Newton Polygons as Lattice Points

by

Ching-Li Chai

Classical Newton Polygons:

(1) DEFINITION

- *Traditional Approach*: a graphic representation of a sequence of rational numbers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

- *Lie Theoretic Approach*: a sequence $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ of non-increasing rational numbers corresponds to a rational point in the Weyl chamber of the group GL_n; or more canonically a Weyl orbit in coroot space $\mathfrak{t}_{\mathbb{R}}$.

Illustrate the two equivalent definitions of Newton polygons in graph no. ?

(2) PARTIAL ORDERING

Say we have a Newton polygon NP_1 with slopes $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and a Newton polygon NP_2 with slopes $\mu_1 \geq \mu_1 \geq \ldots \geq \mu_n$, corresponding to points x_1, x_2 in the Weyl chamber of GL_n.

Then the convex hull of the Weyl orbit of x_1 contains the convex hull of the Weyl orbit of x_2 if and only if

$$\sum_{j=1}^{k} \lambda_j \geq \sum_{j=1}^{k} \mu_j, \quad \forall\, k = 1, \ldots, n-1,$$

$$\text{and} \qquad \sum_{j=1}^{n} \lambda_j = \sum_{j=1}^{n} \mu_j \ .$$

Graphically, this means that NP_1 lies above NP_2. We say that $x_1 \succeq x_2$ if this is the case; this defines a partial ordering on the set of all such Newton polygons $\mathcal{N}$.

Illustrate the two equivalent definitions of the partial ordering in graph no. ?

(3) INTEGRAL NEWTON POLYGONS

The Newton polygons one encounters in application usually satisfy an extra integrality condition, namely the denominator of each 'slope' λ_i divides its multiplicity. If this condition is satisfied, we say that the Newton polygon is *integral*. Denote by $\mathcal{N}_{\mathbb{Z}}$ the set of all integral Newton polygons (for a fixed n).

- The *poset* $\mathcal{N}_{\mathbb{Z}}$ forms a *lattice*.

(Actually, the same is true for the set $\mathcal{N}$ of all Newton polygons. But the "*meet*" operation on $\mathcal{N}_{\mathbb{Z}}$ is not the restriction of the meet operation on $\mathcal{N}$; the "*join*" operation is.)

Notice that the dual meaning of "*lattice*" is reflected here.

(4) Two combinatorial properties of integral Newton polygons (possibly not “well-known”):

• The poset $\mathcal{N}_{\mathbb{Z}}$ is *ranked*, in the sense that any two maximal chains in a given segment have the same length.

• The length of a segment $[x_2, x_1]$ in $\mathcal{N}_{\mathbb{Z}}$ is given by the number of *integral points* which lie strictly above NP_2 and on or below NP_1.

Motivations

Let $Sh(G, X)$ be a Shimura variety, and let $p > 0$ be a prime number where $Sh(G, X)$ has good reduction.

1. Generalize the notion of Newton polygons, so as to predict which "generalized isogeny classes" should occur in the family of motives with G-structure attached to the reduction of $Sh(G, X)$ modulo p. (This has been solved by the works of Kottwitz and Rapoport-Richartz. We shall make things somewhat more explicit.)

2. Find a formula which predicts the dimension of the various Newton strata of the reduction of $Sh(G, X)$ modulo p.

Remark. The answer to the second question above must extrapolate the following theorem of Li and Oort: The supersingular locus in the moduli space $\mathcal{A}_g$ of g-dimensional principally polarized abelian varieties in characteristic p is equal to $\lfloor \frac{g^2}{4} \rfloor$. So part of the question is to find a *group-theoretic* interpretation of the number $\lfloor \frac{g^2}{4} \rfloor$.

Notations

k	—	an algebraically closed field of characteristic $p > 0$.
K	—	the fraction field of the ring of p-adic Witt vectors $W(k)$.
$\overline{K}$	—	an algebraic closure of K.
F	—	a finite extension of $\mathbb{Q}_p$ in $\overline{K}$.
L	—	the compositum of K and F in $\overline{K}$.
σ	—	the Frobenius automorphism of L/F.
Γ	—	the Galois group of $\overline{F}/F$.
$\mathbb{D}$	—	the proalgebraic torus with character group $\mathbb{Q}$.
G	—	a connected reductive group *quasisplit* over F.
$B(G)$	—	the set of all σ-conjugacy classes of $G(L)$.
S	—	a maximally F-split torus in G.
T	—	a maximal torus of G over F which contains S, i.e. $Z_G(S)$.
B	—	a Borel subgroup of G over F which contains T.
$X_*(T)$	—	cocharacters of T.
$X^*(T)$	—	characters of T.
Φ	—	$\Phi(G,T)$, the root system of T-roots of G.
$\Phi^\vee$	—	$\Phi^\vee(G,T)$, the dual root system of $\Phi(G,T)$ consisting of coroots.

Φ^+	—	$\Phi^+(G,T)$, the B-positive roots in Φ.
Φ_F	—	$\Phi(G,S)$, the relative root system of S-roots of G.
$\Phi_F^\vee$	—	$\Phi^\vee(G,S)$, the relative dual root system of S-coroots of G.
Φ_F^+	—	the B-positive roots in Φ_F.
Δ	—	the simple roots in Φ.
$\Delta^\vee$	—	the simple coroots in $\Phi^\vee$.
Δ_F	—	the simple roots in Φ_F.
$\Delta_F^\vee$	—	simple coroots in $\Phi_F^\vee$.
W	—	the Weyl group of $\Phi(G,T)$.
W_F	—	the Weyl group of Φ_F.
C	—	the Weyl chamber in $X_*(T)_{\mathbb{R}}$, with edges given by the the fundamental coweights if G is semisimple. In general it is the inverse image of the Weyl chamber for G^{ad}.
$C^\vee$	—	the obtuse Weyl chamber in $X_*(T)_{\mathbb{R}}$, or the dual cone to C. It has the simple coroots as edges if G is semisimple. In general it is the image of the obtuse Weyl chamber for G^{der}.
C_F	—	the Weyl chamber in $X_*(S)_{\mathbb{R}}$.
$C_F^\vee$	—	the obtuse Weyl chamber in $X_*(S)_{\mathbb{R}}$.

Definition 1. Let $B(G)$ be the set of all σ-conjugacy classes of elements of $G(L)$. Two elements $x, y \in G(L)$ represent the same element in $B(G)$ if and only if $x = gy\sigma(g)^{-1}$ for some $g \in G(L)$, or equivalently iff the two elements $x\sigma, y\sigma \in G(L) \rtimes \langle\sigma\rangle$ are conjugate under $G(L)$. An element $x \in G(L)$ gives, for each finite dimension F-rational representation V of G, a structure of σ-L-isocrystal on the space $V \otimes_F K$ via the action of $x\sigma \in G(L) \rtimes \langle\sigma\rangle$.

Remark. We assumed that G *is quasisplit over* F. The main reason is that a convenient description of the whole set $B(G)$ is available under this assumption.

For applications we are mostly interested in the case when $F = \mathbb{Q}_p$, because then we will be dealing with the usual F-isocrystals, and also because the reductive groups attached to Shimura varieties are defined over $\mathbb{Q}$.

Definition 2. The Newton cone $\mathcal{N}(G)$ is defined to be

$$\begin{aligned}\mathcal{N}(G) &= (\text{Int } G(L)\backslash \text{Hom}_L(\mathbb{D}, G))^{\langle\sigma\rangle} \\ &\cong \Big(X_*(T)_{\mathbb{Q}}/W\Big)^{\Gamma}\end{aligned}$$

Since G is quasisplit over F, one can also identify $\mathcal{N}(G)$ with $X_*(S)_{\mathbb{Q}}/W_F$, or as the set of all rational points in C_F.

Thus $\mathbb{N}(G)$ has a canonical structure as the set of all rational points in a simplicial cone. Its faces are indexed by subsets of Δ_F, or equivalently Γ-stable subsets of Δ.

Fact: There is a canonical map

$$\bar{\nu}_G : B(G) \to \mathcal{N}(G)$$

defined by Kottwitz, which assigns every σ-conjugacy class $\bar{b} \in B(G)$ its associated Newton point $\bar{\nu}(\bar{b}) \in \mathcal{N}(G)$.

Notation

(1) Let θ_F be a subset of Δ_F and let θ be the corresponding subset of Δ.

(2) Let $T_\theta = \bigcap_{\alpha\in\theta}(\mathrm{Ker}\,\chi_\alpha)^0$ be the largest subtorus of T killed by all characters χ_α with $\alpha \in \theta$, and let $S_{\theta_F} = (T_\theta \cap S)^0$.

(3) Let $M_\theta = Z_G(T_\theta) = Z_G(S_{\theta_F})$, the standard Levi subgroup indexed by θ; T_θ is the neutral component of M_θ. Reflections about the root hyperplanes in $X_*(T)_\mathbb{R}$ indexed by elements in θ generate a subgroup W_θ of W, which is canonically isomorphic to the Weyl group of M_θ.

(4) The interior of $C_F \cap X_*(S_{\theta_F})_\mathbb{R}$ in $X_*(S_{\theta_F})_\mathbb{R}$ is an open face of C_F; denote it by $C_F^{\theta,0}$. The closed face $C_F \cap X_*(S_{\theta_F})_\mathbb{R}$ will be denoted by C_F^θ.

Definition 3. For each subset θ_F of Δ, we have a canonical projection

$$\pi_{\theta_F} : X_*(T)_{\mathbb{Q}} \twoheadrightarrow X_*(S_{\theta_F})$$

defined in several equivalent ways.

(1) First form: The Galois group Γ operates on $X_*(T)_{\mathbb{Q}}$ via a finite quotient; the subspace of fixed vectors is $X_*(S)_{\mathbb{Q}}$. This gives us a projection

$$\mathrm{pr}^{\Gamma} : X_*(T)_{\mathbb{Q}} \twoheadrightarrow X_*(S)_{\mathbb{Q}}$$

The finite reflection group W_θ also operated on $X_*(T)_{\mathbb{Q}}$, with $X_*(T_\theta)_{\mathbb{Q}}$ as the subspace of fixed vectors. This gives us a projection

$$\mathrm{pr}^{W_\theta} : X_*(T)_{\mathbb{Q}} \twoheadrightarrow X_*(T_\theta)_{\mathbb{Q}}$$

Clearly $X_*(S_{\theta_F})_{\mathbb{Q}} = X_*(S)_{\mathbb{Q}} \cap X_*(T_\theta)_{\mathbb{Q}}$; moreover $X_*(T_\theta)_{\mathbb{Q}}$ is stable under Γ since θ is. In fact the action of W_θ on $X_*(T)_{\mathbb{Q}}$ is normalized by the action of Γ. We define π_{θ_F} to be

$$\pi_{\theta_F} = \mathrm{pr}^{\Gamma} \circ \mathrm{pr}^{W_\theta} = \int_{W_\theta \cdot \Gamma} .$$

(2) Second form: The $\mathbb{Q}$-vector subspace of $X_*(T)_{\mathbb{Q}}$ generated by coroots $\{\alpha^\vee|\alpha \in \theta\}$ and $\{\gamma \cdot \beta - \beta|\beta \in \Delta, \beta \notin \theta\}$ is a complement to $X_*(S_{\theta_F})_{\mathbb{Q}}$. We define π_{θ_F} to be the projection to $X_*(S_{\theta_F})$ with respect to this direct sum decomposition.

(3) Third form: Choose and fix an admissible inner product $(\cdot|\cdot)$ on $X_*(T)_{\mathbb{R}}$, i.e. it is invariant under both Γ and W. Then we define π_{θ_F} to be the orthogonal projection to $X_*(S_{\theta_F})$ with respect to $(\cdot|\cdot)$.

Definition 4. (i) For each subset θ_F of Δ_F, let $\Lambda_{\theta_F} = \pi_{\theta_F}(X_*(T))$ be the projection of the coweight lattice of $X_*(T)$ under π_{θ_F}, and let $R^\vee_{\theta_F}$ be the projection of the coroot module $R^\vee(G,T)$ in $X_*(T)_\mathbb{Q}$ under π_{θ_F}. Here $R^\vee(G,T)$ is the $\mathbb{Z}$-submodule of $X_*(T)$ generated by $\Phi^\vee$. Clearly $R_{\theta_F} \subseteq \Lambda_{\theta_F}$; Λ_{θ_F} is a lattice in $X_*(S_\theta)_\mathbb{Q}$, while $R^\vee_{\theta_F}$ is a lattice in $X_*(S_\theta)_\mathbb{Q}$ if T/S is anisotropic over F.

(ii) Define $C^{\theta,0}_{F,\mathbb{Z}}$ (resp. $C^{\theta,0}_{F,R^\vee}$) to be the intersection of Λ_{θ_F} (resp. $R^\vee_{\theta_F}$) with the open face $C^{\theta,0}_F$ of C_F. Similarly let $C^\theta_{F,\mathbb{Z}}$ (resp. $C^\theta_{F,R^\vee}$) be the intersection of Λ_{θ_F} (resp. $R^\vee_{\theta_F}$) with the closed face C^θ_F of C_F.

(iii) Define $C_{F,\mathbb{Z}}$ (resp. $C_{F,R^\vee}$) to be the disjoint union of all $C^{\theta,0}_{F,\mathbb{Z}}$ (resp. $C^{\theta,0}_{F,R^\vee}$), with θ running over all subsets of Δ stable under Γ. Since $\mathcal{N}(G)$ is canonically isomorphic to the set of all rational points of C_F, $C_{F,\mathbb{Z}}$ (resp. $C_{F,R^\vee}$) can be identified with a discrete subset of $\mathcal{N}(G)$; denote it by $\mathcal{N}(G)_\mathbb{Z}$ (resp. $\mathcal{N}(G)_{R^\vee}$).

Definition 5. For a give element $\nu \in \mathcal{C}_{F,\mathbb{Z}}$, define

$$C^{\nu}_{F,\mathbb{Z}} = \left\{ x \in C_{F,\mathbb{Z}} \mid x \preceq \nu \right\}$$

$$C^{\nu}_{F,R^\vee} = \left\{ x \preceq \nu \;\middle|\; \begin{array}{c} \exists\ \theta_F \subseteq \Delta_F \text{ s.t. } x \in C^{\theta,0}_{F,\mathbb{Z}} \\ x - \pi_{\theta_F}(\nu) \in \pi_{\theta_F}(R^\vee(G,T)) \end{array} \right\}$$

The corresponding subsets in $\mathcal{N}(G)_{\mathbb{Z}}$ will be denoted by $\mathcal{N}(G)^{\nu}_{\mathbb{Z}}$ and $\mathcal{N}(G)^{\nu}_{\mathbb{Z},R^\vee}$ respectively.

The following proposition explains why the set $C^{\nu}_{F,R^\vee}$ is relevant.

Proposition 1. *Let $b_1 \in B(G)$ be a σ-conjugacy class, $\bar{\nu}(b_1) \in C_{F,\mathbb{Z}}$ be the representative of the Newton point of b in C_F. Then $C^{\bar{\nu}(b_1)}_{F,R^\vee}$ is equal to the image of*

$$\{b \in B(G) | \bar{\nu}(b) \preceq \bar{\nu}(b_1), \gamma(b) = \gamma(b_1)\}$$

under the Newton map

$$\bar{\nu}_G : B(G) \to \mathcal{N}(G)_{\mathbb{Z}} \cong C_{F,\mathbb{Z}}$$

Theorem 1. *Let μ be a miniscule dominant coweight, that is $\langle\alpha,\mu\rangle \in \{0,1\}$ for each root $\alpha \in \Phi^+$. Let $\mu^\natural \in C_{F,\mathbb{Z}}$ be the projection of μ to C_F, that is the average of $\Gamma \cdot \mu$. Then every $y \in C^{\mu^\natural}_{F,R^\vee}$ is the projection of some element of the Weyl orbit $W \cdot \mu$ under π_{θ_F} for a suitable subset $\theta_F \subseteq \Delta_F$.*

Remark. In the situation of a Shimura variety $Sh(G,X)$, take μ to be the coweight of G attached to X. Theorem 1 says the prediction of the generalized Grothendieck conjecture of Rapoport-Richartz as to which Newton points will appear in the reduction modulo p of $Sh(G,X)$ coincides with that of the *complex multiplication* theory and the philosophy of motives.

Theorem 2. *Let ν be an element of $C_{F,\mathbb{Z}}$. (i) The poset $C^{\nu}_{F,R^{\vee}}$ is ranked. In other words every maximal chain between two comparable elements have the same length.*

(i) Let y, z be as in (i). Then

$$\begin{aligned}&\mathrm{length}_{C^{\nu}_{F,R^{\vee}}}([y,z]) \\ &= \#\left(E^{\nu}_{F,R^{\vee}}(z) - E^{\nu}_{F,R^{\vee}}(y)\right) \\ &= \textstyle\sum_{i=1}^{l} \begin{pmatrix} \lceil \langle \omega_{F,i}, \nu \rangle - \langle \omega_{F,i}, y \rangle \rceil \\ -\lceil \langle \omega_{F,i}, \nu \rangle - \langle \omega_{F,i}, z \rangle \rceil \end{pmatrix}\end{aligned}$$

where $\omega_{F,1}, \dots, \omega_{F,l}$ are the fundamental F-coweights. Especially

$$\mathrm{length}_{C^{\nu}_{F,R^{\vee}}}([x,\nu]) = \sum_{i=1}^{l} \lceil \langle \omega_{F,i}, \nu \rangle - \langle \omega_{F,i}, x \rangle \rceil$$

for any $x \in C^{\nu}_{F,R^{\vee}}$.

Dimension of the Newton strata

Let (G, X) be a Shimura data. Assume that G is quasisplit over $\mathbb{Q}_p$ and splits over an unramified extension of $\mathbb{Q}_p$. Let μ be the dominant coweight with respect to a $\mathbb{Q}_p$-rational Borel subgroup B attached to X. Let $\mu^\natural$ be the $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$-average of μ, and let $M_1 = Z_G(\mu^\natural)$. Let b_{M_1} be the basic element in $B(M_1)_{\text{basic}} \cong \pi_1(M_1)$ which corresponds to the image of μ in $\pi_1(M_1)$. Denote by $b_1 \in B(G)$ the image of b_{M_1} under the canonical map $B(M_1) \to B(G)$. Let $b_0 \in B(G)_{\text{basic}} \cong \pi_1(G)$ be the basic element in $B(G)$ corresponding to the image of μ in $\pi_1(G)$. Notice that the Newton point $\bar{\nu}(b_1)$ of b_1 is represented by $\mu^\natural$.

Let K_p be a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_p)$. Assume furthermore that $\mathrm{Sh}_{K_p}(G,X)$ over the reflex field $E(G,X)$ has good reduction at a place v over p. Let $\mathcal{S}_{\text{basic}}$ be the locus in the reduction of $\mathrm{Sh}_{K_p}(G,X)$ at v consisting of points with type $b_0 \in B(G)$. More generally, for each $b \in B(G)$ such that $b \preceq b_1$ and $\gamma(b) = \gamma(b_1)$, let $\mathcal{S}_b$ be the stratum of the reduction of $\mathrm{Sh}_{K_p}(G,X)$ at v consisting of points with type b.

Question. (i) Is the codimension of $\mathcal{S}_{\text{basic}}$ in the reduction of $\mathrm{Sh}_{K_p}(G,X)$ equal to the length of the poset $C^{\mu^\natural}_{\mathbb{Q}_p,R^\vee}$?

(ii) More generally, suppose that b is an element of $B(G)$ such that $b \leq b_1$ and $\gamma(b) = \gamma(b_1)$. Is the codimension $\mathrm{codim}\,(\mathcal{S}_b)$ of $\mathcal{S}_b$ equal to

$$\text{length}_{C^{\mu^\natural}_{\mathbb{Q}_p,R^\vee}}\left([\bar{\nu}(b), \mu^\natural]\right)$$

?

Remark. (i) In the Siegel case the Question (i) is answered affirmatively by the result of Li and Oort. The thesis work of Chia-Fu Yu confirms the question (i) in many cases of Shimura varieties of PEL-type.

(ii) One expects that the generic stratum of the reduction of $\mathrm{Sh}_{K_p}(G, X)$ at v has type b_1. The work of Jeff Achter confirms that this for many cases of Shimura varieties of PEL-type *even when the polarization in question is not principal*.

(iii)The stratum with type b_1 generalizes the ordinary locus in the moduli space of principally polarized varieties. It has long been observed that abelian varieties of dimension g with f-rank $g-1$ share many desirable properties with ordinary abelian varieties of dimension g. They are called "almost ordinary" abelian varieties by some authors. An analog of "almost ordinary" type in the present setting in terms of the Newton points exists when $\mu^\natural$ is an edge element: If this is the case, then there exists a unique maximal element in $C^{\mu^\natural}_{\mathbb{Q}_p,R^\vee}-\{\mu^\natural\}$, namely $\sup\left(E^{\mu^\natural}_{F,R^\vee}(\mu^\natural)-\{\mu^\natural\}\right)$. This occurs often; for instance when G is absolutely simple, or when G is $\mathbb{Q}$-simple and the reflex field is equal to $\mathbb{Q}$.

(iv) Along the same train of thought, one may ask whether there exists a unique minimal element in $C^{\mu^\natural}_{\mathbb{Q}_p,R^\vee} - \{0\}$. The answer is yes in some situations, for instance when G is absolutely simple of type B or C in the Dynkin classification. But the answer is no in many cases, for instance when G is absolutely simple of type A.

EXAMPLES

splict C_2, C_3, C_4

The fundamental coweights are

$$\omega_1^\vee = e_1, \omega_2^\vee = e_2, \ldots$$
$$\omega_{n-1}^\vee = e_1 + \cdots + e_{n-1}$$
$$\mu = \omega_n^\vee = \frac{1}{2}(e_1 + \cdots + e_n)$$

and the simple coroots are

$$\alpha_1^\vee = e_1 - e_2, \ldots, \alpha_{n-1}^\vee = e_{n-1} - e_n, \alpha_n = e_n$$

split C_2

$x \in C^{\mu}_{\mathbb{Z}}$	slopes of x	$y \in W \cdot \mu$ with $\pi_{\theta_F} = x$
$\mu = \omega_2^{\vee}$	1,1,0,0	μ
$\frac{1}{2}\omega_2^{\vee}$	$1,\frac{1}{2},\frac{1}{2},0$	μ
0	$\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$	any elt. in $W \cdot \mu$

We have

$$\mu \succ \frac{1}{2}\omega_2^{\vee} \succ 0$$

In the present case $C^{\mu}_{F,R^{\vee}}$ coincides with $C^{\mu}_{F,\mathbb{Z}}$.

split C_3

$x \in C^{\mu}_{\mathbb{Z}}$	slopes of x	$y \in W \cdot \mu$ with $\pi_{\theta_F} = x$
$\mu = \omega_3^\vee$	1,1,1,0,0,0	μ
$\frac{1}{2}\omega_2^\vee$	$1,1,\frac{1}{2},\frac{1}{2},0,0$	μ
$\frac{1}{2}\omega_1^\vee$	$1,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	μ
$\frac{2}{3}\omega_3^\vee$	$\frac{5}{6},\frac{5}{6},\frac{5}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6}$	does not exist
$\frac{1}{3}\omega_3^\vee$	$\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}$	$\frac{1}{2}(e_1 + e_2 - e_3)$
0	$\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$	any elt. in $W \cdot \mu$

We have

$$\mu \succ \frac{1}{2}\omega_2^\vee \begin{matrix} \succ & \frac{1}{2}\omega_1^\vee \\ \succ & \frac{2}{3}\omega_3^\vee \end{matrix} \succ \frac{1}{3}\omega_3^\vee \succ 0$$

Except $\frac{2}{3}\omega_3^\vee$, all other elements of $C^{\mu}_{F,\mathbb{Z}}$ are in $C^{\mu}_{F,R^\vee}$. Clearly the Newton point $\frac{2}{3}\omega_3^\vee$ does not appear in the moduli space $\mathcal{A}_3$ of principally polarized abelian threefolds (when $F = \mathbb{Q}_p$), since its slopes sequence is not integral.

split C_4

$x \in C^\mu_{\mathbb{Z}}$	slopes of x	$y \in W\cdot\mu$ with $\pi_{\theta_F}=x$
$\mu = \omega_4^\vee$	1,1,1,1,0,0,0,0	μ
$\frac{1}{2}\omega_3^\vee$	$1,1,1,\frac{1}{2},\frac{1}{2},0,0,0$	μ
$\frac{1}{2}\omega_2^\vee$	$1,1,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0$	μ
$\frac{1}{3}\omega_1^\vee+\frac{1}{3}\omega_4^\vee$	$1,\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},0$	$\frac{1}{2}(e_1-e_2+e_3+e_4)$
$\frac{1}{2}\omega_1^\vee$	$1,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	μ
$\frac{1}{2}\omega_4^\vee$	$\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}$	$\frac{1}{2}(e_1-e_2+e_3+e_4)$
$\frac{1}{6}\omega_3^\vee$	$\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{1}{3},\frac{1}{3}$	$\frac{1}{2}(e_1-e_2+e_3-e_4)$
$\frac{1}{3}\omega_3^\vee$	$\frac{5}{6},\frac{5}{6},\frac{5}{6},\frac{1}{2},\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}$	does not exist
0	$\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$	any elt. of $W\cdot\mu$

The partial ordering of $C^\mu_{F,\mathbb{Z}}$ is as follows:

$$\mu \succ \tfrac{1}{2}\omega_3^\vee \succ \tfrac{1}{2}\omega_2^\vee \succ \tfrac{1}{3}\omega_1^\vee + \tfrac{1}{3}\omega_4^\vee \succ$$

$$\begin{matrix} \succ \tfrac{1}{2}\omega_1^\vee \\ \succ \tfrac{1}{3}\omega_3^\vee \succ \tfrac{1}{2}\omega_4^\vee \end{matrix} \succ \tfrac{1}{6}\omega_3^\vee \succ 0$$

The element $\frac{1}{3}\omega_3^\vee$ is not in $C^{\mu}_{F,R^\vee}$; all others are. Notice that $C^{\mu}_{F,\mathbb{Z}}$ is not ranked as a partially ordered set: Both $\frac{1}{3}\omega_1^\vee + \frac{1}{3}\omega_4^\vee \succ \frac{1}{2}\omega_1^\vee \succ \frac{1}{6}\omega_3^\vee$ and $\frac{1}{3}\omega_1^\vee + \frac{1}{3}\omega_4^\vee \succ \frac{1}{3}\omega_3^\vee \succ \frac{1}{2}\omega_4^\vee \succ \frac{1}{6}\omega_3^\vee$ are maximal chains between $\frac{1}{3}\omega_1^\vee + \frac{1}{3}\omega_4^\vee$ and $\frac{1}{6}\omega_3^\vee$. But $C^{\mu}_{F,R^\vee}$ is ranked, once the offending element $\frac{1}{3}\omega_3^\vee$ is removed from $C^{\mu}_{F,\mathbb{Z}}$.

quasisplit non-split A_2, A_3, A_4

quasisplit A_2

$x \in C_{\mathbb{Z}}^{\nu_1}$	slopes of x	$y{\in}W{\cdot}\mu$ with $\pi_{\theta_F}{=}x$
ν_1	$1{,}1{,}\frac{1}{2}{,}\frac{1}{2}{,}0{,}0$	μ
0	$\frac{1}{2}{,}\frac{1}{2}{,}\frac{1}{2}{,}\frac{1}{2}{,}\frac{1}{2}{,}\frac{1}{2}$	any elt. in $W \cdot \mu$

quasisplit A_3

$x \in C_{\mathbb{Z}}^{\nu_1}$	slopes of x	$y{\in}W{\cdot}\mu$ with $\pi_{\theta_F}{=}x$
ν_1	$1{,}1{,}\frac{1}{2}{,}\frac{1}{2}{,}\frac{1}{2}{,}\frac{1}{2}{,}0{,}0$	μ
$\frac{1}{2}\omega_2^\vee$	$\frac{3}{4}{,}\frac{3}{4}{,}\frac{3}{4}{,}\frac{3}{4}{,}\frac{1}{4}{,}\frac{1}{4}{,}\frac{1}{4}{,}\frac{1}{4}$	μ
0	$\frac{1}{2}{,}\frac{1}{2}{,}\frac{1}{2}{,}\frac{1}{2}{,}\frac{1}{2}{,}\frac{1}{2}{,}\frac{1}{2}{,}\frac{1}{2}$	any elt. in $W \cdot \mu$

$$\nu_1 \ \succ \ \tfrac{1}{2}\omega_2^\vee \ \succ \ 0$$

quasisplit A_4

$x \in C_{\mathbb{Z}}^{\nu_1}$	slopes of x	$y \in W \cdot \mu$ with $\pi_{\theta_F} = x$
ν_1	$1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0$	μ
$\frac{1}{4}(\omega_2^\vee + \omega_3^\vee)$	$\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$	μ
0	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	any elt. in $W \cdot \mu$

$$\nu_1 \succ \frac{1}{4}(\omega_2^\vee + \omega_3^\vee) \succ 0$$