TATE-LINEAR FORMAL VARIETIES AND ORBITAL RIGIDITY

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Central foliation on moduli spaces

The modular variety $\mathscr{A}_{g,\overline{\mathbb{F}_p}}$ classifying g-dimensional principally polarized abelian varieties in char. $p > 0$ is a disjoint union of smooth locally closed subvarieties $\mathscr{C}_j$, called *central leaves*.

- Each $\mathscr{C}_j$ consists of all points $[(A,\lambda)]$ in $\mathscr{A}_{g,\overline{\mathbb{F}_p}}$ where the geometric isomorphism type of the polarized p-divisible group $[(A[p^\infty],\lambda[p^\infty])]$ is fixed.
- Central leaves are geometric structures supporting prime-to-p Hecke symmetries on $\mathscr{A}_{g,\overline{\mathbb{F}_p}}$.
- The dimensions of these leaves range between $\dim(\mathscr{A}_{g,\overline{\mathbb{F}_p}}) = g(g+1)/2$ (the ordinary locus) and 0 (supersingular leaves). There are infinitely many leaves (and "have moduli") if $g \geq 2$.

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Local structure of central leaves

$\mathscr{C}^{/x_0} :=$ completion of a central $\mathscr{C}$ at a closed point x_0.

- In a simple case (2 slopes), $\mathscr{C}^{/x_0}$ has a natural structure as a p-divisible formal group.
- In another case (split 3 slopes) $\mathscr{C}^{/x_0}$ has a natural structure as a bi-extension of p-divisible formal groups.
- Analysis with the same technique shows that in general, $\mathscr{C}^{/x_0}$ is up to isogeny "assembled" (often in more than one way) from a finite number of p-divisible formal groups, via a collection of torsors for p-divisible formal groups.

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Goal 1 (as an optimistist)

Quest 1. Formulate a notion of "Tate-linear formal varieties" which accurately captures this "Tate-linearity" phenomenon of $\mathscr{C}^{/x_0}$, so that

- Formal completions $\mathscr{C}^{/x_0}$ of central leaves in $\mathscr{A}_{g,\overline{\mathbb{F}}_p}$, and equal-characteristic deformation spaces $\mathscr{D}ef(X)_{\overline{\mathbb{F}}_p}$ of a p-divisible group $X_{/\overline{\mathbb{F}}_p}$, are *standard examples* of Tate-linear formal varieties.
- Formal completions of central leaves in Shimura varieties over $\overline{\mathbb{F}}_p$ are Tate-linear formal varieties.
- Every Tate-linear formal variety can be Tate-linearly embedded (up to isogney) in a standard Tate-linear formal variety.

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Is $\mathscr{C}^{/x_0}$ also rigid, just as p-div. formal groups?

In the early 2000s, investigations of the prime-to-p Hecke symmetries led to the discovery of a rigidity property of p-divisible formal groups over fields of characteristic p, analogous in spirit to the Mostow–Margulis–Ratner rigidity.

Quest 2. (an optimist's dream). Orbital rigidity of $\mathscr{C}^{/x_0}$
Let G be a p-adic Lie group acting on $\mathscr{C}^{/x_0}$, such that every open subgroup of G induces non-trivial action on each Jordan–Hölder component of $\mathscr{C}^{/x_0}$. Suppose that W is an irreducible closed formal subvariety of $\mathscr{C}^{/x_0}$ closed under the action of G.

Desired conclusion:

W is a Tate-linear formal subvariety of $\mathscr{C}^{/x_0}$.

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Example: orbital rigidity of formal tori

k = an algebraically closed field of characteristic $p > 0$.

Let $\hat{T} = \mathrm{Spf}(k[[x_1, \ldots, x_d]])$ be a formal torus over k, with group law $\Delta : x_i \mapsto x_i \otimes 1 + 1 \otimes x_i + x_i \otimes x_i \ \ \forall i$. Let $[1+p^2]$ be the automorphism of $\hat{T}$ s.t. $[1+p^2]^*(x_i) = (1+x_i)^{1+p^2} - 1 \ \ \forall i$.

Theorem. *Suppose that $W = \mathrm{Spf}(k[[x_1, \ldots, x_d]]/P)$ is closed formal subvariety of $\hat{T}$ stable under $[1+p^2]$, i.e. P is a prime ideal of $k[[x_1, \ldots, x_d]]$ stable under $[1+p^2]^*$. Then W is a formal subtorus of $\hat{T}$, i.e.*

$$\Delta(P) \subseteq P \hat{\otimes}_k k[[x_1, \ldots, x_d]] + k[[x_1, \ldots, x_d]] \hat{\otimes}_k P.$$

Remark. This statement is false (and off the mark) in char. 0.

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Theorem. *Suppose that* $W = \mathrm{Spf}(k[[x_1, \ldots, x_d]]/P)$ *is closed formal subvariety of* $\hat{T}$ *stable under* $[1+p^2]$*, i.e.* P *is a prime ideal of* $k[[x_1, \ldots, x_d]]$ *stable under* $[1+p^2]^*$*. Then* W *is a formal subtorus of* $\hat{T}$*, i.e.*

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Preview

It turned out that optimists prevailed: both questions are answered in the affirmative.

In addition, in view of the existence of Euler flows on all Tate-linear formal varieties, the two affirmative answers uniquely determine the notion of Tate-linear formal varieties.

Application to the Hecke orbit conjecture (D'Addezio–van Hoften): The above results effectively linearize the Hecke orbit conjecture, reducing it to a monodromy problem, so that Crew's parabolicity conjecture for F-isocrystals, proved by D'Addezio, can be brought to bear.

Organization of the rest of this talk

- sustained p-divisible groups and stabilized Aut groups
- local structures of central leaves in $\mathscr{A}_{g,\overline{\mathbb{F}_p}}$ and $\mathscr{D}\!e\!f(X)_{\overline{\mathbb{F}_p}}$, as motivation for the notion of Tate-linear structures
- Tate-linear formal varieties: definition and first properties
- orbital rigidity of Tate-linear formal varieties: method of proof

Review of p-divisible groups

- A p-divisible group $X \to S$ over a base scheme S is a family $X = \big(X_n \to S\big)_{n\in\mathbb{N}}$ of finite locally free group schemes X_n killed by $[p^n]$, plus closed embeddings $\iota_{n+1,n} : X_n \to X_{n+1}$ and faithfully flat homomorphisms $q_{n,n+1} : X_{n+1} \to X_n$ such that $q_{n,n+1} \circ \iota_{n+1,n} = [p]_{X_n}$ $\forall n$.

  Idea: $X = \varinjlim_n X_n$ is divisible, $X_n = \ker([p^n]_X) =: X[p^n]$.
- The height of X/S is the function $S \xrightarrow{h} \mathbb{N}$ s.t. $\operatorname{order}(X_s[p^n]) = p^{hn}$ $\forall s \in S, \forall n$.
- A polarization of X is an S-isogeny $\mu : X \to X^t$, $X^t =$ Serre dual of X, such that $\mu^t = -\mu$.

Slopes of a p-divisible group

slopes: a p-divisible group X of height h over a field $K \supseteq \mathbb{F}_p$ has a unique slope filtration $X = X_1 \supsetneqq X_2 \supsetneqq \cdots \supsetneqq X_m \supsetneqq X_{m+1} = 0$ by p-divisible subgroups X_i such that X_i/X_{i+1} is isoclinic of slope s_i, $s_i \in \mathbb{Q}$, $0 \le s_1 < s_2 < \cdots < s_m \le 1$, s.t.

$\mathrm{Ker}\big(\mathrm{Fr}^N_{X_i/X_{i+1}}\big) \sim (X_i/X_{i+1})[p^{s_i N}]$ asymptotically, with uniformly bounded difference, for $N \gg 0$.

X is a p-divisible formal group iff slopes $s_i > 0$ for all i; then $\varinjlim X_n$ is a smooth formal group over K, and

$$\dim(X) := \dim(\varinjlim X_n) = \sum_i \mathrm{ht}(X_i/X_{i+1}) s_i.$$

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Sustained p-divisible groups

Let κ be a field of characteristic p, and let S be a scheme over κ. Let X_0, Y_0 be p-divisible groups over κ, and let λ_0 be a polarization of Y_0.

- a p-divisible group $X \to S$ is strongly sustained modeled on X_0 if for every $n \geq 1$, $X[p^n] \to S$ is S-locally isomorphic to $X_0[p^n] \times_{\mathrm{Spec}(k)} S$ for the flat topology of S.
- a polarized p-divisible group $(Y, \lambda) \to S$ is strongly sustained modeled on (Y_0, λ_0) if for every $n \geq 1$, $(Y[p^n], \lambda[p^n]) \to S$ is S-locally isomorphic to $(Y_0[p^n], \lambda_0[p^n]) \times_{\mathrm{Spec}(k)} S$ for the flat topology of S.

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Central leaves in moduli spaces

Let $\mathscr{A}_{g,\overline{\mathbb{F}_p}}$ be the moduli space of g-dimensional principally polarized abelian varieties in characteristic $p > 0$. Let $x_0 = [(A_0, \lambda_0)]$ be an $\overline{\mathbb{F}_p}$-point of $\mathscr{A}_{g,\overline{\mathbb{F}_p}}$.

- The central leaf $\mathscr{C}(x_0)$ is the largest subscheme of $\mathscr{A}_{g,\overline{\mathbb{F}_p}}$ such that the restriction to $\mathscr{C}(x_0)$ of the universal principally polarized p-divisible group is strongly sustained modeled on $(A_0[p^\infty], \lambda_0[p^\infty])$.
- $\mathscr{C}(x_0) \subseteq \mathscr{A}_{g,\overline{\mathbb{F}_p}}$ is smooth, locally closed and stable under all prime-to-p Hecke correspondences on $\mathscr{A}_{g,\overline{\mathbb{F}_p}}$.
- $\mathscr{C}(x_0)^{/x_0}$, the formal completion of $\mathscr{C}(x_0)$ at x_0, is canonically isomorphic to the sustained locus $\mathscr{D}ef(A_0[p^\infty], \lambda_0[p^\infty])_{\mathrm{sus}}$ in the char. p deformation space $\mathscr{D}ef(A_0[p^\infty], \lambda_0[p^\infty])_{/\overline{\mathbb{F}_p}}$.

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Stabilized Hom schemes

The structure of $\mathscr{C}(x_0)^{/x_0}$ is best explained in terms of the (projective system of) stabilized Aut group schemes $\mathscr{AUT}^{\mathrm{st}}(A_{x_0}[p^\infty], \lambda_{x_0}[p^\infty])$

Let Y, Z be p-divisible groups over a field κ of char. p.

- $\mathscr{HOM}^{\mathrm{st}}(Y,Z)_n := \mathrm{Image}\big(\mathscr{HOM}(Y[p^{n+N}], Z[p^{n+N}]) \to \mathscr{HOM}(Y[p^n], Z[p^n])\big)$, $N \gg 0$.
- $\mathscr{HOM}^{\mathrm{st}}(Y,Z) := \big(\mathscr{HOM}^{\mathrm{st}}(Y,Z)_n\big)_{n\geq 1}$ corresponds to a p-divisible group over κ.
- If the slope multisets of Y and Z are $\{s_i\}$ and $\{t_j\}$ respective, then the slope multiset of $\mathscr{HOM}^{\mathrm{st}}(Y,Z)$ is $\{t_j - s_i \mid t_j \geq s_i\}$.

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- $\mathscr{AUT}^{\mathrm{st}}(Y,\lambda)^0$ is the projective system of neutral components $\mathscr{AUT}^{\mathrm{st}}(Y,\lambda)^0_n$ of $\mathscr{AUT}^{\mathrm{st}}(Y,\lambda)_n$.
- Let $\mathbf{T}_p\mathscr{AUT}^{\mathrm{st}}(Y,\lambda) := \varprojlim_n \mathscr{AUT}^{\mathrm{st}}(Y,\lambda)_n$ and $\mathbf{T}_p\mathscr{AUT}^{\mathrm{st}}(Y,\lambda)^0 := \varprojlim_n \mathscr{AUT}^{\mathrm{st}}(Y,\lambda)^0_n$, in the category of sheaves of groups on $\mathrm{Spec}(\kappa)$ for the fpqc topology.
- $\mathbf{T}_p\mathscr{AUT}^{\mathrm{st}}(Y,\lambda)^0$ is an fpqc sheaf of torsion free nilpotent groups on $\mathrm{Spec}(k)$; it is uniquely ℓ-divisible $\forall$ prime $\ell \neq p$.

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Stabilized Aut schemes continued

- Strongly sustained polarized p-divisible groups modeled on (Y,λ) over a base scheme S correspond to (right) torsors for $\mathscr{A}\mathscr{U}\mathscr{T}^{\mathrm{st}}(Y,\lambda)$ over S. Similarly strong sustained deformations of (Y,λ) correspond to torsors for $\mathbf{T}_p\mathscr{A}\mathscr{U}\mathscr{T}^{\mathrm{st}}(Y,\lambda)^0$.
- The slope filtration on $\mathscr{E}\mathscr{N}\mathscr{D}^{\mathrm{st}}(Y)$ induces a slope filtration by normal subgroups $\mathrm{Fil}_{\mathrm{sl}}^{\bullet}\mathbf{T}_p\mathscr{A}\mathscr{U}\mathscr{T}^{\mathrm{st}}(Y,\lambda)^0$ on $\mathbf{T}_p\mathscr{A}\mathscr{U}\mathscr{T}^{\mathrm{st}}(Y,\lambda)^0$, indexed by $(0,1]$, s.t. $[\mathrm{Fil}_{\mathrm{sl}}^s,\mathrm{Fil}_{\mathrm{sl}}^t\subseteq \mathrm{Fil}_{\mathrm{sl}}^{s+t}]_{\mathrm{grp}}\ \forall s,t$, where $[x,y]_{\mathrm{grp}}=x^{-1}y^{-1}xy$
- Each of the (finitely many) non-trivial associated graded pieces $\mathrm{gr}^s_{\mathrm{Fil}_{\mathrm{sl}}}\mathbf{T}_p\mathscr{A}\mathscr{U}\mathscr{T}^{\mathrm{st}}(Y,\lambda)^0$ is isomorphic to $\mathbf{T}_pZ_s=\varprojlim_n Z_s[p^n]$, for some isoclinic p-divisible group Z_s of slope s over κ.

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- The slope filtration on $\mathscr{END}^{\mathrm{st}}(Y)$ induces a slope filtration by normal subgroups $\mathrm{Fil}^{\bullet}_{\mathrm{sl}}\mathbf{T}_p\mathscr{AUT}^{\mathrm{st}}(Y,\lambda)^0$ on $\mathbf{T}_p\mathscr{AUT}^{\mathrm{st}}(Y,\lambda)^0$, indexed by $(0,1]$, s.t.
$[\mathrm{Fil}^s_{\mathrm{sl}},\mathrm{Fil}^t_{\mathrm{sl}} \subseteq \mathrm{Fil}^{s+t}_{\mathrm{sl}}]_{\mathrm{grp}}\ \forall s,t$, where $[x,y]_{\mathrm{grp}} = x^{-1}y^{-1}xy$
- Each of the (finitely many) non-trivial associated graded pieces $\mathrm{gr}^s_{\mathrm{Fil}_{\mathrm{sl}}}\mathbf{T}_p\mathscr{AUT}^{\mathrm{st}}(Y,\lambda)^0$ is isomorphic to $\mathbf{T}_pZ_s = \varprojlim_n Z_s[p^n]$, for some isoclinic p-divisible group Z_s of slope s over κ.

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Lie theory of $\mathbf{T}_p\mathscr{AUT}^{\mathrm{st}}(Y,\lambda)^0$

- Let $\mathbf{V}_p\mathscr{END}^{\mathrm{st}}(Y)^0 := \mathbf{T}_p\mathscr{END}^{\mathrm{st}}(Y)^0 \otimes \mathbb{Q}$, $\mathbf{V}_p\mathscr{AUT}^{\mathrm{st}}(Y,\lambda)^0 := \left(\mathbf{V}_p\mathscr{END}^{\mathrm{st}}(Y)^0\right)^{\tau_\lambda=-1}$,
- $\mathrm{U}\left(\mathbf{V}_p\mathscr{END}^{\mathrm{st}}(Y)^0, \tau_\lambda\right)$ $:= \{\alpha \in \mathbf{V}_p\mathscr{END}^{\mathrm{st}}(Y)^0 \mid \alpha\tau(\alpha) = 1 = \tau(\alpha)\alpha\}$
- The exponential and logarithm maps

$$\mathrm{U}\left(\mathbf{V}_p\mathscr{END}^{\mathrm{st}}(Y)^0, \tau_\lambda\right) \underset{\exp}{\overset{\log}{\underset{\longleftarrow}{\overset{\longrightarrow}{\cong}}}} \left(\mathbf{V}_p\mathscr{END}^{\mathrm{st}}(Y)^0\right)^{\tau_\lambda=-1}$$

establish bijections relating the nilpotent group $\mathrm{U}\left(\mathbf{V}_p\mathscr{END}^{\mathrm{st}}(Y)^0, \tau_\lambda\right)$ and the nilpotent Lie algebra $\left(\mathbf{V}_p\mathscr{END}^{\mathrm{st}}(Y)^0\right)^{\tau_\lambda=-1}$.

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Review: Mal'cev completion

The natural embedding $\mathbf{T}_p\mathscr{AUT}^{\mathrm{st}}(Y,\lambda)^0 \hookrightarrow \mathrm{U}\left(\mathbf{V}_p\mathscr{END}^{\mathrm{st}}(Y)^0, \tau_\lambda\right)$ identifies $\mathrm{U}\left(\mathbf{V}_p\mathscr{END}^{\mathrm{st}}(Y)^0, \tau_\lambda\right)$ as the Mal'cev completion of $\mathbf{T}_p\mathscr{AUT}^{\mathrm{st}}(Y,\lambda)^0$.

Recall that every torsion free nilpotent group N admits a *Mal'cev completion* $N \hookrightarrow N_{\mathbb{Q}}$, characterized by the following properties.

- $N_{\mathbb{Q}}$ is nilpotent, torsion free and uniquely divisible.
- $\forall z \in N_{\mathbb{Q}}, \exists m \in \mathbb{N}$ s.t. $z^{p^m} \in N$.

Moreover there is a nilpotent Lie algebra $\mathfrak{n}_{\mathbb{Q}}$ attached to $N_{\mathbb{Q}}$, which is a vector space over $\mathbb{Q}$, and bijective logarithm and exponential maps

$$N_{\mathbb{Q}} \underset{\exp}{\overset{\log}{\rightleftarrows}} \mathfrak{n}_{\mathbb{Q}} \quad (\cong)$$

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The structure of $\mathscr{C}(x_0)^{/x_0}$

Prop. $\mathscr{C}_{\mathscr{A}_{g,\overline{\mathbb{F}}_p}}(x_0)^{/x_0}$ is naturally isomorphic to

$$\mathrm{U}\big(\mathbf{V}_p\,\mathrm{End}(A_{x_0}[p^\infty])^0, \tau_{\lambda_{x_0}}\big)/\mathbf{T}_p\big(\mathscr{AUT}^{\mathrm{st}}(A_{x_0}[p^\infty], \lambda_{x_0}[p^\infty])^0\big).$$

Why is this proposition plausible?

- A (functorial) point of the sheaf of cosets in the above proposition determines a (right) torsor for $\mathbf{T}_p\big(\mathscr{AUT}^{\mathrm{st}}(A_{x_0}[p^\infty], \lambda_{x_0}[p^\infty])^0\big)$.
- A torsor for $\mathbf{T}_p\big(\mathscr{AUT}^{\mathrm{st}}(A_{x_0}[p^\infty], \lambda_{x_0}[p^\infty])^0\big)$ over a complete local base scheme S in char. p determines a strongly sustained polarized p-divisible group modeled on $(A_{x_0}[p^\infty], \lambda_{x_0}[p^\infty])$.

Rmk. This proposition says that $\mathscr{C}_{\mathscr{A}_{g,\overline{\mathbb{F}}_p}}(x_0)^{/x_0}$ is the Tate-linear formal variety attached to the Tate unipotent group $\mathbf{T}_p\big(\mathscr{AUT}^{\mathrm{st}}(A_{x_0}[p^\infty], \lambda_{x_0}[p^\infty])^0\big)$.

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Tate unipotent groups

Let κ be a field of char. p.

Definition. A Tate unipotent group over κ is a sheaf $\mathbf{N}$ of nilpotent groups on $\mathrm{Spec}(k)_{fpqc}$ which admits a decreasing filtration $\mathrm{Fil}_{\mathrm{sl}}^{\bullet}\mathbf{N}$ of normal subgroups indexed by $\mathbb{R}$, s.t.

- $\mathrm{Fil}_{\mathrm{sl}}^{0}\mathbf{N} = \mathbf{N}$, $\mathrm{Fil}_{\mathrm{sl}}^{t}\mathbf{N} = (1)\ \forall t > 1$.
- $\exists$ a finite subset $\mathrm{sl}(\mathbf{N}) \subseteq (0,1] \cap \mathbb{Q}$ s.t. $\mathrm{gr}^{s}_{\mathrm{Fil}_{\mathrm{sl}}}\mathbf{N} \neq (1)$ iff $s \in \mathrm{sl}(\mathbf{N})$.
- $[\mathrm{Fil}_{\mathrm{sl}}^{s}\mathbf{N}, \mathrm{Fil}_{\mathrm{sl}}^{t}\mathbf{N}]_{\mathrm{grp}} \subseteq \mathrm{Fil}_{\mathrm{sl}}^{s+t}\mathbf{N}$
- $\forall s \in \mathrm{sl}(\mathbf{N})\ \exists$ a non-trivial isoclinic p-divisible group Y_s over κ of slope s s.t. $\mathrm{gr}^{s}_{\mathrm{Fil}_{\mathrm{sl}}}\mathbf{N} \cong \mathbf{T}_p Y_s$.
  (Recall: $\mathbf{T}_p Y_s = \varprojlim_n Y_s[p^n]$ as a sheavef on $\mathrm{Spec}(\kappa)_{\mathrm{fpqc}}$.)

Properties of Tate unipotent groups

- Denote by $\mathbf{N}_{\mathbb{Q}}$ the Mal'cev completion of $\mathbf{N}$, a sheaf of uniquely divisible nilpotent groups on $\mathrm{Spec}(\kappa)_{\mathrm{fpqc}}$.
- Let $\mathfrak{n}_{\mathbb{Q}} := \mathrm{Lie}(\mathbf{N}_{\mathbb{Q}})$, which is a p-divisible group up to isogeny over κ, endowed with a Lie bracket.
- If κ is perfect, then the Lie bracket on $\mathfrak{n}_{\mathbb{Q}}$ induces a Lie bracket on the (covariant) Dieudonné module $\mathbb{D}(\mathfrak{n}_{\mathbb{Q}})$ s.t. $[Vx, Vy] = V \cdot [x, y], \ [Fx, y] = F \cdot [x, Vy], \ [x, Fy] = F \cdot [Vx, y]$ for all $x, y \in \mathbb{D}(\mathfrak{n}_{\mathbb{Q}})$.

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Tate-linear formal varieties

Proposition. The fpqc sheaf $\mathbf{N}_{\mathbb{Q}}/\mathbf{N}$ over κ is representable by smooth formal scheme, i.e. a formal spectrum $\mathrm{Spf}(\kappa[[t_1,\ldots,t_d]])$, $d \in \mathbb{N}$.

Definition. Let κ be a field of char. p.
(1) The Tate-linear formal variety $\mathrm{TL}(\mathbf{N})$ attached to a Tate unipotent group $\mathbf{N}$ over κ is the smooth formal scheme representing $\mathbf{N}_{\mathbb{Q}}/\mathbf{N}$. A Tate-linear structure on a formal scheme T over κ is an isomorphism $\mathbf{N}_{\mathbb{Q}}/\mathbf{N} \xrightarrow{\sim} T$ over κ.

(2) A Tate-linear morphism between Tate-linear formal varieties $\mathrm{TL}(\mathbf{N}_i)$, $i = 1,2$, is a morphism induced by a homomorphism $h : \mathbf{N}_1 \to \mathbf{N}_2$ of sheaves of nilpotent groups on $\mathrm{Spec}(\kappa)_{\mathrm{fpqc}}$.

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Tate-linear formal varieties continued

Examples of Tate-linear formal varieties

- p-divisible formal groups over κ,
- bi-extensions of p-divisible formal groups (X,Y) over κ by a third p-divisible formal group Z over κ,
- formal completions $\mathscr{C}(x_0)^{/x_0}$ of central leaves $\mathscr{C}(x_0)$ in $\mathscr{A}_{g,\overline{\mathbb{F}_p}}$,
- sustained deformation spaces $\mathscr{D}\!ef(Y)_{\mathrm{sus}}$, $\mathscr{D}\!ef(Y,\lambda)_{\mathrm{sus}}$ of (polarized) p-divisible groups over κ,
- (with a grain of salt) formal completions of central leaves in Shimura varieties over $\overline{\mathbb{F}_p}$, and local analogs.

Orbital rigidity of Tate-linear formal varieties

Let κ be an algebrically closed field of characteristic p, let $\mathbf{N}$ be a Tate unipotent group over κ, and let G be a compact p-adic group operating on $\mathbf{N}$.

Definition. The action of G on $\mathrm{TL}(\mathbf{N})$ is strongly non-trivial if every Jordan–Hölder component of the representation of $\mathrm{Lie}(G)$ on $\mathbb{D}(\mathfrak{n}_{\mathbb{Q}})$ is non-trivial. In other words, every open subgroup U operates non-trivially on every U-invariant subquotient of $\mathbf{N}_{\mathbb{Q}}$.

Theorem. Let W be a reduced irreducible closed formal subscheme of $\mathrm{TL}(\mathbf{N})$. If G acts strongly nontrivially on $\mathrm{TL}(\mathbf{N})$ and W is stable under G, then W is a Tate-linear formal subvariety of $\mathrm{TL}(\mathbf{N})$.

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Methods for proving orbital rigidity

- Hypocotyl elongation
  This technique emerged from the proof of orbital rigidity for p-divisible formal groups. It produces polynomial equations $F(x_1,\ldots,x_a,y_1,\ldots,y_a)=0$ in two sets of variables, $(x_1,\ldots,x_a,y_1,\ldots,y_a)$, from an infinite family of congruences relations $f_n(x_1,\ldots,x_a)\equiv(\underline{x}^{d_n})$ (before "doubling"), with $f_n(x_1,\ldots,x_a)=F(x_1,\ldots,x_a,h_{1,n}(\underline{x}),\ldots,h_{a,n}(\underline{x}))$, for suitable polynomials $h_{i,n}(x_1,\ldots,x_a)$).
- Tempered perfection
  Every complete noetherian local domain R over a perfect field of characteristic has a (filtering) family of complete local rings R_α, consisting of $R\subseteq R_\alpha\subseteq\widehat{R^{\mathrm{perf}}}$.

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Features of tempered perfections

- Think of elements of R_α as “tempered generallized functions” w.r.t. R.
- The tempered perfections R_α of R are non-noetherian, but they satisfy a weak finiteness condition.
- Many torsors admit “trivializations with coefficients in tempered generalized functions”.

Hypocotyl elongation

Let k be an algebraically closed field of characteristic p.
Let $\mathbf{u} = (u_1, \ldots, u_a)$, $\mathbf{v} = (v_1, \ldots, v_b)$ be two sets of variables.
Let $f \in k[[\mathbf{u}, \mathbf{v}]]$ be a formal power series.
Let $(R, \mathfrak{m})$ be an augmented complete Noetherian local domains with residue field k.

Let $g_1, \ldots, g_a, h_1, \ldots, h_b \in \mathfrak{m}$, $q = p^r$, $r \in \mathbb{N}_{>0}$.

Theorem (Hypocotyl elongation for noetherian local domains)
Suppose that $\exists\, n_0 > 0$ *and a sequence of positive integers* (d_n) *such that* $\lim_{n\to\infty} \frac{q^n}{d_n} = 0$ *and*

$$f(g_1, \ldots, g_a, h_1^{q^n}, \ldots, h_b^{q^n}) \equiv 0 \pmod{\mathfrak{m}^{d_n}} \text{ for all } n \geq n_0.$$

Then $f(g_1 \otimes 1, \ldots, g_a \otimes 1, 1 \otimes h_1, \ldots, 1 \otimes h_b) = 0$ *in* $R \hat{\otimes}_k R$.

Hope inspired by previous strategy

Let E be bi-extension of $X \times Y$ by Z, let Z' be the isoclinic p-divisible subgroup of Z s.t. with slope $>$ slopes(Z/Z').

Hope (establish rigidity for biextensions by hypocotyl elongation):
For any element $v = (A, B, C) \in \mathrm{Lie}(\mathrm{Aut}_{\mathrm{biext}}(E))$, $A \in \mathrm{End}(X)$, $B \in \mathrm{End}(Y)$, $C \in \mathrm{End}(Z)$ with

$$e_E(Ax, y) + e_E(x, By) = C(e_E(x, y)) \quad \forall x \in \mathbb{D}_*(X), \forall y \in \mathbb{D}_*(Y)$$

$\exists\, n_0 \in \mathbb{N}$ and a morphism $\eta_v : E \to Z'$ s.t.

$$\eta_v(z' * u) = \eta_v(u) * p^{n_0} C(z') \quad \forall z' \in Z', \forall u \in E.$$

$$\exp(p^n v)(u) \;\sim\; u * p^{n-n_0} \eta_v(u) \quad \forall u \in E, \forall n >> 0.$$

We will win if such a "weak projection" η_v attached to v if such a map exists. But that's *not* the case.

Tempered perfections

However $\exists$ *such weak projections* $\eta_v : E \dashrightarrow Z'$ *with coefficients in a suitable ring of "generalized formal functions"*
(an extension ring of $\Gamma(\mathscr{O}_E)$ contained in the completion of $\Gamma(\mathscr{O}_E)^{\mathrm{perf}}$).

Definition. Let $(R, \mathfrak{m})$ be an augmented complete Noetherian local domain over k. Let $A, b, d \in \mathbb{R}$, $A, b > 0, d \geq 0$. Define an augmented complete local domain $(R, \mathfrak{m})^{\mathrm{perf},\flat}_{A,b,d}$ over k as follows.

(i) R^{perf} has a decreasing filtration indexed by $\mathbb{R}$:
$\mathrm{Fil}^u_{R^{\mathrm{perf}}}$ consists of all $x \in R^{\mathrm{perf}}$ s.t. $\exists j \in \mathbb{N}$ with $x^{p^j} \in \mathfrak{m}^{\lceil u \cdot p^j \rceil}$ $\forall u \geq 0$,
and $\mathrm{Fil}^u_{R^{\mathrm{perf}}} = R^{\mathrm{perf}}$ $\forall u \leq 0$.

(ii) $(R, \mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d} :=$ completion of $\sum_{n \in \mathbb{N}} \left(\phi^{-n} R \cap \mathrm{Fil}^{b \cdot p^{An} - d}_{R^{\mathrm{perf}}}\right)$,
where ϕ is the Frobenius automorphism $x \mapsto x^p$ of R^{perf}.

Elements of these non-noetherian complete local domains are called tempered virtual functions on $\mathrm{Spf}(R)$.

Examples of tempered perfections

Consider the formal power series ring $k[[t_1, \ldots, t_m]]$. Let $C, E > 0, d \geq 0$ be real numbers.

Definition. Define an augmented complete local domain $k\langle\langle t_1^{p^{-\infty}}, \ldots, t_1^{p^{-\infty}}\rangle\rangle_{C;d}^{E,\flat}$ over k to be the set of all formal power series $\sum_{I \in \mathbb{N}[1/p]^m} b_I \mathbf{t}^I$ with supports contained in the subset

$$\mathrm{supp}(m : \flat : E, C, d) := \left\{(i_1, \ldots, i_m) \,:\, \max\left(p^{-\mathrm{ord}_p(i_j)}\right) \leq \max\left(C \cdot (d + \textstyle\sum_j i_j)^E, 1\right)\right\}$$

Note. The p-adic norm of a typical element I in $\mathrm{supp}(m : \flat : E, C, d)$ is bounded by a *polynomial* of the archimedean norm of I in.