} \Big(\sum_{\pi(e_j) = w_t} c_{ij}\Big) w_t$$

$$\rightsquigarrow \quad d_{st} = \sum_{\substack{\pi(e_i) = w_s \\ \pi(e_j) = w_t}} c_{ij}$$

QED

Composition algebra $(C/k, N)$ N: quad. form on C s.t. its polar form $\langle\, ,\rangle$ is nondegenerate

$N(xy) = N(x)\cdot N(y)$ $\forall x,y \in C$

$+$ $e \in C$ s.t. $e\cdot x = xe = x$ $\forall x \in C$ $\rightsquigarrow$ $N(e)=1$

$$\langle x_1 y, x_2 y\rangle = \langle x_1, x_2\rangle N(y)$$

$$\langle x y_1, x y_2\rangle = N(x) \langle y_1, y_2\rangle$$

$$\langle x_1 y_1, x_2 y_2\rangle + \langle x_1 y_2, x_2 y_1\rangle = \langle x_1, x_2\rangle \langle y_1, y_2\rangle$$

$$\langle x, yx\rangle + \langle x^2, y\rangle = \langle x, y\rangle\langle e, x\rangle$$

$\Rightarrow$ $x^2 - \langle x, e\rangle x + N(x) e = 0$ $\forall x \in C$

$\rightsquigarrow$ define $x^{i+1} = x^i \cdot x$ $\forall i \in \mathbb{N}$

then $x^i \cdot x^j = x^{i+j}$ $\forall i,j \in \mathbb{N}$ $\forall x \in C$ "power associative"

Conjugation: $\bar{x} := \langle x, e\rangle - x$ $\forall x$

$x+\bar{x} = \langle x, e\rangle$, $x\bar{x} = \bar{x} x = N(x)\cdot e$ $\forall x \in C$

$\overline{xy} = \bar{y}\bar{x}$, $\bar{\bar{x}} = x$. $N(\bar{x}) = N(x)$, $\langle \bar{x}, \bar{y}\rangle = \langle x, y\rangle$
($\because$ $N(\bar{x}) = N(x)$ $\forall x \in C$)

Further identities

$$\begin{cases} \langle xy, z\rangle = \langle y, \bar{x}z\rangle = \langle x, z\bar{y}\rangle \\ \langle xy, \bar{z}\rangle = \langle yz, \bar{x}\rangle = \langle zx, \bar{y}\rangle \end{cases}$$

$$\begin{cases} x(\bar{x}y) = N(x) y \\ (x\bar{y})y = N(y)x \\ x(\bar{y}z) + y(\bar{x}z) = \langle x, y\rangle z \\ (x\bar{y})z + (x\bar{z})y = \langle y, z\rangle x \end{cases}$$

$N(a) \neq 0 \Rightarrow a^{-1} = N(a)^{-1}\bar{a}$ $\quad l_{a^{-1}} = (l_a)^{-1}$, $r_{a^{-1}} = (r_a)^{-1}$

$$\begin{cases} (ax)(ya) = a((xy)a) \\ a(x(ay)) = (a(xa))y \\ x(a(ya)) = ((xa)y)a \end{cases}$$ Moufang identitie

Doubling $j \in D^{\perp}$, $N(j) \neq 0$

$\rightsquigarrow$ For $D_1 = D \oplus Dj$ sub-composition alg.

$$\begin{cases} x(vj) = (vx)j \\ (yj)u = (y\bar{u})j \\ (yj)(vj) = -N(j)\bar{v}y \end{cases}$$

$$(jb_2)(jb_4) = -N(j)\, b_4 \bar{b}_2$$

$$(jb_2)\, b_3 = j\,(b_3 b_2)$$

$$b_1 (jb_4) = j\,(\bar{b}_1 b_4)$$

Matricial split form $\left\{ \begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix} : \xi, \eta \in k, \ x, y \in k^3 \right\}$

$$\begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix} \cdot \begin{pmatrix} \xi' & x' \\ y', & \eta' \end{pmatrix} = \begin{pmatrix} \xi\xi' + \langle x, y' \rangle & \xi x' + \eta' y + y \wedge y' \\ \eta y' + \xi' y + x \wedge x' & \eta\eta' + \langle y, x' \rangle \end{pmatrix}$$