1/14/2015

Elliptic functions, elliptic curves and modular forms

elliptic integrals

$$\int_{\infty}^{z} \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}}$$ multi-valued function on $\mathbb{P}^1(\mathbb{C}) \setminus \{e_1, e_2, e_3, \infty\}$

inversion of elliptic integral

$z \longmapsto x = \wp(z)$ is a single-valued meromorphic function on the complex plane $\mathbb{C}$

and $\left(\frac{d}{dz}\wp(z)\right)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3(z)$ $\longleftarrow$ inverse function theorem (implicit differentiation)

also: $\wp(z)$ is doubly periodic:

$$\exists \Lambda \underset{\text{lattice}}{\subset} \mathbb{C} \text{ s.t. } (\wp(z), \wp'(z)) : \mathbb{C}/\Lambda \hookrightarrow \mathbb{P}^2(\mathbb{C})$$

Weistrass functions: starting with a lattice $\Lambda \subseteq \mathbb{C}$

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \underbrace{\left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right]}_{O\left(\frac{1}{|\omega|^3}\right)}$$

aside: why not $\sum \frac{1}{(z-\omega)^2}$

To show periodicity for Λ, direct proof is cumbersome

better: 1) $\frac{d}{dz}\wp(z;\Lambda) = -2\sum_{\omega\in\Lambda} \frac{1}{(z-\omega)^3}$ is obviously doubly periodic

2) $\wp(z;\Lambda) = \wp(-z;\Lambda)$; in particular

$\wp(\omega/2;\Lambda) = \wp(-\frac{\omega}{2};\Lambda)$ $\quad \forall \omega \in \Lambda)$

$\Longrightarrow \wp(z+\omega;\Lambda) = \wp(z;\Lambda) \ \forall \omega \in \Lambda)$

Fourier/Laurent expansion at 0:

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda\setminus\{0\}\\ m\geq 1}} (m+1)\left(\frac{z}{\omega}\right)^m \cdot \frac{1}{\omega} = \frac{1}{z^2} + \sum_{m=1}^{\infty}\Bigg[\underbrace{(m+1)\cdot\sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{\omega^{m+2}}}_{}\Bigg] z^m$$

an Eisenstein series for $SL_2(\mathbb{Z})$

Proof that $\left[\frac{d}{dz}\wp(z)\right]^2 = 4\,\wp(z)^3 - \underbrace{g_2}_{\substack{\| \\ 60 s_4}}\,\wp(z) - \underbrace{g_3}_{\substack{\| \\ 140 s_6}}(z)$

where $s_n(\Lambda) = \sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{\omega^n}$ Note $s_n(\Lambda) = 0$ if n is odd

$$\left(\text{so: } \wp(z) = \frac{1}{z^2} + \sum_{m=1}^{\infty} (m+1)\, s_{m+2}\, z^m = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1)\, s_{2n+2}\, z^{2n} \right.$$

$$\left. = \frac{1}{z^2} + 3 s_4 z^2 + 4 s_6 z^4 + \cdots \right)$$

compare the first few Laurent expansion coefficients.

addition theorem:

$$\boxed{\begin{aligned} \wp(u_1+u_2) &= -\wp(u_1) - \wp(u_2) + \frac{1}{4}\left[\frac{\wp'(u_1)-\wp'(u_2)}{\wp(u_1)-\wp(u_2)}\right]^2 \\ \wp(2u) &= -2\,\wp(u) + \frac{1}{4}\left[\frac{\wp''(u)}{\wp'(u)}\right]^2 \end{aligned}}$$

(use the cubic equation & geometry of cubic curves: intersecting with a line in $\mathbb{P}^2$)

Weights. g_2 = modular form of weight 4 i.e. $g_2(\lambda\Lambda) = \lambda^{-4} g_2(\Lambda)$

g_3 = modular form of weight 6 $\quad g_3(\lambda\cdot\Lambda) = \lambda^{-6} g_3(\Lambda)$

$\forall \lambda \in \mathbb{C}^*$

$\wp(z)$ - weight 2 $\quad \wp'(z)$: weight 3

Formulas summary so far

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}}\left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right]$$

$$g_2(\Lambda) = 60 \sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{\omega^4}$$

$$g_3(\Lambda) = 140 \sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{\omega^6}$$

$$\Delta(\Lambda) = g_2^3(\Lambda) - 27\, g_3^2(\Lambda) \neq 0 \quad \left(\because \wp\left(\tfrac{\omega_1}{2}\right), \wp\left(\tfrac{\omega_2}{2}\right) \text{ and } \wp\left(\tfrac{\omega_1+\omega_2}{2}\right) \text{ are distinct}\right)$$

$$j(\Lambda) = 1728\, \frac{g_2(\Lambda)^3}{g_2(\Lambda)^3 - 27\, g_3(\Lambda)^2}$$

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1)\cdot \overbrace{\sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{\omega^{2n+2}}}^{=S_{2n+2}(\Lambda)} \cdot z^{2n}$$

$$= \frac{1}{z^2} + \frac{1}{20}\, g_2(\Lambda)\cdot z^2 + \frac{1}{28}\, g_3(\Lambda)\, z^4 + O(z^6)$$

$$\zeta(z;\Lambda) = \frac{1}{z} + \sum_{\omega\in\Lambda\setminus\{0\}}\left[\frac{1}{(z-\omega)} + \frac{1}{\omega} + \frac{z}{\omega^2}\right]; \quad \zeta(z+\omega;\Lambda) = \zeta(z;\Lambda) + \eta(z;\Lambda)$$

$$\sigma(z;\Lambda) = z\cdot \prod_{\omega\in\Lambda\setminus\{0\}} \left(1-\frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2}$$

$$\sigma(z+\omega;\Lambda) = \psi(\omega)\cdot e^{\eta(\omega)\left(z+\frac{\omega}{2}\right)} \cdot \sigma(z;\Lambda) \quad \forall z\in\mathbb{C},\ \forall \omega\in\Lambda$$

$$\psi(\omega) = \begin{cases} 1 & \text{if } \omega\in 2\Lambda \\ -1 & \text{if } \omega\notin 2\Lambda \end{cases}$$

$$\wp(u+v) + \wp(u) + \wp(v) = \frac{1}{4}\left[\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right]^2$$

$$\wp(2u) + 2\wp(u) = \frac{1}{4}\left[\frac{\wp''(u)}{\wp'(u)}\right]^2$$

$$\wp(z) - \wp(y) = -\frac{\sigma(z+y)\cdot\sigma(z-y)}{\sigma^2(z)\cdot\sigma^2(y)} \qquad \wp'(z) = -\frac{\sigma(2z)}{\sigma^4(z)}$$

$$= 2\,\frac{\sigma\left(z-\frac{\omega_1}{2}\right)\cdot\sigma\left(z-\frac{\omega_2}{2}\right)\cdot\sigma\left(z-\frac{\omega_3}{2}\right)}{\sigma^3(z)\cdot\sigma\left(\frac{\omega_1}{2}\right)\cdot\sigma\left(\frac{\omega_2}{2}\right)\cdot\sigma\left(\frac{\omega_3}{2}\right)}$$

$$\frac{1}{2}\,\frac{\wp'(z)-\wp'(y)}{\wp(z)-\wp(y)} = \zeta(z+y) - \zeta(z) - \zeta(y)$$

$$\frac{\wp'(z)}{\wp(z)-\wp(y)} = \zeta(z+y) + \zeta(z-y) - 2\zeta(z); \quad \frac{\wp'(y)}{\wp(z)-\wp(y)} = \zeta(z-y) + \zeta(z-y) + 2\zeta(y)$$

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[Note: $\wp(z) = z^{-2} + \sum_{n=1}^{\infty} (2n+1)\cdot s_{2n+2}(\Lambda)\cdot z^{2n}$
$= z^{-2} + \frac{1}{20} g_2 z^2 + \frac{1}{28} g_3 z^4 + O(z^6)$ near 0]

the j-invariant:

$y^2 = 4x^3 - g_2 x - g_3$ $\qquad g_2 = g_2(\Lambda) = 60 \cdot \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^4}$

$g_3 = g_3(\Lambda) = 140 \cdot \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^6}$

$\Delta = \Delta(\Lambda) = g_2^3 - 27 g_3^2 \quad (\neq 0)$

$j = 1728\, J = 1728 \frac{g_2^3}{g_2^3 - 27 g_3^2}$

special cases: $g_2 = 0$ (CM by $\mathbb{Z}(\mu_3)$) $\quad j = 0 \quad \Lambda \sim \mathbb{Z}[\frac{-1+\sqrt{-3}}{2}]$
$\mathbb{C}/\Lambda$: has an autom. of order 6

$g_3 = 0$ (CM by $\mathbb{Z}(\sqrt{-1})$) $\quad j = 1728 \quad \Lambda \sim \mathbb{Z}[\sqrt{-1}]$
$\mathbb{C}/\Lambda$: has an automorphism of order 4)

Weber function(s)

Let

case $g_2 \cdot g_3 \neq 0$: $h = h_1 = \frac{g_2 \cdot g_3}{\Delta} x$ $\qquad$ note: $g_2 \cdot g_3 \cdot x$: has weight 12

case $g_3 = 0$ $\quad h = h_2 = \frac{g_2^2}{\Delta} x^2$ $\qquad g_2^2 \cdot x$ has weight 12

case $g_2 = 0$ $\quad h = h_2 = \frac{g_3}{\Delta} x^3$ $\qquad g_3^2 \cdot x^3$

Prop: (a) The Weber function on an elliptic curve $\mathbb{C}/\Lambda$ is intrinsic:
$\forall$ isomorphism $\mathbb{C}/\Lambda_1 \xrightarrow[\sim]{\beta} \mathbb{C}/\Lambda_2$.
have $h_{\mathbb{C}/\Lambda_2} \circ \beta = h_{\mathbb{C}/\Lambda_1}$

(b) $\forall$ two points P, Q on an elliptic curve $\mathbb{C}/\Lambda$.
$h_{\mathbb{C}/\Lambda}(P) = h_{\mathbb{C}/\Lambda}(Q)$ iff $\exists$ automorphism β s.t. $Q = \beta(P)$

Equivalently: $h_{\mathbb{C}/\Lambda}$ defines a bijection $E_\Lambda / \mathrm{Aut}(E_\Lambda) \xrightarrow[\text{bij.}]{\sim} \mathbb{P}^1(\mathbb{C})$

(Note: $E_\Lambda / \mathrm{Aut}(E_\Lambda) \xrightarrow[\sim]{h_{E_\Lambda}} \mathbb{P}^1$ schematically)

General properties of meromorphic functions on $\mathbb{C}/\Lambda$

Prop f: non-const. meromorphic function on $\mathbb{C}/\Lambda = E_\Lambda$

(i) $\sum \mathrm{Res}(f) = 0$

(ii) $\sum \mathrm{ord}(f) = 0$ Apply (i) to f'/f

(iii) Let $(f) = \sum_i m_i [z_i]$ (formal sum as a divisor on $\mathbb{C}/\Lambda$)

Then: $\sum_i m_i z_i \equiv 0 \pmod{\Lambda}$ $\left(\int_{\text{fund. parallegram}} z \cdot \frac{f'(z)}{f(z)} \, dz \right.$ gives $\left. \sum m_i \cdot z_i \right)$

addition theorem:

(i) $0 = \begin{vmatrix} \wp(x) & \wp'(x) & 1 \\ \wp(y) & \wp'(y) & 1 \\ \underset{\wp(x+y)}{\wp(-x-y)} & \underset{-\wp'(x+y)}{\wp'(-x-y)} & 1 \end{vmatrix}$ equivalent form: $\boxed{\wp(x+y) = \frac{1}{4}\left[\frac{\wp'(x) - \wp'(y)}{\wp(x) - \wp(y)}\right]^2 - \wp(x) - \wp(y)}$

Key: $\exists A, B$ s.t.

$\wp'(\zeta) - A\wp(\zeta) - B$ has 3 roots: $x, y, -x-y$ (mod Λ)

$\rightsquigarrow$ $4\wp(\zeta)^3 - g_2\wp(\zeta) - g_3 = (A\wp(\zeta) + B)^2$

holds when $\zeta = x, y$ or $-x-y$ mod Λ

$0 = 4T^3 - A^2 z - (2AB + g)z - (B^2 + g_3)$

has roots $\wp(x), \wp(y), \wp(x+y)$

$\rightsquigarrow$ $\wp(x) + \wp(y) + \wp(x+y) = \frac{A^2}{4} = \frac{1}{4}\left[\frac{\wp'(x) - \wp'(y)}{\wp(x) - \wp(y)}\right]^2$

(ii) $\boxed{\wp(2z) = \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 - 2\wp(z)}$ $\Leftarrow$ duplication formula

Alternative proof of: $0 = \frac{1}{4}\left[\frac{\wp'(z) - \wp'(y)}{\wp(z) - \wp(y)}\right]^2 - \wp(z) - \wp(z+y) - \wp(y)$

Alternative proof, continued

Consider the meromorphic function $\qquad\qquad y$: generic

$$g(z) = \frac{1}{4}\left[\frac{\wp'(z) - \wp'(y)}{\wp(z) - \wp(y)}\right]^2 - \wp(z) - \wp(z+y) - \wp(y)$$

Clearly: holomorphic at $z = y$

At $z=0$: $g(z) = \frac{1}{4}\left[\frac{-2z^{-3}(1+O(z^4))}{z^{-2}(1+O(z^4))}\right]^2 - z^{-2} + O(1)$ = holomorphic at $z=0$

Better: $g(z) = [1+O(z^4)]\cdot z^{-2} - z^{-2} + O(z^4) - \wp'(y)z + O(z^2)$
$= -\wp'(y)z + O(z^2)$

At $z=-y$, write $z=-y-w$

$$g(z) = \underbrace{\frac{1}{4}\left[\frac{-\wp'(y+w) - \wp'(y)}{\wp(y+w) - \wp(y)}\right]^2}_{\frac{1}{4}\left[\frac{-2\wp'(y)}{\wp'(y)w}\cdot(1+O(w))\right]^2} - \wp(y+w) - \underbrace{\wp(w)}_{\frac{1}{w^2}+O(w^2)} - \wp(y) = \text{holomorphic at } z=-y \text{ as well}$$

Conclude: $g(z)$ is holomorphic on $\mathbb{C}/\Lambda$, hence is a constant (depending on y)
But the Taylor series expansion of $g(z)$ at $z=0$ shows that $g(z)=0$ identically!

The half-periods (2-torsion points)

Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Let $\omega_3 = -\omega_1 - \omega_2$

Choose normalization/orientation: $\mathrm{Im}(\omega_1/\omega_2) > 0$

$$\Longleftrightarrow \begin{vmatrix} \omega_1 & \omega_2 \\ \overline{\omega_1} & \overline{\omega_2} \end{vmatrix} \in \sqrt{-1}\cdot\mathbb{R}_{>0} \Longleftrightarrow \omega_2, \omega_1$$ is an oriented basis for the standard orientation of $\mathbb{C}$

$e_i = \wp(\frac{\omega_i}{2};\Lambda)$

$\rightsquigarrow$ e_1, e_2, e_3 are the three distinct roots of $x^3 - g_2x - g_3$

$[-1] \leftrightarrow ((x,y)\mapsto(x,-y))$

$\rightsquigarrow$ $e_1+e_2+e_3=0$, $e_1e_2+e_2e_3+e_3e_1 = -g_2/4$, $e_1e_2e_3 = \frac{1}{4}g_3$

elliptic integral $\quad$ Suppose $e_1>e_2>e_3$. $e_1, e_2, e_3 \in \mathbb{R}$

$$\rightsquigarrow \frac{\omega_1}{2} = \int_{e_1}^{\infty} (4t^3 - g_2t - g_3)^{-\frac{1}{2}}\,dt > 0 \qquad \text{(Choice of } \omega_1)$$

$$\frac{\omega_3}{2} = \sqrt{-1}\int_{-\infty}^{e_3} (g_3 + g_2t - 4t^3)^{-\frac{1}{2}}\,dt \in \sqrt{-1}\,\mathbb{R}_{>0} \qquad \text{(Choice of } \omega_3)$$

Exer: $\wp(z+\frac{\omega_1}{2}) = \frac{1}{4}\left[\frac{\wp'(z)-\wp'(\frac{\omega_1}{2})}{\wp(z)-\wp(\frac{\omega_1}{2})}\right]^2 - \wp(z) - \wp(\frac{\omega_1}{2})$

$= \frac{[\wp(z)-e_2]\cdot[\wp(z)-e_3]}{\wp(z)-e_1} - \wp(z) - e_1$

$= \underbrace{e_1}_{\wp(\frac{\omega_1}{2})} + \frac{(e_1-e_2)(e_1-e_3)}{\wp(z)-e_1}$

$(\wp(z)-e_2)\cdot(\wp(z)-e_3)$
$= \wp(z)^2 + e_1\wp(z) + e_2e_3$
$= [\wp(z)-e_1][\wp(z)+2e_1] + 2e_1^2 + e_2e_3$

$2e_1^2+e_2e_3 = 2e_2^2+2e_3^2+5e_2e_3 = (2e_2+e_3)(e_2+2e_3) = (e_1-e_2)(e_1-e_3)$

Note: • Used only $e_1+e_2+e_3=0$

• This is the formula for the x-coordinate of translation by a (non-trivial) 2-torsion point

Weistraß ζ-function and Weistraß σ-function:

$$\zeta(z;\Lambda) := -\int_0^z \left(\wp(w)-w^{-2}\right)dw \qquad = -\int_\infty^z \frac{x\,dx}{y}$$

$$= \frac{1}{z} + \sum_{\omega\in\Lambda\setminus\{0\}} \Big[\underbrace{\frac{1}{z-\omega}+\frac{1}{\omega}+\frac{z}{\omega^2}}_{O(\frac{1}{|\omega|^3}) \text{ on compact subsets of } \mathbb{C}\setminus\Lambda}\Big]$$

quasi-periods

$\eta(\omega;\Lambda)$ $\qquad \zeta(z+\omega;\Lambda) = \zeta(z;\Lambda) + \eta(\omega;\Lambda) \qquad \forall z\in\mathbb{C}$, $\forall\omega\in\Lambda)$

$\omega\in\Lambda$

$\mathbb{Z}$-linear function on $\omega\in\Lambda$

defn $\eta(z;\Lambda)$ by: $\mathbb{R}$-linear extension

Legendre relation:

$\eta_2\omega_1 - \eta_1\omega_2 = 2\pi\sqrt{-1}$ $\qquad$ remember: $\mathrm{Im}\left(\omega_1/\omega_2\right) > 0$

Invariant form

Let $\gamma_i = [\omega_i] \in H_1(E(\mathbb{C}); \mathbb{Z}) \qquad i = 1, 2$

i.e. $\omega_i = \langle \underset{\overset{\shortparallel}{dz}}{\frac{dx}{y}}, \gamma_i \rangle = \int_{\gamma_i} \frac{dx}{y}$

α_1, α_2: dual basis of γ_1, γ_2 in $H^1_{dR}(E(\mathbb{C}))$

i.e. $\int_{\gamma_i} \alpha_j = \delta_{ij}$

intersection product: $\gamma_1 \cdot \gamma_2 = 1 \quad \gamma_1 \cdot \gamma_1 = \gamma_2 \cdot \gamma_2 = 0$

$$D: H_1(E(\mathbb{C})) \longrightarrow H^1(E(\mathbb{C}))$$

$$\gamma \mapsto (\underset{H_1}{\delta} \mapsto \delta \cdot \gamma)$$ intersection product

so $D(\gamma_2) = \alpha_1$

$D(\gamma_1) = -\alpha_2$

$$\frac{dx}{y} = \omega_1 \cdot \alpha_1 + \omega_2 \cdot \alpha_2 = \omega_1 \cdot D(\gamma_2) - \omega_2 D(\gamma_1)$$

$$x\frac{dx}{y} = -\eta_1 \alpha_1 - \eta_2 \alpha_2 = -\eta_1 D(\gamma_2) + \eta_2 D(\gamma_1)$$

Legendre relation: $\left[\frac{dx}{y}\right] \underset{\uparrow}{\cup} \left[x\frac{dx}{y}\right] = 2\pi\sqrt{-1}$

topological cup product

(so: $\left[\frac{dx}{y}\right] \underset{\uparrow}{\cdot} \left[x\frac{dx}{y}\right] = 1$)

DR product

How to remember the sign:

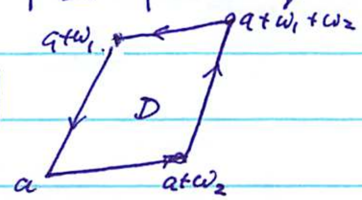
$$\begin{vmatrix} \lambda_1 & \lambda_2 \\ \eta(\lambda_1) & \eta(\lambda_2) \end{vmatrix} \underset{\mathbb{R}^\times_{>0}}{\sim} \begin{vmatrix} \lambda_1 & \lambda_2 \\ \bar\lambda_1 & \bar\lambda_2 \end{vmatrix}$$

$$-\eta \longleftrightarrow \left[\frac{x\,dx}{y}\right]$$

i.e. $-\eta(\omega) = \int_\omega \frac{x\,dx}{y} \quad \forall \omega \in \Lambda$

Proof of the Legendre relation $\omega_1 \eta_2 - \eta_1 \omega_2 = 2\pi\sqrt{-1}$

integrate $\zeta(z)\, dz$ over the boundary of a fund. parallelogram



$$2\pi\sqrt{-1} = \int_{\partial D} \zeta(z)\, dz = \omega_1 \underbrace{\left[\zeta(z+\omega_2) - \zeta(z)\right]}_{z \in \overrightarrow{a \;\; a+\omega_1}} + \omega_2 \underbrace{\left[\zeta(z) - \zeta(z+\omega_1)\right]}_{z \in \overrightarrow{a \;\; a+\omega_2}}$$

only 1 pole with residue 1 in D

$$= \omega_1 \eta_2 - \omega_2 \eta_1$$

$\zeta(z;\Lambda)$ weight 1 i.e. $\zeta(cz;c\Lambda) = \frac{1}{c}\cdot\zeta(z;\Lambda)$ $\forall c \in \mathbb{C}^\times$

η: weight 1 i.e. $\eta(c\omega;\ \Lambda) = \frac{1}{c}\,\eta(\omega;\Lambda)$

<u>Define</u> $\sigma(z;\Lambda) := z\cdot\prod_{\omega\in\Lambda\setminus\{0\}}\left(1-\frac{z}{\omega}\right)e^{z/\omega+\frac{1}{2}(z/\omega)^2}$ entire function on $\mathbb{C}$

Clearly: $\sigma(cz;c\Lambda) = c\cdot\sigma(z;\Lambda)$ $\forall c\in\mathbb{C}^\times$

i.e. $\sigma(z;\Lambda)$ has weight -1

<u>Note</u>: $\log\left(1-\frac{z}{\omega}\right)+\frac{z}{\omega}+\frac{1}{2}\left(\frac{z}{\omega}\right)^2 = O\left(\left|\frac{z}{\omega}\right|^3\right)$. <u>so</u>: the above product is unif convergent on compact subsets of $\mathbb{C}$

<u>Prop</u>

$$\boxed{\begin{array}{l} (1)\ \frac{d}{dz}\log\sigma(z;\Lambda) = \zeta(z;\Lambda) \quad \text{(obvious)} \\ (2)\ \sigma(z+\omega) = \sigma(z)\cdot\psi(\omega)\,e^{\eta(\omega)\left(z+\frac{\omega}{2}\right)} \\ \text{where } \psi(\omega) = \begin{cases} 1 & \text{if } \omega\in 2\Lambda \\ -1 & \text{if } \omega\notin 2\Lambda\end{cases}\end{array}}$$

$\forall\omega\in\Lambda$

Remark

<u>Analogy</u> with $\sin(z) = z\cdot\prod_{m\in\mathbb{Z}\setminus\{0\}}\left(1-\frac{z}{m\pi}\right)e^{\frac{z}{m\pi}}$

<u>Proof</u> of functional equation for $\sigma(z;\Lambda)$

$$\frac{d}{dz}\log\frac{\sigma(z+\omega)}{\sigma(z)} = \eta(\omega) \quad \forall z\in\mathbb{C}\quad\forall\omega\in\Lambda$$

$\rightsquigarrow$ $\sigma(z+\omega) = \sigma(z)\,e^{\eta(\omega)z+c(\omega)}$ where $c(\omega)\in\mathbb{C}$ <u>Write</u> $e^{c(\omega)} = \psi(\omega)\cdot e^{\eta(\omega)\cdot\frac{\omega}{2}}$

To evaluate $c(\omega)$: evaluate the above equation at $z=-\frac{\omega}{2}$

$\rightsquigarrow$ $\sigma\left(\frac{\omega}{2}\right) = \sigma\left(-\frac{\omega}{2}\right)e^{-\eta(\omega)\frac{\omega}{2}+c(\omega)}$

<u>So</u> $\omega\notin 2\Lambda \Rightarrow \psi(\omega) = -1$

$\forall\lambda_1,\lambda_2\in\Lambda$

<u>Note</u>

$$e^{\eta(\lambda_1)\left(z+\lambda_2+\frac{\lambda_1}{2}\right)+\eta(\lambda_2)\left(z+\frac{\lambda_2}{2}\right)} = e^{\eta(\lambda_1+\lambda_2)\left(z+\frac{\lambda_1+\lambda_2}{2}\right)}\cdot \underbrace{e^{\frac{1}{2}[\eta(\lambda_1)\lambda_2-\eta(\lambda_2)\lambda_1]}}_{\in\, e^{\pi\sqrt{-1}\,\mathbb{Z}}}$$

For $\omega = 2\cdot\omega'$ $\omega'\in\Lambda$: Apply the F.E. for ω' twice <u>q.e.d.</u>

More

Addition theorems:

(1) $$\boxed{\wp(z)-\wp(y) = -\frac{\sigma(z+y)\cdot\sigma(z-y)}{\sigma^2(z)\cdot\sigma^2(y)}}$$

proof: For generic y, both sides have simple zeros at $z \equiv \pm y \mod \Lambda$, double pole at $z \equiv 0 \mod \Lambda$. Determine the constant by evaluating at $z=0$.

(2) $$\frac{\wp'(z)}{\wp(z)-\wp(y)} = \zeta(z+y)+\zeta(z-y)-2\zeta(z)$$

$\frac{d}{dz}\log$ (1)

$$\frac{\wp'(y)}{\wp(z)-\wp(y)} = -\zeta(z+y)+\zeta(z-y)+2\zeta(y)$$

interchange y and z

$$\boxed{\frac{1}{2}\,\frac{\wp'(z)-\wp'(y)}{\wp(z)-\wp(y)} = \zeta(z+y)-\zeta(z)-\zeta(y)}$$

(3) Recall: $$\frac{1}{4}\left[\frac{\wp'(z)-\wp'(z)}{\wp(z)-\wp(y)}\right]^2 = \wp(z+y)+\wp(z)+\wp(y)$$

(1) $\Rightarrow$ (4) $$\wp'(z) = -\frac{\sigma(2z)}{\sigma^4(z)}$$ (limit of (1) as $z \to y$)

$$= 2\,\frac{\sigma(z-\frac{\omega_1}{2})\cdot\sigma(z-\frac{\omega_2}{2})\cdot\sigma(z-\frac{\omega_3}{2})}{\sigma^3(z)\cdot\sigma(\frac{\omega_1}{2})\cdot\sigma(\frac{\omega_2}{2})\cdot\sigma(\frac{\omega_3}{2})}$$

Both sides have the same divisor.

and $\lim_{z\to 0^+} z^3\cdot\wp'(z) = -2 = \lim_{z\to 0^+} z^3\cdot(\text{RHS})$

Define $$\boxed{\sigma_r(z;\omega_1,\omega_2) = e^{-\eta(\frac{\omega_r}{2};\Lambda)z}\,\frac{\sigma(z+\frac{\omega_r}{2})}{\sigma(\frac{\omega_r}{2})}}$$

entire, has simple zeros at $\frac{\omega_r}{2}+\Lambda$

Will see:
$$\begin{cases} \wp(z)-e_r = \left(\dfrac{\sigma_r(z)}{\sigma(z)}\right)^2 \\[2ex] e_r-e_s = \left(\dfrac{\sigma_s(\frac{\omega_r}{2})}{\sigma(\frac{\omega_r}{2})}\right)^2 \end{cases}$$

Apply (1) on p. 11:

$$\wp(z) - e_r = -\frac{\sigma(z+\frac{\omega_r}{2})\cdot\sigma(z-\frac{\omega_r}{2})}{\sigma^2(z)\cdot\sigma^2(\frac{\omega_r}{2})} = \frac{\sigma^2(z+\frac{\omega_r}{2})}{\sigma^2(z)\cdot\sigma^2(\frac{\omega_r}{2})}.$$

Now: $\sigma(z-\frac{\omega_r}{2}) = \sigma(z+\frac{\omega_r}{2})\cdot(-1)\cdot e^{-\eta(\omega_r)z}$.

$$\rightsquigarrow \wp(z) - e_r = e^{-\eta(\omega_r)z}\cdot\frac{\sigma^2(z+\frac{\omega_r}{2})}{\sigma^2(z)\cdot\sigma^2(\frac{\omega_r}{2})} = \left[\frac{\sigma_r(z)}{\sigma(z)}\right]^2$$

i.e. $\sqrt{\wp(z)-e_r} = \pm\frac{\sigma_r(z)}{\sigma(z)}$

F.E. for $\sigma_r(z)$: $\forall\, \omega\in\Lambda$

$$\frac{\sigma_r(z+\omega)}{\sigma_r(z)} = \psi(\omega)e^{-\eta(\frac{\omega_r}{2})\omega}\cdot e^{\eta(\omega)(z+\frac{\omega_r}{2}+\frac{\omega}{2})}$$

$$\frac{\sigma(z+\omega)}{\sigma(z)} = \psi(\omega)e^{-\eta(\omega)(z+\frac{\omega}{2})}$$

$$\rightsquigarrow \frac{\sigma_r(z+\omega)}{\sigma(z+\omega)}\Big/\frac{\sigma_r(z)}{\sigma(z)} = e^{\frac{1}{2}[\omega_r\cdot\eta(\omega)-\omega\cdot\eta(\omega_r)]}$$

$= \pm 1$ depending on the parity of $\omega_r\wedge\omega$ in $\det(\Lambda)$
$\uparrow$ 2^{nd} exterior product of Λ

Aside: (i) What happens if we use $-\frac{\omega_r}{2}$ insted of $\frac{\omega_r}{2}$ in the def^n of $\sigma_r(z)$?

Get: $$e^{\eta(\frac{\omega_r}{2})z}\frac{\sigma(z-\frac{\omega_r}{2})}{\sigma(-\frac{\omega_r}{2})}\Big/ e^{\eta(-\frac{\omega_r}{2})z}\frac{\sigma(z+\frac{\omega_r}{2})}{\sigma(\frac{\omega_r}{2})}$$

$$= e^{\eta(\omega_r)z}\cdot\psi(-\omega_r)\cdot e^{-\eta(\omega_r)z}\cdot(-1) = 1$$

Ans. get the same function $\sigma_r(z)$

(ii) What about replacing ω_r with $\omega_r+2\omega$, $\omega\in\Lambda$?

$$e^{-\eta(\frac{\omega_r}{2}+\omega)z}\frac{\sigma(z+\frac{\omega_r}{2}+\omega)}{\sigma(\frac{\omega_r}{2}+\omega)}\Big/ e^{-\eta(\frac{\omega_r}{2})z}\frac{\sigma(z+\frac{\omega_r}{2})}{\sigma(\frac{\omega_r}{2})}$$

$$= e^{-\eta(\omega)z}\cdot e^{\eta(\omega)(z+\frac{\omega_r}{2}+\frac{\omega}{2})-\eta(\omega)(\frac{\omega_r}{2}+\frac{\omega}{2})} = 1$$

Again, get the same function $\sigma_r(z)$

Definition (Jacobi elliptic functions)

$$\boxed{\begin{aligned} sn(u) &= \left(\frac{e_2-e_1}{\wp((e_2-e_1)^{-1}u)-e_1}\right)^{1/2} = \frac{\sigma_1(\frac{\omega_2}{2})}{\sigma(\frac{\omega_2}{2})}\cdot\frac{\sigma((e_2-e_1)^{-1}u)}{\sigma_1((e_2-e_1)^{-1}u)} \\ cn(u) &= \left(\frac{\wp((e_2-e_1)^{-1}u)-e_2}{\wp((e_2-e_1)^{-1}u)-e_1}\right)^{1/2} = \frac{\sigma_2((e_2-e_1)^{-1}u)}{\sigma_1((e_2-e_1)^{-1}u)} \\ dn(u) &= \left(\frac{\wp((e_2-e_1)^{-1}u)-e_3}{\wp((e_2-e_1)^{-1}u)-e_1}\right)^{\frac{1}{2}} = \frac{\sigma_3((e_2-e_1)^{-1}u)}{\sigma_1((e_2-e_1)^{-1}u)} \end{aligned}}$$

They are meromorphic functions on $\mathbb{C}/2\Lambda$
(but not on $\mathbb{C}/\Lambda$)

Exer Show that

$$\int_{[\omega]} \frac{\wp'(y)}{\wp(z)-\wp(y)}\,dz \equiv 2\omega\cdot\zeta(y) - 2\eta(\omega)\,y \pmod{2\pi\sqrt{-1}\,\mathbb{Z}}.$$

$\omega \in \Lambda$

and explain this (Note: LHS seems to be indep. of y mod Λ, but not actually so $\because$ the integrand is not a holomorphic differential
RHS: Only well-defined mod $2\pi\sqrt{-1}\,\mathbb{Z}$)

Function theory on $\mathbb{C}/\Lambda$

1. Expressing any meromorphic function on $\mathbb{C}/\Lambda$ in terms of $\wp(z;\Lambda)$ and $\wp'(z;\Lambda)$:

$$\mathbb{C}(E_\Lambda) = \mathbb{C}(\wp(z;\Lambda)) + \wp'(z;\Lambda)\cdot \mathbb{C}(\wp(z;\Lambda)) \cong \mathrm{frac}\left(\mathbb{C}[x,y]/(y^2-4x^3-g_2x-g_3)\right)$$

2. Expressing elements of $\mathbb{C}(E_\Lambda)$ as linear combinations of translations of $\left(\left(\frac{d}{dz}\right)^n \zeta(z;\Lambda)\right)_{n\geq 0}$

<u>Prop</u> If $f(z)$ is a meromorphic function on $\mathbb{C}/\Lambda$ with poles at $[a_1],\dots,[a_n]$

$$f(z) = \frac{c_{k,1}}{z-a_k} + \frac{c_{k,2}}{(z-a_k)^2} + \dots + \frac{c_{k,r_k}}{(z-a_k)^{r_k}} + (\text{holomorphic at } z=a_k)$$

$k = 1,\dots,n.$

$\exists$ constant A_2 s.t

$$\Rightarrow \quad f(z) = A_2 + \underbrace{\sum_{k=1}^{n} \sum_{i=1}^{r_k} \frac{(-1)^{i-1}}{(r_i-1)!} c_{k,i} \left(\frac{d}{dz}\right)^{i-1} \zeta(z-a_k)}_{g(z)} = A_2 + \sum_{i\geq 1}\sum_{k} \frac{(-1)^{i-1}}{(i-1)!} c_{k,i}\cdot\left(\frac{d}{dz}\right)^{i-1}\zeta(z-a_k)$$

Pf:

$$g(z+\omega) - g(z) = \left(\sum_{k=1}^{n} c_{k,1}\right)\cdot \eta(\omega;\Lambda) \quad \forall \omega\in\Lambda$$

<u>and</u> $\displaystyle\sum_{k=1}^{n} c_{k,1} = \sum$ residues of $f(z)\,dz = 0$. q.e.d.

3. Expressing non-const elements of $\mathbb{C}(E_\Lambda)$ as a quotient of products of $\sigma(z-a;\Lambda)$ $a\in\mathbb{C}$:

<u>Prop</u>: $\forall$ non-nonstant element $f(z)$ on $\mathbb{C}/\Lambda$ of degree n (i.e. $f: \mathbb{C}/\Lambda \to \mathbb{P}^1(\mathbb{C})$ has degree n)

$\exists\ a_1,\dots,a_n\in\mathbb{C}$, $\exists\ b_1,\dots,b_n\in\mathbb{C}$ s.t. $a_i \not\equiv b_j \pmod{\Lambda}$ $\forall i,j$

s.t. $a_1+\dots+a_n = b_1+\dots+b_n$

<u>and</u>

$$f(z) = \frac{\prod_{i=1}^{n}\sigma(z-a_i;\Lambda)}{\prod_{i=1}^{n}\sigma(z-b_i;\Lambda)}$$

The converse of these statements also hold: $[a_1], \cdots, [a_k] \in \mathbb{C}/\Lambda$ distinct

Prop: (1) $\forall$ data: $a_1, \cdots, a_n \in \mathbb{C}$ $\quad r_1, \cdots, r_k \geq 1$ positive integers

$c_{k1}, \cdots, c_{k,r_k} \in \mathbb{C}$ for each $k = 1, \cdots, n$

s.t. $\displaystyle\sum_{k=1}^{n} c_{k,1} = 0$.

$$\sum_{k=1}^{n} \sum_{i=1}^{r_k} \frac{(-1)^{i-1}}{(i-1)!} c_{k,i} \left(\frac{d}{dz}\right)^{i-1} \zeta(z - a_k ; \Lambda)$$

is a meromorphic function on $\mathbb{C}/\Lambda$ whose polar part

at $z = a_k$ is $\displaystyle\sum_{i=1}^{r_k} \frac{c_{k,i}}{(z-a_k)^i}$ for $k = 1, \cdots, n$

(2) $\forall\, a_1, \cdots, a_n ; b_1, \cdots, b_n \in \mathbb{C}$ s.t. $a_i \not\equiv b_j \pmod{\Lambda}$ $\forall i, j$

$$\frac{\prod_{i=1}^{n} \sigma(z - a_i ; \Lambda)}{\prod_{i=1}^{n} \sigma(z - b_i ; \Lambda)}$$

is a non-const. meromorphic function on $\mathbb{C}/\Lambda$

whose divisor is $\displaystyle\sum_{i=1}^{n} [a_i] - \sum_{i=1}^{n} [b_i]$

$$\lambda(\omega_1,\omega_2) := \frac{e_3-e_1}{e_2-e_1} = \frac{\wp(\frac{\omega_3}{2})-\wp(\frac{\omega_1}{2})}{\wp(\frac{\omega_2}{2})-\wp(\frac{\omega_1}{2})}$$

$$\lambda(\tau) = \frac{\wp(\frac{-1-\tau}{2};\Lambda_\tau)-\wp(\frac{\tau}{2};\Lambda_\tau)}{\wp(\frac{1}{2};\Lambda_\tau)-\wp(\frac{\tau}{2};\Lambda_\tau)}$$

Modular transformation:

$\tau \mapsto \underset{\tau'}{\overset{\shortparallel}{\tau+1}}$ i.e. $\begin{cases}\omega_1' = \tau+1\\ \omega_2' = 1\end{cases} \Rightarrow \begin{cases}\frac{\omega_3'}{2} = -\frac{\omega_1'}{2}-\frac{\omega_2'}{2} = -\frac{\tau}{2} \equiv \frac{\tau}{2} = \frac{\omega_1}{2}\\ \frac{\omega_1'}{2} \equiv \frac{\omega_3}{2}\\ \frac{\omega_2'}{2} = \frac{\omega_2}{2}\end{cases}$

$\leadsto \begin{array}{l} e_3' = e_1\\ e_2' = e_2\\ e_1' = e_3\end{array}$ $\lambda(\tau+1) = \frac{e_3'-e_1'}{e_2'-e_1'} = \frac{e_1-e_3}{e_2-e_3} \underset{\text{check}}{\overset{\uparrow}{=}} \frac{\lambda(\tau)}{\lambda(\tau)-1}$

$\tau \mapsto -\frac{1}{\tau} \iff \begin{array}{l}\omega_2' = \omega_1\\ \omega_1' = -\omega_2\end{array} \leadsto \begin{cases}\frac{\omega_2'}{2} \equiv \frac{\omega_1}{2}\\ \frac{\omega_1'}{2} \equiv \frac{\omega_2}{2}\\ \frac{\omega_3'}{2} \equiv \frac{\omega_3}{2}\end{cases} \Rightarrow \lambda(-\frac{1}{\tau}) = -\frac{e_3-e_2}{e_2-e_1} = 1-\lambda(\tau)$

<u>Conclude</u>

$$\lambda(\tau) \underset{\text{def}}{=} \frac{\wp(\frac{-1-\tau}{2};\Lambda_\tau)-\wp(\frac{\tau}{2};\Lambda_\tau)}{\wp(\frac{1}{2};\Lambda_\tau)-\wp(\frac{\tau}{2};\Lambda_\tau)}$$ satisfies the functional equation

$$\lambda(\tau+1) = \frac{\lambda(\tau)}{\lambda(\tau)-1}, \quad \lambda(-\tfrac{1}{\tau}) = 1-\lambda(\tau)$$

<u>Note</u>: monodromy group $\cong S_3$ (permuting the three non-trivial 2-torsion points)

6 representatives:

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&-1\\1&0\end{pmatrix}, \begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&-1\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\-1&1\end{pmatrix}, \begin{pmatrix}1&0\\1&1\end{pmatrix}$$

q-expansions

Starting point: $\sin(\pi z) = \pi z \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right)\left(1 + \frac{z}{n}\right)$ ①

Recall: Start with
0) $$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}$$

(difference is holomorphic, and go to 0 as $Re(z)$ bounded and $|Im(z)| \to \infty$)

2) integrate to get
$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n \in \mathbb{Z} \setminus \{0\}} \underbrace{\left(\frac{1}{z-n} + \frac{1}{n}\right)}_{= \frac{z}{n(z-n)}} = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n}\right)$$ ②

2) $\xLeftarrow{d \log(\cdot)}$ ①

For $\tau \in \mathfrak{h}$ ⇝ $$\pi \frac{\cos(\pi\tau)}{\sin(\pi\tau)} \underset{\uparrow}{=} \pi\sqrt{-1} \frac{e^{\pi\sqrt{-1}\tau} + e^{-\pi\sqrt{-1}\tau}}{e^{\pi\sqrt{-1}\tau} - e^{-\pi\sqrt{-1}\tau}} = \pi\sqrt{-1} \frac{q+1}{q-1}$$
$q = q_\tau = e^{2\pi\sqrt{-1}\tau}$

$$\frac{1}{\tau} + \sum_{n=1}^{\infty} \left[\frac{1}{\tau-n} + \frac{1}{\tau+n}\right] = \pi\sqrt{-1} + \frac{2\pi\sqrt{-1}}{q-1} = \pi\sqrt{-1} - 2\pi\sqrt{-1} \sum_{\nu=0}^{\infty} q^\nu$$ ④ $\forall \tau \in \mathfrak{h}$

$\left(\frac{d}{d\tau}\right)^{k-1}$ ⇝ $$(-1)^{k-1}(k-1)! \sum_{n \in \mathbb{Z}} \frac{1}{(\tau-n)^k} = -(2\pi\sqrt{-1})^k \sum_{\nu=1}^{\infty} \nu^{k-1} q^\nu$$ ⑤ $\forall k \in \mathbb{N}_{\geq 1}$ $\forall \tau \in \mathfrak{h}$

Get q-expansion for $G_{2k}(\tau) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{(m\tau+n)^{2k}}$ $k \geq 2$

↑ wt $2k$ modular form

Recall $g_2 = 60 G_2$
$g_3 = 140 G_3$

$$G_{2k}(\tau) = \underbrace{2\zeta(2k)}_{\text{sum for } m=0} + 2 \cdot \frac{(2\pi\sqrt{-1})^k}{(2k-1)!} \cdot \sum_{m=1}^{\infty} \sum_{\nu=1}^{\infty} \nu^{2k-1} \cdot q^{m\nu}$$

↑ use ④

<u>Prop</u>: $\forall k \geq 2$ (integer), have

$$\boxed{G_{2k}(\tau) = 2\,\zeta(2k) + 2\cdot\frac{(2\pi\sqrt{-1})^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)\,q^n}$$

where

$$\sigma_{2k-1}(n) \underset{\text{def}}{=} \sum_{\substack{d\in\mathbb{N}_{\geq 1}\\ d\mid n}} d^{2k-1}$$

· <u>Lesson</u>: The constant term of the Eisenstein series $G_{2k}(\tau)$ are L-values.

In particular: Use: $\zeta(4) = \frac{\pi^4}{90}$, $\zeta(6) = \frac{2\pi^6}{945}$

$$g_2(\tau) = 60\,G_4(\tau) = (2\pi)^4\cdot\frac{1}{2^2\cdot 3}\Big(1 + 240\underbrace{X}_{\sum_{n\geq 1}\sigma_3(n)q^n}\Big)$$

$$g_3(\tau) = 140\,G_6(\tau) = (2\pi)^6\cdot\frac{1}{2^3\cdot 3^3}\Big(1 - 504\underbrace{Y}_{\sum_{n\geq 1}\sigma_5(n)q^n}\Big)$$

$$\Delta(\tau) = g_2^3 - 27g_3^2 = (2\pi)^{12}\cdot\frac{1}{2^6 3^3}\left[(1+240X)^3 - (1-504Y)^2\right]$$

$$= \frac{(2\pi)^{12}}{2^6\cdot 3^3}\Big(q + \sum_{n\geq 2} d_n q^n\Big) \qquad d_n \in \mathbb{Z} \quad \forall n$$

Claim

i.e. $(1+\underbrace{240}_{2^4\cdot 3\cdot 5}X)^3 - (1-\underbrace{504}_{2^3\cdot 3^2\cdot 7}Y)^2 \equiv 0 \pmod{\underbrace{2^6\cdot 3^3}_{1728}}$

$$\equiv 2^4\cdot 3^2\cdot 5\,X + 2^4\cdot 3^2\cdot 7\,Y \pmod{1728}$$

$$\equiv 2^4\cdot 3^2\,(5X+7Y) \pmod{2^6\cdot 3^2}$$

Must show

$5X+7Y\equiv 0 \pmod 4 \iff X-Y\equiv 0 \pmod 4 \Longleftarrow d^3\equiv d^5 \pmod 4\ \ \forall d\in\mathbb{Z}$

so O.K. mod 4

$5X+7Y\equiv 0 \pmod 3 \iff -X+Y\equiv 0 \pmod 3$

$\Longleftarrow d^3\equiv d\equiv d^5 \pmod 3\ \ \forall d\in\mathbb{Z}$

so O.K. mod 3 as well

$$j(\tau) = 2^6 \cdot 3^3 \cdot \frac{g_2^3}{\Delta} = \frac{1}{q} + \underbrace{\sum_{n\geq 1} a_n q^n}_{\mathbb{Z}[[q]]} = \frac{1}{q} + 744 + 196884q + \cdots$$

(This is the reason to use $1728\frac{g_2^3}{\Delta}$ instead of $\frac{g_2^3}{\Delta}$)

q-expansion of $\wp(z; L_\tau)$ Let $q_z = e^{2\pi\sqrt{-1}z}$

recall:

$$\frac{1}{w} + \sum_{n=1}^{\infty}\left(\frac{1}{w-n} + \frac{1}{w+n}\right) = \pi\sqrt{-1} - 2\pi\sqrt{-1}\sum_{m=0}^{\infty} q_w^m \quad \text{if } |q_w| < 1$$

(5) $$\Rightarrow \sum_{n\in\mathbb{Z}} \frac{1}{(w+n)^2} \underset{\text{if } |q_w|<1}{=} (2\pi\sqrt{-1})^2 \sum_{k=1}^{\infty} k\, q_w^k = (2\pi\sqrt{-1})^2 \cdot \frac{q_w}{(1-q_w)^2}$$

$$\left[\frac{1}{1-t} = 1 + t + t^2 + \cdots \quad \Rightarrow \frac{t}{(1-t)^2} = t + 2t^2 + 3t^3 + \cdots\right]$$

Note: $\sum_{n\in\mathbb{Z}} \frac{1}{(w+n)^2} = (2\pi\sqrt{-1})^2 \frac{q_w}{(1-q_w)^2}$ $\forall w \in \mathbb{C}\setminus\mathbb{Z}$ (analytic continuation)

$$\wp(z;\tau) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(z-m\tau-n)^2} - \frac{1}{(m\tau+n)^2}\right]$$

$$= \underbrace{\frac{1}{z^2} + \sum_{m=0}\sum_{n\neq 0}} + \sum_{m\neq 0}\sum_{n\in\mathbb{Z}}$$

$$= \overbrace{\underbrace{\sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}}_{(2\pi\sqrt{-1})^2 \frac{q_z}{(1-q_z)^2}} - 2\zeta(2)} + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \left[\frac{1}{(z+m\tau+n)^2} + \frac{1}{(-z+m\tau+n)^2}\right] - 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^2}$$

combine terms indexed by m and $-m$

recall $\zeta(2) = \frac{\pi^2}{6}$

$$= (2\pi\sqrt{-1})^2 \frac{q_z}{(1-q_z)^2} - \frac{\pi^2}{3} + (2\pi\sqrt{-1})^2 \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} k \cdot q_\tau^{mk}\left(q_z^k + q_z^{-k}\right) - 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} k\, q_\tau^{mk}$$

if $|q_\tau^m \cdot q_z| < 1$ and $|q_\tau^m \cdot q_z^{-1}| < 1$ $\forall m \geq 1$

(i.e. if $|q_z| < |q_\tau| < |q_z|^{-1}$)

First q-expansion:

$$\boxed{\frac{1}{(2\pi\sqrt{-1})^2}\wp(z;\tau) = \frac{1}{12} + \underbrace{\frac{q_z}{(1-q_z)^2}}_{=\sum_{n=1}^{\infty} n q_z^n} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n\cdot q_\tau^{mn}\left(q_z^{n}+q_z^{-n}\right) - 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n\, q_\tau^{mn}}$$

when $|q_z| < |q_\tau| < |q_z|^{-1}$, $\tau \in \mathfrak{h}$

($|q_z|$ "sufficiently small)

Using: $\displaystyle\sum_{n\in\mathbb{Z}} \frac{1}{(w+n)^2} = (2\pi\sqrt{-1})^2 \frac{q_w}{(1-q_w)^2}$ for all $w \in \mathbb{C}\setminus\mathbb{Z}$

we get

$$= \frac{1}{12} + \frac{q_z}{(1-q_z)^2} + \frac{1}{(2\pi\sqrt{-1})^2}\sum_{m=1}^{\infty}\left[\frac{1}{(z+m\tau+n)^2} + \frac{1}{(z-m\tau+n)^2}\right] - \frac{2}{(2\pi\sqrt{-1})^2}\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\frac{1}{(m\tau+n)^2}$$

$$\wp(z;\tau) = \frac{1}{12} + \sum_{m\in\mathbb{Z}} \frac{q_\tau^m q_z}{(1-q_\tau^m q_z)^2} - 2\underbrace{\sum_{m=1}^{\infty}\frac{q_\tau^m}{(1-q_\tau^m)^2}}$$

Note: $m=0 \iff \frac{q_z}{(1-q_z)^2}$

also $= 2\displaystyle\sum_{n=1}^{\infty}\frac{n q_\tau^n}{1-q_\tau^n}$

Note: $\displaystyle\sum_{m\geq 0}\frac{q_\tau^m q_z}{(1-q_\tau^m q_z)^2}$ and $\displaystyle\sum_{m\geq 1}\frac{q_\tau^{-m} q_z^{-1}}{(1-q_\tau^{-m}\cdot q_z^{-1})^2}$ converge absolutely

Second expansion

$$\boxed{\begin{aligned}\frac{1}{(2\pi\sqrt{-1})^2}\wp(z;\tau) &= \frac{1}{12} + \sum_{m\in\mathbb{Z}}\frac{q_\tau^m q_z}{(1-q_\tau^m q_z)^2} - 2\sum_{m=1}^{\infty}\frac{q_\tau^m}{(1-q_\tau^m)^2} \\ &= \frac{1}{12} + \sum_{m\in\mathbb{Z}}\frac{q_\tau^m q_z}{(1-q_\tau^m q_z)^2} - 2\sum_{n=1}^{\infty}\frac{n q_\tau^n}{1-q_\tau^n}\end{aligned}}$$

for all $\tau \in \mathfrak{h}$, all z

$\frac{d}{dz}\left(2^{nd} \text{ expansion of } \wp(z;\tau)\right)$

$$\rightsquigarrow \boxed{\frac{1}{(2\pi\sqrt{-1})^3}\wp'(z;\tau) = \sum_{m\in\mathbb{Z}} \frac{q_\tau^m q_z (1+q_\tau^m q_z)}{(1-q_\tau^m q_z)^3}}$$

q-expansion of $g_2(\tau)$ and $g_3(\tau)$

$$\frac{1}{(2\pi\sqrt{-1})^4} g_2(\tau) = \frac{1}{12}\left(1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n\right)$$

$$= \frac{1}{12}\left(1+240\sum_{m,n\geq 1} n^3 q^{mn}\right)$$

i.e. $$\boxed{\frac{1}{(2\pi\sqrt{-1})^4} g_2(\tau) = \frac{1}{12}\left[1+240\sum_{n=1}^{\infty}\frac{n^3q^n}{(1-q^n)}\right]}$$

Similarly $$\boxed{\frac{1}{(2\pi\sqrt{-1})^6} g_3(\tau) = \frac{1}{6^3}\left[-1+504\sum_{n=1}^{\infty}\frac{n^5q^n}{(1-q^n)}\right]}$$

Euler computation of ζ (negative integers) $\qquad k \in \mathbb{N}_{\geq 1}$

$$(1-2^{k+1})\,\zeta(-k) = (1-2^{k+1}) \sum_{n\geq 1} n^k = \sum_{n\geq 1} n^k - 2 \cdot \sum_{n\geq 1} (2n)^k$$

$$= \sum_{n\geq 1} (-1)^{n-1} \cdot n^k$$

$$= \sum_{n\geq 1} (-1)^{n-1} n^k\, T^n \Big|_{T=1} = \left(T\frac{d}{dT}\right)^k \left(\frac{T}{1+T}\right)_{T=1}$$

$$\parallel$$

substitute $\quad T \longmapsto \frac{1}{T} = u \qquad \rightsquigarrow (-1)^k \left(u\frac{d}{du}\right)^k \left(\frac{1}{1+u}\right)\Big|_{u=1}$

$\frac{dT}{T} + \frac{du}{u} = 0$

$k \geq 1$

$\rightsquigarrow \zeta(-k) = 0$ if $k \equiv 0\ (2)$

substitution: $\quad T = e^x$

$\frac{dT}{T} = dx \quad \rightsquigarrow \quad T\frac{d}{dT} = \frac{d}{dx}$

$$\left(T\frac{d}{dT}\right)^k \left(\frac{T}{1+T}\right)\Big|_{T=1} = \left(\frac{d}{dx}\right)^k \left(\frac{e^x}{1+e^x}\right)\Big|_{x=0}$$

$$\left[\text{Recall: } \frac{z}{e^z-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \right]$$

$$\parallel$$

$$-\frac{z}{2} + \frac{z}{2}\left(\frac{e^z+1}{e^z-1}\right)$$

$$\zeta(1-2m) = (1-2^{2m})^{-1} \left(t\frac{d}{dt}\right)^{2m-1} \left(\frac{t}{1+t}\right)_{t=1}$$

F.E. $\boxed{\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right) \cdot \zeta(s) = \pi^{-\frac{1-s}{2}}\,\Gamma\left(\frac{1-s}{2}\right) \cdot \zeta(1-s)}$

$$\zeta(2n) = (-1)^{n-1} \frac{2^{2n-1}\pi^{2n}}{(2n)!} B_{2n}$$

Euler's evaluation of $\zeta(1-2m)$ $\quad m \in \mathbb{N}_{\geq 1}$

Have seen: $\zeta(1-2m) = (1-2^{2m})^{-1} \cdot \left(t\frac{d}{dt}\right)^{2m-1}\left(\frac{t}{1+t}\right)\Big|_{t=0}$

$$= (1-2^{2m})^{-1} \left(\frac{d}{dx}\right)^{2m-1}\left(\frac{e^x}{e^x+1}\right)\Big|_{x=0}$$

$$\boxed{\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n} \quad \text{def}^n$$

$$= -(1-2^m)^{-1} \left(\frac{d}{dx}\right)^{2m-1}\left(\frac{1}{e^x+1}\right)\Big|_{x=0}$$

$$\frac{x}{e^x-1} - \frac{2x}{e^{2x}-1} = \sum_{n=0}^{\infty} \frac{(1-2^n)B_n}{n!} x^n$$

$$\underset{\displaystyle \frac{x}{e^x+1}}{\|}$$

i.e. $\displaystyle \frac{1}{e^x+1} = \sum_{n=1}^{\infty} \frac{(1-2^n)B_n}{n!} x^{n-1}$

$$\rightsquigarrow \underset{\zeta(1-2m)}{\underbrace{-(1-2^m)^{-1}\left(\frac{d}{dx}\right)^{2m-1}\left(\frac{1}{e^x+1}\right)\Big|_{x=0}}} = -(1-2^m)^{-1} \cdot \frac{(1-2^m)\cdot B_{2m}}{2m}$$

$$\boxed{\zeta(1-2m) = -\frac{B_{2m}}{2m}} \qquad \forall\, m \geq 1 \qquad \underline{\text{OK.}}$$

$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}$

$$z = \left(\sum_{k=1}^{\infty} \frac{z^k}{k!}\right) \cdot \left(\sum_{i=0}^{\infty} \frac{B_i}{i!} z^i\right) \rightsquigarrow \sum_{i=0}^{n} \frac{B_i}{i!\,(n-i+1)} = 0 \quad \forall_{n\geq 2}$$

say.

Evaluation of $\zeta(2k)$ $\quad k \geq 1 \quad k \in \mathbb{N}$

$$\frac{z}{2} + \frac{z}{e^z - 1} = \frac{z}{2}\left(\frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}}\right) = \sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!} z^{2k}$$

$z = 2\pi\sqrt{-1}\, x$

$$\rightsquigarrow \quad (\pi x) \cdot \cot(\pi x) = \sum_{k=0}^{\infty} \frac{(2\pi\sqrt{-1})^{2k}}{(2k)!} B_{2k} \cdot z^{2k}$$

$$\|$$

$$1 + \sum_{n=1}^{\infty} z\left[\frac{1}{z-n} + \frac{1}{z+n}\right] = 1 - 2\sum_{k=1}^{\infty} \zeta(2k)\, z^{2k}$$

$$\rightsquigarrow \quad \boxed{\zeta(2k) = -\frac{(2\pi\sqrt{-1})^{2k}}{2\,(2k)!} B_{2k}} \qquad k \in \mathbb{N}_{\geq 1}$$

Modular function and modular forms

Prop: (a) $\{ z \in \mathbb{C} \mid -\frac{1}{2} \le Re(z) \le \frac{1}{2},\ |z| \ge 1,\ Im(z) > 0 \} =: D$ is a fundamental domain for $SL_2(\mathbb{Z}) \circlearrowleft \mathfrak{h}$

(b) $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ generate $\Gamma = SL_2(\mathbb{Z})$

pf $\quad Im\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{Im(\tau)}{|c\tau+d|^2} \Rightarrow$ In any given Γ-orbit $\Gamma\cdot\tau_0$
$\{ Im(\gamma\cdot\tau_0) : \gamma \in \Gamma \}$ is bounded above
and goes to 0 as $\max\{|c|,|d|\} \to \infty$

$\rightsquigarrow$ Given any Γ-orbit, $\exists$ a representative τ_1 with $Im(\tau_1)$ maximal

Since $Im\, S\cdot\tau_1 \le Im\,\tau_1 \Rightarrow |\tau_1| \ge 1$

Thus $\langle S,T\rangle \cdot D = \mathfrak{h}$

Suppose $\tau_2, \tau_3 \in D$, and $\Gamma\cdot\tau_2 = \Gamma\cdot\tau_3$

May assume: $Im(\tau_3) > Im(\tau_2)$, $\tau_3 = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\in \Gamma}^{\alpha} \cdot \tau_2 \qquad c \ge 0$

$\rightsquigarrow |c\tau_2 + d| \le 1$ $\qquad$ Use. $Im(\tau_2) \ge \frac{\sqrt{3}}{2}$

$\Longrightarrow c = 0$ or 1

Case A $c = 0 \Rightarrow \alpha = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for some $n \in \mathbb{Z}$

$\rightsquigarrow n = \pm 1$ ($\because \tau_2 + n \in D$)

Case B $c = 1 \Rightarrow \underbrace{\alpha = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}}$ or $\alpha = \begin{pmatrix} a & a-1 \\ 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} a & -a-1 \\ 1 & -1 \end{pmatrix}$

subcase B_2 $\qquad$ subcase B_3

$\tau_3 = a - \frac{1}{\tau_2}$

$\Rightarrow |\tau_2| = 1$

$\Rightarrow \tau_2 = e^{2\pi\sqrt{-1}/3}$ or $-e^{4\pi\sqrt{-1}/3}$

subcase B_2: $\alpha = \begin{pmatrix} a & a-1 \\ 1 & 1 \end{pmatrix}$

$\alpha\cdot\tau_2 = \frac{a\tau_2 + (a-1)}{\tau_2 + 1} = a - \frac{1}{\tau_2+1} \Rightarrow |\tau_2 + 1| = 1 \Rightarrow \tau_2 = e^{2\pi\sqrt{-1}/3}$

TST^{-1}
$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$
$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$

$T^{-1}ST$
$= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
$= \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$

Points with non-trivial stabilizers in D

at $e^{2\pi\sqrt{-1}/3}$: $S\cdot T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

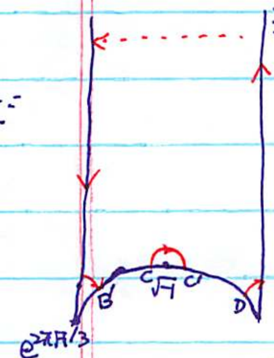
at $\sqrt{-1}$: S

Def: $(f \underset{k}{|} \gamma)(\tau) = f(\gamma\cdot\tau)\cdot(c\tau+d)^{-k}$ — Same formula for action of $GL_2(\mathbb{R})_+$

check
$$\left(\left(f \underset{k}{|} \gamma_1\right) \underset{k}{|} \gamma_2\right)(\tau) = \left(f \underset{k}{|} \gamma_1\right)(\gamma_2\cdot\tau)\cdot(c_2\tau+d_2)^{-k}$$
$$= f(\gamma_1\cdot\gamma_2\tau)\cdot\left(c_1(\gamma_2\cdot\tau)+d\right)^{-k}\cdot(c_2\tau+d)^{-k}$$
$$= \left(f \underset{k}{|} \gamma_1\cdot\gamma_2\right)(\tau)$$

Prop: f: meromorphic modular form of weight $2k$ w.r.t. $SL_2(\mathbb{Z})$
$$\Rightarrow \nu_\infty(f) + \frac{1}{3}\nu_{e^{2\pi\sqrt{-1}/3}}(f) + \frac{1}{2}\nu_{\sqrt{-1}}(f) + \sum_{\substack{P\in \,SL_2(\mathbb{Z})\backslash\mathfrak{h} \\ P\not\sim e^{2\pi\sqrt{-1}/3} \text{ or } \sqrt{-1}}} \nu_P(f) = \frac{k}{6}$$

pf:



$\frac{1}{2\pi\sqrt{-1}}\int \frac{f'(\tau)}{f(\tau)}\,d\tau$

The term $\frac{k}{6}$ comes from:

compare $\overset{\frown}{B'C}$ with $\overset{\frown}{C'D}$

$$f(S\tau) = \tau^{2k} f(\tau)$$

So: $\int_{B'C} + \int_{C'D} = \int_{B'C} \frac{2k}{\tau}\,d\tau$

about $\frac{1}{12}$ of a full circle

Note: g_2 vanishes at $\tau = e^{2\pi\sqrt{-1}/3}$ (CM by $\mathcal{O}_{\mathbb{Q}(\mu_3)}$)

g_3 vanishes at $\tau = \sqrt{-1}$ (CM by $\mathbb{Z}[\sqrt{-1}]$)

Use this formula $\nu_\infty(f) + \frac{1}{3}\nu_{e^{2\pi\sqrt{-1}/3}}(f) + \frac{1}{2}\nu_{\sqrt{-1}}(f) = \frac{k}{6} = \frac{wt}{12}$
for non-const meromorphic modular forms of weight $2k$, get

Prop

$$\underset{k}{\bigoplus} \underbrace{M_k}_{\substack{\text{holomorphic} \\ \text{modular form of weight } k \\ \text{for } SL_2(\mathbb{Z})}} = \mathbb{C}[\underbrace{g_2}_{wt=4}, \underbrace{g_3}_{wt=6}]$$

Prop $j : SL_2(\mathbb{Z}) \backslash \mathfrak{h} \xrightarrow[\text{bijection}]{} \mathbb{C}$

pf: Apply the RR formula to $j - \underbrace{c}_{\text{any } c \in \mathbb{C}}$

Note $j(e^{2\pi\sqrt{-1}/3}) = 0$ $\qquad \nu_{e^{2\pi\sqrt{-1}/3}}(j) = 3$ $\qquad \because j = 1728 \frac{g_2^3}{g_2^3 - 27g_3^2}$

$j(\sqrt{-1}) = 1728$ $\qquad \nu_{\sqrt{-1}}(j - 1728) = 2$

Rmk
This is a proof that $j : SL_2(\mathbb{Z}) \backslash \mathfrak{h} \to \mathbb{C}$ is surjective by classical-style complex analysis.

There are other proofs, e.g. using a Weistrass eq^n with a pre-assigned j-value, plus the fact that a Weistrass eq^n defines an elliptic curve.

q-expansion of Weistraß ~~ζ-function~~ σ-function:

Consider $\sigma(z;\Lambda_\tau)$: Want modify $\sigma(z;\Lambda_\tau)$ by $e^{(\text{quad. poly in } z)}$ so that the resulting function has period 1

(This quadratic poly is uniquely determined by this requirement)

Prop: $\varphi(z;\tau) \stackrel{\text{def}}{=} e^{-\frac{1}{2}\eta(1)z^2} \cdot e^{\pi\sqrt{-1}z} \cdot \sigma(z;\Lambda_\tau)$ satisfies

$$\varphi(z+1) = \varphi(z) \quad ; \quad \varphi(z+\tau) = -\frac{1}{q_z}\varphi(z)$$

Pf: $\varphi(z+1) = (-1) e^{-\frac{1}{2}\eta(1)(z+1)^2} \cdot e^{\pi\sqrt{-1}z} \cdot (-1) \cdot e^{\eta(1)(z+\frac{1}{2})} \cdot \sigma(z;\Lambda_\tau)$

from the factor $e^{\sqrt{-1}z}$; $\psi(1;\Lambda_\tau)$

$= \varphi(z)$ — achieved by construction/definition

$$\varphi(z+\tau) = e^{-\frac{1}{2}\eta(1)(z+\tau)^2} \cdot e^{\pi\sqrt{-1}\tau} \cdot (-1) \cdot e^{\eta(\tau)(z+\frac{\tau}{2})} \cdot \sigma(z;\Lambda_\tau)$$

$$= (-1) \cdot e^{\eta(\tau)\cdot\frac{\tau}{2} - \eta(1)\cdot\tau\cdot z - \frac{1}{2}\eta(1)\tau^2 + \eta(\tau)z + \pi\sqrt{-1}\tau}\varphi(z)$$

Legendre formula / relation:

$$\tau\cdot\eta(1) - \eta(\tau) = 2\pi\sqrt{-1}$$

The factor is: $(-1)\cdot e^{\left[\eta(\tau)\cdot\frac{\tau}{2} - (\eta(\tau)+2\pi\sqrt{-1})z - \frac{1}{2}(\eta(\tau)+2\pi\sqrt{-1})\tau + \eta(\tau)z + \pi\sqrt{-1}\tau\right]}$

The exponent is: $\cancel{\eta(\tau)\cdot\frac{\tau}{2}} - \cancel{\eta(\tau)z} - 2\pi\sqrt{-1}z - \cancel{\frac{1}{2}\eta(\tau)\cdot\tau} - \cancel{\pi\sqrt{-1}\tau} + \cancel{\eta(\tau)z} + \cancel{\pi\sqrt{-1}\tau}$

$= -2\pi\sqrt{-1}\tau$

i.e $\varphi(z+\tau) = -\frac{1}{q_z}$ QED

General Reformulation $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, $\mathrm{Im}(\omega_1/\omega_2) > 0$

Prop' Let

$$\boxed{\varphi(z;\omega_1,\omega_2) := e^{-\frac{1}{2}\eta_\Lambda(\omega_2)\,\omega_2\cdot\left(\frac{z}{\omega_2}\right)^2} \cdot e^{\pi\sqrt{-1}\left(\frac{z}{\omega_2}\right)}\,\sigma(z;\Lambda)}$$

Then $\varphi(z+\omega_2;\omega_1,\omega_2) = \varphi(z;\omega_1,\omega_2)$

and $\varphi(z+\omega_1;\omega_1,\omega_2) = -\frac{1}{q_{z/\omega_2}}\varphi(z;\omega_1,\omega_2)$

(also : $\varphi(\lambda z;\omega_1,\omega_2) = \lambda\varphi(z;\omega_1,\omega_2)$)

$\tau \in \mathfrak{h}$

Prop $\varphi(z;\tau) = e^{-\frac{1}{2}\eta(1) z^2} \cdot e^{\pi\sqrt{-1} z} \cdot \sigma(z; \Lambda_\tau)$ as before

Then $$\varphi(z;\tau) = (2\pi\sqrt{-1})^{-1} \cdot (q_z - 1) \cdot \prod_{n=1}^{\infty} \frac{(1-q_\tau^n q_z)\cdot(1-q_\tau^n q_z^{-1})}{(1-q_\tau^n)^2} \quad ①$$

and ② $$\boxed{\sigma(z;\tau) = (2\pi\sqrt{-1})^{-1} \cdot e^{\frac{1}{2}\eta(1) z^2} \cdot \left(q_z^{\frac{1}{2}} - q_z^{-\frac{1}{2}}\right) \cdot \prod_{n=1}^{\infty} \frac{(1-q_\tau^n q_z)(1-q_\tau^n q_z^{-1})}{(1-q_\tau^n)^2}}$$

These two formulas are clearly equivalent

pf: Let $g(z) = $ RHS of ① Clearly $g(z+1) = g(z)$

$g(z+\tau) = -\frac{1}{q_z} g(z)$

Consider $\frac{\varphi(z)}{g(z)}$: Clearly $\mathrm{div}\, \frac{\varphi(z)}{g(z)} = $ trivial $= 0$.

& doubly periodic

$\rightsquigarrow \frac{\varphi(z)}{g(z)} = $ constant.

Look at $\lim_{z\to 0} \frac{\varphi(z)}{g(z)} = 1$ Q.E.D.

Assume $\omega_1/\omega_2 =: \tau \in \mathfrak{h}$

Reformulation

$$\varphi(z;\omega_1,\omega_2) := e^{-\frac{1}{2}\eta(\omega_2)\omega_2\left(\frac{z}{\omega_2}\right)^2} \cdot e^{\pi\sqrt{-1}\left(\frac{z}{\omega_2}\right)} \sigma(z; \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$$

Then: $$\varphi(z;\omega_1,\omega_2) = \frac{\omega_2}{2\pi\sqrt{-1}} \left(q_{z/\omega_2} - 1\right) \cdot \prod_{n=1}^{\infty} \frac{(1-q_\tau^n q_{z/\omega_2})\cdot(1-q_\tau^n q_{z/\omega_2}^{-1})}{(1-q_\tau^n)^2}$$

$$\sigma(z;\omega_1,\omega_2) = \frac{\omega_2}{2\pi\sqrt{-1}} e^{\frac{1}{2}\eta(\omega_2)\omega_2\cdot\left(\frac{z}{\omega_2}\right)^{-1}} \cdot \prod_{n=1}^{\infty} \frac{(1-q_\tau^n q_{z/\omega_2})(1-q_\tau^n \cdot q_{z/\omega_2}^{-1})}{(1-q_\tau^n)^2}$$

q-expansion of $\zeta(z;\tau) := \frac{d}{dz}\sigma(z;\tau)$

③ $$\zeta(z;\tau) = \eta_{\Lambda_\tau}(1)\,z + \pi\sqrt{-1}\,\frac{q_z^{\frac{1}{2}} + q_z^{-\frac{1}{2}}}{q_z^{1/2} - q_z^{-1/2}} + 2\pi\sqrt{-1}\cdot\sum_{n=1}^{\infty}\left[\frac{q_\tau^n\cdot q_z^{-1}}{1-q_\tau^n\cdot q_z^{-1}} - \frac{q_\tau^n\cdot q_z}{1-q_\tau^n\, q_z}\right]$$

Recall: $$\sigma(z;\tau) = z\cdot\prod_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\left(1-\frac{z}{m\tau+n}\right)\cdot e^{\frac{z}{m\tau+n}+\frac{1}{2}\frac{z^2}{(m\tau+n)^2}}$$

$$\rightsquigarrow\ \zeta(z;\tau) = \frac{1}{z} - \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\ \sum_{k\geq 2}\frac{z^k}{(m\tau+n)^{k+1}}$$

$$= \frac{1}{z} - \sum_{k\geq 2}\Bigg[\underbrace{\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{1}{(m\tau+n)^{k+1}}}_{\text{Only need } k \text{ odd}}\Bigg] z^k \qquad ④$$

$$= \frac{1}{z} - \sum_{\ell\geq 2} G_{2\ell}(\tau)\cdot z^{2\ell-1}$$

compare: $\wp(z;\tau) = \frac{1}{z^2} + \sum_{\ell\geq 1}(2\ell+1)\,G_{2\ell+2}(\tau)\cdot z^{2\ell}$

③': $$\zeta(z;\tau) = \eta(1;\Lambda_\tau)z + \underbrace{\pi\sqrt{-1}\,\frac{q_z+1}{q_z-1}}_{=\pi\frac{\cos(\pi z)}{\sin(\pi z)}} + 2\pi\sqrt{-1}\cdot\sum_{n=1}^{\infty}\left[\frac{q_\tau^n\cdot q_z^{-1}}{1-q_\tau^n\, q_z^{-1}} - \frac{q_\tau^n\cdot q_z}{1-q_\tau^n\, q_z}\right]$$

$$\left(= \pi\sqrt{-1}\left(1-\frac{2}{1-q_z}\right)\right)$$

$$= \frac{1}{z} - \frac{\pi^2 z}{3} - \frac{\pi^4 z^3}{45} - \frac{2\pi^6 z^5}{45\cdot 21} - \cdots \quad \left(\text{Taylor expansion of } \frac{\cos(\pi z)}{\sin(\pi z)}\right)$$

Will get: $\eta(1;\Lambda_\tau)$ by looking at $0 =$ coeff of z in the Taylor expansion of $\zeta(z;\tau)$ at $z=0$ in ③'

Taylor expansion of $\sum_{n=1}^{\infty}\left[\frac{q_\tau^n \cdot q_z^{-1}}{1-q_\tau^n q_z^{-1}} - \frac{q_\tau^n q_z}{1-q_\tau^n q_z}\right]$ $\qquad q_\tau = q,\ q_z = w$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\left(\frac{q^n}{w}\right)^m - (q^n w)^m\right]$$

abs. convergent for $|q|^{-1} < w < |q|^{-1}$

$$= \sum_{m=1}^{\infty}\left(w^{-m} - w^m\right)\cdot\frac{q^m}{1-q^m}$$

$$w^{-m} - w^m = e^{-2\pi\sqrt{-1}mz} - e^{2\pi\sqrt{-1}mz} = -4\pi\sqrt{-1}mz + O(z^3)$$

$$= -4\pi\sqrt{-1}\cdot\sum_{m=1}^{\infty}\frac{m q_\tau^m}{1-q_\tau^m}$$

$$\rightsquigarrow 0 = \eta(1;\Lambda_\tau) - \frac{\pi^2}{3} - 2\cdot(2\pi\sqrt{-1})^2\underbrace{\sum_{m=1}^{\infty}\frac{m q_\tau^m}{1-q_\tau^m}}_{=\sum_{m,n\geq 1} m q_\tau^{mn} = \sum_{n\geq 1}\sigma_1(n) q_\tau^n}$$

Conclude:

$$\boxed{\eta(1;\Lambda_\tau) = \frac{(2\pi\sqrt{-1})^2}{12}\left[-1 + 24\sum_{n=1}^{\infty}\frac{n q_\tau^n}{1-q_\tau^n}\right]} \qquad (5)$$

equiv $\eta(1;\Lambda_\tau) = \frac{(2\pi\sqrt{-1})^2}{12}\left[-1 + 24\sum_{n=1}^{\infty}\sigma_1(n) q_\tau^n\right]$ $\quad$ q-expansion of $\eta_2(\tau,1) = \eta_2(1;\Lambda_\tau)$

Note: $\eta(1;\Lambda_\tau)$ smells like a "modular form of weight 2".
$[\eta(\lambda z;\lambda\Lambda) = \lambda^{-1}\eta(z;\Lambda)]$

Note Legendre relation:

$$\tau\,\eta(1;\Lambda_\tau) - \eta(\tau;\Lambda_\tau) = 2\pi\sqrt{-1}$$

$$\rightsquigarrow \eta(\tau;\Lambda_\tau) = -2\pi\sqrt{-1} + \tau\cdot\eta(1;\Lambda_\tau)$$

$$= -2\pi\sqrt{-1} - \frac{(2\pi\sqrt{-1})^2}{12} + 2(2\pi\sqrt{-1})^2\cdot\tau\cdot\sum_{n=1}^{\infty}\frac{n q_\tau^n}{1-q_\tau^n}$$

Another way:

$$\zeta(z;\tau) = \frac{1}{z} + \sum_{(m,n)\neq\{(0,0)\}} \left[\frac{1}{z-m\tau-n} + \frac{1}{m\tau+n} + \frac{z}{(m\tau+n)^2}\right]$$

$$\rightsquigarrow \quad \eta(1,\Lambda_\tau) = \zeta(z+1;\tau) - \zeta(z;\tau)$$

$$= \frac{1}{z+1} - \frac{1}{z} + \sum_{m\in\mathbb{Z}} \sum_{\substack{n\in\mathbb{Z} \text{ if } m\neq 0 \\ n\in\mathbb{Z}\setminus\{0\} \text{ if } m=0}} \left[\frac{1}{z-m\tau-n+1} - \frac{1}{z-m\tau-n} + \frac{1}{(m\tau+n)^2}\right]$$

$$= \underbrace{\frac{1}{z+1} - \frac{1}{z}}_{} + \sum_{m\in\mathbb{Z}\setminus\{0\}} \sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^2} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \left[\frac{1}{n^2} + \underbrace{\frac{1}{z+1-n} - \frac{1}{z-n}}_{}\right]$$

add to 0

$$= \sum_{m\in\mathbb{Z}} \sum_{\substack{n\in\mathbb{Z} \text{ if } m\neq 0 \\ n\in\mathbb{Z}\setminus\{0\} \text{ if } m=0}} \frac{1}{(m\tau+n)^2}$$

i.e.

$$\boxed{\eta(1;\Lambda_\tau) = \sum_{m\in\mathbb{Z}} \sum_{\substack{n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^2}} \qquad (6)$$

double check:

use
$$(-1)\cdot \sum_{n\in\mathbb{Z}} \frac{1}{(w-n)^2} = \pi\sqrt{-1}\,\frac{q_w+1}{q_w-1} \underset{\text{if } |q_w|<1}{=} -(2\pi\sqrt{-1})^2 \cdot \sum_{k=1}^{\infty} k q_w^k$$

get from (6):

$$\eta(1,\Lambda_\tau) = \underbrace{2\cdot\zeta(2)}_{\frac{\pi^2}{3}} + (2\pi\sqrt{-1})^2\cdot 2\cdot \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n\, q_\tau^{mn}$$

change the order of summation in the last sum

$$= \frac{(2\pi\sqrt{-1})^2}{12}\left[-1 + 24\sum_{n=1}^{\infty} \frac{n\, q_\tau^n}{1-q_\tau^n}\right]$$

This is the same as formula (5) on p. 33

k even
$k \geq 3$

Recall: $G_k(\tau) = \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \frac{1}{(m\tau+n)^2}$ $\qquad k \in \mathbb{N}_{\text{even}, \geq 4}$

$$= 2\zeta(k) + 2\cdot\frac{(2\pi\sqrt{-1})^k}{(k-1)!}\sum_{n\geq 1}\sigma_{k-1}(n)q^n$$

p. 18

also $\zeta(k) = -\frac{(2\pi\sqrt{-1})^k}{2\cdot k!}B_k$

$$= 2\frac{(2\pi\sqrt{-1})^k}{(k-1)!}\cdot\left[-\frac{B_k}{2k} + \sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\right] \qquad \zeta(1-k) = -\frac{B_k}{k}$$

$$= 2\cdot\frac{(2\pi\sqrt{-1})^k}{(k-1)!}\left[\frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\right]$$

We just saw:

$$\eta(1;\Lambda_\tau) = 2\cdot(2\pi\sqrt{-1})^2\cdot\left[-\frac{B_2}{4} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n q_\tau^{mn}\right]$$

recall $B_2 = \frac{1}{6}$

$B_2 = \frac{1}{6}$

$$= 2(2\pi\sqrt{-1})^2\cdot\left(-\frac{B_2}{4} + \sum_{n=1}^{\infty}\sigma_1(n)q_\tau^n\right)$$

So: Formally $\eta(1;\Lambda_\tau)$ can be thought of as "$G_2(\tau)$"

$$\boxed{\eta(1;\Lambda_\tau) = \frac{(2\pi\sqrt{-1})^2}{12}\left[-1 + 24\sum_{n=1}^{\infty}\sigma_1(n)q_\tau^n\right]} \qquad (5)'$$

Compute $\eta(\tau;\Lambda_\tau) = \zeta(z+\tau) - \zeta(z)$ $\qquad \zeta(z) = \frac{1}{z} + \sum_{(m,n)\neq(0,0)}\left[\frac{1}{z-m\tau-n} + \frac{1}{m\tau+n} + \frac{z}{(m\tau+n)^2}\right]$

$$= \frac{1}{z+\tau} - \frac{1}{z} + \sum_n\sum_m{}'\left[\frac{1}{z-(m-1)\tau-n} - \frac{1}{z-m\tau-n} + \frac{\tau}{(m\tau+n)^2}\right]$$

$$= \frac{1}{z+\tau} - \frac{1}{z} + \sum_{n\neq 0}\sum_m\frac{\tau}{(m\tau+n)^2} + \sum_{m\neq 0}\left[\frac{1}{z-(m-1)\tau} - \frac{1}{z-m\tau}\right]$$

$$= \sum_{n\in\mathbb{Z}}\sum_{\substack{m\in\mathbb{Z}\\(m,n)\neq(0,0)}}\frac{\tau}{(m\tau+n)^2}$$

$$\boxed{\eta(\tau,\Lambda_\tau) = \tau\cdot\sum_{n\in\mathbb{Z}}\sum_{\substack{m\in\mathbb{Z}\\(m,n)\neq 0}}\frac{1}{(m\tau+n)^2}} \qquad (7)$$

Have seen

$$\eta(1;\tau) = \sum_{m\in\mathbb{Z}} \sum_{\substack{n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^2} = \frac{(2\pi\sqrt{-1})^2}{12}\left[-1 + 24\sum_{n=1}^{\infty}\sigma_1(n)\, q_\tau^n\right]$$

$$\underset{\substack{\uparrow\\ \Lambda_\tau}}{\eta(\tau;\tau)} = \sum_{n\in\mathbb{Z}} \sum_{\substack{m\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{\tau}{(m\tau+n)^2} \qquad \text{different order of summation}$$

$\rightsquigarrow$

$$\eta(1,-\tfrac{1}{\tau}) = \sum_m \sum_n{}' \frac{1}{(-m\cdot\tau^{-1}+n)^2} = \sum_m \sum_n{}' \frac{\tau^2}{(-m+n\tau)^2} = \tau\cdot\eta(\tau;\tau)$$

Also: Legendre relation

$$\tau\cdot\eta(1;\tau) - \eta(\tau;\tau) = 2\pi\sqrt{-1}$$

$$\rightsquigarrow \quad \eta(1,-\tfrac{1}{\tau}) = \tau\cdot\left[\tau\cdot\eta(1;\tau) - 2\pi\sqrt{-1}\right]$$

$$\boxed{\eta(1,-\tfrac{1}{\tau}) = \tau^2\,\eta(1;\tau) - 2\pi\sqrt{-1}\,\tau} \qquad (8)$$

General: Let $f(\tau) = \eta(1;\Lambda_\tau)$

$$(f\underset{2}{|}\gamma)(\tau) = f(\tau) - 2\pi\sqrt{-1}\,\frac{c}{c\tau+d} \qquad \gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$$

True for $\gamma_1 = \begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix}$, $\gamma_2 = \begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix}$ $\Longrightarrow$ true for $\gamma_1\cdot\gamma_2$

$$(f\underset{2}{|}\gamma_1\gamma_2)(\tau) = \left((f\underset{2}{|}\gamma_1)\underset{2}{|}\gamma_2\right)(\tau) = \underbrace{(f\underset{2}{|}\gamma_2)(\tau)}_{=\, f(\tau) - 2\pi\sqrt{-1}\frac{c_2}{(c_2\tau+d_2)}} - 2\pi\sqrt{-1}\,\frac{c_1}{c_1\left(\frac{a_2\tau+b_2}{c_2\tau+d_2}\right)+d_1}\cdot\frac{1}{(c_2\tau+d_2)^2}$$

$$\gamma_1\cdot\gamma_2 = \begin{pmatrix} a_1a_2+b_1c_2 & a_1b_2+b_1d_2\\ c_1a_2+d_1c_2 & c_1b_2+d_1d_2\end{pmatrix}$$

$$\frac{c_1c_2}{\underbrace{c_1(a_2\tau+b_2)+d_1(c_2\tau+d_2)}_{=\,(c_1a_2+d_1c_2)\tau+(c_1b_2+d_1d_2)}} = \frac{c_1}{(c_1a_2+d_1c_2)\tau+(c_1b_2+d_1d_2)}$$

$$\frac{c_2}{(c_2\tau+d_2)} + \frac{c_1}{\left[(c_1a_2+d_1c_2)\tau+(c_1b_2+d_1d_2)\right](c_2\tau+d_2)}$$

$$= \frac{c_1 + c_2\left[(c_1a_2+d_1c_2)\tau+(c_1b_2+d_1d_2)\right]}{\left[(c_1a_2+d_1c_2)\tau+(c_1b_2+d_1d_2)\right](c_2\tau+d_2)}$$

$$\underline{\text{numerator}} - (c_2\tau+d_2)(c_1a_2+d_1c_2)$$

$$\shortparallel$$

$$\cancel{c_1c_2a_2\tau} + \cancel{c_2c_2d_1\tau} + c_1c_2b_2 + \cancel{c_2d_1d_2} + c_1 - \cancel{c_1c_2a_2\tau} - \cancel{c_2\cdot c_2d_1\tau} - c_1a_2d_2 - \cancel{d_1d_2c_2}$$

$$= c_1(c_2b_2 + 1 - a_2d_2) = 0$$

$\underline{\text{i.e}}$

$$\frac{c_2}{(c_2\tau+d_2)} + \frac{c_1}{c_1(\gamma_2\cdot\tau)+d_1}\cdot\frac{1}{(c_2\tau+d_2)^2}$$

$$= \frac{c_1a_2+d_1c_2}{(c_1a_2+d_1c_2)\tau+(c_1b_2+d_1d_2)}$$

Conclude :

$\forall\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, have

$$\boxed{\eta\left(1;\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2\,\eta(1;\tau) - 2\pi\sqrt{-1}\,c(c\tau+d)} \qquad (9)$$

nearly / almost holomorphic weight 2 Eisenstein series

Define $G_{2,\varepsilon}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m\tau+n)^2\,|m\tau+n|^{2\varepsilon}}$ $\varepsilon>0$, $\tau\in\mathfrak{h}$

absolutely convergent, and

$$G_{2,\varepsilon}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2\,|c\tau+d|^{2\varepsilon}\cdot G_{2,\varepsilon}(\tau) \quad \forall \begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})$$

Claim: $\lim_{\varepsilon\to 0^+} G_{2,\varepsilon}(\tau) = \eta(1;\tau) - \frac{\pi}{y}$ $\quad y = \mathrm{Im}(\tau)$

double check:

$$\eta(1,-\tfrac{1}{\tau}) - \frac{\pi\cdot\tau\bar\tau}{y} = \left[\eta(1,\tau) - \frac{\pi}{y}\right]\cdot\tau^2$$

$$\mathrm{Im}\left(\frac{a\tau+b}{c\tau+d}\right) = \mathrm{Im}(\gamma\cdot\tau) = \frac{1}{2\sqrt{-1}}\left[\frac{a\tau+b}{c\tau+d} - \frac{a\bar\tau+b}{c\bar\tau+d}\right] = \frac{1}{2\sqrt{-1}}\,\frac{\tau-\bar\tau}{|c\tau+d|^2} = \frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}$$

$$\mathrm{Im}(-\tfrac{1}{\tau}) = \frac{\mathrm{Im}(\tau)}{|\tau|^2}$$

$$\eta(1,-\tfrac{1}{\tau}) - \tau^2\eta(1,\tau) = \frac{\pi\tau}{y}(\bar\tau-\tau) = -2\pi\sqrt{-1}\,\tau$$

Claim $\Rightarrow$ $\lim_{\varepsilon\to0^+} G_{2,\varepsilon}\left(\frac{a\tau+b}{c\tau+d}\right) = \left[\lim_{\varepsilon\to0^+} G_{2,\varepsilon}(\tau)\right]\cdot(c\tau+d)^2$ $\quad \forall\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})$

$\Longleftrightarrow$

$$\eta\left(1;\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2\eta(1,\tau) - 2\pi\sqrt{-1}\,\frac{c}{c\tau+d} \quad \forall\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})$$

proof of Claim

Define: $I_\varepsilon(\tau) = \int_{-\infty}^{\infty} \frac{dt}{(\tau+t)^2\cdot|\tau+t|^{\varepsilon}}$ $\quad$ approximate $\sum_n$ by $\int$

Have

(10) $$\frac{1}{2}G_{2,\varepsilon}(\tau) - \sum_{m=1}^{\infty} I_\varepsilon(m\tau) = \sum_{n=1}^{\infty}\frac{1}{n^{2+2\varepsilon}} + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\left[\frac{1}{(m\tau+n)^2|m\tau+n|^{2\varepsilon}} - \int_n^{n+1}\frac{dt}{(m\tau+t)^2|m\tau+t|^{2\varepsilon}}\right]$$

terms $m=0$ in $\sum'_{m,n}$ for $G_{2,\varepsilon}$

Claim A In the double sum of RHS of (10),

$$\left| \frac{1}{(m\tau+n)^2 |m\tau+n|^{2\varepsilon}} - \int_n^{n+1} \frac{dt}{(m\tau+t)^2 |m\tau+t|^{2\varepsilon}} \right| = O(|m\tau+n|^{-3-2\varepsilon})$$

∵ Mean value theorem

$$\lim_{\varepsilon\to 0^+} (\text{RHS of (10)}) \text{ exists},$$

$$= \frac{1}{2}\eta(1;\tau) \qquad \Big(= \overset{\eta(1,\Lambda_\tau)}{\sum_m \sum_n{}' \frac{1}{(m\tau+n)^2}}\Big)$$

Claim B $\displaystyle \lim_{\varepsilon\to 0^+} \sum_{m=1}^{\infty} I_\varepsilon(m\tau) = -\frac{\pi}{2y}$ $\qquad y = \mathrm{Im}(\tau)$

B1 $$I_\varepsilon(x+\sqrt{-1}y) = \int_{-\infty}^{\infty} \frac{dt}{(x+t+\sqrt{-1}y)^2 \left((x+t)^2+y^2\right)^\varepsilon}$$

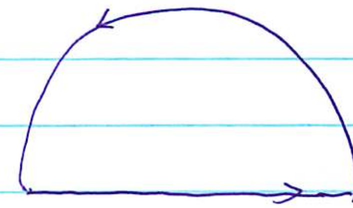
$$= \int_{-\infty}^{\infty} \frac{dt}{(t+\sqrt{-1}y)^2 (t^2+y^2)^\varepsilon}$$

$$\underset{\text{scaling}}{=} \frac{I(\varepsilon)}{y^{1+2\varepsilon}}, \quad \text{where } I(\varepsilon) = \int_{-\infty}^{\infty} \frac{dt}{(t+\sqrt{-1})^2 (t^2+1)^\varepsilon}$$

$$\lim_{\varepsilon\to 0^+} \sum_{m=1}^{\infty} I_\varepsilon(\underset{mx+\sqrt{-1}my}{\underset{\parallel}{m\tau}}) = \lim_{\varepsilon\to 0^+} \sum_{m=1}^{\infty} \frac{I(\varepsilon)}{(my)^{1+2\varepsilon}} = \zeta(1+2\varepsilon)\cdot I(\varepsilon)\cdot \underbrace{y^{-(1+2\varepsilon)}}_{\downarrow \text{ as } \varepsilon\to 0^+ \; y^{-1}}$$

Have $I(0) = 0$

$$I'(\varepsilon) = \frac{d}{d\varepsilon} I(\varepsilon) = -\int_{-\infty}^{\infty} \frac{\log(t^2+1)}{(t+\sqrt{-1})^2}\, dt = \underbrace{\frac{\log(t^2+1)}{t+\sqrt{-1}}\Big|_{-\infty}^{\infty}}_{\overset{\parallel}{0}} - \int_{-\infty}^{\infty} \frac{2t}{(t+\sqrt{-1})(t^2+1)}\, dt$$



$$\operatorname{Res}_{t=\sqrt{-1}} \frac{2t}{(t+\sqrt{-1})^2(t-\sqrt{-1})} = \frac{2\sqrt{-1}}{(2\sqrt{-1})^2} = \frac{1}{2\sqrt{-1}}$$

$$\rightsquigarrow \int_{-\infty}^{\infty} \frac{2t\,dt}{(t+\sqrt{-1})^2(t-\sqrt{-1})} = 2\pi\sqrt{-1}\cdot\frac{1}{2\sqrt{-1}} = \pi$$

$$\zeta(1+2\varepsilon) = \frac{1}{2\varepsilon} + O(1) \qquad I(\varepsilon) = -\pi\varepsilon + O(1)$$

$$\Longrightarrow \lim_{\varepsilon\to 0^+} \sum_{m=1}^{\infty} I_\varepsilon(m\tau) = \frac{1}{y}\cdot\lim_{\varepsilon\to 0^+}\left(\zeta(1+2\varepsilon)\cdot I(\varepsilon)\right) = -\frac{\pi}{2y}$$

Claim B proved.

Claim B $\Rightarrow$ $\lim_{\varepsilon\to 0^+} G_{2,\varepsilon}(\tau)$ exists,

and $= \eta(1;\tau) - \frac{\pi}{y}$

Conclusion : another proof of :

$$\boxed{\eta\left(1;\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2\cdot\eta(1;\tau) - 2\pi\sqrt{-1}\,c\cdot(c\tau+d)} \qquad \forall \begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})$$

$$\Longleftrightarrow \lim_{\varepsilon\to 0^+} \sum_{(m,n)\in\mathbb{Z}^2\smallsetminus\{(0,0)\}} \frac{1}{(m\tau+n)^2\,|m\tau+n|^{2\varepsilon}} = \eta(1;\tau) - \frac{\pi}{\operatorname{Im}(\tau)}$$

Also recall

$$\boxed{\begin{aligned}\eta(1;\tau) &= 2\cdot(2\pi\sqrt{-1})^2\Big[\underbrace{-\frac{B_2}{4}}_{\frac{\zeta(-1)}{2}} + \sum_{n=1}^{\infty}\sigma_1(n)\cdot q_\tau^n\Big] \\ &= \frac{(2\pi\sqrt{-1})^2}{12}\Big[-1 + 24\sum_{n=1}^{\infty}\sigma_1(n)\,q_\tau^n\Big]\end{aligned}}$$

Elementary / classical application of the q-expansion:
modular equations & $j(\tau)$ = an algebraic integer
$\tau \in$ an imaginary quadratic field

Rmk: Converse is also true:
if $\tau \in \overline{\mathbb{Q}} \subseteq \mathbb{C}$ and $j(\tau) \in \overline{\mathbb{Q}}$, then $\tau \in$ an imaginary quadratic field.

Prop: f: $\Gamma(1)$-modular function on $\mathfrak{h}$, meromorphic at ∞ and holomorphic on $\mathfrak{h}$
$f = \sum_{n \geq n_0} c_n q^n$: q-expansion of f

Let $M = \sum_{n \leq 0} \mathbb{Z} \cdot c_n$ = the $\mathbb{Z}$-submodule of $\mathbb{C}$ generated by the Fourier expansion coefficients of f.

Then $f \in M[j]$

(Proof: induction on $-\mathrm{ord}_\infty(f)$, use the fact that the coeff q-expansion of j are integers: $f \in \sum_{m \geq 0} M \cdot j^m$)

Note: This is a strengthened statement of Lang. Elliptic functions, Chap 5. §2. Thm 2

(Theorem on the existence and main properties of the modular polynomials)

Prop (1) $\forall n \in \mathbb{N}_{>0}$, $\Phi_n(X) = \prod_{n=1}^{\psi(n)} (X - j \circ \alpha_i) \in \mathbb{Z}[X, j]$
where $\psi(n) = [\underbrace{\Gamma(1) \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \Gamma(1)}_{\Delta_n^*} : \Gamma(1)] = n \cdot \prod_{p | n} (1 + \frac{1}{p})$
primitive in $GL_2(\mathbb{Q})^+ \cap M_2(\mathbb{Z})$
and $\{\alpha_1, \dots, \alpha_n\}$ = a system of repr. of $\Gamma(1) \backslash \Delta_n^*$

(2) $\forall n > 0$, $\Phi_n(X)$ is irred. over $\mathbb{C}(j)$, of degree $\psi(n)$

(3) If n is not a square, then $\Phi_n(j,j) \in \mathbb{Z}[j]$ is monic, of degree > 1

Note: a convenient system of repr. of $\Gamma(1)\backslash\Delta_n^*$ is
$$\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : \begin{array}{l} a,d>0, \quad ad=n \\ 0 \le b < d \end{array} \right\}$$

Cor: Suppose that $\tau \in \mathfrak{h}$ is imaginary quadratic. Then $j(\tau)$ is an algebraic integer.

Pf: Consider $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$

- $\exists m \in \mathbb{N}_{>0}$ s.t. $\left(\mathbb{Z} + m\cdot\mathcal{O}_{\mathbb{Q}(\tau)}\right)\cdot\Lambda_\tau \subseteq \Lambda_\tau$

$\rightsquigarrow \exists \lambda \in \mathbb{Z}+m\cdot\mathcal{O}_{\mathbb{Q}(\tau)}$ s.t. $\begin{array}{l} \lambda\cdot\tau = a\tau+b \\ \lambda = c\tau+d \end{array}$

$\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \cap GL_2(\mathbb{Q})^+$, $\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: primitive
i.e. $\beta \in \Delta_n^*$ $\quad n = \det(\beta)$

Have $\beta\cdot\tau = \tau \quad \beta \in \Delta_n^*$

$\rightsquigarrow \Phi_n(j(\tau), j(\tau)) = 0$, so $j(\tau)$ is an algebraic integer.

Survey of some other facts about fields of modular functions.

Use $$f_{r,s}(\tau) = -2^7 3^5 \cdot \frac{g_2(\tau)\cdot g_3(\tau)}{\Delta}\, \wp\left(\frac{r\tau+s}{N}; \Lambda_\tau\right)$$
$r,s \in \mathbb{N} \quad (r,s) \not\equiv (0,0) \bmod N$

Equivalently:
$$f_{r,s}(\omega_1,\omega_2) = -2^7 3^5 \frac{g_2(\omega_1\mathbb{Z}+\omega_2\mathbb{Z})\cdot g_3(\omega_1\mathbb{Z}+\omega_3\mathbb{Z})}{\Delta(\omega_1\mathbb{Z}+\omega_3\mathbb{Z})}\, \wp\left(\frac{r\omega_1+s\omega_2}{N}; \omega_1\mathbb{Z}+\omega_2\mathbb{Z}\right)$$

<u>Better</u>: $a = (a_1, a_2) \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$

$$f_0(z;\tau) = -2^7 \cdot 3^5 \cdot \frac{g_2(\tau)\cdot g_3(\tau)}{\Delta(\tau)}\, \wp(z;\Lambda_\tau)$$

"first Weber function"

$$f_a(\tau) = f_0(a_1\tau + a_2; \tau)$$

$\uparrow$

Fricke functions

(<u>Recall</u> up to $2^? 3^?$

2nd Weber function: $\frac{g_2(\tau)}{\Delta}\, \wp(z;\Lambda_\tau)^2$ if $g_3(\tau) = 0$

3rd Weber function $\frac{g_3(\tau)}{\Delta}\, \wp(z;\Lambda_\tau)^3$ if $g_2(\tau) = 0$)

<u>Clearly</u>

$$\boxed{f_{a\cdot\gamma}(\tau) = f_a(\gamma\cdot\tau)} \quad \forall \gamma \in SL_2(\mathbb{Z})$$

<u>Thm</u> 1) Let $F_{N,\mathbb{C}} = \{\text{all modular functions for } \Gamma_1(N)\}$

<u>Then</u> $$F_{N,\mathbb{C}} = \mathbb{C}(j)\,(f_{r,s}(\tau))_{\substack{r,s\in\mathbb{N} \\ (r,s)\not\equiv(0,0) \pmod N}}$$

<u>and</u> $$\mathrm{Gal}(F_{N,\mathbb{C}}/\mathbb{C}(j(\tau))) = SL_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$$

2) Let $F_N = \mathbb{C}(j)\,(f_{r,s})_{\substack{r,s\in\mathbb{N} \\ (r,s)\not\equiv(0,0) \pmod N}}$

$$\begin{array}{ccc} & F_N & \\ GL_2(\mathbb{Z}/N\mathbb{Z})/\{\pm1\} & | & SL_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\} \\ & \mathbb{Q}(\zeta_N)(j) & \\ & | & \\ & \mathbb{Q}(j) & \end{array}$$

2/21/2015

1. (Congruence relation for CM elliptic curves)

$\kappa \cong \mathbb{F}_q$ a finite field, E/κ: elliptic curve. K: an imaginary quadratic field

$\alpha: \mathcal{O}_K \to \mathrm{End}_\kappa(E)$ (i.e. E/κ is CM with action by $\mathcal{O}_K$)

$h: \mathcal{O}_K \to \kappa$ the action of $\mathcal{O}_K$ on $\mathrm{Lie}(E)$; i.e. $\alpha(x)$ acts on $\mathrm{Lie}(E)$ as $h(x)\cdot \mathrm{Id}_{\mathrm{Lie}(E)}$ $\forall x \in \mathcal{O}_K$

$\wp = \mathrm{Ker}(h)$: prime ideal in $\mathcal{O}_K$

$\mathrm{Frob}_{E,\kappa}: E \to E$ geometric Frobenius for the finite field $\kappa \cong \mathbb{F}_q$

$\mathrm{Frob}_{E,\kappa}$ commutes with $\alpha(\mathcal{O}_K) \rightsquigarrow \exists!$ elt. $\pi_E \in \mathcal{O}_K$ s.t. $\mathrm{Frob}_{E,\kappa} = \alpha(\pi_E)$

<u>Proposition</u> $\pi_E \mathcal{O}_K = \wp^{[\kappa : \mathcal{O}_K/\wp]}$, or equivalently $[\mathrm{Lie}(E)] = [\mathcal{O}_K/\pi]$ in the Grothendieck group of finite $\mathcal{O}_K$-modules

pf: (a) general fact: $\deg(x)$ (as a morphism from E to E) $= \#(\mathcal{O}_E/x\mathcal{O}_E)$ $\forall x \in \mathcal{O}_E$

$\rightsquigarrow \#(\mathcal{O}_E/\pi_E\mathcal{O}_E) = q$

(b) Clearly elements of $\mathcal{O}_K \setminus \wp$ acts as étale endomorphisms of E. Since π_E is purely inseparable, we see that $\pi_E\mathcal{O}$ is a power of $\wp$. Q.E.D.

2. <u>Recall</u> / fact:

(a) The dichotomy ordinary vs. supersingular for elliptic curves over a field k of characteristic $p>0$:
several equivalent definition: via (1) $E[p](k^{alg})$, (2) $E[p]$ as a finite group scheme over k^{alg} (c) Hasse invariant (using either $H^1(E, \mathcal{O}_E)$ or the Cartier operator) (d) $E[p^\infty]$

(b) The ordinary / supersingular dichotomy does not change under
$\begin{cases} \cdot \text{ extension of base field} \\ \cdot \text{ modification of } E \text{ by an isogeny} \end{cases}$

3. Corollary: Let $E_{/L}$ be an elliptic curve over a number field L. Suppose E has potential CM by an imag. quadratic field K. Let $\wp$ be a good reduction place of $\mathcal{O}_L$ for E.
Then $E_{/\kappa(\wp)}$ is ordinary if and only if $\wp$ splits in $\mathcal{O}_K$

Pf: By the facts recalled in 2 about the invariance of the ordinary / supersingular dichotomy recalled in 2, we may assume that $K \subseteq L$ and $\mathcal{O}_K$ operates on $E_{/L}$. The assertion follows from the congruence relation 1 and the action of $\pi_{E/\kappa(\wp)}$ on the de Rham cohomology of $E_{/\kappa(\wp)}$ together with its Hodge filtration. (Especially that $\forall x \in \mathcal{O}_K$, $\alpha(x)^*$ acts on $H^1(E, \mathcal{O}_E)$ as $h(\bar{x}) \cdot \mathrm{Id}_{H^1(E,\mathcal{O}_E)}$.) . Q.E.D.

Complex Multiplication of Elliptic Curves

Let $E_{/\bar{\mathbb{Q}}}$ be an elliptic curve over $\bar{\mathbb{Q}}$

Let $\mathcal{O} = \mathrm{End}(E) \underset{\substack{\uparrow \\ \text{finite index}}}{\subseteq} \mathcal{O}_K$ K: imaginary quadratic field.

$\rightsquigarrow j: K \hookrightarrow \bar{\mathbb{Q}}$ via action of $\mathcal{O}_K$ on $\mathrm{Lie}(E)$

<u>Main Theorem</u>: $\forall \sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/K)$

$\forall s \in \mathbb{A}_{K,f}^{\times}$ s.t. $\mathrm{rec}_K(s) = \sigma|_{K^{ab}}$ $\mathrm{rec}_K: \mathbb{A}_{K,f}^{\times} \twoheadrightarrow \mathrm{Gal}(\bar{\mathbb{Q}}/K)$

Artin map, arithmetically normalized.

There exists a unique $\mathcal{O}$-linear isomorphism

$\psi_\sigma: {}^{\sigma}E \longrightarrow s^{-1}\mathcal{O} \otimes_{\mathcal{O}} E$

s.t.
$$\begin{tikzcd} E(\bar{\mathbb{Q}})_{tor} \arrow[r, "{[\sigma]}"] \arrow[rr, bend right, "\psi_\sigma \circ [\sigma]"'] & {}^{\sigma}E(\bar{\mathbb{Q}})_{tor} \arrow[r, "\psi_\sigma"] & s^{-1}\mathcal{O} \otimes_{\mathcal{O}} E(\bar{\mathbb{Q}})_{tor} \end{tikzcd}$$

$\psi_\sigma \circ [\sigma] =$ "multiplication by s^{-1}" $= s^{-1} \otimes \mathrm{id}_{E(\bar{\mathbb{Q}})_{tor}}$

$V(E) = T(E) \otimes_{\mathbb{Z}} \mathbb{Q}$

equivalently:

the composition $T(E) \xrightarrow{\sigma} T({}^{\sigma}E) \xrightarrow{\psi_\sigma} T(s^{-1}\mathcal{O} \otimes_{\mathcal{O}} E) \subseteq V(s^{-1}\mathcal{O} \otimes_{\mathcal{O}} E) = V(E)$

is "multiplication by s^{-1}

<u>CM character</u> $E_{/L_{/K}}$ L = number field (so all $\mathrm{End}^{\circ}(E)$ are defined over L)

(i) $\forall \sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/L)$, $\forall t \in \mathbb{A}_{L,f}^{\times}$ s.t. $\mathrm{rec}_L(t) = \sigma|_{L^{ab}}$

$(\rightsquigarrow \mathrm{rec}_K(\mathrm{Nm}_{L/K}(t)) = \sigma|_{K^{ab}})$

$\exists!$ element $\underset{\substack{\uparrow \\ \text{depending only on } t}}{\mu(t)} \in K^{\times}$ ($\mu_E(t) = \mu(t)$) such that

$\therefore \mathrm{Gal}(\bar{\mathbb{Q}}/L) \xrightarrow{\rho_E} GL(V(E))$ factors through $\mathrm{Gal}(L^{ab}/L)$

$\rho_E(\sigma) = E(\bar{\mathbb{Q}})_{tor} \xrightarrow{[\sigma]} {}^{\sigma}E(\bar{\mathbb{Q}})_{tor} \xrightarrow{\sim} E(\bar{\mathbb{Q}})_{tor}$ (from the L-structure of E)

$\| $

$\mu(t) \cdot \mathrm{Nm}_{L/K}(t)^{-1}$

(ii) $\mu_E: \mathbb{A}_{L,f}^{\times} \longrightarrow K^{\times}$ is a continuous homomorphism

$\mu(t) = \mathrm{Nm}_{L/K}(t) \quad \forall t \in L^{\times}$

Pf of (i):

$$\begin{tikzcd} E(\bar{\mathbb{Q}})_{tor} \arrow[r, "{[\sigma]}"] & {}^{\sigma}E(\bar{\mathbb{Q}})_{tor} \arrow[r, "\psi_\sigma"] \arrow[d, "\eta", "\wr"'] & Nm_{L/K}(t)^{-1}\mathcal{O} \otimes_{\mathcal{O}} E(\bar{\mathbb{Q}})_{tor} \arrow[dl, dotted, "\mu(t)"] \\ & E(\bar{\mathbb{Q}})_{tor} & \end{tikzcd}$$

In other words, $\mu(t) = \eta \circ \psi_\sigma^{-1}$

from the L-structure of $E_{/L}$

i.e. $\eta\circ\psi_\sigma^{-1}$ = the K-linear isogeny from $Nm_{L/K}(t)^{-1} \otimes_{\mathcal{O}} E$ to E defined by the element $\mu(t) \in K^\times$

Pf of (ii): Clearly $\mu(L^\times) = 1$

Must verify that $\mu : \mathbb{A}_{L,f}^\times \longrightarrow K^\times$ is continuous.

Suppose $Nm_{L/K}(t)^{-1}\mathcal{O} = \mathcal{O}$, then $\mu(t) \in \mathcal{O}^\times \subseteq \mu(K)$, a finite group.

$\rightsquigarrow$ Look at action of $Nm_{L/K}(t)^{-1}\cdot\mu(t)$ on $E[N]$, N large

$\Rightarrow$ μ = trivial on an open subgroup of $\mathbb{A}_{L,f}^\times$

CM character

(iii) $\mu_E = \mu_{E'}$ if $E_{/L} \underset{L\text{-isog}}{\sim} E'_{/L}$

(iv) $\mu(t)\cdot\overline{\mu(t)} = \|t\|^{-1}_{\mathbb{A}_{L,f}}$ $\quad\forall t \in \mathbb{A}_{L,f}^\times$

(v) $\mu(t)^{-1}\cdot Nm_{L/K}(t) \in (\mathcal{O} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}})^\times$ $\quad\forall t\in\mathbb{A}_{L,f}^\times$

(This statement places a restriction on the possible fields of definition E when the conductor of $\mathcal{O}$ is high/large)

CM character $\rightsquigarrow$ algebraic Hecke character:

Define: $\chi : \mathbb{A}_L^\times \longrightarrow \mathbb{C}^\times$ by

$(t_\infty, t_f) \longmapsto \mu(t_f)\cdot Nm_{L/K}(t_\infty)^{-1}$

idele class character

χ: a Hecke character

"algebraic Hecke character".

Consequences: $\forall$ finite place v of L s.t. E has good reduction at L

$\rightsquigarrow$ $Frob_{E,\kappa(v)} = \mu_E(\pi_v)$ satisfying $\mu_E(\pi_v)\cdot\mathcal{O}_K = Nm_{L/K}(\pi_v)\cdot\mathcal{O}_K$

π_v: a uniformizer at v

$\kappa(v) \cong \mathbb{F}_p$

In particular: $E_{/\kappa(v)}$ is $\begin{cases} \text{ordinary if } p \text{ splits in } K \\ \text{supersingular if } p \text{ is inert or ramified in } K \end{cases}$

Application to: generating class fields of K by special values of elliptic functions

I. $E(\mathbb{C}) \cong \mathbb{C}/\underbrace{\mathbb{Z}\tau+\mathbb{Z}}_{\Lambda_\tau}$ $\quad \tau \in \underset{K}{\overset{\cap}{\mathfrak{h}}}$ $\quad \mathcal{O} = \mathrm{End}(E) = \{x \in K \mid x\cdot\Lambda_\tau \subseteq \Lambda_\tau\}$
$= \mathbb{Z} + c\cdot\mathcal{O}_K$ i.e. c = conductor of $\mathcal{O}$

$\forall\ \sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/K)$, have $\ {}^\sigma j(\tau) = j(\tau) \overset{c\in\mathbb{Z}_{\geq 1}}{\iff} \exists\, x \in K^\times$ s.t. $x\cdot s^{-1} \in (\mathcal{O}_c \otimes_{\mathbb{Z}} \hat{\mathbb{Z}})^\times$

Pick $s \in \mathbb{A}^\times_{K,f}$ s.t. $\sigma|_{K^{ab}} = \mathrm{rec}_K(s)$

$\Rightarrow K(j(\tau))$ = the abelian extension of K corresponding to the open subgroup $K^\times\cdot K^\times_\infty\cdot(\mathcal{O}_c\otimes_{\mathbb{Z}}\hat{\mathbb{Z}})^\times \subseteq \mathbb{A}^\times_K$
"ring class field of conductor c"

In particular if $\mathcal{O}_K\cdot\Lambda_\tau = \Lambda_\tau$ then $K(j(\tau))$ = Hilbert class field of K

Invariant formulation:

$K(j(\text{any projective rk-one } \mathcal{O}_c\text{-submodule } \Lambda \text{ of } K))$ = ring class field H_c of K of conductor c

$$\begin{array}{ccccc}
\mathrm{Gal}(H_c/K) & \xleftarrow[\text{reciprocity law map}]{\sim} & \left\{\begin{array}{c}\text{isom. classes of projective} \\ \text{rank-one } \mathcal{O}_c\text{-modules}\end{array}\right\} & \ni & [\mathfrak{a}] \\
\Downarrow & & \uparrow & & \uparrow \\
 & & \text{ring class group of conductor } c & & \text{a projective rank-one } \mathcal{O}_c\text{-module} \\
\mathrm{rec}_K(\mathfrak{a}) & \longleftarrow & [\mathfrak{a}] & &
\end{array}$$

Then: $\mathrm{rec}_K(\mathfrak{a})\big(j(\mathfrak{L})\big) = j(\mathfrak{a}^{-1}\otimes_{\mathcal{O}_c}\mathfrak{L})$

$\forall$ projective rk-one $\mathcal{O}_c$-module $\mathfrak{L}$

II Generating generalized ray class fields. $\quad \mathfrak{n}$: an invertible ideal of $\mathcal{O}_c$

$\hat{\mathcal{O}}_c^\times(\mathfrak{n}) := \left\{ y \in (\mathcal{O}_c\otimes_{\mathbb{Z}}\hat{\mathbb{Z}})^\times \mid (y-1)\cdot(\mathcal{O}_c/\mathfrak{n}) = 0 \right\}$

$H_{\mathcal{O}_c,\mathfrak{n}}$ = class field of K corresponding to $K^\times\cdot K^\times_\infty\cdot\hat{\mathcal{O}}_c^\times(\mathfrak{n})$

$\mathfrak{L}$: proj. rk-one $\mathcal{O}_c$-submodule of K

Then $K\left(j(\mathfrak{L}),\ h(u)_{u\in\mathfrak{n}^{-1}\cdot\mathfrak{L}/\mathfrak{L}}\right) = H_{\mathcal{O}_c,\mathfrak{n}}$

Complex multiplication for elliptic curves.

<u>Main Theorem 1</u> Let $(E_{/\bar{\mathbb{Q}}}, \alpha: \mathcal{O} \xrightarrow{\sim} \mathrm{End}(E))$ be a CM elliptic curve $/\bar{\mathbb{Q}}$, $\underset{\text{finite index}}{\mathcal{O} \subseteq \mathcal{O}_K}$, K: imaginary quadratic field. $j: K \longrightarrow \bar{\mathbb{Q}}$ action on $\mathrm{Lie}(E)$

i.e. $\alpha(x)$ acts on $\mathrm{Lie}(E)$ as $j(x)$ $\forall x \in \mathcal{O}$

For every $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/K)$, for every $s \in \mathbb{A}_{K,f}^{\times}$ s.t.

$\mathrm{rec}_K(s) = \sigma|_{K^{ab}}$ (rec_K = Artin map, arithmetically normalized),

$\exists!$ $\mathcal{O}$-linear isomorphism

$$\psi_\sigma: {}^{\sigma}E \xrightarrow{\sim} s^{-1}\mathcal{O} \otimes_{\mathcal{O}} E$$

s.t. the composition $E(\bar{\mathbb{Q}})_{tor} \xrightarrow{[\sigma]} {}^{\sigma}E(\bar{\mathbb{Q}})_{tor} \xrightarrow{\psi_\sigma} s^{-1}\mathcal{O} \otimes_{\mathcal{O}} E(\bar{\mathbb{Q}})_{tor}$

is equal to $s^{-1} \otimes \mathrm{id}_{E(\bar{\mathbb{Q}})_{tor}}$: $E(\bar{\mathbb{Q}})_{tor} = \mathcal{O} \otimes_{\mathcal{O}} E(\bar{\mathbb{Q}})_{tor} \xrightarrow{s^{-1}\otimes \mathrm{id}} s^{-1}\mathcal{O} \otimes_{\mathcal{O}} E(\bar{\mathbb{Q}})_{tor}$

$$s^{-1}\mathcal{O} \otimes_{\mathcal{O}} E(\bar{\mathbb{Q}})_{tor} = (s^{-1}\mathcal{O} \otimes_{\mathcal{O}} E)(\bar{\mathbb{Q}})_{tor}$$

<u>Equivalently</u>: the composition

$$T(E) \xrightarrow{[\sigma]} T({}^{\sigma}E) \xrightarrow{\psi_\sigma} T(s^{-1}\mathcal{O} \otimes_{\mathcal{O}} E) = V(s^{-1}\mathcal{O} \otimes_{\mathcal{O}} E) \xrightarrow{\sim} V(E)$$

is equal to "multiplication by s^{-1}"

<u>Here</u> $T(E) = \varprojlim_n E[n](\bar{\mathbb{Q}})$, $V(E) = T(E) \otimes_{\mathbb{Z}} \mathbb{Q}$

An equivalent statement of the Main Theorem 1, in the isogeny category (apparently weaker but actual equivant K)

<u>Easy Fact</u>: Any two elliptic curves over $\bar{\mathbb{Q}}$ with CM by K, with the same CM-type $j: K \to \bar{\mathbb{Q}}$, are K-linearly isogenous

Def: (i) Let $\mathcal{C}_{K,j}$ = the category consisting of all triples $(E_{/\mathbb{Q}}, \alpha : K \hookrightarrow \overset{\text{CM structure}}{\mathrm{End}^\circ(E)}, \xi)$
where $\xi : V(E) \xrightarrow{\sim} \mathbb{A}_{K,f}^\times$ is a K-linear isomorphism.

- a morphism $(E_1, \alpha_1, \xi_1) \longrightarrow (E_2, \alpha_2, \xi_2)$ is a K-linear isogeny is a K-linear isogeny $\beta : E_1 \to E_2$ s.t.
$\xi_1 = \xi_2 \circ V(\beta)$, $V(\beta) : V(E_1) \to V(E_2)$ induced by β

Note: the group of automorphisms of each object is trivial $\rightsquigarrow$ can use the language of sets

- Every element $t \in \mathbb{A}_{K,f}^\times$ defines a functorial automorphism of $\mathcal{C}_{K,j}$
t : any object (E, α, ξ) to $(E, \alpha, t\cdot\xi)$
isomorphism

$$\begin{array}{ccc} (E_1, \alpha_1, \xi_1) & & (E_1, \alpha_1, t\cdot\xi_1) \\ \downarrow \beta & \text{to} & \downarrow \beta \\ (E_2, \alpha_2, \xi_2) & & (E_2, \alpha_2, t\cdot\xi_2) \end{array}$$

- $\forall$ CM elliptic curve $E_{/\overline{\mathbb{Q}}}$ with CM type $j : K \to \mathbb{Q}$,
$\mathcal{C}_{K,j}(E)$ = the full subcategory of $\mathcal{C}_{K,j}$ consisting of all triples $(E_{/\mathbb{Q}}, \alpha_K : K \hookrightarrow \mathrm{End}^\circ(E), \xi)$

(So that: $\forall$ K-linear isogeny $\beta : (E_1, \alpha_1) \to (E_2, \alpha_2)$ have a pull-back
$$\mathcal{C}_{K,j}(E_2) \longrightarrow \mathcal{C}_{K,j}(E_1)$$
$$(E_2, \alpha_2, \xi_2) \rightsquigarrow (E_2, \alpha_2, \xi_2 \circ V(\beta))$$)

$\mathcal{C}_{K,j}(E)$ is an $\mathbb{A}_{K,f}^\times$-torsor

(ii) $\forall$ two objects $(E_1, \alpha_1, \xi_1), (E_2, \alpha_2, \xi_2)$ in $\mathcal{C}_{K,j}$,
define an invariant $\mathrm{inv}((E_1, \alpha_1, \xi_1), (E_2, \alpha_2, \xi_2)) \in \mathbb{A}_{K,f}^\times / K^\times$
as follows: $\exists$ isogeny $E_1 \xrightarrow{\beta} E_2$
and $t \in \mathbb{A}_{K,f}^\times$ s.t. $\xi_2 \circ V(\beta) = t \cdot \xi_1$

Define: $\mathrm{inv}((E_1, \alpha_1, \xi_1), (E_2, \alpha_2, \xi_2)) = t \pmod{K^\times} \in \mathbb{A}_{K,f}^\times / K^\times$
(well-defined)

Definition/Construction

Let $\left(E_{/\overline{\mathbb{Q}}},\ \alpha : K \xrightarrow{\sim} \mathrm{End}^\circ(E)\right)$ be a CM elliptic curve with CM type $j : K \hookrightarrow \overline{\mathbb{Q}}$

~~Define~~ : $\exists!$ a continuous homom $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \longrightarrow \mathbb{A}_{K,f}^\times / K^\times$

s.t. $\forall$ K-linear isom. $\xi : V(E) \longrightarrow \mathbb{A}_{K,f}^\times$

$\forall \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$,

$$\underbrace{\mathrm{inv}\left((E, \alpha, \xi), ({}^\sigma E, {}^\sigma\alpha, \xi\circ[\sigma]^{-1})\right)}_{C_{K,j}({}^\sigma E)} = \rho(\sigma)$$

$$\begin{tikzcd}
\mathrm{Gal}(\overline{\mathbb{Q}}/K) \arrow[r, "\rho"] \arrow[d] & \mathbb{A}_{K,f}^\times / K^\times \\
\mathrm{Gal}(K^{ab}/K) \arrow[ur, "\overline{\rho}"] & \mathbb{A}_{K,f}^\times / K^\times \arrow[l, "\sim", "\mathrm{rec}_K"'] \arrow[u]
\end{tikzcd}$$

Note : $K^\times \underset{\text{closed}}{\subseteq} \mathbb{A}_{K,f}^\times$

Main Thm 1$'$ for CM elliptic curves

$$\overline{\rho}(\mathrm{rec}_K(s)) = s \qquad \forall s \in \mathbb{A}_{K,f}^\times$$

Remark: The advantage of the formulation approach/ in Thm 1$'$ is :

- it provides a clear picture/overview that Thm 1 is a description of the object $({}^\sigma E, {}^\sigma\alpha, \xi\circ[\sigma]^{-1})$ modulo isomorphism
- Makes it clear that (E, α, ξ) is defined over K^{ab} (in a strong sense: fine moduli)

In particular, $\forall n \in \mathbb{N}_{>0}$, $\exists$ a number field $L_n \subseteq K^{ab}$ s.t. $\left(E, \alpha, E[n] \underset{\xi_n}{\xrightarrow{\simeq}} n^{-1}T(E)/T(E)\right)$ is defined over L_n

(for $n \geq 3$, there is a smallest such n : fine moduli)

and, to be safe, add: at primes v dividing the conductor : the open subgroup of $\mathcal{O}_v^\times$ corresponding to L_n is contained in $1 + n\cdot c\mathcal{O}_v$

Note: In Shimura's proof, one needs to take L_n to be number field over which $\left(E, \alpha, E[n] \underset{\xi_n}{\xrightarrow{\sim}} n^{-1}T(E)/T(E)\right)$ is defined, without knowing at this stage that $L_n \subseteq K^{ab}$

Proof of Main Theorem: n large enough

Given $\sigma \in Gal(\bar{\mathbb{Q}}/K)$, $s \in \mathbb{A}_{K,f}^{\times}$ st. $\sigma|_{K^{ab}} = rec_K(s)$

(Clearly, given that (E, α, ξ) is defined over K^{ab}, $\sigma|_{K^{ab}}$ is all that matters)

Will show:

$\exists$ $\mathcal{O}$-linear isom. $\psi_{\sigma,n}: {}^{\sigma}E \xrightarrow{\sim} s^{-1}\mathcal{O} \otimes_{\mathcal{O}} E$ such that

the composition $$\begin{array}{l} E[n](\bar{\mathbb{Q}}) \xrightarrow{\sigma} {}^{\sigma}E[n](\bar{\mathbb{Q}}) \xrightarrow{\psi_\sigma} (s^{-1}\mathcal{O} \otimes_{\mathcal{O}} E)[n](\bar{\mathbb{Q}}) \\ \parallel \\ E[n](K^{ab}) \end{array}$$

is equal to $s^{-1} \otimes id_{E[n](\bar{\mathbb{Q}})} : E[n](\bar{\mathbb{Q}}) \longrightarrow (s^{-1}\mathcal{O} \otimes_{\mathcal{O}} E)[n](\bar{\mathbb{Q}})$

Note: $\psi_{\sigma,n}$ is unique if $n \geq 3$ $\rightsquigarrow$ so ψ_σ is uniquely determined: it works for all n.

Here, use Chebotarev:

$\exists$ a prime ideal $\mathcal{P} \subseteq \mathcal{O}_{L,n}$ st.
- (E, α) has good reduction at $\mathcal{P}$
- $E[n]/\mathcal{O}_{L_n}$ is finite étale at $\mathcal{P}$; $\gcd(n,p)=1$
- $\mathfrak{p} = \mathcal{P} \cap \mathcal{O}_K$ is a degree 1 prime: $\mathcal{O}_K/\mathfrak{p} \cong \mathbb{F}_p$
- $\sigma|_{L_n} = Frob_{\mathcal{P}}$ (arithmetic Frobenius)
- p is prime to the conductor of $\mathcal{O}$: $\mathcal{O} = \mathbb{Z} + c\cdot\mathcal{O}_K$, $\gcd(p,c)=1$

* In classical / Shimura's proof: the j-invariants of all conjugates of E are not congruent modulo $\mathcal{P}$.

Replacement:

E_1, E_2: CM elliptic curves, with good reduction at $\mathcal{P}|\mathfrak{p}$, over $\mathcal{O}_{L,n}$ $\mathcal{O} = End(E) = \mathbb{Z} + c\cdot\mathcal{O}_K$, $\gcd(p,c)=1$, $\mathfrak{p}\bar{\mathfrak{p}} = p$, $\mathfrak{p} \neq \bar{\mathfrak{p}}$

$\bar{\beta}: E_{1/\kappa(\mathcal{P})} \xrightarrow{\sim} E_{2/\kappa(\mathcal{P})}$ $\mathcal{O}$-linear, same Lie type "$\mathfrak{p}$".

$\rightsquigarrow$ β: lifts to an $\mathcal{O}$-linear isom. over $\hat{\mathcal{O}}_{L,\mathcal{P}}$

$\rightsquigarrow$ over L

Use either of the two options

⇝ $\exists\, \mathcal{O}_K$-linear isom

$${}^{\sigma}E \xleftarrow{\ \varphi\ } (1\cdots 1, \pi_{\mathfrak{p}}^{-1}, \cdots 1)\mathcal{O} \underset{\mathcal{O}}{\otimes} E$$

which reduces modulo $\mathfrak{P}$ to

$$\begin{tikzcd} E_{\kappa(\mathfrak{P})}^{(p)} & (1,\cdots,1,\pi_{\mathfrak{p}}^{-1},\cdots 1)\mathcal{O}\underset{\mathcal{O}}{\otimes} E_{\kappa(\mathfrak{P})} \arrow[l] \\ & E_{\kappa(\mathfrak{P})} \arrow[ul, "\mathrm{Frob}^{(p)}"] \arrow[u] \end{tikzcd}$$

⇝ Everything is good if $s = \underbrace{(1\cdots 1\ \pi_{\mathfrak{p}}\ 1\cdots 1)}_{\text{call it } c_{\mathfrak{p}}}$

Now: CFT: $s = c_{\mathfrak{p}} \cdot \underset{K^\times}{b} \cdot u$, u in the open subgroup U of $\underbrace{\prod_{v\in\Sigma_{K,f}} \mathcal{O}_v^\times}$ corresponding to L_n/K; local norms

- the global element $b \in K^\times$: $M \xrightarrow[\sim]{\cdot b} bM$ $\forall$ projective rank-one $\mathcal{O}$-module
- u: No effect on $E[n]$

(by the precaution in our choice of L_n on p.49.)

Kronecker limit formulas

Lang defined for $a, b \in \mathbb{R}^{\times}_{>0}$

(K1) $K_s(a,b) := \int_0^\infty e^{-(a^2 t + \frac{b^2}{t})}\, t^s \cdot \frac{dt}{t}$ $\forall s \in \mathbb{C}$ abs. convergent

(K2) $= \left(\frac{b}{a}\right)^s K_s(a \cdot b)$

(K3) where $K_s(c) := \int_0^\infty e^{-c(t+\frac{1}{t})}\, t^{-s} \frac{dt}{t}$

Note: 1) standard K-Bessel function is defined by / has an integral representation

$$K_\nu(z) = \frac{1}{2}\left(\frac{1}{2}z\right)^\nu \int_0^\infty e^{-(t+\frac{z^2}{4t})} t^{-\nu} \frac{dt}{t}$$

for $|ph(z)| < \frac{\pi}{4}$

$\rightsquigarrow$ Lang's $K_s^{Lang}(c) = \cdot 2 \cdot \underbrace{K_s(2c)}$

↑ The standard K-Bessel

2) Recall:

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} - (z^2 + \nu^2) w = 0$$ modified Bessel equation

has 2 solutions,

$$I_\nu(z) = \left(\frac{1}{2}z\right)^\nu \cdot \sum_{k=0}^\infty \frac{(z^2/4)^k}{k! \cdot \Gamma(\nu+k+1)}$$

and $K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} \cdot e^{-z} \cos(\nu\pi)$ as $z \longrightarrow \infty$ in $|ph(z)| < \frac{3\pi}{2} - \delta$

$\parallel$

$$\frac{\pi}{2} \cdot (\sin(\nu\pi))^{-1} \cdot \left[I_{-\nu}(z) - I_\nu(z) \right]$$

characterized by the above asymptotic property.

(K4) $K_s(c) = K_{-s}(c)$ — clear from (K3)

(K5) $K_{\frac{1}{2}}(c) = \left(\frac{\pi}{c}\right)^{\frac{1}{2}} e^{-2c}$

⇓

(K6) $K_{\frac{1}{2}}(a,b) = \frac{\sqrt{\pi}}{a} e^{-2ab}$

Proof of K5: Let $g(x) := K_{\frac{1}{2}}(x) = \int_0^\infty e^{-x(t+\frac{1}{t})} \, t^{\frac{1}{2}} \cdot \frac{dt}{t}$

$$\underset{tx=u}{=} \frac{1}{\sqrt{x}} \int_0^\infty e^{-(u+\frac{x^2}{u})} \cdot u^{\frac{1}{2}} \cdot \frac{du}{u}$$

Let $h(x) := \int_0^\infty e^{-(u+\frac{x^2}{u})} \, u^{\frac{1}{2}} \cdot \frac{du}{u} = \sqrt{x} \cdot g(x)$

$\rightsquigarrow \frac{d}{dx} h(x) = -2x \int_0^\infty e^{-(u+\frac{x^2}{u})} \, u^{-\frac{1}{2}} \cdot \frac{du}{u} \underset{u=\frac{x^2}{v}}{=} -2x \int_0^\infty e^{-(\frac{x^2}{v}+v)} \cdot v^{\frac{1}{2}} \cdot \frac{1}{x} \cdot \frac{dv}{v}$

$$= -2\,h(x)$$

$\rightsquigarrow h(x) = C \cdot e^{-2x}$ for some C

$\rightsquigarrow h(0) = C = \int_0^\infty e^{-u} \cdot u^{\frac{1}{2}} \frac{du}{u} = \Gamma(\frac{1}{2}) = \sqrt{\pi}$ i.e. $h(x) = \sqrt{\pi} \cdot e^{-2x}$

$\Rightarrow K_{\frac{1}{2}}(x) = \frac{\sqrt{\pi}}{\sqrt{x}} \cdot e^{-2x}$, or $K_{\frac{1}{2}}(c) = \left(\frac{\pi}{c}\right)^{\frac{1}{2}} \cdot e^{-2c}$

(K7) $\forall x_0 > 0 \,.\, \forall \sigma_0 \leq \sigma_1, \; \exists C = C(x_0, \sigma_0, \sigma_1)$ s.t.

$$K_\sigma(x) \leq C \cdot e^{-2x} \quad \forall x \geq x_0 \;\forall \sigma \text{ with } \sigma_0 \leq \sigma \leq \sigma_1$$

$\uparrow$

$|K_s(x)| \leq K_\sigma(x)$

Proof of (K7)

$$\int_0^\infty = \int_0^{\frac{1}{8}} + \underbrace{\int_{\frac{1}{8}}^{8}} + \int_8^\infty$$

$\uparrow$

$\int_{\frac{1}{8}}^{8} e^{-x(t+\frac{1}{t})} \, t^\sigma \frac{dt}{t} \leq C_1 \cdot e^{-2x}$ $\because$ $t + \frac{1}{t} \geq 2 \quad \forall t \in \mathbb{R}_{>0}$

$\left[u^2 + \frac{1}{u^2} - 2 = (u - \frac{1}{u})^2 \right]$

$\int_0^{\frac{1}{8}} = \int_0^{\frac{1}{8}} e^{-x(t+\frac{1}{t})} \, t^\sigma \frac{dt}{t}$: use $\frac{1}{t} \geq 4 + \frac{1}{2t} \quad \forall t \leq \frac{1}{8}$

(i.e. $\frac{1}{2t} \geq 4$)

$\leq e^{-2x} \cdot \underbrace{\int_0^{\frac{1}{8}} e^{-x_0(t+\frac{1}{2t})} \, t^\sigma \cdot \frac{dt}{t}}_{\overset{\shortparallel}{C_2}}$

so $e^{-x(t+\frac{1}{t})} \leq e^{-x(2+t+\frac{1}{2t})}$

$\leq e^{-2x} \cdot e^{-x_0(t+\frac{1}{2t})}$

Similarly $\int_8^\infty e^{-x(t+\frac{1}{t})} \, t^\sigma \cdot \frac{dt}{t} \leq C_3 \cdot e^{-2x}$. q.e.d.

Note: $\dfrac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} = \sqrt{\pi} \cdot \left(1-(s-1)\lg 4+O(s-1)^2\right)$
near $s=1$

(K8) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(u^2+1)^s}\, du = \sqrt{\pi} \cdot \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}$ $\forall s$ with $\mathrm{Re}(s) > \frac{1}{2}$

Pf: $\displaystyle\Gamma(s) \cdot \int_{-\infty}^{\infty} \frac{1}{(u^2+1)^s}\, du = \int_{u\in\mathbb{R}} \int_{t\in\mathbb{R}^{*}_{>0}} e^{-t} \frac{1}{(u^2+1)^s} t^s \cdot \frac{dt}{t}\, du$

$\displaystyle\overset{=\!=}{\underset{t=(u^2+1)x}{}} \int_{u\in\mathbb{R}} \int_{x\in(0,\infty)} e^{-(u^2+1)x}\, x^s \cdot \frac{dx}{x} \cdot du$

$\displaystyle\int_{-\infty}^{\infty} e^{-u^2 x}\, du \underset{u=v/\sqrt{x}}{=} \frac{1}{\sqrt{x}} \int_{-\infty}^{\infty} e^{-v^2} \cdot dv = \frac{\sqrt{\pi}}{\sqrt{x}}$

$\displaystyle= \sqrt{\pi} \int_0^{\infty} x^{-\frac{1}{2}} \cdot x^s \cdot \frac{dx}{x}$

$= \sqrt{\pi} \cdot \Gamma(s-\frac{1}{2})$ QED.

(K9) $\displaystyle\Gamma(s) \cdot \int_{-\infty}^{\infty} \frac{e^{\sqrt{-1}\, x u}}{(u^2+1)^s}\, du = 2 \cdot \sqrt{\pi} \cdot \left(\frac{x}{2}\right)^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(x)$ $\forall s$ with $\mathrm{Re}(s) > \frac{1}{2}$

<u>Review</u> Poisson summation

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n), \quad \text{where} \quad \hat{f}(x) = \int_{-\infty}^{\infty} f(y)\, e^{-2\pi\sqrt{-1}\, xy}\, dy$$

f: Schwartz function

<u>Note</u>: $h: x \mapsto e^{-\pi x^2}$ is self-dual: $\hat{h} = h$

$\rightsquigarrow$ The Fourier transform of $x \mapsto e^{-\pi x^2 t}$ is $\frac{1}{\sqrt{t}} \cdot e^{-\pi x^2/t}$

$\left(\text{In general, for } g(x) = f(bx), \ \hat{g}(y) = \frac{1}{b} \hat{f}\left(\frac{y}{b}\right) \right)$

$$\rightsquigarrow \underbrace{\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}}_{=\theta(t)} = \frac{1}{\sqrt{t}} \sum_{n\in\mathbb{Z}} e^{-\pi n^2/t} = t^{-\frac{1}{2}} \cdot \theta(t^{-1})$$

Function Equation for $\theta(t)$

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} e^{-t} \cdot t^{\frac{s}{2}} \frac{dt}{t} = \int_0^{\infty} e^{-\pi n^2 t}\, t^{\frac{s}{2}} \frac{dt}{t} \cdot \left(\pi^{\frac{s}{2}} \cdot n^s\right)$$

$$\rightsquigarrow \pi^{-\frac{s}{2}} \cdot \Gamma\left(\frac{s}{2}\right) \cdot \zeta(s) = \int_0^{\infty} \underbrace{\sum_{n=1}^{\infty} e^{-\pi n^2 t}}_{\varphi(t) = \frac{1}{2}\theta(t) - \frac{1}{2}} \cdot t^{\frac{s}{2}} \cdot \frac{dt}{t}$$

$$\rightsquigarrow \varphi\left(\frac{1}{t}\right) = -\frac{1}{2} + \frac{1}{2} \cdot t^{\frac{1}{2}} \theta(t) = -\frac{1}{2} + \frac{1}{2} t^{\frac{1}{2}} + t^{\frac{1}{2}} \varphi(t)$$

$$= \int_1^{\infty} \varphi(t) \cdot t^{\frac{s}{2}} \frac{dt}{t} + \int_1^{\infty} t^{-s/2} \cdot \varphi\left(\frac{1}{t}\right) \frac{dt}{t}$$

$$= \frac{1}{s-1} - \frac{1}{s} + \int_1^{\infty} \varphi(t) \cdot \left[t^{\frac{s}{2}} + t^{\frac{1-s}{2}}\right] \cdot \frac{dt}{t}$$

Kronecker first limit formula

Def $$E(\tau, s) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{y^s}{|m\tau+n|^{2s}} \qquad Re(s) > 1$$

$$\underset{\text{the formula}}{\overset{\uparrow}{=}} \frac{\pi}{s-1} + 2\pi\left[\gamma - \log 2 - \log\left(\sqrt{y}\,|\eta(\tau)|^2\right)\right] + O(s-1)$$

where γ = Euler constant $= \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n} - \log n\right)$

$$\eta(\tau) = q_\tau^{\frac{1}{24}} \cdot \prod_{n=1}^{\infty}(1-q_\tau^n) \qquad \text{Note } e^{-\gamma} = \prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)e^{-\frac{1}{n}}$$

Proof of the 1^{st} limit formula:

$$\tau = x+\sqrt{-1}y \qquad |m\tau+n|^2 = (n+mx)^2 + m^2y^2$$

$$a^{-s}\cdot\pi^{-s}\,\Gamma(s) = \int_0^\infty e^{-\pi a t}\, t^s\, \frac{dt}{t} \qquad \forall a>0$$

substitute $a = \pi|m\tau+n|^2$

$$\rightsquigarrow \pi^{-s}\cdot\Gamma(s)\cdot y^{-s} E(\tau,s) = \pi^{-s}\cdot\Gamma(s)\cdot\sum_{m,n}{}' \frac{1}{(m\tau+n)^{2s}}$$

$$= 2\cdot\pi^{-s}\cdot\Gamma(s)\,\zeta(2s) + 2\cdot\sum_{m=1}^{\infty}\int_0^\infty \sum_{n\in\mathbb{Z}} e^{-\pi|m\tau+n|^2 t}\, t^s\,\frac{dt}{t}$$

$$= \underbrace{2\pi^{-s}\Gamma(s)\cdot\zeta(2s)}_{\overset{\shortparallel}{\mathrm{I}}} + 2\sum_{m=1}^{\infty}\int_0^\infty \underbrace{\left(\sum_{n\in\mathbb{Z}} e^{-\pi(mx+n)^2 t}\right)}_{\overset{\shortparallel}{\frac{1}{\sqrt{t}}\sum_{n\in\mathbb{Z}} e^{2\pi\sqrt{-1}xmn}\cdot e^{-\pi n^2/t}}} \cdot e^{-\pi y^2 m^2 t}\, t^s\,\frac{dt}{t}$$

$\leftarrow$ Poisson summation, $f(u) = e^{-\pi(mx+u)^2 t}$, $\hat{f}(v) = \frac{1}{\sqrt{t}}\cdot e^{2\pi\sqrt{-1}xmv}\cdot e^{-\pi v^2/t}$

$$= 2\pi^{-s}\Gamma(s)\cdot\zeta(2s) + \mathrm{II}_{n=0} + \mathrm{II}_{n\neq 0}$$

$$\mathrm{II}_{n=0} = 2\cdot\sum_{m=1}^{\infty}\int_0^\infty e^{-\pi y^2 m^2 t}\, t^{s-\frac{1}{2}}\,\frac{dt}{t}$$

$$= 2\cdot\pi^{s-\frac{1}{2}}\, y^{-2(s-\frac{1}{2})}\,\Gamma(s-\tfrac{1}{2})\cdot\zeta(2s-1)$$

$$II_{n\neq 0}(\tau,s) = 2\cdot\sum_{m=1}^{\infty}\sum_{\substack{n\neq 0\\ n\in\mathbb{Z}}} e^{2\pi\sqrt{-1}\,x m n}\cdot\underbrace{\int_0^\infty e^{-(\pi y^2 m^2 t + n^2\cdot\frac{1}{t})}\, t^{s-\frac{1}{2}}\cdot\frac{dt}{t}}_{\substack{\| \\ K_{s-\frac{1}{2}}(\sqrt{\pi}\cdot ym,\ \sqrt{\pi}|n|)}}$$

$\Longrightarrow$ $II_{n\neq 0}$ is an entire holomorphic function in s

(use (K7))

Evaluate $\quad II_{n\neq 0}(\tau,s)\Big|_{s=1}$ $\qquad$ use (K6): $K_{\frac{1}{2}}(a,b) = \frac{\sqrt{\pi}}{a}\cdot e^{-2ab}$

$$= 2\cdot\sum_{m=1}^{\infty}\sum_{\substack{n\neq 0\\ n\in\mathbb{Z}}} e^{2\pi\sqrt{-1}\,x m n}\ \underbrace{K_{\frac{1}{2}}(\sqrt{\pi}ym,\sqrt{\pi}\,n)}_{\substack{\| \\ \frac{1}{my}\cdot e^{-2ym\cdot|n|}}}$$

interchang the order of summation here: sum first over m, then over n

$$= 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{my}\cdot \mathrm{Re}\sum_{n=1}^{\infty} q_\tau^{mn}$$

$$= -\frac{4}{y}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\log\left|1-q_\tau^{\,n}\right|$$

$$= -\frac{4}{y}\left[\log|\eta(\tau)| + \frac{\pi y}{12}\right]$$

$$= -\frac{4}{y}\log|\eta(\tau)| - \frac{\pi}{3}.$$

Combine all:

$$\pi^{-s}\Gamma(s)\,y^{-s}\,E(\tau,s) = \underbrace{2\pi^{-s}\Gamma(s)\,\zeta(2s)}_{\substack{\| \\ 2\pi^{-1}\cdot\underbrace{\zeta(2)}_{\substack{\|\\ \pi^2/6}} + O(s-1) \\ \| \\ \frac{\pi}{3}+O(s-1)}} + 2\cdot\pi^{-(s-\frac{1}{2})}\,y^{-2(s-\frac{1}{2})}\,\Gamma(s-\tfrac{1}{2})\cdot\zeta(2s-1)$$

$$-\frac{4}{y}\log|\eta(\tau)| - \frac{\pi}{3} + O(s-1)$$

$$= -\frac{4}{y}\log|\eta(\tau)| + 2\pi^{-(s-\frac{1}{2})}\,y^{-2(s-\frac{1}{2})}\,\Gamma(s-\tfrac{1}{2})\zeta(2s-1) + O(s-1)$$

$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1)$ Titchmarsh, p. 16

$\rightsquigarrow \zeta(2s-1) = \frac{1}{2(s-1)} + \gamma + O(s-1)$

$y^s \cdot y^{-2(s-\frac{1}{2})} = y^{1-s} = e^{-(s-1)\log y} = 1-(s-1)\log y + O((s-1)^2)$

$\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} = \sqrt{\pi} \cdot \left[1-(s-1)\log 4 + O((s-1)^2)\right]$ ← Lang, Elliptic functions, p. 272, use K8

$2^{2s-1}\Gamma(s)\cdot\Gamma(s+\frac{1}{2}) = \sqrt{\pi}\cdot\Gamma(2s)$ Legendre duplication formula, Ahlfors, Ch. 5, §2.4, p. 199

Need $\Gamma'(1) = -\gamma$ (Whittaker-Watson p. 23: Use $\frac{1}{\Gamma(z)} = z\cdot e^{\gamma z}\cdot\prod_{n=1}^{\infty}\left[(1+\frac{z}{n})\cdot e^{-\frac{z}{n}}\right]$, take $\frac{d}{dz}\log$)

and also $\Gamma'(\frac{1}{2})$: Can use . the duplication formula

Recall $\Gamma(s)\cdot\Gamma(1-s) = \frac{\pi}{s\cdot\sin(\pi s)}$ Use the above infinite product + infinite product expansion for $\sin(\pi s)$

Don't really need

$s\cdot\Gamma(s) = \Gamma(s+1)$

$\rightsquigarrow \Gamma'(2) = \Gamma'(1) + \Gamma(1) = 1-\gamma$, $\Gamma'(0) = \Gamma'(1) = -\gamma$

log der. of duplication formula $\rightsquigarrow$

$$2\cdot\log 2 + \frac{\Gamma'(s)}{\Gamma(s)} + \frac{\Gamma'(s+\frac{1}{2})}{\Gamma(s+\frac{1}{2})} = \frac{2\Gamma'(2s)}{\Gamma(2s)}$$

$\underline{s=0}$: $2\log 2 - \gamma + \frac{\Gamma'(\frac{1}{2})}{\sqrt{\pi}} = -2\gamma$

$\rightsquigarrow \Gamma'(\frac{1}{2}) = \sqrt{\pi}\cdot(-\gamma - 2\log 2)$

$\underline{S_0}$: $$\frac{2\pi^{-(s-\frac{1}{2})}\cdot y^{-2(s-\frac{1}{2})}\cdot\Gamma(s-\frac{1}{2})\,\zeta(2s-1)}{\pi^{-s}\quad y^{-s}\quad\cdot\Gamma(s)}$$

$$= 2\cdot\pi^{\frac{1}{2}}\cdot\left[1-(s-1)\log y + O((s-1)^2)\right]\cdot\pi^{\frac{1}{2}}\left[1-(s-1)\log 4 + O((s-1)^2)\right]\cdot\frac{1}{2}\left[\frac{1}{s-1} + 2\gamma + O(s-1)\right]$$

$$= \pi\cdot\frac{1}{s-1}\cdot\left[1 + (-\log y - \log 4 + 2\gamma)\cdot(s-1) + O((s-1)^2)\right]$$

Combining all:

$$E(\tau,s) = \frac{\pi}{s-1} - \pi \log y - 2\pi \log 2 + 2\pi\cdot\gamma - 4\pi \log|\eta(\tau)| + O(s-1)$$

Kronecker's first limit formula

simple pole at $s=1$, residue $= \pi$

"Constant term": $\underbrace{-2\pi\log 2 + 2\pi\gamma}_{\text{abs. constant}} - \pi \log y - 4\pi \log|\eta(\tau)|$

leading term of $4\pi\log|\eta(\tau)|$ as $q_\tau \to 0$ i.e. $y \to \infty$

$= \frac{4\pi}{24}\cdot 2\pi y = \frac{\pi^2}{3} y$ $\qquad q_\tau = e^{-2\pi y + 2\pi\sqrt{-1}x}$

Kronecker second limit formula

$$E_{u,v}(\tau, s) := \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} e^{2\pi\sqrt{-1}(mu+nv)} \frac{y^s}{|m\tau+n|^{2s}} \qquad (0,0)\neq(u,v)\in\mathbb{R}^2 \qquad \mathrm{Re}(s) > 1$$

extends to an entire function for $s \in \mathbb{C}$

$$E_{u,v}(\tau, 1) = -\pi \log \left| f(u - v\tau; \tau) \cdot q_\tau^{\frac{v^2}{2}} \right|^2$$

where
$$f(z;\tau) := q_\tau^{\frac{1}{12}} \left(q_z^{\frac{1}{2}} - q_z^{-\frac{1}{2}}\right) \prod_{n=1}^{\infty} (1 - q_\tau^n q_z) \cdot (1 - q_\tau^n \cdot q_z^{-1})$$

$$(= -f(-z;\tau))$$

$$f(z; \omega_1, \omega_2) = e^{-\frac{1}{2}\eta_2 \cdot z^2/\omega_2} \cdot \Delta^{\frac{1}{12}}(\omega_1, \omega_2) \cdot \sigma(z; \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$$

Proof:
$$y^{-s} E_{u,v}(\tau, s) = \underbrace{\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{e^{2\pi\sqrt{-1}\,nv}}{n^2}}_{\overset{\text{Lemma 1 below}}{=}\ 2\pi^2\left(v^2 - v + \frac{1}{6}\right)} + \sum_{\substack{m\neq 0\\ m\in\mathbb{Z}}} e^{2\pi\sqrt{-1}\,mu} \sum_{n\in\mathbb{Z}} \frac{e^{2\pi\sqrt{-1}\,nv}}{|m\tau+n|^{2s}} \qquad \begin{pmatrix} 0 \le u < 1 \\ 0 \le v < 1 \end{pmatrix} \qquad (1)$$

$$\sum_{n\in\mathbb{Z}} \frac{e^{2\pi\sqrt{-1}\,nv}}{|m\tau+n|^{2s}} :$$

$$a^{-s}\Gamma(s) = \int_0^\infty e^{-at}\, t^s \frac{dt}{t} \qquad a = \pi|m\tau+n|^2$$

$$\pi^{-s} \frac{1}{|m\tau+n|^{2s}} \cdot \Gamma(s) = \int_0^\infty e^{-\pi|m\tau+n|^2 t}\, t^{-s} \frac{dt}{t}$$

$$\rightsquigarrow \sum_{n\in\mathbb{Z}} \frac{e^{2\pi\sqrt{-1}\,nv}}{|m\tau+n|^{2s}} = \frac{\pi^s}{\Gamma(s)} \cdot \int_0^\infty \sum_{n\in\mathbb{Z}} e^{2\pi\sqrt{-1}\,nv} \underbrace{e^{-\pi|m\tau+n|^2 t}}_{= e^{-\pi((n+mx)^2+m^2y^2)t}} \cdot t^s \frac{dt}{t}$$

$$= \frac{\pi^s}{\Gamma(s)} \cdot \int_0^\infty \sum_{n\in\mathbb{Z}} \left[e^{-\pi t(n+mx-\sqrt{-1}v/t)^2} \cdot e^{-\pi(m^2y^2t+v^2/t)} \cdot e^{-2\pi\sqrt{-1}\,mvx} \right] \cdot t^s \frac{dt}{t}$$

Apply Poisson summation to

$$= \frac{\pi^s}{\Gamma(s)} e^{-2\pi\sqrt{-1}\,mvx} \cdot \int_0^\infty \underbrace{\left(\sum_{n\in\mathbb{Z}} e^{-\pi t(n+mx-\sqrt{-1}v/t)^2} \right)}\cdot e^{-\pi(tm^2y^2+v^2/t)}\, t^s \frac{dt}{t} \qquad (2)$$

<u>Lemma 1</u> $\displaystyle\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{e^{2\pi\sqrt{-1}nv}}{n^2} = 2\pi^2(v^2-v+\tfrac{1}{6}) \qquad \forall v\in[0,1]$

Pf: Let $g(v) = v^2 - v + \frac{1}{6}$: $g(0) = g(1)$, $\int_0^1 g(v)\,dv = 0$

$$\int_0^1 g(v)\, e^{-2\pi\sqrt{-1}nv}\,dv = \frac{1}{2\pi^2 n^2} \quad \forall n\in\mathbb{Z}\setminus\{0\}$$

q.e.d.

Compute the Fourier transform of $\underset{\mathbb{R}}{\overset{\cap}{z}} \mapsto e^{-\pi t\left(z+mx-\sqrt{-1}\frac{v}{t}\right)^2} = f_1(z)$

Let $f_2(z) = e^{-\pi t z^2} \rightsquigarrow \hat{f}_2(z) = \frac{1}{\sqrt{t}}\, e^{-\pi z^2/t}$

$$\hat{f}_1(z) = e^{2\pi\sqrt{-1}\left(mx-\sqrt{-1}\frac{v}{t}\right)z}\cdot \hat{f}_2(z) = \frac{1}{\sqrt{t}}\, e^{2\pi\sqrt{-1}\left(mx-\sqrt{-1}\frac{v}{t}\right)z}\cdot e^{-\pi z^2/t}$$

$\rightsquigarrow$ <u>Lemma 2</u>

$$\sum_{n\in\mathbb{Z}} e^{-\pi t\left(n+mx-\sqrt{-1}\,v/t\right)^2} = \frac{1}{\sqrt{t}}\sum_{n\in\mathbb{Z}} e^{-\pi\frac{n^2}{t}}\cdot e^{2\pi\sqrt{-1}\left(mx-\sqrt{-1}\frac{v}{t}\right)n} \tag{3}$$

$$= \frac{1}{\sqrt{t}}\sum_{n\in\mathbb{Z}} e^{2\pi\sqrt{-1}mx}\cdot e^{-\pi\frac{n^2}{t}+2\pi\frac{nv}{t}} \tag{4}$$

<u>Conclude</u>:

$$\sum_{\substack{m\neq 0\\ m\in\mathbb{Z}}} e^{2\pi\sqrt{-1}mu}\sum_{n\in\mathbb{Z}} \frac{e^{2\pi\sqrt{-1}nv}}{|m\tau+n|^{2s}} = \frac{\pi^s}{\Gamma(s)}\sum_{m\neq 0} e^{2\pi\sqrt{-1}\,m(u-vx)}\cdot\sum_{n\in\mathbb{Z}} e^{2\pi\sqrt{-1}mnx}\int_0^\infty e^{-\pi\left[tm^2y^2+\frac{(n-v)^2}{t}\right]}\, t^{s-\frac{1}{2}}\frac{dt}{t} \tag{5}$$

[<u>Note</u>: The factor $e^{-\pi\frac{v^2}{t}}$ in (2) is combined with the facter $e^{-\pi\frac{n^2}{t}+2\pi\frac{nv}{t}}$ to give $e^{-\pi\frac{(n-v)^2}{t}}$ in (5)]

recall $K_{\frac{1}{2}}(a,b) = \frac{\sqrt{\pi}}{a}e^{-2ab}$

$K_{s-\frac{1}{2}}(\sqrt{\pi}|m|y, \sqrt{\pi}\,|n-v|)$. $K_{\frac{1}{2}}(\sqrt{\pi}|m|y, \sqrt{\pi}\,|n-v|) = \frac{\sqrt{\pi}}{\sqrt{\pi}|m|y}\cdot e^{-2\pi y|m|\cdot|n-v|} = \frac{1}{|m|\cdot y}e^{-2\pi y|m||n-v|}$

<u>At s=1</u>:

$$E_{u,v}(\tau,s)\Big|_{s=1} = 2\pi^2(v^2-v+\tfrac{1}{6})y + \pi\sum_{\substack{m\neq 0\\ m\in\mathbb{Z}}} e^{2\pi\sqrt{-1}m(u-vx)}\cdot\sum_{n\in\mathbb{Z}} e^{2\pi\sqrt{-1}mnx}\cdot\frac{1}{|m|}\cdot e^{-2\pi y|n-v||m|} \tag{6}$$

Reverse order of summation:

sum first over m, then n

$$= 2\pi^2(v^2-v+\tfrac{1}{6}y) + \pi\sum_{n\in\mathbb{Z}}\sum_{m\neq 0}\frac{1}{|m|}\, e^{2\pi\sqrt{-1}\left[m(u-vx)+nmx+\sqrt{-1}\,y\,|n-v|\cdot|m|\right]} \tag{7}$$

Assume first that $v \neq 0$, i.e. $0 < v < 1$

$$\sum_{m \neq 0} = \sum_{m=1}^{\infty} + \sum_{m=-1}^{-\infty}$$

the sum in (7)

Used $v \neq 0$ here

$n=0$:
$$\sum_{m=1}^{\infty} \frac{1}{m} e^{2\pi\sqrt{-1}\, m[(u-vx)+\sqrt{-1}yv]} + \sum_{m=1}^{\infty} \frac{1}{m} e^{2\pi\sqrt{-1}[-(u-vx)+\sqrt{-1}yv]m}$$

$$= -\log\left(1-e^{2\pi\sqrt{-1}(u-v\bar{\tau})}\right) - \log\left(1-e^{2\pi\sqrt{-1}(u-v\tau)}\right)$$

Let $z = u - v\tau$

$$= -\log \underbrace{(1-e^{2\pi\sqrt{-1}z})(1-e^{2\pi\sqrt{-1}\bar{z}})}_{\parallel}$$

$$(e^{\pi\sqrt{-1}z}-e^{-\pi\sqrt{-1}z})(e^{-\pi\sqrt{-1}\bar{z}}-e^{\pi\sqrt{-1}\bar{z}}) \cdot e^{\pi\sqrt{-1}(\bar{z}-z)}$$

$$= \left|q_z^{\frac{1}{2}} - q_z^{-\frac{1}{2}}\right| e^{-2\pi v y}$$

i.e. $n=0$: $\pi \cdot \sum_{m \neq 0}$ in (7) is: $\pi \cdot \left[-\log 2\left|q_z^{\frac{1}{2}} - q_z^{-\frac{1}{2}}\right| + 2\pi v y\right]$

Note: The term $2\pi^2 v y$ cancels the $-2\pi^2 v y$ in the term $2\pi^2(v^2 - v + \frac{1}{6})y$

For $n \neq 0$: there are four sums. $\begin{cases} n>0 \\ n<0 \end{cases} \begin{cases} m>0 \\ m<0 \end{cases}$

For $n>0$, $m>0$, get (sum over m first)

$$\pi \sum_{n>0} \sum_{m>0} \frac{e^{2\pi\sqrt{-1}m\cdot[u-v\tau+n\tau]}}{m} = -\pi \log\left(1-q_\tau^n q_z\right)$$

$z = u - v\tau$

The other three sums are similar

Finally, note that $2\pi^2 y v + \frac{1}{6}\pi^2 y = -\pi \left|q_\tau^{v^2/2} \cdot q_\tau^{1/2}\right|$

$$\left(\because \left|q_\tau^{\frac{1}{12}}\right|^2 = e^{-2\pi y/6}\right)$$

First limit formula and L-series

$k = \mathbb{Q}(\sqrt{-d_k})$ $\quad -d_k = disc(k) < 0$

$\mathfrak{A}$ = an ideal class of $\mathcal{O}_k$

$$\zeta(s,\mathfrak{A}) = \sum_{\substack{\mathfrak{a}\in\mathfrak{A} \\ \mathfrak{a}:\text{ ideal of } \mathcal{O}_k}} \frac{1}{N\mathfrak{a}^s}$$

$N\mathfrak{a}$ = the unique positive integer which generates $\mathfrak{a}\bar{\mathfrak{a}}$

$\mathfrak{b}$ = an invertible $\mathcal{O}_k$-module in $\mathfrak{A}^{-1}$ of the form $\mathfrak{b} = \mathbb{Z}\cdot\tau + 1$, $\tau\in k$

Lemma Let $\mathfrak{b}$ be an invertible $\mathcal{O}_k$-module in the class $\mathfrak{A}^{-1}$. Then

$$\left\{\begin{matrix}\text{ideals of } \mathcal{O}_k \\ \text{in the class } \mathfrak{A}\end{matrix}\right\} \xleftrightarrow[\text{bijection}]{} \left\{\begin{matrix}\text{principal fractional} \\ \mathcal{O}_k\text{-ideals containing } \mathfrak{b}\end{matrix}\right\} \xleftrightarrow{bij} \{\xi\in\mathfrak{b} \mid \xi\neq 0\}/\mathcal{O}_k^\times$$

$$\mathfrak{a} \longmapsto \mathfrak{a}\cdot\mathfrak{b}$$

$\rightsquigarrow$ $\mathfrak{A}\ni\mathfrak{a} \mapsto \mathfrak{a}\cdot\mathfrak{b}$ = principal frac ideal containing $\mathfrak{b}$

This is a bijection

i.e. $\{\mathfrak{a}\in\mathfrak{A}\} \longleftrightarrow \{\text{non-zero elements in } \mathfrak{b}\}/\mathcal{O}_k^\times$

$$\rightsquigarrow \zeta(s,\mathfrak{A}) = \frac{N\mathfrak{b}^s}{w_k}\sum_{\xi\in\mathfrak{b}}\frac{1}{N\xi^s}$$

where $w_k = |\mathcal{O}_k^\times|$, $\mathfrak{b}$ = an invertible $\mathcal{O}_k$-module in the class $\mathfrak{A}^{-1}$ of the form $\mathfrak{b} = \mathbb{Z}\tau + \mathbb{Z}$, $\tau\in k$

$$= \frac{N\mathfrak{b}^s}{w_k}\cdot\sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}}\frac{1}{|m\tau+n|^s}$$

Lemma: $N\mathfrak{b}^2 = |\underset{\text{disc. of } \mathfrak{b}}{disc(\mathfrak{b})}|\cdot|disc(\mathcal{O}_k)|^{-1} = \left|\begin{vmatrix}1 & \tau \\ 1 & \bar\tau\end{vmatrix}\right|^2\cdot d_k^{-1} = (2y)^2\cdot d_k^{-1}$

$$\rightsquigarrow \zeta(s,\mathfrak{A}) = \frac{1}{w_k}\cdot\left(\frac{2}{\sqrt{d_k}}\right)^s\cdot\sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}}\frac{y^s}{|m\tau+n|^{2s}}$$

Def / Notation

$$g(\mathfrak{b}) = (2\pi)^{-12}\, N\mathfrak{b}^6\cdot|\Delta(\mathfrak{b})|$$

$\forall$ invertible $\mathcal{O}_k$-module $\mathfrak{b}$

a well-defined function/quantity/invariant of invertible $\mathcal{O}_k$-modules

$\eta^{24}_{\text{Dedekind}}$

Fact – to be explained later:

Ref
Lang. p. 166
Chap 12, §2

Proposition For every invertible $\mathcal{O}_k$-module $\mathfrak{b}$, the number

$$N\mathfrak{b}^{12} \cdot \frac{|\Delta(\mathfrak{b})|^2}{|\Delta(\mathcal{O}_k)|^2}$$

is a unit (in some algebraic number field)

actually, the Hilbert class field

Equivalently: $\forall$ invertible $\mathcal{O}_k$-modules $\mathfrak{a}, \mathfrak{b}$, the number

$$N\mathfrak{b}^{-12} \; \frac{|\Delta(\mathfrak{a}\cdot\mathfrak{b}^{-1})|^2}{|\Delta(\mathfrak{a})|^2} = \frac{g(\mathfrak{a}\mathfrak{b}^{-1})}{g(\mathfrak{a})}$$

is a unit.

$$\left(\frac{2}{\sqrt{d_k}}\right)^s = \frac{2}{\sqrt{d_k}} \cdot \left(\frac{2}{\sqrt{d_k}}\right)^{s-1} = \frac{2}{\sqrt{d_k}} \cdot \left[1 + (s-1) \log \frac{2}{\sqrt{d_k}} + \cdots \right]$$

Recall from First Limit Formula:

$$\sum_{(m,n)\in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{y^s}{|m\tau+n|^{2s}} = \frac{\pi}{s-1} + 2\pi\left[\gamma - \log 2 - \log\left(\sqrt{y}\cdot|\eta(\tau)|^2\right)\right] + O(s-1)$$

$$\rightsquigarrow \zeta(s,\mathfrak{a}) = \frac{1}{w_k}\cdot\frac{2\pi}{\sqrt{d_k}}\left(\frac{1}{s-1} + 2\gamma - \frac{1}{2}\log d_k - \underbrace{2\log\left(\sqrt{y}\cdot|\eta(\tau)|^2\right)}_{\log|y| + \frac{1}{6}\log|\Delta(\tau)|}\right) + O(s-1)$$

$$N\mathfrak{b} = 2y \cdot d_k^{-\frac{1}{2}}$$

$$g(\mathfrak{b}) = (2\pi)^{-12} \cdot (2y)^6 \cdot d_k^{-3} \cdot |\Delta(\tau)|$$

$$\rightsquigarrow \frac{1}{6}\log|\Delta(\tau)| + 4\log y$$

$$= \frac{1}{6}\left[\log g(\mathfrak{b}) + 3\log d_k - 6\log y - 6\log 2 + 12\log(2\pi)\right] + \log y$$

$$\rightsquigarrow \boxed{\zeta(s,\mathfrak{a}) = \frac{1}{w_k}\cdot\frac{2\pi}{\sqrt{d_k}}\left[\frac{1}{s-1} + 2\gamma - \log d_k - \frac{1}{6}\log g(\mathfrak{b}) - 2\log 2\pi\right] + O(s-1)}$$

<u>Thm</u>: (i) $\forall$ non-trivial character of $Cl(\mathcal{O}_k)$, have

$$L(1,\chi) = -\frac{\pi}{3w_k\cdot\sqrt{d_k}} \sum_{\mathfrak{a}} \chi(\mathfrak{a})\cdot \log g(\mathfrak{a}^{-1})$$

(ii) $L(1, \underset{\chi_{\text{trivial}}}{\underset{\|}{\mathbb{1}}}) = \zeta_k(1) = \dfrac{2\pi h_k}{w_k\cdot\sqrt{d_k}}\cdot\dfrac{1}{s-1} + \dots$

<u>Note</u> The formula on p. 64 gives the constant term.

<u>Recall</u>: $K = k(j(\mathcal{O}_k))$ = Hilbert class field

$$\zeta_K(s) = \frac{(2\pi)^{h_k}\cdot h_K\cdot R_K}{w_K\cdot\sqrt{d_K}}$$

$$\rightsquigarrow \boxed{(2\pi)^{h_k-1}\cdot h_K h_k^{-1} R_K\cdot\frac{w_k}{w_K} = \prod_{\substack{\chi\neq 1\\ \text{non-trivial}\\ \text{char. of } Cl(\mathcal{O}_k)}} -\frac{\pi}{3w_k\cdot\sqrt{d_k}}\cdot\sum_{\mathfrak{a}}\chi(\mathfrak{a})\cdot\log g(\mathfrak{a}^{-1})} \quad (2)$$

<u>Frobenius determinant</u> G: abelian group

$\forall$ function f on G, have

$$\prod_{\chi\in\hat{G}}\left(\sum_{a\in G}\chi(a)\, f(a^{-1})\right) = \det\left[f(a b^{-1})\right]_{a,b\in G}$$

$$= \left[\sum_{a\in G} f(a)\right]\cdot \det\left[f(ab^{-1}) - f(a)\right]_{\substack{a,b\in G\\ a,b\neq 1}}$$

$$\rightsquigarrow \prod_{1\neq\chi\in\hat{G}}\left[\sum_{a\in G}\chi(a)\, f(a^{-1})\right] = \det\left[f(a\cdot b^{-1}) - f(a)\right]_{\substack{a,b\in G\\ a,b\neq 1}}$$

$$\frac{h_K}{h_k} \cdot R_K \cdot \left(-6\sqrt{d_k}\right)^{(h_k-1)} \cdot \frac{w_k^{h_k}}{w_K} = \prod_{\substack{\chi:\text{ non-trivial} \\ \text{char. of } Cl(\mathcal{O}_k)}} \left(\sum_{\mathfrak{a}} \chi(\mathfrak{a}) \cdot \log g(\mathfrak{a}^{-1}) \right)$$

$$= \det \Big[\log g(\mathfrak{a}\mathfrak{b}^{-1}) - \log g(\mathfrak{a}) \Big]_{\substack{[\mathfrak{a}],[\mathfrak{b}] \in Cl(\mathcal{O}_k) \\ [\mathfrak{a}],[\mathfrak{b}] \neq [\mathcal{O}_k]}}$$

<u>Conclusion</u>

<u>Prop</u> $$\left(-6\sqrt{d_k}\right)^{h_k-1} \cdot \frac{w_k^{h_k}}{w_K} \cdot R_K \cdot \frac{h_K}{h_k} = \det \Big[\underbrace{\log g(\mathfrak{a}\mathfrak{b}^{-1}) - \log g(\mathfrak{a})}_{= \log \Big(\underbrace{\frac{g(\mathfrak{a}\mathfrak{b}^{-1})}{g(\mathfrak{a})}}_{\text{a unit}} \Big)} \Big]_{\substack{\mathfrak{a},\mathfrak{b} \in Cl(\mathcal{O}_k) \\ \mathfrak{a},\mathfrak{b} \neq [\mathcal{O}_k]}}$$

Application of the Second Limit Formula to L-values

Goal: $L(1, \chi)$, where $\chi : (\mathcal{O}_k/\underbrace{f}_{\text{ideal of } \mathcal{O}_k})^\times \longrightarrow \mathbb{C}_1^\times$ $\quad k = \mathbb{Q}(\sqrt{-d_k})$

assume: χ is primitive (Lang used "proper") $\quad -d_k < 0$, $-d_k = \mathrm{disc}(k/\mathbb{Q})$

i.e. χ does not descend to a character $(\mathcal{O}_k/g)^\times \longrightarrow \mathbb{C}_1^\times$ for some $g \mid f$ $\quad g \neq f$

or: f = the conductor of χ

Will need to use Gaussian sum for $\mathcal{O}/f$, hence will need an additive character $\psi : \mathcal{O}/f \longrightarrow \mathbb{C}_1^\times$

$\mathcal{D}_k$ = different = $\mathcal{D}(k/\mathbb{Q})$

Classical method:

Choose and fix an element $\gamma \in f^{-1}\mathcal{D}_k^{-1}$ s.t. $\gamma \cdot f \cdot \mathcal{D}_k^{-1}$ is prime to f

(such an element γ exists by weak approximation)

$\forall \alpha \in \mathcal{O}_k$, $\forall \chi : (\mathcal{O}_k/f)^\times$

Def^n: $$G(\chi; \underbrace{\alpha, \gamma}) := \sum_{x \in \mathcal{O}_k/f} \chi(x)\, e^{2\pi\sqrt{-1}\,\mathrm{Tr}_{k/\mathbb{Q}}(x\alpha\gamma)}$$

defines an additive character of $\mathcal{O}_k/f$

Note: (G1) $G(\chi; \alpha\lambda, \gamma) = \bar{\chi}(\lambda) \cdot G(\chi; \alpha, \gamma)$ $\quad \forall \lambda \in \mathcal{O}_k$ s.t. $\lambda \bmod f \in (\mathcal{O}_k/f)^\times$

(G2) $G(\chi; \alpha, \gamma) = 0$ if $\alpha \in \mathcal{O}_k$ is not prime to f

$|G(\chi, \alpha, \gamma)| = \sqrt{Nf}$ if α is prime to f

Idea: Use Gaussian sum to express χ:

$$(G1) \Rightarrow \boxed{\chi(\xi) = \frac{G(\bar{\chi}; \xi, \gamma)}{G(\bar{\chi}; 1, \gamma)} \quad \forall \xi \in \mathcal{O}_k}$$

Def: (ray class characters) abuse notation: use the same character χ

$$RCl(f) = \underset{\text{with conductor } f}{\text{ray class group}} = \left(\begin{array}{l}\text{fractional ideals in } k \\ \text{prime to } f\end{array}\right) \Big/ \left(\begin{array}{l}\text{principal ideals gen.} \\ \text{by elements } \alpha \in k^\times \\ \alpha = \text{prime to } f \\ \alpha \equiv 1 \pmod{f}\end{array}\right)$$

$\chi : RCl(f) \longrightarrow \mathbb{C}_1^\times$ a character

Restricting to $(\mathcal{O}/f)^\times$ via $(\mathcal{O}/f)^\times \longrightarrow RCl(f)$ gives a character

$\chi : (\mathcal{O}/f)^\times \longrightarrow \mathbb{C}_1^\times$

$\chi : RC\ell(\mathfrak{f}) \longrightarrow \mathbb{C}_1^\times$

Def: $L_{\mathfrak{f}}(\chi, s) = \sum_{\substack{\mathfrak{a} = \text{ideals in } \mathcal{O}_k \\ (\mathfrak{a}, \mathfrak{f}) = 1}} \frac{\chi(\mathfrak{a})}{\mathbb{N}\mathfrak{a}^s}$

If χ is primitive, then

Lemma 1 $L_{\mathfrak{f}}(\chi, s) = \frac{1}{W_{\mathfrak{f}} \cdot \mathcal{G}(\bar{\chi}; 1, \gamma)} \sum_{R \in RC\ell(\mathfrak{f})} \bar{\chi}(\mathfrak{b}_R)\, \mathbb{N}\mathfrak{b}_R^s \cdot \sum_{0 \neq \xi \in \mathfrak{b}_R} e^{2\pi\sqrt{-1}\, \mathrm{Tr}_{k/\mathbb{Q}}(\xi \cdot \gamma)} \cdot \frac{1}{\mathbb{N}\xi^s}$

Use Gaussian sum to express $\chi(\xi)$ for $\xi \in \mathcal{O}_k$

where: $\mathfrak{b}_R$ = an ideal in the ray class R, prime to $\mathfrak{f}$

$W_{\mathfrak{f}} = \# \{ \xi \in \underbrace{\mu(k)}_{\text{roots of 1 in } k} \mid \xi \equiv 1 \ (\mathrm{mod}\ \mathfrak{f}) \}$

$\gamma \in \mathfrak{f}^{-1} \cdot \mathfrak{D}_k^{-1}$ s.t. $\gamma \cdot \mathfrak{f} \cdot \mathfrak{D}_k$ is prime to $\mathfrak{f}$ (as on p. 67)
fixed once and for all, after χ is given

Lemma 2 (variant of Lemma 1) recall: $\gamma \cdot \mathfrak{D}_k \cdot \mathfrak{f}$ is prime to $\mathfrak{f}$

χ primitive $\Rightarrow$ $L_{\mathfrak{f}}(\chi, s) = \frac{\chi(\gamma \mathfrak{D}_k \cdot \mathfrak{f})}{W_{\mathfrak{f}} \cdot \mathcal{G}(\bar{\chi}; 1, \gamma)} \sum_{R \in RC\ell(\mathfrak{f})} \bar{\chi}(R) \cdot E_{\mathfrak{f}}(R, s),$

where $E_{\mathfrak{f}}(R, s) := \mathbb{N}(\mathfrak{b}_R \mathfrak{D}_k^{-1} \mathfrak{f}^{-1})^s \cdot \sum_{0 \neq \lambda \in \mathfrak{b} \mathfrak{D}_k^{-1} \mathfrak{f}^{-1}} e^{2\pi\sqrt{-1}\, \mathrm{Tr}_{k/\mathbb{Q}}(\lambda)} \cdot \frac{1}{\mathbb{N}\lambda^s}$

$\mathfrak{b}_R$ = an ideal in the ray class R prime to $\mathfrak{f}$

Note: The RHS in the def^n of $E_{\mathfrak{f}}(R, s)$ is independent of the choice of $\mathfrak{b}_R$

Kronecker's 2nd limit formula

$\Rightarrow$ **Thm:**
$$\boxed{E_{\mathfrak f}(\underset{\substack{\Uparrow\\ \mathrm{RCl}(\mathfrak f)}}{R},1) = -\frac{2\pi}{\sqrt{d_k}\cdot 6N}\log\left|H(\mathfrak b_R\cdot\mathfrak D_k^{-1}\cdot\mathfrak f^{-1})\right| = -\frac{2\pi}{\sqrt{d_k}\cdot 6N}\log\left|\Phi_{\mathfrak f}(R)\right|}$$

<u>where</u>: N = the smallest positive integer contained in R
(some sort of "conductor" for the ray class R)

$\mathfrak b_R$ = (as before) an/any ideal in the ray class R prime to $\mathfrak f$

H: Siegel function, defined for every fractional ideal of k

$\Phi_{\mathfrak f}$: Ramachandra invariant (Lang, Chap. 19, §2)

$$\Phi_{\mathfrak f}(\mathfrak b) = H(\mathfrak b\,\mathfrak D_k^{-1}\mathfrak f^{-1}) \quad \forall\ \underline{\text{ideal}}\ \mathfrak b\subseteq\mathcal O_k,\ \text{prime to } \mathfrak f$$

<u>Recall</u> $\forall$ fractional ideal $\mathfrak a = \mathbb Z\cdot\omega_1+\mathbb Z\cdot\omega_2 \subseteq k$ s.t. $\mathrm{Im}\left(\frac{\omega_1}{\omega_2}\right)>0$

$$\tau = \omega_1/\omega_2$$

$$H(\mathfrak a) = \left[f\left(\mathrm{Tr}(\omega_1)-\mathrm{Tr}(\omega_2)\cdot\tau;\tau\right)e^{\pi\sqrt{-1}\,\mathrm{Tr}(\omega_2)\cdot[\mathrm{Tr}(\omega_2)\tau-\mathrm{Tr}(\omega_1)]}\right]^{12N} \quad \left[f(z;\omega_1,\omega_2) = e^{-\frac12\eta_2 z^2/\omega_2}\Delta^{\frac1{12}}(\omega_1,\omega_2)\cdot\sigma(z,\Lambda)\right.$$

$$\Lambda = \mathbb Z\omega_1+\mathbb Z\omega_2$$

<u>where</u> N = the exact denominator of $(\mathrm{Tr}(\omega_1),\mathrm{Tr}(\omega_2))\in\mathbb Q^2$

<u>Repeat</u>:

This exponent cancels the N in $\frac{2\pi}{\sqrt{d_k}\cdot 6N}$

$$H(\mathfrak a) = \left[f\left(\mathrm{Tr}(\omega_1)-\mathrm{Tr}(\omega_2)\tau;\tau\right)\cdot e^{2\pi\sqrt{-1}\,\mathrm{Tr}(\omega_2)\cdot[\mathrm{Tr}(\omega_2)\cdot\tau-\mathrm{Tr}(\omega_1)]}\right]^{12N}$$

<u>and</u> $f(z;\tau) = e^{-\frac12\eta_2(\tau)\cdot z^2}\cdot\Delta^{\frac1{12}}(\tau)\cdot\sigma(z;\tau)$

value of $\sigma(z;\tau)$ at an element of k.
times Δ-value at an element of k
(times an exponential of (elt of k)$\cdot$(a quasi-period))

Consequence of Thm: If χ is a primitive ray class character of conductor $\mathfrak f$, then

$$L_{\mathfrak f}(1,\chi) = -\frac{2\pi\,\chi(\gamma\cdot\mathfrak D_k\cdot\mathfrak f)}{w_{\mathfrak f}\cdot G(\bar\chi;1,\gamma)\cdot\sqrt{d_k}\cdot 6N}\sum_{R\in\mathrm{RCl}(\mathfrak f)}\bar\chi(R)\cdot\log\left|\Phi_{\mathfrak f}(R)\right|$$

<u>Note</u>: This is related to Stark's conjecture