How to compute the Lubin-Tate action

Ching-Li Chai

Department of Mathematics
University of Pennsylvania
and
Institute of Mathematics
Academia Sinica

NCTS, Hsinchu, September 12, 2012

Outline

How to compute the Lubin-Tate action
Ching-Li Chai
One dimensional formal group laws
The Lubin-Tate action
Honda's formalism
The universal p-typical formal group law
Universal isomorphism p-typical formal group laws
The big picture
Sketch of the steps
The first test

One dimensional formal group *laws*

a one-dimensional formal group law over a comm. ring R

= a one-dimensional comm. smooth formal group G over R
 + a rigidification $\mathrm{Spf}(R[[x]]) \xrightarrow{\sim} G$

= a formal power series $G(x,y) \in R[[x,y]]$ such that

- $G(x,y) = G(y,x)$
- $G(x,y) \equiv x + y \pmod{\text{degree } \geq 2}$
- $G(x, G(y,z)) = G(G(x,y), z)$

A homomorphism from $G(x,y)$ to $F(x,y)$ over R is (represented by) a formal power series $\phi(x) \in R[[x]]$ such that

$$F(\phi(x), \phi(y)) = \phi(G(x,y))$$

The *height* of a one-dimensional formal group

Let $k \supset \mathbb{F}_p$ be a field of char. $p > 0$. Let $G(x,y)$ be a one-dim. formal group law over k.

$$[p]_G(x) = \begin{cases} 0 & \text{height} = \infty \\ x^{p^h} \pmod{x^{p^h+1}} & \text{height} = h \end{cases}$$

If $k = k^{\mathrm{alg}}$, then $G(x,y)$ is determined by its height up to non-unique isomorphisms.

Examples.

- $\mathbb{G}_a(x,y) = x + y$, height = ∞ in char. $p > 0$
- $\mathbb{G}_m(x,y) = x + y + xy$, height = 1 in char. p.

Notation: a Lubin-Tate formal group

Let h be a positive integer, fixed from now on. Let $\kappa_s = \mathbb{F}_{p^h}$.

- $g_h(x) := \sum_{j\in\mathbb{N}} p^{-j} x^{p^{jh}} = x + \frac{x^{p^h}}{p} + \frac{x^{p^{2h}}}{p^2} + \cdots$
- Define $G_{W(\kappa_s)} \in \mathbb{Z}_{(p)}[[x,y]] \subset W(\kappa_s)[[x,y]]$ by

$$G_{W(\kappa_s)}(x,y) := g_h^{-1}(g_h(x) + g_h(y))$$

  Remark. $G_{W(\kappa_s)}$ is a Lubin-Tate formal group for $W(\mathbb{F}_{p^h})$: $\mathrm{End}_{W(\kappa_s)}(G_{W(\kappa_s)}) \simeq W(\mathbb{F}_{p^h})$
- Let G_s be the closed fiber of $G_{W(\kappa_s)}$; it is a one-dimensional formal group (law) over $\mathbb{F}_p$ of height h.
- It is well-known that $\mathrm{End}_{\kappa_s}(G_s)$ is the maximal order of $\mathrm{End}^0_{\kappa_s}(G_s)$ = a central division algebra over $\mathbb{Q}_p$ of dimension h^2. So $\mathrm{Aut}_{\kappa_s}(G_s) = \mathrm{End}_{\kappa_s}(G_s)^\times$ is a compact h^2-dimensional p-adic group with center $\mathbb{Z}_p^\times$.

The Lubin-Tate deformation functor

Let $\mathbf{Art}_{\kappa_s}$ be the category whose objects consists of pairs $(R, \varepsilon : \kappa_s \to \kappa)$, where

- R is an Artinian commutative local ring,
- $\kappa = R/\mathfrak{m}_R$,
- ε is a ring homomorphism.

The deformation functor

$$\mathscr{D}ef(G_s) : \mathbf{Art}_{\kappa_s} \longrightarrow \mathbf{Sets}$$

sends each object $(R, \varepsilon : R \to \kappa)$ of $\mathbf{Art}_{\kappa_s}$ to the set of all isomorphism classes of pairs

$$\Big(\Phi, \psi : \Phi \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\kappa) \xrightarrow{\sim} \Phi_s \times_{\mathrm{Spec}(\kappa_s),\varepsilon} \mathrm{Spec}(\kappa)\Big),$$

where Φ is a one-dimensional formal group over R.

The Lubin-Tate moduli space

Equivalently, $\mathscr{D}ef(G_s)(R,\varepsilon)$ is the set of all $*$-isomorphism classes of one-dimensional formal group laws Φ over R whose closed fiber is $\varepsilon_* G_s$.

Recall: An isomorphism $\phi(x)$ from G_1 to G_2 over R is a $*$-isomorphism if $\phi(x) \equiv x \pmod{\mathfrak{m}_R}$.

Fact. $\mathscr{D}ef(G_s)$ is representable by a formal scheme $\mathscr{M}_h$ which is formally smooth over $W(\kappa_s)$ of relative dimension $h-1$. In other words there is a universal one-dimensional deformation Φ_{univ} over $\mathscr{M}_h$ such that every deformation of G_s over (R,ε) is the pull-back of Φ_{univ} via a unique morphism $\mathrm{Spf}(R) \to \mathscr{M}_h$.

The Lubin-Tate action

How to compute the Lubin-Tate action
Ching-Li Chai
One dimensional formal group laws
The Lubin-Tate action
Honda's formalism
The universal p-typical formal group law
Universal isomorphism p-typical formal group laws
The big picture
Sketch of the steps
The first test

The compact p-adic group $\mathrm{Aut}(G_0) = \mathrm{End}(G_0)^\times$ operates on $\mathscr{M}_h$ by "change of marking", as follows:

$$\gamma\colon [(\Phi,\psi)] \mapsto [(\Phi, \gamma\circ\psi)] \qquad \forall [(\Phi,\psi)] \in \mathscr{D}(G_s)((R,\varepsilon))$$

for any $\gamma \in \mathrm{Aut}(G_s)$ and any object (R,ε) in $\mathbf{Art}_{\kappa_s}$.

Equivalently, applying the above to the universal deformation Φ_{univ}: $\forall\,\gamma \in \mathrm{Aut}(G_s)$, we have a commutative diagram

$$\begin{array}{ccc}
\Phi_{\mathrm{univ}} & \xrightarrow{\tilde{\xi}(\gamma)} & \Phi_{\mathrm{univ}} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
\mathscr{M}_h/\mathrm{Spec}(W(\kappa_s)) & \xrightarrow{\xi(\gamma)} & \mathscr{M}_h/\mathrm{Spec}(W(\kappa_s))
\end{array}$$

where $\xi(\gamma)$ is an automorphism of $\mathscr{M}_h$ and $\tilde{\xi}(\gamma)$ is a formal group isomorphism with $\tilde{\xi}(\gamma)|_{G_s} = \gamma$.

The Lubin-Tate action, continued

Remark. 1. This action $\gamma \mapsto \rho(\gamma)$ of $\mathrm{Aut}(G_0)$ on the Lubin-Tate moduli space $\mathscr{M}_h$ was first studied by Lubin and Tate in 1966. It is also known as (the essential part of) the *Morava stabilizer subgroup* action in chromatic homotopy theory.

2. If one passes to the divided power envelope

$$W(\kappa_s)[[w_1,\ldots,w_{h-1}]][w_i^n/n!]_{n\in\mathbb{N},i\le h-1}$$

of the coordinate ring $W(\kappa_s)[[w_1,\ldots,w_{h-1}]]$, one can "linearize" the action by crystalline theory.
However this is not very useful for studying the action of $\mathrm{Aut}(G_s)$ on the characteristic p fiber of $\mathscr{M}_h$.

Set-up with a Frobenius lifting

Fix a prime number p. Our base ring A is cast in a quadruple $(A, K, \mathfrak{a}, \sigma)$, where

- K is a *commutative* ring with 1,
- A is a subring of K containing 1,
- $\mathfrak{a} \subset A$ is an ideal of A, and
- $\sigma : K \to K$ is a ring endomorphism,

such that conditions (a)–(c) below are satisfied.

(a) $p \in \mathfrak{a}$,

(b) $\sigma(A) \subset A$,

(c) $\sigma(a) \equiv a^q \pmod{\mathfrak{a}}$ for all $a \in A$,

(d) $\sigma((A:\mathfrak{a})) \subset (A:\mathfrak{a})$, where $(A:\mathfrak{a}) := \{y \in K \mid y\cdot\mathfrak{a} \subset A\}$.

A twisted power series ring

Let $K[[\partial]]_\sigma$ be the ring of formal power series in ∂ with coefficients in K such that $\partial b = \sigma(b)\partial$ for all $b \in K$, i.e.

$$\left(\sum_{j\in\mathbb{N}} b_j \cdot \partial^j\right) \cdot \left(\sum_{i\in\mathbb{N}} c_i \cdot \partial^i\right) = \sum_{k\in\mathbb{N}} \left(\sum_{j+i=k} b_j \cdot \sigma^j(c_i)\right) \cdot \partial^k .$$

Define a left action of the ring $K[[\partial]]_\sigma$ on power series rings $K[[\underline{t}]]^n$ by

$$\left(\Big(\sum_{j\in\mathbb{N}} b_j \cdot \partial^j\Big) * g \right)(\underline{t}) = \sum_{j\in\mathbb{N}} b_j \cdot (\sigma_*^j g)(t_1^q, \cdots, t_m^q)$$

Functional equations

- An element $u \in K[[\partial]]_\sigma$ is a *special element* if u has the form
$$u = 1 - \sum_{j\geq 1} s_j \cdot \partial^j, \quad s_j \in K \ \ \forall j \geq 1$$
such that $\mathfrak{a} \cdot s_j \subset A \ \forall j \geq 1$.
- An element $h(x) \in K[[\underline{t}]]$ is *u-integral* if $u * h \in A[[\underline{t}]]$.
- An element $f(x) \in K[[x]]_0$ (i.e. $f(0) = 0$) is said to be *regular u-integral* if $u * f \in A[[x]]$ and $f'(0) \in A^\times$. In other words $f(x)$ satisfies a "functional equation" of the form
$$f(x) = g(x) + \sum_{j\geq 1} s_j \cdot (\sigma_*^j f)(x^{q^j})$$
with $g(x) \in A[[x]]$, $g(0) = 0$, and $g'(0) \in A^\times$.

Functional equation lemma

Proposition. Let $u \in K[[\partial]]_\sigma$ be a special element. Let $f(\underline{x})$ be a *regular* u-integral element in $K[[x]]_0$

(1) The element $F(\underline{x},\underline{y}) := f^{-1}(f(x)+f(y)) \in K[[x,y]]$ belongs to $A[[x,y]]$. Hence $F(x,y)$ is a formal group law over A.

(2) For any u-integral element $h(\underline{t}) \in K[[\underline{t}]]_0 = K[[t_1,\dots,t_m]]_0$, the element $f^{-1}(h(\underline{t})) \in K[[\underline{t}]]_0$ belongs to $A[[\underline{t}]]_0$.

(3) Suppose that $\alpha(\underline{z}) \in A[[\underline{z}]]_0 = A[[z_1,\dots,z_k]]_0$, $\beta(\underline{z}) \in K[[\underline{z}]]_0 = K[[z_1,\dots,z_k]]_0$. Then for all $r \geq 1$

$$\alpha(\underline{z}) \equiv \beta(\underline{z}) \pmod{\mathfrak{a}^r} \quad \text{iff} \quad f(\alpha(\underline{z})) \equiv f(\beta(\underline{z})) \pmod{\mathfrak{a}^r}.$$

(4) $\forall \psi \in A[[\underline{t}]]_0$, the element $f(\psi(\underline{t})) \in K[[\underline{t}]]_0$ is u-integral.

(5) $\forall\, v \in A[[\partial]]_\sigma$ and $\forall\, \psi \in A[[\underline{t}]]_0$ we have

$$v*(f\circ\psi) \equiv (v*f)\circ\psi \pmod{\mathfrak{a}\cdot A[[\underline{t}]]}$$

Essential uniqueness of functional equation

Proposition. Let u be a special element of $K[[\partial]]_\sigma$ and let $f(x) \in K[[x]]$ be regular u-integral. Suppose that

$$v = \sum_{j\in\mathbb{N}} \tilde{s}_j \cdot \partial^j \ \in K[[\partial]]_\sigma, \quad \tilde{s}_j \in (A:\mathfrak{a})\ \forall j$$

and f is v-integral. Then $\exists!\, c \in A[[\partial]]_\sigma$ such that $v = c\cdot u$ in $K[[\partial]]_\sigma$. In particular if v is also a special element, then $v \in A[[\partial]]_\sigma^\times \cdot u$.

How to construct homomorhisms over $A/\mathfrak{a}$

Let u, v be special elements in $K[[\partial]]_\sigma$. Let $f, g \in K[[x]]_0$ be regular u-integral and v-integral respectively. Let F and G be the formal group laws over A with logarithms f and g.

Proposition. For any $c \in A[[\partial]]_\sigma$, let $\phi_{g,c,f}(x) := g^{-1}(c * f)$.

(1) $\phi_{g,c,f}(x) \in A[[x]]_\sigma$ iff $\exists d \in A[[\partial]]_\sigma$ such that $v \cdot c = d \cdot u$.

(2) If $v \cdot c = d \cdot u$ and $d \in A[[\partial]]_\sigma$, then the image of $\phi_{f,g,c}$ in $(A/\mathfrak{a})[[x]]$ defines an $(A/\mathfrak{a})$-homomorphism

$$[\phi_{g,c,f}] : (F \mod \mathfrak{a}) \to (G \mod \mathfrak{a}).$$

(3) Suppose that w-is a special element in $K[[\partial]]_\sigma$, $h \in K[[x]]_0$ is w-regular and H is the formal group law over A with logarithm h. Suppose $c, d \in A[[\partial]]_\sigma$ and $w \cdot c' = d' \cdot v$. Then

$$[\phi_{h,c',g}] \circ [\phi_{g,c,f}] = [\phi_{h,c'c,f}].$$

The universal p-typical formal group law

Let $\tilde{R} = \mathbb{Z}_{(p)}[\underline{v}] = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots]$, and let $\sigma : \tilde{R} \to \tilde{R}$ be the ring homomorphism such that $\sigma(v_j) = v_j^p$ for all $j \geq 1$

Let $G_{\underline{v}}(x) \in \tilde{R}[[x, y]]$ be the one-dimensional p-typical formal group law over $\tilde{R}$ whose logarithm

$$g_{\underline{v}}(x) \in \tilde{R}[1/p][[x]] = \sum_{n \geq 1} a_n(\underline{v}) \cdot x^{p^n}$$

satisfies

$$g_{\underline{v}}(x) = x + \sum_{i=1}^{\infty} \frac{v_i}{p} \cdot g_{\underline{v}}^{(\sigma^i)}(x^{p^i})$$

Remarks on the formal group law $G_{\underline{v}}$

Remarks. (1) The above "functional equation" is a recursive formula for the coefficients $a_n(\underline{v}) \in p^{-n} \cdot \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_n]$ of $g_{\underline{v}}(x)$.

(2) Explicitly:

$$a_n(\underline{v}) = \sum_{\substack{i_1, i_2, \ldots, i_r \geq 1 \\ i_1 + \cdots + i_r = n}} p^{-r} \cdot \prod_{s=1}^{r} v_{i_s}^{p^{i_1 + i_2 + \cdots + i_{s-1}}}$$

$$= \sum_{\substack{i_1, i_2, \ldots, i_r \geq 1 \\ i_1 + \cdots + i_r = n}} p^{-r} \cdot v_{i_1} \cdot v_{i_2}^{p^{i_1}} \cdot v_{i_3}^{p^{i_1 + i_2}} \cdots v_{i_r}^{p^{i_1 + \cdots + i_{r-1}}}$$

Note that $a_n(\underline{v})$ is a homogeneous polynomial in $v_1, \ldots, v_n$ of weight $p^n - 1$ when v_j is given the weight $p^j - 1$ $\forall j \geq 1$.

(3) The formal group law $G_{\underline{v}}$ over $\tilde{R}$ is "the" universal one-dimensional p-typical formal group law.

The universal formal group over $\mathscr{M}_h$ made explicit

Let $R = R_h = W(\overline{\mathbb{F}}_p)[[w_1, w_2, \ldots, w_{h-1}]]$.

Let $\pi = \pi_h : \tilde{R} \to R$ be the ring homomorphism such that

$$\pi(v_i) = \begin{cases} w_i & \text{if} \quad 1 \leq i \leq h-1 \\ 1 & \text{if} \quad i = h \\ 0 & \text{if} \quad i \geq h+1 \end{cases}$$

The classifying morphism $\mathrm{Spf}(R) \to \mathscr{M}_h$ for the deformation $\pi_* G_{\underline{v}}$ of G_0 is an isomorphism.

We will identify $\mathscr{M}_h$ with $\mathrm{Spf}(R)$ and the universal deformation G_{univ} of G_0 with the formal group underlying the formal group law $G_R := \pi_* G_{\underline{v}}$.

The universal strict isomorphism

Let $\mathbb{Z}_{(p)}[\underline{v},\underline{t}] = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots; t_1, t_2, t_3, \ldots]$, and let $\sigma : \mathbb{Z}_{(p)}[\underline{v},\underline{t}] \to \mathbb{Z}_{(p)}[\underline{v},\underline{t}]$ be the obvious Frobenius lifting as before, with $\sigma(v_i) = v_i^p$ and $\sigma(t_i) = t_i^p$ $\forall i \geq 1$.

Let $G_{\underline{v},\underline{t}}(x,y)$ be the one-dimensional formal group law over $\mathbb{Z}_{(p)}[\underline{v},\underline{t}]$ whose logarithm $g_{\underline{v},\underline{t}}(x)$ satisfies

$$g_{\underline{v},\underline{t}}(x) = x + \sum_{i=1}^{\infty} t_i \cdot x^{p^i} + \sum_{j=1}^{\infty} \frac{v_j}{p} \cdot g_{\underline{v},\underline{t}}^{(\sigma^j)}(x^{p^j})$$

The universal strict isomorphism, continued

How to compute the Lubin-Tate action
Ching-Li Chai

It is known that $\alpha_{\underline{v},\underline{t}} := g_{\underline{v},\underline{t}}^{-1} \circ g_{\underline{v}} \in \mathbb{Z}_{(p)}[\underline{v},\underline{t}][[x]]$, and defines a *strict isomorphism*

$$\alpha_{\underline{v},\underline{t}} : G_{\underline{v}} \to G_{\underline{v},\underline{t}}$$

between p-typical formal group laws over $\mathbb{Z}_{(p)}[\underline{v},\underline{t}]$.
(A *strict* isomorphism is an isomorphism between formal group laws which is $\equiv x$ modulo higher degree terms in x.)

Moreover $\alpha_{\underline{v},\underline{t}}$ is "the" universal strict isomorphism between one-dimensional p-typical formal group laws.

Parameters of $G_{\underline{v},\underline{t}}$

By the universality $G_{\underline{v}}$ for p-typical formal group laws, there exists a unique ring homomorphism

$$\eta : \mathbb{Z}_{(p)}[\underline{v}] \to \mathbb{Z}_{(p)}[\underline{v}, \underline{t}]$$

such that

$$\eta_* G_{\underline{v}} = G_{\underline{v},\underline{t}}.$$

The elements

$$\overline{v}_n = \overline{v}_n(\underline{v}, \underline{t}) \in \mathbb{Z}_{(p)}[\underline{v}, \underline{t}], \quad n \in \mathbb{N}_{\geq 1}$$

are the *parameters* of the p-typical formal group law $G_{\underline{v},\underline{t}}$.

A known recursive formula for the parameters of $G_{\underline{v},\underline{t}}$

How to compute the Lubin-Tate action

Ching-Li Chai

$$\begin{aligned}\overline{v}_n = v_n + p\, t_n + \sum_{\substack{i+j=n\\ i,j\geq 1}} \left(v_j\, t_i^{p^j} - t_i\, \overline{v}_j^{p^i}\right) \\ + \sum_{j=1}^{n-1} a_{n-j}(\underline{v}) \cdot \left(v_j^{p^{n-j}} - \overline{v}_j^{p^{n-j}}\right) \\ + \sum_{k=2}^{n-1} a_{n-k}(\underline{v}) \cdot \sum_{\substack{i+j=k\\ i,j\geq 1}} \left(v_j^{p^{n-k}} t_i^{p^{n-i}} - t_i^{p^{n-k}} \overline{v}_j^{p^{n-j}}\right)\end{aligned}$$

(This formula contains high power of p in the denominators. Consequently it is not very useful for our purpose.)

An *integral* recursion formula for $\bar{v}_n(\underline{v}, \underline{t})$

(useful for computing the Lubin-Tate action)

$$
\begin{aligned}
\bar{v}_n = {} & v_n + p\,t_n - \sum_{j=1}^{n-1} t_j \cdot \bar{v}_{n-j}^{p^j} + \\
& + \sum_{l=1}^{n-1} v_l \sum_{k=1}^{n-l-1} \frac{1}{p} \cdot a_{n-k-l}(\underline{v})^{(p^l)} \cdot \Big\{ (\bar{v}_k^{(p^l)})^{p^{n-l-k}} - (\bar{v}_k^{p^l})^{p^{n-l-k}} \\
& \qquad + \sum_{\substack{i+j=k \\ i,j\geq 1}} t_j^{p^{n-k}} \Big[(\bar{v}_i^{(p^l)})^{p^{n-l-i}} - (\bar{v}_i^{p^l})^{p^{n-l-i}} \Big] \Big\} \\
& + \sum_{l=1}^{n-1} v_l \cdot \Big\{ \frac{1}{p} (\bar{v}_{n-l}^{(p^l)} - \bar{v}_{n-l}^{p^l}) + \sum_{\substack{i+j=n-l \\ i,j\geq 1}} t_j^{p^l} \cdot \frac{1}{p} \cdot \Big[(\bar{v}_i^{(p^l)})^{p^j} - (\bar{v}_i^{p^l})^{p^j} \Big] \Big\}
\end{aligned}
$$

for every $n \geq 1$.

The groupoid underlying the universal strict isomorphism

Let $X_0 := \mathrm{Spec}(\tilde{R})$, $X_1 := \mathrm{Spec}(\tilde{\Gamma})$. Consider the diagram

$$X_0 \xleftarrow{\text{source}} X_1 \xrightarrow{\text{target}} X_0,$$

where the *source* arrow corresponds to $\tilde{R} \hookrightarrow \tilde{\Gamma}$ and the *target* arrow corresponds to the ring homomorphism $\eta : \tilde{R} \to \tilde{\Gamma}$. The above diagram is part of a natural groupoid structure such that the (partial) product

$$\mu : X_1 \times_{s, X_0, t} X_1 \to X_1$$

corresponds to composition of strict isomorphisms between p-typical formal group laws.

Three ways to think about the moduli space $\mathscr{M}_h$

1. All p-typical deformations of G_s whose parameters satisfy $v_h = 1$, $v_{h+1} = v_{h+2} = \cdots = 0$.

2. All p-typical deformations of G_s whose parameters satisfy $v_{h+1} = v_{h+2} = \cdots = 0$, up to/modulo *scaling by units*

3. All p-typical deformations of G_s, modulo the equivalence relations generated by
   - strict $*$-isomorphisms, and
   - scaling by units.

Rough idea

Start with an element $\gamma \in \mathrm{Aut}(G_s)$.

1. Use Honda's formalism to construct an isomorphism $\psi_\gamma : F_\gamma \to \phi_* G_{\underline{v}}$ in equi-characteristic p whose closed fiber is equal to γ.

2. Compute the parameters $v_1, v_2, v_3, \ldots$ of F_γ. (By recursive relations).

3. Change F_γ by a strict isomorphism with suitable parameters $t_1, t_2, t_3, \ldots$, to a new p-typical formal group law whose (new) parameters satisfy $v_{h+1} = v_{h+2} = \cdots = 0$.
(Implicit function theorem applied to ∞-dimensional spaces)

4. Rescale to make $v_h = 1$.

Step 1

Given an element $\gamma \in \mathrm{Aut}(G_0)$, construct

- a p-typical one-dimensional formal group *law* $F = F_\gamma$ over R whose closed fiber is equal to G_0, and
- an isomorphism

$$\overline{\psi} = \overline{\psi}_\gamma : F_{\overline{R}} \to G_{\overline{R}}$$

over $\overline{R} := R/pR = \overline{\mathbb{F}}_p[[w_1, \ldots, w_{h-1}]]$ whose restriction to the closed fibers is

$$(\psi|_{G_0} : G_0 \to G_0) = \gamma.$$

Here $F_{\overline{R}} = F \otimes_R \overline{R}$, $G_{\overline{R}} = G_R \otimes_R \overline{R}$.

Note that both the formal group law F over R and the isomorphism ψ over $\overline{R}$ depends on the given element $\gamma \in \mathrm{Aut}(G_0)$.

The formal group law F_c, $c \in W(\mathbb{F}_{p^h})^\times$

For $\gamma = [c] \in W(\mathbb{F}_{p^h})^\times = \mathrm{Aut}(G_1)$, we can take F_c to be the formal group over R whose logarithm $g_c(x)$ satisfies

$$f_c(x) = x + \sum_{i=1}^{h} \frac{c^{-1+\sigma^i} \cdot w_i}{p} \cdot f_c^{(\sigma^i)}(x^{p^i})$$

(w_h=1 by convention).

Let

$$\psi_c(x) = \log_{G_R}^{-1} \circ (c \cdot f_c)$$

We have $\psi_c(x) \in R[[x]]$ and ψ_c defines an isomorphism from F_c to G_R over R (not just over $\overline{R}$!) with $\psi_c|_{G_0} = [c]$.

HOW TO COMPUTE THE LUBIN-TATE ACTION
Ching-Li Chai
One dimensional formal group laws
The Lubin-Tate action
Honda's formalism
The universal p-typical formal group law
Universal isomorphism p-typical formal group laws
The big picture
Sketch of the steps
The first test

Step 2

Compute the parameters

$$(u_i = u_i(w_1, \dots, w_{h-1}))_{i \in \mathbb{N}_{\geq 1}}$$

for the p-typical group law $F = F_\gamma$ over R.

The above condition means that

$$\xi_* G_{\tilde{v}} = F,$$

where

$$\xi = \xi_\gamma : \tilde{R} \to R$$

is the ring homomorphism such that

$$\xi(v_i) = u_i \quad \forall i \geq 1.$$

Parameters for F_c, $c \in W(\mathbb{F}_{p^h})^\times$

In the case when $\gamma \in \mathrm{Aut}(G_0)$ lifts to an element [c] with $c \in W(\mathbb{F}_{p^h})^\times \simeq \mathrm{Aut}(G_1)$, we have the following integral recursive formula for the parameters $u_n = u_n(c; \underline{w})$.

$$\begin{aligned} u_n(c;\underline{w}) = c^{-1+\sigma^n} w_n & \\ + \sum_{j=1}^{n-1} c^{-1+\sigma^j} \cdot \frac{1}{p} \left[u_{n-j}(c;\underline{w})^{(p^j)} - u_{n-j}(c;\underline{w})^{p^j} \right] \cdot w_j & \\ + \sum_{j=1ve}^{n-1} \sum_{i=1}^{n-j-1} \frac{1}{p} a_{n-i-j}(\underline{w})^{(p^j)} \cdot c^{-1+\sigma^{n-i}} \cdot & \\ \left[(u_i(c;\underline{w})^{(p^j)})^{p^{n-i-j}} - (u_i(c;\underline{w})^{p^j})^{p^{n-i-j}} \right] \cdot w_j & \end{aligned}$$

where $w_h = 1$, $w_m = 0 \ \forall m \geq h+1$ by convention.

Parameters for F_c, continued

Remark. The above recursive formula for the parameters $u_n(c;\underline{w})$ can be turned into an explicit "path sum" formula for $u_n(c,\underline{w})$, with terms indexed by "paths".

Step 3

Find/compute the uniquely determined element

$$\tau_n \in \mathfrak{m}_R, \quad n \in \mathbb{N}_{\geq 1}$$

and

$$\hat{u}_1 \in \mathfrak{m}_R, \ldots, \hat{u}_{h-1} \in \mathfrak{m}_R, \hat{u}_h \in 1 + \mathfrak{m}_R$$

such that

$$\overline{v}_n(\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_h, 0, 0, \ldots; \underline{\tau}) = u_n \quad \forall n \geq 1.$$

HOW TO COMPUTE THE LUBIN-TATE ACTION

Ching-Li Chai

One dimensional formal group laws
The Lubin-Tate action
Honda's formalism
The universal p-typical formal group law
Universal isomorphism p-typical formal group laws
The big picture
Sketch of the steps
The first test

Remark. (1) The existence and uniqueness statement above is an application the implicit function theorem for an infinite dimensional space over $\tilde{R}$, applied to the "vector-valued" function with components $\overline{v}_n$ in the integral recursion formula discussed before.

(2) This step is a substitute for the operation *taking the quotient of the group "changes of coordinates"* in a space of formal group laws.

(3) The approximate solution coming from the linear term in the τ_j variables is often good enough for our application.

A congruence formula for $\overline{v}_n$

The follow formula helps to explain the last remark.

$$\begin{aligned}\overline{v}_n \equiv v_n - \sum_{j=1}^{n} t_j \cdot v_{n-j}^{p^j} \\ + \sum_{\substack{i,j,t,s_1,s_2,\ldots,s_t\geq 1 \\ s_1+\ldots+s_t+i+j=n}} (-1)^{t-1}\, t_i \cdot v_j^{p^i} \cdot v_1^{(p^{s_1}+p^{s_2}+\cdots+p^{s_t}-t)/(p-1)} \\ \cdot v_{n-s_1}^{p^{s_1}-1} \cdot v_{n-s_1-s_2}^{p^{s_2}-1} \cdots v_{n-s_1-\cdots-s_t}^{p^{s_t}-1} \\ \mathrm{mod}\ (pt_a, t_a\cdot t_b)_{a,b\geq 1}\mathbb{Z}[\underline{v},\underline{t}]\end{aligned}$$

Step 4

Rescale $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_h$ as follows:
$\exists!\ \tau_0 \in \mathfrak{m}_R$ such that

$$(1+\tau_0)^{p^h-1} \cdot \hat{u}_h = 1.$$

Let
$$\hat{v}_i := (1+\tau_0)^{p^i-1} \cdot \hat{u}_i \ \text{ for } i = 1, \ldots, h-1.$$

Let $\omega : \tilde{R} \to R$ be the ring homomorphism such that

$$\omega(v_i) = \hat{u}_i \quad \forall i \geq 1.$$

Let $\rho : R \to R$ be the $W(\overline{\mathbb{F}}_p)$-linear ring homomorphism such that
$$\rho(w_i) = \hat{v}_i \quad \forall i \geq 1.$$

The meaning of Steps 3 and 4

The universal strict isomorphism $\alpha_{\underline{v},\underline{t}}$ specializes to a strict isomorphism
$$\alpha = \alpha_{\underline{\hat{u}},\underline{\tau}} : F \to \omega_* G_{\underline{v}}$$
with $\alpha|_{G_0} = \mathrm{Id}_{G_0}$.

The rescaling in step 4 gives an isomorphism (not necessarily a strict isomorphism)

$$\beta : \omega_* G_{\underline{v}} \to \rho_* G_R$$

with $\beta|_{G_0} = \mathrm{Id}_{G_0}$.

Conclusion

Combined with $\overline{\psi}$, we obtain an isomorphism

$$\overline{\psi}\circ\overline{\alpha}^{-1}\circ\overline{\beta}^{-1}:\overline{\rho}_{*}G_{\overline{R}}\to G_{\overline{R}}$$

whose restriction to the closed fiber G_0 is equal to the given element $\gamma\in\mathrm{Aut}(G_0)$.
(Here $\overline{\alpha}=\alpha\otimes_R\overline{R}$ and $\overline{\beta}=\beta\otimes_R\overline{R}$.)

Conclusion. The given element $\gamma\in\mathrm{Aut}(G_0)$ operates on the equi-characteristic deformation space $\mathrm{Spf}(\overline{R})$ of G_0 via the ring automorphism ρ.
(Notice that $\overline{\psi}$, α and β all depend on γ.)

Local rigidity for the Lubin-Tate moduli space: the first non-trivial case

Proposition. Let $Z\subset\mathscr{M}_{3\,\overline{\mathbb{F}}_p}=\mathrm{Spf}(\overline{\mathbb{F}}_p[[w_1,w_2]])$ be an irreducible closed formal subscheme of $\mathscr{M}_3$ over $\overline{\mathbb{F}}_p$ corresponding to a hight one prime ideal of $\overline{\mathbb{F}}_p[[w_1,w_2]]$.
If Z is stable under the action of an open subgroup of $W(\mathbb{F}_{p^3})^{\times}$, then $Z=\mathrm{Spf}(\overline{\mathbb{F}}_p[[w_1,w_2]]/(w_1))$.