Hecke Orbits and Canonical Coordinates

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RMK *Fine structures* occur only in char. p.

Some modular varieties

- (Siegel modular variety) $\widetilde{X} = (X_n)$, $X_n = \mathcal{A}_{g,n}$, $(n, p) = 1$, where $\mathcal{A}_{g,n}$= the moduli space of n-dimensional principally polarized abelian varieties with symplectic level-n structure.

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- (Hilbert modular variety) $X_n = \mathcal{M}_{E,n}$, where E is a product of totally real number fields, $(n,p) = 1$, $\mathcal{M}_{E,n}$ classifies $[E:\mathbb{Q}]$-dimensional abelian varieties, with endomorphism by $\mathcal{O}_E$, and an $\mathcal{O}_E$-linear level-n structure.

Hecke symmetries

- The group $G(\mathbb{A}_f^{(p)})$ operates on the tower $\widetilde{X}$:

$$\begin{array}{l} \curvearrowleft G(\mathbb{A}_f^{(p)}) \\ \widetilde{X} = \big(\cdots \longrightarrow \underbrace{X_n \longrightarrow \cdots X_0 = X}_{G(\mathbb{Z}/n\mathbb{Z})}\big)_{(n,p)=1} \end{array}$$

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- On a fixed level, e.g. $X = X_0$, the symmetries from $G(\mathbb{A}_f^{(p)})$ induces *Hecke correspondences*.

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EX 2. (Hilbert) For $x = [(A_x, \lambda_x)] \in \mathcal{M}_E(k)$, A_x an $\mathcal{O}_E$-abelian variety, $\mathcal{H}(x)$ consists of all $[(A_y, \lambda_y)]$ s.t. $\exists$ a prime-to-p $\mathcal{O}_E$-linear quasi-isogeny $A_x \to A_y$ respecting the polarizations.

Barsotti-Tate groups

DEF. A *Barsotti-Tate* group (or, a p-divisible group) G over a scheme S of *height* h is a systems of finite locally free group schemes G_n over S, $n \geq 1$, together with inclusions

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- G_n is killed by $p^n \ \forall\, n$
- G_n is locally free of rank hn over $S \ \forall\, n$
- $[p^m] : G_{n+m} \longrightarrow G_n$ is faithfully flat, with G_m as its kernel, $\forall\, m,\ n$

p-divisible groups attached to abelian varieties

EX. Let $A \to S$ be an abelian scheme, $\dim(A/S) = g$. Then the p^n-torsion subgroups $A[p^n] := \mathrm{Ker}([p^n]_A)$ form a BT-group $A[p^\infty]$ of height $2g$.

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Can recover the formal group attached to $A \to S$ if S is over $\mathbb{Z}_{(p)}$ or $\mathbb{F}_p$.

$A[p^\infty]$ is a form of the p-adic cohomology of A.

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RMK. slope $= 0 \Leftrightarrow$ étale; slope $= 1 \Leftrightarrow$ multiplicative.

Dual BT-groups and abelian varieties

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If A is an abelian variety over a field $k \supset \mathbb{F}_p$, the A and its dual abelian variety A^t have the same slope sequence: $A^t[p^\infty] \cong A[p^\infty]^t$.

Stratification by slopes

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The other possible slope sequences for $\mathcal{A}_3$ are $(0, 0, \frac{1}{2}, \frac{1}{2}, 1, 1)$, $(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1)$ and $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})$. The last NP-stratum in $\mathcal{A}_3$ is 3-dimensional.

Leaves in Siegel modular varieties

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RMK. Can define leaves for any (polarized) Barsotti-Tate group over a noetherian reduced base scheme over k.

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- $\mathcal{C}(x)$ is stable under all prime-to-p Hecke correspondences.

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- $\dim(\mathcal{C}(x)) = 0$ iff x is supersingular.
- Every leaf in the NP-stratum with slope sequence $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ is two-dimensional. So the leaves "have moduli".

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Conj. $(\text{HO})_{\text{dc}}$: The prime-to-p Hecke correspondences operate transitively on the set of geometrically irreducible components of $\mathcal{C}(x)$.

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- Conj. (HO) says that foliation structure on $\mathcal{A}_g$ is determined by the Hecke symmetries.
- Previously known case: When A_x is an ordinary abelian variety (CLC, 1995)
- Similarly, have Conj (HO) for other modular varieties. Known case: PEL-type C, A_x ordinary.

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RMK. Proof of Thm 1 in planned monograph with F. Oort, proof of the cont. part of Thm 2 in preparation with C.-F. Yu.

The slope filtration over a leaf

PROP. The Barsotti-Tate group $A[p^\infty]$ over $\mathcal{C}(x)$ admits a slope filtration

$$A[p^\infty] = G_0 \supset G_1 \supset \cdots \supset G_m \supset G_{m+1} = (0)$$

such that each graded piece $H_i = G_i/G_{i+1}$ is a Barsotti-Tate group over $\mathcal{C}(x)$ with a single slope λ_i, and $\lambda_0 < \ldots < \lambda_m$.

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Analogy: The variation of the Hodge filtration gives the local moduli of abelian varieties.

Canonical coordinates on leaves

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- **a perspective**: Every leaf in a PEL-type modular variety is a char. p analog of a Shimura variety; it is "homogeneous", and has similar group-theoretic properties.

Two-slope case: unpolarized

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THM. $\mathcal{C}(\mathfrak{D}ef(X,Y))$ is the maximal p-divisible formal subgroup $\mathcal{DE}(X,Y)_{\mathrm{pdiv}}$ of $\mathcal{DE}(X,Y)$; it is isoclinic, of slope $\mu_Y - \mu_X$, and height $\mathrm{ht}(X) \cdot \mathrm{ht}(Y)$.

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- $\lambda =$ a principal polarization of $X \times Y$
- $\mathfrak{D}ef(X \times Y, \lambda) =$ local deform. space of $(X \times Y, \lambda)$

Two-slope case: polarized

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- X, Y: isoclinic BT-groups over $k \supset \mathbb{F}_p$, of Frobenius slopes $\mu_X,\ \mu_Y = 1 - \mu_X$, $\mu_X < \mu_Y$, $\mathrm{ht}(X) = \mathrm{ht}(Y)$.
- $\lambda =$ a principal polarization of $X \times Y$
- $\mathfrak{Def}(X \times Y, \lambda) =$ local deform. space of $(X \times Y, \lambda)$
- $\mathcal{C}(\mathfrak{Def}(X \times Y, \lambda)) :=$ the leaf in $\mathfrak{Def}(X \times Y, \lambda)$ through the closed point.

Polarized two-slope case, continued

THM. (i) The polarization λ induces an involution on $\mathcal{DE}(X,Y)_{\mathrm{pdiv}}$, whose fixer subscheme $\mathcal{DE}(X,Y)^{\mathrm{sym}}_{\mathrm{pdiv}}$ is equal to $\mathcal{C}(\mathfrak{D}ef(X \times Y, \lambda))$.

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(iv) If $x=[(A_x,\lambda_x)]\in\mathcal{A}_g(k)$, $(A_x[p^\infty],\lambda_x[p^\infty])\cong(X\times Y,\lambda)$, then

$$\mathcal{C}(x)^{/x}\cong \mathrm{M}(\mathcal{DE}(X,Y)^{\mathrm{sym}}_{\mathrm{pdiv}}).$$

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- $W(k) =$ Witt vectors over k

Cartier theory continued

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- $\mathrm{M}(G)$ is the set of all p-typical formal curves in the smooth formal group G.

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$$\mathrm{rk}_W(H_1^{\mathrm{sym}}) = \frac{\mathrm{ht}(X)(\mathrm{ht}(X)+1)}{2}.$$

Rigidity of subgroups of p-divisible formal groups

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- Z is an irreducible closed formal subscheme of X which is stable under the action of U.

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RMK. A basic case is when X is a formal torus $\mathrm{Spf}(k[[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]])$. The Thm says that if an irreducible formal subvariety Z of a formal torus is stable under $[1 + p^n]$ for some $n \geq 1$, then Z is a formal subtorus.

Proof of a special case of Conj. (HO)

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- $A_x[p^\infty] \cong X \times Y$, with X, Y isoclinic, $\mu_X < \mu_Y = 1 - \mu_X$.

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PROOF. STEP 1. (Local stabilizer subgroup principal) The completion $\overline{\mathcal{H}(x)}^{/x}$ of $\overline{\mathcal{H}(x)}$, smooth over k and closed in $\mathcal{C}(x)^{/x} = \mathcal{DE}(X,Y)^{\mathrm{sym}}_{\mathrm{pdiv}}$, is stable under the action of (the closure of) the local stabilizer subgroup of x in prime-to-p Hecke correspondences.

Proof continued

STEP 2. The local stabilizer subgroup U_x at x is an open subgroup of the unitary group attached to $(\mathrm{End}_k(A_x[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \lambda_x[p^\infty])$, a semisimple algebra with involution.

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STEP 6. Hence $\mathcal{H}(x)^{/x} = \mathcal{DE}(X,Y)^{\mathrm{sym}}_{\mathrm{pdiv}} = \mathcal{C}(x)^{/x}$. Q.E.D.