Geometry of Shimura Varieties in Positive Characteristics

by

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Examples of Shimura Varieties

1. Modular Curves

(i) These are quotients of the upper-half plane by congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

(ii) They parameterize elliptic curves with level structures.

2. Siegel Modular Varieties

(i) They are quotients of the Siegel upper- half space by congruence subgroups of $\mathrm{Sp}_{2g}(\mathbb{Z})$.

(ii) They parameterize polarized abelian varieties with level structures.

(iii) The most important of them is $\mathcal{A}_g$, the moduli space of g-dimensional principally polarized abelian varieties.

3. Hilbert Modular Varieties

(i) They are quotients of copies of the upper-half plane

(ii) They parameterize abelian varieties with real multiplication.

4. Picard Modular Surfaces

(i) They are quotients of the unit ball in $\mathbb{C}^2$

(ii) They parameterize abelian surfaces A with endomorphisms by the ring of integers of an imaginary quadratic field K, which operates on $\mathrm{Lie}(A)$ with type $(2,1)$ w.r.t. K.

Properties of Shimura Varieties

1. Input Data (G, X)

The prototype of Shimura varieties is an arithmetic quotient $\Gamma \backslash X$, where

- X is a hermitian symmetric space
- G is a reductive group defined over $\mathbb{Q}$ which operates on X
- Γ is an arithmetic subgroup of G

2. Algebraic Structure

Each arithmetic quotient $\Gamma\backslash X$ has a natural structure as a quasi-projective variety.
(Baily-Borel, 1966)

3. The Tower $Sh(G,X)$

Allowing the arithmetic subgroup Γ to vary, one gets a whole tower of varieties $Sh(G,X)$

$$Sh(G,X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_{\mathbb{Q},f})$$

4. Hecke Correspondences

The group of all finite adelic points $G(\mathbb{A}_{\mathbb{Q},f})$ operates on the tower $Sh(G, X)$. On each $\Gamma\backslash X$ this action induces a large family of algebraic correspondences.

5. Canonical Models

The tower $Sh(G, X)$ is naturally defined over a number field $E(G, X)$, called the the *reflex field*. (Shimura, Langlands, Deligne, Milne, Borovoi) On each finite level, $\Gamma\backslash X$ is defined over a specific extension of $E(G, X)$.

6. Special Points

Also called *CM Points*; they are abundant on $Sh(G, X)$. On the Siegel space they corresponds to *abelian varieties with complex multiplication*.

7. Reciprocity Law

These come from the action of the Galois group on $\pi_0(Sh(G, X))$ and on the special points.

8. Compactification

There are many compactifications of $M = \Gamma\backslash X$ (when it is non-compact):

(a) the minimal compactification M^* (Baily-Borel); it is a projective variety.

(b) the Borel-Serre compactification $\overline{M} \to M^*$; it is a manifold with corners.

(c) *many* compactifications of Satake-type between $\overline{M}$ and M^*.

(d) *toroidal compactifications* $\widetilde{M}$ of M (Mumford, Ash, Rapoport, Tai); they are explicit resolution of singularities of M^* and form a tower.

In general there are no natural maps between $\overline{M} \to M^*$ and $\widetilde{M} \to M^*$; but consult recent results of Goresky and Tai.

Why Bother?

I. Have Information from Symmetry

- The cohomologies $\mathrm{H}^*(Sh(G,X),\mathbb{C})$ can be computed by Lie algebra cohomology

- Can understand $\mathrm{H}^*(Sh(G,X),\mathbb{C})$ as a representation of $G(\mathbb{A}_{\mathbb{Q},f})$.

- The symmetries from $G(\mathbb{A}_{\mathbb{Q},f})$ give information about the étale cohomologies $\mathrm{H}^*(Sh(G,X),\mathbb{Q}_\ell)$.

II. Targeted Search Area

Cohomologies of Shimura varieties are fertile searching ground for Galois representations. (*Think about Fermat!*)

They supply the source for constructing Galois representations with desirable properties in view of *Langlands Correspondence*.

III. They “Are” Moduli of Motives

Reduction of Shimura Varieties, Motivation

(1) One wants to study Shimura varieties over the ring of integers $\mathcal{O}_E$ of the reflex field.

(2) The ramification behavior of $Sh(X, G)$ near a prime of the $\mathcal{O}_E$ is heavily influenced by the reduction.

(3) The p-adic properties of motives (e.g. abelian varieties) are closely related to their moduli space, often Shimura varieties.

Expected (Basic) Properties of Shimura Varieties over Integers

(1) Have smooth canonical integral models over the integer ring of the reflex field, outside a specific set of ramified primes.

(2) The Hecke correspondences coming from the prime-to-p part $G(\mathbb{A}_{\mathbb{Q},f}^{\mathrm{non-p}})$ extends to the canonical integral models outside of p for *good* primes p

(3) Have a good compactification theory, including both the minimal and the toroidal compactifications.

(4) For a finite place v above p, the reduction of a Shimura variety M modulo v has natural stratification(s).

- This comes from the p-adic properties of motives over M.
- Example: the stratification by slopes of Frobenius.

What Are Known

1. Confirmed for the Siegel modular varieties. The compactification problem was documented by Faltings and CCL.

2. Known for those of "PEL-type" These varieties parametrized polarized abelian varieties with prescribed endomorphism ring and level structures.

General Status:

Not completely settled; lots of recent progress are made.

Fine Structure of the Reduction

I. Properties of the Newton Stratification

The closed (smallest) Newton stratum in $\mathcal{A}_g$ corresponds to supersingular abelian varieties, i.e. all slopes are equal to $\frac{1}{2}$.

Theorem 1. *(Li-Oort) The supersingular stratum in $\mathcal{A}_g$ has dimension $\left\lfloor \frac{g^2}{4} \right\rfloor$.*

Questions:

(1) What does this number $\left\lfloor \frac{g^2}{4} \right\rfloor$ "really mean"?

(2) What about other Shimura varieties?

(3) Is there a simple formula for the dimension of various Newton strata?

The answer is "YES":

There is a simple formula in terms of roots and weights (CCL), which *predicts* the dimension of the Newton strata of Shimura varieties at good reduction primes. For $\mathcal{A}_g$, this formula is given by Oort in a different form (counting lattice points).

Illustration of the formula

$$\frac{g(g+1)}{2} - \left\lfloor \frac{g^2}{4} \right\rfloor \qquad \text{(this is the codim)}$$

$$= \lceil \tfrac{1}{2} \rceil + \lceil \tfrac{2}{2} \rceil + \cdots + \lceil \tfrac{g}{2} \rceil$$

$$= \lceil \langle \omega_1, \mu \rangle \rceil + \cdots + \lceil \langle \omega_g, \mu \rangle \rceil$$

- $\omega_1 = x_1, \ldots, \omega_g = x_1 + \cdots + x_g$ are the fundamental weights.
- $\mu = \frac{1}{2}(e_1 + \cdots + e_g)$ is the fundamental coweight for the Siegel upper-half space.

Status of the Prediction:

(1) Verified for $\mathcal{A}_g$ as a consequence of a general purity result (Oort, de Jong-Oort).

(2) Verified for some Hilbert-Blumenthal varieties (Chia-Fu Yu).

II. Hecke Orbits

- Let M be the reduction of a Shimura variety at a good reduction place v above p.
- Let $\ell \neq p$ be a prime different from p.

Expectation: The ℓ-power Hecke orbit of *any* point in the *generic* (open) Newton stratum of M is Zariski dense in M.

For $\mathcal{A}_g$, the generic Newton stratum corresponds to *ordinary* abelian varieties, i.e. all slopes are either 0 or 1.

Verified Cases (CCL)

- $\mathcal{A}_g$
- PEL-type C
- quasi-split $U(n, 1)$

Tate-Linear Subvarieties of $\mathcal{A}_g$

"*Tate-linear*" $\Longleftrightarrow$ being a formal subtorus of the Serre-Tate formal torus (at an ordinary point)

Properties:

1. (analytic propagation)

Tate-linear at one point $\Longrightarrow$ so at every point.

2. (relation with Hecke orbits) For any Shimura subvariety $M \subset \mathcal{A}_g$, the Zariski closure of the Hecke orbit of any ordinary point of M is Tate-linear.

3. Every Tate-linear subvariety X can be lifted to a (unique) Tate-linear formal subvariety $\widetilde{X}$ over $W(\overline{\mathbb{F}_p})$.

4. (The Main Question) The $\widetilde{X}$ above is **expected** to be algebraic (hence a Shimura subvariety by a theorem of Moonen).

This expectation has been verified for some Shimura varieties, including the Hilbert modular varieties.