Chapter 1

Outline

The goal of this short course is much more modest than what the title *Integral Models of Shimura Varieties* might have suggested. Instead of treating the most general Shimura varieties, we consider only a rather restricted class of Shimura varieties—those of PEL type. They are modular varieties which classify polarized abelian varieties with prescribed endomorphisms and level structures. These modular problems make sense over the integers, and their moduli spaces provides natural integral models for Shimura varieties of PEL type Concentrating on these moduli spaces allows us to go far in pursuing a few threads, with a few memorable results at the end of our short journey.

An important feature of Shimura varieties is that they have lots of symmetries, namely the Hecke correspondences. The Hecke symmetries and their local counterpart help revealing structural properties of good integral models of Shimura varieties; they are part of the running theme of this course.

We present two methods for studying the local structure of integral models. The theory of local models is useful for studying the singularities of integral models; it will be illustrated in the case of Hilbert modular varieties. Barsotti-Tate (or p-divisible) groups, including formal groups and Cartier theory, is *the* basic tool of the trade. The Cartier theory translates problems about formal groups to linear algebra problems over a non-commutative ring. Chap. 2 is a set of notes on Cartier theory, with some exercises.

We prove two theorems. One is about the density of ordinary Hecke orbits, and the other is related to the local structure of certain subvarieties of the reduction of Shimura varieties which are invariant under Hecke symmetries.

Below is a plan for the individual lectures. The materials covered in each lecture are grouped under several categories: definitions, examples, black boxes, properties and applications. Inside the *black boxes* are statements of results which will be used later. Items found under *properties* are either explanation of methods, or links between related concepts. The *applications* is a mixed bag, containing both corollaries of black-boxed results and “star theorems” of the course.

1.1 Lecture 1: Introduction

DEFINITIONS

- modular problems of PEL type
- Hecke symmetries

EXAMPLES

- Siegel modular varieties
- Hilbert modular varieties
- Picard modular varieties

BLACK BOXES

- representability of modular problems of PEL type
- Deformation of abelian varieties (Grothendieck-Messing)

APPLICATIONS

- smoothness of the Siegel modular varieties and Picard modular varieties (good reduction case)
- smoothness of Hilbert modular varieties at points where the Lie type is generic
- local symmetries: action of the stabilizer subgroup on local deformation spaces

1.2 Lecture 2: Barsotti-Tate groups

DEFINITIONS

- Barsotti-Tate groups
- dual of a BT-group

EXAMPLES

- étale and multiplicative BT-groups
- ordinary BT-groups

BLACK BOXES

- Serre-Tate theorem: deformation of abelian schemes and deformation of BT-groups
- local rigidity: case of formal tori

APPLICATIONS

- Serre-Tate coordinates of the local deformation space of ordinary abelian varieties
- local structure of $X_0(p)$ at ordinary points
- Larsen's trick

1.3 Lecture 3: Formal groups and Cartier theory

DEFINITIONS

- formal groups
- the Cartier ring
- the reduced Cartier ring
- p-typical curves
- V-reduced Cartier modules

PROPERTIES

- relation between formal groups and Barsotti-Tate groups
- Cartier modules and formal groups: an equivalence of categories

1.4 Lecture 4: Cartier-Dieudonné theory continued

DEFINITIONS

- stratification by Newton polygons
- a-numbers of a BT-group
- leaves

EXAMPLES

- simple Cartier modules
- minimal Cartier modules

- Cartier modules with endomorphisms
- ordinary points in a Picard modular surface and the splitting behavior of p

BLACK BOXES

- Manin-Dieudonné classification
- Manin's finiteness theorem

PROPERTIES

- the largest unipotent subgroup of a formal group
- the largest p-divisible subgroup of a formal group

APPLICATIONS

- Lubin-Tate moduli space and relation to certain Picard modular varieties
- constructibility and smoothness of leaves

1.5 Lecture 5: Hilbert modular varieties

BLACK BOXES

- toroidal compactification

PROPERTIES

- the local model method

APPLICATIONS

- stratification by Lie types
- irreducibility of Hilbert modular varieties
- application of local rigidity for Hilbert modular varieties

1.6 Lecture 6: Density of ordinary Hecke orbits

BLACK BOXES

- de Jong's theorem: p-adic Tate conjecture for endomorphisms of families of abelian varieties
- Hecke symmetry and the action of the local stabilizer subgroup

APPLICATIONS

- density of ordinary Hecke orbit for Hilbert modular varieties
- density of ordinary Hecke orbit for Siegel modular varieties

1.7 Lecture 7: Formal groups defined by extensions

DEFINITIONS

- the formal group $\mathfrak{DE}(X,Y)$ defined by extensions of BT-groups
- a triple Cartier module: p-typical curves in the reduced Cartier ring functor

EXAMPLES

- the maximal p-divisible subgroup $\mathfrak{DE}(X,Y)_{\mathrm{pdiv}}$ of $\mathfrak{DE}(X,Y)$: examples for small BT-groups X and Y

APPLICATIONS

- reduction to homological algebra for the reduced Cartier ring: a formula for the Cartier module of $\mathfrak{DE}(X,Y)$
- filtrations and structure of the triple Cartier module

1.8 Lecture 8: Formal groups defined by extensions, continued

PROPERTIES

- the Oda-Oort map
- relation with leaves: canonical coordinates

APPLICATIONS

- duality in Cartier theory
- the Cartier module of $\mathfrak{DE}(X,Y)_{\mathrm{pdiv}}$

1.9 Comments on the literature

ABELIAN VARIETIES, ABELIAN SCHEMES

The standard references are Mumford's books [23], [24].

EXISTENCE OF MODULI SPACES

There are two approaches. The first is *geometric invariant theory* [24], the second is *algebraic spaces* [1], [2], [3].

BARSOTTI-TATE GROUPS

The standard reference is [21]; see also the Grothendieck's lectures [13] and the survey article [15].

SERRE-TATE THEOREMS

The exposition in [16] has become the standard reference; see also [21].

CARTIER THEORY

The best book is [35], which follows the treatment in §2 of [33]; see also [19], [14].

MANIN-DIEUDONNÉ CLASSIFICATION

Exposition of the classical (contravariant) Dieudonné theory and the Manin-Dieudonné classification can be found in Manin's thesis [20] and [10]. See also [35]. Minimal BT-groups are determined by their truncations at the first level, see [30].

DEFORMATION OF ABELIAN VARIETIES

The method of construction of deformations over a formal curve first appeared in Mumford's seminal article [22]. The methods was developed in [25] and [26], and reappeared in a more general form in Zink's theory of displays [36].
SIEGEL MODULAR VARIETIES See [11].

HILBERT MODULAR VARIETIES

The standard references are [31] and [9].

LOCAL MODELS

The method of local models was used for Siegel modular varieties in [6], and for Hilbert modular varieties in [9], independently. In [32] the method was explained in the context of the Rapoport-Zink period spaces.

HECKE SYMMETRIES IN CHARACTERISTIC p

The expository article [5] contains further references.

STRATIFICATION AND FOLIATION OF MODULI SPACES OF ABELIAN VARIETIES

That the Newton polygon strata are closed was proved in [17]. Grothendieck's conjecture for the Newton polygon stratification was proved in [7]. See [29] for the Ekedahl-Oort stratification, and [28] for the Oort foliations.

Chapter 2

Cartier-Dieudonné Theory

2.1 Introduction

This set of notes offers an introduction to the Cartier-Dieudonné theory on commutative smooth formal groups. The Cartier theory provides a dictionary, translating most questions about commutative smooth formal groups into questions in linear algebra. The main theorem 2.4.9 says that the category of smooth commutative formal groups over a commutative ring k is equivalent to a suitable full subcategory of the category of left modules over a certain non-commutative ring $\mathrm{Cart}(k)$. The equivalence above is a sort of Morita equivalence. When the ring k is a $\mathbb{Z}_{(p)}$-algebra, where p is a prime number, there is a "local version" of the main theorem, with the ring $\mathrm{Cart}(k)$ replaced by a subring $\mathrm{Cart}_p(k)$ of $\mathrm{Cart}(k)$ defined by an idempotent in $\mathrm{Cart}(k)$.

A key role is played by a smooth commutative formal group Λ, which is a *restricted version* of the formal completion of the group scheme of universal Witt vectors; see 2.2.11 for its definition. This smooth formal group Λ is in some sense a free generator of the additive category of smooth commutative formal groups. The ring $\mathrm{Cart}(k)$ is the opposite ring of $\mathrm{End}_k(\Lambda)$; it is in natural bijection with the set of all formal curves in Λ.

An excellent presentation of Cartier theory can be found in the booklet [35] by T. Zink, where the approach in \$2 of [33] is fully developed. We have followed [35] closely, and we make no claim whatsoever to the originality of the exposition here. Exercises appear throughout; they form an integral part of the notes. The readers are advised to try as many of them as possible. Besides [35], there are two other standard references for Cartier theory. Lazard's monograph [19] is the first complete documentation of Cartier's theory. Hazewinkel's treatment [14] employs the technology of the "functional equation lemma", it is a useful reference, with 573+ix pages and a good indexing system.

Although the main results of Cartier theory does not depend on the Witt vectors, in applications the Witt vectors are indispensable. The basic properties of both the ring of universal Witt vectors and the ring of p-adic Witt vectors can be found in Appendix 2.6; the exposition there is self-contained. The Witt vectors can also be viewed as being a part of the Cartier theory, for they are the Cartier module attached to the formal completion $\widehat{\mathbb{G}_m}$ of $\mathbb{G}_m$ in the two versions of Cartier theory. The group of universal Witt vectors consists of all formal curves in $\widehat{\mathbb{G}_m}$, and the group of p-adic Witt vectors consists of all p-typical formal curves in $\widehat{\mathbb{G}_m}$.

2.2 Formal groups

In this section k denotes a commutative ring with 1. The notion of formal groups adopted here differs slightly from the standard definition, because we consider them as functors on the category of nilpotent algebras.

Definition 2.2.1. Let $\mathfrak{Nilp}_k$ be the category of all nilpotent k-algebras, consisting of all commutative k-algebras N without unit such that $N^n = 0$ for some positive integer n.

Remark 2.2.2. Clearly $\mathfrak{Nilp}_k$ is isomorphic to the category of all augmented k-algebras $k \to R \xrightarrow{\epsilon} k$ such that the augmentation ideal $I = \mathrm{Ker}(\epsilon)$ is nilpotent; the isomorphisms are given by $N \mapsto k \oplus N$ and $(R, \epsilon) \mapsto \mathrm{Ker}(\epsilon)$.

Definition 2.2.3. Let $\mathfrak{ProNilp}_k$ be the category of all filtered projective limits of nilpotent k-algebras. Every functor $G : \mathfrak{Nilp}_k \to \mathfrak{Sets}$ can be uniquely extended to a functor from $\mathfrak{ProNilp}$ to $\mathfrak{Sets}$ which commutes with filtered projective limits; this extension is also denoted by G. The analogous statement holds for functors from $\mathfrak{Nilp}_k$ to $\mathfrak{Ab}$.

Remark 2.2.4. As an example, let $k[[X_1, \ldots, X_n]]$ be the power series ring over k in n variables. Denote by $k[[X_1, \ldots, X_n]]^+$ the subset of $k[[X_1, \ldots, X_n]]$ consisting of all power series whose constant term is 0. Then $k[[X_1, \ldots, X_n]]^+$ is an object in $\mathfrak{ProNilp}_k$, and

$$G(k[[X_1, \ldots, X_n]]^+) = \varprojlim_{i \geq 1} G\left(k[[X_1, \ldots, X_n]]^+ \Big/ ((X_1, \ldots, X_n)k[[X_1, \ldots, X_n]])^i\right)$$

Definition 2.2.5. Let $G : \mathfrak{Nilp}_k \to \mathfrak{Ab}$ be a functor from $\mathfrak{Nilp}_k$ to the category of all abelian groups.

(1) The functor G is *left exact* if it commutes with finite inverse limits and $G(0) = (0)$. (Actually the latter condition is a special case of the first one: take the inverse limit over the empty indexing set.)

(2) The functor G is *formally smooth* if every surjection $N_1 \to N_2$ in $\mathfrak{Nilp}_k$ induces a surjection $G(N_1) \to G(N_2)$.

(3) The functor G is *right exact* if it commutes with finite direct product, and every exact sequence $N_3 \to N_2 \to N_1 \to 0$ in $\mathfrak{Nilp}_k$ induces an exact sequence $G(N_3) \to G(N_2) \to G(N_1) \to 0$ in $\mathfrak{Ab}$.

(4) The functor G is *weakly left exact* if G commutes with finite direct product, and if for every exact sequence

$$0 \to N_1 \to N_2 \xrightarrow{\pi} N_3 \to 0$$

in $\mathfrak{Nilp}_k$ such that $N_3^2 = (0)$ and N_3 is a free k-module, the induced sequence

$$0 \to G(N_1) \to G(N_2) \to G(N_3)$$

is exact.

(5) The functor G is *half exact* if G commutes with finite direct product, and if for every exact sequence $0 \to N_1 \to N \xrightarrow{\pi} N_2 \to 0$ in $\mathfrak{Nilp}_k$ such that $N_1 \cdot N = (0)$, the group $G(N_1)$ operates simply transitively on $G(\pi)^{-1}(\xi)$ for every $\xi \in G(N_2)$ such that $G(\pi)^{-1}(\xi) \neq \emptyset$.

Remark. Left exactness implies weak left exactness and half exactness.

Definition 2.2.6. Let k be a commutative ring with 1 and let $\mathfrak{Mod}_k$ be the category of k-modules. There is a natural embedding of $\mathfrak{Mod}_k$ into $\mathfrak{Nilp}_k$, endowing each k-module M the trivial multiplication structure, i.e. $M \cdot M = (0)$. Let G be a functor from $\mathfrak{Nilp}_k$ to $\mathfrak{Ab}$ which commutes with finite direct sums. The *tangent functor* $\mathrm{t}_G : \mathfrak{Mod}_k \to \mathfrak{Mod}_k$ of G is defined by restricting G to $\mathfrak{Mod}_k$ and endowing $G(M)$ the natural k-module structure for any k-module M. The *Lie algebra* $\mathrm{Lie}(G)$ of G is defined to be $G(k)$, where k is regarded as an object in $(Mod)_k$.

Definition 2.2.7. A functor $G : \mathfrak{Nilp}_k \to \mathfrak{Ab}$ from $\mathfrak{Nilp}_k$ to the category of abelian groups is a *commutative smooth formal group* if G is *left exact*, formally smooth, and *commutes with arbitrary direct sums.*

Definition 2.2.8. Let k be a commutative ring with 1 and let I be an indexing set.

(i) Let $\underline{X} = (X_i)_{i \in I}$ be a set of variables indexed by the set I. Denote by $k[[\underline{X}]] = k[[X_i]]_{i \in I}$ the inverse limit of all formal power series rings $k[[X_j]]_{j \in J}$ where J runs through all finite subsets of I. In other words, $k[[\underline{X}]] = k[[X^I]]$ consists of all formal power series

$$\sum_{\alpha} a_\alpha \, \underline{X}^\alpha \,, \qquad a_\alpha \in k \,, \; \underline{X}^\alpha := \prod_{i \in I} X_i^{\alpha_i}$$

where α runs through all functions $\alpha : I \to \mathbb{N}$ vanishing outside some finite subset of I. Elements of $k[[\underline{X}]] = k[[X^I]]$ are in bijection with k-valued functions on the set of all monomials in the variables $\underline{X}$.

(ii) Denote by $k[[\underline{X}]]^+$ the augmentation ideal of $k[[\underline{X}]]$, consisting of all power series without the constant term. For each $n \geq 1$, the quotient

$$k[[\underline{X}]]^+ / \left(k[[\underline{X}]]^+\right)^n$$

is a nilpotent k-algebra, and $k[[\underline{X}]]^+$ is the filtered inverse limit of these quotients.

(iii) Denote by $\widehat{\mathbb{A}}^{(I)}$ the functor from $\mathfrak{Nilp}_k$ to $\mathfrak{Sets}$ such that

$$\widehat{\mathbb{A}}^{(I)}(N) = \bigoplus_{i \in I} N \,,$$

the set underlying the direct sum of I copies of N. Clearly elements of $k[[X^{(I)}]]$ gives rise to formal functions on $\widehat{\mathbb{A}}^{(I)}$, i.e. maps from $\widehat{\mathbb{A}}^{(I)}$ to $\widehat{\mathbb{A}}^1$.

Definition 2.2.9. Let k be a commutative ring with 1 and let I be a set. A *commutative formal group law* on $\widehat{\mathbb{A}}^{(I)}$ is morphism $\mu : \widehat{\mathbb{A}}^{(I)} \times \widehat{\mathbb{A}}^{(I)} \to \widehat{\mathbb{A}}^{(I)}$ which provides a commutative group law on $\widehat{\mathbb{A}}^{(I)}$. Equivalently, a commutative formal group law is a homomorphism $\mu^* : k[[X^{(I)}]] \to k[[X^{(I)}, Y^{(I)}]]$ which is coassociative, cocommutative, and admits a coinverse. Often we identify μ^* with the its restriction to the free topological generators $\underline{X}$.

It is easy to see that every commutative formal group law on $\widehat{\mathbb{A}}^{(I)}$ defines a commutative smooth formal group.

Some Examples

Example. The formal group $\widehat{\mathbb{G}}_a$ attached to the additive group:

$$\widehat{\mathbb{G}}_a(N) := N,$$

the additive group underlying the nilpotent k-algebra N.

Example. The formal group $\widehat{\mathbb{G}}_m$ attached to the multiplicative group:

$$\widehat{\mathbb{G}}_m(N) := 1 + N \subset (k \oplus N)^\times \qquad \forall\, N \in \mathfrak{Nilp}_k \,.$$

Here $(k \oplus N)^\times$ denotes the group of units of the augmented k-algebra $k \oplus N$, so the group law is $(1+u) \cdot (1+v) = 1 + u + v + uv$ for $u, v \in N$.

2.2.10 The Lubin-Tate formal group

Let $\mathcal{O}$ be a complete discrete valuation ring whose residue field κ is a finite field with $q = p^a$ elements, where p is a prime number. Let π be a uniformizing element of $\mathcal{O}$. Recall that a *Lubin-Tate* formal group law over $\mathcal{O}$ is a one-dimensional smooth formal group $G = \mathrm{Spf}(\mathcal{O}[[X]])$ over $\mathcal{O}$ with an endomorphism $\phi : G \to G$ such that

$$\phi(X) := \phi^*(X) \equiv \pi X + X^q \pmod{(\pi, X^2)}.$$

It is well-known that every polynomial $\phi(X) \in \mathcal{O}[[X]]$ satisfying the above property uniquely determines a formal group law $\Phi_\phi(X,Y)$ on $\widehat{\mathbb{A}^1} = \mathrm{Spf}(\mathcal{O}[[X]])$ such that $\phi(X)$ defines an endomorphism of $\Phi_\phi(X,Y)$. In fact there is a ring homomorphism $\alpha : \mathcal{O} \to \mathrm{End}(\Phi_\phi)$ such that $\alpha(\pi) = \phi(X)$, and $\phi(a)$ induces "multiplication by a" on the Lie algebra, $\forall\, a \in \mathcal{O}$. Moreover for any two Lubin-Tate formal groups $\Phi_{\phi_1}, \Phi_{\phi_2}$ over $\mathcal{O}$, there exists a unique $\mathcal{O}$-equivariant isomorphism $\psi : \Phi_{\phi_1} \xrightarrow{\sim} \Phi_{\phi_2}$ such that $\psi(X) \equiv X \pmod{X^2}$.

Below is another description of the the Lubin-Tate formal group law via its logarithm. Let K be the fraction field of $\mathcal{O}$. Let

$$f_\pi(X) = \sum_{n\geq 0} \frac{X^{q^n}}{\pi^n} = X + \frac{X^q}{\pi} + \frac{X^{q^2}}{\pi^2} + \cdots \in K[[X]].$$

Let $\Phi_\pi(X,Y) = f_\pi^{-1}(f_\pi(X) + f_\pi(Y))$. A priori $\Phi_\pi(X,Y)$ has coefficients in K, but in fact $\Phi_\pi(X,Y) \in \mathcal{O}[[X,Y]]$. This can be proved directly, or one can use the "functional equation lemma" on p.9 of [14], since $f_\pi(X)$ satisfies the functional equation

$$f_\pi(X) = X + \frac{1}{\pi} f_\pi(X^q).$$

It follows that $\Phi_\pi(X,Y)$ is a one-dimensional formal group law, and $f_\pi(X)$ is the logarithm of $\Phi_\pi(X,Y)$. Moreover one checks that the polynomial

$$\phi_\pi(X) := f_\pi^{-1}(\pi\, f_\pi(X))$$

has coefficients in $\mathcal{O}$ and satisfies

$$\phi_\pi(X) \equiv \pi X + X^q \pmod{\pi\, x^2\, \mathcal{O}[[X]]}.$$

Hence $\Phi_\pi(X,Y)$ is a Lubin-Tate formal group law for $(\mathcal{O}, \pi)$.

Definition 2.2.11. We define a *restricted version* of the smooth formal group attached to the universal Witt vector group, denoted by Λ:

$$\Lambda(N) = 1 + t\, k[t] \otimes_k N \subset ((k \oplus N)[t])^\times \quad \forall\, N \in \mathfrak{Nilp}_k.$$

In other words, the elements of $\Lambda(N)$ consists of all polynomials of the form $1 + u_1 t + u_2 t^2 + \cdots + u_r t^r$ for some $r \geq 0$, where $u_i \in N$ for $i = 1, \ldots, r$. The group law of $\Lambda(N)$ comes from multiplication in the polynomial ring $(k \oplus N)[t]$ in one variable t. The formal group Λ will play the role of a free generator in the category of (smooth) formal groups. When we want to emphasize that the polynomial $1 + \sum_{i\geq 1} u_i t^i$ is regarded as an element of $\Lambda(N)$, we denote it by $\lambda(1 + \sum_{i\geq 1} u_i t^i)$.

Remark 2.2.12. (i) It is easy to see that $\Lambda(k[[X]]^+)$ consists of all formal power series in $k[[X,t]]$ of the form

$$1+\sum_{m,n\geq 1} b_{m,n}X^m t^n\,,\quad b_{m,n}\in k$$

such that for every m, there exists an integer $C(m)$ such that $b_{m,n}=0$ for all $n\geq C(m)$.

(ii) The formal completion $\widetilde{W}^\wedge$ of the universal Witt vector group $\widetilde{W}$, defined in §2.6, is given by

$$\widetilde{W}^\wedge(N)=1+t\,N[[t]]\subset((k\oplus N)[[t]])^\times\quad\forall\;\; N\in\mathfrak{Nilp}_k\,.$$

In particular $\widetilde{W}(k[[X]]^+)$ consists of *all* power series of the form

$$1+\sum_{m,n\geq 1} b_{m,n}X^m t^n\,,\quad b_{m,n}\in k\;\forall m,n\geq 1$$

in $k[[X,t]]$. However, this functor $\widetilde{W}^\wedge$ does not commute with infinite direct sums in $\mathfrak{Nilp}_k$, so it is not a commutative smooth formal group according to Def. 2.2.7.

Exercise 2.2.1. Prove that for every nilpotent k-algebra N, every element of $\Lambda(N)$ can be uniquely expressed as a finite product

$$(1-a_1\,t)\,(1-a_2\,t^2)\cdots(1-a_m\,t^m)$$

with $a_1,\ldots,a_m\in N$. Deduce that

$$\Lambda(k[[X]]^+)\;=$$

$$\left\{\prod_{m,n\geq 1}(1-a_{mn}\,X^m\,t^n)\;\middle|\;a_{m,n}\in k,\;\forall m\,\exists\,C_m>0\text{ s.t. }a_{mn}=0\text{ if }n\geq C_m\right\}$$

2.3 The Cartier ring

Definition 2.3.1. Let k be a commutative ring with 1. Let $H:\mathfrak{Nilp}_k\to\mathfrak{Ab}$ be a functor from the category of commutative nilpotent k-algebras to the category of abelian groups, extended to the category of topologically nilpotent k-algebras by filtered inverse limit as in 2.2.3. We say that H is *weakly symmetric*, or equivalently that H satisfies the *weak symmetry condition*, if for every $n\geq 1$, the natural map

$$H((k[[T_1,\ldots,T_n]]^+)^{S_n})\to H(k[[T_1,\ldots,T_n]]^+)^{S_n}$$

induced by the inclusion $k[[T_1,\ldots,T_n]]^{S_n}\hookrightarrow k[[T_1,\ldots,T_n]]$ is an isomorphism. Here S_n is the symmetric group in n letters operating naturally on the power series ring $k[[T_1,\ldots,T_n]]$. Note that $k[[T_1,\ldots,T_n]]^{S_n}$ is the power series ring generated by the elementary symmetric polynomials in the variables $T_1,\ldots,T_n$.

Lemma 2.3.2. *Let k be a commutative ring with 1. Let $H : \mathfrak{Nilp}_k \to \mathfrak{Ab}$ be a functor. Suppose that H is left exact, that is H commutes with finite inverse limits. Then H is weakly symmetric. In particular this is the case if H is a smooth commutative formal group over k.*

Proof. The ring $(k[[T_1, \dots, T_n]]^+)^{S_n}$ is the fiber product of two ring homomorphisms from $k[[T_1, \dots, T_n]]^+$ to $\prod_{\sigma \in S_n} k[[T_1, \dots, T_n]]^+$; one is the diagonal embedding, the other sends $f(\underline{T})$ to $(f(\underline{T}^\sigma))_{\sigma \in S_n}$. Applying the half-exactness of H to this fiber product, one deduces (i). The stronger statement (ii) follows from the same argument. □

Exercise 2.3.1. Prove the following stronger version of 2.3.2: If $H : \mathfrak{Nilp}_k \to \mathfrak{Ab}$ is weakly left exact, then H is weakly symmetric. (Hint: Consider the homomorphism $\alpha : k[[T_1, \dots, T_n]]^+ \times k[[T_1, \dots, T_n]]^+ \to \prod_{\sigma \in S_n} k[[T_1, \dots, T_n]]^+$ used in the proof of 2.3.2. Let α' be the homomorphism of induced by α between the graded k-modules associated to the source and the target of α. First show that each graded piece of $\mathrm{Coker}(\alpha')$ is a free k-module.)

Theorem 2.3.3. *Notation as in 2.3.1, and assume that $H : \mathfrak{Nilp}_k \to \mathfrak{Ab}$ satisfies the weak symmetry condition. Let $\Lambda = \Lambda_k$ be the functor defined in 2.2.11. Then the map*

$$Y_H : \mathrm{Hom}(\Lambda_k, H) \to H(k[[X]]^+)$$

which sends each homomorphism $\alpha : \Lambda \to H$ of group-valued functors to the element $\alpha_{k[[X]]^+}(1 - Xt) \in H(k[[X]]^+)$ is a bijection.

Remark. (i) Thm. 2.3.3 can be regarded as a sort of Yoneda isomorphism. The inverse of Y_H is given in the proof.

(ii) The formal group Λ is in some sense a free generator of the additive category of commutative smooth formal groups, a phenomenon reflected in Thm. 2.3.3.

Proof. Suppose that $\alpha \in \mathrm{Hom}(\Lambda, H)$. Given any nilpotent k-algebra N and any element $f(t) = 1 + u_1 t + u_2 t^2 + \cdots + u_n t^n \in \Lambda(N)$, we explain why the element $\alpha_N(f) \in H(N)$ is determined by the element $h_\alpha := \alpha_{k[[X]]^+}(1 - Xt) \in H(k[[X]]^+)$.

Let $U_1, \dots, U_n$ be variables. Let $\beta = \beta_{f,n} : k[[U_1, \dots, U_n]]^+ \to N$ be the continuous k-linear homomorphism such that $\beta(U_i) = (-1)^i u_i$. Let

$$\delta = \delta_n : k[[U_1, \dots, U_n]]^+ \to k[[X_1, \dots, X_n]]^+$$

be the continuous homomorphism sending each U_i to the i-th elementary polynomial in the variables $X_1, \dots, X_n$. Clearly

$$\alpha_N(f) = H(\beta)\alpha_{k[[U_1, \dots, U_n]]^+}(1 - U_1 t + \cdots + (-1)^n U_n t^n) .$$

Moreover we have

$$\begin{aligned}\alpha_{k[[U_1,\dots,U_n]]^+}(1-U_1\,t+\cdots+(-1)^nU_n\,t^n) &= \sum_{i=1}^n \alpha_{k[[X_1,\dots,X_n]]^+}(1-X_i\,t)\\ &= \sum_{i=1}^n H(\iota_i)(h_\alpha)\end{aligned}$$

in $H(k[[X_1,\dots,X_n]]^+)^{S_n} = H(k[[U_1,\dots,U_n]]^+) \subset H(k[[X_1,\dots,X_n]]^+)$, where $\iota_i : k[[X]]^+ \to k[[X_1,\dots,X_n]]^+$ is the continuous k-algebra homomorphism sending X to X_i. The two displayed formulas shows how to compute $\alpha_N(f)$ for any element $f(t) \in H(N)$ in terms of $\alpha_{k[[X]]}(1-Xt)$. The injectivity of Y_H follows.

Conversely, given an element $h \in H(k[[X]]^+)$, we have to construct a homomorphism of functors $\alpha \in \operatorname{Hom}(\Lambda, H)$ such that $\alpha_{k[[X]]}(1-Xt) = h$. The argument above provides a procedure to get an element $\alpha_N(f) \in H(N)$ for any element $f(t) \in \Lambda(N)$ for a nilpotent k-algebra N. Explicitly, for $f = 1 + u_1\,t + u_2\,t^2 + \cdots + u_n\,t^n \in \Lambda(N)$,

- let $\beta_{f,n} : k[[U_1,\dots,U_n]]^+ \to N$ be the continuous k-linear homomorphism such that $\beta_{f,n}(U_i) = (-1)^i u_i$ for each i,
- let $j_n : k[[U_1,\dots,U_n]]^+ \hookrightarrow k[[X_1,\dots,X_n]]^+$ be the continuous k-linear injection such that $j(U_i)$ is equal to the i-th elementary symmetric polynomial in $X_1,\dots,X_n$,
- let $\iota_i : k[[X]]^+ \to k[[X_1,\dots,X_n]]^+$ be the continuous k-linear homomorphism such that $\iota_i(X) = X_i$, $i = 1,\dots,n$, and
- let $\tilde h_n = \tilde h_{f,n} \in H(k[[U_1,\dots,U_n]]^+)$ be the element of $H(k[[U_1,\dots,U_n]]^+)$ such that $H(j_n)(\tilde h_n) = \sum_{i=1}^n H(\iota_i)(h)$.

Define $\alpha_N(f)$ by

$$\boxed{\alpha_N(f) = H\left(\beta_{f,n}\right)\left(\tilde h_{f,n}\right)}$$

It is not hard to check that the element $\alpha_N(f) \in H(N)$ is independent of the choice of the integer n, so that $\alpha_N(f)$ is well-defined. This is left as an exercise, as well as the fact that the collection of maps α_N defines a functor from $\mathfrak{Nilp}_k$ to $\mathfrak{Ab}$.

Lastly, we verify that $\alpha(f_1 + f_2) = \alpha(f_1) + \alpha(f_2)$ for any $f_1(t), f_2(t) \in H(k[[X]]^+)$. It suffices to check this in the universal case. In other words, it suffices to verify the equality $\alpha(f_1+f_2) = \alpha(f_1)+\alpha(f_2)$ in $H(k[[U_1,\dots,U_n,V_1,\dots,V_m]]^+)$, where $f_1(t) = 1-U_1\,t+\cdots+(-1)^nU_n\,t^n$ and $f_2(t) = 1-V_1\,t+\cdots+(-1)^mV_n\,t^m$. As above we may assume that $U_1,\dots,U_n$ are the elementary symmetric polynomials in the variables $X_1,\dots,X_n$ and $V_1,\dots,V_m$ are the elementary symmetric polynomials in the variables $Y_1, \ldots, Y_m$. Let ι_i (resp. ι'_j) be the continuous homomorphism from $k[[X]]$ to $k[[X_1, \ldots, X_n, Y_1, \ldots, Y_m]]$ such that $\iota_i(X) = X_i$ (resp. $\iota'_j(X) = Y_j$.) Then we have

$$\begin{aligned}\alpha(f_1) = \sum_{i=1}^{n} H(\iota_i)(h) \in H((k[[\underline{X}]]^+)^{S_n}) &= H((k[[\underline{X}]]^+)^{S_n}) \\ &= H(k[[\underline{U}]]^+) \subset H(k[[\underline{U}, \underline{V}]]^+)\end{aligned}$$

$$\begin{aligned}\alpha(f_2) = \sum_{j=1}^{m} H(\iota'_j)(h) \in H((k[[\underline{Y}]]^+)^{S_m}) &= H((k[[\underline{Y}]]^+)^{S_m}) \\ &= H(k[[\underline{V}]]^+) \subset H(k[[\underline{U}, \underline{V}]]^+)\end{aligned}$$

and

$$\begin{array}{rcl}\alpha(f_1 + f_2) & = & \sum_{i=1}^{n} H(\iota_i)(h) + \sum_{j=1}^{m} H(\iota'_j)(h) \\ & \in & H(k[[X_1, \ldots, X_n, Y_1, \ldots, Y_m]]^+)^{S_{n+m}} \\ & = & H((k[[X_1, \ldots, X_n, Y_1, \ldots, Y_m]]^+)^{S_{n+m}}) \\ & \subset & H((k[[\underline{X}, \underline{Y}]]^+)^{S_n \times S_m}) \;=\; H(k[[\underline{U}, \underline{V}]]^+)\,.\end{array}$$

We conclude that $\alpha(f_1 + f_2) = \alpha(f_1) + \alpha(f_2)$. ☐

Corollary 2.3.4. *Let $h = h(X, t)$ be an element of $\Lambda(k[[X]]^+)$, and let $\Phi = \Phi_h$ be the endomorphism of Λ_k such that $\Phi_{k[[X]]}(1 - Xt) = h(X, t)$. For each $n \in \mathbb{N}$, define power series ${}^{n}\!A_{h,1}(U_1, \ldots, U_n), \ldots, {}^{n}\!A_{h,n}(U_1, \ldots, U_n) \in k[[U_1, \ldots, U_n]]^+$ by*

$$\prod_{i=1}^{n} h(X_i, t) = 1 + {}^{n}\!A_{h,1}(\sigma_1(\underline{X}), \ldots, \sigma_n(\underline{X}))\, t + \ldots + {}^{n}\!A_{h,n}(\sigma_1(\underline{X}), \ldots, \sigma_n(\underline{X}))\, t^n\,,$$

where $\sigma_i(\underline{X})$ denotes the i-th elementary symmetric polynomial in $X_1, \ldots, X_n$.

(i) *Let N be a nilpotent k-algebra, and let $f(t) = 1 + a_1 t + a_2 t^2 + \cdots + a_n t^n$ be an element of $\Lambda_k(N)$. Then $\Phi_N(f) = \Phi_{h,N}(f)$ is given by*

$$\begin{aligned}\Phi_N(f) = 1 + {}^{n}\!A_{h,1}(-u_1, u_2, \ldots, (-1)^n u_n)\, t + \cdots \\ + {}^{n}\!A_{h,n}(-u_1, u_2, \ldots, (-1)^n u_n)\, t^n\,.\end{aligned}$$

(ii) *We have ${}^{n+1}\!A_{h,n+1}(U_1, \ldots, U_n, 0) = 0$, and*

$${}^{n+1}\!A_{h,i}(U_1, \ldots, U_n, 0) = {}^{n}\!A_{h,i}(U_1, \ldots, U_n)$$

for each $i = 1, 2, \cdots, n$.

(iii) *Suppose that $h(X, t) \equiv 1 \mod (X^m)$, and let $s = \lceil \frac{m}{n} \rceil$. Then*

$${}^{n}\!A_{h,i}(U_1, \ldots, U_n) \equiv 0 \mod (U_1, \ldots, U_n)^s$$

for $i = 1, \ldots, n$.

(iv) *In the situation of* (i) *above, suppose that* $N^r = (0)$, *then* $\Phi_{h,N}(f) = 0$ *if* $h(X,t) \equiv 1 \mod (X^{(r-1)n+1})$.

Proof. The statements (i), (ii) are special cases of Thm. 2.3.3. The statements (iii), (iv) are easy and left as exercises. □

Definition 2.3.5. Define $\mathrm{Cart}(k)$ to be $(\mathrm{End}(\Lambda_k))^{\mathrm{op}}$, the opposite ring of the endomorphism ring of the smooth formal group Λ_k. According to Thm. 2.3.3, for every weakly symmetric functor $H : \mathfrak{Nilp}_k \to \mathfrak{Ab}$, the abelian group $H(k[[X]]^+) = \mathrm{Hom}(\Lambda_k, H)$ is a *left* module over $\mathrm{Cart}(k)$.

Definition 2.3.6. We define some special elements of the Cartier ring $\mathrm{Cart}(k)$, naturally identified with $\Lambda(k[[X]])$ via the bijection $Y = Y_\Lambda : \mathrm{End}(\Lambda) \xrightarrow{\sim} \Lambda(k[[X]]^+)$ in Thm. 2.3.3.

(i) $V_n := Y^{-1}(1 - X^n t)\,, n \geq 1,$

(ii) $F_n := Y^{-1}(1 - X t^n)\,, n \geq 1,$

(iii) $[c] := Y^{-1}(1 - c X t)\,, c \in k.$

Lemma 2.3.7. *For every positive integer* n, *denote by* $\phi_n : k[[X]] \to k[[X]]$ *the* k*-algebra homomorphism which sends* X *to* X^n. *For every* $c \in k$, *denote by* $\psi_c : k[[X]] \to k[[X]]$ *the* k*-algebra homomorphism which sends* X *to* cX. *Then for every weakly symmetric functor* $H : \mathfrak{Nilp} \to \mathfrak{Ab}$, *we have*

$$V_n \gamma = H(\phi_n)(\gamma)\,, \quad [c]\,\gamma = H(\psi_c)\gamma$$

for every $\gamma \in H(k[[X]]^+)$ *and every* $c \in k$. *Applying the above to* Λ, *we get*

$$V_n\,[c]\,F_m = Y^{-1}(1 - c\,X^n\,t^m)$$

in $\mathrm{Cart}(k)$.

Proof. Exercise. □

Remark 2.3.8. Let $\widetilde{W}$ be the ring scheme of universal Witt vectors defined in §2.6. For each positive integer n we have endomorphisms V_n, F_n of $\widetilde{W}$. Consider the element $\omega(1 - X\,T) \in \widetilde{W}(\mathbb{Z}[[X]])$. Then $V_n(\omega(1 - X\,T)) = \omega(1 - X\,T^n)$, and $F_n(\omega(1 - X\,T)) = \omega(1 - X^n\,T)$. This contrasts with the notation used in the Cartier ring: $V_n = Y^{-1}(1 - X^n\,t)$, $F_n = Y^{-1}(1 - X\,T^n)$, see also Exer. 2.3.4. This kind of "flipping" is inevitable, since $\mathrm{Cart}(\mathbb{Z})$ operates on the *right* of $\widetilde{W}$, and we want the same commutation relation of V_n, F_n with the endomorphisms $[c]$ in 2.3.11 to hold in all situations.

Remark. Often $H(\phi_n)(\gamma)$ is abbreviated as $\gamma(X^n)$, and $H(\psi_c)(\gamma)$ is shortened to $\gamma(cX)$. This is compatible with the standard notation when H is representable as a formal scheme $\mathrm{Spf}\,R$, where R is an augmented k-algebra complete with respect to the augmentation ideal. The elements of $H(k[[X]])$ are identified with continuous homomorphisms $R \to k[[X]]$, thought of as "formal curves" in $\mathrm{Spf}\,R$.

Corollary 2.3.9. *For every commutative ring with* 1 *we have*

$$\mathrm{Cart}(k) = \left\{ \sum_{m,n\geq 1} V_m\,[c_{mn}]\,F_n \;\middle|\; c_{mn} \in k,\ \forall m\,\exists\, C_m > 0 \text{ s.t. } c_{mn} = 0 \text{ if } n \geq C_m \right\}$$

Proof. This is a direct translation of Exer. 2.2.1.

Exercise 2.3.2. Let k be a commutative ring with 1 and let n be an integer. Prove that n is invertible in $\mathrm{Cart}(k)$ if and only if n is invertible in k.

Lemma 2.3.10. *Suppose that* $H : \mathfrak{N}ilp_k \to \mathfrak{A}b$ *is weakly symmetric. Let* $n \geq 1$ *be a positive integer. Denote by* $k[\zeta_n]$ *the* k*-algebra* $k[T]/(T^n - 1)$*, and let* $\zeta = \zeta_n$ *be the image of* T *in* $k[T]/(T^n - 1)$*. Denote by* $k[\zeta][[X^{\frac{1}{n}}]]^+$ *the* $k[\zeta]$*-algebra* $k[\zeta][[X,U]]^+/(U^n - X)k[\zeta][[X,U]]$*, and let* $X^{\frac{1}{n}}$ *be the image of* U *in* $k[\zeta][[X,U]]^+/(U^n - X)k[\zeta][[X,U]]$*. For each* $i = 0, \ldots, n-1$*, let* $\phi_{n,i} : k[[X]]^+ \to k[\zeta][[X^{\frac{1}{n}}]]^+$ *be the homomorphism of* k*-algebras which maps* X *to* $\zeta^i X^{\frac{1}{n}}$*. Then*

$$F_n \cdot \gamma = \sum_{i=0}^{n} H(\phi_{n,i})(\gamma)$$

for every $\gamma \in H(k[[X]]^+)$*; the equality holds in* $H(k[\zeta][[X^{\frac{1}{n}}]]^+)$*. Formally one can write the above formula as* $F_n \cdot \gamma = \sum_{i=0}^{n-1} \gamma(\zeta^i X^{\frac{1}{n}})$*.*

Proof. Use Cor. 2.3.4 and the equality $\prod_{i=1}^{n}(1 - \zeta_n^i X^{\frac{1}{n}} t^n)$. ☐

Proposition 2.3.11. *The following identities hold in* $\mathrm{Cart}(k)$*.*

1. $V_1 = F_1 = 1$, $F_n\,V_n = n$.
2. $[a]\,[b] = [ab]$ *for all* $a, b \in k$
3. $[c]\,V_n = V_n\,[c^n]$, $F_n\,[c] = [c^n]\,F_n$ *for all* $c \in k$*, all* $n \geq 1$.
4. $V_m\,V_n = V_n\,V_m = V_{mn}$, $F_m\,F_n = F_n\,F_m = F_{mn}$ *for all* $m, n \geq 1$.
5. $F_n\,V_m = V_m\,F_n$ *if* $(m,n) = 1$.
6. $(V_n[a]F_n) \cdot (V_m[b]F_m) = r\,V_{\frac{mn}{r}}\left[a^{\frac{m}{r}}\,b^{\frac{n}{r}}\right]F_{\frac{mn}{r}}$, $r = (m,n)$, *for all* $a, b \in k$, $m, n \geq 1$.

Proof. We have seen that $\mathrm{Cart}(k)$ operates on the left of the set $H(k[[X]]^+)$ of all formal curves in H for every weakly symmetric functor $H : \mathfrak{Nilp}_k \to \mathfrak{Ab}$. For each of the above identities for elements in $\mathrm{Cart}(k)$, it suffices to check that the effect of both sides of the equality on the element $1 - Xt \in \Lambda(k[[X]]^+)$, by Thm. (2.3.3. The checking for (1)–(5) is straightforward using 2.3.7 and 2.3.10; it is left to the reader. The statement (6) follows from (1)–(5). ☐

Exercise 2.3.3. Let k be a commutative ring with 1 and let $\widetilde{W}^\wedge$ be the formal completion of the universal Witt vectors, so that $\widetilde{W}^\wedge(N) = 1 + N[[T]] \subset (k \oplus N)[[T]]^\times$.

(i) Prove that the map which sends every element $\Phi \in \mathrm{End}_k(\widetilde{W}^\wedge)$ to $\Phi_{k[[X]]}(1-XT)$ establishes a bijection between $\mathrm{End}_k(\widetilde{W}^\wedge)$ with the set of all power series in $k[[X,T]]$ of the form $1 + \sum_{m,n\geq 1} b_{mn} X^m T^n$, $b_{mn} \in k$.

(ii) Show that $\mathrm{End}_k(\widetilde{W}^\wedge)^{\mathrm{op}}$ can be identified with the set of all expressions

$$\sum_{m,n\geq 1} V_m\,[a_{mn}]\,F_n\,, \quad a_{mn} \in k\,,$$

such that the endomorphism represented by such a sum sends the element $1 - XT \in \widetilde{W}^\wedge(k[[X]]^+)$ to $\prod_{m,n\geq 1}(1 - a_{mn}X^m T^n)$. All identities in Prop. 2.3.11 hold in the ring $\mathrm{End}_k(\widetilde{W}^\wedge)^{\mathrm{op}}$.

Exercise 2.3.4. Let k be a commutative ring with 1. The Cartier ring $\mathrm{Cart}(k)$ operates naturally on the *right* of the formal group functor Λ_k. Let N be a nilpotent k-algebra. For every element $a \in N$, every element $c \in k$ and integers m, $n \geq 1$, prove that

(i) $(1 - a\,t^m)\cdot V_n = (1 - a^{\frac{n}{r}}\,t^{\frac{m}{r}})^r$, where $r = (m,n)$.

(ii) $(1 - a\,t^m)\cdot F_n = (1 - a\,t^{mn})$.

(iii) $(1 - a\,t^m)\cdot [c] = (1 - ac^m\,t^m)$.

(iv) Use (i)–(iii) to prove 2.3.11.

Proposition 2.3.12. *Let k be a commutative ring with* 1. *)*

(i) *The subset S of* $\mathrm{Cart}(k)$ *consisting of all elements of the form*

$$\sum_{n\geq 1} V_n[a_n]F_n\,, \quad a_n \in k\ \forall\, n \geq 1$$

form a subring of $\mathrm{Cart}(k)$.

(ii) *The injective map*

$$\widetilde{W}(k) \hookrightarrow \mathrm{Cart}(k), \quad \omega(\underline{a}) \mapsto \sum_{n\geq 1} V_n\,[a_n]\,F_n$$

is a homomorphism of rings.

Proof. Let S' the subset of the power series ring $k[[X,t]]$ consisting of all elements of the form $1+\sum_{m\geq 1} a_m\, X^m t^m$ such that $a_m \in k$ for all $m \geq 1$. Clearly S' is a subgroup of the unit group $k[[X,t]]^\times$ of $k[[X,t]]$. By definition $\mathrm{Cart}(k) = \Lambda(k[[X]]^+)$ is a subgroup of $k[[X,t]]^\times$, and $S = S' \cap \mathrm{Cart}(k)$. If follows that S is a subgroup of the additive group underlying $\mathrm{Cart}(k)$. The formula 2.3.11 (6) implies that the subset $S \subset \mathrm{Cart}(k)$ is stable under multiplication, hence it is a subring. The definition of multiplication for the universal Witt vectors in 2.6.2 tells us that the bijection in (ii) is an isomorphism of rings. ☐

Corollary 1. *Let $A_n(U,V) \in k[U,V]$ be polynomials defined by*

$$(1-UT)\cdot(1-VT) = (1-(U+V)T)\cdot \prod_{n\geq 1}(1-A_n(U,V)T^n)$$

Then for all $c_1, c_2 \in k$ we have

$$[c_1]+[c_2] = [c_1+c_2] + \sum_{\geq 1} V_n[A_n(c_1,c_2)]F_n\,.$$

Definition 2.3.13. The ring $\mathrm{Cart}(k)$ has a natural filtration $\mathrm{Fil}^\bullet\mathrm{Cart}(k)$ by right ideals $\mathrm{Fil}^j\mathrm{Cart}(k)$, where

$$\left\{ \sum_{m\geq j}\sum_{n\geq 1} V_m[a_{mn}]F_n \;\middle|\; a_{mn}\in k,\ \forall\, m\geq j,\ \exists\, C_m>0 \text{ s.t. } a_{mn}=0 \text{ if } n\geq C_m \right\}$$

for every integer $j \geq 1$. The Cartier ring $\mathrm{Cart}(k)$ is complete with respect to the topology given by the above filtration. Moreover each right ideal $\mathrm{Fil}^j\mathrm{Cart}(k)$ is open and closed in $\mathrm{Cart}(k)$.

Exercise 2.3.5. Prove the following statements.

(i) $[c]\cdot \mathrm{Fil}^j\mathrm{Cart}(k) \subseteq \mathrm{Fil}^j\mathrm{Cart}(k)$ for all $c\in k$, all $j\geq 1$.

(ii) $V_m \cdot \mathrm{Fil}^j\mathrm{Cart}(k) \subseteq \mathrm{Fil}^{mj}\mathrm{Cart}(k)$ for all $m, j \geq 1$.

(iii) $F_n \cdot \mathrm{Fil}^j\mathrm{Cart}(k) \subseteq \mathrm{Fil}^{\lceil \frac{j}{n}\rceil}\mathrm{Cart}(k)$ for all $n, j \geq 1$.

(iv) The right ideal of $\mathrm{Cart}(k)$, generated by all elements V_n with $n \geq j$, is dense in $\mathrm{Fil}^j\mathrm{Cart}(k)$.

(v) The quotient $\mathrm{Cart}(k)/\mathrm{Fil}^2\mathrm{Cart}(k)$ is canonically isomorphic to k.

(vi) Left multiplication by V_j induces a bijection

$$V_j : \mathrm{Cart}(k)/\mathrm{Fil}^2\mathrm{Cart}(k) \xrightarrow{\sim} \mathrm{Fil}^j\mathrm{Cart}(k)/\mathrm{Fil}^{j+1}\mathrm{Cart}(k)\,.$$

Exercise 2.3.6. (i) Show that $\mathrm{Cart}(k)$ is a topological ring, i.e. the multiplication is a continuous map for the topology given by the decreasing filtration $\mathrm{Fil}^{\bullet}\mathrm{Cart}(k)$ on $\mathrm{Cart}(k)$. (Hint: The point is to show that for any $x \in \mathrm{Cart}(k)$, the map $y \mapsto x \cdot y$ is continuous.)

(ii) Show that for any $n \geq 1$, there exists $x \in \mathrm{Cart}(k)$ and $y \in \mathrm{Fil}^n\mathrm{Cart}(k)$ such that $x \cdot y \notin \mathrm{Fil}^2\mathrm{Cart}(k)$.

Exercise 2.3.7. Let k be a commutative ring with 1.

(i) Show that the right $\mathrm{Cart}(k)$-module $T := \mathrm{Cart}(k)/\mathrm{Fil}^2\mathrm{Cart}(k)$ is a free k module with basis x_i, $i \geq 1$, where x_i :=the image of F_i in T.

(ii) Show that the right $\mathrm{Cart}(k)$-module T is naturally isomorphic to the Lie algebra $\mathrm{Lie}(\Lambda)$ of the smooth formal group Λ over k.

(iii) The free right $\mathrm{Cart}(k)$-module T in (i) above gives a ring homomorphism

$$\rho : \mathrm{Cart}(k) \to \mathrm{M}'_{\infty}(k)\,,$$

where $\mathrm{M}'_{\infty}(k)$ denotes the set of all $\mathbb{N}_{\geq 1} \times \mathbb{N}_{\geq 1}$-matrices $(c_{ij})_{i,j\geq 1}$ such that each row has at most a finite number of nonzero entries. The ring $\mathrm{M}'_{\infty}(k)$ operates on the right of the k-module $k^{\oplus \mathbb{N}_{\geq 1}}$, consisting of all row vectors indexed by $\mathbb{N}_{\geq 1}$ with at most a finite number of non-zero entries, and the natural surjection $\mathrm{Cart}(k) \twoheadrightarrow T$ is equivariant with respect to ρ. Prove that for each element $\sum_{m,n\geq 1} V_m[a_{mn}]F_n \in \mathrm{Cart}(k)$, $\rho(\sum_{m,n\geq 1} V_m[a_{mn}]F_n)$ is the matrix $(c_{ij})_{i,j\geq 1}$ with

$$c_{ij} = \sum_{r\,|\,(i,j)} \frac{i}{r} \left(a_{\frac{i}{r},\frac{j}{r}}\right)^r \qquad \forall\, i,j \geq 1.$$

(iv) Prove that ρ is an injection if and only if the natural map $k \to k \otimes_{\mathbb{Z}} \mathbb{Q}$ is an injection, or equivalently k is p-torsion free for every prime number p.

(v) Prove that ρ is an isomorphism if and only if k is a $\mathbb{Q}$-algebra, or equivalently every nonzero integer is invertible in k.

(vi) Use (iii) and the properties of the ghost coordinates of the universal Witt vectors to give another proof of 2.3.12 (ii). See 2.6.4 for the definition of ghost coordinates.

Exercise 2.3.8. Let $\rho : \mathrm{Cart}(k) \to \mathrm{M}'_{\infty}(k)$ be the homomorphism in 2.3.7 (iii).

(i) Show that an element $u \in \mathrm{Cart}(k)$ is in the subring $\widetilde{W}(k)$ if and only if $\rho(u)$ is a diagonal matrix in $\mathrm{M}'_{\infty}(k)$.

(ii) Let u be an element of $\mathrm{Cart}(k)$. Prove that u induces an isomorphism of Λ if and only if u induces an isomorphism on $\mathrm{Lie}(\Lambda)$.

(iii) Show that $\rho^{-1}(\mathrm{M}'_\infty(k)^\times) = \mathrm{Cart}(k)^\times$.

(iv) Let $w = \sum_{n\geq 1} V_n[a_n]F_n$ be an element of $\widetilde{W}(k) \subset \mathrm{Cart}(k)$. Prove that w is a unit in $\mathrm{Cart}(k)$ if and only if every sum of the form

$$\sum_{ij=m,\ i,j\in\mathbb{N}} i\, a_i^j$$

is a unit in k, for every integer $m \geq 1$.

(v) Show that $\widetilde{W}(k) \cap \mathrm{Cart}(k)^\times = \widetilde{W}(k)^\times$.

2.4 The main theorem of Cartier theory

Definition 2.4.1. Let k be a commutative ring with 1. A *V-reduced* left $\mathrm{Cart}(k)$-module is a left $\mathrm{Cart}(k)$-module M together with a separated decreasing filtration of M

$$M = \mathrm{Fil}^1 M \supset \mathrm{Fil}^2 M \supset \cdots \mathrm{Fil}^n M \supset \mathrm{Fil}^{n+1} \supset \cdots$$

such that each $\mathrm{Fil}^n M$ is an abelian subgroup of M and

(i) $(M, \mathrm{Fil}^\bullet M)$ is complete with respect to the topology given by the filtration $\mathrm{Fil}^\bullet M$. In other words, the natural map $\mathrm{Fil}^n M \to \varprojlim_{m\geq n} (\mathrm{Fil}^n M/\mathrm{Fil}^m M)$ is a bijection for all $n \geq 1$.

(ii) $V_m \cdot \mathrm{Fil}^n M \subset \mathrm{Fil}^{mn} M$ for all $m, n \geq 1$.

(iii) The map V_n induces a bijection $V_n : M/\mathrm{Fil}^2 M \xrightarrow{\sim} \mathrm{Fil}^n M/\mathrm{Fil}^{n+1} M$ for every $n \geq 1$.

(iv) $[c] \cdot \mathrm{Fil}^n M \subset \mathrm{Fil}^n M$ for all $c \in k$ and all $n \geq 1$.

(v) For every $m, n \geq 1$, there exists an $r \geq 1$ such that $F_m \cdot \mathrm{Fil}^r M \subset \mathrm{Fil}^n M$.

Definition 2.4.2. A V-reduced left $\mathrm{Cart}(k)$-module $(M, \mathrm{Fil}^\bullet M)$ is *V-flat* if $M/\mathrm{Fil}^2 M$ is a flat k-module. The k-module $M/\mathrm{Fil}^2 M$ is defined to be the *tangent space* of $(M, \mathrm{Fil}^\bullet M)$.

As an example, the free $\mathrm{Cart}(k)$-module $\mathrm{Cart}(k)$ has a filtration with

$$\mathrm{Fil}^n \mathrm{Cart}(k) = \sum_{m\geq n} V_m \mathrm{Cart}(k)\,,$$

making it a V-flat V-reduced left $\mathrm{Cart}(k)$-module. Its tangent space is naturally isomorphic to $k[t]$. See 2.3.5.

Exercise 2.4.1. Let $(M, \mathrm{Fil}^\bullet M)$ be a V-reduced left $\mathrm{Cart}(k)$-module and let n be a positive integer.

(i) For each $n \geq 1$, the subgroup of M generated by all $V_m \cdot M$, $m \geq n$ is dense in $\mathrm{Fil}^n M$. This follows from 2.4.1 (i)–(iii).

(ii) If M is a finitely generated left $\mathrm{Cart}(k)$-module, then $\mathrm{Fil}^n M = \mathrm{Fil}^n \mathrm{Cart}(k) \cdot M$.

(iii) Prove that M is finitely generated as a left $\mathrm{Cart}(k)$-module if and only if $M/\mathrm{Fil}^2 M$ is a finitely generated k-module.

(iv) Use 2.3.5 to show that properties (iv), (v) in Def. 2.4.1 follow from 2.4.1 (i)–(iii).

(vi) Prove the following strengthened form of 2.4.1 (v):

$$F_m \cdot \mathrm{Fil}^n M \subseteq \mathrm{Fil}^{\lceil \frac{n}{m} \rceil} M \quad \forall\, m, n \geq 1\,.$$

Definition 2.4.3. Let $H : \mathfrak{Nilp}_k \to \mathfrak{Ab}$ be a formal group functor as in 2.3.1. The abelian group $\mathrm{M}(H) := H(k[[X]]^+)$ has a natural structure as a left $\mathrm{Cart}(k)$-module according to Thm. 2.3.3 The $\mathrm{Cart}(k)$-module $\mathrm{M}(H)$ has a natural filtration, with

$$\mathrm{Fil}^n \mathrm{M}(H) := \mathrm{Ker}(H(k[[X]]^+) \to H(k[[X]]^+/X^n k[[X]]))\,.$$

We call the pair $(\mathrm{M}(H), \mathrm{Fil}^\bullet \mathrm{M}(H))$ the *Cartier module attached to H.*

Lemma 2.4.4. *Let $H : \mathfrak{Nilp}_k \to \mathfrak{Ab}$ be a functor which is weakly left exact and right exact in the sense of* 2.2.5. *Then* $(\mathrm{M}(H), \mathrm{Fil}^\bullet \mathrm{M}(H))$ *is a V-reduced left* $\mathrm{Cart}(k)$-*module. In particular, this is the case if H is a commutative smooth formal group.*

Proof. Since the functor H is right exact, we have

$$\mathrm{M}(H)/\mathrm{Fil}^{n+1}\mathrm{M}(H) \xrightarrow{\sim} H(k[[X]]^+/X^{n+1}k[[X]])\,,$$

and $\mathrm{Fil}^n \mathrm{M}(H)$ is equal to the image of $H(X^n k[[X]])$ in $H(k[[X]]^+)$ under the map induced by the inclusion $X^n k[[X]] \hookrightarrow k[[X]]^+$. By definition,

$$\mathrm{M}(H) = H(k[[X]]^+) = \varprojlim_n H(k[[X]]^+/X^n k[[x]]) = \varprojlim_n \mathrm{Fil}^n \mathrm{M}(H)\,.$$

Condition (i) follows.

The conditions (ii), (iv) of Definition 2.4.1 are easy to check; it is also easy to verify condition (v) of 2.4.1 holds with $r = mn$. These are left to the reader as exercises. Here we check that V_n induces an isomorphism from $\mathrm{gr}^1\mathrm{M}(H)$ to $\mathrm{gr}^n\mathrm{M}(H)$ for every $n \geq 1$.

Since H is weakly left exact as well, we have a functorial isomorphism

$$\mathrm{Fil}^n\mathrm{M}(H)/\mathrm{Fil}^{n+1}\mathrm{M}(H) \xrightarrow{\sim} H(X^n k[[X]]/X^{n+1}k[[X]])$$

for each $n \geq 1$. The isomorphism

$$k[[X]]^+/X^2k[[X]] \xrightarrow{\sim} X^n k[[X]]/X^{n+1}k[[X]]$$

in $\mathfrak{Nilp}_k$ which sends X to X^n induces an isomorphism $\mathrm{gr}^1\mathrm{M}(H) \xrightarrow{\sim} \mathrm{gr}^n\mathrm{M}(H)$. This isomorphism is equal to the map induced by V_n, so $(\mathrm{M}(H), \mathrm{Fil}^\bullet\mathrm{M}(H))$ is V-reduced. □

Lemma 2.4.5. *Let $H : \mathfrak{Nilp}_k \to \mathfrak{Ab}$ be a group-valued functor. If H is exact, i.e. it is left exact and right exact, then $(\mathrm{M}(H), \mathrm{Fil}^\bullet\mathrm{M}(H))$ is a V-reduced V-flat left $\mathrm{Cart}(k)$-module. In particular, this is the case if H is a commutative smooth formal group.*

Proof. The tangent functor $\mathrm{t}_H : \mathfrak{Mod}_k \to \mathfrak{Mod}_k$ of H, being the restriction to the category $\mathfrak{Mod}_k$ of an exact functor, is exact. The map $N \mapsto \mathrm{Lie}(G) \otimes_k N$ is a right exact functor from $\mathfrak{Mod}_k$ to $\mathfrak{Mod}_k$. These two functors are both right exact, commute with finite direct sums, and take the same value on the free k-module k, hence these two functors coincide on the category $\mathfrak{fpMod}_k$ of all finitely presented k-modules. So the functor $N \mapsto \mathrm{Lie}(G) \otimes_k N$ from $\mathfrak{fpMod}_k$ to $\mathfrak{Mod}_k$ is exact, because the tangent functor is. It is well-known that the last property of $\mathrm{Lie}(G) \cong \mathrm{M}_H/\mathrm{Fil}^2\mathrm{M}(H)$ implies that $\mathrm{M}_H/\mathrm{Fil}^2\mathrm{M}(H)$ if a flat k-module. □

Exercise 2.4.2. Let k be a commutative ring with 1. Let $\widetilde{W}(k)$ be the group of universal Witt vectors with entries in k, endowed with the filtration defined in 2.6.3 and the action of $\mathrm{Cart}(k)$ defined in 2.6.5. Prove that $(\widetilde{W}(k), \mathrm{Fil}|^\bullet\widetilde{W}(k))$ is a V-flat V-reduced left $\mathrm{Cart}(k)$-module. (In fact it is the Cartier module attached to $\widehat{\mathbb{G}}_m$.)

Exercise 2.4.3. Let k be a commutative ring with 1. Let $M = k[[X]]^+$, filtered by $\mathrm{Fil}^n M = X^n k[[X]]$, $n \geq 1$. Define operators $F_n, V_n, [c]$ on M, $n \in \mathbb{N}_{\geq 1}$, $c \in k$ as follows:

- $V_n(\sum_{m\geq 1} a_m X^m) = \sum_{m\geq 1} a_m X^{mn}$,
- $F_n(\sum_{m\geq 1} a_m X^m) = \sum_{m\geq 1} n a_{mn} X^m$,
- $[c](\sum_{m\geq 1} a_m X^m) = \sum_{m\geq 1} a_m c^m X^m$.

Prove that M is a V-reduced V-flat left $\mathrm{Cart}(k)$-module. (In fact it is the Cartier module attached to $\widehat{\mathbb{G}}_a$.)

Lemma 2.4.6. *Let $k \to R$ be a homomorphism between commutative rings with* 1. *Let* $(M, \mathrm{Fil}^\bullet M)$ *be a V-reduced left* $\mathrm{Cart}(k)$*-module. Let* $\mathrm{Fil}^\bullet\big(\mathrm{Cart}(R) \otimes_{\mathrm{Cart}(k)} M\big)$ *be the tensor product filtration on* $\mathrm{Cart}(R) \otimes_{\mathrm{Cart}(k)} M$*, defined by*

$$\begin{aligned}&\mathrm{Fil}^n\big(\mathrm{Cart}(R) \otimes_{\mathrm{Cart}(k)} M\big)\\ &= \textstyle\sum_{i,j\geq 1,\, i+j\geq n} \mathrm{Image}\,\big(\mathrm{Fil}^i\mathrm{Cart}(R) \otimes \mathrm{Fil}^j M \to \mathrm{Cart}(R) \otimes_{\mathrm{Cart}(k)} M\big)\end{aligned}$$

for every $n \geq 1$. Let M_R be the completion of the $\mathrm{Cart}(R) \otimes_{\mathrm{Cart}(k)} M$ *with respect to the topology defined by the filtration* $\mathrm{Fil}^\bullet\big(\mathrm{Cart}(R) \otimes_{\mathrm{Cart}(k)} M\big)$*, and let* $\mathrm{Fil}^\bullet M_R$ *be the induced filtration on M_R.*

(i) *The pair* $(M_R, \mathrm{Fil}^\bullet M_R)$ *is a V-reduced left* $\mathrm{Cart}(R)$*-module.*

(ii) *If* $(M, \mathrm{Fil}^\bullet M)$ *is V-flat, then* $(M_R, \mathrm{Fil}^\bullet M_R)$ *is V-flat.*

(iii) $M_R/\mathrm{Fil}^2 M_R \cong R \otimes_k (M/\mathrm{Fil}^2 M)$.

Proof. Exercise. □

Exercise 2.4.4. Let $(M, \mathrm{Fil}^\bullet M)$ be a V-reduced left $\mathrm{Cart}(k)$-module. Let $R = k[\epsilon]/(\epsilon^2)$. The map $R \twoheadrightarrow k$ induces a surjective ring homomorphism $\mathrm{Cart}(R) \twoheadrightarrow \mathrm{Cart}(k)$, so we can regard M as a left module over $\mathrm{Cart}(R)$. Show that $(M, \mathrm{Fil}^\bullet M)$ is a V-reduced left $\mathrm{Cart}(R)$-module which is not V-flat.

Definition 2.4.7. Let M be a V-reduced left $\mathrm{Cart}(k)$-module and let Q be a right $\mathrm{Cart}(k)$-module.

(i) For every integer $m \geq 1$, let $Q_m := \mathrm{Ann}_Q(\mathrm{Fil}^m\mathrm{Cart}(k))$ be the subgroup of Q consisting of all elements $x \in Q$ such that $x \cdot \mathrm{Fil}^m\mathrm{Cart}(k) = (0)$. Clearly we have $Q_1 \subseteq Q_2 \subseteq Q_3 \subseteq \cdots$.

(ii) For each $m, r \geq 1$, define $Q_m \odot M^r$ to be the image of $Q_m \otimes \mathrm{Fil}^r M$ in $Q \otimes_{\mathrm{Cart}(k)} M$. If $r \geq m$ and $s \geq m$, then $Q_m \odot M^r = Q_m \odot M^s$; see 2.4.5. Hence $Q_m \odot M^m \subseteq Q_n \odot M^n$ if $m \leq n$.

(iii) Define the *reduced tensor product* $Q \overline{\otimes}_{\mathrm{Cart}(k)} M$ by

$$Q \overline{\otimes}_{\mathrm{Cart}(k)} M = Q \otimes_{\mathrm{Cart}(k)} M \Big/ \Big(\bigcup_m (Q_m \odot M^m)\Big)\,.$$

(iv) We say that Q is a *torsion* right $\mathrm{Cart}(k)$-module if

$$Q = \bigcup_m \mathrm{Ann}_Q(\mathrm{Fil}^m\mathrm{Cart}(k))\,.$$

Exercise 2.4.5. Notation as in 2.4.7.

(i) For every $x \in Q_m$ and every $y \in \mathrm{Fil}^n M$ with $n \geq m$, let $y_1 \in M$ and $y_2 \in \mathrm{Fil}^{n+1} M$ be such that $y = V^n y_1 + y_2$. Then $x \otimes y = x \otimes y_2$ in $Q \otimes_{\mathrm{Cart}(k)} M$.

(ii) Show that $Q_m \odot M^r = Q_m \odot M^s$ if $r, s \geq m$.

Exercise 2.4.6. Let $(M, \mathrm{Fil}^{\bullet} M)$ be a V-reduced left $\mathrm{Cart}(k)$-module.

(i) Let N be a nilpotent k-algebra such that $N^2 = (0)$. Prove that

$$\Lambda(N) \overline{\otimes}_{\mathrm{Cart}(k)} M \cong N \otimes_k (M/\mathrm{Fil}^2 M)\,.$$

(ii) Prove that $\Lambda(k[[X]]^+/X^n k[[X]]) \overline{\otimes}_{\mathrm{Cart}(k)} M \cong M/\mathrm{Fil}^n M$.

Lemma 2.4.8. *Let* $0 \to Q' \to Q \to Q'' \to 0$ *be a short exact sequence of torsion right* $\mathrm{Cart}(k)$*-modules. Let* M *be a* V*-reduced left* $\mathrm{Cart}(k)$*-module.*

(i) *The map* $Q \odot M \to Q'' \odot M$ *is surjective.*

(ii) *The sequence* $Q' \overline{\otimes}_{\mathrm{Cart}(k)} M \to Q \overline{\otimes}_{\mathrm{Cart}(k)} M \to Q'' \to \overline{\otimes}_{\mathrm{Cart}(k)} M \to 0$ *is exact.*

Proof. The statement (ii) follows from (i) and the general fact that

$$Q' \otimes_{\mathrm{Cart}(k)} \to Q \otimes_{\mathrm{Cart}(k)} \to Q'' \otimes_{\mathrm{Cart}(k)} \to 0$$

is exact. It remains to prove (i).

Suppose that x'' is an element of $\mathrm{Ann}_{Q''}(\mathrm{Fil}^m \mathrm{Cart}(k))$, and y is an element of $\mathrm{Fil}^m M$. We must show that $x'' \otimes y$ belongs to the image of $Q \odot M$ in $Q'' \otimes_{\mathrm{Cart}(k)} M$. Pick $x \in Q$ which maps to $x'' \in Q''$. Because Q is torsion, there exists an integer $n \geq m$ such that $x \cdot \mathrm{Fil}^n \mathrm{Cart}(k) = 0$. Write y as $y = y_1 + y_2$, with $y_1 \in \mathrm{Fil}^m \mathrm{Cart}(k) \cdot M$ and $y_2 \in \mathrm{Fil}^n M$. Then $x'' \otimes y = x'' \otimes y_2$ in $Q'' \otimes_{\mathrm{Cart}(k)} M$. So the element $x \otimes y_2$ in $Q \odot M$ maps to $x'' \otimes y_2$. ☐

Theorem 2.4.9. *Let* k *be a commutative ring with* 1*. Then there is a canonical equivalence of categories, between the category of smooth commutative formal groups over* k *as defined in* 2.2.7 *and the category of* V*-flat* V*-reduced left* $\mathrm{Cart}(k)$*-modules, defined as follows.*

$$\begin{tikzcd}
\{\text{smooth formal groups over } k\} \arrow[r, "\sim"] & \{V\text{-flat } V\text{-reduced left } \mathrm{Cart}(k)\text{-mod}\} \\
G \arrow[r, mapsto] & \mathrm{M}(G) = \mathrm{Hom}(\Lambda, G) \\
\Lambda \overline{\otimes}_{\mathrm{Cart}(k)} M & M \arrow[l, mapsto]
\end{tikzcd}$$

Recall that $\mathrm{M}(G) = \mathrm{Hom}(\Lambda, G)$ *is canonically isomorphic to* $G(X\,k[[X]])$*, the group of all formal curves in the smooth formal group* G*. The reduced tensor product* $\Lambda \overline{\otimes}_{\mathrm{Cart}(k)} M$ *is the functor whose value at any nilpotent* k*-algebra* N *is* $\Lambda(N) \overline{\otimes}_{\mathrm{Cart}(k)} M$*.*

Proof of Thm. 2.4.9.

The key steps of the proof of are Prop. 2.4.11 and Thm. 2.4.13 below.

Lemma 2.4.10. *Let* $\alpha : (L, \mathrm{Fil}^\bullet L) \to (M, \mathrm{Fil}^\bullet M)$ *be a homomorphism between* V*-reduced left* $\mathrm{Cart}(k)$*-modules, i.e.* $\alpha(\mathrm{Fil}^i L) \subseteq \mathrm{Fil}^i M$ *for all* $i \geq 1$*. Then the following are equivalent.*

(i) $\alpha(\mathrm{Fil}^i L) = \mathrm{Fil}^i M$ *for all* $i \geq 1$.

(ii) $\alpha(L) = M$.

(iii) $\alpha : L/\mathrm{Fil}^2 L \to M/\mathrm{Fil}^2 M$ *is surjective.* □

Exercise 2.4.7. Let k be a commutative ring with 1. Let I be any set. Denote by $\mathrm{Cart}(k)^{(I)}$ the free $\mathrm{Cart}(k)$-module with basis I. Define a filtration on $\mathrm{Cart}(k)^{(I)}$ by

$$\mathrm{Fil}^i \mathrm{Cart}(k)^{(I)} = \left(\mathrm{Fil}^i \mathrm{Cart}(k)\right)^{(I)} .$$

(i) Show that $(\mathrm{Cart}(k)^{(I)}, \mathrm{Fil}^\bullet \mathrm{Cart}(k)^{(I)})$ is a V-reduced left $\mathrm{Cart}(k)$-module if and only if I is finite.

(ii) Let $\left(\widehat{\mathrm{Cart}(k)}^{(I)}, \mathrm{Fil}^\bullet \widehat{\mathrm{Cart}(k)}^{(I)}\right)$ be the completion of the filtered module

$$(\mathrm{Cart}(k)^{(I)}, \mathrm{Fil}^\bullet \mathrm{Cart}(k)^{(I)})$$

with the induced filtration. Prove that $\widehat{\mathrm{Cart}(k)}^{(I)}$ is a V-reduced left $\mathrm{Cart}(k)$-module. We call $\widehat{\mathrm{Cart}(k)}^{(I)}$ the *free* V*-reduced* $\mathrm{Cart}(k)$*-module with basis* I. Formulate a universal property which justifies this terminology.

(iii) Let Q be a torsion right $\mathrm{Cart}(k)$-module, i.e. $Q = \bigcup_n \mathrm{Ann}_Q(\mathrm{Fil}^n \mathrm{Cart}(k))$. Prove that $Q \overline{\otimes}_{\mathrm{Cart}(k)} \widehat{\mathrm{Cart}(k)}^{(I)}$ is naturally isomorphic to $Q^{(I)}$.

Proposition 2.4.11. *Let* $\alpha : (L, \mathrm{Fil}^\bullet L) \to (M, \mathrm{Fil}^\bullet M)$ *be a surjective homomorphism between* V*-reduced left* $\mathrm{Cart}(k)$*-modules as in Lemma* 2.4.10*. Let* K *be the kernel of* α*, with the induced filtration* $\mathrm{Fil}^i K = K \cap \mathrm{Fil}^i L$ *for all* $i \geq 1$*. Then* $(K, \mathrm{Fil}^\bullet K)$ *is a* V*-reduced left* $\mathrm{Cart}(k)$*-module.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K/\mathrm{Fil}^2 K \arrow[r] \arrow[d, "V_n"] & L/\mathrm{Fil}^2 L \arrow[r, "\alpha"] \arrow[d, "V_n", "\simeq"'] & M/\mathrm{Fil}^2 M \arrow[r] \arrow[d, "V_n", "\simeq"'] & 0 \\
0 \arrow[r] & \mathrm{Fil}^n K/\mathrm{Fil}^{n+1} K \arrow[r] & \mathrm{Fil}^n L/\mathrm{Fil}^{n+1} L \arrow[r, "\alpha"'] & \mathrm{Fil}^n M/\mathrm{Fil}^{n+1} M \arrow[r] & 0
\end{tikzcd}$$

The top row is exact, because $\mathrm{Fil}^2 \to \mathrm{Fil}^2 M$ is surjective. The bottom row is also exact, by a similar argument. From the five-lemma we see that V_n induces a bijection $V_n : K/\mathrm{Fil}^2 K \xrightarrow{\sim} \mathrm{Fil}^i K/\mathrm{Fil}^{n+1} K$ for all $n \geq 1$. The rest of the conditions for $(K, \mathrm{Fil}^\bullet K)$ to be V-reduced is easy. □

Definition 2.4.12. According to Prop. 2.4.11, for every V-reduced left $\mathrm{Cart}(k)$-module $(M, \mathrm{Fil}^{\bullet}M)$, there exists a free resolution

$$\cdots \to L_i \xrightarrow{\partial_i} L_{i-1} \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_2} L_1 \xrightarrow{\partial_1} L_0 \xrightarrow{\partial_0} M \to 0$$

of M, where each L_i is a free V-reduced left $\mathrm{Cart}(k)$-modules in the sense of 2.4.7 (ii), each ∂_i is compatible with the filtrations, and $\mathrm{Ker}(\partial_i) = \mathrm{Image}(\partial_{i+1})$ for all $i \geq 0$, and ∂_0 is surjective. Define *reduced torsion functors* $\overline{\mathrm{Tor}}_i^{\mathrm{Cart}(k)}(?, M)$ by

$$\overline{\mathrm{Tor}}_i^{\mathrm{Cart}(k)}(Q, M) = H_i\left(Q \overline{\otimes}_{\mathrm{Cart}(k)}(\cdots \to L_i \xrightarrow{\partial_i} L_{i-1} \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_2} L_1 \xrightarrow{\partial_1} L_0)\right)$$

for any torsion right $\mathrm{Cart}(k)$-module Q.

Exercise 2.4.8. (i) Prove that the functor $\overline{\mathrm{Tor}}_i^{\mathrm{Cart}(k)}$ is well-defined.

(ii) Show that every short exact sequence of torsion right $\mathrm{Cart}(k)$-modules gives rise to a long exact sequence for the functor $\overline{\mathrm{Tor}}^{\mathrm{Cart}(k)}$.

(iii) Formulate and prove a similar statement for the second entry of the reduced torsion functor.

Theorem 2.4.13. *let k be a commutative ring with* 1. *Let $(M, \mathrm{Fil}^{\bullet}M)$ be a V-reduced left* $\mathrm{Cart}(k)$*-module. Let N be a nilpotent k-algebra.*

(i) *Suppose that $(M, \mathrm{Fil}^{\bullet}M)$ is V-flat, i.e. M/Fil^2M is a flat k-module. Then*

$$\overline{\mathrm{Tor}}_i^{\mathrm{Cart}(k)}(\Lambda(N), M) = (0)$$

for all $i \geq 1$.

(ii) *Suppose that N has a finite decreasing filtration*

$$N = \mathrm{Fil}^1N \supseteq \mathrm{Fil}^2N \supseteq \cdots \supseteq \mathrm{Fil}^sN = (0)$$

by ideals of N such that $(\mathrm{Fil}^jN)^2 \subseteq \mathrm{Fil}^{i+1}N$ and $\mathrm{Fil}^jN/\mathrm{Fil}^{j+1}N$ is a flat k-module for $j = 1, \ldots, s-1$. Then $\overline{\mathrm{Tor}}_i^{\mathrm{Cart}(k)}(\Lambda(N), M) = 0$ for all $i \geq 1$.

Proof of (i). Choose an $s \in \mathbb{N}$ such that $N^s = 0$. Then we have a decreasing filtration

$$\Lambda(N) \supseteq \Lambda(N^2) \supseteq \cdots \supseteq \Lambda(N^{s-1}) \supseteq \Lambda(N^s) = (0)$$

of $\Lambda(N)$. For each $j = 1, 2, \ldots, s-1$ we have $\Lambda(N^j)/\Lambda(N^{j+1}) \cong \Lambda(N^j/N^{j+1})$ as right $\mathrm{Cart}(k)$-modules, hence it suffices to show that

$$\overline{\mathrm{Tor}}_i^{\mathrm{Cart}(k)}(\Lambda(N^j/N^{j+1}), M) = 0 \quad \text{for } j = 1, 2, \ldots, s-1\,.$$

Let $L \twoheadrightarrow M$ be a surjection from a free reduced left $\mathrm{Cart}(k)$-module L to M, and let K be the kernel. Then K is also a V-flat V-reduced left $\mathrm{Cart}(k)$-module. Recall from Exer. 2.4.6 (i) that

$$\Lambda(N^j/N^{j+1})\overline{\otimes}_{\mathrm{Cart}(k)}M \cong (N^j/N^{j+1}) \otimes_k (M/\mathrm{Fil}^2M)$$

for each $j = 1$. The long exact sequence attached to the short exact sequence $0 \to K \to L \to M \to 0$ yields isomorphisms

$$\overline{\mathrm{Tor}}_{i+1}^{\mathrm{Cart}(k)}(\Lambda(N^j/N^{j+1}), M) \xrightarrow{\sim} \overline{\mathrm{Tor}}_{i}^{\mathrm{Cart}(k)}(\Lambda(N^j/N^{j+1}), K)\,, \quad i \geq 1$$

and an exact sequence

$$\begin{aligned}(N^j/N^{j+1}) \otimes_k (K/\mathrm{Fil}^2K^{)} &\xrightarrow{\alpha} (N^j/N^{j+1}) \otimes_k (L/\mathrm{Fil}^2L)\\ &\to (N^j/N^{j+1}) \otimes_k (M/\mathrm{Fil}^2M) \to 0\end{aligned}$$

such that the kernel of α is isomorphic to $\overline{\mathrm{Tor}}_{1}^{\mathrm{Cart}(k)}(\Lambda(N^j/N^{j+1}), M)$. Since M/Fil^2M is a flat k-module, we see that $\overline{\mathrm{Tor}}_{1}^{\mathrm{Cart}(k)}(\Lambda(N^j/N^{j+1}), M) = 0$ for every V-flat V-reduced left $\mathrm{Cart}(k)$-module M. Since K is also V-flat,

$$\overline{\mathrm{Tor}}_{2}^{\mathrm{Cart}(k)}(\Lambda(N^j/N^{j+1}), M) = 0$$

as well. An induction shows that $\overline{\mathrm{Tor}}_{i}^{\mathrm{Cart}(k)}(\Lambda(N^j/N^{j+1}), M) = 0$ for all $i \geq 1$ and all $j = 1, 2, \ldots, s-1$. The statement (i) follows.

PROOF OF (ii) In the proof of (i) above, replace the ideals N^j by Fil^jN. The sequence

$$\begin{aligned}(\mathrm{Fil}^jN/\mathrm{Fil}^{j+1}N) \otimes_k (K/\mathrm{Fil}^2K^{)} &\xrightarrow{\alpha} (\mathrm{Fil}^jN/\mathrm{Fil}^{j+1}N) \otimes_k (L/\mathrm{Fil}^2L)\\ &\to (\mathrm{Fil}^jN/\mathrm{Fil}^{j+1}N) \otimes_k (M/\mathrm{Fil}^2M) \to 0\end{aligned}$$

is exact because $\mathrm{Fil}^jN/\mathrm{Fil}^{j+1}N$ is a flat k-module. The rest of the proof of (ii) is the same as the proof of (i). □

Proof of Thm. 2.4.9. Suppose that $(M, \mathrm{Fil}^\bullet M)$ is a V-flat V-reduced left $\mathrm{Cart}(k)$-module. It follows immediately from Thm. 2.4.13 that $G := \Lambda\overline{\otimes}_{\mathrm{Cart}(k)}M$ is a smooth formal group. Conversely given any smooth formal group G, $\mathrm{M}(G)$ is V-reduced and V-flat according to Lemma 2.4.5. By Exer. 2.4.6 (ii), we have a functorial isomorphism

$$\left(\Lambda\overline{\otimes}_{\mathrm{Cart}(k)}M\right)(k[[X]]^+) = \varprojlim_n \Lambda(k[[X]]^+/X^nk[[X]])\overline{\otimes}_{\mathrm{Cart}(k)}M \xleftarrow{\sim} M$$

for each V-flat V-reduced left $\mathrm{Cart}(k)$-module M.

To finish the proof, it remains to produce a functorial isomorphism

$$\beta_G : \Lambda\overline{\otimes}_{\mathrm{Cart}(k)}\mathrm{M}(G) \xrightarrow{\sim} G$$

for each commutative smooth formal group G. For each nilpotent k-algebra N, we have a natural map $\beta_{G,N} : \Lambda\otimes_{\mathrm{Cart}(k)}\mathrm{M}(G) \to G$ such that $\beta_{G,N} : \sum_i f_i \otimes h_i \mapsto \sum_i \Phi_{h_i,N}(f_i) \in G(N)$ in the notation of Cor. 2.3.4, where $f_i \in \Lambda(N)$ and $h_i \in \mathrm{M}(G)$ for each i. The map $\beta_{G,N}$ factors through the quotient $\Lambda(N)\overline{\otimes}_{\mathrm{Cart}(k)}\mathrm{M}(G)$ of $\Lambda(N) \otimes_{\mathrm{Cart}(k)} \mathrm{M}(G)$ by 2.3.4 (iv), and gives the desired map

$$\alpha_{G,N} : \Lambda(N)\overline{\otimes}_{\mathrm{Cart}(k)}\mathrm{M}(G) \to G(N)\,.$$

Since both the source and the target of β_G are exact and commute with arbitrary direct sums, to show that β_G is an isomorphism for every nilpotent k-algebra N it suffices to verify this statement when $N^2 = (0)$ and N is isomorphic to k as a k-module. In that case $\beta_{G,N}$ is the canonical isomorphism $\mathrm{M}(G)/\mathrm{Fil}^2\mathrm{M}(G) \xrightarrow{\sim} \mathrm{t}_G$. ☐

Exercise 2.4.9. (i) Prove that the equivalence of categories in Thm. 2.4.9 extends to an equivalence of categories between the category of V-reduced left $\mathrm{Cart}(k)$-modules and the category of functors $G : \mathfrak{Nilp}_k \to \mathfrak{Ab}$ which are right exact, weakly left exact, and commute with arbitrary direct sums.

(ii) Let $G : \mathfrak{Nilp}_k \to \mathfrak{Ab}$ be a functor which satisfies the conditions in (i) above. Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be a short exact sequence of nilpotent k-algebras such that N_3 satisfies the condition in 2.4.13 (ii). Prove that the sequence $0 \to G(N_1) \to G(N_2) \to G(N_3) \to 0$ is short exact.

Exercise 2.4.10. Let M be a V-reduced V-flat left $\mathrm{Cart}(k)$-module. Let k' be a commutative k algebra with 1. Let $M' = \mathrm{Cart}(k')\widehat{\otimes}_{\mathrm{Cart}(k)}M$, defined as the completion of the left $\mathrm{Cart}(k)$-module $\mathrm{Cart}(k')\otimes_{\mathrm{Cart}(k)} M$ with respect to the filtration given by the image of $\mathrm{Fil}^\bullet\mathrm{Cart}(k')\otimes_{\mathrm{Cart}(k)} M$ in $\mathrm{Cart}(k')\otimes_{\mathrm{Cart}(k)} M$, endowed with the induced filtration.

(i) The pair $(M', \mathrm{Fil}^\bullet M)$ is V-reduced, and $k' \otimes_k (M/\mathrm{Fil}^2 M) \xrightarrow{\sim} M'/\mathrm{Fil}^2 M'$ as k'-modules.

(ii) Prove that there is a canonical isomorphism of functors

$$\Lambda_{k'}\overline{\otimes}_{\mathrm{Cart}(k)}M \xrightarrow{\sim} \overline{\otimes}_{\mathrm{Cart}(k')}M'\Lambda_k\,.$$

In other words, we have a functorial isomorphism

$$\beta_N : \Lambda(N)\overline{\otimes}_{\mathrm{Cart}(k)}M \xrightarrow{\sim} \Lambda(N)\overline{\otimes}_{\mathrm{Cart}(k')}M'\,.$$

for every nilpotent k'-algebra N, compatible with arrows induced by morphisms in $\mathfrak{Nilp}_k$.

2.5 Localized Cartier theory

In this section we fix a prime number p. Let k be a commutative ring with 1 over $\mathbb{Z}_{(p)}$.

Definition 2.5.1. Recall from 2.3.2 that every prime number $\ell \neq p$ is invertible in $\mathrm{Cart}(k)$. Define elements ϵ_p and $\epsilon_{p,n}$ of the Cartier ring $\mathrm{Cart}(k)$ for $n \in \mathbb{N}$, $(n,p)=1$ by

$$\begin{aligned} \epsilon_p = \epsilon_{p,1} &= \sum_{\substack{(n,p)=1 \\ n\geq 1}} \frac{\mu(n)}{n} V_n F_n = \prod_{\substack{\ell\neq p \\ \ell \text{ prime}}} \left(1 - \frac{1}{\ell} V_\ell F_\ell\right) \\ \epsilon_{p,n} &= \frac{1}{n} V_n \epsilon_p F_n \end{aligned}$$

In the above μ is the Möbius function on $\mathbb{N}_{\geq 1}$, characterized by the following properties: $\mu(mn) = \mu(m)\,\mu(n)$ if $(m,n)=1$, and for every prime number ℓ we have $\mu(\ell) = -1$, $\mu(\ell^i) = 0$ if $i \geq 2$.

Proposition 2.5.2. *The following properties hold.*

(i) ${\epsilon_p}^2 = \epsilon_p$.

(ii) $\displaystyle\sum_{\substack{(n,p)=1 \\ n\geq 1}} \epsilon_{p,n} = 1$.

(iii) $\epsilon_p V_n = 0$, $F_n \epsilon_p = 0$ *for all* n *with* $(n,p)=1$.

(iv) ${\epsilon_{p,n}}^2 = \epsilon_{p,n}$ *for all* $n \geq 1$ *with* $(n,p)=1$.

(v) $\epsilon_{p,n}\,\epsilon_{p,m} = 0$ *for all* $m \neq n$ *with* $(mn,p)=1$.

(vi) $[c]\,\epsilon_p = \epsilon_p\,[c]$ *and* $[c]\,\epsilon_{p,n} = \epsilon_{p,n}\,[c]$ *for all* $c \in k$ *and all* n *with* $(n,p)=1$.

(vii) $F_p \epsilon_{p,n} = \epsilon_{p,n} F_p$, $V_p \epsilon_{p,n} = \epsilon_{p,n} V_p$ *for all* n *with* $(n,p)=1$.

Proof. From 2.3.11 (1)–(5), one easily deduces that for every prime number $\ell \neq p$ we have

$$(1 - \frac{1}{\ell} V_\ell F_\ell) V_\ell = 0\,,\ F_\ell (1 - \frac{1}{\ell} V_\ell F_\ell) = 0\,,\ \text{and} (1 - \frac{1}{\ell} V_\ell F_\ell)^2 = (1 - \frac{1}{\ell} V_\ell F_\ell)\,.$$

The statements (i) and (iii) follows. Statement (v) is an easy consequence of (iii). The proof of statement (iv) is an easy computation:

$$(\frac{1}{n} V_n \epsilon_p F_n)^2 = \frac{1}{n^2} V_n \epsilon_p F_n V_n \epsilon_p F_n = \frac{1}{n} V_n \epsilon_p F_n\,.$$

By 2.3.11 (4) and (5), the statement (ii) is a consequence of the following telescoping identity:

$$\sum_{m\geq 0} \frac{1}{\ell^m} V_{\ell^m}(1-\frac{1}{\ell}V_\ell F_\ell)F_{\ell^m} = 1$$

for any prime number $\ell \neq p$.

To prove (vi), observe first that $[c]\epsilon_p = \epsilon_p[c]$ by 2.3.11 (3); this in turn gives

$$[c]\epsilon_{p,n} = \frac{1}{n}V_n[c^n]\epsilon_p F_n = \frac{1}{n}V_n\epsilon_p[c^n]F_n = \epsilon_p[c]\,.$$

Statement (vii) is a consequence of the fact that both V_p and F_p commute with all V_n, F_n with $(n,p)=1$. □

Definition 2.5.3. (i) Denote by $\mathrm{Cart}_p(k)$ the subring $\epsilon_p\mathrm{Cart}(k)\epsilon_p$ of $\mathrm{Cart}(k)$. Note that ϵ_p is the unit element of $\mathrm{Cart}_p(k)$.

(ii) Define elements $F, V \in \mathrm{Cart}_p(k)$ by

$$F = \epsilon_p F_p = F_p\epsilon_p = \epsilon_p F_p \epsilon_p\,, \quad V = \epsilon_p V_p = V_p\epsilon_p = \epsilon_p V_p \epsilon_p\,.$$

(iii) For every element $c \in k$, denote by $\langle c\rangle$ the element $\epsilon_p[c]\epsilon_p = \epsilon_p[c] = [c]\epsilon_p \in \mathrm{Cart}_p(k)$.

Exercise 2.5.1. Let $E(T) \in \mathbb{Q}[[T]]$ be the power series

$$E(T) = \prod_{\substack{n\in\mathbb{N}\\(n,p)=1}} (1-T^n)^{\frac{\mu(n)}{n}} = \exp\left(-\sum_{m\geq 0}\frac{T^{p^m}}{p^m}\right)\,.$$

(i) Verify the second equality in the displayed formula above for $E(T)$, and prove that all coefficients of $E(T)$ lie in $\mathbb{Z}_{(p)}$.

(ii) Recall that the additive group underlying $\mathrm{Cart}(k)$ is a subgroup of $k[[X,t]]^\times$ by definition. Show that for any element $x = \sum_{m,n\geq 1} V^m[a_{mn}]F^n$ in $\mathrm{Cart}(k)$ with $a_{mn} \in k$ for all $m,n \geq 1$, the element $\epsilon_p x \epsilon_p$ is represented by the element

$$\prod_{m,n\geq 1} E(a_{mn}X^{p^m}t^{p^n})\,.$$

Exercise 2.5.2. Notation as above. Prove that for any left $\mathrm{Cart}(k)$-module M, the subgroup $\epsilon_p(M)$ consists of all elements $x \in M$ such that $F_n\,x = 0$ for all $n > 1$ with $(n,p)=1$. Elements of M with the above property will be called *p-typical elements.*

Exercise 2.5.3. Prove the following identities in $\mathrm{Cart}_p(k)$.

(1) $F\,\langle a\rangle = \langle a^p\rangle\,F$ for all $a \in k$.

(2) $\langle a\rangle V = V\,\langle a^p\rangle$ for all $a \in k$.

(3) $\langle a\rangle\,\langle b\rangle = \langle ab\rangle$ for all $a, b \in k$.

(4) $FV = p$.

(5) $VF = p$ if and only if $p = 0$ in k.

(6) Every prime number $\ell \neq p$ is invertible in $\mathrm{Cart}_p(k)$. The prime number p is invertible in $\mathrm{Cart}_p(k)$ if and only if p is invertible in k.

(7) $V^m\langle a\rangle F^m\, V^n\langle b\rangle F^n = p^r\, V^{m+n-r}\langle a^{p^{n-r}} b^{p^{m-r}}\rangle F^{m+n-r}$ for all $a, b \in k$ and all $m, n \in \mathbb{N}$, where $r = \min\{m, n\}$.

Definition 2.5.4. Let k be a commutative $\mathbb{Z}_{(p)}$-algebra with 1. Denote by Λ_p the image of ϵ_p in Λ. In other words, Λ_p is the functor from $\mathfrak{Nilp}_k$ to $\mathfrak{Ab}$ such that

$$\Lambda_p(N) = \Lambda(N) \cdot \epsilon_p$$

for any nilpotent k-algebra N.

Exercise 2.5.4. Let $E(T) \in \mathbb{Z}_{(p)}[[T]]$ be the inverse of the Artin-Hasse exponential as in 2.5.3.

(i) Prove that for any nilpotent k-algebra N, every element of $\Lambda_p(N)$ has a unique expression as a finite product

$$\prod_{i=0}^{m} E(u_i\, t^{p^i})$$

for some $m \in \mathbb{N}$, and $u_i \in N$ for $i = 0, 1, \ldots, m$.

(ii) Prove that Λ_p is a smooth commutative formal group over k.

Proposition 2.5.5. (i) *The local Cartier ring* $\mathrm{Cart}_p(k)$ *is complete with respect to the decreasing sequence of right ideals* $V^i\mathrm{Cart}_p(k)$.

(ii) *Every element of* $\mathrm{Cart}_p(k)$ *can be expressed as a convergent sum in the form*

$$\sum_{m,n\geq 0} V^m\langle a_{mn}\rangle F^n\,, \quad a_{mn} \in k\,, \forall m\ \exists\, C_m > 0 \text{ s.t. } a_{mn} = 0 \text{ if } n \geq C_m$$

in a unique way.

(iii) *The set of all elements of* $\mathrm{Cart}_p(k)$ *which can be represented as a convergent sum of the form*

$$\sum_{m\geq 0} V^m\langle a_m\rangle F^m\,, \quad a_m\in k$$

is a subring of $\mathrm{Cart}_p(k)$. *The map*

$$w_p(\underline{a})\ \mapsto \sum_{m\geq 0} V^m\langle a_m\rangle F^m \quad \underline{a}=(a_0,a_1,a_2,\ldots),\ a_i\in k\ \forall\, i\geq 0$$

establishes an isomorphism from the ring of p-adic Witt vectors $W_p(k)$ *to the above subring of* $\mathrm{Cart}_p(k)$.

Proof. Statement (i) and the existence part of statement (ii) are easy and left as an exercises. To prove the uniqueness part of (ii), according to 2.5.1 it suffices to check that if $(a_{mn})_{m,n\in\mathbb{N}}$ is a family of elements in k such that the infinite product

$$\prod_{m,n\geq 1} E(a_{mn}X^{p^m}t^{p^n})$$

is equal to 1 in $k[[X,t]]$, then $a_{mn}=0$ for all $m,n\geq 1$. This follows from the fact that

$$E(X)\equiv 1+X \pmod{(X^p)}\,.$$

The statement (iii) follows from 2.5.3 (7) and the properties of multiplication in the ring of p-adic Witt vectors. □

Exercise 2.5.5. Let $j_p:\Lambda_p\to\Lambda$ be the homomorphism of smooth commutative formal groups over k induced by the inclusion map, and let $\pi_p:\Lambda\to\Lambda_p$ be the homomorphism induced by ϵ_p. Let $G:\mathfrak{Nilp}_k\to\mathfrak{Ab}$ be a functor. The abelian group $\mathrm{Hom}(\Lambda,G)$ has a natural structure as a left module over $\mathrm{Cart}(k)=\mathrm{End}(\Lambda_{/k})^{\mathrm{op}}$.

(i) Prove that for every homomorphism $h\in\mathrm{Hom}(\Lambda_p,G)$, the composition $h\circ\pi_p\in\mathrm{Hom}(\Lambda,G)$ is a p-typical element of $\mathrm{Hom}(\Lambda,G)$.

(ii) Prove that the map $h\mapsto h\circ\pi_p$ above establishes a bijection from $\mathrm{Hom}(\Lambda_p,G)$ to the set of all p-typical elements in $\mathrm{Hom}(\Lambda,G)$, whose inverse is given by $h'\mapsto h'\circ j_p$.

Exercise 2.5.6. Prove that $\mathrm{Cart}_p(k)$ is naturally isomorphic to $\mathrm{End}(\Lambda_p)^{\mathrm{op}}$, the opposite ring of the endomorphism ring of $\mathrm{End}(\Lambda_p)$.

Exercise 2.5.7. Let k be a commutative algebra over $\mathbb{Z}_{(p)}$. Let T be the right $\mathrm{Cart}_p(k)$-module $\mathrm{Cart}_p(k)/V\mathrm{Cart}_p(k)$.

(i) Show that there is a natural isomorphism from T the Lie algebra of the smooth commutative formal group $\Lambda_p:\mathfrak{Nilp}_k\to\mathfrak{Ab}$.

(ii) Show that the element $x_i :=$ the image of F^i in T, $i = 0, 1, 2, \ldots$, form a k-basis of T.

(iii) The basis x_i, $i \in \mathbb{N}$ of the right $\mathrm{Cart}_p(k)$-module T defines a ring homomorphism $\rho_p : \mathrm{Cart}_p(k) \to \mathrm{M}'_{\mathbb{N}}(k)$, where $\mathrm{M}'_{\mathbb{N}}(k)$ consists of all $\mathbb{N} \times \mathbb{N}$-matrices $(c_{ij})_{i,j\geq 0}$ with at most finitely many non-zero entries in each row, and $c_{ij} \in k$ for all $i, j \geq 0$. Let $x = \sum_{m,n\geq 0} V^m \langle a_{mn} \rangle F^n$ be an element of $\mathrm{Cart}_p(k)$, $a_{mn} \in k$ for all $m, n \geq 0$. The entries $\rho_p(u)_{ij}$ of $\rho_p(u)$ for an element $u \in \mathrm{Cart}_p(k)$ is defined by

$$x_i \cdot u = \sum_{j\in\mathbb{N}} \rho_p(u)_{ij}\, x_j \quad \forall i \in \mathbb{N}\,.$$

Prove that $\rho_p(x)$ is the matrix (c_{ij}) whose (i, j)-th entry is given by

$$c_{ij} = \sum_{r\leq \min\{i,j\}} p^{i-r}\, (a_{i-r,\, j-r})^{p^r}$$

for all $i, j \geq 0$.

(iii) Use the formula in (ii) above to give another proof of 2.5.5 (iii).

Definition 2.5.6. Let k be a commutative $\mathbb{Z}_{(p)}$-algebra.

(i) A *V-reduced* left $\mathrm{Cart}_p(k)$-module M is a left $\mathrm{Cart}_p(k)$-module such that the map $V : M \to M$ is injective and the canonical map $M \to \varprojlim_n (M/V^n M)$ is an isomorphism.

(ii) A V-reduced left $\mathrm{Cart}_p(k)$-module M is *V-flat* if M/VM is a flat k-module.

Definition 2.5.7. Let I be a set. Denote by $\mathrm{Cart}_p(k)^{(I)}$ the direct sum of copies of $\mathrm{Cart}_p(k)$ indexed by I. The completion of the free $\mathrm{Cart}_p(k)$-module $\mathrm{Cart}_p(k)^{(I)}$ with respect to the filtered family of subgroups $V^i\mathrm{Cart}_p(k)^{(I)}$ is a V-reduced $\mathrm{Cart}_p(k)$-module, denoted by $\widehat{\mathrm{Cart}_p(k)}^{(I)}$; we called it the *free V-reduced* $\mathrm{Cart}_p(k)$-*module with basis indexed by I.*

Lemma 2.5.8. *Every element of the subset* $\mathrm{Cart}(k)\epsilon_p$ *of* $\mathrm{Cart}(k)$ *can be expressed as a convergent sum*

$$\sum_{(n,p)=1} V_n x_n\,, \qquad x_n \in \mathrm{Cart}_p(k)\ \forall\, n \text{ with } (n,p) = 1$$

for uniquely determined elements $x_n \in \mathrm{Cart}_p(k)$, $(n, p) = 1$*. Conversely every sequence of elements* $(x_n)_{(n,p)=1}$ *in* $\mathrm{Cart}_p(k)$ *defines an element of* $\mathrm{Cart}(k)\epsilon_p$*:*

$$\mathrm{Cart}(k)\epsilon_p = \widehat{\bigoplus}_{(n,p)=1} V_n \cdot \mathrm{Cart}_p(k)\,.$$

Proof. For $x \in \mathrm{Cart}(k)\epsilon_p$, we have

$$x = \sum_{(n,p)=1} \epsilon_{p,n} \cdot x = \sum_{(n,p)=1} V_n(\frac{1}{n}\epsilon_p F_n x \epsilon_p)\,.$$

On the other hand, suppose that we have $\sum_{(n,p)=1} V_n x_n = 0$ and $x_n \in \mathrm{Cart}_p(k)$ for all $n \geq 1$ with $(n,p) = 1$. For any $m \geq 1$ with $(m,p) = 1$, we have

$$0 = \epsilon_{p,m} \cdot \sum_{(n,p)=1} V_n x_n = V_m x_m$$

because $\epsilon_p V_r = 0$ and $F_r x_n = 0$ for all $r > 1$ with $(r,p) = 1$ and all $n \geq 1$ with $(n,p) = 1$. Hence $x_m = 0$ since left multiplication by V_m on $\mathrm{Cart}(k)$ is injective. □

Lemma 2.5.9. *Let M_p be a V-reduced $\mathrm{Cart}_p(k)$-module. Let $\mathrm{Cart}(k)\epsilon_p\widehat{\otimes}_{\mathrm{Cart}_p(k)}M_p$ be the V-adic completion of the tensor product $\mathrm{Cart}(k)\epsilon_p\otimes_{\mathrm{Cart}_p(k)}M_p$, defined as the completion of $\mathrm{Cart}(k)\epsilon_p\otimes_{\mathrm{Cart}_p(k)}M_p$ with respect to the decreasing family of the subgroups*

$$\mathrm{Image}\left(\mathrm{Fil}^i\mathrm{Cart}(k)\epsilon_p\otimes_{\mathrm{Cart}_p(k)}M_p \to \mathrm{Cart}(k)\epsilon_p\otimes_{\mathrm{Cart}_p(k)}M_p\right)$$

of $\mathrm{Cart}(k)\epsilon_p\otimes_{\mathrm{Cart}_p(k)}M_p$.

(i) *The completed tensor product $\mathrm{Cart}(k)\epsilon_p\widehat{\otimes}_{\mathrm{Cart}_p(k)}M_p$ is the topological direct sum*

$$\widehat{\bigoplus}_{(n,p)=1} V_n \otimes M_p$$

of its subgroups $V_n \otimes M_p$, $(n,p) = 1$, so that we have a natural bijection between $\mathrm{Cart}(k)\epsilon_p\widehat{\otimes}_{\mathrm{Cart}_p(k)}M_p$ and the set of sequences $(x_n)_{(n,p)=1}$ of elements in M_p indexed by positive integers prime to p.

(ii) *Define a decreasing filtration on the completed tensor product by*

$$\begin{aligned}&\mathrm{Fil}^m\left(\mathrm{Cart}(k)\epsilon_p\widehat{\otimes}_{\mathrm{Cart}_p(k)}M_p\right)\\ =\;&\overline{\mathrm{Image}\left(\mathrm{Fil}^m\mathrm{Cart}(k)\epsilon_p\otimes M_p \to \mathrm{Cart}(k)\epsilon_p\widehat{\otimes}_{\mathrm{Cart}_p(k)}M_p\right)}\,,\end{aligned}$$

the closure of the image of $\mathrm{Fil}^m\mathrm{Cart}(k)\epsilon_p\otimes M_p$, $m \geq 1$. Then the completed tensor product $\mathrm{Cart}(k)\epsilon_p\widehat{\otimes}_{\mathrm{Cart}_p(k)}M_p$ is a V-reduced $\mathrm{Cart}(k)$-module.

(iii) *The inclusion map $M_p \hookrightarrow \mathrm{Cart}(k)\epsilon_p\widehat{\otimes}_{\mathrm{Cart}_p(k)}M_p$ induces an isomorphism*

$$M_p/VM_p \xrightarrow{\sim} \left(\mathrm{Cart}(k)\epsilon_p\widehat{\otimes}_{\mathrm{Cart}_p(k)}M_p\right)/\mathrm{Fil}^2\left(\mathrm{Cart}(k)\epsilon_p\widehat{\otimes}_{\mathrm{Cart}_p(k)}M_p\right)\,.$$

(iv) *There is a canonical isomorphism*

$$\epsilon_p \cdot \mathrm{Cart}(k)\epsilon_p\widehat{\otimes}_{\mathrm{Cart}_p(k)}M_p \xrightarrow{\sim} M_p\,.$$

from the set of all p-typical elements in $\mathrm{Cart}(k)\epsilon_p\widehat{\otimes}_{\mathrm{Cart}_p(k)}M_p$ to M_p.

Proof. This Lemma is a corollary of 2.5.8. The isomorphism in (iv) is

$$\epsilon_p \cdot \Big(\sum_{(n,p)=1} V_n \otimes x_n \Big) \mapsto x_1 \, ,$$

whose inverse is induced by the inclusion. □

Lemma 2.5.10. *Let M be a V-reduced left* $\mathrm{Cart}(k)$ *module and let $M_p = \epsilon_p M$ be the set of all p-typical elements in M.*

(i) *The canonical map*

$$\begin{aligned} \widehat{\bigoplus}_{\substack{(n,p)=1 \\ n\geq 1}} M_p &\longrightarrow M & (2.1) \\ (x_n)_{(n,p)=1} &\mapsto \sum_{(n,p)=1} V_n x_n & (2.2) \end{aligned}$$

is an isomorphism.

(ii) *The canonical map* $\mathrm{Cart}(k)\epsilon_p \widehat{\otimes}_{\mathrm{Cart}_p(k)} M_p \to M$ *is an isomorphism.*

Proof. The argument of 2.5.8 proves (i). The statement (ii) follows from (i) and 2.5.9.

Combining 2.5.8, 2.5.9 and 2.5.10, we obtain the following theorem.

Theorem 2.5.11. *Let k be a commutative $\mathbb{Z}_{(p)}$-algebra with* 1.

(i) *There is an equivalence of categories between the category of V-reduced left* $\mathrm{Cart}(k)$*-modules and the category of V-reduced left* $\mathrm{Cart}_p(k)$*-modules, defined as follows.*

$$\begin{tikzcd} \{\ V\text{-reduced left } \mathrm{Cart}(k)\text{-mod}\} \arrow[r, "\sim"] & \{\ V\text{-reduced left } \mathrm{Cart}_p(k)\text{-mod}\} \\ M \arrow[r, mapsto] & \epsilon_p M \\ \mathrm{Cart}(k)\epsilon_p \widehat{\otimes}_{\mathrm{Cart}_p(k)} M_p & M_p \arrow[l, mapsto] \end{tikzcd}$$

(ii) *Let M be a V-reduced left* $\mathrm{Cart}(k)$*-module M, and let M_p be the V-reduced let* $\mathrm{Cart}_p(k)$*-module M_p attached to M as in* (i) *above. Then there is a canonical isomorphism $M/\mathrm{Fil}^2 M \cong M_p/VM_p$. In particular M is V-flat if and only if M_p is V-flat. Similarly M is a finitely generated* $\mathrm{Cart}(k)$*-module if and only if M_p is a finitely generated* $\mathrm{Cart}_p(k)$*-module.*

The next theorem is the local version of the main theorem of Cartier theory. The main ingredients of the proof occupies 2.5.13–2.5.18.

Theorem 2.5.12. *Let k be a commutative $\mathbb{Z}_{(p)}$-algebra with* 1*. Then there is a canonical equivalence of categories, between the category of smooth commutative formal groups over k as defined in 2.2.7 and the category of V-flat V-reduced left* $\mathrm{Cart}_p(k)$*-modules, defined as follows.*

$$\begin{tikzcd}
\{\text{smooth formal groups over } k\} \arrow[r, "\sim"] & \{V\text{-flat } V\text{-reduced left } \mathrm{Cart}_p(k)\text{-mod}\} \\
G \arrow[r, mapsto] & M_p(G) = \epsilon_p \operatorname{Hom}(\Lambda, G) \\
\Lambda_p \otimes_{\mathrm{Cart}_p(k)} M & M \arrow[l, mapsto]
\end{tikzcd}$$

Lemma 2.5.13. *Let k be a commutative $\mathbb{Z}_{(p)}$-algebra with* 1*. Let $\beta_p : L_p \to M_p$ be a surjective homomorphism V-reduced left* $\mathrm{Cart}(k)$*-module. Let K_p be the kernel of β_p. Let K be the kernel of*

$$\mathrm{id} \otimes \beta_p : \mathrm{Cart}(k)\epsilon_p \widehat{\otimes}_{\mathrm{Cart}_p(k)} L_p \to \mathrm{Cart}(k)\epsilon_p \widehat{\otimes}_{\mathrm{Cart}_p(k)} M_p$$

with the induced filtration.

(i) *The* $\mathrm{Cart}_p(k)$*-module K_p is V-reduced.*

(ii) *The pair $(K, \mathrm{Fil}^\bullet K)$ is the V-reduced left* $\mathrm{Cart}(k)$*-module which corresponds to K_p under Thm.* 2.5.11*.*

(iii) *The sequence $0 \to V^iK/V^{i+1}M \to V^iL/V^{i+1}L \to V^iM/V^{i+1}M \to 0$ is short exact for every $i \geq 0$.*

Proof. The argument of 2.4.11 works here as well.

Corollary 2.5.14. *Let M_p be a V-reduced left* $\mathrm{Cart}(k)$*-module. Then there exists an exact sequence*

$$\cdots \xrightarrow{\partial_{i+1}} L_i \xrightarrow{\partial_i} L_{i-1} \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_1} L_0 \xrightarrow{\partial_0} M_p \to 0$$

of $\mathrm{Cart}_p(k)$ *modules such that L_i is a free V-reduced left* $\mathrm{Cart}_p(k)$*-module, and induces an exact sequence on each V-adically graded piece.*

Definition 2.5.15. (1) A *torsion* right $\mathrm{Cart}_p(k)$-module Q is a right $\mathrm{Cart}_p(k)$-module such that for every $x \in Q$, there exists a natural number $n \geq 0$ such that $x \cdot V^n = 0$.

(2) Let M be a V-reduced left $\mathrm{Cart}_p(k)$-module. Let Q be a torsion right $\mathrm{Cart}(k)$-module. Let $L_\bullet$ be a resolution of M by free V-reduced left $\mathrm{Cart}_p(k)$-modules as in 2.5.14. Define $\overline{\mathrm{Tor}}_i^{\mathrm{Cart}_p(k)}(Q, M)$, $i \geq 0$, by

$$\overline{\mathrm{Tor}}_i^{\mathrm{Cart}_p(k)}(Q, M) = \mathrm{H}_i(Q \otimes_{\mathrm{Cart}_p(k)} L_\bullet)\,.$$

Exercise 2.5.8. (i) Show that the *continuous* torsion functors $\overline{\mathrm{Tor}}_{\bullet}^{\mathrm{Cart}(k)}(Q, M)$ are well-defined.

(ii) Let Q be a torsion right $\mathrm{Cart}_p(k)$-module and let $\widehat{\mathrm{Cart}_p(k)}^{(I)}$ be a free V-reduced left $\mathrm{Cart}_p(k)$-module with basis indexed by a set I. Prove that

$$Q \otimes_{\mathrm{Cart}_p(k)} \widehat{\mathrm{Cart}_p(k)}^{(I)}$$

is naturally isomorphic to $Q^{(I)}$, the direct sum of copies of the abelian group Q indexed by I.

(iii) Show that for any V-reduced left $\mathrm{Cart}_p(k)$-module M and any short exact sequence

$$0 \to Q_1 \to Q_2 \to Q_3 \to 0$$

of torsion right $\mathrm{Cart}_p(k)$-modules, one has a long exact sequence consisting of the abelian groups $\overline{\mathrm{Tor}}_{i}^{\mathrm{Cart}_p(k)}(Q_j, M)$.

(iv) Show that for any torsion right $\mathrm{Cart}_p(k)$-module Q and any short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of V-reduced left $\mathrm{Cart}_p(k)$-modules, one has a long exact sequence consisting of the abelian groups $\overline{\mathrm{Tor}}_{i}^{\mathrm{Cart}_p(k)}(Q, M_j)$.

(v) Show that for any torsion right $\mathrm{Cart}_p(k)$-module Q and any V-reduced left $\mathrm{Cart}_p(k)$-module M, the continuous tensor product

$$Q \overline{\otimes}_{\mathrm{Cart}_p(k)} M := \overline{\mathrm{Tor}}_{0}^{\mathrm{Cart}_p(k)}(Q, M)$$

is naturally isomorphic to $Q \otimes_{\mathrm{Cart}_p(k)} M$.

Remark 2.5.16. Let Q be a torsion right $\mathrm{Cart}(k)$-module.

(i) The canonical maps

$$Q \otimes_{\mathrm{Cart}(k)} (\mathrm{Cart}(k)\epsilon_p) \to Q \overline{\otimes}_{\mathrm{Cart}(k)} (\mathrm{Cart}(k)\epsilon_p) \to Q\, \epsilon_p$$

are isomorphisms of torsion right $\mathrm{Cart}_p(k)$-modules; denote these canonically isomorphic right $\mathrm{Cart}_p(k)$-modules by Q_p.

(ii) The subset $Q\, \epsilon_p$ of Q, the image of right multiplication by ϵ_p on Q, consists of all elements $x \in Q$ such that $x \cdot V_n = 0$ for all $n \geq 2$ with $(n, p) = 1$.

Proposition 2.5.17. *Let Q be a torsion right $\mathrm{Cart}(k)$-module, and let M be a V-reduced left $\mathrm{Cart}(k)$-module. Let $Q_p = Q \overline{\otimes}_{\mathrm{Cart}(k)} \mathrm{Cart}(k)\epsilon_p$. Let $M_p = \epsilon_p M$ be the V-reduced left $\mathrm{Cart}_p(k)$-module consisting of all p-typical elements in M.*

(i) *The canonical map*

$$Q_p \otimes_{\mathrm{Cart}_p(k)} M_p = \big(Q \overline{\otimes}_{\mathrm{Cart}(k)} \mathrm{Cart}(k)\epsilon_p\big) \otimes_{\mathrm{Cart}_p(k)} M_p \longrightarrow Q \overline{\otimes}_{\mathrm{Cart}(k)} M$$

is an isomorphism.

(ii) *For all $i \geq 0$ the canonical map*

$$\overline{\mathrm{Tor}}_i^{\mathrm{Cart}_p(k)}(Q_p, M_p) \longrightarrow \overline{\mathrm{Tor}}_i^{\mathrm{Cart}(k)}(Q, M)$$

is an isomorphism.

Proof. Since Q is a torsion right $\mathrm{Cart}(k)$-module, the canonical map

$$Q \otimes_{\mathrm{Cart}(k)} \big(\mathrm{Cart}(k)\epsilon_p \otimes_{\mathrm{Cart}_p(k)} \epsilon_p M\big) \to Q \overline{\otimes}_{\mathrm{Cart}(k)} \big(\mathrm{Cart}(k)\epsilon_p \widehat{\otimes}_{\mathrm{Cart}_p(k)} \epsilon_p M\big)$$

is an isomorphism, and (i) follows from the associativity of tensor product. To prove (ii), let $L_\bullet \to M$ be a resolution of M by free V-reduced left $\mathrm{Cart}(k)$-modules. Then $\epsilon_p L_\bullet \to \epsilon_p M = M_p$ is a resolution of M_p by free V-reduced left $\mathrm{Cart}_p(k)$-modules. By (i) the natural map

$$Q_p \otimes_{\mathrm{Cart}_p(k)} \epsilon_p L_\bullet \xrightarrow{\sim} Q \overline{\otimes}_{\mathrm{Cart}(k)} M$$

is an isomorphism of chain complexes, and the statement (ii) follows. □

Theorem 2.5.18. *Let k be a commutative $\mathbb{Z}_{(p)}$-algebra with 1. Let N be a nilpotent k-algebra. Let M_p be a V-reduced left $\mathrm{Cart}_p(k)$-module. Let $M = \mathrm{Cart}(k)\epsilon_p \widehat{\otimes} M_p$.*

(i) *The canonical map*

$$\Lambda_p(N) \otimes_{\mathrm{Cart}_p(k)} M_p \longrightarrow \Lambda(N) \overline{\otimes}_{\mathrm{Cart}(k)} M$$

is an isomorphism.

(ii) *Assume either that M_p is V-flat, or that N has a finite decreasing filtration*

$$N = \mathrm{Fil}^1 N \supseteq \mathrm{Fil}^2 N \supseteq \cdots \supseteq \mathrm{Fil}^s N = (0)$$

by ideals of N such that $(\mathrm{Fil}^j N)^2 \subseteq \mathrm{Fil}^{j+1} N$ and $\mathrm{Fil}^j N / \mathrm{Fil}^{j+1} N$ is a flat k-module for $j = 1, \ldots, s-1$. Then

$$\overline{\mathrm{Tor}}_i^{\mathrm{Cart}_p(k)}(Q_p, M_p) \cong \overline{\mathrm{Tor}}_i^{\mathrm{Cart}(k)}(Q, M) = (0) \qquad \forall\, i \geq 1\,.$$

Proof. The statement (i) is a corollary of 2.5.17 (i). The statement (ii) follows from 2.5.17 (ii) and Thm. 2.4.13. □

Proof of Thm. 2.5.12. Theorem 2.5.12 follows from Thm. 2.5.18, Thm. 2.5.11 and Thm. 2.4.9. □

Theorem 2.5.19. *Let k be a commutative $\mathbb{Z}_{(p)}$-algebra with 1.*

(i) *Let M be a V-reduced left* $\mathrm{Cart}_p(k)$*-module. Assume that there is a family*

$$\{x_i \,|\, x_i \in M \forall i \in I\}$$

of elements in M indexed by a set I such that M/VM is a free k-module with basis $\{\overline{x_i} \,|\, i \in I\}$, where $\overline{x_i}$ denotes the image of x_i in M/VM. Then

$$M = \left\{ \sum_{\substack{m \geq 0 \\ i \in I}} V^m \langle a_{m\,i} \rangle x_i \;\middle|\; \begin{array}{l} \text{(i) } a_{m\,i} \in k \quad \forall m \geq 0\,, \forall i \in I \\ \text{(ii) } \forall m \ \exists \ \textit{a finite subset} \quad J_m \subset I \ \textit{s.t.} \\ \qquad a_{mi} = 0 \ \textit{or if} \ i \notin J_m \end{array} \right\}$$

In other words, every element of M can be written in the form

$$\sum_{\substack{m \geq 0 \\ i \in I}} V^m \langle a_{m\,i} \rangle x_i$$

satisfying the conditions in the displayed formula above, in a unique way.

(ii) *Notation and assumption as in* (i) *above. There exists uniquely determined elements $a_{m\,i\,j} \in k$, with $(m, i, j) \in \mathbb{N} \times I \times I$ such that*

$$F \cdot x_i = \sum_{\substack{m \in \mathbb{N} \\ j \in I}} V^m \langle a_{m\,i\,j} \rangle x_j, \quad \forall i \in I \,\forall m \in \mathbb{N},$$

and for each $m \geq 0$ and each $i \in I$, $a_{m\,i\,j} = 0$ for all j outside a finite subset of I.

(iii) *Let $\alpha_{m\,i\,j} \in W_p(k)$ be a family of elements in $W_p(k)$ indexed by $\mathbb{N} \times I \times I$ such that for each $m \geq 0$ and each $i \in I$, there exists a finite subset $J_{m\,i} \subset I$ such that $\alpha_{m\,i\,j} = 0$ for all $j \notin J_{m\,i}$. Then the* $\mathrm{Cart}_p(k)$*-module N defined by the short exact sequence* $\mathrm{Cart}_p(k)$*-modules*

$$\begin{tikzcd}
0 \arrow[r] & L_1 = \widehat{\mathrm{Cart}_p(k)}^{(I)} \arrow[r, "\psi"] & L_2 = \widehat{\mathrm{Cart}_p(k)}^{(I)} \arrow[r] & N \arrow[r] & 0 \\
 & f_i \arrow[r, mapsto] & F\,e_i - \sum_{m,j} V^m \alpha_{m\,i\,j} e_j & & \\
 & & e_i \arrow[r, mapsto] & y_i &
\end{tikzcd}$$

is V-reduced. Here $\{f_i \,|\, i \in I\}$, $\{e_i \,|\, i \in I\}$ are the bases of the two free V-reduced left $\mathrm{Cart}_p(k)$*-modules L_1, L_2 respectively. Moreover the image of the elements $\{y_i \,|\, i \in I\}$ in N/VN form a k-basis of N/VN.*

Proof. The statement (i) follows from the definition of V-reduced left $\mathrm{Cart}_p(k)$-modules, and (ii) follows from (i).

To prove (iii), it suffices to show that the sequence in the displayed formula induces an exact sequence

$$0 \to L_1/VL_1 \xrightarrow{\overline{\psi}} L_2/VL_2 \to N/VN \to 0$$

of k-modules, and the elements $(\overline{y_i})_{i\in I}$ form a k-basis of N/VN. Here we used the convention that $\overline{y_i}$ denotes the image of y_i in N/VN; the same convention will be used for L_1/VL_1 and L_2/VL_2. Recall that a typical element $\omega_p(\underline{c}) \in W_p(k)$ is identified with the element $\sum_{m\ge 0} V^m\langle c_m\rangle F^m$ of $\mathrm{Cart}_p(k)$. For $i, j \in I$, let $a_{ij} = w_0(\alpha_{0\,i\,j}) \in k$, so that $\alpha_{0\,i\,j} - \omega_p(a_{ij}, 0, 0, \ldots) \in V(W_p(k))$. We know that

$$L_1/VL_1 \;=\; \bigoplus_{n\in\mathbb{N},\, i\in I} \overline{F^n f_i}\,, \qquad L_2/VL_2 \;=\; \bigoplus_{n\in\mathbb{N},\, i\in I} \overline{F^n e_i}$$

$$\overline{\psi}\Big(\sum_{n\ge 0,\, i\in I} b_{ni}\,\overline{F^n f_i}\,\Big) \;=\; \sum_{n\ge 0,\, j\in I}\left(b_{n-1,j} - \sum_{i\in I} b_{ni} a_{i,j}^{p^n}\right)\overline{F^n e_j}\,.$$

The desired conclusion follows from an easy calculation. ☐

Remark 2.5.20. In the situation of 2.5.19 (i), (ii), we have a short exact sequence of V-reduced left $\mathrm{Cart}_p(k)$-modules

$$\begin{tikzcd}
0 \arrow[r] & L_1 = \widehat{\mathrm{Cart}_p(k)}^{(I)} \arrow[r] & L_2 = \widehat{\mathrm{Cart}_p(k)}^{(I)} \arrow[r] & M \arrow[r] & 0 \\
& f_i \arrow[r, mapsto] & F\,e_i - \sum_{m,j} V^m\langle a_{m\,i\,j}\rangle e_j & & \\
& & e_i \arrow[r, mapsto] & x_i &
\end{tikzcd}$$

The family of equations $Fx_i = \sum_{m,j} V^m\langle a_{m\,i\,j}\rangle x_j$, $i \in I$, are called the *structural equations* of M for the generators $\{x_i \,|\, i \in I\}$.

Proposition 2.5.21. *Let* M_p *be a* V*-reduced* V*-flat left* $\mathrm{Cart}_p(k)$*-module. Let* k' *be a commutative* k *algebra with* 1*. Let* M'_p *be the* V*-adic completion of the left* $\mathrm{Cart}_p(k')$*-module* $\mathrm{Cart}_p(k') \otimes_{\mathrm{Cart}_p(k)} M_p$.

(i) *The* $\mathrm{Cart}_p(k')$*-module* M' *is* V*-reduced, and* $k' \otimes_k (M/VM) \xrightarrow{\sim} M'/VM'$ *as* k'*-modules.*

(ii) *For every nilpotent* k'*-algebra* N*, there is a canonical isomorphism*

$$\Lambda_p(N)\otimes_{\mathrm{Cart}_p(k)} M_p \xrightarrow{\sim} \Lambda_p(N)\otimes_{\mathrm{Cart}_p(k')} M'_p\,.$$

(iii) *Suppose that* $\{x_i \,|\, i \in I\}$ *is a family of elements in* M_p *such that* $\{\overline{x_i} \,|\, i \in I\}$ *form a* k*-basis of* M/VM*. Let* $Fx_i = \sum_{m\ge 0,\, j\in I} V^m\langle a_{mij}\rangle x_j$*.* $i \in I$ *be the structural equation of* M *w.r.t. the generators* $\{x_i \,|\, i \in I\}$*. Then these equations are also the structural equations of* M' *for the generators* $\{1 \otimes x_i \,|\, i \in I\}$*.*

Proof. Exercise. ☐

2.5.22 Exercises.

Exercise 2.5.9. Prove that the equivalence of categories in Thm. 2.5.12 extends to an equivalence of categories between the category of V-reduced V-flat left $\mathrm{Cart}_p(k)$-modules and the category of functors $G : \mathfrak{Nilp}_k \to \mathfrak{Ab}$ which are right exact, weakly left exact, and commute with infinite direct sums.

Exercise 2.5.10. Prove that the left ideal $\mathrm{Cart}(k)\epsilon_p$ of $\mathrm{Cart}(k)$ consists of all elements $x \in \mathrm{Cart}(k)$ such that $xV_n = 0$ for all $n \geq 2$ with $(n,p) = 1$. (Hint: Prove that $x \cdot (1 - \epsilon_p) \in \mathrm{Fil}^m\mathrm{Cart}(k) = 0$ for all $m \geq 1$. Or, use Exer. 2.3.6.)

Exercise 2.5.11. Let x be an element of $\mathrm{Cart}(k)$.

(i) Prove that $x \cdot \epsilon_p = 0$ if and only if x lies in the closure of the sum of left ideals $\sum_{(n,p)=1} \mathrm{Cart}(k)F_n$. (Hint: Use 2.3.6.)

(ii) Prove that $\epsilon_p \cdot x = 0$ if and only if x lies in the convergent sum of right ideals $\sum_{(n,p)=1} V_n\mathrm{Cart}(k)$.

Exercise 2.5.12. Prove that $\epsilon_p\mathrm{Cart}(k)$ consists of all elements of the form

$$\sum_{\substack{i,j\geq 0\\ (n,p)=1}} V^i\langle a_{i,j,n}\rangle F^j F_n$$

satisfying the following conditions:

(i) $a_{i,j,n} \in k\ \forall i,j \geq 0,\ \forall n \geq 1$ with $(n,p) = 1$,

(ii) $\forall i \geq 0,\ \exists C_i > 0$ s.t. $a_{i,j,n} = 0$ if $j > C_i$ or $n > C_i$.

In other words, $\epsilon_p\mathrm{Cart}(k)$ is the V-adic completion of the discrete direct sum of the free $\mathrm{Cart}_p(k)$-modules $\mathrm{Cart}_p(k) \cdot F_n$, where n ranges through all positive integers prime to p.

Exercise 2.5.13. Show that the canonical maps

$$\begin{aligned} \mathrm{Cart}(k)\epsilon_p\widehat{\otimes}_{\mathrm{Cart}_p(k)}\epsilon_p\mathrm{Cart}(k) &\longrightarrow \mathrm{Cart}(k)\\ \epsilon_p\mathrm{Cart}(k) \otimes_{\mathrm{Cart}_p(k)} \epsilon_p\mathrm{Cart}(k) &\longrightarrow \mathrm{Cart}_p(k)\end{aligned}$$

are isomorphisms.

Exercise 2.5.14. Let S be a subset of the set of all prime numbers, and let $\mathbb{Z}_S = \mathbb{Z}[\frac{1}{\ell}]_{\ell\notin S}$ be the subring of $\mathbb{Q}$ generated by $\mathbb{Z}$ and all prime numbers $\ell \notin S$. Generalize the results in this section to the case when the base ring k is a commutative algebra over $\mathbb{Z}_S$ with 1.

Exercise 2.5.15. Assume that k is a field of characteristic p. Let M be a V-reduced left $\mathrm{Cart}_p(k)$-module such that $\dim_k(M/VM) = 1$. Let $e \in M$ be an element of M such that $e \notin VM$, so that $M = \mathrm{Cart}_p(k)\cdot e$. Suppose that $Fe = \sum_{m\geq n} V^m\langle a_m\rangle e$, with $a_m \in k$ for all $m \geq n$, and $a_n \neq 0$.

(i) Suppose that there exists an element $b \in k$ such that $b^{p^{n+1}-1} = a_n$. Prove that there exists a generator x of M such that $Fx - V^n x \in V^{n+1}M$. (Hint: Use a generator of the form $\langle c \rangle e$.)

(ii) Assume that k is perfect and there exists an element $b \in k$ such that $b^{p^{n+1}-1} = a_n$. Prove that there exists a generator y of M such that $Fy = V^n y$.

Exercise 2.5.16. Let k be a field of characteristic p. For $i = 1, 2$, let

$$M_i = \mathrm{Cart}_p(k)/\mathrm{Cart}_p(k) \cdot (F - \sum_{m \geq n_i} V^m \langle a_{im} \rangle),$$

where $a_{im} \in k$ for all $m \geq n_i$, and n_1, n_2 are natural numbers. If $n_1 \neq n_2$, prove that M_1 and M_2 are not isomorphic.

Exercise 2.5.17. Let $r \geq 1$ be a positive integer, and let $q = p^r$. Define formal power series $f(X) \in \mathbb{Q}[[X]]$ and $g(X,Y) \in \mathbb{Q}[[X,Y]]$ by

$$\begin{aligned} f(X) &= \sum_{n \geq 0} \frac{X^{q^n}}{p^n} \\ g(X,Y) &= f^{-1}(f(X) + f(Y)). \end{aligned}$$

It is well-known that $g(X,Y) \in \mathbb{Z}_{(p)}[[X,Y]]$ is a one-dimensional formal group law, a special case of the Lubin-Tate formal group law. The formal group law $g(X,Y)$ defines a smooth commutative formal group $G : \mathfrak{Nilp}_{\mathbb{Z}_{(p)}} \to \mathfrak{Ab}$. By definition, the $\mathrm{Cart}(\mathbb{Z}_{(p)})$-module M attached to G is $G(\mathbb{Z}_{(p)}[[X]]^+) = \mathbb{Z}_{(p)}[[X]]^+$. Let γ be the element of M corresponding to $X \in \mathbb{Z}_{(p)}[[X]]^+$

(i) Prove that γ is a p-typical element of M. (Hint: Change the base ring from $\mathbb{Z}_{(p)}$ to $\mathbb{Q}$.)

(ii) Prove that $M_p := \epsilon_p M$ is generated by γ.

(iii) Prove that $F \cdot \gamma = V^{r-1} \cdot \gamma$.

(iv) Prove that $\mathrm{End}(G) = Z_{(p)}$.

Exercise 2.5.18. Let k be a perfect field of characteristic p. Let n be a natural number. Prove that $\mathrm{Cart}_p(k)/\mathrm{Cart}_p(k) \cdot (F - V^n)$ is a free $W_p(k)$-module of rank $n + 1$.

Exercise 2.5.19. Let $n \geq 0$ be a natural number. Let k be a field of characteristic p. Let $M = \mathrm{Cart}_p(k)/\mathrm{Cart}_p(k) \cdot (F - V^n)$.

(i) If $n = 0$, show that M is a free $W_p(k)$-module of rank one.

(ii) Suppose that $n \geq 1$, $c \in k$. Prove that $(V^j \langle c \rangle F^j) \cdot M \subseteq pM$ if $j \geq 2$.

(iii) Prove that M is *not* a free $W_p(k)$-module if $n \geq 1$ and k is not perfect.

Exercise 2.5.20. Notation as in 2.5.19. Let k_1 be the finite subfield of k consisting of all elements $x \in k$ such that $x^{p^{n+1}} = x$. Let $\operatorname{Card}(k_1) = p^r$.

(i) Show that $r | n+1$.

(ii) Show that $\operatorname{End}_{\operatorname{Cart}(k)}(M)$ is a $W_p(k_1)$-module of rank $(n+1)$.

(iii) Let $D = \operatorname{End}_{\operatorname{Cart}(k)}(M) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Prove that D_p is a division algebra.

(iv) Prove that the center of D is a totally ramified extension of degree $e = \frac{n+1}{r}$, isomorphic to $\mathbb{Q}_p[T]/(T^e - p)$.

(v) Find the Brauer invariant of the division algebra D with center E.

2.6 Appendix: Witt vectors

In this appendix we explain the basic properties of the ring $\widetilde{W}$ of universal Witt vectors and the ring W_p of p-adic Witt vectors. Both are ring schemes over $\mathbb{Z}$, and W_p is a factor of $\widetilde{W}$ over $\mathbb{Z}_p$.

Definition 2.6.1. The *universal Witt vector group* $\widetilde{W}$ is defined as the functor from the category of all commutative algebras with 1 to the category of abelian groups such that

$$\widetilde{W}(R) = 1 + T\,R[[T]] \subset R[[T]]^{\times}$$

for every commutative ring R with 1. It turns out that the $\widetilde{W}$ has a natural structure as a ring scheme. When we regard a formal power series $1+\sum_{m\geq 1} u_m T^m$ in $R[[T]]$ as an element of $\widetilde{W}(R)$, we use the notation $\omega(1 + \sum_{m\geq 1} u_m T^m)$. It is easy to see that every element of $\widetilde{W}(R)$ has a unique expression as

$$\omega\left(\prod_{m\geq 1} (1 - a_m T^m)\right) .$$

Hence $\widetilde{W}$ is isomorphic to Spec $\mathbb{Z}[x_1, x_2, x_3, \ldots]$ as a scheme; the R-valued point such that $x_i \mapsto a_i$ is denoted by $\omega(\underline{a})$, where $\underline{a}$ is short for $(a_1, a_2, a_3, \ldots)$. In other words, $\omega(\underline{a}) = \omega(\prod_{m\geq 1} (1 - a_m T^m))$

Definition 2.6.2. The ring structure of $\widetilde{W}$ is given by

$$\omega(1 - aT^m) \cdot \omega(1 - bT^n) = \omega\left(\left(1 - a^{\frac{n}{r}}\, b^{\frac{m}{r}}\, T^{\frac{mn}{r}}\right)^r\right) , \quad \text{where } r = (m, n) .$$

Exercise 2.6.1. (i) Prove that the recipe above for multiplication indeed defines a ring structure on $\widetilde{W}$. In other words, prove that

$$(\omega(1-a_1T^{n_1})\cdot\omega(1-a_2T^{n_2}))\cdot\omega(1-a_3T^{n_3}) = \\ \omega(1-a_1T^{n_1})\cdot(\omega(1-a_2T^{n_2})\cdot\omega(1-a_3T^{n_3}))$$

for any elements $a_1, a_2, a_3 \in R$ and any $n_1, n_2, n_3 \geq 1$.

(ii) Show that ring structure on $\widetilde{W}$ is uniquely determined by the requirement that

$$\omega(1-aT)\cdot\omega(1-bT) = \omega(1-abT)$$

for every commutative ring R and all elements $a, b \in R$.

Remark 2.6.3. The group scheme $\widetilde{W}$ has a decreasing filtration $\left(\mathrm{Fil}^n\widetilde{W}\right)_{n\geq 1}$, where

$$\mathrm{Fil}^n\widetilde{W}(R) = \omega(1+T^n\,R[[T]]) \subset \widetilde{W}(R)\,.$$

In terms of the coordinates in §2.6.1, $\mathrm{Fil}^n\widetilde{W}(R)$ consists of all R-valued points such that the coordinates $x_1, \ldots, x_{n-1}$ vanish. For every n, $\mathrm{Fil}^n\widetilde{W}(R)$ is an ideal of $\widetilde{W}(R)$. This $W(R)$ is complete with respect to this filtration, and the addition, multiplication, and the operator F_n, V_n defined in 2.6.5 below are continuous with respect to this filtration; see (2.6.3) (9).

Definition 2.6.4. There is a homomorphism of ring schemes

$$\mathrm{ghost} : \widetilde{W} \longrightarrow \prod_{m=1}^{\infty} \mathbb{A}^1 = \mathrm{Spec}\ k[\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \ldots]\,,$$

where the target has the standard ring scheme structure. The coordinates of the map "ghost" are

$$\tilde{w}_m(\underline{a}) = \sum_{d|m} d\cdot a_d^{m/d}\,.$$

Equivalently if we identify an R-valued point $\underline{r}$ of Spec $k[\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \ldots]$ with the power series $1+\sum_{m=1}^{\infty} r_m T^m \in R[[T]]$ for a commutative ring R, then the map ghost on $\widetilde{W}$ is induced by the operator $-t\,\frac{d}{dt}$ log:

$$\mathrm{ghost}(\omega\left(\prod_{m\geq 1}(1-a_m T^m)\right)) = \sum_{m\geq 1}\sum_{d\geq 1} m\,a_m^d\,T^{md} = \sum_{m\geq 1}\tilde{w}_m(\underline{a})\,T^m\,.$$

Exercise 2.6.2. Prove that the map ghost is a homomorphism of ring schemes over $\mathbb{Z}$, and is an isomorphism over $\mathbb{Q}$.

Definition 2.6.5. There are two families of endomorphisms of the group scheme $\widetilde{W}$: V_n and F_n, $n \in \mathbb{N}_{\geq 1}$. Also for each commutative ring R with 1 and each element $c \in R$ we have an endomorphism $[c]$ of $\widetilde{W} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\, R$. They are defined as follows

$$\begin{array}{rcll} V_n & : & \omega(f(T)) \mapsto & \omega(f(T^n)) \\ F_n & : & \omega(f(T)) \mapsto & \sum_{\zeta\in\mu_n} \omega(f(\zeta\, T^{\frac{1}{n}})) \qquad (\text{formally}) \\ [c] & : & \omega(f(T)) \mapsto & \omega(f(cT)) \end{array}$$

The formula for $F_n(\omega(f(T)))$ means that $F_n(\omega(f(T)))$ is defined as the unique element such that $V_n(F_n(\omega(f(T)))) = \sum_{\zeta\in\mu_n} \omega(f(\zeta\, T))$.

Exercise 2.6.3. Prove the following statements.

(1) $V_n(\omega(1 - a\,T^m)) = \omega(1 - a\,T^{mn})$, $F_n(\omega(1 - a\,T^m)) = \omega\left(\left(1 - a^{\frac{n}{r}}\, T^{\frac{m}{r}}\right)^r\right)$, $\forall\, m, n \geq 1$, where $r = (m, n)$.

(2) $F_n\, F_m = F_{mn}$, $V_m\, V_n = V_{mn}$, $\forall\, m, n \geq 1$.

(3) $V_n\, F_m = F_m\, V_n$ if $(m, n) = 1$.

(4) $F_n\, V_n = n$, i.e. $F_n\, V_n\, \omega(1 - a_m\, T^m) = \omega((1 - a_m\, T^m)^n)$ for all $m, n \geq 1$.

(5) Let p be a prime number. Then $V_p\, F_p = p$ on $\widetilde{W}(R)$ if $p = 0$ in R. Conversely if $V_p(1) = V_p(F_p(1)) = p$, then $p = 0$ in R. (Hint: For the "only if" part, show that $V_p\, F_p\, \omega(1 - T) = \omega(1 - T^p)$.)

(6) F_n is a ring homomorphism on $\widetilde{W}$ for all $n \geq 1$. (Hint: Either verify this statement for the set of topological generators $\omega(1 - a_m\, T^m)$, or use the ghost coordinates.)

(7) $x \cdot (V_n\, y) = V_n(F_n(x) \cdot y)$ for all $x, y \in \widetilde{W}(R)$.

(8) If a positive integer N is invertible in R, then N is also invertible in $W(R)$.

(9) We have $V_n(\mathrm{Fil}^m \widetilde{W}) \subseteq \mathrm{Fil}^{mn} \widetilde{W}$, $F_n(\mathrm{Fil}^m \widetilde{W}) \subseteq \mathrm{Fil}^{\lceil \frac{m}{n} \rceil} \widetilde{W}$, and $\mathrm{Fil}^m \widetilde{W} \cdot \mathrm{Fil}^n \widetilde{W} \subseteq \mathrm{Fil}^{\max(m,n)} \widetilde{W}$ for all $m, n \geq 1$.

(10) For all $c \in R$, all $m, n \geq 1$ and all $x \in \widetilde{W}(R)$, we have

$$\tilde{w}_m([c](x)) = c^m\, \tilde{w}_m(x), \quad \tilde{w}_m(F_n(x)) = \tilde{w}_{mn}(x),$$

and

$$\tilde{w}_m(V_n(x)) = \begin{cases} n\, \tilde{w}_{\frac{m}{n}} & \text{if } n | m \\ 0 & \text{if } n \not| m \end{cases}$$

Exercise 2.6.4. Let R be a commutative ring with 1.

(i) Show that every endomorphism Φ of the group scheme $\widetilde{W}$ over R is determined by the element $\Phi_{R[X]}(1 - XT) \in \widetilde{W}(R[X])$.

(ii) Prove that every element $\Phi \in \mathrm{End}_R(\widetilde{W})$ can be expressed as an infinite series in the form
$$\sum_{m,n\geq 1} V_m\,[a_{mn}]\,F_n$$
with $a_{mn} \in R$ for all $m, n \geq 1$, and for every m there exists $C_m \geq 0$ such that $a_{mn} = 0$ if $n \geq C_m$. The elements $a_{mn} \in R$ are uniquely determined by the endomorphism Φ, and every family of elements $\{a_{mn}\}$ in R satisfying the above condition gives an endomorphism of $\widetilde{W}$.

Definition 2.6.6. Over $\mathbb{Z}_{(p)}$ we define a projector
$$\epsilon_p := \sum_{(n,p)=1} \frac{\mu(n)}{n}\, V_n\, F_n = \prod_{\ell\neq p} \left(1 - \frac{1}{\ell} V_\ell\, F_\ell\right) ,$$
where ℓ runs through all prime numbers not equal to p. Note that the factors $(1 - \frac{1}{\ell} V_\ell\, F_\ell)$ commute.

Exercise 2.6.5. Prove that

(i) $(1 - \frac{1}{\ell} V_\ell\, F_\ell) V_\ell = 0 = F_\ell (1 - \frac{1}{\ell} V_\ell\, F_\ell)$ for all prime number $\ell \neq p$. (Use 2.6.3 (4).)

(ii) $(1 - \frac{1}{\ell} V_\ell F_\ell)^2 = 1 - \frac{1}{\ell} V_\ell F_\ell$.

(iii) $\epsilon_p \circ V_\ell = 0 = F_\ell \circ \epsilon_p \quad \forall\, \ell \neq p$.

(iv) $\epsilon_p^2 = \epsilon_p$.

Definition 2.6.7. Denote by W_p the image of ϵ_p, i.e. $W_p(R) := \epsilon_p(\widetilde{W}(R))$ for every $\mathbb{Z}_{(p)}$-algebra R. Equivalently, $W_p(R)$ is the intersection of the kernels $\mathrm{Ker}(F_\ell)$ of the operators F_ℓ on $\widetilde{W}(R)$, where ℓ runs through all prime numbers different from p. The functor W_p has a natural structure as a ring-valued functor induced from that of $\widetilde{W}$; see Exer. 2.6.6 (3); it is represented by the scheme $\mathrm{Spec}\, \mathbb{Z}_{(p)}[y_0, y_1, y_2, \ldots, y_n, \ldots]$ according to the computation below.

For every $\mathbb{Z}_{(p)}$-algebra R and every sequence of elements $a_m \in R$, we have
$$\begin{aligned}\epsilon_p \left(\omega \left(\prod_{m\geq 1} (1 - a_m\, T^m)\right)\right) &= \epsilon_p \left(\omega \left(\prod_{n\geq 0} (1 - a_{p^n}\, T^{p^n})\right)\right) \\ &= \omega \left(\prod_{n\geq 0} E(a_{p^n}\, T^{p^n})\right) ,\end{aligned}$$

where

$$E(X) = \prod_{(n,p)=1} (1 - X^n)^{\frac{\mu(n)}{n}} = \exp\left(-\sum_{n\geq 0} \frac{X^{p^n}}{p^n} \right) \in 1 + X\mathbb{Z}_{(p)}[[X]]$$

is the inverse of the classical Artin-Hasse exponential. It follows that the map

$$\prod_0^\infty R \ni (c_0, c_1, c_2, \ldots) \mapsto \omega\left(\prod_{n=0}^{\infty} E(c_n T^{p^n}) \right) \in W_p(R) = \epsilon_p(\widetilde{W}(R))$$

establishes a bijection between $\prod_0^\infty R$ and $W_p(R)$. Denote the element

$$\omega(\prod_{n=0}^{\infty} E(c_n T^{p^n})) \in W_p(R)$$

by $\omega_p(\underline{c})$. We have shown that the functor $R \mapsto W_p(R)$ is represented by the scheme Spec $k[y_0, y_1, y_2, \ldots]$, such that the element $\omega_p(\underline{c})$ has coordinates $\underline{c} = (c_0, c_1, c_2, \ldots)$.

The ghost coordinates on $\widetilde{W}$ simplifies greatly when restricted to W_p. Most of them vanish: $\tilde{w}_m(\omega_p(\underline{c})) = 0$ if m is not a power of p for all $\underline{c}$. Let $w_n(\underline{c}) = \tilde{w}_{p^n}(\omega_p(\underline{c}))$ for all $n \geq 0$. Then

$$w_n(\underline{c}) = \sum_{i=0}^{n} p^{n-i} c_{n-i}^{p^i},$$

and $\mathrm{ghost}(\omega_p(\underline{c})) = \sum_{n=0}^\infty w_n(\underline{c}) T^{p^n}$. For each n the map $\omega_p(\underline{c}) \mapsto (w_n(\underline{c}))_n$ is a homomorphism of ring schemes $W_p \longrightarrow \prod_0^\infty \mathbb{A}^1$. The endomorphism V_p, F_p of the group scheme $\widetilde{W}$ induces endomorphisms V, F of the group scheme W_p. Clearly $V(\omega_p(c_0, c_1, c_2, \ldots)) = \omega_p(0, c_1, c_2, \ldots)$ for all $\underline{c}$.

Exercise 2.6.6. Verify the following statements.

(1) $\epsilon_p(1) \cdot x = \epsilon_p(x)$ for all $x \in \widetilde{W}(R)$. (Hint: Use 2.6.3 (7).)

(2) $\epsilon_p(1) = \omega(E(T))$, $\epsilon_p(1) \cdot \epsilon_p(1) = \epsilon_p(1)$

(3) $\epsilon_p(x \cdot y) = \epsilon_p(x) \cdot \epsilon_p(y)$ for all $x, y \in \widetilde{W}(R)$. Hence $W_p(R)$ is a subring of $\widetilde{W}(R)$ whose unit element is $\epsilon_p(1)$.

(4) $F\,V = p$ on $W_p(R)$.

(5) $V\,F = p$ on $W_p(R)$ if $p = 0$ in R. Conversely if $V(1) = V(F(1)) = p$ then $p = 0$ in R. (Hint: For the "only if" part, show that $VF(1) = \omega(E(T^p))$, while $p = \omega(E(T)^p)$.

(6) $V(Fx \cdot y) = x \cdot Vy$ for all $x, y \in W_p(R)$.

(7) $F(xy) = F(x) \cdot F(y)$ for all $x, y \in W_p(R)$.

(8) $F(\omega_p(c_0, c_1, c_2, \ldots)) = \omega_p(c_0^p, c_1^p, c_2^p, \ldots)$ if $p = 0$ in R and $c_i \in R$ for all R.

(9) For each $a \in R$, let $\langle a \rangle := \omega_p(a, 0, 0, 0, \ldots) = \omega(E(cT))$. Then

$$\langle a \rangle \cdot \omega_p(\underline{c}) = \omega_p(ac_0, a^p c_1, a^{p^2} c_2, \ldots)$$

for all $\underline{c}$.

(10) $w_n \circ F = w_{n+1}$ for all $n \geq 0$, and

$$w_n \circ V = \begin{cases} p\,w_{n-1} & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$$

(11) $F\langle a \rangle = \langle a^p \rangle$ for any $a \in R$.

Exercise 2.6.7. The group scheme W has a decreasing filtration $\mathrm{Fil}^n W$, $n \geq 0$ defined by $\mathrm{Fil}^n W = V^n W(R)$, that is $\mathrm{Fil}^n W(R)$ consists of all elements of the form $\omega_p(\underline{c})$ such that $c_i = 0$ for all $i < n$. Verify the following properties of this filtration.

(i) For each commutative ring R over $\mathbb{Z}_{(p)}$, the ring $W_p(R)$ is complete with respect to the filtration $\mathrm{Fil}^\bullet W_p(R)$.

(ii) For each $n \geq 0$, $\mathrm{Fil}^n W_p(R)$ is an ideal of $W_p(R)$.

(iii) $V(\mathrm{Fil}^n W_p(R)) \subseteq \mathrm{Fil}^{n+1} W_p(R)$ for all $n \geq 0$.

(iv) $F(\mathrm{Fil}^n W_p(R)) \subseteq \mathrm{Fil}^{n-1} W_p(R)$ for all $n \geq 0$.

Exercise 2.6.8. Show that the universal polynomials defining the ring law for W_p all have coefficients in $\mathbb{Z}$, therefore the ring scheme W_p over $\mathbb{Z}_{(p)}$ has a canonical extension to $\mathbb{Z}$.

Exercise 2.6.9. Suppose that $p = 0$ in R. Prove that the ideal $V\,W_p(R)$ is generated by p if and only if R is perfect; i.e. the Frobenius map $x \mapsto x^p$ for R is surjective.

Exercise 2.6.10. Suppose that k is a perfect field of characteristic p. Prove that $W(k)$ is a complete discrete valuation ring with maximal ideal $V\,W(k) = p\,W(k)$ and residue field k.

2.6.8 Ramified Witt vectors

Let $\mathcal{O}$ is a complete discrete valuation ring such that the residue field is a finite field with q elements. Let π be a uniformizing element of $\mathcal{O}$. The ring scheme of ramified Witt vectors W_π is similar to the p-adic Witt vectors, but the formal completion of $\mathbb{G}_m$ is replaced by the Lubin-Tate formal group. More precisely, the role of the logarithm of the Artin-Hasse exponential is played by the power series $f_\pi(X) := \sum_{n\geq 0} \frac{X^{q^n}}{\pi^n}$ in (2.2.10). Let $E_\pi(X) = f_\pi^{-1}(X)$, the inverse of $f_\pi(X)$; $E_\pi(X)$ has coefficients in $\mathcal{O}$. One can show that there exist polynomials $g_i(\underline{u}, \underline{v})$, $i = 0, 1, 2, \ldots$, where $\underline{u} = (u_0, u_1, u_2, \ldots)$, $\underline{v} = (v_0, v_1, v_2, \ldots)$, such that

$$\sum_{m\geq 0}^{\Phi_\pi} E_\pi(u_m T^{q^m}) + \sum_{m\geq 0}^{\Phi_\pi} E_\pi(v_m T^{q^m}) = \sum_{i\geq 0}^{\Phi_\pi} E_\pi(g_i(\underline{u}, \underline{v}))\, T^{q^i} \,.$$

The above family of polynomials $g_i(\underline{u}, \underline{v})$ defines a group law on $\mathbb{A}^\infty$, denoted by W_π, called the ramified Witt vectors for $(\mathcal{O}, \pi)$. The phantom coordinates are

$$w_{\pi,n}(\underline{u}) := \sum_{i=0}^{n} \pi^{n-i}\, u_{n-i}^{q^i}, \quad n \geq 0 \,.$$

Each $w_{\pi,n}$ defines a group homomorphism from W_π to $\mathbb{G}_a$. Moreover there is a canonical ring scheme structure on W_π such that each $w_{\pi,n}$ is a ring homomorphism from W_π to $\mathbb{A}^1$.

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