CHAPTER I

Schemes and sheaves: definitions

1. Spec(R)

For any commutative ring R, we seek to represent R as a ring of continuous functions on some topological space. This leads us naturally to Spec(R):

DEFINITION 1.1. $\mathrm{Spec}(R) =$ the set of prime ideals $\mathfrak{p} \subset R$ (here R itself is not considered as a prime ideal, but $\{0\}$, if prime is OK). If $\mathfrak{p}$ is a prime ideal, to avoid confusion we denote the corresponding point of Spec(R) by $[\mathfrak{p}]$.

DEFINITION 1.2. For all $x \in \mathrm{Spec}(R)$, if $x = [\mathfrak{p}]$, let

$$\mathbb{k}(x) = \text{the quotient field of the integral domain } R/\mathfrak{p}.$$

For all $f \in R$, define the value $f(x)$ of f at x as the image of f via the canonical maps

$$R \twoheadrightarrow R/\mathfrak{p} \rightarrow \mathbb{k}(x).$$

In this way, we have defined a set Spec(R) and associated to each $f \in R$ a function on Spec(R) — with values unfortunately in fields that vary from point to point. The next step is to introduce a topology in Spec(R):

DEFINITION 1.3. For every subset $S \subset R$, let

$$\begin{aligned} V(S) &= \{x \in \mathrm{Spec}(R) \mid f(x) = 0 \text{ for all } f \in S\} \\ &= \{[\mathfrak{p}] \mid \mathfrak{p} \text{ a prime ideal and } \mathfrak{p} \supseteq S\}. \end{aligned}$$

It is easy to verify that V has the properties:

a) If $\mathfrak{a} =$ the ideal generated by S, then $V(S) = V(\mathfrak{a})$,
b) $S_1 \supseteq S_2 \Longrightarrow V(S_1) \subseteq V(S_2)$,
c) $V(S) = \emptyset \Longleftrightarrow$ [1 is in the ideal generated by S].

PROOF. $\Longleftarrow$ is clear; conversely, if $\mathfrak{a} =$ the ideal generated by S and $1 \notin \mathfrak{a}$, then $\mathfrak{a} \subset \mathfrak{m}$, some maximal ideal $\mathfrak{m}$. Then $\mathfrak{m}$ is prime and $[\mathfrak{m}] \in V(S)$. □

d)

$$\begin{aligned} V(\bigcup_\alpha S_\alpha) &= \bigcap_\alpha V(S_\alpha) \text{ for any family of subsets } S_\alpha \\ V(\sum_\alpha \mathfrak{a}_\alpha) &= \bigcap_\alpha V(\mathfrak{a}_\alpha) \text{ for any family of ideals } \mathfrak{a}_\alpha. \end{aligned}$$

e) $V(\mathfrak{a}_1 \cap \mathfrak{a}_2) = V(\mathfrak{a}_1) \cup V(\mathfrak{a}_2)$.

PROOF. The inclusion $\supseteq$ follows from (b). To prove "$\subseteq$", say $\mathfrak{p} \supset \mathfrak{a}_1 \cap \mathfrak{a}_2$ but $\mathfrak{p} \not\supseteq \mathfrak{a}_1$ and $\mathfrak{p} \not\supseteq \mathfrak{a}_2$. Then $\exists f_i \in \mathfrak{a}_i \setminus \mathfrak{p}$, hence $f_1 \cdot f_2 \in \mathfrak{a}_1 \cap \mathfrak{a}_2$ and $f_1 \cdot f_2 \notin \mathfrak{p}$ since $\mathfrak{p}$ is prime. This is a contradiction. □

f) $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$.

Because of (d) and (e), we can take the sets $V(\mathfrak{a})$ to be the closed sets of a topology on $\mathrm{Spec}(R)$, known as the *Zariski topology.*

DEFINITION 1.4.

$$\begin{aligned}\mathrm{Spec}(R)_f &= \{x \in \mathrm{Spec}(R) \mid f(x) \neq 0\} \\ &= \mathrm{Spec}(R) \setminus V(f).\end{aligned}$$

Since $V(f)$ is closed, $\mathrm{Spec}(R)_f$ is open: we call these the *distinguished* open subsets of $\mathrm{Spec}(R)$.

Note that the distinguished open sets form a basis of the topology closed under finite intersections. In fact, every open set U is of the form $\mathrm{Spec}(R) \setminus V(S)$, hence

$$\begin{aligned}U &= \mathrm{Spec}\, R \setminus V(S) \\ &= \mathrm{Spec}\, R \setminus \bigcap_{f \in S} V(f) \\ &= \bigcup_{f \in S} \big(\mathrm{Spec}\, R \setminus V(f)\big) \\ &= \bigcup_{f \in S} \mathrm{Spec}(R)_f\end{aligned}$$

and

$$\bigcap_{i=1}^{n} \big(\mathrm{Spec}\, R\big)_{f_i} = \big(\mathrm{Spec}\, R\big)_{f_1 \cdots f_n}.$$

DEFINITION 1.5. If $S \subset \mathrm{Spec}\, R$ is any subset, let

$$I(S) = \{f \in R \mid f(x) = 0, \text{ all } x \in S\}.$$

We get a Nullstellensatz-like correspondence between subsets of R and of $\mathrm{Spec}\, R$ given by the operations V and I (cf. Part I [**76**, §1A, (1.5)], Zariski-Samuel [**109**, vol. II, Chapter VII, §3, Theorem 14] and Bourbaki [**26**, Chapter V, §3.3, Proposition 2]):

PROPOSITION 1.6.

(a) *If $\mathfrak{a}$ is any ideal in R, then $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.*
(b) *V and I set up isomorphisms inverse to each other between the set of ideals $\mathfrak{a}$ with $\mathfrak{a} = \sqrt{\mathfrak{a}}$, and the set of Zariski-closed subsets of* $\mathrm{Spec}\, R$.

PROOF. In fact,

$$\begin{aligned}f \in I(V(\mathfrak{a})) &\iff f \in \mathfrak{p} \text{ for every } \mathfrak{p} \text{ with } [\mathfrak{p}] \in V(\mathfrak{a}) \\ &\iff f \in \mathfrak{p} \text{ for every } \mathfrak{p} \supseteq \mathfrak{a}\end{aligned}$$

so

$$\begin{aligned}I(V(\mathfrak{a})) &= \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p} \\ &= \sqrt{\mathfrak{a}}\end{aligned}$$

(cf. Zariski-Samuel [**109**, vol. I, p. 151, Note II] or Atiyah-MacDonald [**19**, p. 9]).

(b) is then a straightforward verification. □

The points of Spec(R) *need not be closed*: In fact,

$$\begin{aligned}\overline{\{[\mathfrak{p}]\}} &= \text{smallest set } V(S), \text{ containing } [\mathfrak{p}], \text{ i.e., } S \subseteq \mathfrak{p} \\ &= V(S), \text{ with } S \text{ the largest subset of } \mathfrak{p} \\ &= V(\mathfrak{p}),\end{aligned}$$

hence:

$$[\mathfrak{p}'] \in \text{closure of } \{[\mathfrak{p}]\} \Longleftrightarrow \mathfrak{p}' \supseteq \mathfrak{p}.$$

Thus $[\mathfrak{p}]$ is closed if and only if $\mathfrak{p}$ is a maximal ideal. At the other extreme, if R is an integral domain then (0) is a prime ideal contained in every other prime ideal, so the closure of $[(0)]$ is the whole space Spec(R). Such a point is called a generic point of Spec(R).

Definition 1.7. If X is a topological space, a closed subset S is *irreducible* if S is not the union of two properly smaller closed subsets $S_1, S_2 \subsetneqq S$. A point x in a closed subset S is called a *generic point of* S if $S = \overline{[\{x\}]}$, and will be written η_S.

It is obvious that the closed sets $\overline{\{x\}}$ are irreducible. For Spec(R), we have the converse:

Proposition 1.8. *If $S \subset \text{Spec}(R)$ is an irreducible closed subset, then S has a unique generic point η_S.*

Proof. I claim S irreducible $\Longrightarrow$ $I(S)$ prime. In fact, if $f \cdot g \in I(S)$, then for all $x \in S$, $f(x) \cdot g(x) = 0$ in $\Bbbk(x)$, hence $f(x) = 0$ or $g(x) = 0$. Therefore

$$S = [S \cap V(f)] \cup [S \cap V(g)].$$

Since S is irreducible, S equals one of these: say $S = S \cap V(f)$. Then $f \equiv 0$ on S, hence $f \in I(S)$. Thus $I(S)$ is prime and

$$\begin{aligned} S &= V(I(S)) \\ &= \text{closure of } [I(S)]. \end{aligned}$$

As for uniqueness, if $[\mathfrak{p}_1]$, $[\mathfrak{p}_2]$ were two generic points of S, then $[\mathfrak{p}_1] \in V(\mathfrak{p}_2)$ and $[\mathfrak{p}_2] \in V(\mathfrak{p}_1)$, hence $\mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq \mathfrak{p}_1$. □

Proposition 1.9. *Let S be a subset of R. Then*

$$\left[\text{Spec}(R) = \bigcup_{f \in S} \text{Spec}(R)_f\right] \Longleftrightarrow \left[1 \in \sum_{f \in S} f \cdot R, \textit{ the ideal generated by } S\right].$$

Proof. In fact,

$$\text{Spec}\, R \setminus \bigcup_{f \in S} \text{Spec}(R)_f = V\left(\sum_{f \in S} f \cdot R\right)$$

so apply (c) in Definition 1.3. □

Notice that $1 \in \sum_{f \in S} f \cdot R$ if and only if there is a finite set $f_1, \ldots, f_n \in S$ and elements $g_1, \ldots, g_n \in R$ such that

$$1 = \sum g_i \cdot f_i.$$

This equation is the algebraic analog of the partitions of unity which are so useful in differential geometry.

Corollary 1.10. Spec R *is quasi-compact*[1]*, i.e., every open covering has a finite subcovering.*

[1] "compact" in the non-Hausdorff space.

PROOF. Because distinguished open sets form a basis, it suffices to check that every covering by distinguished opens has a finite subcover. Because of Proposition 1.9, this follows from the fact that

$$\left[1 \in \sum_{f \in S} f \cdot R\right] \Longrightarrow \left[1 \in \sum_{1=1}^{n} f_i \cdot R, \text{ some finite set } f_1, \ldots, f_n \in S\right].$$

□

When R is noetherian, even more holds:

DEFINITION 1.11. If X is a topological space, the following properties are equivalent:

i) the closed sets satisfy the descending chain condition,
ii) the open sets satisfy the ascending chain condition,
iii) every open set U is quasi-compact.

A space with these properties is called a noetherian topological space.

Because of property (b) of V in Definition 1.3, if R is a noetherian ring, then $\mathrm{Spec}(R)$ is a noetherian space and every open is quasi-compact!

The next big step is to "enlarge" the ring R into a whole sheaf of rings on $\mathrm{Spec}\, R$, written

$$\mathcal{O}_{\mathrm{Spec}\, R}$$

and called the *structure sheaf* of $\mathrm{Spec}\, R$. For background on sheaves, cf. Appendix to this chapter. To simplify notation, let $X = \mathrm{Spec}\, R$. We want to define rings

$$\mathcal{O}_X(U)$$

for every open set $U \subset X$. We do this first for distinguished open sets X_f. Then by Proposition 7 of the Appendix, there is a canonical way to define $\mathcal{O}_X(U)$ for general open sets. The first point is a generalization of Proposition 1.9:

LEMMA 1.12.

$$\left[X_f \subset \bigcup_{i=1}^{n} X_{g_i}\right] \Longleftrightarrow \left[\exists m \geq 1,\ a_i \in R \textit{ such that } f^m = \sum a_i g_i\right].$$

PROOF. The assertion on the left is equivalent to:

$$g_i([\mathfrak{p}]) = 0 \text{ all } i \Longrightarrow f([\mathfrak{p}]) = 0, \text{ for all primes } \mathfrak{p},$$

which is the same as

$$f \in I\Big(V\left(\sum g_i R\right)\Big) = \sqrt{\sum g_i R},$$

which is the assertion on the right. □

We want to define

$$\begin{aligned} \mathcal{O}_X(X_f) &= R_f \\ &= \text{localization of ring } R \text{ with respect to multiplicative system} \\ &\quad \{1, f, f^2, \ldots\};\ \text{or ring of fractions } a/f^n,\ a \in R,\ n \in \mathbb{Z}. \end{aligned}$$

In view of Lemma 1.12, if $X_f \subset X_g$, then $f^m = a \cdot g$ for some $m \geq 1$, $a \in R$, hence there is a canonical map

$$R_g \longrightarrow R_f.$$

(Explicitly, this is the map

$$\frac{b}{g^n} \longmapsto \frac{ba^n}{(ag)^n} = \frac{ba^n}{f^{nm}}.\Big)$$

In particular, if $X_f = X_g$, there are canonical maps $R_f \to R_g$ and $R_g \to R_f$ which are inverse to each other, so we can identify R_f and R_g. Therefore it is possible to define $\mathcal{O}_X(X_f)$ to be R_f. Furthermore, whenever $X_f \subset X_g$, we take the canonical map $R_g \to R_f$ to be the restriction map. Whenever $X_k \subset X_g \subset X_f$, we get a commutative diagram of canonical maps:

$$\begin{tikzcd} R_f \arrow[rr] \arrow[dr] & & R_k \\ & R_g \arrow[ur] & \end{tikzcd}$$

Thus we have defined a presheaf $\mathcal{O}_X$ on the distinguished open sets. We now verify the sheaf axioms:

Key lemma 1.13. *Assume $X_f = \bigcup_{i=1}^N X_{g_i}$. Then*

a) *if $b/f^k \in R_f$ maps to 0 in each localization R_{g_i}, then $b/f^k = 0$,*

b) *if $b_i/g_i^{k_i} \in R_{g_i}$ is a set of elements such that $b_i/g_i^{k_i} = b_j/g_j^{k_j}$ in $R_{g_ig_j}$, then $\exists\, b/f^k \in R_f$ which maps to $b_i/g_i^{k_i}$ for each i.*

Proof. The hypothesis implies that

$$f^m = \sum a_i g_i$$

for some $m \geq 1$ and $a_i \in R$. Raising this to a high power, one sees that for all n, there exists an m' and a_i' such that

$$f^{m'} = \sum a_i' g_i^n$$

too. To prove (a), if $b/f^k = 0$ in R_{g_i}, then $g_i^n \cdot b = 0$ for all i, if n is large enough. But then

$$f^{m'} \cdot b = \sum a_i'(g_i^n b) = 0$$

hence $b/f^k = 0$ in R_f. To prove (b), note that $b_i/g_i^{k_i} = b_j/g_j^{k_j}$ in $R_{g_ig_j}$ means:

$$(g_ig_j)^{m_{ij}} g_j^{k_j} b_i = (g_ig_j)^{m_{ij}} g_i^{k_i} b_j$$

for some $m_{ij} \geq 1$. If $M = \max m_{ij} + \max k_i$, then

$$\frac{b_i}{g_i^{k_i}} = \frac{\overbrace{b_i g_i^{M-k_i}}^{\text{call this } b_i'}}{g_i^M} \quad \text{in } R_{g_i},$$

and

$$\begin{aligned} g_j^M \cdot b_i' &= (g_j^{M-k_j} g_i^{M-k_i}) \cdot g_j^{k_j} b_i \\ &= (g_j^{M-k_j} g_i^{M-k_i}) \cdot g_i^{k_i} b_j, \quad \text{since } M-k_i \text{ and } M-k_j \text{ are } \geq m_{ij} \\ &= g_i^M \cdot b_j'. \end{aligned}$$

Now choose k and a_i' so that $f^k = \sum a_i' g_i^M$. Let $b = \sum a_j' b_j'$. Then I claim b/f^k equals b_i'/g_i^M in R_{g_i}. In fact,

$$\begin{aligned} g_i^M b &= \sum_j g_i^M a_j' b_j' \\ &= \sum_j g_j^M a_j' b_i' \\ &= f^k \cdot b_i'. \end{aligned}$$

$\square$

This means that $\mathcal{O}_X$ is a sheaf on distinguished open sets, hence by Proposition 7 of the Appendix it extends to a sheaf on all open sets of X. Its stalks can be easily computed: if $x = [\mathfrak{p}] \in \operatorname{Spec} R$, then

$$\begin{aligned}
\mathcal{O}_{x,X} &\underset{\text{def}}{=} \varinjlim_{\substack{\text{open } U \\ x \in U}} \mathcal{O}_X(U) \\
&= \varinjlim_{\substack{\text{dist. open } X_f \\ f(x) \neq 0}} \mathcal{O}_X(X_f) \\
&= \varinjlim_{f \in R \setminus \mathfrak{p}} R_f \\
&= R_{\mathfrak{p}}
\end{aligned}$$

where $R_{\mathfrak{p}}$ as usual is the ring of fractions a/f, $a \in R$, $f \in R \setminus \mathfrak{p}$.

Now $R_{\mathfrak{p}}$ is a local ring, with maximal ideal $\mathfrak{p} \cdot R_{\mathfrak{p}}$, and residue field:

$$R_{\mathfrak{p}}/(\mathfrak{p} \cdot R_{\mathfrak{p}}) = (\text{quotient field of } R/\mathfrak{p}) = \Bbbk(x).$$

Thus the stalks of our structure sheaf are local rings and the evaluation of functions $f \in R$ defined above is just the map:

$$R = \mathcal{O}_X(X) \longrightarrow \mathcal{O}_{x,X} \longrightarrow \text{residue field } \Bbbk(x).$$

In particular, the evaluation of functions at x extends to all $f \in \mathcal{O}_X(U)$, for any open neighborhood U of x. Knowing the stalks of $\mathcal{O}_X$ we get the following explicit description of $\mathcal{O}_X$ on all open $U \subset X$:

$$\mathcal{O}_X(U) = \left\{ (s_{\mathfrak{p}}) \in \prod_{[\mathfrak{p}] \in U} R_{\mathfrak{p}} \;\middle|\; \begin{array}{l} U \text{ is covered by distinguished} \\ \text{open } X_{f_i}, \text{ and } \exists s_i \in R_{f_i} \\ \text{inducing } s_{\mathfrak{p}} \text{ whenever } f_i \notin \mathfrak{p} \end{array} \right\}.$$

The pairs $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ are called *affine schemes*. We give a name to one of the most important ones:

$$\begin{aligned}
\mathbb{A}^n_R &= \left(\operatorname{Spec} R[X_1, \ldots, X_n], \mathcal{O}_{\operatorname{Spec} R[X_1,\ldots,X_n]}\right) \\
&= \text{affine } n\text{-space over } R.
\end{aligned}$$

2. $\widetilde{M}$

An important aspect of the construction which defines the structure sheaf $\mathcal{O}_X$ is that it generalizes to a construction which associates a sheaf $\widetilde{M}$ on $\operatorname{Spec}(R)$ to every R-module M. To every distinguished open set X_f, we assign the localized module:

$$\begin{aligned}
M_f &\underset{\text{def}}{=} \left\{ \begin{array}{l} \text{set of symbols } m/f^n,\ m \in M,\ n \in \mathbb{Z}, \\ \text{modulo the identification } m_1/f^{n_1} = m_2/f^{n_2} \text{ iff} \\ f^{n_2+k} \cdot m_1 = f^{n_1+k} \cdot m_2, \text{ some } k \in \mathbb{Z} \end{array} \right\} \\
&= M \otimes_R R_f.
\end{aligned}$$

We check (1) that if $X_f \subset X_g$, then there is a natural map $M_g \to M_f$, (2) that

$$\varinjlim_{[\mathfrak{p}] \in X_f} M_f = M_{\mathfrak{p}}$$

where

$$M_{\mathfrak{p}} \underset{\text{def}}{=} \left\{ \begin{array}{l} \text{set of symbols } m/g,\ m \in M,\ g \in R \setminus \mathfrak{p}, \\ \text{modulo the identification } m_1/g_1 = m_2/g_2 \\ \text{iff } hg_2m_1 = hg_1m_2,\ \text{some } h \in R \setminus \mathfrak{p} \end{array} \right\}$$
$$= M \otimes_R R_{\mathfrak{p}},$$

and (3) that $X_f \mapsto M_f$ is a "sheaf on the distinguished open sets", i.e., satisfies Key lemma 1.13. (The proofs are word-for-word the same as the construction of $\mathcal{O}_X$.) We can then extend the map $X_f \mapsto M_f$ to a sheaf $U \mapsto \widetilde{M}(U)$ such that $\widetilde{M}(X_f) = M_f$ as before. Explicitly:

$$\widetilde{M}(U) = \left\{ s \in \prod_{[\mathfrak{p}] \in U} M_{\mathfrak{p}} \;\middle|\; \text{“}s \text{ given locally by elements of } M_f\text{'s”} \right\}.$$

The sheaf $\widetilde{M}$ that we get is a sheaf of groups. But more than this, it is a sheaf of $\mathcal{O}_X$-modules in the sense of:

Definition 2.1. Let X be a topological space and $\mathcal{O}_X$ a sheaf of rings on X. Then a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on X is a sheaf $\mathcal{F}$ of abelian groups plus an $\mathcal{O}_X(U)$-module structure on $\mathcal{F}(U)$ for all open sets U such that if $U \subset V$, then $\mathrm{res}_{V,U} \colon \mathcal{F}(V) \to \mathcal{F}(U)$ is a module homomorphism with respect to the ring homomorphism $\mathrm{res}_{V,U} \colon \mathcal{O}_X(V) \to \mathcal{O}_X(U)$.

In fact check that the restriction of the natural map

$$\prod_{[\mathfrak{p}] \in U} R_{\mathfrak{p}} \times \prod_{[\mathfrak{p}] \in U} M_{\mathfrak{p}} \longrightarrow \prod_{[\mathfrak{p}] \in U} M_{\mathfrak{p}}$$

maps $\mathcal{O}_X(U) \times \widetilde{M}(U)$ into $\widetilde{M}(U)$, etc.

Moreover, the map $M \mapsto \widetilde{M}$ is a *functor*: given any R-homomorphism of R-modules:

$$\varphi \colon M \longrightarrow N$$

induces by localization:

$$\varphi_f \colon M_f \longrightarrow N_f, \quad \forall f \in R$$

hence

$$\varphi \colon \widetilde{M}(U) \longrightarrow \widetilde{N}(U), \quad \forall \text{ distinguished opens } U.$$

This extends uniquely to a map of sheaves:

$$\widetilde{\varphi} \colon \widetilde{M} \longrightarrow \widetilde{N},$$

which is clearly a homomorphism of these sheaves as $\mathcal{O}_X$-modules.

Proposition 2.2. *Let M, N be R-modules. Then the two maps*

$$\begin{array}{ccc} \mathrm{Hom}_R(M,N) & \rightleftarrows & \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M},\widetilde{N}) \\ \varphi & \longmapsto & \widetilde{\varphi} \\ \left[\begin{array}{l} \psi(X),\ \textit{the map} \\ \textit{on global sections} \end{array} \right] & \longleftarrow & \psi \end{array}$$

are inverse to each other, hence are isomorphisms.

Proof. Immediate. ∎

Corollary 2.3. *The category of R-modules is equivalent to the category of $\mathcal{O}_X$-modules of the form $\widetilde{M}$.*

This result enables us to translate much of the theory of R-modules into the theory of sheaves on $\operatorname{Spec} R$, and brings various geometric ideas into the theory of modules. (See for instance, Bourbaki [**26**, Chapter IV].)

But there are even stronger categorical relations between R-modules M and the sheaves $\widetilde{M}$: in fact, both the category of R-modules M and the category of sheaves of abelian groups on X are *abelian*, i.e., kernels and cokernels with the usual properties exist in both these categories (cf. Appendix to this chapter). In particular one can define exact sequences, etc. The fact is that $\widetilde{\ }$ preserves these operations too:

Proposition 2.4. *Let $f\colon M \to N$ be a homomorphism of R-modules and let $K = \operatorname{Ker}(f)$, $C = \operatorname{Coker}(f)$. Taking $\widetilde{\ }$'s, we get maps of sheaves:*

$$\widetilde{K} \longrightarrow \widetilde{M} \xrightarrow{\widetilde{f}} \widetilde{N} \longrightarrow \widetilde{C}.$$

Then

(a) $\widetilde{K} = \operatorname{Ker}(\widetilde{f})$, *i.e.,* $\widetilde{K}(U) = \operatorname{Ker}[\widetilde{M}(U) \to \widetilde{N}(U)]$ *for all* U.

(b) $\widetilde{C} = \operatorname{Coker}(\widetilde{f})$*: by definition this means* $\widetilde{C}$ *is the sheafification of* $U \to \widetilde{N}(U)/\widetilde{f}(\widetilde{M}(U))$*; but in our case, we get the stronger assertion:*

$$\widetilde{C}(X_a) = \operatorname{Coker}\left(\widetilde{M}(X_a) \to \widetilde{N}(X_a)\right), \quad \textit{all distinguished opens } X_a.$$

Proof. Since $0 \to K \to M \to N \to C \to 0$ is exact, for all $a \in R$ the localized sequence:

$$0 \to K_a \to M_a \to N_a \to C_a \to 0$$

is exact (cf. Bourbaki [**26**, Chapter II, §2.4]; Atiyah-MacDonald [**19**, p. 39]). Therefore

$$0 \to \widetilde{K}(X_a) \to \widetilde{M}(X_a) \to \widetilde{N}(X_a) \to \widetilde{C}(X_a) \to 0$$

is exact for all a. It follows that $\widetilde{K}$ and $\operatorname{Ker}(\widetilde{f})$ are isomorphic on distinguished open sets, hence are isomorphic for all U (cf. Proposition 7 of the Appendix). Moreover it follows that the presheaf $\widetilde{N}(U)/\widetilde{f}(\widetilde{M}(U))$ is already a sheaf on the distinguished open sets X_a, with values $\widetilde{C}(X_a)$; there is only one sheaf on all open sets U extending this, and this sheaf is on the one hand [sheafification of $U \to \widetilde{N}(U)/\widetilde{f}(\widetilde{M}(U))$] or $\operatorname{Coker}(\widetilde{f})$, (see the Appendix) and on the other hand it is $\widetilde{C}$. □

Corollary 2.5. *A sequence*

$$M \longrightarrow N \longrightarrow P$$

of R-modules is exact if and only if the sequence

$$\widetilde{M} \longrightarrow \widetilde{N} \longrightarrow \widetilde{P}$$

of sheaves is exact.

Moreover in both the category of R-modules and of sheaves of $\mathcal{O}_X$-modules there is an internal Hom: namely if M, N are R-modules, $\operatorname{Hom}_R(M, N)$ has again the structure of an R-module; and if $\mathcal{F}$, $\mathcal{G}$ are sheaves of $\mathcal{O}_X$-modules, there is a sheaf of $\mathcal{O}_X$-modules $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ whose global sections are $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ (cf. Appendix to this chapter). In some cases Proposition 2.2 can be strengthened:

Proposition 2.6. *Let M, N be R-modules, and assume M is finitely presented, i.e., $\exists$ an exact sequence:*

$$R^p \longrightarrow R^q \longrightarrow M \longrightarrow 0.$$

Then

$$\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \cong \operatorname{Hom}_R(M, N)\widetilde{\ }.$$

PROOF. There is a natural map on all distinguished opens X_f:

$$\begin{aligned}\operatorname{Hom}_R(M,N)\widetilde{\ }(X_f) &= \operatorname{Hom}_R(M,N)\otimes_R R_f\\ &\to \operatorname{Hom}_{R_f}(M_f,N_f)\\ &\cong \operatorname{Hom}_{\substack{\text{As sheaves of}\\ \mathcal{O}_{X_f}\text{-modules}\\ \text{on } X_f}}(\widetilde{M}|_{X_f},\widetilde{N}|_{X_f}),\ \text{by Proposition 2.2}\\ &= \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})(X_f).\end{aligned}$$

When M is finitely presented, one checks that the arrow on the second line is an isomorphism using:

$$0\longrightarrow \operatorname{Hom}_R(M,N)\longrightarrow \operatorname{Hom}_R(R^q,N)\longrightarrow \operatorname{Hom}_R(R^p,N)$$

hence

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_R(M,N)\otimes_R R_f \arrow[r]\arrow[d] & \operatorname{Hom}_R(R^q,N)\otimes_R R_f \arrow[r]\arrow[d, "\approx"'] & \operatorname{Hom}_R(R^p,N)\otimes_R R_f \arrow[d, "\approx"'] \\ 0 \arrow[r] & \operatorname{Hom}_{R_f}(M_f,N_f) \arrow[r] & \operatorname{Hom}_{R_f}(R_f^q,N_f) \arrow[r] & \operatorname{Hom}_{R_f}(R_f^p,N_f) \end{tikzcd}$$

□

Finally, we will need at one point later that $\widetilde{\ }$ commutes with direct sums, even infinite ones (Proposition-Definition 5.1):

PROPOSITION 2.7. *If $\{M_\alpha\}_{\alpha\in S}$ is any collection of R-modules, then*

$$\widetilde{\sum_{\alpha\in X} M_\alpha} = \sum_{\alpha\in S}\widetilde{M}_\alpha.$$

PROOF. Since each open set X_f is quasi-compact,

$$\begin{aligned}\left(\sum \widetilde{M}_\alpha\right)(X_f) &= \sum\left(\widetilde{M}_\alpha(X_f)\right) \quad \text{cf. remark at the end of Appendix}\\ &= \sum (M_\alpha)_f\\ &= \left(\sum_\alpha M_\alpha\right)_f\\ &= \sum M_\alpha(X_f).\end{aligned}$$

Therefore these sheaves agree on all open sets. □

3. Schemes

We now proceed to the main definition:

DEFINITION 3.1. An *affine scheme* is a topological space X, plus a sheaf of rings $\mathcal{O}_X$ on X isomorphic to $(\operatorname{Spec} R,\mathcal{O}_{\operatorname{Spec} R})$ for some ring R. A *scheme* is a topological space X, plus a sheaf of rings $\mathcal{O}_X$ on X such that there exists an open covering $\{U_\alpha\}$ of X for which each pair $(U_\alpha,\mathcal{O}_X|_{U_\alpha})$ is an affine scheme.

Schemes in general have some of the peculiar topological properties of $\operatorname{Spec} R$. For instance:

PROPOSITION 3.2. *Every irreducible closed subset S of a scheme X is the closure of a unique point $\eta_S\in S$, called its generic point.*

PROOF. Reduce to the affine case, using: U open, $x\in U$, $x\in\overline{\{y\}} \implies y\in U$. □

PROPOSITION 3.3. *If* $(X, \mathcal{O}_X)$ *is a scheme, and* $U \subset X$ *is an open subset, then* $(U, \mathcal{O}_X|_U)$ *is a scheme.*

PROOF. If $\{U_\alpha\}$ is an affine open covering of X, it suffices to show that $U \cap U_\alpha$ is a scheme for all α. But if $U_\alpha = \mathrm{Spec}(R_\alpha)$, then $U \cap U_\alpha$, like any open subset of $\mathrm{Spec}(R_\alpha)$ can be covered by smaller open subsets of the form $\mathrm{Spec}(R_\alpha)_{f_\beta}$, $f_\beta \in R_\alpha$. Therefore we are reduced to proving:

LEMMA 3.4. *For all rings* R *and* $f \in R$,

$$\Big((\mathrm{Spec}\, R)_f, \mathcal{O}_{\mathrm{Spec}\, R}|_{(\mathrm{Spec}\, R)_f}\Big) \cong \Big(\mathrm{Spec}(R_f), \mathcal{O}_{\mathrm{Spec}(R_f)}\Big),$$

hence $(\mathrm{Spec}\, R)_f$ *is itself an affine scheme.*

PROOF OF LEMMA 3.4. Let $i\colon R \to R_f$ be the canonical map. Then if $\mathfrak{p}$ is a prime ideal of R, such that $f \notin \mathfrak{p}$, $i(\mathfrak{p})\cdot R_f$ is a prime ideal of R_f; and if $\mathfrak{p}$ is a prime ideal of R_f, $i^{-1}(\mathfrak{p})$ is a prime ideal of R not containing f. These maps set up a bijection between $\mathrm{Spec}(R)_f$ and $\mathrm{Spec}(R_f)$ (cf. Zariski-Samuel [**109**, vol. I, p. 223]). This is a homeomorphism since the distinguished open sets

$$\mathrm{Spec}(R)_{fg} \subset \mathrm{Spec}(R)_f$$

and

$$\mathrm{Spec}(R_f)_g \subset \mathrm{Spec}(R_f)$$

correspond to each other. But the sections of the structure sheaves $\mathcal{O}_{\mathrm{Spec}(R)}$ and $\mathcal{O}_{\mathrm{Spec}(R_f)}$ on these two open sets are both isomorphic to R_{fg}. Therefore, these rings of sections can be naturally identified with each other and this sets up an isomorphism of (i) the restriction of $\mathcal{O}_{\mathrm{Spec}(R)}$ to $\mathrm{Spec}(R)_f$, and (ii) $\mathcal{O}_{\mathrm{Spec}(R_f)}$ compatible with the homeomorphism of underlying spaces. □

□

Since all schemes are locally isomorphic to a $\mathrm{Spec}(R)$, it follows from §1 that the stalks $\mathcal{O}_{x,X}$ of $\mathcal{O}_X$ are *local* rings. As in §1, define $\Bbbk(x)$ to be the residue field $\mathcal{O}_{x,X}/\mathfrak{m}_{x,X}$ where $\mathfrak{m}_{x,X} =$ maximal ideal, and for all $f \in \Gamma(U, \mathcal{O}_X)$ and $x \in U$, define $f(x) =$ image of f in $\Bbbk(x)$. We can now make the set of schemes into the objects of a category:

DEFINITION 3.5. If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_y)$ are two schemes, a *morphism* from X to Y is a continuous map

$$f\colon X \longrightarrow Y$$

plus a collection of homomorphisms:

$$\Gamma(V, \mathcal{O}_Y) \xrightarrow{f_V^*} \Gamma(f^{-1}(V), \mathcal{O}_X)$$

for every open set $V \subset Y$[2], such that

a) whenever $V_1 \subset V_2$ are two open sets in Y, then the diagram:

$$\begin{array}{ccc} \Gamma(V_2, \mathcal{O}_Y) & \xrightarrow{f_{V_2}^*} & \Gamma(f^{-1}(V_2), \mathcal{O}_X) \\ \Big\downarrow{\scriptstyle \mathrm{res}} & & \Big\downarrow{\scriptstyle \mathrm{res}} \\ \Gamma(V_1, \mathcal{O}_Y) & \xrightarrow{f_{V_1}^*} & \Gamma(f^{-1}(V_1), \mathcal{O}_X) \end{array}$$

commutes, and

[2]Equivalently, a homomorphism of sheaves

$$\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$$

in the notation introduced at the end of the Appendix to this chapter.

b) because of (a), then f_V^*'s pass in the limit to homomorphisms on the stalks:

$$f_x^* \colon \mathcal{O}_{y,Y} \longrightarrow \mathcal{O}_{x,X}$$

for all $x \in X$ and $y = f(x)$; then we require that f_x^* be a local homomorphism, i.e., if $a \in \mathfrak{m}_{y,Y}$ = the maximal ideal of $\mathcal{O}_{y,Y}$, then $f_x^*(a) \in \mathfrak{m}_{x,X}$ = the maximal ideal of $\mathcal{O}_{x,X}$. Equivalently, if $a(y) = 0$, then $f_x^*(a)(x) = 0$.

To explain this rather elaborate definition, we must contrast the situation among schemes with the situation with differentiable or analytic manifolds. In the case of differentiable or analytic manifolds X, X also carries a "structure sheaf" $\mathcal{O}_X$, i.e.,

$$\mathcal{O}_X(U) = \left\{ \begin{array}{l} \text{ring of real-valued differentiable or} \\ \text{complex-valued analytic functions on } U \end{array} \right\}.$$

Moreover, to define a differentiable or analytic map from X to Y, one can ask for a continuous map $f \colon X \to Y$ with the extra property that:

for all open $V \subset Y$ and all $a \in \mathcal{O}_Y(V)$, the compositie function $a \circ f$ on $f^{-1}(V)$ should be in $\mathcal{O}_X(f^{-1}(V))$.

Then we get a homomorphism:

$$\begin{aligned} \Gamma(V, \mathcal{O}_Y) &\longrightarrow \Gamma(f^{-1}(V), \mathcal{O}_X) \\ a &\longmapsto a \circ f \end{aligned}$$

automatically from the map f on the topological spaces. Note that this homomorphism does have properties (a) and (b) of our definition. (a) is obvious. To check (b), note that the stalks $\mathcal{O}_{x,X}$ of the structure sheaf are the rings of *germs* of differentiable or analytic functions at the point $x \in X$. Moreover, $\mathfrak{m}_{x,X}$ is the ideal of germs a such that $a(x) = 0$, and

$$\begin{aligned} \mathcal{O}_{x,X} &\cong \mathfrak{m}_{x,X} \oplus \mathbb{R} \cdot 1_x \quad \text{(differentiable case)} \\ \mathcal{O}_{x,X} &\cong \mathfrak{m}_{x,X} \oplus \mathbb{C} \cdot 1_x \quad \text{(anallytic case)} \end{aligned}$$

where 1_x represents the germ at x of the constant function $a \equiv 1$ (i.e., every germ a equals $a(x) \cdot 1_x + b$, where $b(x) = 0$). Then given a differentiable or analytic map $f \colon X \to Y$, the induced map on stalks $f_x^* \colon \mathcal{O}_{y,Y} \to \mathcal{O}_{x,X}$ is just the map on germs $a \longmapsto a \circ f$, hence

$$\begin{aligned} a \in \mathfrak{m}_{y,Y} &\iff a(y) = 0 \\ &\iff a \circ f(x) = 0 \\ &\iff f_x^* a \in \mathfrak{m}_{x,X}. \end{aligned}$$

The new feature in the case of schemes is that the structure sheaf $\mathcal{O}_X$ is not equal to a sheaf of functions from X to any field k: it is a sheaf of rings, possibly with nilpotent elements, and whose "values" $a(x)$ lie in different fields $\Bbbk(x)$ as x varies. Therefore the continuous map $f \colon X \to Y$ does not induce a map $f^* \colon \mathcal{O}_Y \to \mathcal{O}_X$ automatically. However property (b) does imply that f^* is compatible with "evaluation" of the elements $a \in \mathcal{O}_Y(U)$, i.e., the homomorphism f_x^* induces one on the residue fields:

$$\Bbbk(y) = \mathcal{O}_{y,Y}/\mathfrak{m}_{y,Y} \xrightarrow{f_x^* \text{ modulo maximal ideals}} \mathcal{O}_{x,X}/\mathfrak{m}_{x,X} = \Bbbk(x).$$

Note that it is injective, (like all maps of fields), and that using it (b) can be strengthened to:

(b′) For all $V \subset Y$, and $x \in f^{-1}(V)$, let $y = f(x)$ and identify $\Bbbk(y)$ with its image in $\Bbbk(x)$ by the above map. Then

$$f^*(a)(x) = a(y)$$

for all $a \in \Gamma(V, \mathcal{O}_Y)$.

Given two morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$, we can define their composition $g \circ f\colon X \longrightarrow Z$ in an obvious way. This gives us the *category of schemes*. Also very useful are the related categories of "schemes over S".

DEFINITION 3.6. Fix a scheme S, sometimes referred to as the base scheme. Then a scheme *over* S, written X/S, is a scheme X plus a morphism $p_X\colon X \to S$. If $S = \operatorname{Spec}(R)$, we call this simply a scheme over R or X/R. If X/S and Y/S are two schemes over S, an S-morphism from X/S to Y/S is a morphism $f\colon X \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p_X"'] & & Y \arrow[dl, "p_Y"] \\ & S & \end{tikzcd}$$

commutes.

The following theorem is absolutely crucial in tying together these basic concepts:

THEOREM 3.7. *Let X be a scheme and let R be a ring. To every morphism $f\colon X \to \operatorname{Spec}(R)$, associate the homomorphism:*

$$R \cong \Gamma\left(\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)}\right) \xrightarrow{f^*} \Gamma(X, \mathcal{O}_X).$$

Then this induces a bijection between $\operatorname{Hom}(X, \operatorname{Spec}(R))$ *in the category of schemes and* $\operatorname{Hom}(R, \Gamma(X, \mathcal{O}_X))$ *in the category of rings.*

PROOF. For all f's, let $A_f\colon R \to \Gamma(X, \mathcal{O}_X)$ denote the induced homomorphism. We first show that f is determined by A_f. We must begin by showing how the map of point sets $X \to \operatorname{Spec}(R)$ is determined by A_f. Suppose $x \in X$. The crucial fact we need is that since $\mathfrak{p} = \{a \in R \mid a([\mathfrak{p}]) = 0\}$, a point of $\operatorname{Spec}(R)$ is determined by the ideal of elements of R vanishing at it. Thus $f(x)$ is determined if we know $\{a \in R \mid a(f(x)) = 0\}$. But this equals $\{a \in R \mid f_x^*(a)(x) = 0\}$, and $f_x^*(a)$ is obtained by restricting $A_f(a)$ to $\mathcal{O}_{x,X}$. Therefore

$$f(x) = [\{a \in R \mid (A_f a)(x) = 0\}]\,.$$

Next we must show that the maps f_U^* are determined by A_f for all open sets $U \subset \operatorname{Spec}(R)$. Since f^* is a map of sheaves, it is enough to show this for a basis of open sets (in fact, if $U = \bigcup U_\alpha$ and $s \in \Gamma(U, \mathcal{O}_{\operatorname{Spec}(R)})$, then $f_U^*(s)$ is determined by its restrictions to the sets $f^{-1}(U_\alpha)$, and these equal $f_{U_\alpha}^*(\operatorname{res}_{U,U_\alpha} s)$). Now let $Y = \operatorname{Spec}(R)$ and consider f^* for the distinguished open set Y_b. It makes the diagram

$$\begin{tikzcd} \Gamma(f^{-1}(Y_b), \mathcal{O}_X) & \Gamma(Y_b, \mathcal{O}_Y) = R_b \arrow[l, "f_{Y_b}^*"'] \\ \Gamma(X, \mathcal{O}_X) \arrow[u, "\mathrm{res}"'] & \Gamma(Y, \mathcal{O}_Y) = R \arrow[l, "A_f"] \arrow[u, "\mathrm{res}"'] \end{tikzcd}$$

commutative. Since these are ring homomorphisms, the map on the ring of fractions R_b is determined by that on R: thus A_f determines everything.

Finally any homomorphism $A\colon R \to \Gamma(X, \mathcal{O}_X)$ comes from some morphism f. To prove this, we first reduce to the case when X is affine. Cover X by open affine sets X_α. Then A induces homomorphisms

$$A_\alpha\colon R \longrightarrow \Gamma(X, \mathcal{O}_X) \xrightarrow{\mathrm{res}} \Gamma(X_\alpha, \mathcal{O}_{X_\alpha}).$$

Assuming the result in the affine case, there is a morphism $f_\alpha\colon X_\alpha \to \mathrm{Spec}(R)$ such that $A_\alpha = A_{f_\alpha}$. On $X_\alpha \cap X_\beta$, f_α and f_β agree because the homomorphisms

$$\begin{tikzcd} & \Gamma(X_\alpha, \mathcal{O}_X) \arrow[dr, "\mathrm{res}"] & \\ R \arrow[ur, "A_\alpha"] \arrow[dr, "A_\beta"'] & & \Gamma(X_\alpha \cap X_\beta, \mathcal{O}_X) \\ & \Gamma(X_\beta, \mathcal{O}_X) \arrow[ur, "\mathrm{res}"'] & \end{tikzcd}$$

agree and we know that the morphism is determined by the homomorphism. Hence the f_α patch together to a morphism $f\colon X \to \mathrm{Spec}(R)$, and one checks that A_f is exactly A.

Now let $A\colon R \to S$ be a homomorphism. We want a morphism

$$f\colon \mathrm{Spec}(S) \to \mathrm{Spec}(R).$$

Following our earlier comments, we have no choice in defining f: for all points $[\mathfrak{p}] \in \mathrm{Spec}(S)$,

$$f([\mathfrak{p}]) = [A^{-1}(\mathfrak{p})].$$

This is continuous since for all ideals $\mathfrak{a} \subseteq R$, $f^{-1}(V(\mathfrak{a})) = V(A(\mathfrak{a})\cdot S)$. Moreover if $U = \mathrm{Spec}(R)_a$, then $f^{-1}(U) = \mathrm{Spec}(S)_{A(a)}$, so for f_U^* we need a map $R_a \to S_{A(a)}$. We take the localization of A. These maps are then compatible with restriction, i.e.,

$$\begin{tikzcd} R_a \arrow[r] \arrow[d] & S_{A(a)} \arrow[d] \\ R_{ab} \arrow[r] & S_{A(a)\cdot A(b)} \end{tikzcd}$$

commutes. Hence they determine a sheaf map (in fact, if $U = \bigcup U_\alpha$, U_α distinguished, and $s \in \Gamma(U, \mathcal{O}_{\mathrm{Spec}(R)})$ then the elements $f_{U_\alpha}^*(\mathrm{res}_{U,U_\alpha} s)$ patch together to give an element $f_U^*(s)$ in $\Gamma(f^{-1}(U), \mathcal{O}_{\mathrm{Spec}(S)})$). From our definition of f, it follows easily that f^* on $\mathcal{O}_{[A^{-1}\mathfrak{p}]}$ takes the maximal ideal $\mathfrak{m}_{[A^{-1}\mathfrak{p}]}$ into $\mathfrak{m}_{[\mathfrak{p}]}$. $\square$

COROLLARY 3.8. *The category of affine schemes is equivalent to the category of commutative rings with unit, with arrows reversed.*

COROLLARY 3.9. *If X is a scheme and R is a ring, to make X into a scheme over R is the same thing as making the sheaf of rings $\mathcal{O}_X$ into a sheaf of R-algebras. In particular, there is a unique morphism of every scheme to* $\mathrm{Spec}\,\mathbb{Z}$*: "*$\mathrm{Spec}\,\mathbb{Z}$ *is a final object in the category of schemes"!*

Another point of view on schemes over a given ring A is to ask: what is the "raw data" needed to define a scheme X over $\mathrm{Spec}\,A$? It turns out that such an X can be given by a collection of polynomials with coefficients in A and under suitable finiteness conditions (see Definition II.2.6) this is the most *effective* way to construct a scheme. In fact, first cover X by affine open sets U_α (possibly an infinite set) and let $U_\alpha = \mathrm{Spec}\,R_\alpha$. Then each R_α is an A-algebra. Represent R_α as a quotient of a polynomial ring:

$$R_\alpha = A[\dots, X_\beta^{(\alpha)}, \dots]/(\dots, f_\gamma^{(\alpha)}, \dots)$$

where the $f_\gamma^{(\alpha)}$ are polynomials in the variables $X_\beta^{(\alpha)}$. The scheme X results from glueing a whole lot of isomorphic localizations $(U_{\alpha_1})_{g_{\alpha_1\alpha_2\nu}}$ and $(U_{\alpha_2})_{h_{\alpha_1\alpha_2\nu}}$, and these isomorphisms result from

A-algebra isomorphisms:

$$A\left[\dots, X_\beta^{\alpha_1}, \dots, \frac{1}{g_{\alpha_1\alpha_2\nu}(X_\beta^{(\alpha_1)})}\right] \Big/ (\dots, f_\gamma^{(\alpha_1)}, \dots)$$
$$\cong A\left[\dots, X_\beta^{\alpha_2}, \dots, \frac{1}{h_{\alpha_1\alpha_2\nu}(X_\beta^{(\alpha_2)})}\right] \Big/ (\dots, f_\gamma^{(\alpha_2)}, \dots)$$

given by

$$X_{\beta_2}^{(\alpha_2)} = \frac{\phi_{\alpha_1\alpha_2\nu\beta_2}(\dots, X_\beta^{(\alpha_1)}, \dots)}{(g_{\alpha_1\alpha_2\nu})^{N_{\alpha_1\alpha_2\nu\beta_2}}}$$
$$X_{\beta_1}^{(\alpha_1)} = \frac{\psi_{\alpha_1\alpha_2\nu\beta_1}(\dots, X_\beta^{(\alpha_2)}, \dots)}{(h_{\alpha_1\alpha_2\nu})^{M_{\alpha_1\alpha_2\nu\beta_1}}}.$$

Thus the collection of polynomials f, g, h, ϕ and ψ with coefficients in A explicitly describes X. In reasonable cases, this collection is finite and gives the most effective way of "writing out" the scheme X.

It is much harder to describe explicitly the set of morphisms from $\operatorname{Spec} R$ to X than it is to describe the morphisms from X to $\operatorname{Spec} R$. In one case this can be done however:

PROPOSITION 3.10. *Let R be a local ring with maximal ideal M. Let X be a scheme. To every morphism $f\colon \operatorname{Spec} R \to X$ associate the point $x = f([M])$ and the homomorphism*

$$f_x^*\colon \mathcal{O}_{x,X} \longrightarrow \mathcal{O}_{[M],\operatorname{Spec} R} = R.$$

Then this induces a bijection between $\operatorname{Hom}(\operatorname{Spec} R, X)$ and the set of pairs (x, ϕ), where $x \in X$ and $\phi\colon \mathcal{O}_{x,X} \to R$ is a local homomorphism.

(Proof left to the reader.)

This applies for instance to the case $R = K$ a field, in which case $\operatorname{Spec} K$ consists in only one point $[M] = [(0)]$. A useful example is:

COROLLARY 3.11. *For every $x \in X$, there is a canonical homomorphism*

$$i_x\colon \operatorname{Spec} \mathbb{k}(x) \longrightarrow X$$

defined by requiring that $\operatorname{Image}(i_x) = x$, and that

$$i_x^*\colon \mathcal{O}_{x,X} \to \mathcal{O}_{[(0)],\operatorname{Spec} \mathbb{k}(x)} = \mathbb{k}(x)$$

be the canonial map. For every field k, every morphism

$$f\colon \operatorname{Spec} k \longrightarrow X$$

factors uniquely:

$$\operatorname{Spec} k \xrightarrow{g} \operatorname{Spec} \mathbb{k}(x) \xrightarrow{i_x} X$$

where $x = \operatorname{Image}(f)$ and g is induced by an inclusion $\mathbb{k}(x) \to k$.

4. Products

There is one exceedingly important and very elementary existence theorem in the category of schemes. This asserts that arbitrary fibre products exist:

Recall that if morphisms:

$$\begin{tikzcd} X \arrow[dr, "r"'] & & Y \arrow[dl, "s"] \\ & S & \end{tikzcd}$$

are given a *fibre product* is a commutative diagram

$$\begin{tikzcd} & X \times_S Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[dr, "r"'] & & Y \arrow[dl, "s"] \\ & S & \end{tikzcd}$$

with the obvious universal property: i.e., given any commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ X \arrow[dr, "r"'] & & Y \arrow[dl, "s"] \\ & S & \end{tikzcd}$$

there is a unique morphism $t\colon Z \to X \times_S Y$ such that $q_1 = p_1 \circ t$, $q_2 = p_2 \circ t$. The fibre product is unique up to canonical isomorphism. When S is the final object $\operatorname{Spec} \mathbb{Z}$ in the category of schemes, we drop the S and write $X \times Y$ for the product.

THEOREM 4.1. *If A and B are C-algebras, let the diagram of affine schemes*

$$\begin{tikzcd} & \operatorname{Spec}(A \otimes_C B) \arrow[dl] \arrow[dr] & \\ \operatorname{Spec}(A) \arrow[dr] & & \operatorname{Spec}(B) \arrow[dl] \\ & \operatorname{Spec}(C) & \end{tikzcd}$$

be defined by the canonical homomorphisms $C \to A$, $C \to B$, $A \to A \otimes_C B$ ($a \mapsto a \otimes 1$), $B \to A \otimes_C B$ ($b \mapsto 1 \otimes b$). This makes $\operatorname{Spec}(A \otimes_C B)$ *a fibre product of* $\operatorname{Spec}(A)$ *and* $\operatorname{Spec}(B)$ *over* $\operatorname{Spec}(C)$.

THEOREM 4.2. *Given any morphisms $r\colon X \to S$, $s\colon Y \to S$, a fibre product exists.*

PROOF OF THEOREM 4.1. It is well known that in the diagram (of solid arrows):

$$\begin{tikzcd} & A \arrow[dr] \arrow[drr, dashed] & & \\ C \arrow[ur] \arrow[dr] & & A \otimes_C B \arrow[r] & D \\ & B \arrow[ur] \arrow[urr, dashed] & & \end{tikzcd}$$

the tensor product has the universal mapping property indicated by dotted arrows, i.e., is the "direct sum" in the category of commutative C-algebras, or the "fibre sum" in the category of commutative rings. Dually, this means that $\operatorname{Spec}(A \otimes_C B)$ is the fibre product in the category of *affine* schemes. But if T is an arbitrary scheme, then by Theorem 3.7, every morphism of T into any affine scheme $\mathrm{Spec}(E)$ factors uniquely through $\mathrm{Spec}(\Gamma(T, \mathcal{O}_T))$:

$$\begin{tikzcd} T \arrow[rr] \arrow[dr] & & \mathrm{Spec}(E) \\ & \mathrm{Spec}(\Gamma(T,\mathcal{O}_T)) \arrow[ur, dashed] & \end{tikzcd}$$

Using this, it follows immediately that $\mathrm{Spec}(A \otimes_C B)$ is the fibre product in the category of all schemes. $\square$

Theorem 4.1 implies for instance that:

$$\mathbb{A}^n_R \cong \mathbb{A}^n_{\mathbb{Z}} \times \mathrm{Spec}\, R.$$

Proof of Theorem 4.2. There are two approaches to this. The first is a *patching argument* that seems quite straightforward and "mechanical", but whose details are really remarkably difficult. The second involves the direct construction of $X \times_S Y$ as a local ringed space and then the verification that locally it is indeed the same product as that given by Theorem 4.1. We will sketch both. For the first, the main point to notice is this: suppose

$$\begin{tikzcd} & X \times_S Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[dr, "r"'] & & Y \arrow[dl, "s"] \\ & S & \end{tikzcd}$$

is some fibre product and suppose that $X_\circ \subset X$, $Y_\circ \subset Y$ and $S_\circ \subset S$ are open subsets. Assume that $r(X_\circ) \subset S_\circ$ and $s(Y_\circ) \subset S_\circ$. Then the open subset

$$p_1^{-1}(X_\circ) \cap p_2^{-1}(Y_\circ) \subset X \times_S Y$$

is always the fibre product of $X_\circ$ and $Y_\circ$ over $S_\circ$. This being so, it is clear how we must set about constructing a fibre product: first cover S by open affines:

$$\mathrm{Spec}(C_k) = W_k \subset S.$$

Next, cover $r^{-1}(W_k)$ and $s^{-1}(W_k)$ by open affines:

$$\mathrm{Spec}(A_{k,i}) = U_{k,i} \subset X,$$
$$\mathrm{Spec}(B_{k,j}) = V_{k,j} \subset Y.$$

Then the affine schemes:

$$\mathrm{Spec}(A_{k,i} \otimes_{C_k} B_{k,j}) = \Phi_{k,i,j}$$

must make an open affine covering of $X \times_S Y$ if it exists at all. To patch together $\Phi_{k,i,j}$ and $\Phi_{k',i',j'}$, let p_1, p_2, and p_1', p_2' stand for the canonical projections of $\Phi_{k,i,j}$ and $\Phi_{k',i',j'}$ onto its factors. Then one must next check that the open subsets:

$$p_1^{-1}(U_{k,i} \cap U_{k',i'}) \cap p_2^{-1}(V_{k,j} \cap V_{k',j'}) \subset \Phi_{k,i,j}$$

and

$$(p_1')^{-1}(U_{k',i'} \cap U_{k,i}) \cap (p_2')^{-1}(V_{k',j'} \cap V_{k,j}) \subset \Phi_{k',i',j'}$$

are both fibre products of $U_{k,i} \cap U_{k',i'}$ and $V_{k,j} \cap V_{k',j'}$ over S. Hence they are canonically isomorphic and can be patched. Then you have to check that everything is consistent at triple overlaps. Finally you have to check the universal mapping property. All this is in some sense obvious but remarkably confusing unless one takes a sufficiently categorial point of view. For details, cf. EGA [**1**, Chapter I, pp. 106–107].

The second proof involves explicitly constructing $X \times_S Y$ as a local ringed space. To motivate the construction note that if $z \in X \times_S Y$ lies over $x \in X$, $y \in Y$ and $s \in S$, then the residue fields of the four points lie in a diagram:

$$\begin{tikzcd} & \mathbb{k}(z) & \\ \mathbb{k}(x) \arrow[ur, hook, "p_1^*"] & & \mathbb{k}(y) \arrow[ul, hook', "p_2^*"'] \\ & \mathbb{k}(s) \arrow[ul, hook', "r^*"] \arrow[ur, hook, "s^*"'] & \end{tikzcd}$$

From Theorem 4.1, one sees that the local rings of $X \times_S Y$ are generated by tensor product of the local rings of X and Y and this implies that in the above diagram $\mathbb{k}(z)$ is the quotient field of its subring $\mathbb{k}(x) \cdot \mathbb{k}(y)$, i.e., $\mathbb{k}(z)$ is a compositum of $\mathbb{k}(x)$ and $\mathbb{k}(y)$ over $\mathbb{k}(s)$. We may reverse these conclusions and use them as a basis of a definition of $X \times_S Y$;

i) As a point set, $X \times_S Y$ is the set of 5-tuples (x, y, L, α, β) where

$$x \in X, \quad y \in Y,$$

lie over the same point $s \in S$ and

$$L = \text{a field extension of } \mathbb{k}(s)$$
$$\alpha,\ \beta \text{ are homomorphisms:}$$

$$\begin{tikzcd} & L & \\ \mathbb{k}(x) \arrow[ur, hook, "\alpha"] & & \mathbb{k}(y) \arrow[ul, hook', "\beta"'] \\ & \mathbb{k}(s) \arrow[ul, hook', "r^*"] \arrow[ur, hook, "s^*"'] & \end{tikzcd}$$

such that

$$L = \text{quotient field of } \mathbb{k}(x) \cdot \mathbb{k}(y).$$

Two such points are equal if the points x, y on X and Y are equal and the corresponding diagrams of fields are isomorphic.

ii) As a topological space, a basis of open sets is given by the distinguished open sets

$$\mathcal{U}(V, W, \{f_l\}, \{g_l\})$$

where

$$\begin{aligned} &V \subset X \text{ is affine open} \\ &W \subset Y \text{ is affine open} \\ &f_l \in \mathcal{O}_X(V) \\ &g_l \in \mathcal{O}_Y(W) \\ \mathcal{U} = \{&(x, y, L, \alpha, \beta) \mid x \in V, y \in W, \\ &\quad \sum_l \alpha(f_l) \cdot \beta(g_l) \neq 0 \text{ (this sum taken in } L)\}. \end{aligned}$$

iii) The structure sheaf $\mathcal{O}_{X \times_S Y}$ is defined as a certain sheaf of maps from open sets in $X \times_S Y$ to:

$$\coprod_{x,y,L,\alpha,\beta} \mathcal{O}_{(x,y,L,\alpha,\beta)}$$

where

$$\mathcal{O}_{(x,y,L,\alpha,\beta)} = \left[\begin{array}{l} \text{localization of } \mathcal{O}_{x,X} \otimes_{\mathcal{O}_{s,S}} \mathcal{O}_{y,Y} \\ \text{at } \mathfrak{p} = \mathrm{Ker}(\mathcal{O}_x \otimes_{\mathcal{O}_s} \mathcal{O}_y \xrightarrow[\alpha\otimes\beta]{} L) \end{array} \right]$$

(i.e., the elements of the sheaf will map points $(x, y, L, \alpha, \beta) \in X \times_S Y$ to elements of the corresponding ring $\mathcal{O}_{(x,y,L,\alpha,\beta)}$.) The sheaf is defined to be those maps which locally are given by expressions

$$\begin{aligned} &\frac{\sum f_l \otimes g_l}{\sum f'_l \otimes g'_l} \\ &f_l, f'_l \in \mathcal{O}_X(V) \\ &g_l, g'_l \in \mathcal{O}_Y(W) \\ &\text{on open sets } \mathcal{U}(V, W, \{f'_l\}, \{g'_l\}). \end{aligned}$$

This certainly gives us a local ringed space, but it must be proven to be a scheme and to be the fibre product. We will not give details. For the first, one notes that the construction is local on X and Y and hence it suffices to prove that if $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$ and $S = \operatorname{Spec} A$, then the local ringed space $X \times_S Y$ constructed above is simply $\operatorname{Spec}(R \otimes_A S)$. The first step then is to verify:

LEMMA 4.3. *The set of prime ideals of $R \otimes_A S$ is in one-to-one correspondence with the set of 5-tuples $(\mathfrak{p}_R, \mathfrak{p}_S, L, \alpha, \beta)$ where $\mathfrak{p}_R \subset R$ and $\mathfrak{p}_S \subset S$ are prime ideals with the same inverse image $\mathfrak{p}_A \subset A$ and (L, α, β) is a compositum of the quotient fields of $R/\mathfrak{p}_R$, $S/\mathfrak{p}_S$ over $A/\mathfrak{p}_A$.*

The proof is straightforward.

COROLLARY 4.4 (of proof). *As a point set, $X \times_S Y$ is the set of pairs of points $x \in X$, $y \in Y$ lying over the same point of S, plus a choice of compositum of their residue fields up to isomorphisms:*

$$\begin{tikzcd} & L & \\ \Bbbk(x) \arrow[ur, hook, "\alpha"] & & \Bbbk(y) \arrow[ul, hook', "\beta"'] \\ & \Bbbk(s) \arrow[ul, hook', "r^*"] \arrow[ur, hook, "s^*"'] & \end{tikzcd}$$

□

Summarizing the above proof, we can give in a special case the following "explicit" idea of what fibre product means: Suppose we are in the situation

$$\begin{tikzcd} X \arrow[dr, "r"'] & & \operatorname{Spec}(B) \arrow[dl, "s"] \\ & \operatorname{Spec}(A) & \end{tikzcd}$$

and that $X = \bigcup U_\alpha$, U_α affine. Then each U_α is $\operatorname{Spec} R_\alpha$ and via r^*,

$$R_\alpha = A[\ldots, X_\beta^{(\alpha)}, \ldots]/(\ldots, f_\gamma^{(\alpha)}, \ldots)$$

as in §3, where the $f_\gamma^{(\alpha)}$ are polynomials in the variables $X_\beta^{(\alpha)}$. Represent the glueing between the U_α's by a set of polynomials $g_{\alpha_1,\alpha_2,\nu}$, $h_{\alpha_1,\alpha_2,\nu}$, $\phi_{\alpha_1,\alpha_2,\nu,\beta_2}$ and $\psi_{\alpha_1,\alpha_2,\nu,\beta_1}$ as in §3 again. Let

s correspond to a homomorphism $\sigma\colon A \to B$. If f is a polynomial over A, let ${}^\sigma f$ denote the polynomial over B gotten by applying σ to its coefficients. Then

$$\begin{aligned} X \times_{\operatorname{Spec} A} \operatorname{Spec} B &\cong \bigcup_\alpha U_\alpha \times_{\operatorname{Spec} A} \operatorname{Spec} B \\ &\cong \bigcup_\alpha \operatorname{Spec}\left[\left(A[\dots, X_\beta^{(\alpha)}, \dots]/(\dots, f_\gamma^{(\alpha)}, \dots)\right) \otimes_A B\right] \\ &\cong \bigcup_\alpha \operatorname{Spec}\left[B[\dots, X_\beta^{(\alpha)}, \dots]/(\dots, {}^\sigma f_\gamma^{(\alpha)}, \dots)\right]. \end{aligned}$$

In other words, the new scheme $X \times_{\operatorname{Spec} A} \operatorname{Spec} B$ is gotten by glueing corresponding affines, each defined by the new equations in the same variables gotten by pushing their coefficients from A to B via σ. Moreover, it is easy to see that the identification on $(U_\alpha \times_{\operatorname{Spec} A} \operatorname{Spec} B) \cap (U_\beta \times_{\operatorname{Spec} A} \operatorname{Spec} B)$ is gotten by glueing the distinguished opens ${}^\sigma g_{\alpha_1,\alpha_2,\nu} \neq 0$ and ${}^\sigma h_{\alpha_1,\alpha_2,\nu} \neq 0$ by isomorphisms given by the polynomials ${}^\sigma\phi$ and ${}^\sigma\psi$. Or we may simply say that the collection of polynomials ${}^\sigma f$, ${}^\sigma g$, ${}^\sigma h$, ${}^\sigma\phi$, ${}^\sigma\psi$ with coefficients in B explicitly describes $X \times_{\operatorname{Spec} A} \operatorname{Spec} B$ by the same recipe used for X.

We can illustrate this further by a very important special case of fibre products: suppose $f\colon X \to Y$ is any morphism and $y \in Y$. Consider the fibre product:

$$\begin{tikzcd} X \times_Y \operatorname{Spec} \Bbbk(y) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} \Bbbk(y) \arrow[r, "i_Y"] & Y \end{tikzcd}$$

Definition 4.5. Denote $X \times_Y \operatorname{Spec} \Bbbk(y)$ by $f^{-1}(y)$ and call it the *fibre of f over y*.

To describe $f^{-1}(y)$ explicitly, let $U \subset Y$ be an affine neighborhood of y, let $U = \operatorname{Spec}(R)$, and $y = [\mathfrak{p}]$. It is immediate that the fibre product $X \times_Y U$ is just the open subscheme $f^{-1}(U)$ of X, and by associativity of fibre products, $f^{-1}(y) \cong f^{-1}(U) \times_U \operatorname{Spec} \Bbbk(y)$. Now let $f^{-1}(U)$ be covered by affines:

$$\begin{aligned} V_\alpha &= \operatorname{Spec}(S_\alpha) \\ S_\alpha &\cong R[\dots, X_\beta^{(\alpha)}, \dots]/(\dots, f_\gamma^{(\alpha)}, \dots). \end{aligned}$$

Then $f^{-1}(y)$ is covered by affines

$$\begin{aligned} V_\alpha \cap f^{-1}(y) &= \operatorname{Spec}(S_\alpha \otimes_R \Bbbk(y)) \\ &= \operatorname{Spec}\left[\Bbbk(y)[\dots, X_\beta^{(\alpha)}, \dots]/(\dots, \overline{f}_\gamma^{(\alpha)}, \dots)\right] \end{aligned}$$

($\overline{f} =$ polynomial gotten from f via coefficient homomorphism $R \to \Bbbk(y)$). Notice that the underlying topological space of $f^{-1}(y)$ is just the subspace $f^{-1}(y)$ of X. In fact via the ring homomorphism

$$S_\alpha \xrightarrow{\phi} (S_\alpha/\mathfrak{p}S_\alpha)_{(R/\mathfrak{p}\setminus(0))} \cong S_\alpha \otimes_R \Bbbk(y)$$

the usual maps

$$\begin{aligned} \mathfrak{q} &\longmapsto \phi(\mathfrak{q}) \cdot (S_\alpha/\mathfrak{p}S_\alpha)_{(R/\mathfrak{p}\setminus(0))} \\ \phi^{-1}(\mathfrak{q}) &\longleftarrow\!\shortmid\ \mathfrak{q} \end{aligned}$$

set up a bijection between all the prime ideals of $(S_\alpha/\mathfrak{p}S_\alpha)_{(R/\mathfrak{p}\setminus(0))}$ and the prime ideals $\mathfrak{q} \subset S_\alpha$ such that $\mathfrak{q} \cap R = \mathfrak{p}$, and it is easily seen to preserve the topology. This justifies the notation $f^{-1}(y)$.

5. Quasi-coherent sheaves

For background on kernels and cokernels in the category of sheaves of abelian groups, see the Appendix to this chapter. If $(X, \mathcal{O}_X)$ is a scheme, the sheaves of interest to us are the sheaves $\mathcal{F}$ of $\mathcal{O}_X$-modules (Definition 2.1). These form an abelian category too, if we consider $\mathcal{O}_X$-linear homomorphisms as the maps. (In fact, given $\alpha\colon \mathcal{F} \to \mathcal{G}$, the sheaf $U \mapsto \mathrm{Ker}(\alpha\colon \mathcal{F}(U) \to \mathcal{G}(U))$ is again a sheaf of $\mathcal{O}_X$-modules; and the sheafification of $U \mapsto \mathcal{G}(U)/\alpha\mathcal{F}(U)$ has a canonical $\mathcal{O}_X$-module structure on it.) The most important of these sheaves are the quasi-coherent ones, which are the ones locally isomorphic to the sheaves $\widetilde{M}$ defined in §2:

PROPOSITION-DEFINITION 5.1. *Let X be a scheme and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. The following are equivalent:*

i) *for all $U \subset X$, affine and open, $\mathcal{F}|_U \cong \widetilde{M}$ for some $\Gamma(U, \mathcal{O}_X)$-module M,*

ii) *$\exists$ an affine open covering $\{U_\alpha\}$ of X such that $\mathcal{F}|_{U_\alpha} \cong \widetilde{M_\alpha}$ for some $\Gamma(U_\alpha, \mathcal{O}_X)$-module M_α,*

iii) *for all $x \in X$, there is a neighborhood U of x and an exact sequence of sheaves on U:*

$$(\mathcal{O}_X|_U)^I \to (\mathcal{O}_X|_U)^J \to \mathcal{F}|_U \to 0$$

(where the exponents I, J denote direct sums, possibly infinite).

If $\mathcal{F}$ has these properties, we call it quasi-coherent.

PROOF. It is clear that (i) $\implies$ (ii). Conversely, to prove (ii) $\implies$ (i), notice first that if U is an open affine set such that $\mathcal{F}|_U \cong \widetilde{M}$ for some $\Gamma(U, \mathcal{O}_X)$-module M, then for all $f \in \Gamma(U, \mathcal{O}_X)$, $\mathcal{F}|_{U_f} \cong \widetilde{M_f}$. Therefore, starting with condition (ii), we deduce that there is a *basis* $\{U_i\}$ for the topology of X consisting of open affines such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$. Now if U is any open affine set and $R = \Gamma(U, \mathcal{O}_X)$, we can cover U by a finite number of these U_i's. Furthermore, we can cover each of these U_i's by smaller open affines of the type U_g, $g \in R$. Since $U_g = (U_i)_g$, $\mathcal{F}|_{U_g}$ is isomorphic to $\widetilde{(M_i)_g}$. In other words, we get a finite covering of U by affines U_{g_i} such that $\mathcal{F}|_{U_{g_i}} \cong \widetilde{N_i}$, N_i an R_{g_i}-module.

For every open set $V \subset U$, the sequence

$$0 \longrightarrow \Gamma(V, \mathcal{F}) \longrightarrow \prod_i \Gamma(V \cap U_{g_i}, \mathcal{F}) \longrightarrow \prod_{i,j} \Gamma(V \cap U_{g_i} \cap U_{g_j}, \mathcal{F})$$

is exact. Define new sheaves $\mathcal{F}_i^*$ and $\mathcal{F}_{i,j}^*$ by:

$$\Gamma(V, \mathcal{F}_i^*) = \Gamma(V \cap U_{g_i}, \mathcal{F})$$
$$\Gamma(V, \mathcal{F}_{i,j}^*) = \Gamma(V \cap U_{g_i} \cap U_{g_j}, \mathcal{F}).$$

Then the sequence of sheaves:

$$0 \longrightarrow \mathcal{F} \longrightarrow \prod_i \mathcal{F}_i^* \longrightarrow \prod_{i,j} \mathcal{F}_{i,j}^*$$

is exact, so to prove that $\mathcal{F}$ is of the form $\widetilde{M}$, it suffices to prove this for $\mathcal{F}_i^*$ and $\mathcal{F}_{i,j}^*$. But if M_i° is M_i viewed as an R-module, then $\mathcal{F}_i^* \cong \widetilde{M_i^\circ}$. In fact, for all distinguished open sets U_g,

$$\begin{aligned}\Gamma(U_g, \mathcal{F}_i^*) &= \Gamma(U_g \cap U_{g_i}, \mathcal{F}) \\ &= \Gamma((U_{g_i})_g, \mathcal{F}|_{U_{g_i}}) \\ &= (M_i)_g \\ &= \Gamma(U_g, \widetilde{M_i^\circ}).\end{aligned}$$

The same argument works for the $\mathcal{F}_{i,j}^*$'s.

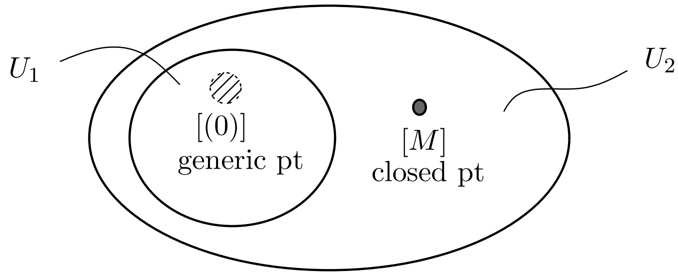

FIGURE I.1. The Spectrum of a discrete valuation ring

Next, (ii) $\implies$ (iii) because if $\mathcal{F}|_{U_\alpha} \cong \widetilde{M_\alpha}$, write $\widetilde{M_\alpha}$ by generators and relations:

$$R_\alpha^I \longrightarrow R_\alpha^J \longrightarrow M_\alpha \longrightarrow 0$$

where $R_\alpha = \Gamma(U_\alpha, \mathcal{O}_X)$. By Corollary 2.5

$$\widetilde{(R_\alpha^I)} \longrightarrow \widetilde{(R_\alpha^J)} \longrightarrow \widetilde{M_\alpha} \longrightarrow 0$$

is exact. But $\widetilde{R_\alpha} \cong \mathcal{O}_X|_{U_\alpha}$ since U_α is affine and $\widetilde{\quad}$ commutes with direct sums (even infinite ones by Proposition 2.7) so we get the required presentation of $\mathcal{F}|_{U_\alpha}$.

Finally (iii) $\implies$ (ii). Starting with (iii), we can pass to smaller neighborhoods so as to obtain an *affine* open covering $\{U_\alpha\}$ of X in which presentations exist:

$$\begin{array}{ccccc} (\mathcal{O}_X|_{U_\alpha})^I & \xrightarrow{h} & (\mathcal{O}_X|_{U_\alpha})^J & \longrightarrow \mathcal{F}|_{U_\alpha} & \longrightarrow 0 \\ \| & & \| & & \\ (R_\alpha^I)^{\widetilde{\ }} & & (R_\alpha^J)^{\widetilde{\ }}. & & \end{array}$$

By Proposition 2.2, h is induced by an R_α-homomorphism $k\colon R_\alpha^I \to R_\alpha^J$. Let $M_\alpha = \operatorname{Coker}(k)$. Then by Proposition 2.4, $\widetilde{M_\alpha} \cong \mathcal{F}|_{U_\alpha}$. $\square$

COROLLARY 5.2. *If $\alpha\colon \mathcal{F} \to \mathcal{G}$ is an $\mathcal{O}_X$-homomorphism of quasi-coherent sheaves, then* $\operatorname{Ker}(\alpha)$ *and* $\operatorname{Coker}(\alpha)$ *are quasi-coherent.*

PROOF. Use characterization (i) of quasi-coherent and Proposition 2.4. $\square$

We can illustrate the concept of quasi-coherent quite clearly on $\operatorname{Spec} R$, R a discrete valuation ring. R has only two prime ideals, (0) and M the maximal ideal. Thus $\operatorname{Spec} R$ has two points, one in the closure of the other as in Figure I.1: and only two non-empty sets: U_1 consisting of $\{(0)\}$ alone, and U_2 consisting of the whole space. M is principal and if π is a generator, then U_1 is the distinguished open set $(\operatorname{Spec} R)_\pi$. Thus:

a) the structure sheaf is:

$$\begin{aligned} \mathcal{O}_{\operatorname{Spec} R}(U_2) &= R, \\ \mathcal{O}_{\operatorname{Spec} R}(U_1) &= R[\frac{1}{\pi}] \\ &= \text{quotient field } K \text{ of } R \end{aligned}$$

b) general sheaf of abelian groups is a pair of abelian groups

$$\mathcal{F}(U_1),\ \mathcal{F}(U_2) \text{ plus a homomorphism } \operatorname{res}\colon \mathcal{F}(U_2) \to \mathcal{F}(U_1),$$

c) general sheaf of $\mathcal{O}_{\operatorname{Spec} R}$-modules is an R-module $\mathcal{F}(U_2)$, a K-vector space $\mathcal{F}(U_1)$ plus an R-linear homomorphism $\operatorname{res}\colon \mathcal{F}(U_2) \to \mathcal{F}(U_1)$,

d) quasi-coherence means that $\mathcal{F} = \widetilde{\mathcal{F}(U_2)}$, i.e., res factors through an isomorphism:

$$\mathcal{F}(U_2) \longrightarrow \mathcal{F}(U_2) \otimes_R K \xrightarrow{\approx} \mathcal{F}(U_1).$$

The next definition gives the basic finiteness properties of quasi-coherent sheaves:

DEFINITION 5.3. A quasi-coherent sheaf $\mathcal{F}$ is *finitely generated* if every $x \in X$ has a neighborhood U in which there is a surjective $\mathcal{O}_X$-homomorphism:

$$(\mathcal{O}_X|_U)^n \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

some $n \geq 1$. $\mathcal{F}$ is *finitely presented*, or *coherent*[3] if every $x \in X$ has a neighborhood U in which there is an exact sequence:

$$(\mathcal{O}_X|_U)^m \longrightarrow (\mathcal{O}_X|_U)^n \longrightarrow \mathcal{F}|_U \longrightarrow 0.$$

$\mathcal{F}$ is *locally free* (of finite rank) if every $x \in X$ has a neighborhood U in which there is an isomorphism

$$(\mathcal{O}_X|_U)^n \xrightarrow{\approx} \mathcal{F}|_U.$$

The techniques used in the proof of Proposition 5.1 show easily that if $U \subset X$ is affine and open and $\mathcal{F}$ is finitely generated (resp. coherent), then $\mathcal{F}|_U = \widetilde{M}$ where M is finitely generated (resp. finitely presented) as module over $\Gamma(U, \mathcal{O}_X)$.

DEFINITION 5.4. Let $\mathcal{F}$ be a quasi-coherent sheaf on a scheme X. Then for all $x \in X$, in addition to the stalk of $\mathcal{F}$ at x, we get a vector space over $\Bbbk(x)$ the residue field:

$$\mathcal{F}(x) = \mathcal{F}_x \otimes \Bbbk(x)$$
$$\operatorname{rk}_x \mathcal{F} = \dim_{\Bbbk(x)} \mathcal{F}(x).$$

A very important technique for finitely generated quasi-coherent sheaves is Nakayama's lemma:

PROPOSITION 5.5 (Nakayama). *Let $\mathcal{F}$ be a finitely generated quasi-coherent sheaf on a scheme X. Then*

i) *if $x \in X$ and if the images of $s_1, \ldots, s_n \in \mathcal{F}_x$ in $\mathcal{F}(x)$ span the vector space $\mathcal{F}(x)$, then the s_i extend to a neighborhood of x on which they define a surjective homomorphism*

$$(\mathcal{O}_X|_U)^n \xrightarrow{(s_1,\ldots,s_n)} \mathcal{F}|_U \longrightarrow 0$$

on U. When this holds, we say that $s_1, \ldots, s_n$ generate $\mathcal{F}$ over U.

ii) *if $\operatorname{rk}_x \mathcal{F} = 0$, then x has a neighborhood U such that $\mathcal{F}|_U = \{0\}$.*

iii) *$\operatorname{rk}\colon x \mapsto \operatorname{rk}_x \mathcal{F}$ is upper-semi-continuous, i.e., for all $k \geq 0$, $\{x \in X \mid \operatorname{rk}_x \mathcal{F} \leq k\}$ is open.*

PROOF. (i) is the geometric form of the usual Nakayama lemma. Because of its importance, we recall the proof. (i) reduces immediately to the affine case where it says this:

> R any commutative ring, $\mathfrak{p}$ a prime ideal, M an R-module, generated by $m_1, \ldots, m_k$. If $n_1, \ldots, n_l \in M$ satisfy
>
> $$\overline{n_1}, \ldots, \overline{n_l} \text{ generate } M_\mathfrak{p} \otimes \Bbbk(\mathfrak{p}) \text{ over } \Bbbk(\mathfrak{p})$$
>
> then $\exists f \in R \setminus \mathfrak{p}$ such that
>
> $$n_1, \ldots, n_l \text{ generate } M_f \text{ over } R_f.$$

[3]If X is locally noetherian, i.e., X is covered by $\operatorname{Spec} R$'s with R noetherian (see §II.2), then it is immediate that a quasi-coherent finitely generated $\mathcal{F}$ is also coherent; and that sub- and quotient-sheaves of coherent $\mathcal{F}$'s are automatically coherent. The notion of coherent will not be used except on noetherian X's. (What about §IV.4?)

But the hypothesis gives us immediately:

$$a_i m_i = \sum_{j=1}^{l} b_{ij} n_j + \sum_{j=1}^{k} c_{ij} m_j, \quad 1 \leq i \leq k$$

for some $a_i \in R \setminus \mathfrak{p}$, $b_{ij} \in R$, $c_{ij} \in \mathfrak{p}$. Solving these k equations for the m_i by Cramer's rule, we get

$$\left(\det_{p,q}(a_p \delta_{pq} - c_{pq}) \right) \cdot m_i = \sum_{j=1}^{l} b'_{ij} n_j.$$

Let f be this determinant. Then $f \notin \mathfrak{p}$ and $n_1, \ldots, n_l$ generate M_f over R_f.

(ii) and (iii) are immediate consequences of (i). □

The following Corollary is often useful:

Corollary 5.6. *Let X be a quasi-compact scheme, $\mathcal{F}$ a finitely generated quasi-coherent sheaf of $\mathcal{O}_X$-modules. Suppose that for each $x \in X$, there exists a finite number of global sections of $\mathcal{F}$ which generate $\mathcal{F}(x)$. Then there exists a finite number of global sections of $\mathcal{F}$ that generate $\mathcal{F}$ everywhere.*

An important construction is the tensor product of quasi-coherent sheaves. The most general setting for this is when we have

$$\begin{tikzcd} & X \times_S Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[dr, "r"'] & & Y \arrow[dl, "s"] \\ & S & \end{tikzcd}$$

$\mathcal{F}$ quasi-coherent on X

$\mathcal{G}$ quasi-coherent on Y.

Then we can construct a quasi-coherent sheaf $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G}$ on $X \times_S Y$ analogously to our definition and construction of $X \times_S Y$ itself—viz.

Step I: characterize $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G}$ by a universal mapping property: consider all quisi-coherent[4] sheaves of $\mathcal{O}_{X \times_S Y}$-modules $\mathcal{H}$ plus collections of maps:

$$\mathcal{F}(U) \times \mathcal{G}(V) \to \mathcal{H}(p_1^{-1} U \cap p_2^{-1} V)$$

($U \subset X$ and $V \subset Y$ open) which are $\mathcal{O}_X(U)$-linear in the first variable and $\mathcal{O}_Y(V)$-linear in the second and which commute with restriction. $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G}$ is to be the universal one.

Step II: Show that when $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $S = \operatorname{Spec} C$, $\mathcal{F} = \widetilde{M}$, $\mathcal{G} = \widetilde{N}$, then $(M \otimes_C N)\widetilde{\ }$ on $\operatorname{Spec}(A \otimes_C B)$ has the required property.

Step III: "Glue" these local solutions $(M_\alpha \otimes_{C_\alpha} N_\alpha)\widetilde{\ }$ together to form a sheaf $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G}$.

We omit the details. Notice that the stalks of $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G}$ are given by:

If $z \in X \times_S Y$ has images $x \in X$, $y \in Y$ and $s \in S$,

$$(\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G})_z \cong \left\{ \begin{array}{l} \text{localization of the } \mathcal{O}_{x,X} \otimes_{\mathcal{O}_{s,S}} \mathcal{O}_{y,Y}\text{-module} \\ \mathcal{F}_x \otimes_{\mathcal{O}_{s,S}} \mathcal{G}_y \text{ with respect to the} \\ \text{prime ideal } \mathfrak{m}_{x,X} \otimes \mathcal{O}_{y,Y} + \mathcal{O}_{x,X} \otimes \mathfrak{m}_{y,Y} \end{array} \right\}.$$

(Use the description of $\otimes$ in the affine case.) Two cases of this construction are most important:

[4]In fact, $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G}$ is universal for non-quasi-coherent $\mathcal{H}$'s too.

i) $X = Y = S$: Given two quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{G}$, we get a third one $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ with stalks $\mathcal{F}_x \otimes_{\mathcal{O}_{x,X}} \mathcal{G}_x$. On affines, it is given by:

$$\widetilde{M} \otimes_{\operatorname{Spec} R} \widetilde{N} \cong (M \otimes_R N)^{\sim}.$$

ii) $Y = S$, $\mathcal{F} = \mathcal{O}_X$: Given a morphism $r\colon X \to Y$ and a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G}$, we get a quasi-coherent $\mathcal{O}_X$-module $\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{G}$. This is usually written $r^*(\mathcal{G})$ and has stalks $(r^*\mathcal{G})_x = \mathcal{O}_{x,X} \otimes_{\mathcal{O}_{y,Y}} \mathcal{G}_y$ ($y = r(x)$). If X and Y are affine, say $X = \operatorname{Spec}(R)$, $Y = \operatorname{Spec}(S)$, then it is given by:

$$r^*(\widetilde{M}) \cong (M \otimes_S R)^{\sim}.$$

The general case can be reduced to these special cases by formula:

$$\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G} \cong p_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} p_2^*\mathcal{G}.$$

Also iterating (i), we define $\mathcal{F}_1 \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{F}_k$; symmetrizing or skew-symmetrizing, we get $\operatorname{Symm}^k \mathcal{F}$ and $\bigwedge^k \mathcal{F}$ just like the operations $\operatorname{Symm}^k M$, $\bigwedge^k M$ on modules.

We list a series of properties of quasi-coherent sheaves whose proofs are straightforward using the techniques already developed. These are just a sample from the long list to be found in EGA [**1**].

5.7. *If $\mathcal{F}$ is a quasi-coherent sheaf on X and $\mathcal{I} \subset \mathcal{O}_X$ is a quasi-coherent sheaf of ideals, then the sheaf*

$$\mathcal{I} \cdot \mathcal{F} \underset{def}{=} \left[\begin{array}{l} \textit{subsheaf of } \mathcal{F} \textit{ generated by} \\ \textit{the submodules } \mathcal{I}(U) \cdot \mathcal{F}(U) \end{array} \right]$$

is quasi-coherent and for U affine

$$\mathcal{I} \cdot \mathcal{F}(U) = \mathcal{I}(U) \cdot \mathcal{F}(U).$$

5.8. *If $\mathcal{F}$ is quasi-coherent and $U \subset V \subset X$ are two affines, then*

$$\mathcal{F}(U) \cong \mathcal{F}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U).$$

5.9. *Let X be a scheme and let*

$$U \longmapsto \mathcal{F}(U)$$

be a presheaf. Suppose that for all affine U and all $f \in R = \Gamma(U, \mathcal{O}_X)$, the map

$$\mathcal{F}(U) \otimes_R R_f \longrightarrow \mathcal{F}(U_f)$$

is an isomorphism. Then the sheafification $\operatorname{sh}(\mathcal{F})$ *of $\mathcal{F}$ is quasi-coherent and*

$$\operatorname{sh}(\mathcal{F})(U) \cong \mathcal{F}(U)$$

for all affine *U.*

5.10. *If $\mathcal{F}$ is coherent and $\mathcal{G}$ is quasi-coherent, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent, with a canonical homomorphism*

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{G}.$$

(cf. Appendix to this chapter and Proposition 2.6.)

5.11. *Let $f\colon X \to Y$ be a morphism of schemes, $\mathcal{F}$ a quasi-coherent sheaf on X and $\mathcal{G}$ a quasi-coherent sheaf on Y. Then*

$$\operatorname{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).$$

(See (ii) above for the definition of $f^*\mathcal{G}$.)

5.12. *Let R be an S-algebra and let $f\colon \operatorname{Spec} R \to \operatorname{Spec} S$ be the corresponding morphism of affine schemes. Let M be an R-module. Then M can be considered as an S-module too and we can form $\widetilde{M_R}$, $\widetilde{M_S}$ the corresponding sheaves on $\operatorname{Spec} R$ and $\operatorname{Spec} S$. Then*

$$f_*(\widetilde{M_R}) = \widetilde{M_S}.$$

(cf. Appendix to this chapter for the definition of f_*.)

6. The functor of points

We have had several indications that the underlying point set of a scheme is peculiar from a geometric point of view. Non-closed points are odd for one thing. Another peculiarity is that the point set of a fibre product $X \times_S Y$ does not map injectively into the set-theoretic product of X and Y. The explanation of these confusing facts is that there are really two concepts of "point" in the language of schemes. To see this in its proper setting, look at some examples in other categories:

Example. Let $\mathcal{C}$ = category of differentiable manifolds. Let $\mathbf{z}$ be the manifold with *one* point. Then for any manifold X,

$$\operatorname{Mor}_{\mathcal{C}}(\mathbf{z}, X) \cong X \quad \text{as a point set.}$$

Example. Let $\mathcal{C}$ = category of groups. Let $\mathbf{z} = \mathbb{Z}$. Then for any group G

$$\operatorname{Mor}_{\mathcal{C}}(\mathbf{z}, G) \cong G \quad \text{as a point set.}$$

Example. Let $\mathcal{C}$ = category of rings with 1 (and homomorphisms f such that $f(1) = 1$). Let $\mathbf{z} = \mathbb{Z}[X]$. Then for any ring R,

$$\operatorname{Mor}_{\mathcal{C}}(\mathbf{z}, R) \cong R \quad \text{as a point set.}$$

This indicates that if $\mathcal{C}$ is any category, whose objects may not be point sets to begin with, and $\mathbf{z}$ is an object, one can try to conceive of $\operatorname{Mor}_{\mathcal{C}}(\mathbf{z}, X)$ as the underlying set of points of the object X. In fact:

$$X \longmapsto \operatorname{Mor}_{\mathcal{C}}(\mathbf{z}, X)$$

extends to a functor from the category $\mathcal{C}$ to the category (Sets), of sets. But, it is not satisfactory to call $\operatorname{Mor}_{\mathcal{C}}(\mathbf{z}, X)$ the set of points of X unless this functor is *faithful*, i.e., unless a morphism f from X_1 to X_2 is determined by the map of sets:

$$\tilde{f}\colon \operatorname{Mor}_{\mathcal{C}}(\mathbf{z}, X_1) \longrightarrow \operatorname{Mor}_{\mathcal{C}}(\mathbf{z}, X_2).$$

Example. Let (Hot) be the category of CW-complexes, where

$$\operatorname{Mor}(X, Y)$$

is the set of homotopy-classes of continuous maps from X to Y. If $\mathbf{z}$ = the 1 point complex, then

$$\operatorname{Mor}_{(\mathrm{Hot})}(\mathbf{z}, X) = \pi_0(X), \quad \text{(the set of components of } X\text{)}$$

and this does *not* give a faithful functor.

Example. Let $\mathcal{C}$ = category of schemes. Take for instance $\mathbf{z}$ to be the *final* object of the category $\mathcal{C}$: $\mathbf{z} = \operatorname{Spec}(\mathbb{Z})$. Now

$$\operatorname{Mor}_{\mathcal{C}}(\operatorname{Spec}(\mathbb{Z}), X)$$

is absurdly small, and does not give a faithful functor.

Grothendieck's ingenious idea is to remedy this defect by considering (for arbitrary categories $\mathcal{C}$) not *one* $\mathbf{z}$, but *all* $\mathbf{z}$: attach to X the whole set:

$$\bigcup_{\mathbf{z}} \mathrm{Mor}_{\mathcal{C}}(\mathbf{z}, X).$$

In a natural way, this always give a faithful functor from the category $\mathcal{C}$ to the category (Sets). Even more than that, the "extra structure" on the set $\bigcup_{\mathbf{z}} \mathrm{Mor}_{\mathcal{C}}(\mathbf{z}, X)$ which characterizes the object X, can be determined. It consists in:

i) the decomposition of $\bigcup_{\mathbf{z}} \mathrm{Mor}_{\mathcal{C}}(\mathbf{z}, X)$ into subsets $S_{\mathbf{z}} = \mathrm{Mor}_{\mathcal{C}}(\mathbf{z}, X)$, one for each $\mathbf{z}$.

ii) the natural maps from one set $S_{\mathbf{z}}$ to another $S_{\mathbf{z}'}$, given for each morphism $g\colon \mathbf{z}' \to \mathbf{z}$ in the category.

Putting this formally, it comes out like this:

Attach to each X in $\mathcal{C}$, the *functor* h_X (contravariant, from $\mathcal{C}$ itself to (Sets)) via

($*$) $h_X(\mathbf{z}) = \mathrm{Mor}_{\mathcal{C}}(\mathbf{z}, X)$, $\mathbf{z}$ an object in $\mathcal{C}$.

($**$) $h_X(g) = \{\text{induced map from } \mathrm{Mor}_{\mathcal{C}}(\mathbf{z}, X) \text{ to } \mathrm{Mor}_{\mathcal{C}}(\mathbf{z}', X)\}$, $g\colon \mathbf{z}' \to \mathbf{z}$ a morphism in $\mathcal{C}$.

Now the functor h_X is an object in a category too: viz.

$$\mathrm{Funct}(\mathcal{C}^\circ, (\mathrm{Sets})),$$

(where Funct stands for functors, $\mathcal{C}^\circ$ stands for $\mathcal{C}$ with arrows reversed). It is also clear that if $g\colon X_1 \to X_2$ is a morphism in $\mathcal{C}$, then one obtains a morphism of functors $h_g\colon h_{X_1} \to h_{X_2}$. All this amounts to one big functor:

$$h\colon \mathcal{C} \longrightarrow \mathrm{Funct}(\mathcal{C}^\circ, (\mathrm{Sets})).$$

PROPOSITION 6.1. *h is fully faithful, i.e., if X_1, X_2 are objects of $\mathcal{C}$, then, under h,*

$$\mathrm{Mor}_{\mathcal{C}}(X_1, X_2) \xrightarrow{\sim} \mathrm{Mor}_{\mathrm{Funct}}(h_{X_1}, h_{X_2}).$$

PROOF. Easy. □

The conclusion, heuristically, is that an object X of $\mathcal{C}$ can be *identified* with the functor h_X, which is basically just a structured set.

Return to algebraic geometry! What we have said motivates I hope:

DEFINITION 6.2. If X and K are schemes, a *K-valued point* of X is a morphism $f\colon K \to X$; if $K = \mathrm{Spec}(R)$, we call this an R-valued point of X. If X and K are schemes over a third scheme S, i.e., we are given morphisms $p_X\colon X \to S$, $p_K\colon K \to S$, then f is a K-valued point of X/S if $p_X \circ f = p_K$; if $K = \mathrm{Spec}(R)$, we call this an R-valued point of X/S. The set of all R-valued points of a scheme X, or of X/S, is denoted $X(R)$.

Proposition 3.10, translated into our new terminology states that if R is a local ring, there is a bijection between the set of R-valued points of X and the set of pairs (x, ϕ), where $x \in X$ and $\phi\colon \mathcal{O}_{x,X} \to R$ is a local homomorphism. Corollary 3.11 states that for every point $x \in X$ in the usual sense, there is a canonical $\Bbbk(x)$-valued point i_x of X in our new sense. In particular, suppose X is a scheme over $\mathrm{Spec}\, k$: then there is a bijection

$$\left\{\begin{array}{l}\text{set of } k\text{-valued points}\\ \text{of } X/\mathrm{Spec}\, k\end{array}\right\} \cong \left\{\begin{array}{l}\text{set of points } x \in X \text{ such that}\\ \text{the natural map } k \to \Bbbk(x)\\ \text{is surjective}\end{array}\right\}$$

given by associating i_x to x. Points $x \in X$ with $k \xrightarrow{\approx} \Bbbk(x)$ are called *k-rational points* of X.

K-valued points of a scheme are compatible with products. In fact, if K, X, Y are schemes over S, then the set of K-valued points of $(X \times_S Y)/S$ is just the (set-theoretic) product of the set of K-valued points of X/S and the set of K-valued points of Y/S. This is the definition of the fibre product.

The concept of an R-valued point generalizes the notion of a solution of a set of diophantine equations in the ring R. In fact, let:

$$\begin{aligned} &f_1, \dots, f_m \in \mathbb{Z}[X_1, \dots, X_n] \\ &X = \mathrm{Spec}(\mathbb{Z}[X_1, \dots, X_n]/(f_1, \dots, f_m)). \end{aligned}$$

I claim an R-valued point of X is the "same thing" as an n-tuple $a_1, \dots, a_n \in R$ such that

$$f_1(a_1, \dots, a_n) = \dots = f_m(a_1, \dots, a_n) = 0.$$

But in fact a morphism

$$\mathrm{Spec}(R) \xrightarrow{g} \mathrm{Spec}(\mathbb{Z}[X_1, \dots, X_n]/(f_1, \dots, f_m))$$

is determined by the n-tuple $a_i = g^*(X_i)$, $1 \le i \le n$, and those n-tuples that occur are exactly those such that $h \mapsto h(a_1, \dots, a_n)$ defines a homomorphism

$$R \xleftarrow{g^*} \mathbb{Z}[X_1, \dots, X_n]/(f_1, \dots, f_m),$$

i.e., solutions of $f_1, \dots, f_m$.

An interesting point is that a scheme is actually determined by the functor of its R-valued points as well as by the larger functor of its K-valued points. To state this precisely, let X be a scheme, and let $h_X^{(\circ)}$ be the *covariant* functor from the category (Rings) of commutative rings with 1 to the category (Sets) defined by:

$$h_X^{(\circ)}(R) = h_X(\mathrm{Spec}(R)) = \mathrm{Mor}(\mathrm{Spec}(R), X).$$

Regarding $h_X^{(\circ)}$ as a functor in X in a natural way, one has:

PROPOSITION 6.3. *For any two schemes X_1, X_2,*

$$\mathrm{Mor}(X_1, X_2) \xrightarrow{\sim} \mathrm{Mor}(h_{X_1}^{(\circ)}, h_{X_2}^{(\circ)}).$$

Hence $h^{(\circ)}$ is a fully faithful functor from the category of schemes to

$$\mathrm{Funct}((\mathrm{Rings}), (\mathrm{Sets})).$$

This result is more readily checked privately than proven formally, but it may be instructive to sketch how a morphism $F\colon h_{X_1}^{(\circ)} \to h_{X_2}^{(\circ)}$ will induce a morphism $f\colon X_1 \to X_2$. One chooses an affine open covering $U_i \cong \mathrm{Spec}(A_i)$ of X_1; let

$$I_i\colon \mathrm{Spec}(A_i) \cong U_i \to X_1$$

be the inclusion. Then I_i is an A_i-valued point of X_1. Therefore $F(I_i) = f_i$ is an A_i-valued point of X_2, i.e., f_i defines

$$U_i \cong \mathrm{Spec}(A_i) \to X_2.$$

Modulo a verification that these f_i patch together on $U_i \cap U_j$, these f_i give the morphism f via

$$\begin{tikzcd} U_i \arrow[r, "f_i"] \arrow[d, phantom, "\cap"] & X_2. \\ X_1 \arrow[ur, "f"'] & \end{tikzcd}$$

Proposition 6.3 suggests a whole new approach to the foundations of the theory of schemes. Instead of defining a scheme as a space X plus a sheaf of rings $\mathcal{O}_X$ on X, why not define a scheme as a covariant functor F from (Rings) to (Sets) which satisfies certain axioms strong enough to show that it is isomorphic to a functor $h_X^{(\circ)}$ for some scheme in the usual sense? More precisely:

DEFINITION 6.4. A covariant functor $F\colon (\text{Rings}) \to (\text{Sets})$ is a *sheaf in the Zariski topology* if for all rings R and for all equations

$$1 = \sum_{i=1}^{n} f_i g_i,$$

then

a) the natural map $F(R) \to \prod_{i=1}^{n} F(R_{f_i})$ is injective
b) for all collections $s_i \in F(R_{f_i})$ such that s_i and s_j have the same image in $F(R_{f_i f_j})$, there is an $s \in F(R)$ mapping onto the s_i's.

If F is a functor and $\xi \in F(R)$, we get a morphism of functors:

$$\phi_\xi\colon h_R \longrightarrow F$$

i.e., a set of maps

$$\phi_{\xi,S}\colon h_R(S) \underset{\text{def}}{=} \operatorname{Hom}(R,S) \to F(S)$$

given by:

$$\begin{aligned} &\forall R \xrightarrow{\alpha} S \\ &\phi_{\xi,S}(\alpha) = F(\alpha)(\xi). \end{aligned}$$

If $\mathfrak{a} \subset R$ is an ideal, define the subfunctor

$$h_R^{\mathfrak{a}} \subset h_R$$

by

$$h_R^{\mathfrak{a}}(S) = \left\{ \begin{array}{l} \text{set of all homomorphisms } \alpha\colon R \to S \\ \text{such that } \alpha(\mathfrak{a}) \cdot S = S \end{array} \right\}.$$

DEFINITION 6.5. Let $F\colon (\text{Rings}) \to (\text{Sets})$ be a functor. An element $\xi \in F(R)$ is an *open subset* if

a) $\phi_\xi\colon h_R \to F$ is injective
b) for all rings S and all $\eta \in F(S)$, consider the diagram:

$$\begin{tikzcd} h_R \arrow[r, hook, "\phi_\xi"'] & F \\ & h_S \arrow[u, "\phi_\eta"'] \end{tikzcd}$$

Then there is an ideal $\mathfrak{a} \subset S$ such that $\phi_\eta^{-1}(h_R) =$ subfunctor $h_S^{\mathfrak{a}}$ of h_S.

DEFINITION 6.6. A functor $F\colon (\text{Rings}) \to (\text{Sets})$ is a *scheme-functor* if

a) it is a sheaf in the Zariski-topology,
b) there exist open subsets $\xi_\alpha \in F(R_\alpha)$ such that for all fields k,

$$F(k) = \bigcup_{\alpha} \phi_{\xi_\alpha} h_{R_\alpha}(k).$$

We leave it to the reader now to check that the scheme-functors F are precisely those given by

$$F(R) = \mathrm{Mor}(\mathrm{Spec}\, R, X)$$

for some scheme X. This point of view is worked out in detail in Demazure-Gabriel [**34**]. It is moreover essential in a very important generalization of the concept of scheme which arose as follows. One of the principal goals in Grothendieck's work on schemes was to find a characterization of scheme-functors by weak general properties that could often be checked in practice and so lead to many existence theorems in algebraic geometry (like Brown's theorem[5] in (Hot)). It seemed at first that this program would fail completely and that scheme-functors were really quite special[6];but then Artin discovered an extraordinary approximation theorem which showed that there was a category of functors F only a "little" larger than the scheme-functors which can indeed be characterized by weak general properties. Geometrically speaking, his functors F are like spaces gotten by dividing affines by étale equivalence relations (cf. Chapter V) and then glueing. He called these *algebraic spaces* (after algebraic functions, i.e., meromorphic functions on $\mathbb{C}$ satisfying a polynomial equation; see Artin [**15**], [**16**], [**17**], [**18**], Knutson [**63**])[7].

7. Relativization

The goal of this section is to extend the concept of Spec in a technical but very important way. Instead of starting with a *ring* R and defining a *scheme* $\mathrm{Spec}\, R$, we want to start with a *sheaf of rings* $\mathcal{R}$ on an arbitrary scheme X and define a *scheme over* X, $\pi\colon \mathcal{S}pec_X\, \mathcal{R} \to X$. More precisely, $\mathcal{R}$ must be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. We may approach the definition of $\mathcal{S}pec_X\, \mathcal{R}$ by a universal mapping property as follows:

THEOREM-DEFINITION 7.1. *Let X be a scheme and let $\mathcal{R}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. Then there is a scheme over X:*

$$\pi\colon \mathcal{S}pec_X\, \mathcal{R} \to X$$

and an isomorphism of $\mathcal{O}_X$-algebras:

$$\mathcal{R} \xrightarrow{\approx} \pi_*(\mathcal{O}_{\mathcal{S}pec_X \mathcal{R}})$$

uniquely characterized by the property:

For all morphisms

$$f\colon Y \to X$$

plus homomorphisms of $\mathcal{O}_X$-algebras

$$\alpha\colon \mathcal{R} \to f_*(\mathcal{O}_Y)$$

there is a unique factorization:

$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr, "f"'] & & \mathcal{S}pec_X\, \mathcal{R} \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

for which α is given by g^*:

$$\mathcal{R} \xrightarrow{\approx} \pi_*(\mathcal{O}_{\mathcal{S}pec_X \mathcal{R}}) \xrightarrow{g^*} f_*(\mathcal{O}_Y).$$

[5]See Spanier [**98**, Chapter 7, §7].
[6]See for instance Hironaka [**54**] and Mumford [**72**, p. 83].
[7](Added in publication) For later developments see, for instance, FAG [**3**].

The situation is remarkably similar to the construction of fibre products:

Firstly, if X is affine, then this existence theorem has an immediate solution:

$$\mathcal{S}pec_X\,\mathcal{R} = \mathrm{Spec}(\mathcal{R}(X)).$$

The universal mapping property is just a rephrasing of Theorem 3.7 and (5.12).

Secondly, we can use the solution in the affine case to prove the general existence theorem modulo a patching argument. In fact, let U_α be an affine open covering of X. Then the open subset

$$\pi^{-1}(U_\alpha) \subset \mathcal{S}pec_X(\mathcal{R})$$

will have to be

$$\mathcal{S}pec_{U_\alpha}(\mathcal{R}|_{U_\alpha})$$

(just restrict the universal mapping property to those morphisms $f\colon Y \to X$ which factor through U_α). Therefore $\mathcal{S}pec_X(\mathcal{R})$ must be the union of affine open pieces $\mathrm{Spec}(\mathcal{R}(U_\alpha))$. To use this observation as a construction for all α, β, we must identify the open subsets below:

$$\begin{tikzcd}[column sep=small]
\mathrm{Spec}(\mathcal{R}(U_\alpha)) \arrow[dr, "\pi_\alpha"'] & \supset & \pi_\alpha^{-1}(U_\alpha \cap U_\beta) \arrow[rr, equal, "\overset{??}{\sim}"] \arrow[dr] & & \pi_\beta^{-1}(U_\alpha \cap U_\beta) \arrow[dl] & \subset & \mathrm{Spec}(\mathcal{R}(U_\beta)) \arrow[dl, "\pi_\beta"] \\
& U_\alpha & \supset & U_\alpha \cap U_\beta & \subset & U_\beta &
\end{tikzcd}$$

Note that

$$\pi_{\alpha,*}(\mathcal{O}_{\mathrm{Spec}\,\mathcal{R}(U_\alpha)}) \cong \mathcal{R}|_{U_\alpha}$$

by (5.12) hence

$$\Gamma(\pi_\alpha^{-1}(U_\alpha \cap U_\beta), \mathcal{O}_{\mathrm{Spec}\,\mathcal{R}(U_\alpha)}) \cong \mathcal{R}(U_\alpha \cap U_\beta).$$

Composing this with

$$\mathcal{R}(U_\beta) \xrightarrow[\mathrm{res}]{} \mathcal{R}(U_\alpha \cap U_\beta)$$

and using Theorem 3.7, we get a morphism

$$\pi_\alpha^{-1}(U_\alpha \cap U_\beta) \to \mathrm{Spec}\,\mathcal{R}(U_\beta)$$

that factors through $\pi_\beta^{-1}(U_\alpha \cap U_\beta)$. Interchanging α and β, we see that we have an isomorphism.

Thirdly, we can also give a totally explicit construction of $\mathcal{S}pec_X\,\mathcal{R}$ as follows:

i) as a point set, $\mathcal{S}pec_X\,\mathcal{R}$ is the set of pairs $(x, \mathfrak{p})$, where $x \in X$ and $\mathfrak{p} \subset \mathcal{R}_x$ is a prime ideal such that if

$$i\colon \mathcal{O}_x \to \mathcal{R}_x$$

is the given map, then

$$i^{-1}(\mathfrak{p}) = \mathfrak{m}_x$$

ii) as a topological space, we get a basis of open sets:

$$\{\mathcal{U}(V, f) \mid V \subset X \text{ open affine, } f \in \mathcal{R}(V)\}$$

where

$$\mathcal{U}(V, f) = \{(x, \mathfrak{p}) \mid x \in V,\ f \notin \mathfrak{p}\}.$$

iii) the structure sheaf is a certain sheaf of functions from open sets in $\mathcal{S}pec_X\,\mathcal{R}$ to

$$\coprod_{x,\mathfrak{p}} (\mathcal{R}_x)_{\mathfrak{p}},$$

namely the functions which are locally given by f/f', $f, f' \in \mathcal{R}(V)$, on $\mathcal{U}(V, f')$.

Corollary 7.2 (of proof). *π has the property that for all affine open sets $U \subset X$, $\pi^{-1}(U)$ is affine.*

In fact, we can formulate the situation as follows:

PROPOSITION-DEFINITION 7.3. *Let $f\colon Y \to X$ be a morphism of schemes. Then the following are equivalent:*

i) *for all affine open $U \subset X$, $f^{-1}(U)$ is affine,*
ii) *there is an affine open covering $\{U_\alpha\}$ of X such that $f^{-1}(U_\alpha)$ is affine,*
iii) *there is a quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{R}$ such that*

$$Y \cong \mathcal{S}pec_X(\mathcal{R}).$$

Such an f is called an affine morphism.

PROOF. (i) $\Longrightarrow$ (ii) is obvious.
(iii) $\Longrightarrow$ (i) has just been proven.
(ii) $\Longrightarrow$ (iii): let $\mathcal{R} = f_*\mathcal{O}_Y$. Note that if $V_\alpha = f^{-1}(U_\alpha)$ and f_α is the restriction of f to

$$f_\alpha\colon V_\alpha \longrightarrow U_\alpha,$$

then $f_*\mathcal{O}_{V_\alpha}$ is quasi-coherent by (5.10). But $\mathcal{R}|_{U_\alpha} = f_*\mathcal{O}_{V_\alpha}$, so $\mathcal{R}$ is quasi-coherent. Now compare Y and $\mathcal{S}pec_X\,\mathcal{R}$. Using the isomorphism

$$f_*\mathcal{O}_Y = \mathcal{R} = \pi_*(\mathcal{O}_{\mathcal{S}pec_X\,\mathcal{R}})$$

the universal mapping property for $\mathcal{S}pec_X\,\mathcal{R}$ gives us a morphism ϕ

$$\begin{tikzcd} Y \arrow[rr, "\phi"] \arrow[dr, "f"'] & & \mathcal{S}pec_X\,\mathcal{R} \arrow[dl, "\pi"] \\ & X. & \end{tikzcd}$$

But $f^{-1}(U_\alpha)$ is affine, so

$$\begin{aligned} f^{-1}(U_\alpha) &\cong \mathcal{S}pec_{U_\alpha}(f_*\mathcal{O}_Y|_{U_\alpha}) \\ &\cong \mathcal{S}pec_{U_\alpha}(\mathcal{R}|_{U_\alpha}) \\ &\cong \pi^{-1}(U_\alpha) \end{aligned}$$

hence ϕ is an isomorphism. □

8. Defining schemes as functors $\boxed{\text{Added}}$

(Added in publication)

To illustrate the power of Grothendieck's idea (cf. FGA [**2**]) referred to in §6, we show examples of schemes defined as functors.

For any category $\mathcal{C}$ we defined in §6 a fully faithful functor

$$h : \mathcal{C} \longrightarrow \mathrm{Funct}(\mathcal{C}^\circ, (\mathrm{Sets})).$$

Here is a result slightly more general than Proposition 6.1:

PROPOSITION 8.1 (Yoneda's lemma). *For any $X \in \mathcal{C}$ and any $F \in \mathrm{Funct}(\mathcal{C}^\circ, (\mathrm{Sets}))$, we have a natural bijection*

$$F(X) \xrightarrow{\sim} \mathrm{Mor}_{\mathrm{Funct}}(h_X, F).$$

The proof is again easy, and can be found in EGA [**1**, Chapter 0 revised, Proposition (1.1.4)]. From this we easily get the following:

Proposition-Definition 8.2. *$F \in \mathrm{Funct}(\mathcal{C}^\circ, (\mathrm{Sets}))$ is said to be* representable *if it is isomorphic to h_X for some $X \in \mathcal{C}$. This is the case if and only if there exists $X \in \mathcal{C}$ and $u \in F(X)$, called the* universal element*, such that*

$$\mathrm{Mor}(Z, X) \ni \varphi \longmapsto F(\varphi)(u) \in F(Z)$$

is a bijection for all $Z \in \mathcal{C}$. The pair (X, u) is determined by F up to unique isomorphism.

Let us now fix a scheme S and restrict ourselves to the case

$$\mathcal{C} = (\mathrm{Sch}/S) = \text{the category of schemes over } S \text{ and } S\text{-morphisms.}$$

For schemes X and Y over S, denote by $\mathrm{Hom}_S(X, Y)$ the set of S-morphisms. (cf. Definition 3.6.)

A representable $F \in \mathrm{Funct}((\mathrm{Sch}/S), (\mathrm{Sets}))$ thus defines a scheme over S.

Suppose F is represented by X. Then for any open covering $\{U_i\}_{i\in I}$ of Z, the sequence

$$F(Z) \longrightarrow \prod_{i\in I} F(U_i) \overset{\longrightarrow}{\longrightarrow} \prod_{i,j\in I} F(U_i \cap U_j)$$

is an exact sequence of sets, that is, for any $(f_i)_{i\in I} \in \prod_{i\in I} F(U_i)$ such that the images of f_i and f_j in $F(U_i \cap U_j)$ coincide for all $i, j \in I$, there exists a unique $f \in F(Z)$ whose image in $F(U_i)$ coincides with f_i for all $i \in I$. This is because a morphism $f \in F(Z) = \mathrm{Hom}_S(Z, X)$ is obtained uniquely by glueing morphisms $f_i \in F(U_i) = \mathrm{Hom}_S(U_i, X)$ satisfying the compatibility condition $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for all $i, j \in I$. Another way of looking at this condition is that F is a *sheaf of sets* (cf. Definition 3 in the Appendix below).

Actually, a representable functor satisfies a stronger necessary condition: it is a sheaf of sets in the "faithfully flat quasi-compact topology". (See §IV.2 for related topics. See also FAG [**3**].)

Example 8.3. Let X and Y be schemes over S. The functor

$$\begin{aligned} F(Z) &= \mathrm{Hom}_S(Z, X) \times \mathrm{Hom}_S(Z, Y) \\ &= \{(q_1, q_2) \mid q_1 \colon Z \to X,\ q_s \colon Z \to Y \text{ are } S\text{-morphisms}\}, \end{aligned}$$

with obvious maps $F(f) \colon F(Z) \to F(Z')$ for S-morphisms $f \colon Z' \to Z$, is represented by the fibre product $X \times_S Y$ by Theorem 4.2. The universal element is $(p_1, p_2) \in F(X \times_S Y)$, where $p_1 \colon X \times_S Y \to X$ and $p_2 \colon X \times_S Y \to Y$ are projections.

Example 8.4. The functor

$$\begin{aligned} F(Z) &= \Gamma(Z, \mathcal{O}_Z), \quad \text{for } Z \in (\mathrm{Sch}/S) \\ F(f) &= f^* \colon \Gamma(Z, \mathcal{O}_Z) \to \Gamma(Z', \mathcal{O}_{Z'}), \quad \text{for } f \in \mathrm{Hom}_S(Z', Z) \end{aligned}$$

is represented by the relatively affine S-scheme $\mathcal{S}pec_S(\mathcal{O}_S[T])$ by Theorem-Definition 7.1, where $\mathcal{O}_S[T]$ is the polynomial algebra over $\mathcal{O}_S$ in one variable T. The universal element is $T \in \Gamma(S, \mathcal{O}_S[T])$. This S-scheme is a commutative group scheme over S in the sense defined in §VI.1.

More generally, we have (cf. EGA [**1**, Chapter I, revised, Proposition (9.4.9)]):

Example 8.5. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module on S. Then the relatively affine S-scheme

$$\mathcal{S}pec_S(\mathrm{Symm}(\mathcal{F})),$$

where $\mathrm{Symm}(\mathcal{F})$ is the symmetric algebra of $\mathcal{F}$ over $\mathcal{O}_S$, represents the functor F defined as follows: For any S-scheme $\varphi \colon Z \to S$, denote by $\varphi^*\mathcal{F} = \mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F}$ the inverse image of $\mathcal{F}$ by the morphism $\varphi \colon Z \to S$ (cf. §5).

$$F(Z) = \mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F}, \mathcal{O}_Z), \quad \text{for } Z \in (\mathrm{Sch}/S)$$

with the obvious map

$$F(f) = f^* : \mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{F}_Z, \mathcal{O}_Z) \to \mathrm{Hom}_{\mathcal{O}_{Z'}}(\mathcal{O}_{Z'} \otimes_{\mathcal{O}_S} \mathcal{F}, \mathcal{O}_{Z'}) = \mathrm{Hom}_{\mathcal{O}_{Z'}}(f^*(\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F}), f^*\mathcal{O}_Z)$$

for $f \in \mathrm{Hom}_S(Z', Z)$. If we denote by $\pi\colon X = \mathcal{S}pec_S(\mathrm{Symm}(\mathcal{F})) \to S$ the canonical projection, then the universal element is $\pi^*\mathcal{F} \to \mathcal{O}_X$ corresponding to the canonical injection $\mathcal{F} \to \pi_*\mathcal{O}_X = \mathrm{Symm}(\mathcal{F})$. This S-scheme is a commutative group scheme over S in the sense defined in §VI.1.

Similarly to Example 8.4, we have:

Example 8.6. The functor

$$\begin{aligned} F(Z) &= \Gamma(Z, \mathcal{O}_Z)^*, \quad \text{for } Z \in (\mathrm{Sch}/S) \\ F(f) &= f^*\colon \Gamma(Z, \mathcal{O}_Z)^* \to \Gamma(Z, \mathcal{O}_{Z'})^*, \quad \text{for } f \in \mathrm{Hom}_S(Z', Z), \end{aligned}$$

where the asterisk denotes the set of invertible elements, is represented by the relatively affine S-scheme

$$\mathcal{S}pec_S(\mathcal{O}_S[T, T^{-1}]).$$

The universal element is again $T \in \Gamma(S, \mathcal{O}_S[T, T^{-1}])$. This S-scheme is a commutative group scheme over S in the sense defined in §VI.1.

More generally:

Example 8.7. Let n be a positive integer. The relatively affine S-scheme defined by

$$\mathrm{GL}_{n,S} = \mathcal{S}pec_S\left(\mathcal{O}_S\left[T_{11}, \dots, T_{nn}, \frac{1}{\det(T)}\right]\right),$$

where $T = (T_{ij})$ is the $n \times n$-matrix with indeterminates T_{ij} as entries, represents the functor

$$F(Z) = \mathrm{GL}_n(\Gamma(Z, \mathcal{O}_Z)), \qquad \text{for } Z \in (\mathrm{Sch}/S),$$

the set of $n \times n$-matrices with entries in $\Gamma(Z, \mathcal{O}_Z)$, with obvious maps corresponding to S-morphisms. This S-scheme is a group scheme over S in the sense defined in §VI.1.

Even more generally, we have (cf. EGA [**1**, Chapter I, revised, Proposition (9.6.4)]):

Example 8.8. Let $\mathcal{E}$ be a locally free $\mathcal{O}_S$-module of finite rank (cf. Definition 5.3). The functor F defined by

$$F(Z) = \mathrm{Aut}_{\mathcal{O}_Z}(\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{E}) \quad \text{for } Z \in (\mathrm{Sch}/S)$$

with obvious maps corresponding to S-morphisms is represented by a relatively affine S-scheme $\mathrm{GL}(\mathcal{E})$. (cf. EGA [**1**, Chapter I, revised, Proposition (9.6.4)].) This S-scheme is a group scheme over S in the sense defined in §VI.1. Example 8.7 is a special case with

$$\mathrm{GL}_{n,S} = \mathrm{GL}(\mathcal{O}_S^{\oplus n}).$$

Example 8.9. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module, and r a positive integer. For each S-scheme Z exact sequences of $\mathcal{O}_Z$-modules

$$\begin{aligned} &\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0 \\ &\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F} \longrightarrow \mathcal{E}' \longrightarrow 0, \end{aligned}$$

where $\mathcal{E}$ and $\mathcal{E}'$ are locally free $\mathcal{O}_Z$-modules of rank r, are said to be *equivalent* if there exists an $\mathcal{O}_Z$-isomorphism $\alpha\colon \mathcal{E} \xrightarrow{\sim} \mathcal{E}'$ so that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{O}_Z \otimes \mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d, "\alpha"'] & 0 \\ \mathcal{O}_Z \otimes \mathcal{F} \arrow[r] & \mathcal{E}' \arrow[r] & 0. \end{tikzcd}$$

For each S-scheme Z, let

$$F(Z) = \{\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F} \to \mathcal{E} \to 0 \mid \text{ exact with } \mathcal{E} \text{ locally free } \mathcal{O}_Z\text{-module of rank } r\}/\sim$$

($\sim$ denotes the set of equivalence classes). For each S-morphism $f\colon Z' \to Z$ and an exact sequence $\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F} \to \mathcal{E} \to 0$, the inverse image by f

$$\mathcal{O}_{Z'} \otimes_{\mathcal{O}_S} \mathcal{F} = f^*(\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F}) \longrightarrow f^*\mathcal{E} \longrightarrow 0$$

defines an element of $F(Z')$, since the inverse image preserves surjective homomorphisms and local freeness. Thus we have a functor $F\colon (\mathrm{Sch}/S)^\circ \to (\mathrm{Sets})$. This functor turns out to be representable. The proof can be found in EGA [**1**, Chapter I, revised, Proposition (9.7.4)]. The S-scheme representing it is denoted by $\pi\colon \mathrm{Grass}^r(\mathcal{F}) \to S$ and is called the *Grassmannian* scheme over S. The universal element is given by an exact sequence

$$\pi^*\mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0$$

with a locally free $\mathcal{O}_{\mathrm{Grass}^r(\mathcal{F})}$-module $\mathcal{Q}$ of rank r called the *universal quotient*.

Locally free $\mathcal{O}_S$-modules of rank one are called *invertible* $\mathcal{O}_S$-modules. (cf. Definition III.1.1.) As a special case for $r = 1$ we have the following:

Example 8.10. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. The functor

$$F(Z) = \{\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F} \to \mathcal{L} \to 0 \mid \text{ exact with } \mathcal{L} \text{ invertible } \mathcal{O}_Z\text{-module}\}/\sim$$

with the map $F(f)\colon F(Z) \to F(Z')$ defined by the inverse image by each $f\colon Z' \to Z$ is represented by an S-scheme

$$\pi\colon \mathbb{P}(\mathcal{F}) = \mathcal{P}roj_S(\mathrm{Symm}(\mathcal{F})) \longrightarrow S$$

with the universal element given by the universal quotient invertible sheaf

$$\pi^*\mathcal{F} \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) \longrightarrow 0.$$

(cf. Definition II.5.7, Theorem III.2.8.)

When $S = \mathrm{Spec}(k)$ with k an algebraically closed field, $\mathrm{Grass}^r(k^{\oplus n})$ is (the set of k-rational points of) the Grassmann variety parametrizing the r-dimenensional quotient spaces of $k^{\oplus n}$, hence parametrizing $(n-r)$-dimensional subspaces of $k^{\oplus n}$ that are the kernels of the quotient maps. In particular $\mathbb{P}(k^{\oplus n})$ is (the set of k-rational points of) the $(n-1)$-dimensional projective space parametrizing the *one-dimensional quotient spaces* of $k^{\oplus n}$. To have a functor in the general setting, however, the subspace approach does not work, since tensor product is not left exact.

S-morphisms between representable functors can be defined as morphisms of functors by Proposition 6.1. Here are examples:

Example 8.11. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. Then the *Plücker S-morphism*

$$\mathrm{Grass}^r(\mathcal{F}) \longrightarrow \mathbb{P}(\bigwedge^r \mathcal{F})$$

is defined in terms of the functors they represent as follows: For any S-Scheme Z and

$$\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0 \quad \text{exact with locally free } \mathcal{O}_Z\text{-module } \mathcal{E} \text{ of rank } r,$$

the r-th exterior product gives rise to an exact sequence

$$\mathcal{O}_Z \otimes_{\mathcal{O}_S} \bigwedge^r \mathcal{F} \longrightarrow \bigwedge^r \mathcal{E} \longrightarrow 0,$$

with $\bigwedge^r \mathcal{E}$ an invertible $\mathcal{O}_Z$-module, hence a morphism $Z \to \mathbb{P}(\bigwedge^r \mathcal{F})$. EGA [**1**, Chapter I, revised, §9.8] shows that the Plücker S-morphism is a closed immersion (cf. Definition II.3.2).

For quasi-coherent $\mathcal{O}_S$-modules $\mathcal{F}$ and $\mathcal{F}'$, the *Segre S-morphism*

$$\mathbb{P}(\mathcal{F}) \times_S \mathbb{P}(\mathcal{F}') \longrightarrow \mathbb{P}(\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{F}')$$

is defined in terms of the functors they represent as follows: For any S-scheme Z and exact sequences

$$\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F} \longrightarrow \mathcal{L} \longrightarrow 0 \quad \text{and} \quad \mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F}' \longrightarrow \mathcal{L}' \longrightarrow 0,$$

with invertible $\mathcal{O}_Z$-modules $\mathcal{L}$ and $\mathcal{L}'$, the tensor product gives rise to an exact sequence

$$\mathcal{O}_Z \otimes_{\mathcal{O}_S} (\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{F}') \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_Z} \mathcal{L}' \longrightarrow 0,$$

with $\mathcal{L} \otimes_{\mathcal{O}_Z} \mathcal{L}'$ an invertible $\mathcal{O}_Z$-module, hence a morphism $Z \to \mathbb{P}(\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{F}')$. The Segre S-morphism also turns out to be a closed immersion (cf. EGA [**1**, Chapter I, revised, §9.8]).

Some of the important properties of schemes and morphisms can be checked in terms of the functors and morphisms of functors representing them: for instance, valuative criterion for properness (cf. Proposition II.6.8) and criterion for smoothness (cf. Criterion V.4.10).

In some cases, the tangent space of a scheme over a field at a point can be defined in terms of the funtor representing it (cf. §V.1).

Example 8.12. The Picard group $\mathrm{Pic}(X)$ of a scheme X is the set of isomorphism classes of invertible $\mathcal{O}_X$-modules forming a commutative group under tensor product (cf. Definition III.1.2). The inverse image by each morphism $f\colon X' \to X$ gives rise to a homomorphism $f^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(X')$. The contravariant functor thus obtained is far from being representable. Here is a better formulation: For each S-scheme X define a functor $\mathrm{Pic}_{X/S}\colon (\mathrm{Sch}/S)^\circ \to (\mathrm{Sets})$ by

$$\mathrm{Pic}_{X/S}(Z) = \mathrm{Coker}[\varphi^*\colon \mathrm{Pic}(Z) \longrightarrow \mathrm{Pic}(X \times_S Z)], \quad \text{for each } S\text{-scheme } \varphi\colon Z \to S.$$

The inverse image by each S-morphism $f\colon Z' \to Z$ gives rise to the map $f^*\colon \mathrm{Pic}_{X/S}(Z) \to \mathrm{Pic}_{X/S}(Z')$. The representability of (modified versions of) the relative Picard functor $\mathrm{Pic}_{X/S}$ has been one of the important issues in algebraic geometry. The reader is referred to FGA [**2**] as well as Kleiman's account on the interesting history (before and after FGA [**2**]) in [**3**, Chapter 9]. When representable, the S-scheme $\mathrm{Pic}_{X/S}$ representing it is called the *relative Picard scheme* of X/S and the universal invertible sheaf on $X \times_S \mathrm{Pic}_{X/S}$ is called the *Poincaré* invertible sheaf. It is a commutative group scheme over S in the sense defined in §VI.1.

Example 8.13. Using the notion of flatness to be defined in Definition IV.2.10 and §IV.4, the *Hilbert functor* for an S-scheme X, is defined by

$$\mathrm{Hilb}_{X/S}(Z) = \{Y \subset X \times_S Z \mid \text{closed subschemes flat over } Z\}$$

with the maps induced by the inverse image by S-morphisms.

Giving a closed subscheme $Y \subset X \times_S Z$ is the same as giving a surjective homomomorphism

$$\mathcal{O}_{X \times_S Z} \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

of $\mathcal{O}_{X \times_S Z}$-modules. Thus the Hilbert functor is a special case of the more general functor defined for a quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$ on an S-scheme X by

$$\mathrm{Quot}_{\mathcal{E}/X/S}(Z) = \{\mathcal{O}_{X \times_S Z} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{F} \to 0 \mid \text{ with } \mathcal{F} \text{ flat over } \mathcal{O}_Z\}/\sim$$

with the maps induced by the inverse image by S-morphisms.

The representability of $\mathrm{Hilb}_{X/S}$ and $\mathrm{Quot}_{\mathcal{E}/X/S}$ has been another major issues. See, for instance, FGA [**2**] and Nitsure's account in FAG [**3**, Chapters 5 and 7].

There are many other important schemes that could be defined as functors such as $\mathrm{Aut}_S(X)$ for an S-scheme, $\mathrm{Hom}_S(X, Y)$ for S-schemes X and Y, moduli spaces, etc. introduced in FGA [**2**].

Appendix: Theory of sheaves

DEFINITION 1. Let X be a topological space. A presheaf $\mathcal{F}$ on X consists in:

a) for all open sets $U \subset X$, a set $\mathcal{F}(U)$,
b) whenever $U \subset V \subset X$, a map
$$\mathrm{res}_{V,U} \colon \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$$
called the restriction map,

such that

c) $\mathrm{res}_{U,U} = \text{identity}$
d) if $U \subset V \subset W$, then $\mathrm{res}_{V,U} \circ \mathrm{res}_{W,V} = \mathrm{res}_{W,U}$.

DEFINITION 2. If $\mathcal{F}$, $\mathcal{G}$ are presheaves on X, a map $\alpha \colon \mathcal{F} \to \mathcal{G}$ is a set of maps
$$\alpha(U) \colon \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$$
one for each open $U \subset X$, such that for all $U \subset V \subset X$,
$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r, "\alpha(V)"] \arrow[d, "\mathrm{res}_{V,U}"'] & \mathcal{G}(V) \arrow[d, "\mathrm{res}_{V,U}"] \\
\mathcal{F}(U) \arrow[r, "\alpha(U)"] & \mathcal{G}(U)
\end{tikzcd}$$
commutes.

DEFINITION 3. A presheaf $\mathcal{F}$ is a sheaf if for all open $V \subset X$ and all open coverings $\{U_\alpha\}_{\alpha \in S}$ of V the two properties hold:

a) if $s_1, s_2 \in \mathcal{F}(V)$ and $\mathrm{res}_{V,U_\alpha}(s_1) = \mathrm{res}_{V,U_\alpha}(s_2)$ in each set $\mathcal{F}(U_\alpha)$, then $s_1 = s_2$.
b) if $s_\alpha \in \mathcal{F}(U_\alpha)$ is a set of elements such that for all $\alpha, \beta \in S$,
$$\mathrm{res}_{U_\alpha, U_\alpha \cap U_\beta}(s_\alpha) = \mathrm{res}_{U_\beta, U_\alpha \cap U_\beta}(s_\beta) \text{ in } \mathcal{F}(U_\alpha \cap U_\beta),$$
then there exists an $s \in \mathcal{F}(V)$ such that $\mathrm{res}_{V,U_\alpha}(s) = s_\alpha$ for all α.

(Thus $\mathcal{F}(V)$ can be reconstructed from the local values $\mathcal{F}(U_\alpha)$, $\mathcal{F}(U_\alpha \cap U_\beta)$ of the sheaf.) If $\mathcal{F}$ is a sheaf, we will sometimes write $\Gamma(U, \mathcal{F})$ for $\mathcal{F}(U)$ and call it the set of sections of $\mathcal{F}$ over U.

DEFINITION 4. If $\mathcal{F}$ is a sheaf on X and $x \in X$, then with respect to the restriction maps, one can form
$$\mathcal{F}_x = \varinjlim_{\substack{\text{all open } U \\ \text{with } x \in U}} \mathcal{F}(U).$$
$\mathcal{F}_x$ is called the stalk of $\mathcal{F}$ at x.

Thus $\mathcal{F}_x$ is the set of germs of sections of $\mathcal{F}$ at x — explicitly, $\mathcal{F}_x$ is the set of all $s \in \Gamma(U, \mathcal{F})$, for all neighborhoods U of x, modulo the equivalence relation:
$$s_1 \sim s_2 \quad \text{if} \quad \mathrm{res}_{U_1, U_1 \cap U_2}(s_1) = \mathrm{res}_{U_2, U_1 \cap U_2}(s_2).$$
The usefulness of stalks is due to the proposition:

PROPOSITION 5.

i) *For all sheaves $\mathcal{F}$ and open sets U, if $s_1, s_2 \in \mathcal{F}(U)$, then $s_1 = s_2$ if and only if the images of s_1, s_2 in $\mathcal{F}_x$ are equal for all $x \in U$.*

ii) *Let $\alpha\colon \mathcal{F} \to \mathcal{G}$ be a map of sheaves. Then $\alpha(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all U (resp. bijective for all U), if and only if the induced map on stalks $\alpha_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$ (resp. bijective for all $x \in X$).*

(Proof left to the reader.)

Definition 6. A sheaf $\mathcal{F}$ is a sheaf of groups, rings, etc., if its values $\mathcal{F}(U)$ are groups, rings, etc., and its restriction maps are homomorphisms.

A typical example of a sheaf is the following: Let X and Y be topological spaces and define, for all open $U \subset X$:

$$\mathcal{F}(U) = \{\text{continuous maps from } U \text{ to } Y\}.$$

If $Y = \mathbb{R}$, $\mathcal{F}$ is a sheaf of rings whose stalks $\mathcal{F}_x$ are the rings of germs of continuous real functions at x.

In our applications to schemes, we encounter the situation where we are given a *basis* $\mathfrak{B} = \{U_\alpha\}$ for the open sets of a topological space X, closed under intersection, and a "sheaf" only on $\mathfrak{B}$, i.e., satisfying the properties in Definition 3 for open sets and coverings of $\mathfrak{B}$ — call this a $\mathfrak{B}$-sheaf. In such a situation, we have the facts:

Proposition 7. *Every $\mathfrak{B}$-sheaf extends canonically to a sheaf on all open sets. If $\mathcal{F}$ and $\mathcal{G}$ are two sheaves, every collection of maps*

$$\phi(U_\alpha)\colon \mathcal{F}(U_\alpha) \to \mathcal{G}(U_\alpha) \quad \textit{for all } U_\alpha \in \mathfrak{B}$$

commuting with restriction extends uniquely to a map $\phi\colon \mathcal{F} \to \mathcal{G}$ of sheaves.

Idea of Proof. Given $\mathcal{F}(U_\alpha)$ for $U_\alpha \in \mathfrak{B}$, define stalks

$$\mathcal{F}_x = \varinjlim_{\substack{U_\alpha \in \mathfrak{B} \\ x \in U_\alpha}} \mathcal{F}(U_\alpha)$$

as before. Then for all open U, set

$$\mathcal{F}(U) = \left\{ (s_x) \in \prod_{x \in U} \mathcal{F}_x \;\middle|\; \begin{array}{l} \exists \text{ a covering } \{U_{\alpha_i}\} \text{ of } U,\ U_{\alpha_i} \in \mathfrak{B}, \\ \text{and } s_i \in \mathcal{F}(U_{\alpha_i}) \text{ such that} \\ s_x = \operatorname{res} s_i \text{ whenever } x \in U_{\alpha_i} \end{array} \right\}.$$

□

If $\mathcal{F}$ is a presheaf, we can define several associated presheaves:

a) $\forall U$, $\forall s_1, s_2 \in \mathcal{F}(U)$, say

$$s_1 \sim s_2 \qquad \begin{array}{l} \text{if } \exists \text{ a covering } \{U_\alpha\} \text{ of } U \text{ such that} \\ \operatorname{res}_{U,U_\alpha}(s_1) = \operatorname{res}_{U,U_\alpha}(s_2), \text{ for all } \alpha. \end{array}$$

This is an equivalence relation, so we may set

$$\mathcal{F}^{(\mathrm{a})}(U) = \mathcal{F}(U)/(\text{the above equivalence relation } \sim).$$

Then $\mathcal{F}^{(\mathrm{a})}$ is a presheaf satisfying (a) for sheaves.

b) $\forall U$, consider sets $\{U_\alpha, s_\alpha\}$ where $\{U_\alpha\}$ is a covering of U and $s_\alpha \in \mathcal{F}^{(\mathrm{a})}(U_\alpha)$ satisfy

$$\operatorname{res}_{U_\alpha, U_\alpha \cap U_\beta}(s_\alpha) = \operatorname{res}_{U_\beta, U_\alpha \cap U_\beta}(s_\beta), \quad \text{all } \alpha, \beta.$$

Say

$$\{U_\alpha, s_\alpha\} \sim \{V_\alpha, t_\alpha\} \quad \text{if } \operatorname{res}_{U_\alpha, U_\alpha \cap V_\beta}(s_\alpha) = \operatorname{res}_{V_\beta, U_\alpha \cap V_\beta}(t_\beta), \text{ all } \alpha, \beta.$$

Let

$$\operatorname{sh}(\mathcal{F})(U) = \left\{ \begin{array}{l} \text{the set of sets } \{U_\alpha, s_\alpha\} \text{ modulo} \\ \text{the above equivalence relation} \end{array} \right\}.$$

Then $\operatorname{sh}(\mathcal{F})$ is in fact a sheaf.

Definition 8. $\operatorname{sh}(\mathcal{F})$ is the sheafification of $\mathcal{F}$.

It is trivial to check that the canonical map

$$\mathcal{F} \longmapsto \operatorname{sh}(\mathcal{F})$$

is universal with respect to maps of $\mathcal{F}$ to sheaves, i.e., $\forall \mathcal{F} \xrightarrow{\alpha} \mathcal{G}$, $\mathcal{G}$ a sheaf, $\exists\, \operatorname{sh}(\mathcal{F}) \xrightarrow{\beta} \mathcal{G}$ such that

$$\begin{tikzcd} & \operatorname{sh}(\mathcal{F}) \arrow[d, "\beta"] \\ \mathcal{F} \arrow[ur] \arrow[r, "\alpha"'] & \mathcal{G} \end{tikzcd}$$

commutes. A useful connection between these concepts is:

Proposition 9. *Let $\mathfrak{B}$ be a basis of open sets and $\mathcal{F}$ a presheaf defined on all open sets, but which is already a sheaf on $\mathfrak{B}$. Then the unique sheaf that extends the restriction to $\mathfrak{B}$ of $\mathcal{F}$ is the sheafification of the full $\mathcal{F}$.*

(Proof left to the reader)

The set of all sheaves of abelian groups on a fixed topological space X forms an *abelian category* (cf. ??? Eilenberg [**36**, p. 254] or Bass [**20**, p. 21]). In fact

a) the set of maps $\operatorname{Hom}(\mathcal{F}, \mathcal{G})$ from one sheaf $\mathcal{F}$ to another $\mathcal{G}$ is clearly an abelian group because we can add two maps; and composition of maps is bilinear.
b) the 0-sheaf, $0(U) = \{0\}$ for all U, is a 0-object (i.e., $\operatorname{Hom}(0, \mathcal{F}) = \operatorname{Hom}(\mathcal{F}, 0) = \{0\}$, for all $\mathcal{F}$),
c) sums exist, i.e., if $\mathcal{F}$, $\mathcal{G}$ are two sheaves, define $(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$. This is a sheaf which is categorically both a sum and a product (i.e., $\operatorname{Hom}(\mathcal{H}, \mathcal{F} \oplus \mathcal{G}) = \operatorname{Hom}(\mathcal{H}, \mathcal{F}) \oplus \operatorname{Hom}(\mathcal{H}, \mathcal{G})$ and $\operatorname{Hom}(\mathcal{F} \oplus \mathcal{G}, \mathcal{H}) = \operatorname{Hom}(\mathcal{F}, \mathcal{H}) \oplus \operatorname{Hom}(\mathcal{G}, \mathcal{H})$).

(This means we have an *additive* category.)

d) Kernels exist: if $\alpha \colon \mathcal{F} \to \mathcal{G}$ is any homomorphism, define

$$\operatorname{Ker}(\alpha)(U) = \{s \in \mathcal{F}(U) \mid \alpha(s) = 0 \text{ in } \mathcal{G}(U)\}.$$

Then one checks immediately that $\operatorname{Ker}(\alpha)$ is a sheaf and is a categorical kernel, i.e.,

$$\operatorname{Hom}(\mathcal{H}, \operatorname{Ker}(\alpha)) = \{\beta \in \operatorname{Hom}(\mathcal{H}, \mathcal{F}) \mid \alpha \circ \beta = 0\}.$$

e) Cokernels exist: if $\alpha \colon \mathcal{F} \to \mathcal{G}$ is any homomorphism, look first at the *presheaf*:

$$\text{Pre-Coker}(\alpha)(U) = \text{quotient of } \mathcal{G}(U) \text{ by } \alpha(\mathcal{F}(U)).$$

This is not usually a sheaf, but set

$$\operatorname{Coker}(\alpha) = \text{sheafification of Pre-Coker}(\alpha).$$

One checks that this is a categorical cokernel, i.e.,

$$\operatorname{Hom}(\operatorname{Coker}(\alpha), \mathcal{H}) = \{\beta \in \operatorname{Hom}(\mathcal{G}, \mathcal{H}) \mid \beta \circ \alpha = 0\}.$$

f) Finally, the main axiom: given $\alpha\colon \mathcal{F} \to \mathcal{G}$, then

$$\operatorname{Ker}(\mathcal{G} \to \operatorname{Coker}\alpha) \cong \operatorname{Coker}(\operatorname{Ker}\alpha \to \mathcal{F}).$$

PROOF. By definition

$$\begin{aligned}
&\operatorname{Coker}(\operatorname{Ker}\alpha \to \mathcal{F})\\
&\quad= \text{sheafification of } \{U \mapsto \mathcal{F}(U)/\operatorname{Ker}(\alpha)(U)\}\\
&\quad= \text{sheafification of } \{U \mapsto \text{Image of } \mathcal{F}(U) \text{ in } \mathcal{G}(U)\}.
\end{aligned}$$

Since the presheaf $U \mapsto \alpha\mathcal{F}(U)$ satisfies the first condition for a sheaf, and is contained in a sheaf $\mathcal{G}$, its sheafification is simply described as:

$$\begin{aligned}
&\operatorname{Coker}(\operatorname{Ker}\alpha \to \mathcal{F})(U)\\
&\quad= \left\{ s \in \mathcal{G}(U) \;\middle|\; \begin{array}{l} \exists \text{ a covering } \{U_\alpha\} \text{ of } U \\ \text{such that } \operatorname{res}_{U,U_\alpha}(s) \in \alpha\mathcal{F}(U) \end{array} \right\}.
\end{aligned}$$

But

$$\begin{aligned}
&\operatorname{Ker}(\mathcal{G} \to \operatorname{Coker}\alpha)(U)\\
&\quad= \{s \in \mathcal{G}(U) \mid s \mapsto 0 \text{ in } \operatorname{Coker}(\alpha)(U)\}\\
&\quad= \left\{ s \in \mathcal{G}(U) \;\middle|\; \begin{array}{l} \text{image of } s \text{ in the presheaf} \\ U \mapsto \mathcal{G}(U)/\alpha\mathcal{F}(U) \text{ is killed by} \\ \text{process (a) of sheafification} \end{array} \right\}\\
&\quad= \left\{ s \in \mathcal{G}(U) \;\middle|\; \begin{array}{l} \exists \text{ a covering } \{U_\alpha\} \text{ of } U \text{ such} \\ \text{that } s \mapsto 0 \text{ in } \mathcal{G}(U_\alpha)/\alpha\mathcal{F}(U_\alpha) \end{array} \right\}\\
&\quad= \left\{ s \in \mathcal{G}(U) \;\middle|\; \begin{array}{l} \exists \text{ a covering } \{U_\alpha\} \text{ of } U \text{ such} \\ \text{that } \operatorname{res}_{U,U_\alpha}(s) \in \alpha\mathcal{F}(U_\alpha) \end{array} \right\}.
\end{aligned}$$

□

The essential twist in the theory of abelian sheaves is that if

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is an exact sequence, then:

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U) \quad \text{is exact}$$

but

$$\mathcal{G}(U) \longrightarrow \mathcal{H}(U) \quad \text{is } \textit{not in general surjective}.$$

In fact, to test the surjectivity of a sheaf homomorphism $\alpha\colon \mathcal{G} \to \mathcal{H}$, one must see whether the presheaf $U \longmapsto \mathcal{H}(U)/\mathcal{G}(U)$ dies when it is sheafified, i.e.,

$$[\alpha\colon \mathcal{G} \to \mathcal{H} \text{ surjective}] \iff \left[\begin{array}{l} \forall s \in \mathcal{H}(U),\ \exists \text{ covering} \\ \{U_\alpha\} \text{ of } U \text{ such that} \\ \operatorname{res}_{U,U_\alpha}(s) \in \text{Image of } \mathcal{G}(U_\alpha) \end{array} \right].$$

As one easily checks this is equivalent to the induced map on stalks $\mathcal{G}_x \to \mathcal{H}_x$ being surjective for all $x \in X$.

The category of abelian sheaves also has *infinite* sums and products but one must be a little careful: if $\{\mathcal{F}_\alpha\}_{\alpha \in S}$ is any set of sheaves, then

$$U \longmapsto \prod_{\alpha \in S} \mathcal{F}_\alpha(U)$$

is again a sheaf, and it is categorically the product of the $\mathcal{F}_\alpha$'s but

$$U \longmapsto \sum_{\alpha \in S} \mathcal{F}_\alpha(U)$$

need not be a sheaf. It has property (a) but not always property (b), so we must define the sheaf $\sum \mathcal{F}_\alpha$ to be its sheafification, i.e.,

$$\sum_{\alpha \in S} \mathcal{F}_\alpha(U) = \left\{ s \in \prod_{\alpha \in S} \mathcal{F}_\alpha(U) \;\middle|\; \begin{array}{l} \exists \text{ a covering } \{U_\beta\} \text{ of } U \text{ such that} \\ \text{for all } \beta,\ \mathrm{res}_{U,U_\beta}(s) \text{ has only a} \\ \text{finite number of non-zero components} \end{array} \right\}.$$

This $\sum_{\alpha \in S} \mathcal{F}_\alpha$ is a categorical sum. But note that if U is quasi-compact, i.e., all open coverings have finite subcoverings, then clearly

$$\sum_{\alpha \in S} \mathcal{F}_\alpha(U) = \sum_{\alpha \in S} \left(\mathcal{F}_\alpha(U) \right).$$

There are several more basic constructions that we will use:

a) given $\mathcal{F}$, $\mathcal{G}$ abelian sheaves on X, we get a new abelian sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ by

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \{\text{homomorphisms over } U \text{ from } \mathcal{F}|_U \text{ to } \mathcal{G}|_U\}.$$

b) given a continuous map $f\colon X \to Y$ of topological spaces and a sheaf $\mathcal{F}$ on X, we get a sheaf $f_*\mathcal{F}$ on Y by

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)).$$

It is trivial to check that both of these are indeed sheaves.

CHAPTER II

Exploring the world of schemes

1. Classical varieties as schemes

Having now defined the category of schemes, we would like to see how the principal objects of classical geometry—complex projective varieties—fit into the picture. In fact a variety is essentially a very special kind of scheme and a regular correspondence between two varieties is a morphism. I would like first to show very carefully how a variety is made into a scheme, and secondly to analyze step by step what special properties these schemes have and how we can characterize varieties among all schemes.

I want to change notation slightly to bring it in line with that of the last chapter and write $\mathbb{P}^n(\mathbb{C})$ for complex projective n-space, the set of non-zero $(n+1)$-tuples $(a_0, \ldots, a_n)$ of complex numbers modulo $(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n)$ for $\lambda \in \mathbb{C}^*$. Let

$$X(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$$

be a complex projective variety, i.e., the set of zeroes of the homogeneous equations $f \in \mathfrak{p}$, $\mathfrak{p} \subset \mathbb{C}[X_0, \ldots, X_n]$ being a homogeneous prime ideal. Next for every irreducible subvariety:

$$W(\mathbb{C}) \subset X(\mathbb{C}), \qquad \dim W(\mathbb{C}) \geq 1$$

let η_W be a new point. Define X to be the union of $X(\mathbb{C})$ and the set of these new points $\{\ldots, \eta_W, \ldots\}$. This will be the underlying point set of a scheme with $X(\mathbb{C})$ as its closed points and the η_W's as the non-closed points. Extend the topology from $X(\mathbb{C})$ to X as follows:

$$\begin{aligned}&\text{for all Zariski open } U(\mathbb{C}) \subset X(\mathbb{C}),\\ &\text{let } U = U(\mathbb{C}) \cup \{\eta_W \mid W(\mathbb{C}) \cap U(\mathbb{C}) \neq \emptyset\}.\end{aligned}$$

One sees easily that the map $U(\mathbb{C}) \mapsto U$ preserves arbitrary unions and finite intersections, hence it defines a topology on X. Moreover, in this topology:

a) $\forall x \in X(\mathbb{C})$, $x \in \overline{\{\eta_W\}} \Longleftrightarrow x \in W(\mathbb{C})$
b) $\forall V(\mathbb{C}) \subset X(\mathbb{C})$, $\eta_V \in \overline{\{\eta_W\}} \Longleftrightarrow V(\mathbb{C}) \subset W(\mathbb{C})$,

hence $\overline{\{\eta_W\}}$ is just W, i.e., η_W is a generic point of W. You can picture $\mathbb{P}^2$ for instance, something like that in Figure II.1:

To put a sheaf on X, we can proceed in two ways:

Method (1). Recall that we have defined in Part I [**76**, Chapter 2], a function field $\mathbb{C}(X)$ and for every $x \in X(\mathbb{C})$, a local ring $\mathcal{O}_{x,X}$ with quotient field $\mathbb{C}(X)$. Now for every open set $U \subset X$, define

$$\mathcal{O}_X(U) = \bigcap_{x \in U(\mathbb{C})} \mathcal{O}_{x,X}$$

and whenever $U_1 \subset U_2$, note that $\mathcal{O}_X(U_2)$ is a subring of $\mathcal{O}_X(U_1)$: let

$$\mathrm{res}_{U_2,U_1} \colon \mathcal{O}_X(U_2) \longrightarrow \mathcal{O}_X(U_1)$$

be the inclusion map. In this way we obviously get a sheaf; in fact a subsheaf of the constant sheaf with value $\mathbb{C}(X)$ on every U.

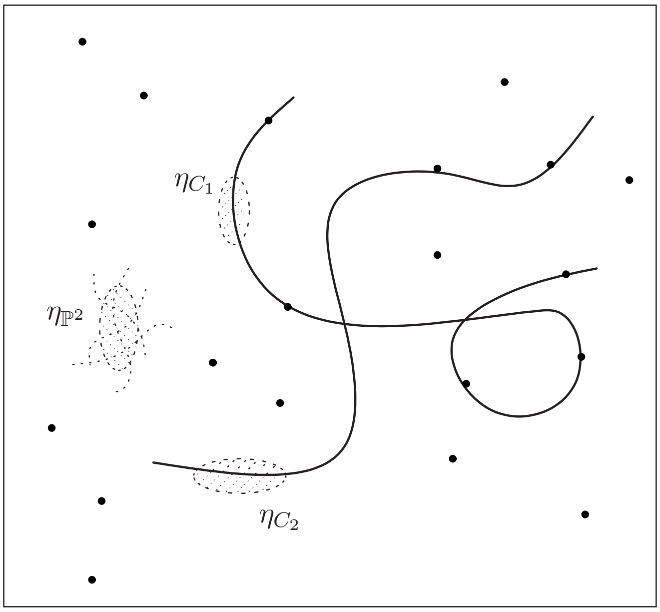

FIGURE II.1. $\mathbb{P}^2$

METHOD (2). Instead of working inside $\mathbb{C}(X)$, we can instead work inside the sheaf of functions from the closed points of X to $\mathbb{C}$:

$$\mathbb{C}^X(U) = \{\text{set of functions } f\colon U(\mathbb{C}) \longrightarrow \mathbb{C}\}$$

restriction now being just restriction of functions. Then define

$$\mathcal{O}_X(U) = \left\{ \begin{array}{l} \text{subset of } \mathbb{C}^X(U) \text{ of functions } f \text{ such that for every} \\ x \in U(\mathbb{C}), \text{ there is a neighborhood } U_x \text{ of } x \text{ in } U \text{ and a} \\ \text{rational function } a(x_0,\ldots,x_n)/b(x_0,\ldots,x_n),\ a \text{ and } b \\ \text{homogeneous of the same degree, such that} \\ f(y_0,\ldots,y_n) = \dfrac{a(y_0,\ldots,y_n)}{b(y_0,\ldots,y_n)}, \quad b(y_0,\ldots,y_n) \neq 0 \\ \text{for every } y \in U_x \end{array} \right\}.$$

This is clearly a subsheaf of $\mathbb{C}^X$. To see that we have found the same sheaf twice, call these two sheaves $\mathcal{O}_X^{\mathrm{I}}$, $\mathcal{O}_X^{\mathrm{II}}$ for a minute and observe that we have maps:

$$\mathcal{O}_X^{\mathrm{I}}(U) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathcal{O}_X^{\mathrm{II}}(U)$$

$$\alpha(f) = \left\{ \begin{array}{l} \text{the function } x \mapsto f(x) \text{ (OK since } f(x) \text{ is defined)} \\ \text{whenever } f \in \mathcal{O}_{x,X} \end{array} \right\}$$

$$\beta(f) = \left\{ \begin{array}{l} \text{the element of } \mathbb{C}(X) \text{ represented by any of the} \\ \text{rational functions } a(x_0,\ldots,x_n)/b(x_0,\ldots,x_n) \\ \text{which equal } f \text{ in a Zariski open subset of } U. \\ \text{(OK since if } a/b \text{ and } c/d \text{ have the same values in a} \\ \text{non-empty Zariski-open } U \cap V, \text{ then } ad - bc \equiv 0 \\ \text{on } X, \text{ hence } a/b = c/d \text{ in } \mathbb{C}(X).) \end{array} \right\}$$

From now on, we identify these two sheaves and consider the structure sheaf $\mathcal{O}_X$ either as a subsheaf of the constant sheaf $\mathbb{C}(X)$ or of $\mathbb{C}^X$, whichever is appropriate. The main point now is that $(X, \mathcal{O}_X)$ is indeed a scheme. To see this it is easiest first to note that we can make all the above definitions starting with a complex affine variety $Y(\mathbb{C}) \subset \mathbb{C}^n$ instead of with a projective variety. And moreover, just as $X(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$ is covered by affine varieties

$$Y_i(\mathbb{C}) = X(\mathbb{C}) \setminus X(\mathbb{C}) \cap V(X_i)$$

so too the pair $(X, \mathcal{O}_X)$ is locally isomorphic at every point to $(Y_i, \mathcal{O}_{Y_i})$ for some i. Therefore it is enough to show that $(Y_i, \mathcal{O}_{Y_i})$ is a scheme. But if the affine $Y(\mathbb{C})$ equals $V(\mathfrak{p})$, $\mathfrak{p} \subset \mathbb{C}[X_1, \ldots, X_n]$ a prime ideal, then I claim:

$$(Y, \mathcal{O}_Y) \cong (\operatorname{Spec} \mathbb{C}[X_1, \ldots, X_n]/\mathfrak{p}, \mathcal{O}_{\operatorname{Spec} \mathbb{C}[X]/\mathfrak{p}}). \tag{1.1}$$

PROOF. The prime ideals $\overline{\mathfrak{q}} \subset \mathbb{C}[X_1, \ldots, X_n]/\mathfrak{p}$ are in one-to-one correspondence with the prime ideals $\mathfrak{q}$:

$$\mathfrak{p} \subset \mathfrak{q} \subset \mathbb{C}[X_1, \ldots, X_n],$$

and these are in one-to-one correspondence with the set of irreducible closed subsets of $V(\mathfrak{p})$, i.e., to the points of $X(\mathbb{C})$ plus the positive dimensional subvarieties of $X(\mathbb{C})$. Therefore there is a canonical bijection:

$$Y \cong \operatorname{Spec} \mathbb{C}[X_1, \ldots, X_n]/\mathfrak{p}$$

via

$$\begin{aligned} \eta_{V(\mathfrak{q})} &\longleftrightarrow [\overline{\mathfrak{q}}] \text{ for } \overline{\mathfrak{q}} \text{ not maximal} \\ Y(\mathbb{C}) \ni a &\longleftrightarrow [(X_1 - a_1, \ldots, X_n - a_n) \mod \mathfrak{p}] \end{aligned}$$

[Recall that the maximal ideals of $\mathbb{C}[X_1, \ldots, X_n]/\mathfrak{p}$ are the ideals $I(a)$ of all functions vanishing at a point $a \in X(\mathbb{C})$, i.e., the ideals $(X_1 - a_1, \ldots, X_n - a_n)/\mathfrak{p}$.] It is seen immediately that this bijection is a homeomorphism. To identify the sheaves, note that for all $f \in \mathbb{C}[X_1, \ldots, X_n]$,

$$\begin{aligned} \mathcal{O}_Y(Y_f) &\underset{\text{def}}{=} \bigcap_{a \in Y_f(\mathbb{C})} \mathcal{O}_{a,Y} \\ &= \bigcap_{\substack{a \in Y(\mathbb{C}) \\ f(a) \neq 0}} (\text{localization of } \ \mathbb{C}[X_1, \ldots, X_n]/\mathfrak{p} \ \text{ at } \ \overbrace{(X_1 - a_1, \ldots, X_n - a_n)}^{I(a)}) \end{aligned}$$

while

$$\mathcal{O}_{\operatorname{Spec} \mathbb{C}[X]/\mathfrak{p}}(Y_f) \underset{\text{def}}{=} \text{localization } (\mathbb{C}[X_1, \ldots, X_n]/\mathfrak{p})_f.$$

These are both subrings of $\mathbb{C}(Y)$, the quotient field of $\mathbb{C}[X_1, \ldots, X_n]/\mathfrak{p}$. Now since $f(a) \neq 0 \Longrightarrow f \in (\mathbb{C}[X]/\mathfrak{p}) \setminus I(a)$, we see that

$$(\mathbb{C}[X]/\mathfrak{p})_f \subset \bigcap_{\substack{a \in Y(\mathbb{C}) \\ f(a) \neq 0}} (\mathbb{C}[X]/\mathfrak{p})_{I(a)}.$$

And if

$$g \in \bigcap_{\substack{a \in Y(\mathbb{C}) \\ f(a) \neq 0}} (\mathbb{C}[X]/\mathfrak{p})_{I(a)},$$

let

$$\mathfrak{a} = \{h \in \mathbb{C}[X]/\mathfrak{p} \mid gh \in \mathbb{C}[X]/\mathfrak{p}\}.$$

If $f(a) \neq 0$, then $\exists g_a, h_a \in \mathbb{C}[X]/\mathfrak{p}$ and $h_a \notin I(a)$ such that $g = g_a/h_a$, hence $h_a \in \mathfrak{a}$. Thus $a \notin V(\mathfrak{a})$. Since this holds for all $a \in Y(\mathbb{C})_f$, we see that $V(\mathfrak{a}) \subset V(f)$, hence by the Nullstellensatz (cf. Part I [**76**, §1A, (1.5)], Zariski-Samuel [**109**, vol. II, Chapter VII, §3, Theorem 14] and Bourbaki [**26**, Chapter V, §3.3, Proposition 2]) $f^N \in \mathfrak{a}$ for some $N \geq 1$. This means precisely that $g \in (\mathbb{C}[X]/\mathfrak{p})_f$. Thus the sheaves are the same too. $\square$

To simplify terminology, we will now call the scheme X attached to $X(\mathbb{C})$ a complex projective variety too. Next, if

$$X(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$$
$$Y(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$$

are two complex projective varieties and if

$$Z(\mathbb{C}) \subset X(\mathbb{C}) \times Y(\mathbb{C})$$

is a regular correspondence from X to Y, we get a canonical morphism

$$f_Z : X \longrightarrow Y.$$

In fact, as a map of sets, define the following.

$$\text{If } x \in X(\mathbb{C})\text{: } f_Z(x) = \text{the unique } y \in Y(\mathbb{C}) \text{ such that } (x, y) \in Z(\mathbb{C})$$

$$\text{If } W(\mathbb{C}) \subset X(\mathbb{C})\text{: } f_Z(\eta_W) = \begin{cases} \eta_V & \text{if } \dim V \geq 1 \\ v & \text{if } V(\mathbb{C}) = \{v\} \end{cases}$$
$$\text{where} \quad V(\mathbb{C}) = p_2\left[(W(\mathbb{C}) \times Y(\mathbb{C})) \cap Z(\mathbb{C})\right].$$

One checks immediately that this map is continuous. To define the map backwards on sheaves, proceed in either of two ways:

METHOD (1). Recall that Z defined a map $Z^* : \mathbb{C}(Y) \to \mathbb{C}(X)$ and the fact that Z is regular implies that for all $x \in X(\mathbb{C})$, if $y = f_Z(x)$, then

$$Z^*(\mathcal{O}_{y,Y}) \subset \mathcal{O}_{x,X}.$$

Therefore, for every open set $U \subset Y$,

$$\begin{aligned} Z^*(\mathcal{O}_Y(U)) &= Z^*\left(\bigcap_{y \in U(\mathbb{C})} \mathcal{O}_{y,Y}\right) \\ &\subset \bigcap_{x \in f_Z^{-1}U(\mathbb{C})} \mathcal{O}_{x,X} \\ &= \mathcal{O}_X(f_Z^{-1}U) \end{aligned}$$

giving a map of sheaves.

METHOD (2). Define a map

$$f_Z^* : \mathbb{C}^Y(U) \longrightarrow \mathbb{C}^X(f_Z^{-1}U)$$

by composition with f_Z, i.e., if $\alpha : U(\mathbb{C}) \to \mathbb{C}$ is a function, then $\alpha \circ f_Z$ is a function $f_Z^{-1}U(\mathbb{C}) \to \mathbb{C}$. One checks immediately using the regularity of Z that f_Z^* maps functions α in the subring $\mathcal{O}_Y(U)$ to functions $\alpha \circ f_Z \in \mathcal{O}_X(f_Z^{-1}(U))$.

There is one final point in this direction which we will just sketch. That is:

PROPOSITION 1.2. *Let $X(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$ and $Y(\mathbb{C}) \subset \mathbb{P}^m(\mathbb{C})$ be complex projective varieties. Let $Z(\mathbb{C}) \subset \mathbb{P}^{nm+n+m}(\mathbb{C})$ be their set-theoretic product, embedded by the Segre embedding as third complex projective variety (cf. Part I* [**76**, Chapter 2]*). Then the scheme Z is canonically isomorphic to the fibre product $X \times_{\mathrm{Spec}(\mathbb{C})} Y$ of the schemes X and Y.*

Idea of proof. Let $X_0, \ldots, X_n, Y_0, \ldots, Y_m$ and Z_{ij} $(0 \le i \le n, 0 \le j \le m)$ be homogeneous coordinates in $\mathbb{P}^n(\mathbb{C})$, $\mathbb{P}^m(\mathbb{C})$ and $\mathbb{P}^{nm+n+m}(\mathbb{C})$. Then by definition $Z(\mathbb{C})$ is covered by affine pieces $Z_{i_0j_0} \neq 0$ which are set-theoretically the product of the affine $X_{i_0} \neq 0$ in $X(\mathbb{C})$ and $Y_{j_0} \neq 0$ in $Y(\mathbb{C})$. The Segre embedding is given in this piece by

$$\frac{Z_{ij}}{Z_{i_0j_0}} = \frac{X_i}{X_{i_0}} \cdot \frac{Y_j}{Y_{j_0}}$$

so the affine ring of Z comes out:

$$\begin{aligned}\mathbb{C}[\ldots, \frac{Z_{ij}}{Z_{i_0j_0}}, \ldots]/\{\text{functions } 0 \text{ on } Z(\mathbb{C})\} \\ = \mathbb{C}[\ldots, \frac{X_i}{X_{i_0}}, \ldots, \frac{Y_j}{Y_{j_0}}, \ldots]/\{\text{functions } 0 \text{ on } X(\mathbb{C}) \times Y(\mathbb{C})\}.\end{aligned}$$

To see that this is the tensor product of the affine rings of X and Y:

$$\Big(\mathbb{C}[\ldots, \frac{X_i}{X_{i_0}}, \ldots]/\{\text{functions } 0 \text{ on } X(\mathbb{C})\}\Big) \otimes_{\mathbb{C}} \Big(\mathbb{C}[\ldots, \frac{Y_j}{Y_{j_0}}, \ldots]/\{\text{functions } 0 \text{ on } Y(\mathbb{C})\}\Big)$$

one uses the ordinary Nullstellensatz (cf. Part I [**76**, §1A, (1.5)], Zariski-Samuel [**109**, vol. II, Chapter VII, §3, Theorem 14] and Bourbaki [**26**, Chapter V, §3.3, Proposition 2]) plus:

Lemma 1.3. *If R and S are k-algebras with no nilpotents, k a perfect field, then $R \otimes_k S$ has no nilpotent elements.*

(cf. §IV.2 below.) □

Corollary 1.4. *Let $X(\mathbb{C})$, $Y(\mathbb{C})$ be complex projective varieties. Then the set of regular correspondences from $X(\mathbb{C})$ to $Y(\mathbb{C})$ and the set of $\mathbb{C}$-morphisms from the scheme X to the scheme Y are the same.*

Idea of proof. Starting from $f\colon X \to Y$, we get a morphism

$$f \times 1_Y : X \times_{\mathrm{Spec}(\mathbb{C})} Y \longrightarrow Y \times_{\mathrm{Spec}(\mathbb{C})} Y.$$

If $\Delta(\mathbb{C}) \subset Y(\mathbb{C}) \times Y(\mathbb{C})$ is the diagonal, which is easily checked to be closed, define $\Gamma = (f \times 1_Y)^{-1}(\Delta)$, then $\Gamma(\mathbb{C})$ is closed in $X(\mathbb{C}) \times Y(\mathbb{C})$ and is the graph of $\mathrm{res}(f)$. Therefore $\Gamma(\mathbb{C})$ is a single-valued correspondence and a local computation shows that it is regular. □

2. The properties: reduced, irreducible and finite type

The goal of this section is to analyze some of the properties that make classical varieties special in the category of schemes. We shall do two things:

a) Define for general schemes, and analyze the first consequences of three basic properties of classical varieties: being *irreducible*, *reduced*, and *of finite type over a field k*. A scheme with these properties will be defined to be a *variety* over k.
b) Show that for reduced schemes X of finite type over any algebraically closed field k, the structure sheaf $\mathcal{O}_X$ can be considered as a sheaf of k-valued functions and a morphism is determined by its map of points. Thus varieties over algebraically closed k's form a truly geometric category which is quite parallel to differentiable manifolds/analytic spaces/classical varieties.

Property 1. *A complex projective variety X is irreducible, or equivalently has a generic point η_X.*

This is obvious from the definition. To put this property in its setting, we can prove that every scheme has a unique irredundant decomposition into irreducible components. In fact:

DEFINITION 2.1. A scheme X is locally *noetherian* if every $x \in X$ has an affine neighborhood U which is $\mathrm{Spec}(R)$, R noetherian. A scheme is noetherian if it is locally noetherian and quasi-compact; or equivalently, if it has a finite covering by Spec's of noetherian rings.

PROPOSITION 2.2. *Every scheme X has a unique decomposition*

$$X = \bigcup_\alpha Z_\alpha, \quad Z_\alpha \text{ irreducible closed, } Z_\alpha \not\subset Z_\beta \quad \text{if } \alpha \neq \beta.$$

If X is locally noetherian, this decomposition is locally finite. If X is noetherian, then the decomposition is finite.

PROOF. The general case is immediate, and the noetherian cases from the fact that in a noetherian ring R, $\sqrt{(0)}$ is a finite intersection of prime ideals. □

An important point concerning the definition of locally noetherian is:

PROPOSITION 2.3. *If X is locally noetherian, then for every affine open* $\mathrm{Spec}(R) \subset X$, *$R$ is noetherian.*

Without this proposition, "locally noetherian" would be an awkward artificial concept. This proposition is the archetype of a large class of propositions that "justify" a definition by showing that if some property is checked for a covering family of open affines, then it holds for all open affines.

PROOF OF PROPOSITION 2.3. Let $U_\alpha = \mathrm{Spec}(R_\alpha)$ be an open cover of X with R_α noetherian. Then $\mathrm{Spec}(R)$ is covered by distinguished open subsets of the U_α, and each of these is of the form $\mathrm{Spec}((R_\alpha)_{f_\alpha})$, i.e., Spec of another noetherian ring. But now when $f \in R$ is such that:

$$\mathrm{Spec}(R_f) \subset \mathrm{Spec}(R_\alpha)_{f_\alpha}$$

then

$$\mathrm{Spec}(R_f) \cong \mathrm{Spec}\left(((R_\alpha)_{f_\alpha})_{\mathrm{res}\, f}\right), \quad \text{via res}\colon R \to (R_\alpha)_{f_\alpha},$$

hence

$$R_f \cong ((R_\alpha)_{f_\alpha})_{\mathrm{res}\, f}$$

hence R_f is noetherian. Therefore we can cover $\mathrm{Spec}(R)$ by distinguished opens $\mathrm{Spec}(R_{f_i})$ with R_{f_i} noetherian. Since $\mathrm{Spec}(R)$ is quasi-compact, we can take this covering finite. This implies that if $\mathfrak{a}_\alpha$ is an ascending chain of ideals in R, $\mathfrak{a}_\alpha \cdot R_{f_i}$ is stationary for all i if α is large enough, and then

$$\mathfrak{a}_{\alpha+1} = \bigcap_{i=1}^n \mathfrak{a}_{\alpha+1} R_{f_i} = \bigcap_{i=1}^n \mathfrak{a}_\alpha R_{f_i} = \mathfrak{a}_\alpha.$$

□

PROPERTY 2. *A complex projective variety X is reduced, in the sense of:*

DEFINITION 2.4. A scheme X is *reduced* if all its local rings $\mathcal{O}_{x,X}$ have no non-zero nilpotent elements.

It is easy to check that a ring R has non-zero nilpotents if and only if at least one of its localizations $R_\mathfrak{p}$ has nilpotents: therefore a scheme X is reduced if and only if it has an affine covering U_α such that $\mathcal{O}_X(U_\alpha)$ has no non-zero nilpotents, or if and only if this holds for all affine $U \subset X$. Moreover, it is obvious that a complex projective variety is reduced.

Reduced and irreducible schemes in general begin to look a lot like classical varieties. In fact:

PROPOSITION 2.5. *Let X be a reduced and irreducible scheme with generic point η. Then the stalk $\mathcal{O}_{\eta,X}$ is a field which we will denote $\mathbb{R}(X)$, the function field of X. Then*

i) *for all affine open $U \subset X$, (resp. all points $x \in X$), $\mathcal{O}_X(U)$ (resp. $\mathcal{O}_{x,X}$) is an integral domain with quotient field $\mathbb{R}(X)$,*
ii) *for all open $U \subset X$,*
$$\mathcal{O}_X(U) = \bigcap_{x \in X} \mathcal{O}_{x,X}$$
(the intersection being taken inside $\mathbb{R}(X)$) and if $U_1 \subset U_2$, then $\mathrm{res}_{U_2,U_1} \colon \mathcal{O}_X(U_2) \to \mathcal{O}_X(U_1)$ is the inclusion map between subrings of $\mathbb{R}(X)$.

PROOF. If $U = \operatorname{Spec} R$ is an affine open of X and $\eta = [\mathfrak{p}]$, $\mathfrak{p}$ a prime ideal of R, then $\overline{\{\eta\}} \supset U$ implies that $\mathfrak{p}$ is contained in all prime ideals of R, hence $\mathfrak{p} = \sqrt{(0)}$ in R. But R has no nilpotents so $\mathfrak{p} = (0)$, i.e., R is an integral domain. Moreover $\mathcal{O}_{\eta,X} = \mathcal{O}_{[\mathfrak{p}],\operatorname{Spec} R} = R_{\mathfrak{p}} =$ quotient field of R. Thus $\mathcal{O}_{\eta,X} \underset{\text{def}}{=} \mathbb{R}(X)$ is a field and is the common quotient field both of the affine rings R of X and of all localizations R_S of these such as the local rings $R_{\mathfrak{q}} = \mathcal{O}_{[\mathfrak{q}],X}$ ($\mathfrak{q} \subset R$ any prime ideal). This proves (i). Now if $U \subset X$ is any open set, consider
$$\mathrm{res} \colon \mathcal{O}_X(U) \longrightarrow \mathcal{O}_{\eta,X} = \mathbb{R}(X).$$
For all $s \in \mathcal{O}_X(U)$, $s \neq 0$, there is an affine $U' = \operatorname{Spec} R' \subset U$ such that $\mathrm{res}_{U,U'}(s)$ is not 0 in R'. Since $R' \subset \mathbb{R}(X)$, $\mathrm{res}(s) \in \mathbb{R}(X)$ is not 0. Thus res is injective. Since it factors through $\mathcal{O}_{x,X}$ for all $x \in U$, this shows that
$$\mathcal{O}_X(U) \subset \bigcap_{x \in U} \mathcal{O}_{x,X}.$$
Conversely, if $s \in \bigcap_{x \in X} \mathcal{O}_{x,X}$, then there is an open covering $\{U_\alpha\}$ of U and $s_\alpha \in \mathcal{O}_X(U_\alpha)$ mapping to s in $\mathbb{R}(X)$. Then $s_\alpha - s_\beta \in \mathcal{O}_X(U_\alpha \cap U_\beta)$ goes to 0 in $\mathbb{R}(X)$, so it is 0. Since $\mathcal{O}_X$ is a sheaf, then s_α's patch together to an $s \in \mathcal{O}_X(U)$. This proves (ii). □

PROPERTY 3. *A complex projective variety X is a scheme of finite type over $\mathbb{C}$, meaning:*

DEFINITION 2.6. A morphism $f \colon X \to Y$ is locally of finite type (resp. locally finitely presented) if X has an affine covering $\{U_\alpha\}$ such that $f(U_\alpha) \subset V_\alpha$, V_α an affine of Y, and the ring $\mathcal{O}_X(U_\alpha)$ is isomorphic to $\mathcal{O}_Y(V_\alpha)[t_1, \ldots, t_n]/\mathfrak{a}$ (resp. same with finitely generated $\mathfrak{a}$). f is *quasi-compact* if there exists an affine covering $\{V_\alpha\}$ of Y such that each $f^{-1}(V_\alpha)$ has a finite affine covering; f is of finite type (resp. finitely presented) if it is locally of finite type (resp. locally finitely presented) and quasi-compact.

It is clear that the canonical morphism of a complex projective variety to $\operatorname{Spec}(\mathbb{C})$ has all these properties. As above with the concept of noetherian, these definitions should be "justified" by checking:

PROPOSITION 2.7. *If f is locally of finite type, then for every pair of affine opens $U \subset X$, $V \subset Y$ such that $f(U) \subset V$, $\mathcal{O}_X(U)$ is a finitely generated $\mathcal{O}_Y(V)$-algebra; if f is quasi-compact, then for every quasi-compact open subset $S \subset Y$, $f^{-1}(S)$ is quasi-compact. (Analogous results hold for the concept "locally finitely presented".)*

PROOF. The proof of the first assertion parallels that of Proposition 2.3. We are given U_α's, V_α's with $\mathcal{O}_X(U_\alpha)$ finitely generated over $\mathcal{O}_Y(V_\alpha)$. Using the fact that $R_f \cong R[x]/(1 - xf)$, hence is finitely generated over R, we can replace U_α, V_α by distinguished opens to get new

U_β's, V_β's such that $\mathcal{O}_X(U_\beta)$ is still finitely generated over $\mathcal{O}_Y(V_\beta)$, but now $U_\beta \subset U$, $V_\beta \subset V$ and $U = \bigcup U_\beta$. Next make another reduction until U_γ (resp. V_γ) is a distinguished open in U (resp. V). Then since $\mathcal{O}_X(U_\gamma)$ is finitely generated over $\mathcal{O}_Y(V_\gamma)$ and $\mathcal{O}_Y(V_\gamma) \cong \mathcal{O}_Y(V)_{f_\gamma}$ is finitely generated over $\mathcal{O}_Y(V)$, we may replace V_α by V. We come down to the purely algebraic lemma:

$$\left.\begin{array}{l} S \textit{ is an } R\textit{-algebra} \\ 1 = \sum_{i=1}^{n} f_i g_i, \quad f_i, g_i \in S \\ S_{f_i} \textit{ finitely generated over } R \end{array}\right\} \Longrightarrow S \textit{ finitely generated over } R.$$

PROOF. Take a finite set of elements x_λ of S including the f_i's, g_i's and elements whose images in S_{f_i} plus $1/f_i$ generate S_{f_i} over R. These generate S, because if $k \in S$, then

$$k = \frac{P_i(x_\lambda)}{f_i^N} \quad \text{in} \quad S_{f_i}$$
$$P_i = \text{polynomial over } R.$$

Thus $f_i^{N+M} k = f_i^M P_i(x_\lambda)$ in S. But

$$\begin{aligned} 1 &= \left(\sum f_i g_i\right)^{n(N+M)} \\ &= \sum_{i=1}^{n} Q_i(f,g) \cdot f_i^{N+M} \end{aligned}$$

hence

$$k = \sum_{i=1}^{n} Q_i(f,g) f_i^N P_i(x_\lambda).$$

□

We leave the proof of the second half of Proposition 2.7 to the reader. □

A morphism of finite type has good topological properties generalizing those we found in Part I [**76**, (2.31)]. To state these, we must first define:

DEFINITION 2.8. If X is a scheme, a constructible subset $S \subset X$ is an element of the Boolean algebra of subsets generated by the open sets: in other words,

$$S = S_1 \cup \cdots \cup S_t$$

where S_i is locally closed, meaning it is an intersection of an open set and a closed subset.

THEOREM 2.9 (Chevalley's Nullstellensatz). *Let $f\colon X \to Y$ be a morphism of finite type and Y a noetherian scheme. Then for every constructible $S \subset X$, $f(S) \subset Y$ is constructible.*

PROOF. First of all, we can reduce the theorem to the special case where X and Y are affine: in fact there are finite affine covering $\{U_i\}$ of X and $\{V_i\}$ of Y such that $f(U_i) \subset V_i$. Let $f_i = \operatorname{res} f\colon U_i \to V_i$. Then for every $S \subset X$ constructible, $f(S) = \bigcup f_i(S \cap U_i)$ so if $f_i(S \cap U_i)$ is constructible, so is $f(S)$. Secondly if $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$, we can reduce the theorem to the case $R = S[x]$. In fact, if $R = S[x_1, \ldots, x_n]$, we can factor f:

$$\begin{aligned} X = \operatorname{Spec} S[x_1, \ldots, x_n] &\to \operatorname{Spec} S[x_1, \ldots, x_{n-1}] \to \cdots \\ \cdots &\to \operatorname{Spec} S[x_1] \to \operatorname{Spec} S = Y. \end{aligned}$$

Now a basic fact is that every closed subset $V(\mathfrak{a})$ of an affine scheme $\operatorname{Spec}(R)$ is homeomorphic to the affine scheme $\operatorname{Spec}(R/\mathfrak{a})$. In fact there is a bijection between the set of prime ideals $\overline{\mathfrak{q}} \subset R/\mathfrak{a}$ and the set of prime ideals $\mathfrak{q} \subset R$ such that $\mathfrak{q} \supset \mathfrak{a}$ and this is readily seen to be a homeomorphism (we will generalize this in §3). Also, since $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$, $V(\mathfrak{a})$ is homeomorphic to the reduced scheme $\operatorname{Spec}(R/\sqrt{\mathfrak{a}})$ too. We use this first to make a third reduction to the case

$$f\colon \operatorname{Spec} S[X] \longrightarrow \operatorname{Spec} S.$$

In fact, if R is generated over S by one element, then $R \cong S[X]/\mathfrak{a}$ and via the diagram:

$$\begin{tikzcd} \operatorname{Spec} R \cong V(\mathfrak{a}) \arrow[dr, "f'"'] & \subset & S[X] \arrow[dl, "f"] \\ & \operatorname{Spec} S & \end{tikzcd}$$

the theorem for f implies the theorem for f'. Fourthly, we make a so-called "noetherian induction": since the closed subsets $V(\mathfrak{a}) \subset \operatorname{Spec} S$ satisfy the descending chain condition, if the theorem is false, there will be a minimal $V(\mathfrak{a}) \subset \operatorname{Spec}(S)$ such that

$$\operatorname{res} f\colon f^{-1}(V(\mathfrak{a})) \longrightarrow V(\mathfrak{a})$$

does not take constructibles to constructibles. Since $f^{-1}(V(\mathfrak{a})) = V(\mathfrak{a} \cdot S[X])$, we can replace $\operatorname{Spec} S$ by $\operatorname{Spec} S/\mathfrak{a}$ and $\operatorname{Spec} S[X]$ by $\operatorname{Spec}(S/\mathfrak{a})[X]$ and reduce to the case:

(∗) for all constructible sets $C \subset \operatorname{Spec} S[X]$, if $\overline{f(C)} \subsetneqq \operatorname{Spec}(S)$, then $f(C)$ is constructible.

Of course we can assume in this reduction that $\mathfrak{a} = \sqrt{\mathfrak{a}}$, so that the new S has no nilpotents. $\operatorname{Spec} S$ in fact must be irreducible too: if not,

$$\operatorname{Spec} S = Z_1 \cup Z_2, \qquad Z_i \subsetneqq \operatorname{Spec} S, \quad Z_i \text{ closed.}$$

Then if $C \subset \operatorname{Spec} S[X]$ is constructible, so are $C \cap f^{-1}(Z_i)$, hence by (∗) so are $f(C \cap f^{-1}(Z_i)) = f(C) \cap Z_i$; hence $f(C) = (f(C) \cap Z_1) \cup (f(C) \cap Z_2)$ is constructible. Thus S is an integral domain. In view of (∗), it is clear that the whole theorem is finally reduced to:

Lemma 2.10. *Let S be an integral domain and let $\eta \in \operatorname{Spec} S$ be its generic point. Let $C \subset \operatorname{Spec} S[X]$ be an irreducible closed set and $C_0 \subset C$ an open subset. Consider the morphism:*

$$f\colon \operatorname{Spec} S[X] \longrightarrow \operatorname{Spec} S.$$

Then there is an open set $U \subset \operatorname{Spec} S$ such that either $U \subset f(C_0)$ or $U \cap f(C_0) = \emptyset$.

Proof of Lemma 2.10. Let K be the quotient field of S. Note that $f^{-1}(\eta) \cong \operatorname{Spec} K[X] = \mathbb{A}^1_K$, which consists only of a generic point η^* and its closed points. $C \cap f^{-1}(\eta)$ is a closed irreducible subset of $f^{-1}(\eta)$, hence there are three possibilities:

Case i) $C \supset f^{-1}(\eta)$, so $C = \operatorname{Spec} S[X]$,
Case ii) $C \cap f^{-1}(\eta) = \{\zeta\}$, ζ a closed point of $f^{-1}(\eta)$, and
Case iii) $C \cap f^{-1}(\eta) = \emptyset$.

In case (i), C_0 contains some distinguished open $\operatorname{Spec} S[X]_g$, where $g = a_0X^n + a_1X^{n-1} + \cdots + a_n$, $a_0 \neq 0$. Let $U = \operatorname{Spec} S_{a_0}$. For all $x \in \operatorname{Spec} S$, $f^{-1}(x) \cong \operatorname{Spec} \Bbbk(x)[X] = \mathbb{A}^1_{\Bbbk(x)}$ and:

$$C_0 \cap f^{-1}(x) \supset \left\{ y \in \mathbb{A}^1_{\Bbbk(x)} \;\middle|\; \begin{array}{l} \overline{g}(y) \neq 0, \text{ where } \overline{g} = \overline{a}_0X^n + \cdots + \overline{a}_n \\ \text{and } \overline{a}_i = \text{image of } a_i \text{ in } \Bbbk(x) \end{array} \right\}.$$

So if $x \in U$, $\overline{a}_0 \neq 0$, hence $\overline{g} \not\equiv 0$, hence the generic point of $f^{-1}(x)$ is in $C_0 \cap f^{-1}(x)$, hence $x \in f(C_0)$. In case (ii), let $C = V(\mathfrak{p})$. Then

$$\mathfrak{p} \cdot K[X] = g \cdot K[X], \qquad g \text{ irreducible.}$$

We may assume that $g = a_0 X^n + \cdots + a_n$ is in $\mathfrak{p}$, hence $a_i \in S$. Then

$$V(g) \supset C \supset C_0,$$

but all three sets intersect the generic fibre $f^{-1}(\eta)$ in only one point ζ. Thus $V(g) \setminus C_0$ is a constructible set disjoint from $f^{-1}(\eta)$. Let:

$$\overline{V(g) \setminus C_0} = W_1 \cup \cdots \cup W_t, \quad W_i \text{ irreducible with generic points } w_i \notin f^{-1}(\eta).$$

Then $f(W_i) \subset \overline{\{f(w_i)\}}$ and $\overline{\{f(w_i)\}}$ is a closed proper subset of $\operatorname{Spec} S$. Thus

$$f(V(q) \setminus C_0) \subset \bigcup_{i=1}^{t} \overline{\{f(w_i)\}} \subset \text{some subset } V(\alpha) \text{ of } \operatorname{Spec} S$$

$(\alpha \in S,\ \alpha \neq 0)$. Now let $U = \operatorname{Spec} S_{a_0\alpha}$. Then if $x \in U$,

$$\begin{aligned} \alpha(x) \neq 0 \Longrightarrow f^{-1}(x) \cap C_0 &= f^{-1}(x) \cap V(g) \\ &= \left\{ y \in \mathbb{A}^1_{\Bbbk(x)} \;\middle|\; \overline{g}(y) = 0 \right\}. \end{aligned}$$

Since $a_0(x) \neq 0$, $\overline{g} \not\equiv 0$, hence $\overline{g}$ has an irreducible factor $\overline{g}_1$ and the prime ideal $\overline{g}_1 \cdot \Bbbk(x)[X]$ defines a point of $f^{-1}(x)$ where g is zero. Thus $x \in f(C_0)$, which proves $U \subset f(C_0)$. In case (iii), let ζ be the generic point of C. Then

$$f(C) \subset \overline{\{f(\zeta)\}}$$

hence $U = \operatorname{Spec} S \setminus \overline{\{f(\zeta)\}}$ is an open set disjoint from $f(C_0)$. □

□

Corollary 2.11. *Let k be a field and X a scheme of finite type over k. If $x \in X$ then*

$$[x \textit{ is closed}] \Longleftrightarrow \begin{bmatrix} x \textit{ is an algebraic point} \\ \textit{i.e., } \Bbbk(x) \textit{ is an algebraic extension of } k. \end{bmatrix}$$

Proof. First assume x closed and let $U = \operatorname{Spec} R$ be an affine neighborhood of x. Then x is closed in U and hence $\{x\}$ is a constructible subset of U. Let $R \cong k[X_1, \ldots, X_n]/\mathfrak{a}$. Each X_i defines a morphism $p_i \colon U \to \mathbb{A}^1_k$ by Theorem I.3.7 p_i is clearly of finite type so by Theorem 2.9 $p_i(x)$ is a constructible point of $\mathbb{A}^1_k$. Now apply:

Lemma 2.12 (Euclid). *For any field k, $\mathbb{A}^1_k$ contains an infinite number of closed points.*

Proof. $\mathbb{A}^1_k = \operatorname{Spec} k[X]$ and its closed points are of the form $[(f)]$, f monic and irreducible. If $f_1, \ldots, f_N$ is any finite set of such irreducible polynomials, then an irreducible factor g of $\prod_{i=1}^N f_i + 1$ cannot divide any of the f_i, hence $[(g)] \neq [(f_i)]$ for any i. □

It follows that the generic point of $\mathbb{A}^1_k$ is not a constructible set! Thus $k(p_i(x))$ is algebraic over k. Since the residue field $\Bbbk(x)$ is generated over k by the values of the coordinates X_i, i.e., by the subfields $k(p_i(x))$, $\Bbbk(x)$ is algebraic over k. Conversely, if x is algebraic but not closed, let $y \in \overline{\{x\}}$, $y \neq x$. Let $U = \operatorname{Spec} R$ be an affine neighborhood of y. Then $x \in U$ too, so x is not closed in U. Let $x = [\mathfrak{p}]$ and use the fact that if ξ is algebraic over k, then $k[\xi]$ is already a field. Since $\Bbbk(x) \supset R/\mathfrak{p} \supset k$, all elements of $R/\mathfrak{p}$ are algebraic over k, hence $R/\mathfrak{p}$ is already a field. Therefore $\mathfrak{p}$ is maximal and x must be closed in U — contradiction. □

Corollary 2.13. *Let k be a field and X a scheme of finite type over k. Then:*

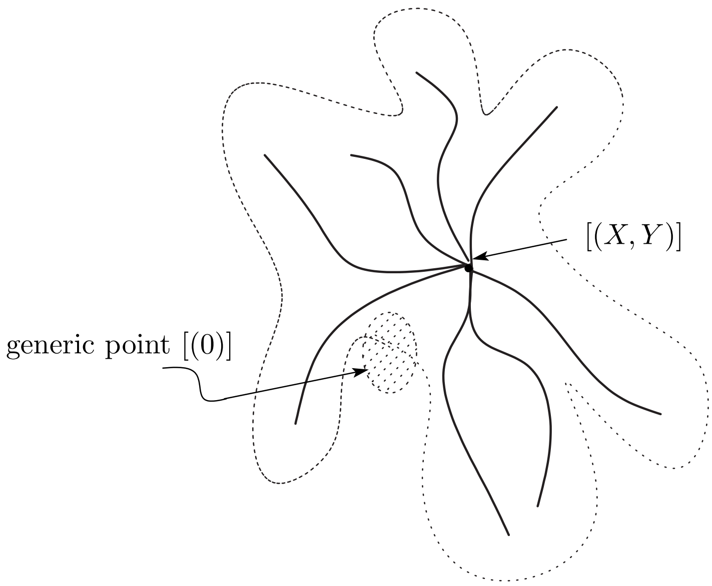

FIGURE II.2. "Parody of $\mathbb{P}^1_k$"

a) *If* $U \subset X$ *is open, and* $x \in U$*, then* x *is closed in* U *if and only if* x *is closed in* X*.*
b) *For all closed subsets* $S \subset X$*, the closed points of* S *are dense in* S*.*
c) *If* $\mathrm{Max}(X)$ *is the set of closed points of* X *in its induced topology, then there is a natural bijection beween* X *and the set of irreducible closed subsets of* $\mathrm{Max}(X)$ *(i.e.,* X *can be reconstructed from* $\mathrm{Max}(X)$ *as schemes were from classical varieties).*

PROOF. (a) is obvious by Corollary 2.11. To prove (b), we show that for every affine open $U \subset X$, if $U \cap S \neq \emptyset$, then $U \cap S$ contains a point closed in X. But if $U = \operatorname{Spec} R$, and $U \cap S = V(\mathfrak{p})$, then in the ring R, let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{p}$. Then $[\mathfrak{m}]$ is a closed point of U in $U \cap S$. By (a), $[\mathfrak{m}]$ is closed in X. Finally (c) is a formal consequence of (b) which we leave to the reader. $\square$

To illustrate what might go wrong here, contrast the situation with the case

$$X = \operatorname{Spec}(\mathcal{O}), \quad \mathcal{O} \text{ local noetherian, maximal ideal } \mathfrak{m}.$$

If

$$U = X \setminus [\mathfrak{m}],$$

then U satisfies the descending chain condition for closed sets so it has lots of closed points. But none of them can be closed in X, since $[\mathfrak{m}]$ is the only closed point of X. Take the case $\mathcal{O} = k[X,Y]_{(X,Y)}$: its prime ideals are $\mathfrak{m} = (X, Y)$, principal prime ideals f, (f irreducible) and (0). In this case, U has only closed points and one generic point and is a kind of parody of $\mathbb{P}^1_k$ as in Figure II.2 (cf. §5 below).

We have now seen that any scheme of finite type over a field shares many properties with classical projective varieties and when it is reduced and irreducible the resemblance is even closer. We canonize this similarity with a very important definition:

DEFINITION 2.14. Let k be a field. A variety X over k is a reduced and irreducible scheme X plus a morphism $p\colon X \to \operatorname{Spec} k$ making it of finite type over k. The dimension of X over k is $\operatorname{tr.deg}_k \mathbb{R}(X)$.

We want to finish this section by showing that when k is algebraically closed, the situation is even more classical.

Proposition 2.15. *Let k be an algebraically closed field and let X be a scheme of finite type over k. Then:*

a) *For all $x \in X$*

$$[x \text{ is closed}] \iff [x \text{ is rational, i.e., } \Bbbk(x) \cong k].$$

Let $X(k)$ denote the set of such points.

b) *Evaluation of functions define a homomorphism of sheaves:*

$$\mathcal{O}_X \longrightarrow k^{X(k)}$$

where

$$k^{X(k)}(U) = \text{ring of } k\text{-valued functions on } U(k).$$

If X is reduced, this is injective.

Now let X and Y be two schemes of finite type over k and $f\colon X \to Y$ a k-morphism. Then:

c) *$f(X(k)) \subset Y(k)$.*

d) *If X is reduced, f is uniquely determined by the induced map $X(k) \to Y(k)$, hence by its graph*

$$\{(x, f(x)) \mid x \in X(k)\} \subset X(k) \times Y(k).$$

Proof. (a) is just Corollary 2.11 in the case k algebraically closed. To check (b), let $U = \operatorname{Spec} R$ be an affine. If $f \in R$ is 0 at all closed points of U, then $U \setminus V(f)$ has no closed points in it, hence is empty. Thus

$$f \in \bigcap_{\mathfrak{p} \text{ prime of } R} \mathfrak{p} = \sqrt{(0)}$$

and if X is reduced, $f = 0$. (c) follows immediately from (a) since for all $x \in X$, we get inclusions of fields:

$$\Bbbk(x) \hookleftarrow \Bbbk(f(x)) \hookleftarrow k.$$

As for (d), it follows immediately from the density of $X(k)$ in X, plus (b). $\square$

3. Closed subschemes and primary decompositions

The deeper properties of complex projective varieties come from the fact that they are closed subschemes of projective space. To make this precise, in the next two sections we will discuss two things—closed subschemes and a construction called Proj. At the same time that we make the definitions necessary for characterizing complex projective varieties we want to study the more general classes of schemes that naturally arise.

Definition 3.1. Let X be a scheme. A *closed subscheme* $(Y, \mathcal{I})$ consists in two things:

a) a closed subset $Y \subset X$

b) a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ such that

$$\mathcal{I}_x \subsetneqq \mathcal{O}_{X,x} \quad \text{iff} \quad x \in Y$$

and such that Y, plus the sheaf of rings $\mathcal{O}_X/\mathcal{I}$ supported by Y is a scheme.

Definition 3.2. Let $f\colon Y \to X$ be a morphism of schemes. Then f is a *closed immersion* if

a) f is an injective closed map,

b) the induced homomorphisms

$$f_y^*\colon \mathcal{O}_{X,f(y)} \longrightarrow \mathcal{O}_{Y,y}$$

are surjective, for every $y \in Y$.

It is clear that

a) if you start from a closed subscheme $(Y, \mathcal{I})$, then the morphism $(Y, \mathcal{O}_X/\mathcal{I}) \to (X, \mathcal{O}_X)$ defined by the inclusion of Y in X and the surjection of $\mathcal{O}_X$ to $\mathcal{O}_X/\mathcal{I}$ is a closed immersion;
b) conversely if you start with a closed immersion $f\colon Y \to X$, then the closed subset $f(Y)$ and the sheaf $\mathcal{I}$:

$$\mathcal{I}(U) = \operatorname{Ker}\big(\mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}U)\big)$$

is a closed subscheme.

Thus these two concepts are essentially equivalent. A *locally closed subscheme* or simply *subscheme* (resp. *immersion*) in general is defined to be a closed subscheme of an open set $U \subset X$ (resp. a morphism f such that $f(Y) \subset U$ open and $\operatorname{res} f\colon Y \to U$ is a closed immersion). The simplest example of a closed immersion is the morphism

$$f\colon \operatorname{Spec}(R/\mathfrak{a}) \longrightarrow \operatorname{Spec}(R)$$

where $\mathfrak{a}$ is any ideal in R. In fact, as noted in the proof of Theorem 2.9 above, f maps $\operatorname{Spec}(R/\mathfrak{a})$ homeomorphically onto the closed subset $V(\mathfrak{a})$ of $\operatorname{Spec}(R)$. And if $\overline{\mathfrak{q}} \subset R/\mathfrak{a}$ is a prime ideal, $\overline{\mathfrak{q}} = \mathfrak{q}/\mathfrak{a}$, then the induced map on local rings is clearly surjective:

$$\begin{array}{ccccc} (R/\mathfrak{a})_{\mathfrak{q}} & \cong & R_{\mathfrak{q}}/\mathfrak{a}\cdot R_{\mathfrak{q}} & \twoheadleftarrow & R_{\mathfrak{q}} \\ \| & & & & \| \\ \mathcal{O}_{\operatorname{Spec}(R/\mathfrak{a}),[\overline{\mathfrak{q}}]} & & & & \mathcal{O}_{\operatorname{Spec}(R),[\mathfrak{q}]} \end{array}$$

We will often say for short, "consider the closed subscheme $\operatorname{Spec}(R/\mathfrak{a})$ of $\operatorname{Spec}(R)$". What we want to check is that these are the only closed subschemes of $\operatorname{Spec} R$.

We prove first:

PROPOSITION 3.3. *If $(Y, \mathcal{I})$ is a closed subscheme of X, then $\mathcal{I}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules.*

PROOF. On the open set $X \setminus Y$, $\mathcal{I} \cong \mathcal{O}_X$ so it is quasi-coherent. If $y \in Y$, we begin by finding an affine neighborhood $U \subset X$ of x such that $U \cap Y$ is affine in Y. To find U, start with any affine neighborhood U_1 and let $V_1 \subset U_1 \cap Y$ be an affine neighborhood of x in Y. Then choose some $\alpha \in \Gamma(U_1, \mathcal{O}_X)$ such that $\alpha \equiv 0$ on $U_1 \cap Y \setminus V_1$, while $\alpha(x) \neq 0$. Let $U = (U_1)_\alpha$. Since $U \cap Y = (U_1 \cap Y)_{\operatorname{res}\alpha} = (V_1)_{\operatorname{res}\alpha}$, $U \cap Y$ is affine in Y too. Next, suppose that $U = \operatorname{Spec} R$, $U \cap Y = \operatorname{Spec} S$ and let the inclusion of $U \cap Y$ into U correspond to $\phi\colon R \to S$. Let $I = \operatorname{Ker}(\phi)$: I claim then that

$$\mathcal{I}|_U \cong \widetilde{I}$$

hence $\mathcal{I}$ is quasi-coherent. But for all $\beta \in \Gamma(U, \mathcal{O}_X)$,

$$\begin{aligned} \widetilde{I}(U_\beta) &= I_\beta \\ &\cong \operatorname{Ker}(R_\beta \to S_\beta) \\ &\cong \operatorname{Ker}\left(\mathcal{O}_X(U_\beta) \to \mathcal{O}_Y(Y \cap U_\beta)\right) \\ &= \mathcal{I}(U_\beta) \end{aligned}$$

hence $\widetilde{I} \cong \mathcal{I}|_U$. $\square$

COROLLARY 3.4. *If* $(Y, \mathcal{I})$ *is a closed subscheme of* X*, then for all affine open* $U \subset X$*,* $U \cap Y$ *is affine in* Y *and if* $U = \operatorname{Spec} R$*, then* $U \cap Y \cong \operatorname{Spec}(R/\mathfrak{a})$ *for some ideal* $\mathfrak{a} \subset R$*, i.e.,* $Y \cong \mathcal{S}pec_X(\mathcal{O}_X/\mathcal{I})$*.*

PROOF. Since $\mathcal{I}$ is quasi-coherent, $\mathcal{I}|_U = \widetilde{\mathfrak{a}}$ for some ideal $\mathfrak{a} \subset R$. But then

$$\begin{aligned} \mathcal{O}_Y|_U &= \operatorname{Coker}\left(\mathcal{I}|_U \to \mathcal{O}_X|_U\right) \\ &= \operatorname{Coker}(\widetilde{\mathfrak{a}} \to \widetilde{R}) \\ &= \widetilde{R/\mathfrak{a}} \end{aligned}$$

hence

$$(Y, \mathcal{O}_Y) = (V(\mathfrak{a}), \widetilde{R/\mathfrak{a}}) \cong (\operatorname{Spec}(R/\mathfrak{a}), \mathcal{O}_{\operatorname{Spec} R/\mathfrak{a}}).$$

□

COROLLARY 3.5. *Let* $f \colon Y \to X$ *be a morphism. Then* f *is a closed immersion if and only if:*

$$(*) \qquad \begin{array}{l} \exists \text{ an affine covering } \{U_i\} \text{ of } X \text{ such that } f^{-1}(U_i) \text{ is affine} \\ \text{and } \Gamma(U_i, \mathcal{O}_X) \to \Gamma(f^{-1}(U_i), \mathcal{O}_Y) \text{ is surjective.} \end{array}$$

PROOF. Immediate. □

We want to give some examples of closed subschemes and particularly of how one can have many closed subschemes attached to the same underlying subset.

EXAMPLE 3.6. *Closed subschemes of* $\operatorname{Spec}(k[t])$, k algebraically closed. Since $k[t]$ is a PID, all non-zero ideals are of the form

$$\mathfrak{a} = \left(\prod_{i=1}^{n} (t - a_i)^{r_i}\right).$$

The corresponding subscheme Y of $\mathbb{A}^1_k = \operatorname{Spec}(k[t])$ is supported by the n points $a_1, \ldots, a_n$, and at a_i its structure sheaf is

$$\mathcal{O}_{a_i,Y} = \mathcal{O}_{a_i,\mathbb{A}^1_k}/\mathfrak{m}_i^{r_i},$$

where $\mathfrak{m}_i = \mathfrak{m}_{a_i,\mathbb{A}^1} = (t - a_i)$. Y is the union of the a_i's "with multiplicity r_i". The real significance of the multiplicity is that if you restrict a function f on $\mathbb{A}^1_k$ to this subscheme, the restriction can tell you not only the value $f(a_i)$ but the first $(r_i - 1)$-derivatives:

$$\frac{d^l f}{dt^l}(a_i), \qquad l \leq r_i - 1.$$

In other words, Y contains the $(r_i - 1)$st-order normal neighborhood of $\{a_i\}$ in $\mathbb{A}^1_k$.

Consider all possible subschemes supported by $\{0\}$. These are the subschemes

$$Y_n = \operatorname{Spec}\left(k[t]/(t^n)\right).$$

Y_1 is just the point as a reduced scheme, but the rest are not reduced. Corresponding to the fact that the defining ideals are included in each other:

$$(t) \supset (t^2) \supset (t^3) \supset \cdots \supset (t^n) \supset \cdots \supset (0),$$

the various schemes are subschemes of each other:

$$Y_1 \subset Y_2 \subset Y_3 \subset \cdots \subset Y_n \subset \cdots \subset \mathbb{A}^1_k.$$

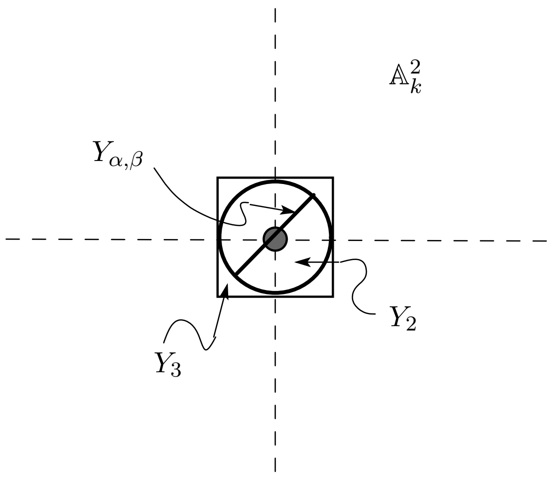

FIGURE II.3. 0-dimensional subschemes of $\mathbb{A}_k^2$

EXAMPLE 3.7. *Closed subschemes of* $\mathrm{Spec}(k[x,y])$, k algebraically closed. Every ideal $\mathfrak{a} \subset k[x,y]$ is of the form:

$$(f) \cap Q$$

for some $f \in k[x,y]$ and Q of finite codimension (to check this use noetherian decomposition and the fact that prime ideals are either maximal or principal). Let $Y = \mathrm{Spec}(k[x,y]/\mathfrak{a})$ be the corresponding subscheme of $\mathbb{A}_k^2$. First, suppose $\mathfrak{a} = (f)$. If $f = \prod_{i=1}^n f_i^{r_i}$, with f_i irreducible, then the subscheme Y is the union of the irreducible curves $f_i = 0$, "with multiplicity r_i". As before, if g is a function on $\mathbb{A}_k^2$, then one can compute solely from the restriction of g to Y the first $r_i - 1$ normal derivatives of g to the curve $f_i = 0$. Second, look at the case $\mathfrak{a}$ of finite codimension. Then

$$\mathfrak{a} = Q_1 \cap \cdots \cap Q_t$$

where $\sqrt{Q_i}$ is the maximal ideal $(x - a_i, y - b_i)$. Therefore, the support of Y is the finite set of points (a_i, b_i), and the stalk of Y at (a_i, b_i) is the finite dimensional algebra $k[x,y]/Q_i$. For simplicity, look at the case $\mathfrak{a} = Q_1$, $\sqrt{Q_1} = (x,y)$. The lattice of such ideals $\mathfrak{a}$ is much more complicated than in the one-dimensional case. Consider, for example, the ideals:

$$(x,y) \supset (\alpha x + \beta y, x^2, xy, y^2) \supset (x^2, xy, y^2) \supset (x^2, y^2) \supset (0).$$

These define subschemes:

$$\{(0,0) \text{ with reduced structure}\} \subset Y_{\alpha,\beta} \subset Y_2 \subset Y_3 \subset \mathbb{A}_k^2.$$

Since $(\alpha x + \beta y, x^2, xy, y^2) \supset (\alpha x + \beta y)$, $Y_{\alpha,\beta}$ is a subscheme of the reduced line $\ell_{\alpha,\beta}$ defined by $\alpha x + \beta y = 0$: $Y_{\alpha,\beta}$ is the point and *one* normal direction. But Y_2 is not a subscheme of any reduced line: it is the full double point and is invariant under rotations. Y_3 is even bigger, is *not* invariant under rotations, but still does not contain the second order neighborhood of $(0,0)$ along any line. If g is a function on $\mathbb{A}_k^2$, $g|_{Y_{\alpha,\beta}}$ determines *one* directional derivative of g at $(0,0)$, $g|_{Y_2}$ determines both partial derivatives of g at $(0,0)$ and $g|_{Y_3}$ even determines the mixed partial $\frac{\partial^2 g}{\partial x \partial y}(0,0)$ (cf. Figure II.3). As an example of the general case, look at $\mathfrak{a} = (x^2, xy)$. Then $\mathfrak{a} = (x) \cap (x^2, xy, y^2)$. Since $\sqrt{\mathfrak{a}} = (x)$, the support of Y is y-axis. The stalk $\mathcal{O}_{z,Y}$ has no nilpotents in it except when $z = (0,0)$. This is an "embedded point", and if a function g on $\mathbb{A}_k^2$ is cut down to Y, the restriction determines both partials of g at $(0,0)$, but only $\frac{\partial}{\partial y}$ at other points (cf. Figure II.4):

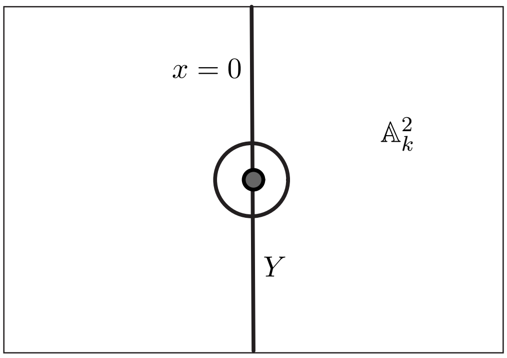

FIGURE II.4. Subschemes of $\mathbb{A}^2_k$

EXAMPLE 3.8. The theory of the primary decomposition of an ideal is an attempt to describe more "geometrically" a general closed subscheme of $\operatorname{Spec} R$, when R is noetherian. In fact, if

$$Z = \operatorname{Spec} R/\mathfrak{a} \subset \operatorname{Spec} R$$

is a closed subscheme, then the theory states that we can write:

$$\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$$

where $\mathfrak{q}_i$ is primary with $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ prime. Then geometrically:

$$\begin{aligned} Z =& \text{scheme-theoretic union of (i.e., smallest closed} \\ & \text{subscheme containing) } W_1, \ldots, W_t \\ \text{where} \quad W_i =& \operatorname{Spec} R/\mathfrak{q}_i \\ =& \text{set-theoretically } V(\mathfrak{p}_i)\text{, the closure of } [\mathfrak{p}_i] \\ & \text{but with some infinitesimal thickening.} \end{aligned}$$

The property which distinguishes the W_i's is described as follows:

$$\begin{aligned} \mathfrak{q} \text{ is } \mathfrak{p}\text{-primary} \iff& \mathfrak{q} = \sqrt{\mathfrak{q}} \text{ and } \mathfrak{q} = R \cap \mathfrak{q} \cdot (R_{\mathfrak{p}}) \\ \iff& \text{set-theoretically } \operatorname{Spec} R/\mathfrak{q} \text{ is } V(\mathfrak{p}) \text{ and the map} \\ & \Gamma(\mathcal{O}_{\operatorname{Spec} R/\mathfrak{q}}) \to \big(\text{the generic stalk } \mathcal{O}_{\operatorname{Spec} R/\mathfrak{q},[\mathfrak{p}]}\big) \\ & \text{is injective.} \end{aligned}$$

(In other words, a "function" $f \in R$ is to have 0 restriction everywhere to $\operatorname{Spec} R/\mathfrak{q}$ if it restricts to 0 at the generic point of $\operatorname{Spec} R/\mathfrak{q}$.) The unfortunate thing about the primary decomposition is that it is not unique: if W_i is an "embedded component", i.e., set-theoretically $W_i \subsetneqq W_j$, then the scheme structure on W_i is not unique. However the subsets W_i are uniquely determined by Z. By far the clearest treatment of this is in Bourbaki [**26**, Chapter 4] who considers the problem module-theoretically rather than ideal-theoretically. His theory globalizes immediately to give:

THEOREM 3.9. *Let X be a noetherian scheme, $\mathcal{F}$ a coherent sheaf on X. Then there is a finite set of points $x_1, \ldots, x_t \in X$ such that*

i) $\forall U \subset X$, $\forall s \in \mathcal{F}(U)$, $\exists I \subset \{1, \ldots, t\}$ *such that:*

$$\operatorname{Supp}(s) \underset{def}{=} \{x \in U \mid \textit{the image } s_x \in \mathcal{F}_x \textit{ is not } 0\} = \bigcup_{i \in I} \overline{\{x_i\}} \cap U$$

ii) *if U is affine, then any subset of U of the form $\bigcup_{i \in I} \overline{\{x_i\}} \cap U$ occurs as the support of some $s \in \mathcal{F}(U)$.*

These x_i are called the associated points of $\mathcal{F}$, or $\mathrm{Ass}(\mathcal{F})$.

PROOF. Note that if $U = \operatorname{Spec} R$, $\mathcal{F}|_U = \widetilde{M}$, $s \in M$, and $\mathrm{Ann}(s) = \{a \in R \mid as = 0\}$, then

$$\begin{aligned} s_x \neq 0 \text{ in } \mathcal{F}_x \quad &\iff s \not\mapsto 0 \quad \text{in } M_{\mathfrak{p}} \\ \text{where } x = [\mathfrak{p}] \quad & \\ &\iff \forall a \in R \setminus \mathfrak{p},\ a \cdot s \neq 0 \\ &\iff \mathrm{Ann}(s) \subset \mathfrak{p} \\ &\iff x \in V(\mathrm{Ann}(s)) \end{aligned}$$

so that $\mathrm{Supp}(s) = V(\mathrm{Ann}(s))$. It follows from the results in Bourbaki [**26**, Chapter 4, §1] that in this case his set of points $\mathrm{Ass}(M) \subset \operatorname{Spec} R$ has our two required properties[1]. Moreover, he proves in [**26**, §1.3] that $\mathrm{Ass}(M_f) = \mathrm{Ass}(M) \cap \operatorname{Spec} R_f$: hence the finite subsets $\mathrm{Ass}(M)$ all come from one set $\mathrm{Ass}(\mathcal{F})$ by $\mathrm{Ass}(M) = \mathrm{Ass}(\mathcal{F}) \cap \operatorname{Spec} R$. □

Note that $\mathrm{Ass}(\mathcal{F})$ must include the generic points of $\mathrm{Supp}(\mathcal{F})$ but may also include in addition embedded associated points.

COROLLARY 3.10. *If $Z \subset \operatorname{Spec} R$ is a closed subscheme and*

$$Z = W_1 \cup \cdots \cup W_t$$

is a primary decomposition, then

$$\mathrm{Ass}(\mathcal{O}_Z) = \{w_1, \ldots, w_t\},$$

where w_i = generic point of W_i.

PROOF. Let $Z = \operatorname{Spec} R/\mathfrak{a}$, $W_i = \operatorname{Spec} R/\mathfrak{q}_i$, so that $\mathfrak{a} = \bigcap \mathfrak{q}_i$. A primary decomposition is assumed irredundant, i.e., $\forall i$,

$$\mathfrak{q}_i \not\supset \bigcap_{j \neq i} \mathfrak{q}_j.$$

This means $\exists f \in \bigcap_{j \neq i} \mathfrak{q}_j \setminus \mathfrak{q}_i$, i.e., the "function" f is identically 0 on the *subschemes* W_j, $j \neq i$, but it is not 0 at the generic point of W_i, i.e., in $\mathcal{O}_{w_i, W_i}$. Therefore as a section of $\mathcal{O}_Z$, $\mathrm{Supp}(f) = W_i$. On the other hand, we get natural maps:

$$R/\mathfrak{a} \hookrightarrow \bigoplus_{i=1}^{t} R/\mathfrak{q}_i \hookrightarrow \bigoplus_{i=1}^{t} R_{\mathfrak{q}_i}/\mathfrak{q}_i R_{\mathfrak{q}_i}$$

hence

$$\mathcal{O}_Z \hookrightarrow \bigoplus_{i=1}^{t} \mathcal{O}_{W_i} \hookrightarrow \bigoplus_{i=1}^{t} (\text{constant sheaf on } W_i \text{ with value } \mathcal{O}_{w_i, W_i})$$

from which it follows readily that the support of any section of $\mathcal{O}_Z$ is a union of various W_i's. □

For instance, in the example $R = k[x, y]$, $\mathfrak{a} = (x^2, xy)$,

$$\begin{aligned} (\text{Supp in } R/\mathfrak{a})(y) &= \text{whole subset } V(\mathfrak{a}) \\ (\text{Supp in } R/\mathfrak{a})(x) &= \text{embedded point} V(x, y). \end{aligned}$$

In order to globalize the theory of primary decompositions, or to analyze the uniqueness properties that it has, the following result is very useful:

[1] In fact, if $s \in M$, then $R/\mathrm{Ann}(s) \hookrightarrow M$ by multiplication by s, hence $\mathrm{Ass}(R/\mathrm{Ann}(s)) \subset \mathrm{Ass}(M)$; if $\mathrm{Supp}(s) = S_1 \cup \cdots \cup S_k$, S_i irreducible and $S_i \not\subset S_j$, then $S_i = V(\mathfrak{p}_i)$, and $\mathfrak{p}_i$ are the minimal primes in $\mathrm{Supp}(R/\mathrm{Ann}(s))$, hence by his [**26**, Chapter 4, §1, Proposition 7], are in $\mathrm{Ass}(R/\mathrm{Ann}(s))$. This gives our assertion (ii). Conversely, for all $\mathfrak{p} \in \mathrm{Ass}(M)$, there is an $s \in M$ with $\mathrm{Ann}(s) = \mathfrak{p}$, hence $\mathrm{Supp}(s) = V(\mathfrak{p})$. Adding these, we get our assertion (ii).

PROPOSITION 3.11. *If X is locally noetherian*[2] *and $Y \subset X$ is a locally closed subscheme, then there is a smallest closed subscheme $\overline{Y} \subset X$ containing Y as an open subscheme, called the scheme-theoretic closure of Y. The ideal sheaf $\mathcal{I}$ defining $\overline{Y}$ is given by:*

$$\mathcal{I}(U) = \operatorname{Ker}[\mathcal{O}_X(U) \longrightarrow \mathcal{O}_Y(Y \cap U)],$$

and the underlying point set of $\overline{Y}$ is the topological closure of Y. $\overline{Y}$ can be characterized as the unique closed subscheme of X containing Y as an open subscheme such that $\operatorname{Ass}(\mathcal{O}_{\overline{Y}}) = \operatorname{Ass}(\mathcal{O}_Y)$.

PROOF. Everything is easy except the fact that $\mathcal{I}$ is quasi-coherent. To check this, it suffices to show that if $U = \operatorname{Spec} R$ is an affine in X and $U_f = \operatorname{Spec} R_f$ is a distinguished affine subset, then:

$$\operatorname{Ker}(R \to \mathcal{O}_Y(Y \cap U)) \cdot R_f = \operatorname{Ker}(R_f \to \mathcal{O}_Y(Y \cap U_f))$$

because then $\widetilde{\operatorname{Ker}}(R \to \mathcal{O}_Y(Y \cap U))$ agrees with $\mathcal{I}$ on all U_f's, hence agrees with $\mathcal{I}$ on U. Since " $\subset$ " is obvious, we mush check that if $a \in R$ and $a/f^n = 0$ in $\mathcal{O}_Y(Y \cap U_f)$, then $\exists m$, $f^m(a/f^n) = 0$ in $\mathcal{O}_Y(Y \cap U)$. Now U is noetherian so $Y \cap U$ is quasi-compact, hence is covered by a *finite* number of affines V_i. For each i,

$$a/f^n = 0 \text{ in } \mathcal{O}_Y((V_i)_f) \Longrightarrow \exists m_i,\ f^{m_i}(a/f^n) = 0 \text{ in } \mathcal{O}_Y(V_i)$$

and taking $m = \max(m_i)$

$$\Longrightarrow \exists m,\ f^m(a/f^n) = 0 \text{ in } \mathcal{O}_Y(Y \cap U).$$

□

We can apply Proposition 3.11 to globalize Example 3.8:

THEOREM 3.12. *Let X be a noetherian scheme, let Z be a subscheme and let $\operatorname{Ass}(\mathcal{O}_Z) = \{w_1, \ldots, w_t\}$. Then there exist closed subschemes $W_1, \ldots, W_t \subset Z$ such that*

a) *W_i is irreducible with generic point w_i and for all open $U_i \subset W_i$,*

$$\mathcal{O}_{W_i}(U_i) \longrightarrow \mathcal{O}_{w_i, W_i}$$

is injective (i.e., $\operatorname{Ass}(\mathcal{O}_{W_i}) = \{w_i\}$).

b) *Z is the scheme-theoretic union of the W_i's, i.e., set-theoretically $Z = W_1 \cup \cdots \cup W_t$ and*

$$\mathcal{O}_Z \longrightarrow \bigoplus_{i=1}^{t} \mathcal{O}_{W_i}$$

is surjective.

PROOF. For each i, let $U_i = \operatorname{Spec} R_i$ be an affine neighborhood of w_i, let $Z \cap U_i = \operatorname{Spec} R_i/\mathfrak{a}_i$, let $w_i = [\mathfrak{p}_i]$ and let $\mathfrak{q}_i$ be a $\mathfrak{p}_i$-primary component of $\mathfrak{a}_i$. Let

$$W_i = \text{scheme-theoretic closure of } \operatorname{Spec} R_i/\mathfrak{q}_i \text{ in } X.$$

(a) and (b) are easily checked. □

[2]Actually all we need here is that the inclusion of Y in X is a quasi-compact morphism. (cf. Definition 4.9 below.)

Proposition 3.11 can also be used to strip off various associated points from a subscheme. For instance, returning to Example 3.8:

$$\operatorname{Spec} R \supset Z = W_1 \cup \cdots \cup W_t, \qquad \text{a primary decomposition,}$$

and applying the proposition with $X = \operatorname{Spec} R$, $Y = Z \cap U$ where U is an open subset of $\operatorname{Spec} R$, we get

$$\overline{Z \cap U} = \bigcup_{\substack{i \text{ such that} \\ W_i \cap U \neq \emptyset}} W_i,$$

and hence these unions of the W_i's are schemes independent of the primary decomposition chosen.

Two last results are often handy:

PROPOSITION 3.13. *Let X be a scheme and $Z \subset X$ a closed subset. Then among all closed subschemes of X with support Z, there is a unique one $(Z, \mathcal{O}_X/\mathcal{I})$ which is reduced. It is a subscheme of any other subscheme $(Z, \mathcal{O}_X/\mathcal{I}')$ with support Z, i.e., $\mathcal{I} \supset \mathcal{I}'$.*

PROOF. In fact define $\mathcal{I}$ by

$$\mathcal{I}(U) = \{s \in \mathcal{O}_X(U) \mid s(x) = 0,\ \forall x \in U \cap Z\}\,.$$

The rest of the proof is left to the reader. □

PROPOSITION 3.14. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two morphisms of schemes. If $g \circ f$ is a closed immersion (resp. immersion), then f is a closed immersion (resp. immersion).*

PROOF. Apply Definition 3.2 directly and everything is easy. For instance, if $S \subset X$ is closed, then $f(S)$ is closed because $f(S) = g^{-1}((g \circ f)(S))$, etc. □

4. Separated schemes

In the theory of topological spaces, the concept of a Hausdorff space plays an important role. Recall that a topological space X is called Hausdorff if for any two points $x, y \in X$, there are disjoint open sets $U, V \subset X$ such that $x \in U$, $y \in V$. This very rarely holds in the Zariski topology so it might seem as if the Hausdorff axiom has no relevance among schemes. But if the product topology is given to the set-theoretic product $X \times X$, then the Hausdorff axiom for X is equivalent to the diagonal $\Delta \subset X \times X$ being closed. In the category of schemes, the product scheme $X \times X$ is neither set-theoretically nor topologically the simple Cartesian product of X by itself so the closedness of the diagonal gives a way to interpret the Hausdorff property for schemes. The most striking way to introduce this property is by proving a theorem that asserts the equivalence of a large number of properties of X, one of them being that the diagonal Δ is closed in $X \times X$.

Before giving this theorem, we need some preliminaries. We first introduce the concept of the *graph of a morphism*. Say we have an S-morphism f of two schemes X, Y over S, i.e., a diagram:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

Then f induces a section of the projection:

$$\begin{tikzcd} X \times_S Y \arrow[d, shift right, "p_1"'] \\ X \arrow[u, shift right, bend right, "\Gamma_f"'] \end{tikzcd}$$

defined by $\Gamma_f = (1_X, f)$. I claim that Γ_f is an immersion. In fact, choose affine coverings $\{U_i\}$ of X, $\{V_i\}$ of Y and $\{W_i\}$ of S such that $f(U_i) \subset V_i$ and $q(V_i) \subset W_i$. Then

$$\Gamma_f^{-1}(U_i \times_S V_i) = U_i$$

and if $U_i = \operatorname{Spec} R_i$, $V_i = \operatorname{Spec} S_i$, $W_i = \operatorname{Spec} T_i$, then

$$\operatorname{res} \Gamma_f \colon U_i \longrightarrow U_i \times_S V_i$$

corresponds to the ring map

$$\begin{aligned} R_i \otimes_{T_i} S_i &\longrightarrow R_i \\ a \otimes b &\longmapsto a \cdot f^* b \end{aligned}$$

which is surjective. Therefore if $\mathcal{U} = \bigcup_i (U_i \times_S V_i)$, then Γ_f factors

$$X \underset{\substack{\text{closed}\\ \text{immersion}}}{\hookrightarrow} \mathcal{U} \underset{\substack{\text{open}\\ \text{subscheme}}}{\subset} X \times_S Y.$$

This proves:

PROPOSITION 4.1. *If X and Y are schemes over S and $f \colon X \to Y$ is an S-morphism, then $\Gamma_f = (1_X, f) \colon X \to X \times_S Y$ is an immersion.*

The simplest example of Γ_f arises when $X = Y$ and $f = 1_X$. Taking $S = \operatorname{Spec} \mathbb{Z}$, we get the *diagonal map*

$$\delta = (1_X, 1_X) \colon X \longrightarrow X \times X.$$

We have proven that if $\{U_i\}$ is an open cover of X, then δ is an isomorphism of X with a closed subscheme $\delta(X)$ of $\mathcal{U} \subset X \times X$, where

$$\mathcal{U} = \bigcup_i (U_i \times U_i).$$

But is $\delta(X)$ closed in $X \times X$? This leads to:

PROPOSITION 4.2. *Let X be a scheme. The following properties are equivalent:*

i) *$\delta(X)$ is closed in $X \times X$.*
ii) *There is an open affine covering $\{U_i\}$ of X such that for all i, j, $U_i \cap U_j$ is affine and $\mathcal{O}_X(U_i)$, $\mathcal{O}_X(U_j)$ generate $\mathcal{O}_X(U_i \cap U_j)$.*
iii) *For all open affines $U, V \subset X$, $U \cap V$ is affine and $\mathcal{O}_X(U)$, $\mathcal{O}_X(V)$ generate $\mathcal{O}_X(U \cap V)$.*

PROOF. (i) $\implies$ (iii): Given open affines U, V, note that $U \times V$ is an open affine subset of $X \times X$ such that $\mathcal{O}_{X \times X}(U \times V)$ is $\mathcal{O}_X(U) \otimes \mathcal{O}_X(V)$. If $\delta(X)$ is closed in $X \times X$, δ is a closed immersion. Therefore $\delta^{-1}(U \times V)$ is affine and its ring is generated by $\mathcal{O}_{X \times X}(U \times V)$. But $\delta^{-1}(U \times V) = U \cap V$ so this proves (iii).

(iii) $\implies$ (ii) is obvious.

(ii) $\implies$ (i): Note that if $\{U_i\}$ is an open affine covering of X, then $\{U_i \times U_j\}$ is an open affine covering of $X \times X$. Since $\delta^{-1}(U_i \times U_j) = U_i \cap U_j$, (ii) is exactly the hypothesis $(*)$ of Corollary 3.5. The corollary says that then δ is a closed immersion, hence (i) holds. $\square$

DEFINITION 4.3. X is a separated scheme if the equivalent properties of Proposition 4.2 hold.

Here's the simplest example of a non-separated scheme X:

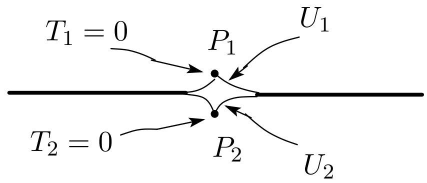

FIGURE II.5. $\mathbb{A}^1$ with "duplicate origin"

EXAMPLE 4.4. Take $X = U_1 \cup U_2$ where

$$U_1 = \operatorname{Spec} k[T_1]$$
$$U_2 = \operatorname{Spec} k[T_2]$$

and where U_1 and U_2 are identified along the open sets:

$$(U_1)_{T_1} = \operatorname{Spec} k[T_1, T_1^{-1}]$$
$$(U_2)_{T_2} = \operatorname{Spec} k[T_2, T_2^{-1}]$$

by the isomorphism

$$i\colon \operatorname{Spec} k[T_1, T_1^{-1}] \xrightarrow{\sim} \operatorname{Spec} k[T_2, T_2^{-1}]$$
$$i(T_1) = T_2.$$

This "looks" like Figure II.5, i.e., it is $\mathbb{A}^1_k$ except that the origin occurs twice!

The same construction with the real line gives a simple non-Hausdorff one-dimensional manifold. It is easy to see $\delta(X)$ is not closed in $U_1 \times U_2$ or $U_2 \times U_1$ because $(P_1, P_2) \in U_1 \times U_2$ and $(P_2, P_1) \in U_2 \times U_1$ will be in its closure.

Once a scheme is known to be separated, many other intuitively reasonable things follow. For example:

PROPOSITION 4.5. *Let $f\colon X \to Y$ be a morphism and assume X is separated. Then*

$$\Gamma_f\colon X \longrightarrow X \times Y$$

is a closed immersion. Hence for all $U \subset X$, $V \subset Y$ affine, $U \cap f^{-1}(V)$ is affine and its ring is generated by $\mathcal{O}_X(U)$ and $\mathcal{O}_Y(V)$.

PROOF. Consider the diagram:

$$\begin{tikzcd}
X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X \times Y \arrow[d, "(f \times 1_Y)"] \\
Y \arrow[r, "\delta_Y"'] & Y \times Y
\end{tikzcd}$$

It is easy to see that this diagram makes X into the fibre product of Y and $X \times Y$ over $Y \times Y$, so the proposition follows from the following useful result: □

PROPOSITION 4.6. *If $X \to S$ is a closed immersion and $Y \to S$ is any morphism, then $X \times_S Y \to Y$ is a closed immersion.*

PROOF. Follows from Corollary 3.5 and (using the definition of fibre product) the fact that $(A/I) \otimes_A B \cong B/I \cdot B$. □

Before giving another useful consequence of separation, recall from §I.6, that two morphisms

$$\operatorname{Spec} k \underset{f_2}{\overset{f_1}{\rightrightarrows}} X$$

are equal if and only if $f_1(\operatorname{Spec} k) = f_2(\operatorname{Spec} k)$ — call this point x — and the induced maps

$$f_1^*\colon \Bbbk(x) \longrightarrow k$$
$$f_2^*\colon \Bbbk(x) \longrightarrow k$$

are equal. Now given two morphisms

$$Z \underset{f_2}{\overset{f_1}{\rightrightarrows}} X$$

one can consider the "subset of Z where $f_1 = f_2$": the way to define this is:

$$\operatorname{Eq}(f_1, f_2) = \left\{ z \in Z \;\middle|\; \begin{array}{l} f_1(z) = f_2(z) \text{ and the induced maps} \\ f_1^*, f_2^*\colon \Bbbk(f_1(z)) \to \Bbbk(z) \text{ are equal} \end{array} \right\}.$$

Using this concept, we have:

PROPOSITION 4.7. *Given two morphisms $f_1, f_2\colon Z \to X$ where X is separated,* $\operatorname{Eq}(f_1, f_2)$ *is a closed subset of Z.*

PROOF. f_1 and f_2 define

$$(f_1, f_2)\colon Z \longrightarrow X \times X$$

and it is straightforward to check that $\operatorname{Eq}(f_1, f_2) = (f_1, f_2)^{-1}(\delta(X))$. □

Looking at reduced and irreducible separated schemes, another useful perspective is that such schemes are characterized by the set of their affine rings, i.e., the glueing need not be given explicitly. The precise statement is this:

PROPOSITION 4.8. *Let X and Y be two reduced and irreducible separated schemes with the same function field $K = \mathbb{R}(X) = \mathbb{R}(Y)$. Suppose $\{U_i\}$ and $\{V_i\}$ are affine open coverings of X and Y such that for all i, $\mathcal{O}_X(U_i) = \mathcal{O}_Y(V_i)$ as subrings of K. Then $X \cong Y$.*

PROOF. Left to the reader. □

Another important consequence of separation is the quasi-coherence of direct images. More precisely:

DEFINITION 4.9. A morphism $f\colon X \to Y$ of schemes is quasi-compact if for all $U \subset Y$ quasi-compact, $f^{-1}U$ is quasi-compact. Equivalently, for all affine open $U \subset Y$, $f^{-1}U$ is covered by a finite set of affine open subsets of X.

PROPOSITION 4.10. *Let $f\colon X \to Y$ be a quasi-compact morphism of separated schemes and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then $f_*\mathcal{F}$ is quasi-coherent.*

PROOF. The assertion is local on Y so we may assume $Y = \operatorname{Spec} R$. Let $\{U_i\}$ be a finite affine open cover of X and let $f_i\colon U_i \to Y$ be the restriction of f to U_i. Since X is separated, $U_i \cap U_j$ is also affine. Let $f_{ij}\colon U_i \cap U_j \to Y$ be the restriction of f to $U_i \cap U_j$. Then consider the homomorphisms:

$$0 \longrightarrow f_*\mathcal{F} \xrightarrow{\alpha} \prod_i f_{i,*}\mathcal{F} \xrightarrow{\beta} \prod_{j,k} f_{jk,*}\mathcal{F}$$

where α is just restriction and β is the difference of restrictions, i.e.,

$$\beta(\{s_i\})_{jk} = \operatorname{res}(s_j) - \operatorname{res}(s_k).$$

By the sheaf property of $\mathcal{F}$ and the definition of direct images, this sequence is exact! But f_i and f_{jk} are affine morphisms by Proposition 4.5 and the products are *finite* so the second and third sheaves are quasi-coherent by Lemma (I.5.12). Therefore $f_*\mathcal{F}$ is quasi-coherent. $\square$

> $(**)$ **In the rest of this book, we will always assume that all our schemes are separated** $(**)$

5. Proj R

The essential idea behind the construction of $\mathbb{P}^n$ can be neatly generalized. Let

$$R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$$

be any graded ring (i.e., $R_i \cdot R_j \subset R_{i+j}$), and let

$$R_+ = \bigoplus_{i=1}^{\infty} R_i$$

be the ideal of elements of positive degree. We define a scheme Proj R as follows:

(I) As a point set:

$$\operatorname{Proj} R = \left\{ \mathfrak{p} \subset R \;\middle|\; \begin{array}{l} \mathfrak{p} \text{ a homogeneous prime ideal} \\ (\text{i.e., } \mathfrak{p} = \bigoplus_{i=0}^{\infty} \mathfrak{p} \cap R_i) \text{ and } \mathfrak{p} \not\supset R_+ \end{array} \right\}.$$

(II) As a topological space:

$$\text{for all subsets } S \subset R, \text{ let} \quad V(S) = \{[\mathfrak{p}] \in \operatorname{Proj} R \mid S \subset \mathfrak{p}\}.$$

If $\mathfrak{a}$ is the homogeneous ideal generated by the homogeneous parts of all $f \in S$, then

$$V(S) = V(\mathfrak{a}).$$

It follows easily that the $V(S)$ are the closed sets of a topology and that the "distinguished open subsets"

$$(\operatorname{Proj} R)_f = \{[\mathfrak{p}] \in \operatorname{Proj} R \mid f \notin \mathfrak{p}\}, \quad \text{where } f \in R_k, \text{ some } k \geq 1$$

form a basis of open sets.
[Problem for the reader: check that if $f \in R_0$, then

$$\{[\mathfrak{p}] \in \operatorname{Proj} R \mid f \notin \mathfrak{p}\} = \bigcup_{k \geq 1} \bigcup_{g \in R_k} (\operatorname{Proj} R)_{fg}.]$$

(III) The structure sheaf:

$$(*) \qquad \text{for all } f \in R_k,\ k \geq 1, \text{ let} \quad \mathcal{O}_{\operatorname{Proj} R}((\operatorname{Proj} R)_f) = (R_f)_0,$$

where $(R_f)_0 =$ degree 0 component of the localization R_f. This definition is justified in a manner quite parallel to the construction of Spec, resting in this case however on:

Proposition 5.1. *Let f, $\{g_i\}_{i \in S}$ be homogeneous elements of R, with* $\deg f > 0$. *Then*

$$\left[(\operatorname{Proj} R)_f = \bigcup_{i \in S} (\operatorname{Proj} R)_{g_i} \right] \Longleftrightarrow \left[f^n = \sum a_i g_i,\ \text{some } n \geq 1,\ \text{some } a_i \in R \right].$$

PROOF. The left hand side means

$$\forall[\mathfrak{p}] \in \operatorname{Proj} R,\ f \notin \mathfrak{p} \Longrightarrow \exists i \text{ such that } g_i \notin \mathfrak{p}$$

which is equivalent to saying

$$f \in \left\{ \bigcap \mathfrak{p} \;\middle|\; \begin{array}{l} \mathfrak{p} \text{ homogeneous prime ideal such that} \\ \mathfrak{p} \supset \sum g_i R \text{ but } \mathfrak{p} \not\supset R_+ \end{array} \right\}.$$

Since $\mathfrak{p} \supset R_+$ implies $f \in \mathfrak{p}$, we can ignore the second restriction on $\mathfrak{p}$ in the parentheses, and what we need is:

LEMMA 5.2. *If $\mathfrak{a} \subset R$ is a homogeneous ideal, then*

$$\sqrt{\mathfrak{a}} = \bigcap_{\substack{\mathfrak{p} \textit{ homogeneous} \\ \mathfrak{p} \supset \mathfrak{a}}} \mathfrak{p}.$$

PROOF OF LEMMA 5.2. Standard, i.e., if $f \notin \sqrt{\mathfrak{a}}$, choose a homogeneous ideal $\mathfrak{q} \supset \mathfrak{a}$ maximal among those such that $f^n \notin \mathfrak{q}$, all $n \geq 1$. Check that $\mathfrak{q}$ is prime. □

□

COROLLARY 5.3. *If* $\deg f, \deg g > 0$, *then*

$$\begin{aligned}(\operatorname{Proj} R)_f \subset (\operatorname{Proj} R)_g &\Longrightarrow f^n = a \cdot g, \ \textit{some } n,\, a \\ &\Longrightarrow \exists \textit{ canonical map } (R_g)_0 \to (R_f)_0.\end{aligned}$$

COROLLARY 5.4. *If* $\deg g_i > 0$, $\forall i \in S$, *then*

$$\left[\operatorname{Proj} R = \bigcup_{i \in S} (\operatorname{Proj} R)_{g_i}\right] \Longleftrightarrow \left[R_+ = \sqrt{\sum g_i R}.\right]$$

We leave to the reader the details in checking that there is a unique sheaf $\mathcal{O}_{\operatorname{Proj} R}$ satisfying $(*)$ and with restriction maps coming from Corollary 5.3. The fact that we get a scheme in this way is a consequence of:

PROPOSITION 5.5. *Let $f \in R_k$, $k \geq 1$. Then there is a canonical isomorphism:*

$$((\operatorname{Proj} R)_f, \operatorname{res} \mathcal{O}_{\operatorname{Proj} R}) \cong \left(\operatorname{Spec}\left((R_f)_0\right), \mathcal{O}_{\operatorname{Spec}\left((R_f)_0\right)}\right).$$

PROOF. For all homogeneous primes $\mathfrak{p} \subset R$ such that $f \notin \mathfrak{p}$, let

$$\mathfrak{p}' = \{a/f^n \mid a \in \mathfrak{p} \cap R_{nk}\} = \mathfrak{p} \cdot R_f \cap (R_f)_0.$$

This is a prime ideal in $(R_f)_0$. Conversely, if $\mathfrak{p}' \subset (R_f)_0$ is prime, let

$$\mathfrak{p} = \bigoplus_{n=0}^{\infty} \left\{ a \in R_n \;\middle|\; a^k/f^n \in \mathfrak{p}' \right\}.$$

It follows readily that there are inverse maps which set up the set-theoretic isomorphism $(\operatorname{Proj} R)_f \cong \operatorname{Spec}(R_f)_0$. It is straightforward to check that it is a homeomorphism and that the two structure sheaves are canonically isomorphic on corresponding distinguished open sets. □

Moreover, just as with Spec, the construction of the structure sheaf carries over to modules too. In this case, for every *graded* R-module M, we can define a quasi-coherent sheaf of $\mathcal{O}_{\operatorname{Proj}(R)}$-modules $\widetilde{M}$ by the requirement:

$$\widetilde{M}\left((\operatorname{Proj} R)_f\right) = (M_f)_0.$$

We give next a list of fairly straightforward properties of the operations Proj and of $\widetilde{\ }$:

a) The homomorphisms $R_0 \to (R_f)_0$ for all $f \in R_k$, all $k \geq 1$ induce a morphism

$$\mathrm{Proj}(R) \longrightarrow \mathrm{Spec}(R_0).$$

b) If R is a finitely generated R_0-algebra, then $\mathrm{Proj}(R)$ is of finite type over $\mathrm{Spec}(R_0)$.

c) If S_0 is an R_0-algebra, then

$$\mathrm{Proj}(R \otimes_{R_0} S_0) \cong \mathrm{Proj}(R) \times_{\mathrm{Spec}\, R_0} \mathrm{Spec}\, S_0.$$

d) If $d \geq 1$ and $R\langle d\rangle = \bigoplus_{k=0}^{\infty} R_{dk}$, then $\mathrm{Proj}\, R \cong \mathrm{Proj}\, R\langle d\rangle$.
[Check that for all $f \in R_k$, $k \geq 1$, the rings $(R_f)_0$ and $(R\langle d\rangle_{f^d})_0$ are canonically isomorphic; this induces isomorphisms $(\mathrm{Proj}\, R)_f \cong (\mathrm{Proj}\, R\langle d\rangle)_{f^d}, \ldots$.]

e) Because of (c), it is possible to *globalize* Proj just as Spec was globalized in §I.7. If X is a scheme, and

$$\mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \cdots$$

is a quasi-coherent graded sheaf of $\mathcal{O}_X$-algebras where each $\mathcal{R}_i$ is quasi-coherent, then one can construct a scheme over X:

$$\pi\colon\ \mathcal{P}\!\mathit{roj}_X\Big(\bigoplus \mathcal{R}_i\Big) \longrightarrow X$$

as follows: for all $U \subset X$ open affine, take the scheme $\mathrm{Proj}(\bigoplus \mathcal{R}_i(U))$, which lies over $\mathrm{Spec}\, \mathcal{O}_X(U)$, i.e., U. For any two open affines $U_1, U_2 \subset X$ construct an isomorphism ϕ_{12}:

$$\begin{tikzcd}
\mathrm{Proj}(\bigoplus \mathcal{R}_i(U)) \arrow[r, "\pi_1"] & U_1 \\
\pi_1^{-1}(U_1 \cap U_2) \arrow[u, phantom, "\cup"] \arrow[dd, "\phi_{12}"', "\approx"] \arrow[dr] & \\
& U_1 \cap U_2 \arrow[uu, phantom, "\cup"] \arrow[dd, phantom, "\cap"] \\
\pi_2^{-1}(U_1 \cap U_2) \arrow[ur] \arrow[d, phantom, "\cap"] & \\
\mathrm{Proj}(\bigoplus \mathcal{R}_i(U_2)) \arrow[r, "\pi_2"] & U_2
\end{tikzcd}$$

by covering $U_1 \cap U_2$ by open affine U_3, and noting that:

$$\begin{aligned}
\pi_1^{-1}(U_3) &\cong \mathrm{Proj}\left(\bigoplus \mathcal{R}_i(U_1) \otimes_{\mathcal{O}_X(U_1)} \mathcal{O}_X(U_3)\right) \\
&\cong \mathrm{Proj}\left(\bigoplus \mathcal{R}_i(U_3)\right) \\
&\cong \mathrm{Proj}\left(\bigoplus \mathcal{R}_i(U_2) \otimes_{\mathcal{O}_X(U_2)} \mathcal{O}_X(U_3)\right) \\
&\cong \pi_2^{-1}(U_3).
\end{aligned}$$

f) If $\mathfrak{a} \subset R$ is a homogeneous ideal, then there is a canonical closed immersion:

$$\mathrm{Proj}\, R/\mathfrak{a} \hookrightarrow \mathrm{Proj}\, R.$$

A somewhat harder result is the converse in the case when R is finitely generated over R_0: that every closed subscheme Z of $\mathrm{Proj}\, R$ is isomorphic to $\mathrm{Proj}\, R/\mathfrak{a}$ for some $\mathfrak{a}$. The proof uses the remark that if $f_1, f_2, g \in R_n$ and g/f_1 vanishes on $Z \cap (\mathrm{Proj}\, R)_{f_1}$ then for some k, $g f_1^k / f_2^{k+1}$ vanishes on $Z \cap (\mathrm{Proj}\, R)_{f_2}$.

g) $\mathrm{Proj}\, R$ is separated.

PROOF. Use Criterion (ii) of Proposition 4.2, applying it to a covering of $\mathrm{Proj}\, R$ by distinguished affines. □

h) The map taking R to $\operatorname{Proj} R$ is not a functor but it does have a partial functoriality. To be precise, let R and R' be two graded rings and let

$$\phi\colon R \longrightarrow R'$$

be a homomorphism such that for some $d > 0$,

$$\phi(R_n) \subset R'_{nd}, \quad \text{all } n.$$

(Usually $d = 1$ but this isn't necessary.) Let

$$R_+ = \sum_{n>0} R_n$$
$$\mathfrak{a} = \phi(R_+) \cdot R'.$$

Then ϕ induces a natural map:

$$f\colon \operatorname{Proj} R' \setminus V(\mathfrak{a}) \longrightarrow \operatorname{Proj} R.$$

In fact,

$$\operatorname{Proj} R' \setminus V(\mathfrak{a}) = \bigcup_{n\geq 1} \bigcup_{a \in R_n} (\operatorname{Proj} R')_{\phi a}.$$

Define the restriction of f to $(\operatorname{Proj} R')_{\phi a}$ to be the morphism from $(\operatorname{Proj} R')_{\phi a}$ to $(\operatorname{Proj} R)_a$ induced by the ring homomorphism

$$\phi\colon (R_a)_0 \longrightarrow (R'_{\phi a})_0$$
$$\phi\left(\frac{b}{a^l}\right) = \frac{\phi(b)}{\phi(a)^l}, \quad b \in R_{nl}.$$

It is easy to check that these morphisms agree on intersections hence glue together to the morphism f.

i) If R and R' are two graded rings with the same degree 0 piece: $R_0 = R'_0$, then

$$\operatorname{Proj} R \times_{\operatorname{Spec} R_0} \operatorname{Proj} R' = \operatorname{Proj} R''$$

where

$$R'' = \bigoplus_{n=0}^{\infty} R_n \otimes_{R_0} R'_n.$$

PROOF. This follows easily from noting that for all $f \in R_n$, $f' \in R'_n$,

$$(R_f)_0 \otimes_{R_0} (R'_{f'})_0 \cong (R''_{f\otimes f'})_0$$

hence

$$(\operatorname{Proj} R)_f \times_{\operatorname{Spec} R_0} (\operatorname{Proj} R')_{f'} \cong (\operatorname{Proj} R'')_{f\otimes f'}$$

and glueing. ☐

j) $M \longmapsto \widetilde{M}$ is an exact functor; more precisely $\forall \phi\colon M \to N$ preserving degrees, we get an $\mathcal{O}_{\operatorname{Proj} R}$-homomorphism $\widetilde{\phi}\colon \widetilde{M} \to \widetilde{N}$ and if

$$0 \longrightarrow M \xrightarrow{\phi} N \xrightarrow{\psi} P \longrightarrow 0$$

is a sequence with $\psi \circ \phi = 0$ and such that

$$0 \longrightarrow M_k \longrightarrow N_k \longrightarrow P_k \longrightarrow 0$$

is exact if $k >> 0$, then

$$0 \longrightarrow \widetilde{M} \longrightarrow \widetilde{N} \longrightarrow \widetilde{P} \longrightarrow 0$$

is exact.

k) There is a natural map:

$$M_0 \longrightarrow \Gamma(\operatorname{Proj} R, \widetilde{M})$$

given by

$$m \longmapsto \text{element } m/1 \in (M_f)_0 = \widetilde{M}((\operatorname{Proj} R)_f).$$

There is a natural relationship between Spec and Proj which generalizes the fact that ordinary complex projective space $\mathbb{P}^n$ is the quotient of $\mathbb{C}^{n+1} \setminus (0)$ by homotheties. If R is any graded ring, let

$$R_+ = \sum_{n>0} R_n.$$

Then there is a canonical morphism

$$\pi\colon \operatorname{Spec} R \setminus V(R_+) \longrightarrow \operatorname{Proj} R.$$

In fact, for all $n \geq 1$, $a \in R_n$, the restriction of π to $(\operatorname{Spec} R)_a$ will be the morphism

$$\begin{array}{ccc} (\operatorname{Spec} R)_a & \longrightarrow & (\operatorname{Proj} R)_a \\ \wr\| & & \wr\| \\ \operatorname{Spec} R_a & & \operatorname{Spec}(R_a)_0 \end{array}$$

given by the inclusion of $(R_a)_0$ in R_n.

The most important Proj is:

Definition 5.6. $\mathbb{P}^n_R = \operatorname{Proj} R[X_0, \ldots, X_n]$.

Note that since $X_0, \ldots, X_n$ generate the ideal of elements of positive degree, this Proj is covered by the distinguished affines $(\operatorname{Proj} R[X_0, \ldots, X_n])_{X_i}$, i.e., by the $n+1$ copies of $\mathbb{A}^n_R$:

$$U_i = \operatorname{Spec} R\left[\frac{X_0}{X_i}, \ldots, \frac{X_n}{X_i}\right], \quad 0 \leq i \leq n$$

glued in the usual way. Moreover if $R = \bigoplus_{i=0}^{\infty} R_i$ is any graded ring generated over R_0 by elements of R_1 and with R_1 finitely generated as R_0-module, then R is a quotient of $R_0[X_0, \ldots, X_n]$ for some n: just choose generators $a_0, \ldots, a_n$ of R_1 and define

$$R_0[X_0, \ldots, X_n] \longrightarrow R$$
$$\text{by} \qquad X_i \longmapsto a_i.$$

Therefore by (f), $\operatorname{Proj} R$ is a closed subscheme of $\mathbb{P}^n_{R_0}$.

More generally, let X be any scheme and let $\mathcal{F}$ be a finitely generated quasi-coherent sheaf of $\mathcal{O}_X$-modules. Then we can construct symmetric powers $\operatorname{Symm}^n(\mathcal{F})$ by

$$\operatorname{Symm}^n(\mathcal{F})(U) = \operatorname{Symm}^n(\mathcal{F}(U)), \quad \text{all affine open } U$$

and hence a quasi-coherent graded $\mathcal{O}_X$-algebra:

$$\operatorname{Symm}^* \mathcal{F} = \mathcal{O}_X \oplus (\mathcal{F}) \oplus (\operatorname{Symm}^2 \mathcal{F}) \oplus (\operatorname{Symm}^3 \mathcal{F}) \oplus \cdots.$$

Definition 5.7. $\mathbb{P}_X(\mathcal{F}) = \mathcal{P}roj_X(\operatorname{Symm}^* \mathcal{F})$.

Note that by (f) above, if $\mathcal{R}$ is any quasi-coherent graded $\mathcal{O}_X$-algebra with

$$\begin{array}{l} \mathcal{R}_0 = \mathcal{O}_X \\ \mathcal{R}_1 \text{ finitely generated} \\ \mathcal{R}_n \text{ generated by } \mathcal{R}_1,\ n \geq 2, \end{array}$$

then we get a surjection

$$\operatorname{Symm}^* \mathcal{R}_1 \twoheadrightarrow \mathcal{R}$$

hence a closed immersion

$$\mathcal{P}roj_X(\mathcal{R}) \hookrightarrow \mathbb{P}_X(\mathcal{R}_1).$$

This motivates:

DEFINITION 5.8. Let $f\colon X \to Y$ be a morphism of schemes.

a) f is projective if $X \cong \mathcal{P}roj_Y(\bigoplus \mathcal{R}_i)$, some quasi-coherent graded $\mathcal{O}_Y$-algebra $\bigoplus \mathcal{R}_i$ but such that $\mathcal{R}_0 = \mathcal{O}_Y$, $\mathcal{R}_1$ finitely generated as $\mathcal{O}_Y$-module and $\mathcal{R}_1$, multiplied by itself n times generates $\mathcal{R}_n$, $n \geq 2$. Equivalently $\exists$ a diagram:

$$\begin{tikzcd} X \arrow[rr, hook, "\substack{\text{closed}\\ \text{immersion}}"] \arrow[dr, "f"'] & & \mathbb{P}_Y(\mathcal{F}) \arrow[dl] \\ & Y & \end{tikzcd}$$

where $\mathcal{F}$ is quasi-coherent, finitely generated.

b) f is quasi-projective[3] if $\exists$ a diagram:

$$\begin{tikzcd} X \arrow[rr, hook, "\text{open}"'] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd}$$

where f' is projective.

Note that if $Y = \operatorname{Spec} R$, say, then

$$f \text{ projective} \iff X \text{ is a closed subscheme of } \mathbb{P}^n_R, \text{ some } n$$
$$f \text{ quasi-projective} \iff X \text{ is a subscheme of } \mathbb{P}^n_R, \text{ some } n.$$

We can now make the final link between classical geometry and the theory of schemes: when $R = \mathbb{C}$ it is clear that $\mathbb{P}^n_R$ becomes the scheme that we associated earlier to the classical variety $\mathbb{P}^n(\mathbb{C})$. Moreover the reduced and irreducible closed subschemes of $\mathbb{P}^n_{\mathbb{C}}$ are precisely the schemes $\operatorname{Proj}(\mathbb{C}[X_0, \ldots, X_n]/\mathfrak{p})$, which are the schemes that we associated earlier to the classical varieties $V(\mathfrak{p}) \subset \mathbb{P}^n(\mathbb{C})$. In short, "complex projective varieties" as in Part I [**76**] define "complex projective varieties" in the sense of Definition 5.8, and, up to isomorphism, they all arise in this way.

Note too that for $\mathbb{P}^n_R$, the realization of $\mathbb{P}^n_R \times_{\operatorname{Spec} R} \mathbb{P}^m_R$ as a Proj in (h) above is identical to the Segre embedding studied in Part I [**76**]. In fact, the construction of (h) shows:

$$\begin{aligned} \mathbb{P}^n_R \times_{\operatorname{Spec} R} \mathbb{P}^m_R &= \operatorname{Proj} R[X_0, \ldots, X_n] \times_{\operatorname{Spec} R} \operatorname{Proj} R[Y_0, \ldots, Y_m] \\ &\cong \operatorname{Proj} \begin{bmatrix} \text{subring of } R[X] \otimes_R R[Y] \text{ generated by} \\ \text{polynomials of degrees } (k,k), \text{ some } k \end{bmatrix} \\ &\cong \operatorname{Proj} \begin{bmatrix} \text{subring of } R[X_0, \ldots, X_n, Y_0, \ldots, Y_m] \\ \text{generated by elements } X_iY_j \end{bmatrix}. \end{aligned}$$

Let U_{ij}, $0 \leq i \leq n$, $0 \leq j \leq m$, be new indeterminates. Then for some homogeneous prime ideal $\mathfrak{p} \subset R[U]$,

$$R[U_{00}, \ldots, U_{nm}]/\mathfrak{p} \cong \begin{bmatrix} \text{subring of } R[X_0, \ldots, Y_m] \\ \text{generated by elements } X_iY_j \end{bmatrix}$$
$$\text{via} \quad U_{ij} \longmapsto X_iY_j.$$

[3]Grothendieck's definition agrees with this only when Y is quasi-compact. I made the above definition only to avoid complications and have no idea which works better over non-quasi-compact bases.

Thus $\mathbb{P}^n_R \times_{\operatorname{Spec} R} \mathbb{P}^m_R$ is isomorphic to a closed subscheme of $\mathbb{P}^{nm+n+m}_R$. This is clearly the Segre embedding from a new angle:

The really important property of Proj is that the fundamental theorem of elimination theory (Part I [**76**, Chapter 2]), can be generalized to it.

Veronese should be mentioned too.

Theorem 5.9 (Elimination theory for Proj). *If R is a finitely generated R_0-algebra, then the map*

$$\pi\colon \operatorname{Proj} R \longrightarrow \operatorname{Spec} R_0$$

is closed.

Proof. Every closed subset of $\operatorname{Proj} R$ is isomorphic to $V(\mathfrak{a})$ for some homogeneous ideal $\mathfrak{a} \subset R_+$. But $V(\mathfrak{a}) \cong \operatorname{Proj} R/\mathfrak{a}$, so to show that $\pi(V(\mathfrak{a}))$ is closed in $\operatorname{Spec} R_0$, we may as well replace R by $R/\mathfrak{a}$ to start with and reduce the theorem to simply showing that Image π is closed. Also, we may reduce the theorem to the case when R is generated over R_0 by elements of degree 1. This follows because of $\operatorname{Proj} R \cong \operatorname{Proj} R\langle d\rangle$ and the amusing exercise:

Lemma 5.10. *Let R be a graded ring, finitely generated over R_0. Then for some d, $R\langle d\rangle$ is generated over R_0 by $R\langle d\rangle_1 = R_d$.*

(Proof left to the reader).

After these reductions, take $\mathfrak{p}_0 \subset R_0$ a prime ideal. Then

$$[\mathfrak{p}_0] \notin \operatorname{Image}\pi \iff \left[\begin{array}{l}\nexists \text{ homogeneous prime } \mathfrak{p} \subset R \text{ such} \\ \text{that } \mathfrak{p} \cap R_0 = \mathfrak{p}_0,\ \mathfrak{p} \not\supset R_+\end{array}\right].$$

Let $R' = R \otimes_{R_0} (R_0)_{\mathfrak{p}_0}$. Then homogeneous primes $\mathfrak{p}$ in R such that $\mathfrak{p} \cap R_0 = \mathfrak{p}_0$ are in one-to-one correspondence with homogeneous primes $\mathfrak{p}'$ in R' such that $\mathfrak{p}' \supset \mathfrak{p}_0 \cdot R'$. Therefore

$$\begin{aligned}
[\mathfrak{p}_0] \notin \operatorname{Image}\pi &\iff \left[\begin{array}{l}\nexists \text{ homogeneous prime } \mathfrak{p}' \subset R' \text{ such} \\ \text{that } \mathfrak{p}' \supset \mathfrak{p}_0 \cdot R' \text{ and } \mathfrak{p}' \not\supset R'_+\end{array}\right] \\
&\iff \sqrt{\mathfrak{p}_0 \cdot R'} \supset R'_+ \\
&\iff \exists n,\ \mathfrak{p}_0 \cdot R' \supset (R'_+)^n \text{ (since } R_+ \text{ is a finitely generated ideal)} \\
&\iff \exists n,\ \mathfrak{p}_0 \cdot R'_n \supset R'_n \text{ (since } R_+ \text{ is generated by } R_1) \\
&\iff \left[\begin{array}{l}\exists n,\ R'_n = (0) \\ \text{(by Nakayama's lemma since } R'_n \\ \text{is a finitely generated } R'\text{-module)}\end{array}\right].
\end{aligned}$$

Now for any finitely generated R-module M,

$$M_{\mathfrak{p}} = (0) \Longrightarrow M_f = (0), \quad \text{some } f \in R \setminus \mathfrak{p},$$

hence

$$\{[\mathfrak{p}] \in \operatorname{Spec} R \mid M_{\mathfrak{p}} = (0)\}$$

is the maximal open set of $\operatorname{Spec} R$ on which $\widetilde{M}$ is trivial, i.e.,

$$\operatorname{Supp} \widetilde{M} = \{[\mathfrak{p}] \in \operatorname{Spec} R \mid M_{\mathfrak{p}} \neq (0)\}$$

and this is a *closed* set. What we have proven is:

$$\begin{aligned}
[\mathfrak{p}_0] \in \operatorname{Image}\pi &\iff \forall n,\ R_n \otimes_{R_0} (R_0)_{\mathfrak{p}_0} \neq (0) \\
&\iff [\mathfrak{p}_0] \in \bigcap_{n=1}^{\infty} \operatorname{Supp} \widetilde{R_n}.
\end{aligned}$$

Thus Image π is closed. □

6. Proper morphisms

Theorem 5.9 motivates one of the main non-trivial definitions in scheme-theory:

Definition 6.1. Let $f\colon X \to Y$ be a morphism of schemes. Then f is *proper* if

a) f is of finite type,
b) for all $Y' \to Y$, the canonical map
$$X \times_Y Y' \longrightarrow Y'$$
is closed.

When $Y = \operatorname{Spec} k$, X is complete over k if $f\colon X \to \operatorname{Spec}(k)$ is proper.

Since "proper" is defined by such an elementary requirement, it is easy to deduce several general properties—

Suppose we are given $X \xrightarrow{f} Y \xrightarrow{g} Z$. Then

i) f, g proper $\implies$ $g \circ f$ proper
ii) $g \circ f$ proper $\implies$ f proper
iii)
$$\left.\begin{array}{l} g\circ f \text{ proper} \\ f \text{ surjective} \\ g \text{ of finite type}\end{array}\right\} \implies g \text{ proper}$$
iv) Proper morphisms are "maximal" in the following sense: given
$$\begin{tikzcd} X \arrow[r, phantom, "\subset"] \arrow[dr, "f"'] & X' \arrow[d, "f'"] \\ & Y \end{tikzcd}$$
where X is open and dense in X',
$$f \text{ proper} \implies X = X'.$$

For instance, take (ii) which is perhaps subtler. One notes that f can be gotten as a composition:
$$\begin{tikzcd} X \arrow[r, hook, "{(1,f)}"] & X \times_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "g\circ f"'] & Z \end{tikzcd}$$
where $(1, f)$ is a closed immersion.

Using the concept proper, the Elimination Theorem (Theorem 5.9) now reads:

Corollary 6.2. *A projective morphism $f\colon X \to Y$ is proper.*

Proof. Note that $f\colon X \to Y$ is closed if there exists an open cover $\{U_i\}$ of Y such that $f^{-1}U_i \to U_i$ is closed. Therefore Corollary 6.2 follows from the Theorem 5.9, the definition of $\mathcal{P}roj_Y$ and Property (c) of Proj. $\square$

On the other hand, "projective" is the kind of explicit constructive property that gives one a very powerful hold on such morphisms, whereas "proper" is just an abstraction of the main qualitative property that projective morphisms possess. Now there exist varieties over k that are complete but not projective—even non-singular complex varieties—so proper is certainly weaker than projective. But what makes proper a workable concept is that it is not too much weaker than projective because of the following:

THEOREM 6.3 ("Chow's lemma"). *Let $f\colon X \to Y$ be a morphism of finite type between noetherian schemes. Then there exists*

a) *a surjective projective morphism $\pi\colon X' \to X$, "birational" in the sense that there is an open set U such that:*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, phantom, "\underset{dense}{\subset}"] \arrow[d, "\approx", "isomorphism"'] & X' \arrow[d, "\pi"] \\ U \arrow[r, phantom, "\underset{dense}{\subset}"] & X \end{tikzcd},$$

b) *a factorization of $f \circ \pi$:*

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ X' \arrow[ur, "\pi"] \arrow[dr, "i"'] & & Y \\ & \mathbb{P}^n \times Y \arrow[ur, "p_2"'] & \end{tikzcd}$$

where i is an immersion, so that $f \circ \pi$ is quasi-projective.

If f is proper, then i is a closed immersion, so we have π and $f \circ \pi$ projective, i.e., f is a "factor" of projective morphisms!

PROOF. We do this in several steps:

STEP (I). $\exists$ a finite affine covering $\{U_i\}$ of X such that $\bigcap U_i$ is dense in X.

PROOF. Let $X = X_1 \cup \cdots \cup X_t$ be the components of X and let $\{V_i\}$ be any finite affine covering of X. For all s, $1 \leq s \leq t$, let X_s° be an affine open subset of X_s such that

a) $X_s^\circ \cap X_r = \emptyset$ if $r \neq s$
b) $X_s^\circ \subset V_i$ whenever $X_s \cap V_i \neq \emptyset$.

Then define U_i to be the union of V_i and those X_s° such that $V_i \cap X_s = \emptyset$. Since V_i and all these X_s° are disjoint, U_i is affine too. Moreover $\bigcap U_i \supset \bigcup X_s^\circ$, hence is dense in X. $\square$

STEP (II). For each i, $\operatorname{res} f\colon U_i \to Y$ can be factored

$$U_i \underset{I_i}{\hookrightarrow} \mathbb{A}^{\nu_i} \times Y \underset{p_2}{\longrightarrow} Y$$

where I_i is a closed immersion.

PROOF. Let $\{V_j\}$ be an affine covering of Y. Then $U_i \cap f^{-1}(V_j)$ is affine and its ring is generated by $\mathcal{O}_X(U_i) \otimes \mathcal{O}_Y(V_j)$. Let $f_1, \ldots, f_{\nu_i} \in \mathcal{O}_X(U_i)$ be enough elements to generate the affine rings of $U_i \cap f^{-1}(V_j)$ over $\mathcal{O}_Y(V_j)$ for all j. Define $I_{i,1}\colon U_i \to \mathbb{A}^{\nu_i}$ by $I_{i,1}^*(X_k) = f_k$ and define $I_i = (I_{i,1}, \operatorname{res} f)$. One sees easily that I_i is a closed immersion. $\square$

STEP (III). Consider the immersions:

$$I_i'\colon U_i \hookrightarrow \mathbb{P}^{\nu_i} \times Y$$

gotten by composing I_i with the usual inclusion of $\mathbb{A}^{\nu_i}$ in $\mathbb{P}^{\nu_i}$. Let $\overline{U_i}$ be the scheme-theoretic closure:

$$U_i \underset{\substack{\text{open}\\ \text{dense}}}{\subset} \overline{U_i} \underset{\substack{\text{closed}\\ \text{immersion}}}{\longrightarrow} \mathbb{P}^{\nu_i} \times Y.$$

Let $U = \bigcap_{i=1}^N U_i$. Consider the immersion:

$$U \hookrightarrow \mathbb{P}^{\nu_1} \times \cdots \times \mathbb{P}^{\nu_N} \times Y$$

given by $(I'_1, \ldots, I'_N)$ and the inclusion of U in each U_i. Let $\overline{U}$ be the scheme-theoretic closure of the image here. Via the Segre embedding, we get:

$$\begin{tikzcd} U \arrow[r, hook, "\substack{\text{open}\\ \text{dense}}"'] \arrow[dr, "f"'] & \overline{U} \arrow[r, hook, "\substack{\text{closed}\\ \text{immersion}}"'] \arrow[d] & \mathbb{P}^n \times Y \arrow[dl, "p_2"] \\ & Y & \end{tikzcd}$$

Note that by projecting $\mathbb{P}^{\nu_1} \times \cdots \times \mathbb{P}^{\nu_N}$ on its i-th factor, we get morphisms:

$$\begin{tikzcd} U \arrow[r, hook, "\substack{\text{open}\\ \text{dense}}"'] \arrow[d, phantom, "\cap"] & \overline{U} \arrow[d, "p_i"] \\ U_i \arrow[r, hook, "\substack{\text{open}\\ \text{dense}}"'] & \overline{U_i} \end{tikzcd}$$

Define X' to be the open subscheme of $\overline{U}$ which is the union of the open subschemes $p_i^{-1}(U_i)$. Finally, define $\pi\colon X' \to X$ by:

$$\begin{tikzcd} X' \arrow[r, dashed, "\pi"] \arrow[d, phantom, "\cup"] & X \arrow[d, phantom, "\cup"] \\ p_i^{-1}U_i \arrow[r, "p_i"] & U_i \end{tikzcd}$$

Note that this is OK because on the open set $U \subset \bigcap p_i^{-1}U_i$, all these morphisms p_i equal the inclusion morphism $U \hookrightarrow X$, and hence $p_i = p_j$ on $p_i^{-1}U_i \cap p_j^{-1}U_j$ since U is scheme-theoretically dense in $p_i^{-1}U_i \cap p_j^{-1}U_j$.

Step (IV). $\pi\colon X' \to X$ is projective. In fact note that $p_i\colon \overline{U} \to \overline{U_i}$ is the restriction of the projection $\mathbb{P}^{\nu_1} \times \cdots \times \mathbb{P}^{\nu_n} \times Y \to \mathbb{P}^{\nu_i} \times Y$ to $\overline{U}$, hence it is projective, hence it is proper. Therefore $\operatorname{res} p_i\colon p_i^{-1}(U_i) \to U_i$ is proper. We are now in the abstract situation:

Lemma 6.4. *If $\pi\colon X \to Y$ is a morphism, $U_i \subset X'$, $V_i \subset X$ open dense sets covering X and Y such that $\pi(U_i) \subset V_i$, $\operatorname{res}\pi\colon U_i \to V_i$ proper, then $\pi^{-1}(V_i) = U_i$ and π is proper.*

(Proof left to the reader.)

But now consider the morphism

$$j\colon X' \longrightarrow \mathbb{P}^n \times X$$

induced by a) $X' \subset \overline{U} \hookrightarrow \mathbb{P}^n \times Y \xrightarrow{p_1} \mathbb{P}^n$ and b) $\pi\colon X' \to X$. It is an immersion since the composite $X' \xrightarrow{j} \mathbb{P}^n \times X \xrightarrow{1\times f} \mathbb{P}^n \times Y$ is an immersion. Since $\pi\colon X' \to X$ is proper, j is proper too, hence $j(X')$ is closed, hence j is a closed immersion. Thus π is projective. Finally, if f is proper too, then $f \circ \pi\colon X' \to Y$ is proper, hence the immersion $X' \to \mathbb{P}^n \times Y$ is proper, hence it is a *closed* immersion, hence $f \circ \pi$ is projective. □

Interestingly, when this result first appeared in the context of varieties, it was considered quite clear and straightforward. It is one example of an idea which got much harder when transported to the language of schemes.

Proper morphisms arise in another common situation besides Proj:

Proposition 6.5. *Let $\phi\colon A \to B$ be a homomorphism of rings where B is a finite A-module (equivalently, B is a finitely generated A-algebra and B is integrally dependent on A). Then the induced morphism f*

$$f\colon \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$$

is proper.

PROOF. This is simply the "going-up" theorem (Zariski-Samuel [**109**, vol. I, Chapter V, §2, Theorem 3]). It suffices to show f is closed. Let $Z = V(\mathfrak{a}) \subset \operatorname{Spec} B$ be a closed set. I claim

$$f(Z) = V(\phi^{-1}(\mathfrak{a})).$$

We must show that if $\mathfrak{p}$ is a prime ideal:

$$\phi^{-1}(\mathfrak{a}) \subset \mathfrak{p} \subset A,$$

then there is a prime ideal $\mathfrak{q}$:

$$\mathfrak{a} \subset \mathfrak{q} \subset B, \qquad \phi^{-1}(\mathfrak{q}) = \mathfrak{p}.$$

Apply the going-up theorem to $\mathfrak{p}/\phi^{-1}(\mathfrak{a})$ and the inclusion:

$$A/\phi^{-1}(\mathfrak{a}) \subset B/\mathfrak{a}.$$

□

One globalizes this situation via a definition:

DEFINITION 6.6. A morphism $f\colon X \to Y$ is called finite if $X \cong \mathcal{S}pec_Y\, \mathcal{R}$ where $\mathcal{R}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras which is finitely generated as $\mathcal{O}_Y$-modules.

COROLLARY 6.7. *A finite morphism is proper.*

There is a very important criterion for properness known as the "valuative criterion":

PROPOSITION 6.8. *Let $f\colon X \to Y$ be a morphism of finite type. Then f is proper if and only if the "valuative criterion" holds:*

> *For all valuation rings R, with quotient field K, every K-valued point α of X extends to an R-valued point if the K-valued point $f \circ \alpha$ of Y extends, i.e., given the solid arrows:*
>
> $$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "\alpha"] \arrow[d, hook, "i"'] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$
>
> *the dotted arrow exists.*

PROOF. It's obvious that the criterion is necessary: just make the base change by the extended morphism $\beta\colon \operatorname{Spec} R \to Y$:

$$\begin{tikzcd} & X' = X \times_Y \operatorname{Spec} R \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ \operatorname{Spec} K \arrow[r, hook] \arrow[ur, dashed, "\alpha' = (\alpha, i)"] & \operatorname{Spec} R \arrow[r, "\beta"] & Y \end{tikzcd}$$

Then α defines a morphism $\alpha' = (\alpha, i)$ from $\operatorname{Spec} K$ to $X \times_Y \operatorname{Spec} R$, i.e., a section of f' over $\operatorname{Spec} K$. Let $z \in X \times_Y \operatorname{Spec} R$ be the image of α' and let $Z = \overline{\{z\}}$. Then if f is proper:

$$f'(Z) = \overline{f'(Z)} = \overline{\{f'(z)\}} = \operatorname{Spec} R.$$

Let $w \in Z$ lie over the closed point of $\operatorname{Spec} R$. Then we get homomorphisms

$$\begin{tikzcd} & \mathcal{O}_{w,Z} \arrow[dl, hook', "\alpha^*"'] \\ K & R \arrow[l, phantom, "\supset"] \arrow[u, "(f')^*"'] \end{tikzcd}$$

Since R is a valuation ring and $(f')^*$ is a local homomorphism, this can only hold if $R = \mathcal{O}_{w,Z}$ (a valuation ring is a maximal subring of its quotient field with respect to local homomorphisms: Zariski-Samuel [**109**, vol. II, Chapter VI, §2]). Then

$$\operatorname{Spec} \mathcal{O}_{w,Z} \longrightarrow Z$$

defines the required extension:

$$\operatorname{Spec} R \longrightarrow Z \subset X' \longrightarrow X$$

of α.

The converse is only a bit harder. Assume f satisfies the criterion. Then so does $p_2 \colon X \times_Y Y' \to Y'$ after every base change $Y' \to Y$, so replacing f by p_2, it suffices to check that f itself is closed. Everything is local over Y so we may also assume Y is affine: say $Y = \operatorname{Spec} S$. Since f is of finite type, X is the union of finitely many affines X_α: say $X_\alpha = \operatorname{Spec} R_\alpha$. Now let $Z \subset X$ be closed. Then

$$Z = \bigcup_\alpha \overline{(Z \cap X_\alpha)}$$

so if $f(\overline{Z \cap X_\alpha})$ is closed for every α, so is $f(Z)$. We can therefore also replace Z by $\overline{Z \cap X_\alpha}$ for some α, i.e., we can assume $Z \cap X_\alpha$ dense in Z for some α. There are two steps:

a) for every irreducible component W of $\overline{f(Z)}$, the generic point η_W equals $f(z)$, some $z \in Z$,
b) for every $z \in Z$ and $y \in \overline{\{f(z)\}}$, there is a point $x \in \overline{\{z\}}$ such that $f(x) = y$.

Together, these prove that $f(Z)$ is closed.

Proof of (a). The affine morphism

$$Z \cap X_\alpha \longrightarrow \overline{f(Z)} = \overline{f(Z \cap X_\alpha)}$$

corresponds to an injective ring homomorphism

$$R_\alpha/\mathfrak{b}_\alpha \xhookleftarrow{f^*} S/\mathfrak{a}$$

between rings without nilpotents. η_W corresponds to a minimal prime ideal $\mathfrak{p} \subset S/\mathfrak{a}$. Localizing with respect to $M = ((S/\mathfrak{a}) \setminus \mathfrak{p})$, we still get an injection $(\operatorname{res} f)^*$ in the diagram

$$\begin{tikzcd}
(R_\alpha/\mathfrak{b}_\alpha)_M & (S/\mathfrak{a})_M \arrow[l, hook', "(\operatorname{res} f)^*"'] \\
R_\alpha/\mathfrak{b}_\alpha \arrow[u, "j"] & S/\mathfrak{a} \arrow[l, hook', "f^*"] \arrow[u]
\end{tikzcd}$$

(Zariski-Samuel [**109**, vol. I, Chapter IV, §9] and Bourbaki [**26**, Chapter II, §2.4, Theorem 1]). But $(S/\mathfrak{a})_M$ is the field $\Bbbk(\eta_W)$, so if $\mathfrak{q} \subset (R_\alpha/\mathfrak{b}_\alpha)_M$ is any prime ideal, $((\operatorname{res} f)^*)^{-1}(\mathfrak{q}) = (0)$. Then $j^{-1}(\mathfrak{q})$ defines a point $z = [j^{-1}(\mathfrak{q})] \in Z \cap X_\alpha$ such that $f(z) = \eta_W$. □

Proof of (b). In the notation of (b), let $W = \overline{\{f(z)\}}$. Then we have a diagram of rings

$$\mathcal{O}_{y,W} \hookrightarrow \text{its quotient field } \Bbbk(f(z)) \subset \Bbbk(z).$$

We use the fundamental valuation existence theorem (Zariski-Samuel [**109**, vol. II, Chapter VI, Theorem 5]) which states that there is a valuation ring $R \subset \Bbbk(z)$ with quotient field $K = \Bbbk(z)$

such that $\mathcal{O}_{y,W} \to R$ is a local homomorphism. This gives us maps:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, hook] & \overline{\{z\}} \arrow[d, "\operatorname{res} f"] & \subset & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] & W & \subset & Y \end{tikzcd}$$

By the criterion, a lifting $\operatorname{Spec} R \to X$ exists, and this must factor through $\overline{\{z\}}$ (since $\operatorname{Spec} K$ is dense in $\operatorname{Spec} R$). Then x, the image under this lifting of the closed point of $\operatorname{Spec} R$, is the required point of $\overline{\{z\}}$. □

□

An amusing exercise that shows one way the definition of properness can be used is:

Proposition 6.9. *Let k be a field and let X be a scheme proper over* $\operatorname{Spec} k$. *Then the algebra $\Gamma(X, \mathcal{O}_X)$ is integrally dependent on k.*

Proof. Let $a \in \Gamma(X, \mathcal{O}_X)$. Define a morphism

$$f_a \colon X \longrightarrow \mathbb{A}^1_k$$

by the homomorphism

$$k[T] \longrightarrow \Gamma(X, \mathcal{O}_X)$$
$$T \longmapsto a.$$

Let $i \colon \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$ be the inclusion. Consider the diagram

$$\begin{tikzcd} X \arrow[rr, "i \circ f_a"] \arrow[dr, "\pi_1"'] & & \mathbb{P}^1_k \arrow[dl, "\pi_2"] \\ & \operatorname{Spec} k & \end{tikzcd}$$

where π_1, π_2 are the canonical maps. Since π_1 is proper, so is $i \circ f_a$ (cf. remarks following Definition 6.1). Therefore the image of $i \circ f_a$ is closed. But $\infty \notin \operatorname{Image}(i \circ f_a)$, so the image must be a proper subscheme of $\mathbb{A}^1_k$. Since $k[T]$ is a principal ideal domain, the image is contained in $V(p)$, some monic polynomial $p(T)$. Therefore the function

$$p(a) \in \Gamma(X, \mathcal{O}_X)$$

is everywhere zero on X. On each affine, such a function is nilpotent (an element in every prime ideal of a ring is nilpotent) and X is covered by a finite number of affines. Thus

$$p(a)^N = 0$$

some $N \geq 1$, and a is integral over k. □

Corollary 6.10. *Let k be an algebraically closed field and let X be a complete k-variety. Then $\Gamma(X, \mathcal{O}_X) = k$.*

The following result, given in a slightly strong form in EGA [**1**, Chapter III, §3.1], will be needed in the proof of Snapper's theorem (Theorem VII.11.1).

Definition 6.11. Let $\mathbf{K}$ be an abelian category, and denote by $\mathrm{Ob}(\mathbf{K})$ the set of its objects. A subset $\mathbf{K}' \subset \mathrm{Ob}(\mathbf{K})$ is said to be *exact* if $0 \in \mathbf{K}'$ and if the following is satisfied: In an exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathbf{K}$, if two among A, A' and A'' belong to $\mathbf{K}'$, then the third belongs to $\mathbf{K}'$.

THEOREM 6.12 ("Lemma of dévissage"). *Let* $\mathbf{K}$ *be the abelian category of coherent* $\mathcal{O}_X$*-modules on a noetherian scheme* X*, and* $\mathbf{K}' \subset \mathrm{Ob}(\mathbf{K})$ *an exact subset. We have* $\mathbf{K}' = \mathrm{Ob}(\mathbf{K})$*, if for any closed irreducible subset* $Y \subset X$ *with generic point* y *there exists* $\mathcal{G} \in \mathbf{K}'$ *with support* Y *such that* $\mathcal{G}_y$ *is a one-dimensional* $\Bbbk(y)$*-vector space.*

PROOF. For simplicity, a closed subset $Y \subset X$ is said to have property $\mathbf{P}(Y)$ if any $\mathcal{S} \in \mathrm{Ob}(\mathbf{K})$ with $\mathrm{Supp}(\mathcal{S}) \subset Y$ satisfies $\mathcal{S} \in \mathbf{K}'$.

We need to show that X has property $\mathbf{P}(X)$.

By noetherian induction, it suffices to show that a closed subset $Y \subset X$ has property $\mathbf{P}(Y)$ if any closed subset $Y' \subsetneqq Y$ has property $\mathbf{P}(Y')$.

Thus we now show $\mathcal{F} \in \mathrm{Ob}(\mathbf{K})$ satisfies $\mathcal{F} \in \mathbf{K}'$ if $\mathrm{Supp}(\mathcal{F}) \subset Y$. Endow Y with the unique structure of closed reduced subscheme of X with the ideal sheaf $\mathcal{J}$. Since $\mathcal{J} \supset \mathrm{Ann}(\mathcal{F})$, there exists $n > 0$ such that $\mathcal{J}^n\mathcal{F} = (0)$. Looking at successive quotients in the filtration

$$\mathcal{F} \supset \mathcal{J}\mathcal{F} \supset \mathcal{J}^2\mathcal{F} \supset \cdots \supset \mathcal{J}^{n-1}\mathcal{F} \supset \mathcal{J}^n\mathcal{F} = (0),$$

we may assume $n = 1$, that is, $\mathcal{J}\mathcal{F} = (0)$, in view of the exactness of $\mathbf{K}'$. Let $j\colon Y \to X$ be the closed immersion so that $\mathcal{F} = j_*j^*\mathcal{F}$.

Suppose Y is reducible and $Y = Y' \cup Y''$ with closed reduced subschemes $Y', Y'' \subsetneqq Y$. Let $\mathcal{J}'$ and $\mathcal{J}''$ be the ideal sheaves of $\mathcal{O}_X$ defining Y' and Y'', respectively, so that $\mathcal{J} = \mathcal{J}' \cap \mathcal{J}''$. Let $\mathcal{F}' = \mathcal{F} \otimes (\mathcal{O}_X/\mathcal{J}')$ and $\mathcal{F}'' = \mathcal{F} \otimes (\mathcal{O}_X/\mathcal{J}'')$, both of which belong to $\mathbf{K}'$ by assumption. Regarding the canonical $\mathcal{O}_X$-homomorphism

$$u\colon \mathcal{F} \longrightarrow \mathcal{F}' \oplus \mathcal{F}'',$$

we have $\mathcal{F}' \oplus \mathcal{F}'' \in \mathbf{K}'$ by exactness, while $\mathrm{Ker}(u), \mathrm{Coker}(u) \in \mathbf{K}'$ by assumption, since the induced homomorphism of the stalks at each $z \notin Y' \cap Y''$ is obviously bijective. Hence we have $\mathcal{F} \in \mathbf{K}'$ by exactness.

It remains to deal with the case Y irreducible. Endowing Y with the unique integral scheme structure, let y be the generic point of Y. Since $\mathcal{O}_{y,Y} = \Bbbk(y)$ and $j^*\mathcal{F}$ is $\mathcal{O}_Y$-coherent, $\mathcal{F}_y = (j^*\mathcal{F})_y$ is a $\Bbbk(y)$-vector space of finite dimensio m, say. By assumption there exists $\mathcal{G} \in \mathbf{K}'$ with $\mathrm{Supp}(\mathcal{G}) = Y$ and $\dim_{\Bbbk(y)} \mathcal{G}_y = 1$. Hence there is a $\Bbbk(y)$-isomorphism $(\mathcal{G}_y)^{\oplus m} \to \mathcal{F}_y$ which extends to an $\mathcal{O}_Y$-isomorphism in a neighborhood W in X of y. Let $\mathcal{H}$ be the graph in $(\mathcal{G}^{\oplus m} \oplus \mathcal{F})|_W$ of the $\mathcal{O}_Y|_W$-isomorphism $\mathcal{G}^{\oplus m}|_W \to \mathcal{F}|_W$. The projections from $\mathcal{H}$ to $\mathcal{G}^{\oplus m}|_W$ and $\mathcal{F}|_W$ are isomorphisms. Hence there exists a coherent $\mathcal{O}_X$-submodule $\mathcal{H}_0 \subset \mathcal{G}^{\oplus m} \oplus \mathcal{F}$ such that $\mathcal{H}_0|_W = \mathcal{H}$ and that $\mathcal{H}_0|_{X\setminus Y} = (0)$, since $\mathrm{Supp}(\mathcal{G}), \mathrm{Supp}(\mathcal{F}) \subset Y$. The projections from $\mathcal{H}_0$ to $\mathcal{G}^{\oplus m}$ and $\mathcal{F}$ are homomorphisms of $\mathcal{O}_X$-modules which are isomorphisms on W and $X \setminus Y$. Thus their kernels and cokernels have support in $Y \setminus (Y \cap W) \subsetneqq Y$, hence belong to $\mathbf{K}'$. Since $\mathcal{G} \in \mathbf{K}'$, we thus have $\mathcal{H}_0 \in \mathbf{K}'$, hence $\mathcal{F} \in \mathbf{K}'$. □

CHAPTER III

Elementary global study of $\operatorname{Proj} R$

1. Intertible sheaves and twists

Definition 1.1. Let X be a scheme. A sheaf $\mathcal{L}$ of $\mathcal{O}_X$-modules is called *invertible* if $\mathcal{L}$ is locally free of rank one. This means that each point has an open neighborhood U such that

$$\mathcal{L}|_U \approx \mathcal{O}_X|_U;$$

or equivalently, there exists an open covering $\{U_\alpha\}$ of X such that for each α,

$$\mathcal{L}|_{U_\alpha} \approx \mathcal{O}_X|_{U_\alpha}.$$

The reason why invertible sheaves are called invertible is that their isomorphism classes form a group under the tensor product over $\mathcal{O}_X$ for multiplication, as we shall now see.

(a) If $\mathcal{L}$, $\mathcal{L}'$ are invertible, so is $\mathcal{L} \otimes \mathcal{L}'$.

Proof. For each point we can find an open neighborhood U such that both $\mathcal{L}$, $\mathcal{L}'$ are isomorphic to $\mathcal{O}_X$ when restricted to U, so $\mathcal{L} \otimes \mathcal{L}'$ is isomorphic to $\mathcal{O}_X \otimes \mathcal{O}_X = \mathcal{O}_X$ when restricted to U. □

(b) It is clear that $\mathcal{L} \otimes \mathcal{O}_X \approx \mathcal{L} \approx \mathcal{O}_X \otimes \mathcal{L}$, so $\mathcal{O}_X$ is a unit element for the multiplication, up to isomorphism.

(c) Let $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$. Then $\mathcal{L}^\vee$ is invertible.

Proof. Restricting to a suitable open set U we may assume that $\mathcal{L} \approx \mathcal{O}_X$, in which case

$$\mathcal{H}om(\mathcal{L}, \mathcal{O}_X) \approx \mathcal{H}om(\mathcal{O}_X, \mathcal{O}_X) \approx \mathcal{O}_X.$$

□

(d) The natural map

$$\mathcal{L} \otimes \mathcal{H}om(\mathcal{L}, \mathcal{O}_X) \longrightarrow \mathcal{O}_X$$

is an isomorphism.

Proof. Again restricting to an appropriate open set U, we are reduced to proving the statement when $\mathcal{L} = \mathcal{O}_X$, in which case the assertion is immediate. □

Thus $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$, which is call the *dual sheaf*, is an inverse for $\mathcal{L}$ up to isomorphism. This proves that isomorphism classes of invertible sheaves over $\mathcal{O}_X$ form a group.

We also have the property:

(e) Let $f\colon X \to Y$ be a morphism and $\mathcal{L}$ an invertible sheaf on Y. Then $f^*\mathcal{L}$ is an invertible sheaf on X.

Definition 1.2. Let X be a scheme. We let $\operatorname{Pic}(X)$, the *Picard group*, be the group of all isomorphism classes of invertible sheaves.

Invertible sheaves and Proj are closely related because under a certain hypothesis, Proj R carries a canonical invertible sheaf, known as $\mathcal{O}_{\operatorname{Proj} R}(1)$.

Let R be a graded ring,

$$R = \bigoplus_{n \geq 0} R_n.$$

Then R is an algebra over R_0. *The hypothesis that allows us to define $\mathcal{O}_{\operatorname{Proj} R}(1)$ is that R is generated by R_1 over R_0, that is*

$$R = R_0[R_1]$$

(cf. Proposition II.5.1). *We shall make this hypothesis throughout this section.*

Example 1.3. The most basic ring of this type is obtained as in Definition II.5.6 as follows. Let A be any commutative ring, and let

$$R = A[T_0, \dots, T_r]$$

be the polynomial ring in $r+1$ variables. Then $R_0 = A$, and R_n consists of the homogeneous polynomials of degree n with coefficients in A. Furthermore R_1 is the free module over A, with basis $T_0, \dots, T_r$.

For simplicity, we abbreviate

$$\mathbb{P} = \operatorname{Proj} R.$$

To define $\mathcal{O}_{\mathbb{P}}(1)$, start with any graded module M. Then for all integer $d \in \mathbb{Z}$ we may define the d-twist $M(d)$ of M, which is the module M but with the new grading

$$M(d)_n = M_{d+n}.$$

Then we define

$$\mathcal{O}_{\mathbb{P}}(1) = \widetilde{R(1)}$$

where the $\widetilde{\ }$ is the projective $\widetilde{\ }$. If $f \in R$ is a homogeneous element, we abbreviate the open subset

$$(\operatorname{Proj} R)_f = \mathbb{P}_f \text{ or } U_f.$$

Proposition 1.4. *The sheaf $\mathcal{O}_{\mathbb{P}}(1)$ is invertible on* Proj R. *In fact: Given $f \in R_1$, the multiplication by f*

$$m_f \colon R \longrightarrow R(1)$$

is a graded homomorphism of degree 0, *whose induced sheaf homomorphism*

$$\widetilde{m_f} \colon \widetilde{R} = \mathcal{O}_{\mathbb{P}} \longrightarrow \widetilde{R(1)} = \mathcal{O}_{\mathbb{P}}(1)$$

restricts to an isomorphism on U_f. Let $\varphi_f = \widetilde{m_f}$. For $f, g \in R_1$, the sheaf map $\varphi_f^{-1} \circ \varphi_g$ is multiplication by g/f on $U_f \cap U_g$.

Proof. By definition

$$\mathcal{O}_{\mathbb{P}}(1)|_{U_f} = (\widetilde{R(1)_f})_0,$$

and we have an isomorphism

$$\text{multiplication by } f \colon R_f \longrightarrow R(1)_f.$$

This induces an isomorphism on the parts of degree 0, whence taking the affine $\widetilde{\ }$, it induces the isomorphism

$$\mathcal{O}_{\mathbb{P}}|_{U_f} \longrightarrow \mathcal{O}_{\mathbb{P}}(1)|_{U_f}.$$

In fact, the module associated with $\mathcal{O}_{\mathbb{P}}(1)$ on U_f is just given by

$$(R_f)_0 \cdot f,$$

and is consequently free of rank 1 over the affine coordinate ring of $\operatorname{Spec}(R_f)_0$. Since R is generated by R_1, the U_f's cover $\operatorname{Proj} R$, and this shows that $\mathcal{O}_{\mathbb{P}}(1)$ is invertible. □

PROPOSITION 1.5. *Let M be a graded R-module. Then the isomorphism*

$$M \otimes_R R(1) \longrightarrow M(1)$$

induces an isomorphism

$$\widetilde{M} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}(1) \longrightarrow \widetilde{M(1)}.$$

PROOF. Let $f \in R_1$. On $\mathbb{P}_f$ the isomorphism of graded modules induces the corresponding isomorphism of $(R_f)_0$-modules

$$(M_f)_0 \otimes (R(1)_f)_0 \longrightarrow (M(1))_0,$$

where the tensor product is taken over $(R_f)_0$. Taking the affine tilde yields the desired sheaf isomorphism. □

DEFINITION 1.6. For every integer d we define

$$\mathcal{O}_{\mathbb{P}}(d) = \widetilde{R(d)},$$

and for any sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathbb{P}}$-modules, we define

$$\mathcal{F}(d) = \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}(d).$$

PROPOSITION 1.7.

(i) *For $d, m \in \mathbb{Z}$ we have $\mathcal{F}(d+m) \approx \mathcal{F}(d) \otimes \mathcal{O}_{\mathbb{P}}(m)$.*

(ii) *For d positive,*

$$\mathcal{O}_{\mathbb{P}}(d) \approx \mathcal{O}_{\mathbb{P}}(1) \otimes \cdots \otimes \mathcal{O}_{\mathbb{P}}(1) \quad (\textit{product taken } d \textit{ times}).$$

(iii) *For $d \in \mathbb{Z}$ the natural pairing*

$$\mathcal{O}_{\mathbb{P}}(d) \otimes \mathcal{O}_{\mathbb{P}}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}}$$

identifies $\mathcal{O}_{\mathbb{P}}(-d)$ with the dual sheaf $\mathcal{O}_{\mathbb{P}}(d)^\vee$.

(iv) *For a graded module M, we have $\widetilde{M(d)} \approx \widetilde{M}(d)$.*

PROOF. The first assertion follows from the formula

$$(\widetilde{M \otimes_R} N) \approx \widetilde{M} \otimes_{\mathcal{O}_{\mathbb{P}}} \widetilde{N}$$

for any two graded R-modules M and N, because R is generated by R_1. Indeed, for $f \in R_1$ we have

$$(M \otimes_R N)_f = M_f \otimes_{R_f} N_f.$$

The other assertions are immediate. □

The collection of sheaves $\widetilde{M}(d)$ attached to M allows us to interpret globally each graded piece of the module M. In fact, for each d, we get a canonical homomorphism (cf. §II.5)

$$M_d = M(d)_0 \longrightarrow \Gamma(\mathbb{P}, \widetilde{M(d)}) \xrightarrow{\approx} \Gamma(\mathbb{P}, \widetilde{M}(d)).$$

For any sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathbb{P}}$-modules, we define

$$\Gamma_*(\mathcal{F}) = \bigoplus_{m \in \mathbb{Z}} \Gamma(\mathbb{P}, \mathcal{F}(m)).$$

Then we obtain a canonical homomorphism

$$M \longrightarrow \Gamma_*(\widetilde{M}).$$

In particular, when $M = R$, we get a *ring* homomorphism

$$R \longrightarrow \bigoplus_{d=0}^{\infty} \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)) = \Gamma_*(\widetilde{R}) = \Gamma_*(\mathcal{O}_{\mathbb{P}}),$$

where multiplication on the right hand side is defined by the tensor product.

We also note that $\Gamma_*(\mathcal{F})$ is a graded R-module as follows. We have the inclusion of R_d in $\Gamma(\mathbb{P}, \widetilde{R(d)})$, and the product of R_d on $\Gamma(\mathbb{P}, \mathcal{F}_{\mathbb{P}}(m))$ is induced by the tensor product

$$\Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)) \otimes \Gamma(\mathbb{P}, \mathcal{F}(m)) \longrightarrow \Gamma(\mathbb{P}, \mathcal{F}(m+d)).$$

It is not always the case that there is an isomorphism

$$\Gamma_*(\mathcal{O}_{\mathbb{P}}) \approx R,$$

so for some positive integer d, it may happen that the module of sections $\Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d))$ is larger than R_d. We now give an example when these are equal.

PROPOSITION 1.8. *Let A be a ring and $R = A[T_0, \dots, T_r]$, $r \geq 1$. Let $\mathbb{P} = \operatorname{Proj} R = \mathbb{P}^r_A$. Then for all integers $d \in \mathbb{Z}$ we have*

$$R_d \approx \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)) \quad \textit{so} \quad R \approx \Gamma_*(\mathcal{O}_{\mathbb{P}}).$$

PROOF. For $i = 0, \dots, r$ let $U_i = U_{T_i}$, so U_i is the usual affine open subscheme of $\operatorname{Proj} R$, complement of the hyperplanes $T_i = 0$. A section $s \in \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n))$ is the same as a family of sections $s_i \in \mathcal{O}_{\mathbb{P}}(n)(U_i)$ for all i, such that $s_i = s_j$ on $U_i \cap U_j$ for all i, j. But a section in $\mathcal{O}_{\mathbb{P}}(n)(U_i)$ is simply an element

$$s_i = \frac{f_i(T)}{T_i^{k(i)}}$$

where $k(i)$ is an integer and $f_i(T)$ is a homogeneous polynomial of degree $k(i)+n$. The restriction to $U_i \cap U_j$ is the image of that element in the localization $R_{T_iT_j}$. Since the elements $T_0, \dots, T_r$ are not zero-divisors in R, the natural maps

$$R \longrightarrow R_{T_i} \quad \text{and} \quad R_{T_i} \longrightarrow R_{T_iT_j}$$

are injective, and all such localized rings can be viewed as subrings of $R_{T_0 \cdots T_r}$. Hence $\Gamma_*(\mathcal{O}_{\mathbb{P}})$ is the intersection $\bigcap R_{T_i}$ taken inside $R_{T_0 \cdots T_r}$. Any homogeneous element of $R_{T_0 \cdots T_r}$ can be written in the form

$$f(T_0, \dots, T_r) T_0^{k(0)} \cdots T_r^{k(r)}$$

where $f(T_0, \dots, T_r)$ is a homogeneous polynomial not divisible by any T_i ($i = 0, \dots, r$) and $k(0), \dots, k(r) \in \mathbb{Z}$. Such an element lies in R_{T_i} if and only if $k(j) \geq 0$ for all $j \neq i$. Hence the intersection of all the R_{T_i} for $i = 0, \dots, r$ is equal to R. This proves the proposition. $\square$

The proposition both proves a result and gives an example of the previous constructions. In particular, we see that the elements $T_0, \dots, T_r$ form a basis of R_1 over A, and can be viewed as a basis of the A-module of sections $\Gamma(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}}(1))$.

Next we look at the functoriality of twists with respect to graded ring homomorphisms. As in §II.5 we let R' be a graded ring which we now assume generated by R'_1 over R'_0. Let

$$\varphi \colon R \longrightarrow R'$$

be a graded homomorphism of degree 0. Let V be the subset of $\operatorname{Proj} R'$ consisting of those primes $\mathfrak{p}'$ such that $\mathfrak{p}' \not\supset \varphi(R_+)$. Then we saw that V is open in $\operatorname{Proj} R'$, and that the inverse image map on prime ideals

$$f \colon V \longrightarrow \operatorname{Proj} R = \mathbb{P}$$

defines a morphism of schemes.

PROPOSITION 1.9. *Let* $\mathbb{P}' = \operatorname{Proj} R'$. *Then*

$$f^*\mathcal{O}_{\mathbb{P}}(d) = \mathcal{O}_{\mathbb{P}'}(d)|_V \quad \textit{and} \quad f_*(\mathcal{O}_{\mathbb{P}'}(d)|_V) = (f_*\mathcal{O}_V)(d).$$

PROOF. These assertions about the twists hold more generally for any graded R-module M, because

$$f^*(\widetilde{M}) \approx (\widetilde{M \otimes_R R'})|_V$$

and for any graded R'-module N, we have

$$f_*(\widetilde{N}|_V) \approx \widetilde{(N_R)},$$

where N_R is N viewed as R-module via φ. The proof is routine and left to the reader. $\square$

To conclude this section we note that everything we have said extends to the global Proj without change. Instead of $\operatorname{Proj} R$, we can consider $\mathcal{P}roj_X\,\mathcal{R}$ where $\mathcal{R}$ is a quasi-coherent graded sheaf of $\mathcal{O}_X$-algebras. We need to make the hypothesis that $\mathcal{R}_n$ is generated by $\mathcal{R}_1$ over $\mathcal{R}_0$, i.e., the multiplication map

$$\operatorname{Symm}^n_{\mathcal{R}_0} \mathcal{R}_1 \longrightarrow \mathcal{R}_n$$

is surjective. Let $\mathbb{P} = \mathcal{P}roj_X\,\mathcal{R}$. Then if $\mathcal{M}$ is a quasi-coherent graded sheaf of $\mathcal{R}$-modules, we define $\mathcal{M}(d)$ by

$$\mathcal{M}(d)_n = \mathcal{M}_{d+n}.$$

Then let

$$\mathcal{O}_{\mathbb{P}}(d) = \widetilde{\mathcal{R}(d)}$$

and for every quasi-coherent $\mathcal{F}$ on $\mathbb{P}$, let

$$\mathcal{F}(d) = \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}(d).$$

As before, $\mathcal{O}_{\mathbb{P}}(1)$ is invertible, with powers $\mathcal{O}_{\mathbb{P}}(d)$ and

$$\widetilde{\mathcal{M}(d)} = (\widetilde{\mathcal{M}})(d).$$

The extension of the definition of $\Gamma_*(\mathcal{F})$ to the global case is:

$$\pi_*^{\mathrm{gr}}\mathcal{F} = \bigoplus_{m\in\mathbb{Z}} \pi_*\mathcal{F}(m)$$

where π is the projection of $\mathcal{P}roj_X\,\mathcal{R}$ to X. This is quasi-coherent provided that $\mathcal{R}_1$ is finitely generated as $\mathcal{R}_0$-modules, since this implies that π is quasi-compact, hence Proposition II.4.10 applies. As above, we have a natural graded homomorphism

$$\mathcal{M} \longrightarrow \pi_*^{\mathrm{gr}}(\widetilde{\mathcal{M}}).$$

Finally Proposition 1.8 globalizes immediately to:

PROPOSITION 1.10. *Let* $\mathcal{E}$ *be a locally free sheaf of* $\mathcal{O}_X$*-module and consider* $\mathbb{P}(\mathcal{E}) = \mathcal{P}roj_X(\operatorname{Symm}^* \mathcal{E})$. *Then the natural homomorphism*

$$\operatorname{Symm}^d \mathcal{E} \longrightarrow \pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(d)$$

is an isomorphism. In particular, $\operatorname{Symm}^* \mathcal{E} \approx \pi_*^{\mathrm{gr}}\mathcal{O}_{\mathbb{P}(\mathcal{E})}$.

2. The functor of $\operatorname{Proj} R$

Throughout this section we let R be a graded ring, generated by R_1 over R_0. We let $S = \operatorname{Spec}(R_0)$, $\mathbb{P} = \operatorname{Proj} R$ and let $\pi\colon \mathbb{P} \to S$ be the canonical projection.

An important example of a graded ring R as above is $\operatorname{Symm}_{R_0}(R_1)$, namely the symmetric algebra, but we shall meet other cases, so we do not restrict our attention to this special case.

We are interested in schemes X over S, and in morphisms of X into $\operatorname{Proj} R$ over S:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathbb{P} = \operatorname{Proj} R \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

In the simplest case, $\mathbb{P} = \mathbb{P}^r_{R_0}$ and f becomes a morphism of X into projective space.

Given such a morphism $f\colon X \to \mathbb{P}$, we can take the inverse image $f^*\mathcal{O}_{\mathbb{P}}(1)$, which is an invertible sheaf on X. By the general formalism of inverse images of sheaves, this induces a natural map on global sections

$$f^*\colon \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \longrightarrow \Gamma(X, f^*\mathcal{O}_{\mathbb{P}}(1)),$$

and in light of the natural map $R_1 \to \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$ induces a homomorphism

$$\varphi_f = \varphi\colon R_1 \longrightarrow \Gamma(X, f^*\mathcal{O}_{\mathbb{P}}(1)).$$

Thus to each morphism $f\colon X \to \mathbb{P}$ we have associated a pair $(\mathcal{L}, \varphi)$ consisting of an invertible sheaf $\mathcal{L}$ (in this case $f^*\mathcal{O}_{\mathbb{P}}(1)$) and a homomorphism

$$\varphi\colon R_1 \longrightarrow \Gamma(X, \mathcal{L}).$$

To describe an additional important property of this homomorphism, we need a definition.

Definition 2.1. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Let $\{s_i\}$ be a family of sections. We say that this family *generates* $\mathcal{F}$ if any one of the following conditions is satisfied:

(1) For every point $x \in X$ the family of images $\{(s_i)_x\}$ generates $\mathcal{F}_x$ as an $\mathcal{O}_x$-module, or equivalently (by Nakayama's lemma Proposition I.5.5) $\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$.

(2) For each point $x \in X$ there exists some open neighborhood U of x such that the sections $\{s_i|_U\}$ generate $\mathcal{F}(U)$ over $\mathcal{O}_X(U)$.

Note that by Proposition 1.4, if $g \in R_1$, then over the open set $(\operatorname{Proj} R)_g$ of $\mathbb{P} = \operatorname{Proj} R$ the section $g \in \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$ generates the sheaf $\mathcal{O}_{\mathbb{P}}(1)$. Since these open sets cover the scheme $\mathbb{P}$, it follows that the collection of global sections R_1 of $\mathcal{O}_{\mathbb{P}}(1)$ generates $\mathcal{O}_{\mathbb{P}}(1)$ everywhere (see Nakayama's lemma Proposition I.5.5), or equivalently that

$$\pi^* R_1 \longrightarrow \mathcal{O}_{\mathbb{P}}(1)$$

is surjective.

From the definition of the inverse image f^*, which is locally given by the tensor product, it follows that the inverse image f^*R_1 generates $f^*\mathcal{O}_{\mathbb{P}}(1)$.

Thus finally, to each morphism $f\colon X \to \operatorname{Proj} R$ we have associated a pair $(\mathcal{L}, \varphi)$ consisting of an invertible sheaf $\mathcal{L}$ on X and a homomorphism

$$\varphi\colon R_1 \longrightarrow \Gamma(X, \mathcal{L})$$

such that $\varphi(R_1)$ generates $\mathcal{L}$, or equivalently, the homomorphism

$$f^*R_1 \longrightarrow \mathcal{L}$$

is surjective.

Theorem 2.2. *Let $\mathbb{P} = \operatorname{Proj} R$. Assume that $R = \operatorname{Symm}_{R_0}(R_1)$. Let $S = \operatorname{Spec}(R_0)$. Let $p\colon X \to S$ be a scheme over S and let $(\mathcal{L}, \varphi)$ be a pair consisting of an invertible sheaf $\mathcal{L}$ on X and a homomorphism*

$$\varphi\colon R_1 \longrightarrow \Gamma(X, \mathcal{L})$$

which generates $\mathcal{L}$. Then there exists a unique pair (f, ψ) consisting of a morphism $f\colon X \to \operatorname{Proj} R$ and a homomorphism $\psi\colon f^*\mathcal{O}_{\mathbb{P}}(1) \to \mathcal{L}$ making the following diagram commutative:

$$\begin{tikzcd} R_1 \arrow[r, "\varphi"] \arrow[d] & \Gamma(X, \mathcal{L}) \\ \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \arrow[r, "f^*"'] & \Gamma(X, f^*\mathcal{O}_{\mathbb{P}}(1)) \arrow[u, "\Gamma(\psi)"'] \end{tikzcd}$$

Furthermore, the homomorphism ψ is an isomorphism.

Before giving the proof, we make some comments. An important special case occurs when R_1 is a free module of finite rank $r+1$ over R_0. Then $\mathbb{P} = \mathbb{P}^r_{R_0}$. The R_0-module R_1 then has a basis $T_0, \ldots, T_r$. Let $s_0, \ldots, s_r$ be sections of $\mathcal{L}$ which generate $\mathcal{L}$. There is a unique homomorphism $\varphi\colon R_1 \to \Gamma(X, \mathcal{L})$ such that $\varphi(T_i) = s_i$. The theorem asserts that there is a unique morphism $f\colon X \to \mathbb{P}^r_{R_0}$ such that $f^*\mathcal{O}_{\mathbb{P}}(1)$ is isomorphic to $\mathcal{L}$, and the sections s_i correspond to f^*T_i under this isomorphism. This is the formulation of the theorem in terms of the homogeneous coordinates $T_0, \ldots, T_r$.

The proof of Theorem 2.2 will require some lemmas. We first consider the uniqueness, and for this the hypothesis that $R = \operatorname{Symm}_{R_0}(R_1)$ will *not* be used.

Let s be a section of an invertible sheaf $\mathcal{L}$ over the scheme X. Let s_x be the value of the section in $\mathcal{L}_x$, and let $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}_x$. Then s_x generates L_x if and only if $s_x \notin \mathfrak{m}_x\mathcal{L}_x$.

Lemma 2.3. *Let $\mathcal{L}$ be an invertible sheaf on the scheme X. Let $s \in \Gamma(X, \mathcal{L})$ be a global section of $\mathcal{L}$. Then the set of points $x \in X$ such that s_x generates $\mathcal{L}_x$ is an open set which we denote by X_s. Multiplication by s, that is,*

$$m_s\colon \mathcal{O}_X|_U \longrightarrow \mathcal{L}|_U$$

is an isomorphism on this open set.

Proof. We may suppose that $X = \operatorname{Spec}(A)$, and $\mathcal{L} = \mathcal{O}_X$ since the conclusions of the lemma are local. Then $s \in A$. The first assertion is then obvious from the definition of $\operatorname{Spec}(A)$. As to the second, s is a unit in A_s so multiplication by s induces an isomorphism on the sheaf on the open subset $\operatorname{Spec}(A_s)$. This proves the lemma. (No big deal.) $\square$

To show uniqueness, we suppose given the pair

$$f\colon X \longrightarrow \operatorname{Proj} R \quad \text{and} \quad \varphi\colon R_1 \longrightarrow \Gamma(X, \mathcal{L}),$$

and investigate the extent to which f is determined by φ. Note that for all $a \in R_1$ the map f restricts to a morphism

$$X_{\varphi(a)} = f^{-1}((\operatorname{Proj} R)_a) \longrightarrow (\operatorname{Proj} R)_a$$

and

$$(\operatorname{Proj} R)_a = \operatorname{Spec}(R_a)_0.$$

If $b \in R_1$, then the map φ sends b, a to $\varphi(b)$, $\varphi(a)$ respectively, and so

$$f^*\colon \frac{b}{a} \longmapsto \frac{\varphi(b)}{\varphi(a)} = m_a^{-1}(\varphi(b)).$$

But the set of elements b/a with $b \in R_1$ generates $(R_a)_0$. Consequently the ring homomorphism

$$(R_a)_0 \longrightarrow \Gamma(X_{\varphi(a)}, \mathcal{O}_X)$$

is uniquely determined by φ. This proves the uniqueness.

Next we wish to show existence. The next lemma still does not use that $R = \mathrm{Symm}_{R_0}(R_1)$.(??)

LEMMA 2.4. *Let R be a graded ring generated by R_1 over R_0. Let $a \in R_1$. Then there is a unique (not graded) ring homomorphism*

$$R/(a-1) \xrightarrow{\approx} (R_a)_0$$

such that for $b \in R_1$ we have

$$b \longmapsto \frac{b}{a}$$

PROOF OF LEMMA 2.4. The map $b \mapsto b/a$ defines an additive homomorphism of R_1 into $(R_a)_0$. Consequently this additive maps extends uniquely to a ring homomorphism (??)

$$h\colon R \longrightarrow (R_a)_0,$$

and $a-1$ is in the kernel. Since a becomes invertible under the map $R \to R/(a-1)$, we can factor h as follows:

$$R \longrightarrow R_a \longrightarrow R/(a-1) \longrightarrow (R_a)_0.$$

The first map is the natural map of R into the localization of R by a. Since R_1 generates R, any element of the homogeneous component R_n can be written as a sum of elements in the form $b_1 \cdots b_n$ for some $b_i \in R_1$, so an element of $(R_a)_0$ is a sum of elements of the form

$$\frac{b_1 \cdots b_n}{a^n} = \left(\frac{b_1}{a}\right) \cdots \left(\frac{b_n}{a}\right).$$

Since $(R_a)_0$ is contained in R_a, it follows that the composite map

$$(R_a)_0 \xrightarrow{\text{inclusion}} R_a \longrightarrow R/(a-1) \longrightarrow (R_a)_0$$

is the identity. Furthermore given an element in $R/(a-1)$ represented by a product $b_1 \cdots b_n$ with $b_i \in R_1$, it is the image of an element in $(R_a)_0$ since $a \equiv 1 \mod (a-1)$. Hence the map

$$(R_a)_0 \longrightarrow R/(a-1)$$

is an isomorphism. This concludes the proof of Lemma 2.4. $\square$

We revert to the existence part of Theorem 2.2. Given the data $(\mathcal{L}, \varphi)$ we wish to construct the morphism

$$f\colon X \longrightarrow \mathrm{Proj}\, R.$$

For each $a \in R_1$ we let $X_{\varphi(a)}$ be the open set of points $x \in X$ such that $\varphi(a)(x) \neq 0$ (we are using Lemma 2.3). Since $\varphi(R_1)$ generates $\mathcal{L}$, it follows that the sets $X_{\varphi(a)}$ cover X for $a \in R_1$. On the other hand,

$$\mathrm{Proj}\, R = \bigcup_{a \in R_1} \mathrm{Spec}(R_a)_0.$$

It will suffice to construct for each $a \in R_1$ a morphism

$$X_{\varphi(a)} \longrightarrow \mathrm{Spec}(R_a)_0 \subset \mathrm{Proj}\, R$$

such that this family is compatible on the intersections of the open sets $X_{\varphi(a)}$. The construction is done for the corresponding rings of global sections. By restriction from X to $X_{\varphi(a)}$ the map φ gives rise to a map

$$\varphi_a\colon R_1 \longrightarrow \Gamma(X_{\varphi(a)}, \mathcal{L}).$$

Composing with the multiplication m_a^{-1} as in Lemma 2.3, we obtain a homomorphism $R_1 \to \Gamma(X_{\varphi(a)}, \mathcal{O}_X)$ as in the following triangle:

$$\begin{tikzcd} R_1 \arrow[rr] \arrow[dr] & & \Gamma(X_{\varphi(a)}, \mathcal{L}) \arrow[dl, "m_a^{-1}"] \\ & \Gamma(X_{\varphi(a)}, \mathcal{O}_X) & \end{tikzcd}$$

But m_a^{-1} sends $\varphi(a)$ to the section represented by 1. By the assumption that $R = \mathrm{Symm}_{R_0}(R_1)$, the additive R_0-homomorphism

$$R_1 \longrightarrow \Gamma(X_{\varphi(a)}, \mathcal{O}_X)$$

induces a ring homomorphism

$$\psi_a\colon R/(a-1) = (R_a)_0 \longrightarrow \Gamma(X_{\varphi(a)}, \mathcal{O}_X).$$

This is the homomorphism of global sections that we wanted. Then ψ_a induces a morphism

$$f_a\colon X_{\varphi(a)} \longrightarrow \mathrm{Spec}(R_a)_0.$$

We now leave to the reader the verification that these morphisms are compatible on the intersections of two open subschemes $X_{\varphi(a)} \cap X_{\varphi(b)}$. From the construction, it is also easy to verify that the morphism

$$f\colon X \longrightarrow \mathrm{Proj}\, R$$

obtained by glueing the morphisms f_a together has the property that

$$f^*\mathcal{O}_{\mathbb{P}}(1) = \mathcal{L},$$

and that the original map φ is induced by f^*. This proves the existence.

Finally, the fact that ψ is an isomorphism results from the following lemma.

LEMMA 2.5. *Let $\psi\colon \mathcal{L}' \to \mathcal{L}$ be a surjective homomorphism of invertible sheaves. Then ψ is an isomorphism.*

PROOF. The proof is immediate and will be left to the reader. □

We used the assumption that $R = \mathrm{Symm}_{R_0}(R_1)$ only once in the proof. In important applications, like those in the next section, we deal with a ring R which is not $\mathrm{Symm}_{R_0}(R_1)$, and so we give another stronger version of the result with a weaker, but slightly more complicated hypothesis.

The symmetric algebra had the property that a module homomorphism on R_1 induces a ring homomorphism on R. We need a property similar to this one. We have the graded ring

$$\Gamma_*(\mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^n),$$

where $\mathcal{L}^n = \mathcal{L}^{\otimes n}$ is the tensor product of $\mathcal{L}$ with itself n times. The R_0-homomorphism $\varphi\colon R_1 \to \Gamma(\mathcal{L})$ induces a graded algebra homomorphism

$$\mathrm{Symm}(\varphi)\colon \mathrm{Symm}_{R_0}(R_1) \longrightarrow \Gamma_*(\mathcal{L}).$$

We say that $\mathrm{Symm}(\varphi)$ *factors through* R if there is a commutative diagram of graded algebras

$$\begin{tikzcd} \mathrm{Symm}_{R_0}(R_1) \arrow[rr] \arrow[dr] & & \Gamma_*(\mathcal{L}) \\ & R \arrow[ur] & \end{tikzcd}$$

so for each n we have a commutative diagram:

$$\begin{tikzcd} \mathrm{Symm}^n_{R_0}(R_1) \arrow[rr, "\mathrm{Symm}^n(\varphi)"] \arrow[dr] & & \Gamma(\mathcal{L}^n) \\ & R_n \arrow[ur] & \end{tikzcd}$$

THEOREM 2.6. *Theorem 2.2 is valid without change except that instead of assuming* $R = \mathrm{Symm}_{R_0}(R_1)$ *we need only assume that* $\mathrm{Symm}(\varphi)$ *factors through* R.

PROOF. The proof is the same, since the hypothesis that $\mathrm{Symm}(\varphi)$ factors through R can be used instead of $R = \mathrm{Symm}_{R_0}(R_1)$. □

COROLLARY 2.7. *Let* $\mathcal{E}$ *be a locally free sheaf on the scheme* X. *Then sections* $s\colon X \to \mathbb{P}_X\mathcal{E} = \mathcal{P}roj_X(\mathrm{Symm}_{\mathcal{O}_X}(\mathcal{E}))$ *are in bijection with surjective homomorphisms*

$$\mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$$

of $\mathcal{E}$ *onto invertible sheaves over* X.

PROOF. Take $X = S$ in Theorem 2.2. □

Let $\mathcal{R}$ be a quasi-coherent graded sheaf of $\mathcal{O}_X$-algebras, and let $\mathbb{P} = \mathcal{P}roj_X\,\mathcal{R}$. We have a canonical homomorphism

$$\mathcal{R}_1 \longrightarrow \pi_*\mathcal{O}_{\mathbb{P}}(1)$$

or equivalently (cf. Lemma (I.5.11))

$$\pi^*\mathcal{R}_1 \longrightarrow \mathcal{O}_{\mathbb{P}}(1)$$

which is surjective. This leads to the following generalization of Theorem 2.2:

THEOREM 2.8. *Let* $p\colon Z \to X$ *be a scheme over* X *and let* $\mathcal{L}$ *be an invertible sheaf on* Z. *Let*

$$h\colon p^*\mathcal{R}_1 \longrightarrow \mathcal{L}$$

be a surjective homomorphism. Assume in addition that $\mathcal{R} = \mathrm{Symm}_{\mathcal{R}_0}(\mathcal{R}_1)$ *or that* $\mathrm{Symm}(h)$ *factors through* $\mathcal{R}$. *Then there exists a unique pair* (f, ψ) *consisting of a morphism*

$$f\colon Z \longrightarrow \mathcal{P}roj_X(\mathcal{R}) = \mathbb{P}$$

and a homomorphism

$$\psi\colon f^*\mathcal{O}_{\mathbb{P}}(1) \longrightarrow \mathcal{L}$$

making the following diagram commutative:

$$\begin{tikzcd} f^*\pi^*(\mathcal{R}_1) = p^*(\mathcal{R}_1) \arrow[rr, "f^*(\text{canonical})"] \arrow[dr, "h"'] & & f^*\mathcal{O}_{\mathbb{P}}(1) \arrow[dl, "\psi"] \\ & \mathcal{L} & \end{tikzcd}$$

In other words, $h\colon p^*(\mathcal{R}_1) \to \mathcal{L}$ *is obtained from* $\pi^*(\mathcal{R}_1) \to \mathcal{O}_{\mathbb{P}}(1)$ *by applying* f^* *and composing with* ψ. *Furthermore, this homomorphism* ψ *is an isomorphism.*

3. Blow ups

This section provides examples for Proj of some graded rings, in one of the major contexts of algebraic geometry.

Throughout this section, we let X be a scheme.

Let $\mathcal{I}$ be a quasi-coherent sheaf of ideals of $\mathcal{O}_X$. We may then form the sheaf of graded algebras

$$\mathcal{R} = \bigoplus_{n \geq 0} \mathcal{I}^n$$

where by definition $\mathcal{I}^0 = \mathcal{O}_X$. Then $\mathcal{R}$ satisfies the hypotheses stated at the beginning of §2, so the results of §2 apply to such $\mathcal{R}$. The sheaf of ideals $\mathcal{I}$ defines a closed subscheme Y whose structure sheaf is

$$\mathcal{O}_Y = \mathcal{O}_X / \mathcal{I}.$$

We define the *blow up of X along Y*, or *with respect to $\mathcal{I}$*, to be:

$$\mathrm{Bl}_Y(X) = \mathcal{P}roj_X\, \mathcal{R}.$$

Let

$$\pi \colon \mathrm{Bl}_Y(X) \longrightarrow X$$

be the structural morphism.

We recall that we defined the inverse image of a sheaf in §I.5. Let

$$f \colon X' \longrightarrow X$$

be a morphism. Let $\mathcal{I}$ be a sheaf of ideals of $\mathcal{O}_X$. Then we have homomorphism

$$f^*\mathcal{I} \longrightarrow f^*\mathcal{O}_X = \mathcal{O}_{X'}$$

as defined in §I.5. We let

$$f^{-1}(\mathcal{I})\mathcal{O}_{X'} \quad \text{or also} \quad \mathcal{I}\mathcal{O}_{X'}$$

to be the image of this homomorphism. Then $\mathcal{I}\mathcal{O}_{X'}$ is a quasi-coherent sheaf of ideals of $\mathcal{O}_{X'}$.

THEOREM 3.1. *Let $X' = \mathrm{Bl}_Y(X)$ be the blow up of X along Y, where Y is the closed subscheme defined by a sheaf of ideals $\mathcal{I}$, and let $\pi \colon X' \to X$ be the structural morphism.*

i) *The morphism π gives an isomorphism*

$$X' \setminus \pi^{-1}(Y) \xrightarrow{\approx} X \setminus Y.$$

ii) *The inverse image sheaf $\mathcal{I}\mathcal{O}_{X'}$ is invertible, and in fact*

$$\mathcal{I}\mathcal{O}_{X'} = \mathcal{O}_{X'}(1).$$

PROOF. The first assertion is immediate since $\mathcal{I} = \mathcal{O}_X$ on the complement of Y by definition. So if we put $U = X \setminus Y$, then

$$\pi^{-1}(U) = \mathcal{P}roj_U\, \mathcal{O}_U[T] = U.$$

For (ii), we note that for any affine open set V in X, the sheaf $\mathcal{O}_{X'}(1)$ on $\mathrm{Proj}(\mathcal{R}(V))$ is the sheaf associated to the graded $\mathcal{R}(V)$-module

$$\mathcal{R}(V)(1) = \bigoplus_{n \geq 0} \mathcal{I}^{n+1}(V).$$

But this is equal to the ideal $\mathcal{I}\mathcal{R}(V)$ generated by $\mathcal{I}(V)$ in $\mathcal{R}(V)$. This proves (ii), and the concludes the proof of the theorem. $\square$

THEOREM 3.2 (Universality of Blow-ups). *Let*

$$\pi\colon \mathrm{Bl}_Y(X) \longrightarrow X$$

be the blow up of a sheaf of ideals $\mathcal{I}$ in X. Let

$$f\colon Z \longrightarrow X$$

be a morphism such that $\mathcal{I}\mathcal{O}_Z$ is an invertible sheaf of ideals on Z. Then there exists a unique morphism $f_1\colon Z \to \mathrm{Bl}_Y(X)$ such that the following diagram is commutative.

$$\begin{tikzcd} Z \arrow[rr, "f_1"] \arrow[dr, "f"'] & & \mathrm{Bl}_Y(X) \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

PROOF. To construct f_1 we use Theorem 2.8, taking $\mathcal{L} = \mathcal{I}\mathcal{O}_Z$ and h to be the natural map

$$h\colon f^*\mathcal{R}_1 = f^*\mathcal{I} \longrightarrow \mathcal{I}\mathcal{O}_Z = \mathcal{L}.$$

Note that $\mathrm{Symm}(h)$ factors through $\bigoplus \mathcal{L}^n$.

To see that f_1 is unique, take a sufficiently small affine open piece $\mathrm{Spec}(R)$ of Z in which $\mathcal{I}\mathcal{O}_Z$ is $\widetilde{(aR)}$, $a \in \mathcal{I}$. Then a is non-zero divisor in R by hypothesis. Now $\mathrm{Spec}(R_a)$ lies over $X \setminus Y$, over which π is an isomorphism:

$$\begin{tikzcd} \mathrm{Spec}(R_a) \arrow[dr] & & \mathrm{Bl}_Y(X) \setminus \pi^{-1}(Y) \arrow[dl, "\approx"', "\pi"] \\ & X \setminus Y & \end{tikzcd}$$

Therefore f_1 is unique on $\mathrm{Spec}(R_a)$. But since a is not a zero-divisor, any morphism on $\mathrm{Spec}(R_a)$ has at most one extension to $\mathrm{Spec}(R)$. This is because $R \to R_a$ is injective and hence a homomorphism $S \to R$ is determined by the composition $S \to R_a$. This concludes the proof. $\square$

THEOREM 3.3. *Let Y' be the restriction of* $\mathrm{Bl}_Y(X)$ *to Y, or in other words*

$$Y' = Y \times_X \mathrm{Bl}_Y(X).$$

Then $Y' = \mathcal{P}roj_Y \mathrm{Gr}_{\mathcal{I}}(\mathcal{O}_X)$ where $\mathrm{Gr}_{\mathcal{I}}(\mathcal{O}_X) = \bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$. In other words we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{P}roj_Y \mathrm{Gr}_{\mathcal{I}}(\mathcal{O}_X) = Y' \arrow[r] \arrow[d] & \mathrm{Bl}_Y(X) = \mathcal{P}roj_X(\bigoplus \mathcal{I}^n) \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

PROOF. Let $\mathcal{R} = \bigoplus_{n\geq 0} \mathcal{I}^n$ as before. Then $\mathcal{I}\mathcal{R} = \bigoplus_{n\geq 0} \mathcal{I}^{n+1}$, where $\mathcal{I}^{n+1}$ is the n-th graded component, and is a homogeneous ideal sheaf of $\mathcal{R}$. The restriction to Y is given by the graded ring homomorphism

$$\mathcal{R} \longrightarrow \mathcal{R}/\mathcal{I}\mathcal{R},$$

which induces the restriction of $\mathcal{P}roj_X(\mathcal{R})$ to Y. Hence this restriction is equal to $\mathcal{P}roj_Y(\mathcal{R}/\mathcal{I}\mathcal{R})$, viewing $\mathcal{R}/\mathcal{I}\mathcal{R}$ as an $\mathcal{O}_X/\mathcal{I} = \mathcal{O}_Y$-sheaf of graded algebras. But

$$\mathcal{R}/\mathcal{I}\mathcal{R} = \bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}.$$

This proves the theorem. $\square$

In general, nothing much more can be said about the sheaf

$$\mathrm{Gr}_{\mathcal{I}}(\mathcal{O}_X) = \bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}.$$

However, under some conditions, this sheaf is the symmetric algebra of $\mathcal{I}/\mathcal{I}^2$. Assume that A is a noetherian ring and I an ideal of A. We say that a sequence of elements $(a_1, \dots, a_r)$ is a *regular sequence* in I if a_1 is not a divisor of 0, and if a_{i+1} is not a divizor of 0 in $I/(a_1, \dots, a_i)$ for all $i \geq 1$.

LEMMA 3.4. *Assume that I is generated by a regular sequence of length r. Then there is a natural isomorphism*

$$\mathrm{Symm}_{A/I}(I/I^2) \approx \bigoplus_{n\geq 0} I^n/I^{n+1}$$

and I/I^2 is free of dimension r over A/I.

PROOF. See Matsumura [**69**, Chapter 6]. □

Now suppose X is a noetherian scheme and $\mathcal{I}$ is a sheaf of ideals as before, defining the subscheme Y. We say that Y is *locally complete intersection in X of codimension* r if each point $y \in Y$ has an affine open neighborhood $\mathrm{Spec}(A)$ in X, such that if I is the ideal corresponding to $\mathcal{I}$ over $\mathrm{Spec}(A)$, then I is generated by a regular sequence of length r. The elementary commutative algebra of regular sequences shows that if this condition is true over $\mathrm{Spec}(A)$, then it is true over $\mathrm{Spec}(A_f)$ for any element $f \in A$. Lemma 3.4 then globalizes to an isomorphism

$$\mathrm{Symm}_Y(\mathcal{I}/\mathcal{I}^2) \approx \mathrm{Gr}_{\mathcal{I}}(\mathcal{O}_X) = \bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}.$$

Furthermore $\mathcal{I}/\mathcal{I}^2$ is locally free of rank r over $\mathcal{O}_Y$. Therefore we may rephrase Theorem 3.3 as follows:

THEOREM 3.5. *Suppose that Y is a local complete intersection of codimension r in X, and is defined by the sheaf of ideals $\mathcal{I}$. Let Y' be the restriction of $\mathrm{Bl}_Y(X)$ to Y. Then we have a commutative diagram:*

$$\begin{tikzcd} Y' = \mathbb{P}_Y(\mathcal{I}/\mathcal{I}^2) \arrow[r] \arrow[d] & \mathrm{Bl}_Y(X) \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

In particular, if y is a closed local complete intersection point, then

$$\mathbb{P}_y(\mathcal{I}/\mathcal{I}^2) = \mathbb{P}^r_k$$

where k is the residue class field of the point. Thus the fibre of the blow up of such a point is a projective space.

We shall now apply blow ups to resolve indeterminacies of rational maps.

Let X be a noetherian scheme and let $\mathcal{L}$ be an invertible sheaf on X. Let $s_0, \dots, s_r$ be global sections of $\mathcal{L}$. By Lemma 2.3, the set of points $x \in X$ such that $(s_0)_x, \dots, (s_r)_x$ generate $\mathcal{L}_x$ is an open set U_s, and these sections generate $\mathcal{L}$ over U_s. Here s denotes the r-tuple

$$s = (s_0, \dots, s_r).$$

Then s defines a morphism

$$f_s \colon U_s \longrightarrow \mathbb{P}^r_X$$

of U_s into projective r-space over X, in line with Theorem 2.2 and the remarks following it. We shall now define a closed subscheme of X whose support is the complement of U_s, and we shall define a canonical blow up (depending on the given sections) so that the morphism f_s extends to a morphism of this blow up.

Let $s_0, \dots, s_r$ be sections of $\mathcal{L}$. We shall define an associated sheaf of ideals $\mathcal{I}_s$ as follows. Let U be an open affine set where $\mathcal{L}$ is free, and so

$$\mathcal{L}|_U \approx \mathcal{O}_X|_U.$$

Under this isomorphism, the sections become sections of $\mathcal{O}_X$ over U. We let $\mathcal{I}_U$ be the sheaf of ideals generated by these sections over U. If $U = \operatorname{Spec}(A)$, then the sections can be identified with elements of A, and the ideal corresponding to this sheaf is the ideal $(s_0, \dots, s_r)$ generated by these elements. It is immediately verified that this ideal is independent of the trivialization of $\mathcal{L}|_U$, and that the sheaf $\mathcal{I}_U$ agrees with the similarly defined sheaf $\mathcal{L}|_V$ on the intersection $U \cap V$ of two affine open sets U and V. This is the *sheaf of ideals* which we call $\mathcal{I}_s$, *determined by* or *associated with the family of sections* s.

Since X is assumed noetherian, $\mathcal{I}_s$ is a coherent sheaf of ideals, or in other words, it is locally finitely generated.

U_s is the open subset of X which is the complement of the support of $\mathcal{I}_s$. Thus $\mathcal{I}_s$ defines a closed subscheme Y, and U_s is the complement of Y. We view U_s as a scheme, whose structure sheaf is $\mathcal{O}_X|_{U_s}$.

PROPOSITION 3.6. *Let $s = (s_0, \dots, s_r)$ be sections of an invertible sheaf $\mathcal{L}$ over X as above. Let $\mathcal{I} = \mathcal{I}_s$ be the associated sheaf of ideals, defining the subscheme Y, and let $\pi\colon X' \to X$ be the blow up of X along Y. Then the sections $\pi^*s_0, \dots, \pi^*s_r$ generate an invertible subsheaf of $\pi^*\mathcal{L}$, and thus define a morphism*

$$f_{\pi^*s}\colon X' \longrightarrow \mathbb{P}^r_X,$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \pi^{-1}(U_s) & \xrightarrow{f_{\pi^*s}} & \mathbb{P}^r_X \\ \text{isomorphism} \updownarrow & & \uparrow \text{inclusion} \\ U_s & \xrightarrow[f_s]{} & \mathbb{P}^r_{U_s} \end{array}$$

PROOF. By Theorem 3.1 we know that $\mathcal{I}\mathcal{O}_X$ is invertible, and the sections $\pi^*s_0, \dots, \pi^*s_r$ generate this subsheaf of $\pi^*\mathcal{L}$.

Thus the assertion of the proposition is immediate. $\square$

In this manner, we have a globally defined morphism on the blow up X' which "coincides" with f_s on the open set U_s.

4. Quasi-coherent sheaves on Proj R

Throughout this section we let R be a graded ring, generated by R_1 over R_0 with R_0 noetherian (?). We let $\mathbb{P} = \operatorname{Proj} R$. We assume moreover that R_1 is a finitely generated R_0-module, hence $\mathbb{P}$ is quasi-compact.

The purpose of this section is to classify quasi-coherent sheaves in terms of graded modules on projective schemes in a manner analogous to the classification of quasi-coherent sheaves in terms of ordinary modules over affine schemes. We start with a lemma.

Let $\mathcal{L}$ be an invertible sheaf on a scheme X. Let $f \in \Gamma(X, \mathcal{L})$ be a section. We let:

$$X_f = \text{set of points } x \text{ such that } f(x) \neq 0.$$

We recall that $f(x)$ is the value of f in $\mathcal{L}_x/\mathfrak{m}_x\mathcal{L}_x$, as distinguished from $f_x \in \mathcal{L}_x$.

LEMMA 4.1. *Let $\mathcal{L}$ be an invertible sheaf on the scheme X. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Assume X is quasi-compact.*

i) *Let $s \in \Gamma(X, \mathcal{F})$ be a section whose restriction to X_f is 0. Then for some $n > 0$ we have $f^n s = 0$, where $f^n s \in \Gamma(\mathcal{L}^n \otimes \mathcal{F}) \approx \Gamma(\mathcal{F} \otimes \mathcal{L}^n)$.*
ii) *Suppose X has a finite covering by open affine subsets U_j such that $\mathcal{L}|_{U_j}$ is free for each j. Let $t \in \Gamma(X_f, \mathcal{F})$ be a section over X_f. Then there exists $n > 0$ such that the section $f^n t \in \Gamma(X_f, \mathcal{F} \otimes \mathcal{L}^n)$ extends to a global section of $\mathcal{F} \otimes \mathcal{L}^n$ over X.*

PROOF. There is a covering of X by affine open sets on which $\mathcal{L}$ is free, and since X is assumed quasi-compact, we can take this covering to be finite. Hence it suffices to prove that if $U = \operatorname{Spec}(A)$ is affine open such that $\mathcal{L}|_U$ is free, then there is some $n > 0$ such that $f^n s = 0$ on U. But $\mathcal{F}|_U = \widetilde{M}$ with some A-module M by Proposition-Definition I.5.1. Then we can view s as an element of M, and f as an element of A under an isomorphism $\mathcal{L}|_U \approx \mathcal{O}_X|_U$. By definition of the localization, the fact that the restriction of s to X_f is 0 means that s is 0 in M_f, and so there is some n such that $f^n s = 0$. This has an intrinsic meaning in $\mathcal{L}^n \otimes \mathcal{F}$, independently of the choice of trivialization of $\mathcal{L}$ over U, whence (i) follows.

For (ii), let $t \in \Gamma(X_f, \mathcal{F})$. We can cover X by a finite number of affine open $U_i = \operatorname{Spec}(A_i)$ such that $\mathcal{L}|_{U_i}$ is free. On each U_i there is an A_i-module M_i such that $\mathcal{F}|_{U_i} = \widetilde{M}_i$. The restriction of t to $X_f \cap U_i = (U_i)_f$ is in $(M_i)_{f_i}$, where $f_i = f|_{U_i}$ can be viewed as an element of A_i since $\mathcal{L}|_{U_i}$ is free of rank one. By definition of the localization, for each i there is an integer n and a section $t_i \in \Gamma(U_i, \mathcal{F})$ such that the restriction of t_i to $(U_i)_{f_i}$ is equal to $f^n t$ (that is $f^n \otimes t$) over $(U_i)_{f_i}$. Since we are dealing with a finite number of such open sets, we can select n large to work for all i. On $U_i \cap U_j$ the two sections t_i and t_j are defined, and are equal to $f^n t$ when restricted to $X_f \cap U_i \cap U_j$. By the first part of the lemma, there is an integer m such that $f^m(t_i - t_j) = 0$ on $U_i \cap U_j$ for all i, j, again using the fact that there is only a finite number of pairs (i, j). Then the section $f^m t_i \in \Gamma(U_i, \mathcal{L}^m \otimes \mathcal{F})$ define a global section of $\mathcal{L}^m \otimes \mathcal{F}$, whose restriction to X_f is $f^{n+m}t$. This concludes the proof of the lemma. $\square$

We turn to the application in the case of sheaves over $\mathbb{P} = \operatorname{Proj}(R)$. The sheaf $\mathcal{L}$ of Lemma 4.1 will be $\mathcal{O}_{\mathbb{P}}(1)$.

Let M be a graded module over R. Then $\widetilde{M}$ is a sheaf on $\mathbb{P}$. Suppose that N is a graded module such that $N_d = M_d$ for all $d \geq d_0$. Then

$$\widetilde{M} = \widetilde{N}.$$

This is easily seen, because for $f \in R_1$, we know that $\mathbb{P}$ is covered by the affine open sets $\mathbb{P}_f$. Then any section of $\widetilde{M}$ over $\mathbb{P}_f$ can be written in the form x/f^n for some $x \in M_n$, but we can also write such an element in the form

$$\frac{x}{f^n} = \frac{f^m x}{f^{m+n}}$$

so we can use only homogeneous elements of arbitrarily high degree. Hence changing a finite number of graded components in M does not affect M_f, nor $\widetilde{M}$.

If M is finitely generated, it is therefore natural to say that M is *quasi-equal* to N if $M_d = N_d$ for all d sufficiently large. Quasi-equality is an equivalence relation. Two graded homomorphisms

$$f, g \colon M \longrightarrow N$$

are called *quasi-equal* if $f_d = g_d$ for all d sufficiently large. ($f_d, g_d \colon M_d \to N_d$ are the restrictions of f, g.) More generally, we define:

$$\mathrm{Hom}_{\mathrm{qe}}(M, N) = \varinjlim_n \mathrm{Hom}(M_{\geq n}, N_{\geq n}),$$

where $M_{\geq n}$ denotes the submodule of M of components of degree $\geq n$. This defines a category which we call the category of *graded modules modulo quasi-equality*, and denote by $\mathrm{GrMod}_{\mathrm{qe}}(R)$.

The association

$$M \longmapsto \widetilde{M} \quad \text{(projective tilde)}$$

is a functor from this category to the category of quasi-coherent sheaves on $\mathbb{P}$.

Our object is now to drive toward Theorem 4.8, which states that under suitable finiteness assumptions, this functor establishes an equivalence of categories. Some of the arguments do not use all the assumptions, so we proceed stepwise. The first thing to show is that every quasi-coherent sheaf is some $\widetilde{M}$. Let $\mathcal{F}$ be quasi-coherent over $\mathbb{P}$. Then in §1 we had defined

$$\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(\mathbb{P}, \mathcal{F}(n)).$$

Proposition 4.2. *Let $\mathcal{F}$ be a quasi-coherent sheaf over $\mathbb{P}$. Let $M = \Gamma_*(\mathcal{F})$. Then $\mathcal{F} \approx \widetilde{M}$.*

Proof. Let $f \in R_1$. We want to establish an isomorphism

$$(M_f)_0 \xrightarrow{\approx} \mathcal{F}(\mathbb{P}_f).$$

The left hand side is the module of sections of $\widetilde{M}$ over $\mathbb{P}_f$. The compatibility as f varies will be obvious from the definition, and this isomorphism will give the desired isomorphism of $\widetilde{M}$ with $\mathcal{F}$. Multiplication by f gives a homomorphism

$$\mathcal{F}(n) \xrightarrow{f} \mathcal{F}(n+1)$$

whence a corresponding homomorphism on global sections. There is a natural isomorphism

$$(M_f)_0 \approx \varinjlim_n (M_n, f) \approx \varinjlim_n (\Gamma\mathcal{F}(n), f)$$

where the right hand side is the direct limit of the system:

$$M_0 \xrightarrow{f} M_1 \xrightarrow{f} M_2 \xrightarrow{f} \cdots \xrightarrow{f} M_n \xrightarrow{f} \cdots$$

Indeed, an element of $(M_f)_0$ can be represented as a quotient x/f^n with $x \in \Gamma\mathcal{F}(n)$. There is an equality

$$\frac{x}{f^n} = \frac{y}{f^m}$$

with $y \in \Gamma\mathcal{F}(m)$ if and only if there is some power f^d such that

$$f^{d+m}x = f^{d+n}y.$$

This means precisely that an element of $(M_f)_0$ corresponds to an element of the direct limit as stated.

On the other hand, let $\mathcal{O} = \mathcal{O}_{\mathbb{P}}$. We have an isomorphism

$$\mathcal{O}|_{\mathbb{P}_f} \xrightarrow[\approx]{f^n} \mathcal{O}(n)|_{\mathbb{P}_f}$$

and since $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{O}(n)$ by definition, we get an isomorphism

$$\mathcal{F}|_{\mathbb{P}_f} \xrightarrow[\approx]{f^n} \mathcal{F}(n)|_{\mathbb{P}_f}.$$

Now we look at the directed system and commutative diagrams:

$$\begin{tikzcd}
\Gamma\mathcal{F}(n) \arrow[r, "\mathrm{res}"] & \mathcal{F}(n)(\mathbb{P}_f) & \mathcal{F}(\mathbb{P}_f) \arrow[l, "f^n"', "\approx"] \arrow[dl, "\approx"', "f^{n-1}"] \\
\Gamma\mathcal{F}(n-1) \arrow[u, "f"] \arrow[r, "\mathrm{res}"'] & \mathcal{F}(n-1)(\mathbb{P}_f) \arrow[u, "f"] &
\end{tikzcd}$$

The top row gives a homomorphism

$$f^{-n} \circ \mathrm{res}\colon \Gamma\mathcal{F}(n) \longrightarrow \mathcal{F}(\mathbb{P}_f).$$

The commutativity of the square and triangle induces a homomorphism on the direct limit

$$(M_f)_0 \approx \varinjlim_n (\Gamma\mathcal{F}(n), f) \longrightarrow \mathcal{F}(\mathbb{P}_f).$$

The first part of Lemma 4.1 shows that this map is injective. Using the quasi-compactness of $\mathbb{P}$, the second part shows that this map is surjective, whence the desired isomorphism. We leave to the reader the verification of the compatibility condition as f varies in R_1, to conclude the proof. $\square$

Theorem 4.3 (Serre). *Let $\mathcal{F}$ be a finitely generated quasi-coherent sheaf on $\mathbb{P}$. Then there is some n_0 such that for all $n \geq n_0$, the sheaf $\mathcal{F}(n)$ is generated by a finite number of global sections.*

Proof. Let $f_0, \dots, f_r$ generate R_1 over R_0, and let $\mathbb{P}_i = \mathbb{P}_{f_i}$. For each i there is a finitely generated module M_i over $\mathcal{O}(\mathbb{P}_i)$ such that $\mathcal{F}|_{\mathbb{P}_i} = \widetilde{M_i}$. For each i, let s_{ij} be a finite number of sections in M_i generating M_i over $\mathcal{O}(\mathbb{P}_i)$. By Lemma 4.1 there is an integer n such that for all i, j the sections $f_i^n s_{ij}$ extend to global sections of $\mathcal{F}(n)$. But for fixed i, the global sections $f_i^n s_{ij}$ (j variable) generate M_i over $\mathcal{O}(\mathbb{P}_i)$ since f_i^n is invertible over $\mathcal{O}(\mathbb{P}_i)$. Since the open sets $\mathbb{P}_i$ ($i = 0, \dots, r$) cover $\mathbb{P}$, this concludes the proof. $\square$

Proposition 4.4. *Let $\mathcal{F}$ be a finitely generated quasi-coherent sheaf on $\mathbb{P}$. Then there is a finitely generated R-submodule N of $\Gamma_*\mathcal{F}$ such that $\mathcal{F} = \widetilde{N}$.*

Proof. As in Proposition 4.2, let $M = \Gamma_*\mathcal{F}$, so $\widetilde{M} = \mathcal{F}$. By Theorem 4.3, there exists n such that $\mathcal{F}(n)$ is generated by global sections in $\Gamma(\mathbb{P}, \mathcal{F}(n))$. Let N be the R-submodule of M generated by this finite number of global sections. The inclusion $N \hookrightarrow M$ induces an injective homomorphism of sheaves

$$0 \longrightarrow \widetilde{N} \longrightarrow \widetilde{M} = \mathcal{F}$$

whence an injective homomorphism obtained by twisting n times

$$0 \longrightarrow \widetilde{N}(n) \longrightarrow \widetilde{M}(n) = \mathcal{F}(n).$$

This homomorphism is an isomorphism because $\mathcal{F}(n)$ is generated by the global sections in N. Twisting back by $-n$ we get the isomorphism $\widetilde{N} \approx \mathcal{F}$, thereby concluding the proof. $\square$

We have now achieved part of our objective to relate quasi-equal graded modules with coherent sheaves. We proceed to the inverse construction, and we consider the morphisms.

Proposition 4.5. *Assume that M is a finitely presented graded module over R. Let N be a graded module. Then we have an isomorphism*

$$\varinjlim_n \mathrm{Hom}(M_{\geq n}, N_{\geq n}) \xrightarrow{\approx} \mathrm{Hom}(\widetilde{M}, \widetilde{N}).$$

Proof. Consider a finite presentation

$$R^p \longrightarrow R^q \longrightarrow M \longrightarrow 0.$$

In such a presentation, the homomorphism are not of degree 0, and we rewrite it in the form

$$F \longrightarrow E \longrightarrow M \longrightarrow 0$$

where each of F, E is a direct sum of free graded module of type $R(d)$ with $d \in \mathbb{Z}$. We then obtain an exact and commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Hom}(\widetilde{M}, \widetilde{N}) & \longrightarrow & \operatorname{Hom}(\widetilde{E}, \widetilde{N}) & \longrightarrow & \operatorname{Hom}(\widetilde{F}, \widetilde{N}) \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \varinjlim_n \operatorname{Hom}(M_{\geq n}, N_{\geq n}) & \longrightarrow & \varinjlim_n \operatorname{Hom}(E_{\geq n}, N_{\geq n}) & \longrightarrow & \varinjlim_n \operatorname{Hom}(F_{\geq n}, N_{\geq n}). \end{array}$$

It will suffice to prove that the two vertical arrows on the right are isomorphisms. In light of the direct sum structure of E and F, it suffices to prove that

$$\varinjlim_n \operatorname{Hom}(R(d)_{\geq n}, N_{\geq n}) \longrightarrow \operatorname{Hom}(\widetilde{R(d)}, \widetilde{N})$$

is an isomorphism, and twisting by $-d$, it suffices to prove that

$$\varinjlim_n \operatorname{Hom}(R_{\geq n}, N_{\geq n}) \longrightarrow \operatorname{Hom}(\widetilde{R}, \widetilde{N})$$

is an isomorphism for any graded module N. But $\widetilde{R} = \mathcal{O}_{\mathbb{P}}$ and thus

$$\operatorname{Hom}(\widetilde{R}, \widetilde{N}) = \operatorname{Hom}(\mathcal{O}_{\mathbb{P}}, \widetilde{N}) = \Gamma\widetilde{N}.$$

Thus it suffices to prove the following lemma.

Lemma 4.6. *Let N be a graded R-module. Then we have an isomorphism*

$$\varinjlim_n \operatorname{Hom}(R_{\geq n}, N) \longrightarrow \Gamma\widetilde{N}.$$

Proof of Lemma 4.6. Corresponding to a finite set of generators of R_1 over R_0, we have a graded surjective homomorphism

$$R_0[T_0, \ldots, T_r] \longrightarrow R_0[R_1] = R \longrightarrow 0,$$

which makes $\mathbb{P} = \operatorname{Proj} R$ into a closed subscheme of $\mathbb{P}^r_A$ where $A = R_0$. We can view the module N as graded module over $\mathbb{P}^r_A$, and the sheaves are sheaves over $\mathbb{P}^r_A$. We also view R as graded module over the polynomial ring $A[T_0, \ldots, T_r]$. The relation to be proved is then concerned with objects on $\mathbb{P}^r_A$. We have to prove the surjectivity and injectivity of the arrow. For surjectivity, let $x \in \Gamma\widetilde{N}$. Let $\mathbb{P}_i$ be the complement of the hyperplane $T_i = 0$ as usual. Then

$$\operatorname{res}_{\mathbb{P}_i}(x) = \frac{x_i}{T_i^n}$$

with n sufficiently large and some $x_i \in N_n$. Increasing n further, we may assume that

$$T_i^n x_j = T_j^n x_i$$

because $x_i/T_i^n = x_j/T_j^n$ in $N_{T_iT_j}$ for all i, j. Therefore there exists a homomorphism of the ideal $(T_0^n, \ldots, T_r^n)$ into N,

$$\varphi\colon (T_0^n, \ldots, T_r^n) \longrightarrow N$$

sending $T_i^n \mapsto x_i$ for each i, and this homomorphism maps on x by the arrow

$$\operatorname{Hom}(R_{\geq m}, N) \longrightarrow \Gamma\widetilde{N},$$

for m sufficiently large, because $R_{\geq m} \subset (T_0^n, \dots, T_r^n)$ for m large compared to n. In fact, the ideals $(T_0^n, \dots, T_r^n)$ are cofinal with the modules $R_{\geq m}$ as m, n tend to infinity. This shows that the map

$$\varinjlim_n \operatorname{Hom}(R_{\geq n}, N) \longrightarrow \Gamma \widetilde{N}$$

is surjective. The injectivity is proved in the same way. This concludes the proof of the lemma, and also the proof of Proposition 4.5. □

□

The proof of the next proposition relies on the following:

FACT. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P} = \operatorname{Proj} R$ with R_0 noetherian. Then $\Gamma_* \mathcal{F}$ is a finitely presented R-module.*

The proof of this fact will be given as a consequence of theorems in cohomology, by descending induction, and is therefore postponed to Chapter VII (cf. Theorem VII.6.1, which is the fundamental theorem of Serre [**87**], and its proof.)

PROPOSITION 4.7. *Let M be a finitely presented graded module over R with R_0 noetherian. Then the natural map*

$$M \longrightarrow \Gamma_* \widetilde{M}$$

is an isomorphism modulo quasi-equality.

PROOF. By Proposition 4.2 we have an isomorphism

$$\varphi \colon \widetilde{(\Gamma_* \widetilde{M})} \xrightarrow{\approx} \widetilde{M},$$

so by Proposition 4.5, and the "Fact" above:

$$\varphi \in \operatorname{Hom}(\widetilde{(\Gamma_* \widetilde{M})}, \widetilde{M}) \approx \varinjlim_n \operatorname{Hom}((\Gamma_* \widetilde{M})_{\geq n}, M_{\geq n}).$$

Therefore φ comes from a homomorphism

$$h_n \colon (\Gamma_* \widetilde{M})_{\geq n} \longrightarrow M_{\geq n}$$

for n sufficiently large since M is finitely presented over R, that is $\varphi = \widetilde{h_n}$. But since φ is an isomorphism, it follows from applying Proposition 4.5 to φ^{-1} that h_n has to be an isomorphism for n large. This concludes the proof. □

We can now put together Propositions 4.2 and 4.7 to obtain the goal of this section.

THEOREM 4.8. *If R_0 is noetherian, then the association*

$$M \longmapsto \widetilde{M}$$

is an equivalence of categories between finitely presented graded modules over R modulo quasi-equality and coherent sheaves on $\mathbb{P}$. The inverse functor is given by

$$\mathcal{F} \longmapsto \Gamma_* \mathcal{F}.$$

This theorem now allows us to handle sheaves like graded modules over R. For example we have the immediate application:

COROLLARY 4.9. *Let $\mathcal{F}$ be a coherent sheaf on* Proj R *with R_0 noetherian. Then there exists a presentation*

$$\mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

where $\mathcal{E}$ is a finite direct sum of sheaves $\mathcal{O}_{\mathbb{P}}(d)$ with $d \in \mathbb{Z}$.

PROOF. The corresponding assertion is true for graded modules, represented as quotients of finite direct sums of modules $R(d)$ with $d \in \mathbb{Z}$. Taking the tilde gives the result for coherent sheaves. $\square$

5. Ample invertible sheaves

There will be two notions of ampleness, one absolute and the other relative. We start with the absolute notion. For simplicity, we develop the theory only in the notherian case.

DEFINITION 5.1. Let X be a noetherian scheme. An invertible sheaf $\mathcal{L}$ on X is called *ample* if for all coherent sheaves $\mathcal{F}$ on X there exists n_0 such that $\mathcal{F} \otimes \mathcal{L}^n$ is generated by its global sections if $n \geq n_0$.

EXAMPLE. Serre's Theorem 4.3 gives the fundamental example of an ample $\mathcal{L}$, namely $\mathcal{O}_{\mathbb{P}}(1)$ where $\mathbb{P} = \operatorname{Proj} R$ with R noetherian.

It is obvious that if $\mathcal{L}$ is ample, then $\mathcal{L}^m$ is ample for any positive integer m. It is convenient to have a converse version of this fact.

LEMMA 5.2. *If $\mathcal{L}^m$ is ample for some positive integer m, then $\mathcal{L}$ is ample.*

PROOF. Let $\mathcal{F}$ be a coherent sheaf on X. Then $\mathcal{F} \otimes \mathcal{L}^{mn}$ is generated by global sections for all $n \geq n_0$. Furthermore, for each $i = 0, \ldots, m-1$ the sheaf

$$\mathcal{F} \otimes \mathcal{L}^i \otimes \mathcal{L}^{mn}$$

is generated by global sections for $n \geq n_i$. We let N be the maximum of $n_0, \ldots, n_{m-1}$. Then $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections for $n \geq N$, thus proving the lemma. $\square$

DEFINITION 5.3. Let $\varphi\colon X \to Y$ be a morphism of finite type over a noetherian base Y. Let $\mathcal{L}$ be an invertible sheaf on X. We say that $\mathcal{L}$ is *relatively very ample with respect to* φ, or φ-*relatively very ample*, if there exists a coherent sheaf $\mathcal{F}$ on Y and an immersion (not necessarily closed)

$$\iota\colon X \longrightarrow \mathbb{P}_Y(\mathcal{F})$$

over Y, i.e., making the following diagram commutative

$$\begin{tikzcd} X \arrow[rr, "\iota"] \arrow[dr, "\varphi"'] & & \mathbb{P}_Y(\mathcal{F}) \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

such that $\mathcal{L} = \iota^*\mathcal{O}_{\mathbb{P}}(1)$. We say that $\mathcal{L}$ is *relatively ample* if for some $n \geq 1$, $\mathcal{L}^{\otimes n}$ is relatively very ample.

The definition is adjusted to be able to deal with a wide assortment of base scheme Y. However, when $Y = \operatorname{Spec}(A)$ is affine, then it turns out that one can replace $\mathbb{P}_Y(\mathcal{F})$ by $\mathbb{P}^r_A$ for some r, as in the following theorem. Observe that in the affine case, we have

$$\mathbb{P}^r_A = \mathbb{P}_Y(\mathcal{F}) \quad \text{with} \quad \mathcal{F} = \mathcal{O}_Y^{r+1}.$$

THEOREM 5.4. *Let X be a scheme of finite type over a noetherian ring A and let $\mathcal{L}$ be an invertible sheaf on X. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}$ is relatively ample over* $\operatorname{Spec}(A)$. *Moreover, when this holds the immersion $\iota\colon X \to \mathbb{P}_A(\mathcal{F})$ such that $\mathcal{L} = \iota^*\mathcal{O}_{\mathbb{P}}(1)$ can be taken into projective space $\mathbb{P}^r_A$.*

REMARK. Serre's cohomological criterion for ampleness will be given in Theorem VII.8.2.

PROOF. Suppose that there is an immersion $\iota\colon X \to \mathbb{P}^r_A$. The only problem to show that $\mathcal{L}$ is ample is that X need not be closed in $\mathbb{P}^r_A$, because if X is closed then we can apply Theorem 4.3. The next result is designed to take care of this problem.

PROPOSITION 5.5. *Let $\mathcal{F}$ be a quasi-coherent sheaf on a noetherian scheme X. Let U be an open subscheme of X, and let $\mathcal{G}_U$ be a coherent subsheaf of $\mathcal{F}|_U$. Then there exists a coherent subsheaf $\mathcal{G}$ of $\mathcal{F}$ on X such that*

$$\mathcal{G}|_U = \mathcal{G}_U.$$

PROOF. Consider all pairs $(\mathcal{G}, W)$ consisting of an open subscheme W of X and a coherent subsheaf $\mathcal{G}$ of $\mathcal{F}|_W$ extending $(\mathcal{G}_U, U)$. Such pairs are partially ordered by inclusion of W's and are in fact inductively ordered because the notion of a coherent sheaf is local, so the usual union over a totally ordered subfamily gives a pair dominating every element of the family. By Zorn's lemma, there exists a maximal element, say $(\mathcal{G}, W)$. We reduce the proposition to the affine case as follows. If $W \neq X$, then there is an affine open subscheme $V = \mathrm{Spec}(A)$ in X such that $V \not\subset W$. Then $W \cap V$ is an open subscheme of V, and if we have the proposition in the affine case, then we extend $\mathcal{G}$ from $W \cap V$ to V, thus extending $\mathcal{G}$ to a larger subscheme than W, contradicting the maximality.

We now prove the proposition when X is affine. In that case, we note that the coherent subsheaves of $\mathcal{G}_U$ satisfy the ascending chain condition. We let $\mathcal{G}_1$ be a maximal coherent subsheaf which admits a coherent extension $\mathcal{G}$ which is a subsheaf of $\mathcal{F}$. We want to prove that $\mathcal{G}_1 = \mathcal{G}_U$. If $\mathcal{G}_1 \neq \mathcal{G}_U$ then there exists an affine open $X_f \subset U$ and a section $s \in \mathcal{G}_U(X_f)$ such that $s \notin \mathcal{G}_1(X_f)$. By Lemma 4.1 (ii), there exists n such that $f^n s$ extends to a section $s' \in \mathcal{F}(X)$ and the restriction of s' to U is in $\mathcal{F}(U)$. By Lemma 4.1 (i) there exists a still higher power f^m such that

$$f^m(s'|_U) = 0 \quad \text{in} \quad (\mathcal{F}/\mathcal{G})(U).$$

Then $\mathcal{G}_1 + f^m s' \mathcal{O}_X$ is a coherent subsheaf of $\mathcal{F}$ which is bigger than $\mathcal{G}_1$, contradiction. This concludes the proof of the proposition. $\square$

COROLLARY 5.6. *Let X be a noetherian scheme. Let U be an open subscheme, and let $\mathcal{G}$ be a coherent sheaf on U. Then $\mathcal{G}$ has a coherent extension to X, and this coherent extension may be taken as a subsheaf of $\iota_*\mathcal{G}$, where $\iota\colon U \to X$ is the open immersion.*

PROOF. By Proposition II.4.10 we know that $\iota_*\mathcal{G}$ is quasi-coherent, and so we can apply Proposition 5.5 to finish the proof. $\square$

We can now finish one implication in Theorem 5.4. Assuming that we have the projective immersion $\iota\colon X \to \mathbb{P}^r_A$, we consider the closure $\overline{X}$ and apply Theorem 4.3 to an extension $\overline{\mathcal{F}}$ of a coherent sheaf $\mathcal{F}$ on X. Then $\overline{\mathcal{F}} \otimes \mathcal{O}_{\overline{X}}(n)$ is generated by global sections for $n \geq n_0$, and the restrictions of these sections to $\mathcal{F}$ generate $\mathcal{F}$, thus concluding the proof of one half of the theorem.

To prove the converse, we need a lemma.

LEMMA 5.7. *Let $\mathcal{L}$ be an ample sheaf on a noetherian scheme X. Then there exists an open affine covering of X by subschemes defined by the property $s(x) \neq 0$, for some global section s of $\mathcal{L}^n$, some n.*

PROOF. Given a point $x \in X$, there is an open affine neighborhood U of x such that $\mathcal{L}|_U$ is free. Let $Y = X \setminus U$ be the complement of U, with the reduced scheme structure, so that Y is a closed subscheme, defined by a sheaf of ideals $\mathcal{I}_Y$, which is coherent on X. There exists n such that $\mathcal{I}_Y \otimes \mathcal{L}^n$ is generated by global sections, and in particular, there is a section s of $\mathcal{I}_Y \otimes \mathcal{L}^n$

such that $s(x) \neq 0$, or equivalently, $s_x \notin \mathfrak{m}_x(\mathcal{I}_Y \otimes \mathcal{L}^n)_x$. Since $\mathcal{L}^n$ is free, we can view $\mathcal{I}_Y \otimes \mathcal{L}^n$ as a subsheaf of $\mathcal{L}^n$. Then by Lemma 2.3 the set X_s of points z such that $s(z) \neq 0$ is open and is contained in U because $s(y) \in \mathfrak{m}_y \mathcal{L}^n_y$ for $y \in Y$. The section s restricted to U can be viewed as an element of $\mathcal{L}^n(U)$, and since $\mathcal{L}$, so $\mathcal{L}^n$, are free over U, it follows that s corresponds to a section f of $\mathcal{O}_U$ and that $X_s = U_f$ so X_s is affine. □

Thus we have proved that for each point $x \in X$ there is an affine open neighborhood X_s defined by a global section s of $\mathcal{L}^{n(x)}$ such that $s(x) \neq 0$. Since X is quasi-compact, we can cover X by a finite number of such affine open sets, and we let m to be the least common multiple of the finite number of exponents $n(x)$.

Since we wish to prove that $\mathcal{L}^n$ is very ample for sufficiently large n, we may now replace $\mathcal{L}$ by $\mathcal{L}^m$ without loss of generality. We are then in the situation when we have a finite number of global sections $s_1, \dots, s_r$ of $\mathcal{L}$ which generate $\mathcal{L}$, such that X_{s_i} is affine for all i, and such that the open sets X_{s_i} cover X. We abbreviate X_{s_i} by X_i.

Let B_i be the affine algebra of X_i over A. By assumption X is of finite type over A, so B_i is finitely generated as A-algebra, say by elements b_{ij}. By Lemma 4.1 there exists an integer N such that for all i, j the section $s_i^N b_{ij}$ extends to a global section t_{ij} of $\mathcal{L}^N$. The family of sections s_i^N, t_{ij} for all i, j generates $\mathcal{L}^N$ since already the sections $s_1^N, \dots, s_r^N$ generate $\mathcal{L}^N$, and hence they define a morphism

$$\psi\colon X \longrightarrow \mathbb{P}^M_A$$

for some integer M. It will now suffice to prove that ψ is a closed embedding. Let T_i, T_{ij} be the homogeneous coordinates of $\mathbb{P}^M_A$, and put $\mathbb{P} = \mathbb{P}^M_A$ for simplicity. If $\mathbb{P}_i$ is the complement of the hyperplane $T_i = 0$ then $X_i = \psi^{-1}(\mathbb{P}_i)$. The morphism induces a morphism

$$\psi_i\colon X_i \longrightarrow \mathbb{P}_i$$

which corresponds to a homomorphism of the corresponding affine algebras

$$A[z_k, z_{kj}] \longrightarrow B_i,$$

where z_k, z_{kj} are the affine coordinates: $z_k = T_k/T_i$ and $z_{kj} = T_{kj}/T_i$. We see that z_{ij} maps on $t_{ij}/s_i^N = b_{ij}$ so the affine algebra homomorphism is surjective. This means that ψ_i is a closed embedding of X_i in $\mathbb{P}_i$. Since X is covered by the finite number of affine open sets $X_1, \dots, X_r$ it follows by Corollary II.3.5 that ψ itself is a closed embedding. This concludes the proof of Theorem 5.4. □

Next we want to investigate the analogous situation when the base Y is not affine.

Proposition 5.8. *Let U be open in X and $\mathcal{L}$ ample on X. Then $\mathcal{L}|_U$ is ample on U.*

Proof. By Proposition 5.5, a coherent sheaf $\mathcal{F}$ on U has an extension to a coherent sheaf on X. Global sections which generate this extension restrict to sections of $\mathcal{F}$ on U which generate $\mathcal{F}$ on U, so the proposition is immediate. □

Now comes the globalized version of Theorem 5.4.

Theorem 5.9. *Let $\varphi\colon X \to Y$ be of finite type with X, Y noetherian. The following conditions are equivalent.*

i) *There exists a positive integer n such that $\mathcal{L}^n$ is relatively very ample for φ.*
ii) *There exists an open affine covering $\{V_i\}$ of Y such that $\mathcal{L}|_{\varphi^{-1}V_i}$ is ample for all i.*
iii) *For all affine open subsets V of Y the restriction $\mathcal{L}|_{\varphi^{-1}V}$ is ample.*

Proof. The implication (iii) $\implies$ (ii) is trivial and (i) implies (iii) follows immediately from Theorem 5.4.

We must show that (ii) implies (i). We have done this when the base Y is affine in Theorem 5.4, and we must globalize the construction. When Y is affine, we could take the embedding of X into a projective space, but now we must use $\mathbb{P}_Y(\mathcal{F})$ with some sheaf $\mathcal{F}$ which need not be locally free.

Applying Theorem 5.4 to $\mathcal{L}|_{\varphi^{-1}V_i}$, we get coherent sheaves $\mathcal{F}_i$ on V_i and immersions ψ_i

$$\begin{tikzcd} \varphi^{-1}(V_i) \arrow[rr, "\psi_i"] \arrow[dr, "\operatorname{res}\varphi"'] & & \mathbb{P}_{V_i}(\mathcal{F}_i) \arrow[dl] \\ & V_i & \end{tikzcd}$$

satisfying $\psi_i^*(\mathcal{O}(1)) \approx \mathcal{L}^{n_i}|_{\varphi^{-1}V_i}$. We first make two reductions. First of all, we may assume the n_i are equal because if $n = \mathrm{l.c.m}(n_i)$ and $m_i = n/n_i$ then

$$\begin{aligned} \mathbb{P}_{V_i(\mathcal{F}_i)} &= \mathcal{P}roj_{V_i}\left(\mathrm{Symm}(\mathcal{F}_i)\right) \\ &\approx \mathcal{P}roj_{V_i}\left(\bigoplus_k \mathrm{Symm}^{m_i k}(\mathcal{F}_i)\right) \\ &= \mathcal{P}roj_{V_i}\left(\mathrm{Symm}\left(\mathrm{Symm}^{m_i}(\mathcal{F}_i)\right)/I_i\right) \quad \text{for some ideal } I_i \\ &\subset \mathbb{P}_{V_i}\left(\mathrm{Symm}^{m_i}(\mathcal{F}_i)\right). \end{aligned}$$

Replacing $\mathcal{F}_i$ by $\mathrm{Symm}^{m_i}(\mathcal{F}_i)$, we find $\psi_i^*(\mathcal{O}(1)) \approx \mathcal{L}^n|_{\varphi^{-1}V_i}$ for the new ψ_i.

Secondly, ψ_i gives us the canonical surjective homomorphisms

$$\alpha_i \colon (\operatorname{res}\varphi)^*(\mathcal{F}_i) \twoheadrightarrow \mathcal{L}^n|_{\varphi^{-1}V_i}$$

hence

$$\beta_i \colon \mathcal{F}_i \longrightarrow (\operatorname{res}\varphi)_*(\mathcal{L}^n|_{\varphi^{-1}V_i}) \quad \text{(cf. (I.5.11))}.$$

We may assume that β_i is injective. In fact, let $\mathcal{F}_i'$ be the image of $\mathcal{F}_i$ in $(\operatorname{res}\varphi)_*(\mathcal{L}^n|_{\varphi^{-1}V_i})$. Then $\mathcal{F}_i'$ is still coherent because $(\operatorname{res}\varphi)_*(\mathcal{L}^n|_{\varphi^{-1}V_i})$ is quasi-coherent (cf. Proposition II.4.10), and the morphism ψ_i factors

$$\varphi^{-1}(V_i) \to \mathbb{P}_{V_i}(\mathcal{F}_i') \hookrightarrow \mathbb{P}_{V_i}(\mathcal{F}_i).$$

We now apply Corollary 5.6 to choose a coherent subsheaf $\mathcal{G}_i \subset \varphi_*\mathcal{L}^n$ such that $\mathcal{G}_i|_{V_i} \approx \mathcal{F}_i$. Now the homomorphism

$$\beta \colon \bigoplus \mathcal{G}_i \longrightarrow \varphi_*\mathcal{L}^n$$

defines

$$\alpha \colon \varphi^*\left(\bigoplus \mathcal{G}_i\right) \longrightarrow \mathcal{L}^n$$

(cf. (I.5.11)) and α is surjective because on each V_i,

$$\varphi^*\mathcal{G}_i|_{V_i} \longrightarrow \mathcal{L}^n|_{V_i}$$

is surjective. By the universal mapping property of $\mathbb{P}_Y$, $\mathcal{L}^n$ and α define a morphism:

$$\begin{tikzcd} X \arrow[rr, "\psi"] \arrow[dr, "\varphi"'] & & \mathbb{P}_Y\left(\bigoplus \mathcal{G}_i\right) \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

I claim this is an immersion. In fact, restrict the morphisms to $\varphi^{-1}(V_i)$. The functoriality of Proj (cf. §II.5, Remark h)) plus the homomorphism

$$\mathrm{Symm}(\mathcal{G}_i|_{V_i}) \hookrightarrow \mathrm{Symm}(\bigoplus \mathcal{G}_j|_{V_i})$$

gives us an open set $W_i \subset \mathbb{P}_Y(\bigoplus \mathcal{G}_j)$ and a "projection" morphism:

$$\begin{tikzcd} W_i \arrow[d] & \subset & \mathbb{P}_Y(\bigoplus \mathcal{G}_j) \\ \mathbb{P}_Y(\mathcal{G}_i) & & \end{tikzcd}$$

It is not hard to verify that $\psi(\varphi^{-1}(V_i)) \subset W_i$, and that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\psi"] & \mathbb{P}_Y(\bigoplus \mathcal{G}_j) \\ \cup & \cup \\ \varphi^{-1}(V_i) \arrow[r, "\mathrm{res}\,\varphi"] \arrow[ddr, "\psi_i"'] & W_i \cap \pi^{-1}(V_i) \arrow[d] \\ & \mathbb{P}_{V_i}(\mathcal{G}_i|_{V_i}) \arrow[d, "\approx"] \\ & \mathbb{P}_{V_i}(\mathcal{F}_i) \end{tikzcd}$$

Since ψ_i is an immersion, so is $\mathrm{res}\,\psi$ (cf. Proposition II.3.14), and since this holds for all i, it follows that ψ is an immersion. $\square$

A final result explains further why relatively ample is the relative version of the concept ample.

THEOREM 5.10. *Let $f\colon X \to Y$ be of finite type with X, Y noetherian. Let $\mathcal{L}$ be relatively ample on X with respect to f, and $\mathcal{M}$ ample on Y. Then $\mathcal{L} \otimes f^*\mathcal{M}^k$ is ample on X for all k sufficiently large.*

PROOF. The first step is to fix a coherent sheaf $\mathcal{F}$ on X and to show that for all n_1 sufficiently large, there exists n_2 such that

$$\mathcal{F} \otimes \mathcal{L}^{n_1} \otimes f^*\mathcal{M}^{n_2}$$

is generated by global sections. This goes as follows: because $\mathcal{M}$ is ample, Y can be covered by affine open sets Y_{s_i}, with $s_i \in \Gamma(Y, \mathcal{M}^{m_1})$ for suitable m_1 by Lemma 5.7. Then $\mathcal{L}|_{f^{-1}(Y_{s_i})}$ is ample by Theorem 5.4. Thus $\mathcal{F} \otimes \mathcal{L}^{n_1}|_{f^{-1}(Y_{s_i})}$ is generated by sections $t_{i1}, \ldots, t_{iN}$ if n_1 is sufficiently large. But by Lemma 4.1, for large m_2 all the sections

$$s_i^{m_2} t_{ij}$$

extend from X_{s_i} to X as sections of $\mathcal{F} \otimes \mathcal{L}^{n_1} \otimes f^*(\mathcal{M}^{m_1 m_2})$. Let $n_2 = m_1 m_2$. Then this collection of global sections generates

$$\mathcal{F} \otimes \mathcal{L}^{n_1} \otimes f^*(\mathcal{M}^{n_2}).$$

There remains to "rearrange the order of the quantifiers", i.e., to pick an upper bound of n_2/n_1 independent of $\mathcal{F}$. The simplest way to do this is to consider the set:

$$S = \{(n_1, n_2) \mid \mathcal{L}^{n_1} \otimes f^*(\mathcal{M}^{n_2}) \text{ is generated by global sections}\}.$$

Note that:

(a) S is a semi-group;
(b) $S \supset (0) \times (n_0 + \mathbb{N})$ for some n_0 because $\mathcal{M}$ is ample on Y ($\mathbb{N}$ is the set of positive integers);

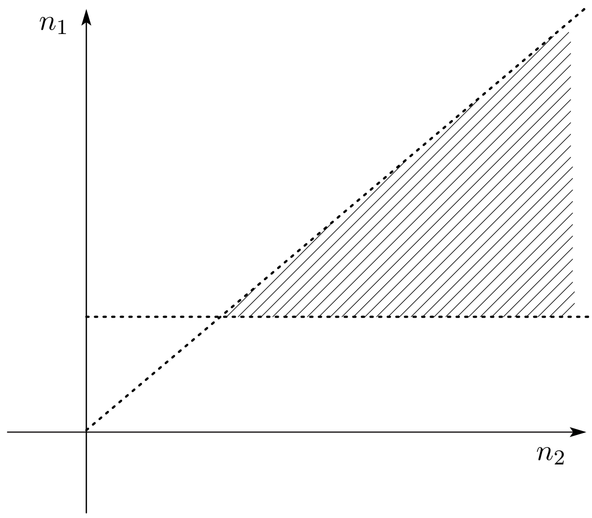

FIGURE III.1

(c) there exists n'_0 such that if $n_1 \geq n'_0$ then

$$(n_1, n_2) \in S \quad \text{for some } n_2.$$

For this last part, apply Step I with $\mathcal{F} = \mathcal{O}_X$.

A little juggling will convince you that such an S must satisfy

$$S \supset \{(n_1, n_2) \mid n_2 \geq k_0 n_1 \geq n_0\}$$

for suitable k_0, n_0 (see Figure III.1). Now take any $k > k_0$ (strictly greater). Then I claim $\mathcal{L} \otimes f^*\mathcal{M}^k$ is ample. In fact, for any $\mathcal{F}$,

$$\mathcal{F} \otimes \mathcal{L}^{n_1} \otimes f^*\mathcal{M}^{n_2}$$

is generated by its sections for some n_1, n_2. Then so is

$$\mathcal{F} \otimes \mathcal{L}^{n_1+n'_1} \otimes f^*\mathcal{M}^{n_2+n'_2} \quad \text{if } (n'_1, n'_2) \in S.$$

But $(n, nk) - (n_1, n_2) \in S$ if $n \gg 0$, so we are OK. This concludes the proof of the theorem. □

6. Invertible sheaves via cocycles, divisors, line bundles

There is a natural correspondence between the four objects occurring in the title of this section. We have already met the invertible sheaves. We shall define the other three and establish this correspondence. We then relate these to Weil divisors.

Basic to all the constructions is the following definition. Let X be a scheme. We define the subsheaf of *units* $\mathcal{O}_X^*$ of $\mathcal{O}_X$ to be the sheaf such that for any open U we have

$$\begin{aligned} \mathcal{O}_X^*(U) = \mathcal{O}_X(U)^* &= \text{units in } \mathcal{O}_X(U) \\ &= \{f \in \mathcal{O}_X(U) \text{ such that } f(x) \neq 0 \text{ for all } x \in U\}. \end{aligned}$$

1-cocycles of units. Let X be a scheme and let $\mathcal{L}$ be an invertible sheaf of $\mathcal{O}_X$-modules or as we also say, an invertible sheaf over X. Let $\{U_i\} = \mathcal{U}$ be an open covering such that the restriction $\mathcal{L}|_{U_i}$ is isomorphic to $\mathcal{O}_X|_{U_i}$ for each i. Thus we have isomorphisms

$$\varphi_i \colon \mathcal{L}|_{U_i} \longrightarrow \mathcal{O}_X|_{U_i}.$$

It follows that

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} \colon \mathcal{O}_X|_{(U_i \cap U_j)} \longrightarrow \mathcal{O}_X|_{(U_i \cap U_j)}$$

is an automorphism, which is $\mathcal{O}_X$-linear, and so is given by multiplication with a unit in $\mathcal{O}_X(U_i \cap U_j)^*$. We may therefore identify φ_{ij} with such a unit. The family of such units $\{\varphi_{ij}\}$ satisfies the condition

$$\varphi_{ij}\varphi_{jk} = \varphi_{ik}.$$

A family of units satisfying this condition is called a 1-*cocycle*. The group of these is denoted $Z^1(\mathcal{U}, \mathcal{O}_X^*)$. By a *coboundary* we mean a cocycle which can be written in the form $f_i f_j^{-1}$, where $f_i \in \mathcal{O}_X(U_i)^*$. These form a subgroup of $Z^1(\mathcal{U}, \mathcal{O}_X^*)$ written $B^1(\mathcal{U}, \mathcal{O}_X^*)$. The factor group $Z^1(\mathcal{U}, \mathcal{O}_X^*)/B^1(\mathcal{U}, \mathcal{O}_X^*)$ is called $H^1(\mathcal{U}, \mathcal{O}_X^*)$. If $\mathcal{U}'$ is a refinement of $\mathcal{U}$, i.e., for each $U_i' \in \mathcal{U}'$, there is a $U_j \in \mathcal{U}$ such that $U_i' \subset U_j$, then there is a natural homomorphism

$$H^1(\mathcal{U}, \mathcal{O}_X^*) \longrightarrow H^1(\mathcal{U}', \mathcal{O}_X^*),$$

(for details, see §VII.1). The direct limit taken over all open coverings $\mathcal{U}$ is called the *first Čech cohomology group* $H^1(X, \mathcal{O}_X^*)$.

Suppose

$$f \colon \mathcal{L} \longrightarrow \mathcal{M}$$

is an isomorphism of invertible sheaves. We can find a covering $\mathcal{U}$ by open sets such that on each U_i of $\mathcal{U}$, $\mathcal{L}$ and $\mathcal{M}$ are free. Then f is represented by an isomorphism

$$f_i \colon \mathcal{O}_X|_{U_i} \longrightarrow \mathcal{O}_X|_{U_i}$$

which can be identified with an element of $\mathcal{O}_X(U_i)^*$. We then see that the cocycles φ_{ij} and φ_{ij}' associated to $\mathcal{L}$ and $\mathcal{M}$ with respect to this covering differ by multiplication by $f_i f_j^{-1}$. This yields a homomorphism (cf. Definition 1.2)

$$\mathrm{Pic}(X) \longrightarrow H^1(X, \mathcal{O}_X^*).$$

PROPOSITION 6.1. *This map* $\mathrm{Pic}(X) \to H^1(X, \mathcal{O}_X^*)$ *is an isomorphism.*

PROOF. The map is injective, for if two cocycles associated with $\mathcal{L}$, $\mathcal{M}$ give the same element in $H^1(X, \mathcal{O}_X^*)$, then the quotient of these cocycles is a coboundary which can be used to define an isomorphism between the invertible sheaves. Conversely, given a cocycle $\varphi_{ij} \in Z^1(\mathcal{U}, \mathcal{O}_X^*)$ it constitutes glueing data in the sense of §I.5 and there exists a unique sheaf $\mathcal{L}$ which corresponds to this glueing data. □

Cartier divisors. Let X be a scheme. Let $U = \mathrm{Spec}(A)$ be an open affine subset of X. Let S be the multiplicative subset of elements of A which are not zero-divisors, and let $K(U) = S^{-1}A$ be the localization of A with this subset. We call $K(U)$, also denoted by $K(A)$, the *total quotient ring* of A. If A has no divisors of 0, then $K(A)$ is the usual quotient field.

The association $U \mapsto K(U)$ defines a presheaf, whose associated sheaf is the sheaf of total quotient rings of $\mathcal{O}_X$, and is denoted by $\mathcal{K}_X$. If X is integral, then all the rings $\mathcal{O}_X(U)$ for affine open U can be identified as subrings of the same quotient field K and $\mathcal{K}_X$ is the constant sheaf with global sections K. ($K = \mathbb{R}(X)$, the function field of X, in the notation of Proposition II.2.5.)

We now consider pairs (U, f) consisting of an open set U and an element $f \in \mathcal{K}^*(U)$, where $\mathcal{K}^*(U)$ is the group of invertible elements of $\mathcal{K}(U)$. We say that two such pairs (U, f) and (V, g) are *compatible* if $fg^{-1} \in \mathcal{O}(U \cap V)^*$, that is, fg^{-1} is a unit in the sheaf of rings over $U \cap V$. Let $\{(U_i, f_i)\}$ be a family of compatible pairs such that the open sets U_i cover X. Two such families are called *compatible* if each pair from one is compatible with all the pairs from the other. A compatibility class of such covering families is a *Cartier divisor* D. As usual, we can say that a Cartier divisor is a maximal family of compatible pairs, covering X. If $f \in \mathcal{K}^*(U)$ and (U, f)

belongs to the compatibility class, then we say that the divisor is *represented* by f over U, and we write $D|_U = (f)$. We also say that $f = 0$ is a *local equation* for D over U.

This amounts to saying that a Cartier divisor is a global section of the sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$. We can define the *support* of a Cartier divisor D, and denote by $\mathrm{Supp}(D)$, the set of points x such that if D is represented by (U, f) on an open neighborhood of x, then $f \notin \mathcal{O}_x^*$. It is easy to see that the support of D is closed.

A Cartier divisor is called *principal* if there exists an element $f \in \Gamma(X, \mathcal{K}^*)$ such that for every open set U, the pair (U, f) represents the divisor. We write (f) for this principal divisor.

Let D, E be Cartier divisors. Then there exists a unique Cartier divisor $D + E$ having the following property. If (U, f) represents D and (U, g) represents E, then (U, fg) represents $D + E$. This is immediate, and one then sees that Cartier divisors form a group $\mathrm{Div}(X)$ having the principal divisors as subgroup. The group is written additively, so $-D$ is represented by (U, f^{-1}). We can take f^{-1} since $f \in \mathcal{K}^*(U)$ by definition.

We introduce a partial ordering in the group of divisors. We say that a divisor D is *effective* if for every representative (U, f) of the divisor, the function f is a morphism on U, that is, $f \in \mathcal{O}_X(U)$. The set of effective divisors is closed under addition. We write $D \geq 0$ if D is effective, and $D \geq E$ if $D - E$ is effective. *Note*: although sometimes one also calls D positive, there are other positive cones which can be introduced in the group of divisors, such as the ample cone. The word “positive” is usually reserved for these other cones.

Remark. It may be that the function f is not on $\mathcal{O}_X(U)$ but is integral over $\mathcal{O}_X(U)$. Thus the function f may be finite over a point, without being a morphism. If X is integral, and all the local rings $\mathcal{O}_x$ for $x \in X$ are integrally closed, then this cannot happen. See below, where we discuss divisors in this context. In this case, the support of D turns out to be the union of the codimension one subschemes where the representative function f has a zero or a pole. This difference in behavior is one of the main differences between Cartier divisors and the other divisors discussed below.

Let D be an effective Cartier divisor. If (U, f) is a representative of D, then f generates a principal ideal in $\mathcal{O}_X(U)$, and this ideal does not depend on the choice of f. In this way we can define a sheaf of ideals, denoted by $\mathcal{I}_D$. It defines a closed subscheme, which is often identified with D.

Two Cartier divisors D, E are called *linearly equivalent*, and we write $D \sim E$, if there exists $f \in \Gamma(X, \mathcal{K}_X^*)$ such that

$$D = E + (f).$$

In other words, $D - E$ is principal. We define the group of *divisor classes*

$$\mathrm{DivCl}(X) = \mathrm{Div}(X)/\mathcal{K}_X^*(X)$$

to be the factor group of Cartier divisors mod principal divisors.

To each Cartier divisor D we shall now associate an invertible sheaf $\mathcal{O}_X(D) = \mathcal{O}(D)$ as follows. If $\{(U_i, f_i)\}$ is a covering family of pairs representing D, then there is a unique subsheaf $\mathcal{L}$ of $\mathcal{K}_X$ such that

$$\mathcal{L}(U_i) = \mathcal{O}(U_i) f_i^{-1}.$$

This subsheaf is denoted by $\mathcal{O}(D)$. Since f_i is a unit in $\mathcal{K}_X(U_i)$, it follows that $\mathcal{L}(U_i)$ is free of rank one over $\mathcal{O}(U_i)$, so $\mathcal{O}(D)$ is invertible. $\mathcal{I}_D = \mathcal{O}_X(-D)$ if D is effective.

Proposition 6.2. *The association*

$$D \longmapsto \mathcal{O}(D)$$

is an isomorphism between Cartier divisors and invertible subsheaves of $\mathcal{K}_X$ *(under the tensor product).*

It induces an injective homomorphism on the classes

$$0 \longrightarrow \operatorname{DivCl}(X) \longrightarrow \operatorname{Pic}(X),$$

where $\operatorname{Pic}(X)$ *is the group of isomorphism classes of invertible sheaves. In other words,* $D \sim E$ *if and only if* $\mathcal{O}(D) \approx \mathcal{O}(E)$*. If* X *is an integral scheme, then this homomorphism is surjective, so we have a natural isomorphism*

$$\operatorname{DivCl}(X) \approx \operatorname{Pic}(X).$$

Proof. The fact that the map $D \mapsto \mathcal{O}(D)$ is homomorphic is immediate from the definitions. From an invertible subsheaf of $\mathcal{K}_X$ we can define a Cartier divisor by the inverse construction that we used to get $\mathcal{O}(D)$ from D. That is, D is represented by f on U if and only if $\mathcal{O}(D)$ is free with basis f^{-1} over U. If $D \sim E$, say $D = E + (f)$, then multiplication by f induces an isomorphism from $\mathcal{O}(D)$ to $\mathcal{O}(E)$. Conversely suppose $\mathcal{O}(D)$ is isomorphic to $\mathcal{O}(E)$. Then $\mathcal{O}(D - E)$ is isomorphic to $\mathcal{O} = \mathcal{O}_X$, so we must prove that if $\mathcal{O}(D) \approx \mathcal{O}$ then $D = 0$. But the image of the global section $1 \in \mathcal{K}^*(X)$ then represents D as a principal divisor.

Finally, suppose X integral. We must show that every invertible sheaf is isomorphic to $\mathcal{O}(D)$ for some divisor D. Let

$$\varphi_i \colon \mathcal{L}|_{U_i} \longrightarrow \mathcal{O}|_{U_i}$$

be an isomorphism and let $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} \in \mathcal{O}(U_i \cap U_j)^*$ be the associated cocycle. We have seen already that this constitutes glueing data to define an invertible sheaf. But now we may view all rings $\mathcal{O}(U_i)$ or $\mathcal{O}(U_i \cap U_j)$ as contained in the quotient field K of X since X is integral. We fix an index j, and define the divisor D by the covering $\{U_i\}$, and the local equation φ_{ij}. In other words, the family of pairs (U_i, φ_{ij}) (with j fixed) is a compatible family, defining a Cartier divisor D. Then it is immediately verified that $\mathcal{O}(D)$ is isomorphic to $\mathcal{L}$. This concludes the proof. □

Line bundles. Let $L \to X$ be a scheme over X. Let $\mathbb{A}^1$ be the affine line. We shall say that L is a *line bundle* over X if one is given an open affine covering $\{U_i\}$ of X and over each U_i an isomorphism of schemes

$$f_i \colon L|_{U_i} \longrightarrow U_i \times \mathbb{A}^1$$

over U_i such that the automorphism

$$f_i \circ f_j^{-1} \colon (U_i \cap U_j) \times \mathbb{A}^1 \longrightarrow (U_i \cap U_j) \times \mathbb{A}^1$$

over $U_i \cap U_j$ is given by an $\mathcal{O}(U_i \cap U_j)$-linear map. Such a map is then represented by a unit $\varphi_{ij} \in \mathcal{O}(U_i \cap U_j)^*$, and such units satisfy the cocycle condition. Consequently, there is an invertible sheaf $\mathcal{L}$ corresponding to this cocycle.

One defines an isomorphism of line bundles over X in the obvious way, so that they are linear on the affine line when given local representations as above.

Proposition 6.3. *The above association of a cocycle to a line bundle over* X *induces a bijection between isomorphism classes of line bundles over* X *and* $H^1(X, \mathcal{O}_X^*)$*. If* $\mathcal{L}$ *is an invertible sheaf corresponding to the cocycle, then we have an isomorphism*

$$L \approx \mathcal{S}pec_X(\operatorname{Symm}^*(\mathcal{L})).$$

Proof. Left to the reader. □

Weil divisors. The objects that we have called Cartier divisors are rather different from the divisors that we defined in Part I [**76**, §1C]. In good cases we can bring these closer together. The problem is: for which integral domain R can we describe the structure of K^*/R^* more simply?

Definition 6.4. A (not necessarily integral) scheme X is called *normal* if all its local rings $\mathcal{O}_{x,X}$ are integral domains, integrally closed in their quotient field (integrally closed, for short); *factorial* if all its local rings $\mathcal{O}_{x,X}$ are unique factorization domains (UFD).

In particular, note that:

$$\begin{aligned} &X \text{ factorial} &&\Longrightarrow X \text{ normal} \\ & &&\quad \text{(all UFD's are integrally closed,} \\ & &&\quad \text{see Zariski-Samuel [\textbf{109}, vol. I, Chapter V, §3, p. 261])} \\ &X \text{ normal} &&\Longrightarrow X \text{ reduced.} \end{aligned}$$

Now the fundamental structure theorem for integrally closed ring states:

Theorem 6.5 (Krull's Structure Theorem). *Let R be a noetherian integral domain. Then*

$$R \textit{ integrally closed} \Longleftrightarrow \begin{cases} \text{a)} & \forall \textit{(non-zero) minimal prime ideal } \mathfrak{p} \subset R, \\ & R_{\mathfrak{p}} \textit{ is a discrete valuation ring,} \\ \text{b)} & R = \bigcap_{\mathfrak{p} \textit{ (non-zero) minimal}} R_{\mathfrak{p}} \end{cases}$$

(cf. Zariski-Samuel [**109**, vol. I, Chapter V, §6]; Bourbaki [**26**, Chapter 7]).

Corollary 6.6. *Assume R noetherian and integrally closed. Let*

$$\mathcal{S} = \textit{set of (non-zero) minimal prime ideals of } R$$

$$Z^1(R) = \textit{free abelian group generated by } \mathcal{S}.$$

If $\mathfrak{p} \in \mathcal{S}$

$$\mathrm{ord}_{\mathfrak{p}} = \left\{ \begin{array}{l} \textit{valuation on } K^* \textit{ defined by the valuation ring } R_{\mathfrak{p}} \\ \textit{i.e., if } \pi \cdot R_{\mathfrak{p}} = \textit{ maximal ideal, } f = \pi^{\mathrm{ord}_{\mathfrak{p}} f} \cdot u,\ u \in R_{\mathfrak{p}}^* \end{array} \right\}.$$

Then the homomorphism:

$$\mathrm{ord}\colon K^*/R^* \longrightarrow Z^1(R)$$

given by $\mathrm{ord}(f) = \sum_{\mathcal{S}} (\mathrm{ord}_{\mathfrak{p}} f) \cdot \mathfrak{p}$ *is injective.* ord *is surjective if and only if R is a UFD.*

Proof. Everything is a straightforward consequence of Theorem 6.5 except for the last assertion. This follows from the well known characterization of UFD's among all noetherian domains—that the (non-zero) minimal prime ideals should be principal, i.e., that Image(ord) $\ni$ the cycle $\mathfrak{p}$ (cf. Zariski-Samuel [**109**, vol. I, Chapter IV, §14, p. 238]). □

Corollary 6.7. *Assume X is a normal irreducible noetherian scheme. Let*

$$\mathcal{S} = \textit{set of maximal closed irreducible subsets } Z \subsetneqq X$$

$$Z^1(X) = \textit{free abelian group generated by } \mathcal{S}.$$

$Z^1(X)$ is called the group of Weil divisors on X. If $Z \in \mathcal{S}$, let

$$\mathrm{ord}_Z = \left\{ \begin{array}{l} \textit{valuation on } \mathbb{R}(X) \textit{ defined by the valuation ring} \\ \mathcal{O}_{z,X},\ z = \textit{generic point of } Z \end{array} \right\}.$$

Then there is a well-defined homomorphism:

$$\mathrm{ord}\colon \mathrm{Div}(X) \longrightarrow Z^1(X)$$

given by $\mathrm{ord}(D) = \sum_{\mathcal{S}} (\mathrm{ord}_Z(f_z)) \cdot Z$ *(where f_z = local equation of D near the generic point $z \in Z$), and it is injective.* ord *is surjective if and only if X is factorial.*

Proof. Straightforward. □

Remark. Let X be a normal irreducible noetherian scheme with the function field $\mathbb{R}(X)$, and let D be a Cartier divisor on X. Then for $f \in \mathbb{R}(X)^*$, one has $(f) + D \geq 0$ if and only if $f \in \Gamma(X, \mathcal{O}_X(D))$. Thus the set of effective Cartier divisors linearly equivalent to D is controlled by the space $\Gamma(X, \mathcal{O}_X(D))$ of global sections of the invertible sheaf $\mathcal{O}_X(D)$.

Materials yet to be covered found among the loose notes for Chapter III.

- A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is said to be locally free of rank r if each point $x \in X$ has a neighborhood U such that there is an isomorphism
$$(\mathcal{O}_X|_U)^r \xrightarrow{\approx} \mathcal{F}|_U.$$
Already in Definition I.5.3. Such an $\mathcal{F}$ may be explicitly described in terms of $H^1(X, GL_r(\mathcal{O}_X))$.
- It is also possible to associate vector bundles to locally free sheaves and vice versa which are inverse to each other up to canonical isomorphism: Given a locally free $\mathcal{O}_X$-module $\mathcal{F}$ of rank r, let $\check{\mathcal{F}} = \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ be the dual $\mathcal{O}_X$-module. Let
$$\mathbb{V}(\mathcal{F}) = \mathcal{S}pec_X(\bigoplus_{n=0}^{\infty} \mathrm{Symm}^n(\check{\mathcal{F}})),$$
and let $\pi\colon \mathbb{V}(\mathcal{F}) \to X$ be the projection. $\pi\colon \mathbb{V}(\mathcal{F}) \to X$ is the vector bundle of rank r over X, and $\mathcal{F}$ is the sheaf of germs of sections of π.

Exercise.

(1) Prove that the Segre embedding (cf. Example I.8.11 and Proposition II.1.2)
$$i\colon \mathbb{P}^{n_1}_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{P}^{n_2}_{\mathbb{Z}} \hookrightarrow \mathbb{P}^{n_1 n_2 + n_1 + n_2}_{\mathbb{Z}}$$
corresponds in Theorem 2.2 to the invertible sheaf $\mathcal{O}_{\mathbb{P}^{n_1}_{\mathbb{Z}}}(1) \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{P}^{n_2}_{\mathbb{Z}}}(1)$ and the surjective homomorphism
$$(\mathcal{O}_{\mathbb{P}^{n_1}_{\mathbb{Z}}})^{n_1+1} \otimes_{\mathbb{Z}} (\mathcal{O}_{\mathbb{P}^{n_2}_{\mathbb{Z}}})^{n_2+1} \longrightarrow \mathcal{O}_{\mathbb{P}^{n_1}_{\mathbb{Z}}}(1) \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{P}^{n_2}_{\mathbb{Z}}}(1)$$
obtained as the tensor product over Z of the canonical surjective homomorphisms
$$(\mathcal{O}_{\mathbb{P}^{n_1}_{\mathbb{Z}}})^{n_1+1} \longrightarrow \mathcal{O}_{\mathbb{P}^{n_1}_{\mathbb{Z}}}(1)$$
$$(\mathcal{O}_{\mathbb{P}^{n_2}_{\mathbb{Z}}})^{n_2+1} \longrightarrow \mathcal{O}_{\mathbb{P}^{n_2}_{\mathbb{Z}}}(1).$$

(2) Let X be of finite type over R. Prove that if $\mathcal{L}_1$, $\mathcal{L}_2$ are very ample (resp. ample) invertible sheaves on X, then $\mathcal{L}_1 \otimes \mathcal{L}_2$ is very ample (resp. ample).

(3) Let k be a field and consider $\mathbb{P}^n_k$.
  a) All maximal irreducible subsets of $\mathbb{P}^n_k$ are of the form $V(f)$, $f \in k[X_0, \dots, X_n]$ homogeneous and irreducible.
  b) All effective Cartier divisors D on $\mathbb{P}^n_k$, considered via (a) above as subschemes of $\mathbb{P}^n_k$, are equal to $V(f)$, some homogeneous $f \in k[X_0, \dots, X_n]$.
  c) Two effective divisors $D_1 = V(f_1)$ and $D_2 = V(f_2)$ are linearly equivalent if and only if $\deg f_1 = \deg f_2$; hence the set of all effective divisors D given by subschemes $V(f)$, $\deg f = d$, is a complete linear system; the canonical map
$$k[X_0, \dots, X_n]_d \longrightarrow \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d))$$
is an isomorphism and $\mathrm{Pic}(\mathbb{P}^n_k) \cong \mathbb{Z}$, with $\mathcal{O}_{\mathbb{P}^n_k}(1)$ being a generator.

d) If $\sigma\colon \mathbb{P}^n_k \to \mathbb{P}^n_k$ is an automorphism over k, then $\sigma^*(\mathcal{O}_{\mathbb{P}^n_k}(1)) \cong \mathcal{O}_{\mathbb{P}^n_k}(1)$. Using the induced action on $\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1))$, show that σ is induced by the linear change of homogeneous coordinates $A \in GL_{n+1}(k)$.

(4) Maybe in Chapter II? Let $f\colon X \to Y$ be a finite morphism. If the fibre $f^{-1}(y)$ over one point $y \in Y$ is isomorphic to $\operatorname{Spec} \Bbbk(y)$, show that $\operatorname{res} f\colon f^{-1}(U) \to U$ is a closed immersion for some neighborhood U of y.

(5) Should be in Chapter V after étale is defined. (Kummer theory. cf. Theorem VIII.4.2 for the case $n = \operatorname{char} k$.) Let X be a noetherian scheme with $1/n, \zeta \in \Gamma(\mathcal{O}_X)$, $\zeta =$ primitive n-th root of unity, and consider pairs (π, ϕ):

$$\begin{tikzcd} Y \arrow[loop right, "\phi"] \arrow[d, "\pi"'] \\ X \end{tikzcd}$$

π étale and proper, $\pi = \pi \circ \phi$, $\phi^n = 1_Y$ and for all geometric points:

$$\lambda\colon \operatorname{Spec} k \longrightarrow X, \qquad k \text{ algebraically closed},$$

we assume

$$Y \times_X \operatorname{Spec} k = n \text{ points permuted cyclically by } \phi \times 1_y.$$

We call this an *n-cyclic étale covering* of X. Prove that $\exists$ an invertible sheaf $\mathcal{L}$ on X and an isomorphism $\alpha\colon \mathcal{L}^n \xrightarrow{\approx} \mathcal{O}_X$ such that

$$Y = \mathcal{S}pec_X\, \mathcal{A}$$

$$\mathcal{A} = \mathcal{O}_X \oplus \mathcal{L} \oplus \mathcal{L}^2 \oplus \cdots \oplus \mathcal{L}^{n-1}$$

with multiplication

$$\mathcal{L}^i \times \mathcal{L}^j \longrightarrow \begin{cases} \mathcal{L}^{i+j} & i+j < n \\ \mathcal{L}^{i+j-n} & i+j \geq n \text{ via } \alpha. \end{cases}$$

Hint: Write $Y = \mathcal{S}pec_X\, \mathcal{A}$ (cf. Proposition-Definition I.7.3) and show that $\mathcal{A}$ decomposes into eigensheaves under the action of ϕ^*:

$$\mathcal{A} = \bigoplus_{\nu=0}^{n-1} \mathcal{L}_\nu, \qquad \phi^*(x) = \zeta^\nu \cdot x, \quad x \in \mathcal{L}_\nu(U).$$

Use the fact: flat + finite presentation over a local ring $\Longrightarrow$ free to deduce that the $\mathcal{L}_\nu$ are locally free. Then show by computing geometric fibres that $\operatorname{rk} \mathcal{L}_\nu = 1$ and multiplication induces an isomorphism $\mathcal{L}_i \otimes \mathcal{L}_j \xrightarrow{\approx} \mathcal{L}_{i+j}$ or $\mathcal{L}_{i+j-n}$. Show conversely that for any $\mathcal{L}$, α, we obtain an n-cyclic étale covering Y. Deduce that if X is a complete variety over an algebraically closed field k, then:

$$\{\text{Set of } n\text{-cyclic étale coverings}\} \cong \{\lambda \in \operatorname{Pic}(X) \mid n\lambda = 0\}.$$

(6) Maybe in another chapter? Let $f\colon X \to Y$ be a morphism of finite type with Y noetherian such that $f^{-1}(y)$ is finite for all $y \in Y$. Show that $\exists$ an open dense $U \subset Y$ such that

$$\operatorname{res} f\colon f^{-1}(U) \longrightarrow U$$

is finite.

CHAPTER IV

Ground fields and base rings

1. Kronecker's big picture

For all schemes X, there is a unique morphism:

$$\pi \colon X \longrightarrow \operatorname{Spec} \mathbb{Z}.$$

This follows from Theorem I.3.7, since there is a unique homomorphism

$$\pi^* \colon \mathbb{Z} \longrightarrow \Gamma(\mathcal{O}_X).$$

Categorically speaking, $\operatorname{Spec} \mathbb{Z}$ is the final object in the category of schemes. $\operatorname{Spec} \mathbb{Z}$ itself is something like a line, but in which the variable runs not over constants in a fixed field but over primes p. In fact $\mathbb{Z}$ is a principal ideal domain like $k[X]$ and its prime ideals are (p) or $p \cdot \mathbb{Z}$, p a prime number, and (0). (cf. Figure IV.1) The stalk of the structure sheaf at $[(p)]$ is the discrete valuation ring $\mathbb{Z}_{(p)} = \{m/n \mid p \nmid n\}$ and at $[(0)]$ is the field $\mathbb{Q}$. $\operatorname{Spec} \mathbb{Z}$ is reduced and irreducible with "function field" $\mathbb{R}(\operatorname{Spec} \mathbb{Z}) = \mathbb{Q}$. The non-empty open sets of $\operatorname{Spec} \mathbb{Z}$ are gotten by throwing away finitely many primes $p_1, \ldots, p_n$. If $m = \prod p_i$, then this is a distinguished open set:

$$\operatorname{Spec}(\mathbb{Z})_m, \qquad \text{with ring} \quad \mathbb{Z}_m = \{\frac{a}{m^n} \mid a, n \in \mathbb{Z}\}.$$

The residue fields are:

$$\begin{aligned} \Bbbk([(p)]) &= \mathbb{Z}/p\mathbb{Z} \\ \Bbbk([(0)]) &= \mathbb{Q}, \end{aligned}$$

i.e., each prime field occurs exactly once.

If X is an arbitrary scheme, then set-theoretically the morphism

$$\pi \colon X \longrightarrow \operatorname{Spec} \mathbb{Z}$$

is just the map

$$x \longmapsto \operatorname{char} \Bbbk(x),$$

because if $\pi(x) = y$, then we get

$$\Bbbk(x) \xhookleftarrow[\pi_x^*]{} \Bbbk(y) = \begin{cases} \mathbb{Z}/p\mathbb{Z} \\ \text{or} \\ \mathbb{Q} \end{cases}$$

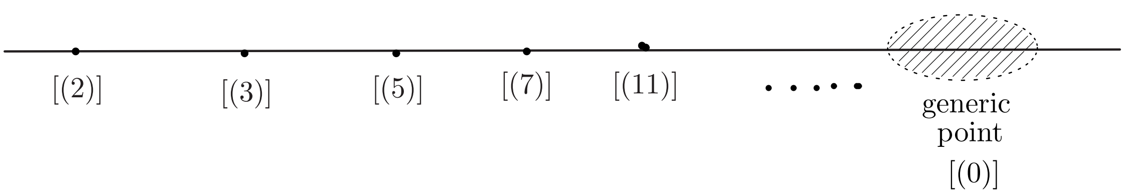

Figure IV.1. $\operatorname{Spec} \mathbb{Z}$

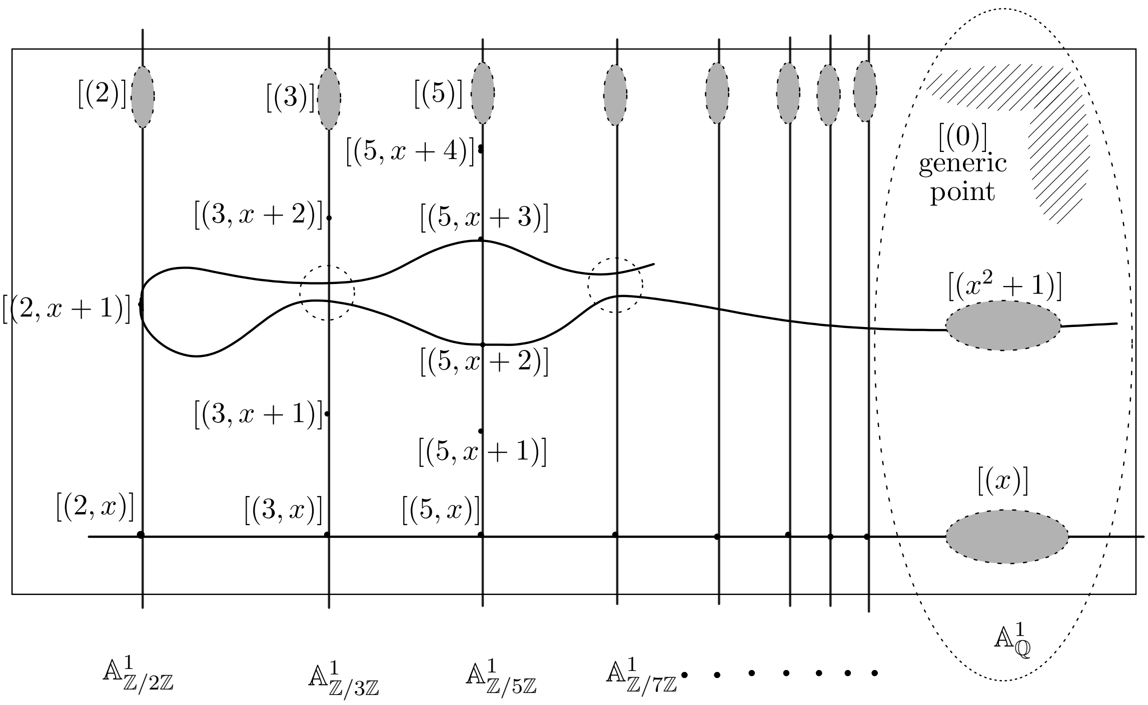

FIGURE IV.2. $\mathbb{A}^1_{\mathbb{Z}}$

hence

$$\begin{aligned}\operatorname{char} \mathbb{k}(x) = p > 0 &\Longrightarrow \pi(x) = [(p)]\\ \operatorname{char} \mathbb{k}(x) = 0 &\Longrightarrow \pi(x) = [(0)].\end{aligned}$$

Thus every scheme X is a kind of fibred object, made up out of separate schemes (possibly empty),

$$X \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \begin{cases} \mathbb{Z}/p\mathbb{Z} \\ \text{or} \\ \mathbb{Q}, \end{cases}$$

of each characteristic! For instance, we can "draw" a sort of picture of the scheme $\mathbb{A}^1_{\mathbb{Z}}$, showing how it is the union of the affine lines $\mathbb{A}^1_{\mathbb{Z}/p\mathbb{Z}}$ and $\mathbb{A}^1_{\mathbb{Q}}$. The prime ideals in $\mathbb{Z}[X]$ are:

i) (0),
ii) principal prime ideals (f), where f is either a prime number p, or a $\mathbb{Q}$-irreducible integral polynomial written so that its coefficients have greatest common divisor 1,
iii) maximal ideals (p, f), p a prime and f a monic integral polynomial irreducible modulo p.

The whole should be pictured as in Figure IV.2. (The picture is misleading in that $\mathbb{A}^1_{\mathbb{Z}/p\mathbb{Z}}$ for any p has actually an infinite number of closed points: i.e., in addition to the maximal ideals $(p, X-a)$, $0 \le a \le p-1$, with residue field $\mathbb{Z}/p\mathbb{Z}$, there will be lots of others $(p, f(x))$, $\deg f > 1$, with residue fields $\mathbb{F}_{p^n} =$ finite field with p^n elements, $n > 1$.)

An important property of schemes of finite type over $\mathbb{Z}$ is:

PROPOSITION 1.1. *Let X be of finite type over $\mathbb{Z}$ and let $x \in X$. Then*

$$[x \textit{ is closed}] \Longleftrightarrow [\mathbb{k}(x) \textit{ is finite}].$$

PROOF. Let $\pi\colon X \to \operatorname{Spec}\mathbb{Z}$ be the morphism. By Theorem II.2.9 (Chevalley's Nullstellensatz),

$$x \text{ closed} \Longrightarrow \{\pi(x)\} \text{ constructible} \Longrightarrow \pi(x) \text{ closed}.$$

If $\pi(x) = [(p)]$, then $x \in X \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}/p\mathbb{Z}$ — call this scheme X_p. Then x is a closed point of X_p, so by Corollary II.2.11, x is an algebraic point, i.e., $\Bbbk(x)$ is algebraic over $\mathbb{Z}/p\mathbb{Z}$, so $\Bbbk(x)$ is finite. Conversely, if $\Bbbk(x)$ is finite, let p be its characteristic. Then $x \in X_p$ and by Corollary II.2.11, x is closed in X_p and since X_p is closed in X, x is closed in X. $\square$

From the point of view of arithmetic, schemes of finite type over $\mathbb{Z}$ are the basic objects. The classical problem in Diophantine equations is always to find all $\mathbb{Z}$- or $\mathbb{Q}$-valued points of various schemes X (recall Definition I.6.2). For instance, if $f \in \mathbb{Z}[X_1, \dots, X_n]$, the solutions

$$f(a_1, \dots, a_n) = 0$$

with a_i in any ring R are just the R-valued points of the affine scheme

$$\operatorname{Spec}\mathbb{Z}[X_1, \dots, X_n]/(f)$$

(see Theorem I.3.7). Because of its homogeneity, however, Fermat's last theorem may also be interpreted via the "plane curve"

$$V(X_1^n + X_2^n - X_0^n) \subset \mathbb{P}^2_{\mathbb{Z}}$$

and the conjecture[1] asserts that if $n \geq 3$, its only $\mathbb{Q}$-valued points are the trivial ones, where either X_0, X_1, or X_2 is 0. Moreover, it is for such schemes that a zeta-function can be introduced formally:

$$\zeta_X(s) = \prod_{\substack{\text{closed}\\ \text{points}\\ x\in X}} \left(1 - \frac{1}{(\#\Bbbk(x))^s}\right)^{-1}, \qquad \# = \text{cardinality} \tag{1.2}$$

which one expands formally to the Dirichlet series

$$\begin{aligned} \zeta_X(s) &= \sum_{n=1}^{\infty} \frac{a_n}{n^s} \\ a_n &= \left\{ \begin{array}{l} \text{number of 0-cycles } \mathfrak{a} = \sum n_i x_i \text{ on } X, \\ \text{where } n_i > 0,\ x_i \in X \text{ closed and } \deg \mathfrak{a} \overset{\text{def}}{=} \sum n_i \#\Bbbk(x_i) \text{ is } n. \end{array} \right\} \end{aligned} \tag{1.3}$$

This is known to converge if $\operatorname{Re} s \gg 0$ and is conjectured to be meromorphic in the whole s-plane—cf. Serre's talk [**94**] for a general introduction.

But these schemes also play a fundamental role for many geometric questions because of the following simple but very significant observation:

> Suppose $X \subset \mathbb{A}^n_{\mathbb{C}}$ (resp. $X \subset \mathbb{P}^n_{\mathbb{C}}$) is a complex affine (resp. projective) variety. Let its ideal be generated by polynomials (resp. homogeneous polynomials) $f_1, \dots, f_k$. Let $R \subset \mathbb{C}$ be a subring finitely generated over $\mathbb{Z}$ containing the coefficients of the f_i: Then $f_1, \dots, f_k$ define $X_0 \subset \mathbb{A}^n_R$ (resp. $X_0 \subset \mathbb{P}^n_R$) such that
> a) $X \cong X_0 \times_{\operatorname{Spec} R} \operatorname{Spec}\mathbb{C}$
> b) X_0 is of finite type over R, hence is of finite type over $\mathbb{Z}$.

More generally, we have:

[1] (Added in publication) The conjecture has since been settled affirmatively by Wiles [**107**].

Proposition 1.4. *Let X be a scheme of finite type over $\mathbb{C}$. Then there is a subring $R \subset \mathbb{C}$, finitely generated over $\mathbb{Z}$ and a scheme X_0 of finite type over R such that*

$$X \cong X_0 \times_{\operatorname{Spec} R} \operatorname{Spec} \mathbb{C}.$$

Proof. Let $\{U_i\}$ be a finite affine open covering of X and write

$$U_i = \operatorname{Spec} \mathbb{C}[X_1, \dots, X_{n_i}]/(f_{i,1}, \dots, f_{i,k_i}) = \operatorname{Spec} R_i.$$

For each i, j, cover $U_i \cap U_j$ by open subsets which are distinguished affines in U_i and U_j and let each of these subsets define an isomorphism

$$\phi_{ij,l} \colon (R_i)_{g_{ij,l}} \xrightarrow{\approx} (R_j)_{g_{ji,l}}.$$

The fact that

$$(U_i)_{g_{ij,l} \cdot \phi^{-1}_{ij,l}(g_{jk,l'})} \subset U_i \cap U_j \cap U_k \subset \bigcup_{l''} (U_i)_{g_{ik,l''}}$$

means that

$$(*) \qquad \left[g_{ij,l} \cdot \phi^{-1}_{ij,l}(g_{jk,l'}) \right]^N = \sum_{l''} a_{ijkll'l''} g_{ikl''}, \quad \text{suitable } a\text{'s in } R_i.$$

Let R be generated by the coefficients of the f_{ij}'s, the g's and a's (lifted to $\mathbb{C}[X]$) and of the polynomials defining the $\phi_{ij,l}$'s. Define

$$U_{i,0} = \operatorname{Spec} R[X_1, \dots, X_{n_i}]/I_i = \operatorname{Spec} R_{i,0}$$

where $I_i = \operatorname{Ker} [R[X] \to \mathbb{C}[X]/(f_{i,1}, \dots, f_{i,k_i})]$, i.e., I_i consists of the f_{ij}'s plus enough other polynomials to make $R_{i,0}$ into a subring of R_i. Clearly $R_i \cong R_{i,0} \otimes_R \mathbb{C}$. Then $g_{ij,l}$ is in the subring $R_{i,0}$ and $\phi_{ij,l}$ restricts to an isomorphism $(R_{i,0})_{g_{ij,l}} \xrightarrow{\approx} (R_{j,0})_{g_{ji,l}}$, hence ϕ defines:

$$(U_{i,0})_{g_{ij,l}} \xrightarrow{\approx} (U_{j,0})_{g_{ji,l}}.$$

Let $U^{(j)}_{i,0} = \bigcup_l (U_{i,0})_{g_{ij,l}}$ and glue $U^{(j)}_{i,0}$ to $U^{(i)}_{j,0}$ by these ϕ's: the fact that $\phi_{ij,l} = \phi_{ij,l'}$ on overlaps is guaranteed by the fact that $R_{i,0} \subset R_i$. Moreover the identity $(*)$ still holds because we smartly put the coefficients of the a's in R, hence points of $U_{i,0}$ which are being glued to points of $U_{j,0}$ which in turn are being glued to points of $U_{k,0}$ are being *directly* glued to points of $U_{k,0}$; Moreover the direct and indirect glueing maps again agree because $R_{i,0} \subset R_i$. Thus an X_0 can be constructed by glueing all the $U_{i,0}$'s and clearly $X \cong X_0 \times_{\operatorname{Spec} R} \operatorname{Spec} \mathbb{C}$. □

The idea of Kroneckerian geometry is that when you have $X \cong X_0 \times_{\operatorname{Spec} R} \operatorname{Spec} \mathbb{C}$, then (a) classical geometric properties of X over $\mathbb{C}$ may influence Diophantine problems on X_0, and (b) Diophantine properties of X_0, even for instance the characteristic p fibres of X_0, may influence the geometry on X. In order to go back and forth in this way between schemes over $\mathbb{C}$, $\mathbb{Z}$ and finite fields, one must make use of all possible homomorphisms and intermediate rings that nature gives us. These "God-given" natural rings form a diagram as in Figure IV.3 (with various Galois groups acting too), where the completion $\widehat{\overline{\mathbb{Q}_p}}$ of the algebraic closure $\overline{\mathbb{Q}_p}$ of the p-adic number field $\mathbb{Q}_p$ is known to be algebraically closed, $\widehat{\widetilde{\mathbb{Z}_p}}$ is the completion of the integral closure $\widetilde{\mathbb{Z}_p}$ in $\overline{\mathbb{Q}_p}$ of the ring of p-adic integers $\mathbb{Z}_p$, the field of algebraic numbers $\overline{\mathbb{Q}}$ is the algebraic closure of the rational number field $\mathbb{Q}$, and $\overline{\mathbb{Z}/p\mathbb{Z}}$ is the algebraic closure of $\mathbb{Z}/p\mathbb{Z}$: Thus given any $X \to \operatorname{Spec} \mathbb{Z}$, say of finite type, one gets a big diagram of schemes as in Figure IV.4 (where we have written X_R for $X \times \operatorname{Spec} R$, and $\overline{R}$ for of the algebraic closure or integral closure of R, or completions thereof.)

In order to use the diagram (1.6) effectively, there are two component situations that must first be studied in detail:

(1.5)

$$\begin{tikzcd}
\mathbb{C} \arrow[rrr, dashed, "\text{non-canonical isomorphism}"] & & & \widehat{\overline{\mathbb{Q}}_p} & \\
\cup & \overline{\mathbb{Q}} \arrow[ul, hook] \arrow[urr, hook] & & & \\
\mathbb{R} & \cup & \mathbb{Q}_p \arrow[uur, hook] & \cup & \\
& \mathbb{Q} \arrow[ul, hook] \arrow[ur, hook] & \cup & \widehat{\overline{\mathbb{Z}}_p} \arrow[dr, two heads] & \\
& \cup & \mathbb{Z}_p \arrow[ur, hook] \arrow[dr, two heads] & & \overline{\mathbb{Z}/p\mathbb{Z}} \\
& \mathbb{Z} \arrow[ur, hook] & & \mathbb{Z}/p\mathbb{Z} \arrow[ur, hook] &
\end{tikzcd}$$

Figure IV.3. The diagram formed by "God-given" natural rings

(1.6)

$$\begin{tikzcd}
X_{\mathbb{C}} \arrow[rrr, dash, dashed, "\text{non-canonical isomorphism}"] \arrow[dd] \arrow[dr] & & & X_{\overline{\mathbb{Q}_p}} \arrow[dll] \arrow[ddl] \arrow[ddd] & \\
& X_{\overline{\mathbb{Q}}} \arrow[dd] & & & \\
X_{\mathbb{R}} \arrow[dr] & & X_{\mathbb{Q}_p} \arrow[dl] \arrow[dd] & & \\
& X_{\mathbb{Q}} \arrow[dd] & & X_{\overline{\mathbb{Z}_p}} \arrow[dl] \arrow[dr] & \\
& & X_{\mathbb{Z}_p} \arrow[dl] \arrow[dr] & & X_{\overline{\mathbb{Z}/p\mathbb{Z}}} \arrow[dl] \\
& X_{\mathbb{Z}} & & X_{\mathbb{Z}/p\mathbb{Z}} &
\end{tikzcd}$$

Figure IV.4. The big diagram of schemes

1.7. *Given*

$$\begin{cases} k \text{ a field} \\ \overline{k} = \text{algebraic closure of } k \\ X \text{ of finite type over } k \end{cases}$$

consider:

$$\begin{tikzcd}
\overline{X} \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} \overline{k} \arrow[r] & \operatorname{Spec} k
\end{tikzcd}$$

where $\overline{X} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$. *Compare* X *and* $\overline{X}$.

1.8. *Given*

$$\begin{cases} R \text{ a valuation ring} \\ K \text{ its quotient field} \\ k \text{ its residue field} \\ X \text{ of finite type over } R \end{cases}$$

consider:

$$\begin{tikzcd}
X_\eta \arrow[r] \arrow[d] & X \arrow[d] & X_0 \arrow[l] \arrow[d] \\
\operatorname{Spec} K \arrow[r] & \operatorname{Spec} R & \operatorname{Spec} k \arrow[l]
\end{tikzcd}$$

where $X_\eta = X \times_{\operatorname{Spec} R} \operatorname{Spec} K$, $X_0 = X \times_{\operatorname{Spec} R} \operatorname{Spec} k$. *Compare* X_0 *and* X_η.

We take these situations up in §§2–3 and §§4–6 separately. In §VIII.5 we will give an illustration of how the big picture is used.

Classical geometry was the study of varieties over $\mathbb{C}$. But it did not exploit the fact that the defining equations of a variety can have coefficients in a *subfield* of $\mathbb{C}$. This possibility leads us directly to the analysis of schemes over non-algebraically closed fields (1.7), and to the relation between schemes over two different fields given by (1.8).

2. Galois theory and schemes

For this whole part, fix a field k and an algebraic closure $\overline{k}$. We write $\mathrm{Gal}(\overline{k}/k)$ for the Galois group, and for each scheme X over k, we write $\overline{X}$ for $X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \overline{k}$. First consider the action of $\mathrm{Gal}(\overline{k}/k)$ on $\overline{k}^n$ by conjugation:

1. For $\sigma \in \mathrm{Gal}(\overline{k}/k)$
$$(a_1, \dots, a_n) \longmapsto (\sigma a_1, \dots, \sigma a_n), \qquad \overline{k}^n \longrightarrow \overline{k}^n.$$
If we identify $\overline{k}^n$ with the set of closed points of $\mathbb{A}^n_{\overline{k}}$, then this map *extends in fact to an automorphism of* $\mathbb{A}^n_{\overline{k}}$:
2. Define $\sigma_{\mathbb{A}^n} \colon \mathbb{A}^n_{\overline{k}} \to \mathbb{A}^n_{\overline{k}}$ by
$$(\sigma_{\mathbb{A}^n})^* \colon \overline{k}[X_1, \dots, X_n] \longrightarrow \overline{k}[X_1, \dots, X_n]$$
where
$$\sigma^*_{\mathbb{A}^n}(X_i) = X_i, \qquad \sigma^*_{\mathbb{A}^n}(a) = \sigma^{-1} a, \qquad a \in \overline{k}.$$
In fact, for all prime ideals $\mathfrak{p} \subset \overline{k}[X_1, \dots, X_n]$,
$$\sigma_{\mathbb{A}^n}([\mathfrak{p}]) = [(\sigma^*_{\mathbb{A}^n})^{-1}\mathfrak{p}]$$
and if $\mathfrak{p} = (X_1 - a_1, \dots, X_n - a_n)$, then since $\sigma^*_{\mathbb{A}^n}(X_i - \sigma a_i) = X_i - a_i$, we find $(\sigma^*_{\mathbb{A}^n})^{-1}\mathfrak{p} \supset (X_1 - \sigma a_1, \dots, X_n - \sigma a_n)$; since $(X_1 - \sigma a_1, \dots, X_n - \sigma a_n)$ is maximal, $(\sigma^*_{\mathbb{A}^n})^{-1}\mathfrak{p} = (X_1 - \sigma a_1, \dots, X_n - \sigma a_n)$.

   Note that $\sigma_{\mathbb{A}^n}$ is a k-morphism but *not a $\overline{k}$-morphism.* For this reason, $\sigma_{\mathbb{A}^n}$ will have, for instance, a graph in
$$\mathbb{A}^n_{\overline{k}} \times_{\mathrm{Spec}\, k} \mathbb{A}^n_{\overline{k}} = \mathrm{Spec}\left((\overline{k} \otimes_k \overline{k})[X_1, \dots, X_n, Y_1, \dots, Y_n]\right),$$
but not in $\mathbb{A}^n_{\overline{k}} \times_{\mathrm{Spec}\, \overline{k}} \mathbb{A}^n_{\overline{k}} = \mathbb{A}^{2n}_{\overline{k}}$. Thus when $\overline{k} = \mathbb{C}$, $\sigma_{\mathbb{A}^n}$ will not be a correspondence nor will it act at all continuously in the classical topology (with the one exception $\sigma =$ complex conjugation).
3. Now we may also define $\sigma_{\mathbb{A}^n}$ as:
$$\sigma_{\mathbb{A}^n} = 1_{\mathbb{A}^n_k} \times \sigma_k \colon \mathbb{A}^n_k \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \overline{k} \longrightarrow \mathbb{A}^n_k \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \overline{k}$$
where $\sigma_k \colon \mathrm{Spec}\, \overline{k} \to \mathrm{Spec}\, \overline{k}$ is defined by $(\sigma_k)^* a = \sigma^{-1} a$.

The third form clearly generalizes to all schemes of the form X:

Definition 2.1. For every k-scheme X, define the conjugation action of $\mathrm{Gal}(\overline{k}/k)$ on $\overline{X}$ to be:
$$\sigma_X = 1_X \times \sigma_k \colon \overline{X} \to \overline{X}, \quad \text{all } \sigma \in \mathrm{Gal}(\overline{k}/k).$$

Then σ_X is *not* a $\overline{k}$-morphism, but rather fits into a diagram:
$$\begin{array}{ccc} \overline{X} & \xrightarrow{\sigma_X} & \overline{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, \overline{k} & \xrightarrow{\sigma_k} & \mathrm{Spec}\, \overline{k}. \end{array}$$

$$\begin{tikzcd}
 & & & & \Omega \\
 & (R\otimes_k \overline{k})/\mathfrak{p}_1 \arrow[urrr, dashed, "\alpha_1"] & & (R\otimes_k \overline{k})/\mathfrak{p}_2 \arrow[ur, dashed, "\alpha_2"'] & \\
 \overline{k} \arrow[ur, hook] \arrow[urrr, hook] & & & & R/\mathfrak{p} \arrow[ulll, hook] \arrow[ul, hook] \arrow[uu, hook] \\
 & & k \arrow[ull, hook] \arrow[urr, hook] & &
\end{tikzcd}$$

FIGURE IV.5

What this means is that if $f \in \mathcal{O}_{\overline{X}}(U)$ then $\sigma_X^* f \in \mathcal{O}_{\overline{X}}(\sigma_X^{-1}U)$ has value at a point $x \in \sigma_X^{-1}U$ given by:

$$(\sigma_X^* f)(x) = \sigma^{-1} \cdot f(\sigma_X \cdot x), \tag{2.2}$$

i.e., set-theoretically, σ_X^* is *not* "pull-back" on functions. This can be proven as follows:

$$\begin{aligned}
f - f(\sigma_X \cdot x) \in \mathfrak{m}_{\sigma_X x, \overline{X}} &\Longrightarrow \sigma_X^*(f - f(\sigma_X \cdot x)) \in \mathfrak{m}_{x,\overline{X}} \\
&\Longrightarrow \sigma_X^* f - \sigma^{-1} \cdot f(\sigma_X \cdot x) \in \mathfrak{m}_{x,\overline{X}} \\
&\Longrightarrow \sigma_X^* f(x) = \sigma^{-1} \cdot f(\sigma_X \cdot x).
\end{aligned}$$

I want next to analyze the relationship between X and $\overline{X}$. The first point is that topologically X is the quotient of $\overline{X}$ by the action of $\mathrm{Gal}(\overline{k}/k)$.

THEOREM 2.3. *Let X be a scheme of finite type over k, let*

$$\overline{X} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$$

and let $p\colon \overline{X} \to X$ be the projection. Then

1) *p is surjective and both open and closed (i.e., maps open (resp. closed) sets to open (resp. closed) sets);*
2) *$\forall x, y \in \overline{X}$, $p(x) = p(y)$ iff $x = \sigma_X(y)$ for some $\sigma \in \mathrm{Gal}(\overline{k}/k)$;*
3) *$\forall x \in X$, let $Z =$ closure of $\{x\}$. Then $p^{-1}(x) =$ the set of generic points of the components of $p^{-1}(Z)$. In particular, $p^{-1}(x)$ is finite.*

PROOF. Since all these results are local on X, we may as well replace X by an open affine subset U, and replace $\overline{X}$ by $p^{-1}U$. Therefore assume $X = \operatorname{Spec} R$, $\overline{X} = \operatorname{Spec} R \otimes_k \overline{k}$. First of all, p is surjective by Corollary I.4.4. Secondly, p is closed because $R \otimes_k \overline{k}$ is integrally dependent on R (cf. Proposition II.6.5; this is an easy consequence of the Going-up theorem). Thirdly, let's prove (2). If $\mathfrak{p}_1, \mathfrak{p}_2 \subset R \otimes_k \overline{k}$ are two prime ideals, we must show:

$$\mathfrak{p}_1 \cap R = \mathfrak{p}_2 \cap R \Longleftrightarrow \exists \sigma \in \mathrm{Gal}(\overline{k}/k), \quad \mathfrak{p}_1 = (1_R \otimes \sigma)\mathfrak{p}_2.$$

$\Longleftarrow$ is obvious, so assume $\mathfrak{p}_1 \cap R = \mathfrak{p}_2 \cap R$. Call this prime $\mathfrak{p}$. Let Ω be an algebraically closed field containing $R/\mathfrak{p}$. Consider the *solid* arrows in Figure IV.5. It follows that there exist injective k-homomorphisms α_1, α_2 as indicated. Then $\alpha_1(\overline{k})$ and $\alpha_2(\overline{k})$ both equal the algebraic closure of k in Ω, so for some $\sigma \in \mathrm{Gal}(\overline{k}/k)$, $\alpha_2 = \alpha_1 \circ \sigma$ on $\overline{k}$. But then if $x_i \in R$, $y_i \in \overline{k}$:

$$\begin{aligned} \sum x_i \otimes y_i \in \mathfrak{p}_2 &\iff \sum x_i \cdot \alpha_2(y_i) = 0 \text{ in } \Omega \\ &\iff \sum x_i \cdot \alpha_1(\sigma(y_i)) = 0 \text{ in } \Omega \\ &\iff \sum x_i \otimes \sigma(y_i) \in \mathfrak{p}_1, \end{aligned}$$

so $(1_R \otimes \sigma)\mathfrak{p}_2 = \mathfrak{p}_1$. Fourthly, p is an open map. In fact, let $U \subset \overline{X}$ be open. Then

$$U' = \bigcup_{\sigma \in \mathrm{Gal}} \sigma_X(U)$$

is also open, and by (2), $p(U) = p(U')$ and $U' = p^{-1}(p(U'))$. Therefore $X \backslash p(U) = p(\overline{X} \backslash U')$ which is closed since p is a closed map. Therefore $p(U)$ is open. Finally, let $x \in X$, $Z =$ closure of $\{x\}$. Choose $w \in p^{-1}(x)$ and let $W =$ closure of $\{w\}$. Since p is closed, $p(W)$ is a closed subset of Z containing x, so $p(W) = Z$. Therefore $\bigcup_{\sigma \in \mathrm{Gal}} \sigma_X(W)$ is Gal-invariant and maps onto Z, so by (2):

$$\bigcup_{\sigma \in \mathrm{Gal}} \sigma_X(W) = p^{-1}Z.$$

Therefore every component of $p^{-1}Z$ equals $\sigma_X(W)$ for some σ, and since they are all conjugate, the $\sigma_X(W)$'s are precisely the components of $p^{-1}Z$. (3) now follows easily. □

Suppose now X is a k-variety. Is $\overline{X}$ necessarily a $\overline{k}$-variety?

THEOREM 2.4. *Let X be a k-variety and let $\overline{X} = X \times_k \overline{k}$.*

i) *Let*

$$L = \{x \in \mathbb{R}(X) \mid x \text{ separable algebraic over } k\}.$$

Then L is a finite algebraic extension of k. Let $U \subset X$ be an open set such that the elements of L extend to sections of $\mathcal{O}_X$ over U. Then the basic morphism from X to $\mathrm{Spec}\, k$ *factors:*

$$\begin{tikzcd} U \arrow[d, "f"'] & \subset & X \arrow[dd] \\ \mathrm{Spec}\, L \arrow[drr] & & \\ & & \mathrm{Spec}\, k \end{tikzcd}$$

and taking fibre products with $\mathrm{Spec}\,\overline{k}$*, we get:*

$$\begin{tikzcd} \overline{U} \arrow[d, "\overline{f}"'] & \subset & \overline{X} \arrow[dd] \\ \mathrm{Spec}\, L \otimes_k \overline{k} \arrow[drr] & & \\ & & \mathrm{Spec}\,\overline{k}. \end{tikzcd}$$

Then

$$L \otimes_k \overline{k} \cong \prod_{i=1}^{t} \overline{k}$$

$\mathrm{Spec}\, L \otimes_k \overline{k} =$ *disjoint union of t reduced closed points $P_1, \ldots, P_t$*

$\overline{U} =$ *disjoint union of t irreducible pieces $\overline{U}_i = \overline{f}^{-1}(P_i)$*

$\overline{X} =$ *union of t irreducible components $\overline{X}_i$, with $\overline{X}_i =$ closure$(\overline{U}_i)$.*

This induces an isomorphism of sets:

$$(\textit{Components of } \overline{X}) \cong \operatorname{Hom}_k(L, \overline{k}),$$

commuting with the action of the Galois group $\operatorname{Gal}(\overline{k}/k)$.

ii) *If* $y_i =$ *generic point of* $\overline{X}_i$, *then* y_i *maps to the generic point of* X *and*

$$\prod_{i=1}^{t} \mathcal{O}_{y_i,\overline{X}} \cong \mathbb{R}(X) \otimes_k \overline{k}$$

hence $\dim \overline{X}_i = \dim X$ *for all* i, *and:*

$$\begin{aligned} \overline{X} \textit{ is reduced} &\Longleftrightarrow \mathcal{O}_{y_i,\overline{X}} \textit{ has no nilpotents, for all } i \\ &\Longleftrightarrow \mathbb{R}(X) \textit{ is separable over } k. \end{aligned}$$

Proof of Theorem 2.4, (i). Let $L_1 \subset L$ be a subfield which is finite algebraic over k. Then $L_1 \otimes_k \overline{k}$ is a finite-dimensional separable $\overline{k}$-algebra, hence by the usual Wedderburn theorems,

$$L_1 \otimes_k \overline{k} \cong \prod_{i=1}^{t} \overline{k}, \quad \text{where } t = [L_1 : k]$$

and $\operatorname{Spec} L_1 \otimes_k \overline{k} = \{P_1, \dots, P_t\}$ as asserted. Elements of a basis of L_1 extend to sections of $\mathcal{O}_X$ over some open set U_1, and we get a diagram

$$\begin{tikzcd}
\overline{U}_1 \arrow[d, "\overline{f}_1"'] & \subset & \overline{X} \arrow[dd] \\
\{P_1, \dots, P_t\} \arrow[drr] & & \\
& & \operatorname{Spec} \overline{k}.
\end{tikzcd}$$

Therefore $\overline{U}_1$ is the disjoint union of open sets $\overline{f}_1^{-1}(P_i)$. Therefore $\overline{X}$ has *at least* t components, i.e., components of the closure of $\overline{f}_1^{-1}(P_i)$ in $\overline{X}$. But $\overline{X}$ has only a finite number of components, hence t is bounded above. Therefore L itself is finite over k. Now take $L_1 = L$. The main step consists in showing that $\overline{f}^{-1}(P_i)$ is irreducible. In fact

$$\begin{aligned} \overline{f}^{-1}(P_i) &\cong \overline{U} \times_{\operatorname{Spec} L \otimes_k \overline{k}} \operatorname{Spec} \overline{k} \\ &\cong U \times_{\operatorname{Spec} L} \operatorname{Spec} \overline{k}, \quad \text{via } L \to L \otimes_k \overline{k} \xrightarrow[\text{i-th factor}]{\text{projection on}} \overline{k} \end{aligned}$$

so in effect this step amounts to checking the special case:

$$k \text{ separable algebraically closed in } \mathbb{R}(X) \Longrightarrow \overline{X} \text{ is irreducible.}$$

The rest of part (i) follows from two remarks: first, by Theorem 2.3, (3), each component of $\overline{X}$ is the closure of a component of $\overline{U}$; secondly, there is an isomorphism of sets commuting with Gal:

$$\begin{aligned} \{\text{Maximal ideals of } L \otimes_k \overline{k}\} &\cong \{\text{Kernels of the various projections } L \otimes_k \overline{k} \to \overline{k}\} \\ &\cong \operatorname{Hom}_k(L, \overline{k}). \end{aligned}$$

Now consider the special case. If $\overline{X} = \bigcup_{i=1}^{t} \overline{X}_i$ is reducible, we can find an affine open $U \subset X$ such that the sets $p^{-1}(U) \cap \overline{X}_i = \overline{U}_i$ are disjoint. Let $U = \operatorname{Spec} R$, so that

$$\coprod_{i=1}^{t} \overline{U}_i = \operatorname{Spec} R \otimes_k \overline{k}.$$

Let ϵ_i be the function which equals 1 on $\overline{U}_i$ and 0 on the other $\overline{U}_j$. Then $\epsilon_i^n = \epsilon_i$ for all n and $\epsilon_i \in R \otimes_k \overline{k}$. Write

$$\epsilon_i = \sum_j \beta_{ij} \otimes \gamma_{ij}, \quad \beta_{ij} \in R, \gamma_{ij} \in \overline{k}.$$

Then if the characteristic is $p > 0$,

$$\epsilon_i = \epsilon_i^{p^n} = \sum \beta_{ij}^{p^n} \otimes \gamma_{ij}^{p^n}$$

and if $n \gg 0$, $\gamma_{ij}^{p^n} \in k_s$ = separable closure of k. Thus if $p > 0$, we find $\epsilon_i \in R \otimes_k k_s$ too. Let L_s be the k_s-subalgebra of $R \otimes_k \overline{k}$ that the ϵ_i generate. The Galois group, acting on $\overline{X}$, permutes the $\overline{X}_i$; hence acting on $R \otimes_k \overline{k} = \Gamma(\coprod \overline{U}_i, \mathcal{O}_{\overline{X}})$, it permutes the ϵ_i. Therefore L_s is a Gal-invariant subspace of $R \otimes_k k_s$. Now apply:

Lemma 2.5. *Let V be a k-vector space and let $W' \subset V \otimes_k k_s$ be a k_s-subspace. Then*

$$\left[\begin{array}{l} W' = W \otimes_k k_s \textit{ for} \\ \textit{some } k\textit{-subspace } W \subset V \end{array}\right] \iff \left[\begin{array}{l} W' \textit{ is invariant} \\ \textit{under } \mathrm{Gal}(k_s/k) \end{array}\right].$$

Proof of Lemma 2.5. "$\Longrightarrow$" is obvious. To prove "$\Longleftarrow$", first note that any $w \in W'$ has only a finite number of conjugates w^σ, $\sigma \in \mathrm{Gal}(k_s/k)$, hence $\sum_\sigma k_s \cdot w^\sigma$ is a *finite-dimensional* Gal-invariant subspace of W' containing w. Thus it suffices to prove "$\Longleftarrow$" when $\dim W' < \infty$. Let $\{e_\alpha\}_{\alpha \in S}$ be a basis of V and let $f_1, \ldots, f_t$ be a basis of W'. Write $f_i = \sum c_{i\alpha} e_\alpha$, $c_{i\alpha} \in k_s$. Since the f's are independent, some $t \times t$-minor of the matrix $(c_{i\alpha})$ is non-zero: say $(c_{i,\alpha_j})_{1 \le i,j \le t}$. Then W' has a *unique* basis f_i' of the form

$$f_i' = e_{\alpha_i} + \sum_{\beta \notin \{\alpha_1, \ldots, \alpha_t\}} c_{i\beta}' e_\beta.$$

Since $\forall \sigma \in \mathrm{Gal}$, $W'^\sigma = W'$, it follows that $(f_i')^\sigma = f_i'$, hence $(c_{i\beta}')^\sigma = c_{i\beta}'$, hence $c_{i\beta}' \in k$, hence $f_i' \in V$. If $W = \sum k f_i'$, then $W' = W \otimes_k k_s$. □

By the lemma, $L_s = L' \otimes_k k_s$ for some subspace $L' \subset R$. But L' is clearly unique and since for all $a \in L'$, $a \cdot L_s \subset L_s$, therefore $a \cdot L' \subset L'$. So L' is a subalgebra of R and hence of $\mathbb{R}(X)$ of dimension t, separable over k because L_s is separable over k_s. Therefore $L' = k$ and $t = 1$. This proves Theorem 2.4, (i). □

Proof of Theorem 2.4, (ii). Let $U = \operatorname{Spec} R$ be any open affine in X so that $p^{-1}(U) = \operatorname{Spec} R \otimes_k \overline{k}$. Since $R \subset \mathbb{R}(X)$, $R \otimes_k \overline{k} \subset \mathbb{R}(X) \otimes_k \overline{k}$. Thus if $\overline{X}$ is not reduced, some ring $R \otimes_k \overline{k}$ has nilpotents, hence $\mathbb{R}(X) \otimes_k \overline{k}$ must have nilpotent elements in it. On the other hand, if U is small enough as we saw above, $p^{-1}(U) = \coprod \overline{U}_i$, where $\overline{U}_i$ is irreducible, open and $y_i \in \overline{U}_i$. Therefore

$$R \otimes_k \overline{k} = \prod_{i=1}^t \mathcal{O}_{\overline{X}}(\overline{U}_i).$$

Replacing R by R_f, U by U_f and $\overline{U}_i$ by $(\overline{U}_i)_f$, and passing to the limit over f's, this shows that

$$\mathbb{R}(X) \otimes_k \overline{k} \cong \prod_{i=1}^t \varinjlim_{\substack{\text{distinguished} \\ \text{open sets } (\overline{U}_i)_f}} \mathcal{O}_{\overline{X}}((\overline{U}_i)_f) = \prod_{i=1}^t \mathcal{O}_{y_i,\overline{X}}.$$

In particular, if $\mathbb{R}(X) \otimes_k \overline{k}$ has nilpotents, so does one of the rings $\mathcal{O}_{y_i,\overline{X}}$ and hence $\overline{X}$ is not reduced. Now recall that the separability of $\mathbb{R}(X)$ over k means by definition that one of the equivalent properties holds:

Let $k^{p^{-\infty}}$ = perfect closure of k.

a) $\mathbb{R}(X)$ and $k^{p^{-\infty}}$ are linearly disjoint over k.
b) $\mathbb{R}(X) \otimes_k k^{p^{-\infty}} \longrightarrow \mathbb{R}(X)^{p^{-\infty}}$ is injective.
c) $\mathbb{R}(X)$ and $k^{p^{-1}}$ are linearly disjoint over k.
d) $\mathbb{R}(X) \otimes_k k^{p^{-1}} \longrightarrow \mathbb{R}(X)^{p^{-1}}$ is injective.

(cf. Zariski-Samuel [**109**, vol. I, pp. 102–113]; or Lang [**65**, pp. 264–265]. A well known theorem of MacLane states that these are also equivalent to $\mathbb{R}(X)$ being separable *algebraic* over a purely transcendental extension of k.)

Note that the kernel of $\mathbb{R}(X) \otimes_k k^{p^{-\infty}} \longrightarrow \mathbb{R}(X)^{p^{-\infty}}$ is precisely the ideal $\sqrt{(0)}$ of nilpotent elements in $\mathbb{R}(X) \otimes_k k^{p^{-\infty}}$: because if $a_i \in \mathbb{R}(X)$, $b_i \in k^{p^{-n}}$, then

$$\begin{aligned} \sum a_i b_i = 0 \text{ in } \mathbb{R}(X)^{p^{-\infty}} &\Longrightarrow \sum a_i^{p^n} b_i^{p^n} = 0 \text{ in } \mathbb{R}(X) \\ &\Longrightarrow (\sum a_i \otimes b_i)^{p^n} = \sum a_i^{p^n} b_i^{p^n} \otimes 1 = 0. \end{aligned}$$

Now if $N =$ ideal of nilpotents in $\mathbb{R}(X) \otimes_k \overline{k}$, then N is Gal-invariant, so by Lemma 2.5 applied to $\overline{k}$ over $k^{p^{-\infty}}$, $N = N_0 \otimes_{(k^{p^{-\infty}})} \overline{k}$ for some $N_0 \subset \mathbb{R}(X) \otimes_k k^{p^{-\infty}}$. Hence

$$N \neq (0) \Longleftrightarrow N_0 \neq (0) \Longleftrightarrow \operatorname{Ker}\left(\mathbb{R}(X) \otimes_k k^{p^{-\infty}} \to \mathbb{R}(X)^{p^{-\infty}}\right) \neq (0).$$

□

Corollary 2.6 (Zariski). *If X is a k-variety, then $\overline{X}$ is a $\overline{k}$-variety if and only if $\mathbb{R}(X)$ is separable over k and k is algebraically closed in $\mathbb{R}(X)$.*

Corollary 2.7. *Let X be any scheme of finite type over k and let $p\colon \overline{X} \to X$ be as before. Then for any $x \in X$, if $L = \{a \in \Bbbk(x) \mid a \text{ separable algebraic over } k\}$, $\exists$ an isomorphism of sets:*

$$p^{-1}x \cong \operatorname{Hom}_k(L, \overline{k})$$

commuting with $\operatorname{Gal}(\overline{k}/k)$*, and the scheme-theoretic fibre is given by:*

$$p^{-1}x \cong \operatorname{Spec} \Bbbk(x) \otimes_k \overline{k},$$

hence is reduced if and only if $\Bbbk(x)$ is separable over k.

Proof. If we let $Z = \overline{\{x\}}$ with reduced structure, then we can replace X by Z and so reduce to the case X a k-variety, $x =$ generic point. Corollary 2.7 then follows from Theorem 2.4. □

Corollary 2.8. *Let X be any scheme of finite type over k and let $p\colon \overline{X} \to X$ be as before. Let $x \in \overline{X}$ be a closed or $\overline{k}$-rational point. By k-coordinates near x, we mean: take an affine neighborhood U of $p(x)$, generators $f_1, \ldots, f_n$ of $\mathcal{O}_X(U)$, and then define a closed immersion:*

$$\begin{array}{ccc} p^{-1}U & \longrightarrow & \mathbb{A}^n_{\overline{k}} \\ \cap & & \\ \overline{X} & & \end{array}$$

by the functions $f_1, \ldots, f_n$. Then

i) *$\#$(Galois orbits of x) $= [\Bbbk(p(x)) : k]_s$*
ii) *The following are equivalent:*
   a) *$p^{-1}(\{p(x)\}) =$ the reduced closed subscheme $\{x\}$,*
   b) *$p(x)$ is a k-rational point of X,*
   c) *In k-coordinates, x goes to a point in $k^n \subset \mathbb{A}^n_{\overline{k}}$.*

   If these hold, we say that x is defined over k.
iii) *If k is perfect, these are equivalent to*
   d) *x is a fixed point of the Galois action on $\overline{X}$.*

Proof. (i) and the equivalence of (a) and (b) are restatements of Corollary 2.7 for closed points; as for (c), note that the values of the "proper coordinates" at x are $f_1(x), \dots, f_n(x)$ and that $\Bbbk(p(x)) = k(f_1(x), \dots, f_n(x))$, hence (b) $\iff$ (c). (iii) is clear. $\square$

In case k is perfect, Corollary 2.8 suggests that there are further ties between X and $\overline{X}$:

Theorem 2.9. *Let k be a perfect field and $p\colon \overline{X} \to X$ as before. Then*

i) *$\forall U \subset X$ open,*

$$\mathcal{O}_X(U) = \left\{ f \in \mathcal{O}_{\overline{X}}(p^{-1}U) \mid \sigma_X^* f = f,\ \forall \sigma \in \mathrm{Gal}(\overline{k}/k) \right\}.$$

ii) *$\forall$ closed subschemes $\overline{Y} \subset \overline{X}$*

$$\overline{Y} \text{ is Gal-invariant} \iff \exists \text{ closed subschemes } Y \subset X \text{ with } \overline{Y} = Y \otimes_k \overline{k}$$

and if this holds, Y is unique, and one says that $\overline{Y}$ is defined over k.

iii) *If $x \in \overline{X}$ and $H = \{\sigma \in \mathrm{Gal} \mid \sigma_X(x) = x\}$, then $\Bbbk(p(x)) = \Bbbk(x)^H$.*

iv) *If Y is another scheme of finite type over k and $\overline{Y} = Y \times_k \overline{k}$, then every $\overline{k}$-morphism $\overline{f}\colon \overline{X} \to \overline{Y}$ that commutes with the Galois action (i.e., $\sigma_Y \circ \overline{f} = \overline{f} \circ \sigma_X$, for all $\sigma \in \mathrm{Gal}$) is of the form $f \times 1_{\overline{k}}$ for a unique k-morphism $f\colon X \to Y$.*

Proof of (i). Let

$$\mathcal{F}(U) = \left\{ f \in \mathcal{O}_{\overline{X}}(p^{-1}U) \mid \sigma_X^* f = f, \text{ for all } \sigma \right\}.$$

Then $\mathcal{F}$ is easily seen to be a sheaf and whenever U is affine, say $U = \operatorname{Spec} R$, then

$$\begin{aligned} \mathcal{F}(U) &= \left\{ f \in R \otimes_k \overline{k} \mid (1_R \otimes \sigma) f = f, \text{ for all } \sigma \right\} \\ &= R, \quad \text{since } k \text{ is perfect} \\ &= \mathcal{O}_X(U). \end{aligned}$$

Thus $\mathcal{F} \cong \mathcal{O}_X$. $\square$

Proof of (ii). Suppose $\overline{Y} \subset \overline{X}$ is Gal-invariant. Then for all open affine $U = \operatorname{Spec} R$ in X, $\overline{Y} \cap p^{-1}U$ is defined by an ideal $\overline{\mathfrak{a}} \subset R \otimes_k \overline{k}$. Then $\overline{\mathfrak{a}}$ is Gal-invariant so by Lemma 2.5, $\overline{\mathfrak{a}} = \mathfrak{a} \otimes_k \overline{k}$ for some k-subspace $\mathfrak{a} \subset R$. Since $a\overline{\mathfrak{a}} \subset \overline{\mathfrak{a}}$ for all $a \in R$, it follows that $a\mathfrak{a} \subset \mathfrak{a}$ and so $\mathfrak{a}$ is an ideal. It is easy to see that these ideals $\mathfrak{a}$ define the unique $Y \subset X$ such that $\overline{Y} = Y \times_k \overline{k}$. $\square$

Proof of (iii). As in Corollary 2.7 above, we can replace X by the closure of $p(x)$ and so reduce to the case where X is a variety with generic point $p(x)$ and $x = x_1$ is one of the generic points $x_1, \dots, x_t$ of $\overline{X}$. By Theorem 2.4, X is reduced and we have

$$\prod_{i=1}^{t} \Bbbk(x_i) = \prod_{i=1}^{t} \mathbb{R}(\overline{X}_i) \cong \mathbb{R}(X) \otimes_k \overline{k} = \Bbbk(p(x)) \otimes_k \overline{k}.$$

Thus

$$\begin{aligned} \Bbbk(p(x)) &\cong \left\{ (x_1', \dots, x_t') \in \prod \Bbbk(x_i) \mid (x_1', \dots, x_t') \text{ Gal-invariant} \right\} \\ &\cong \left\{ x_1' \in \Bbbk(x_1) \mid x_1' \text{ is } H\text{-invariant} \right\}. \end{aligned}$$

$\square$

Proof of (iv). Left to the reader. $\square$

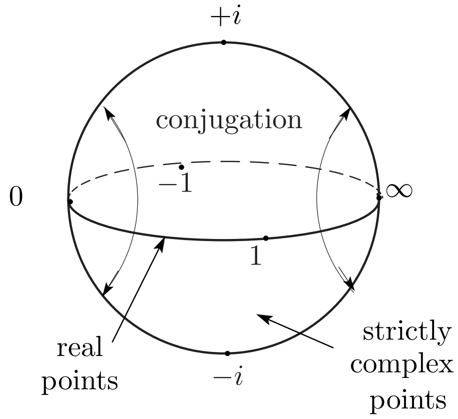

FIGURE IV.6. $\overline{X} = \mathbb{P}^1_{\mathbb{C}}$

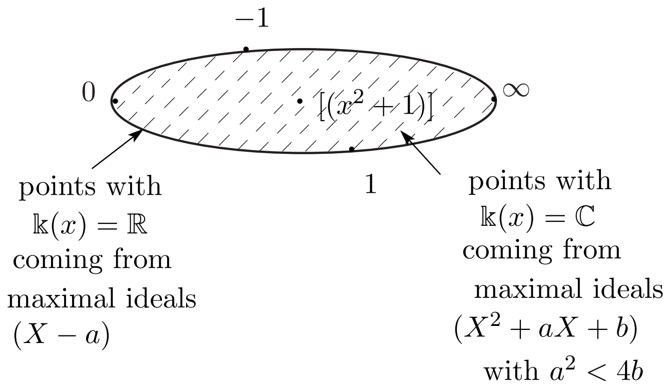

FIGURE IV.7. $X = \mathbb{P}^1_{\mathbb{R}}$

Note that when $\overline{Y}$ = one point x, then $\{x\}$ is defined over k as in Theorem 2.9 above if and only if it is defined over k as in Corollary 2.8.

When k is not perfect, the theorem is false. One still says "$\overline{Y}$ is defined over k" if $\overline{Y} = Y \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ for some closed subscheme $Y \subset X$, and Y is still unique if it exists. But being Gal-invariant is not strong enough to guarantee being defined over k. For instance, if $\overline{Y}$ is a reduced Gal-invariant subscheme, one can try by setting $Y' = p(\overline{Y})$ with reduced structure. Then $\overline{Y}' = Y' \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ will be a subscheme of $\overline{X}$ defined over k, with the same point set as $\overline{Y}$ and $\overline{Y} \subset \overline{Y}'$ but in general $\overline{Y}'$ need not be reduced: i.e., the *subset* $\overline{Y}$ is defined over k but the *subscheme* $\overline{Y}$ is not (cf. Example 4 below).

The theory can be illustrated with very pretty examples in the case:

$$k = \mathbb{R}$$
$$\overline{k} = \mathbb{C}$$
$$\operatorname{Gal}(\overline{k}/k) = \{\mathrm{id}, *\}, \quad * = \text{complex conjugation.}$$

In this case, $*_X : \overline{X} \to \overline{X}$ is continuous in the classical topology and can be readily visualized.

EXAMPLE. 1. Let $X = \mathbb{P}^1_{\mathbb{R}}$, $\overline{X} = \mathbb{P}^1_{\mathbb{C}}$. Ignoring the generic point, $\mathbb{P}^1_{\mathbb{C}}$ looks like Figure IV.6. Identifying conjugate points, $\mathbb{P}^1_{\mathbb{R}}$ looks like Figure IV.7.

EXAMPLE. 2. Let $\overline{X} = \mathbb{P}^1_{\mathbb{C}}$ again. Then in fact there are exactly two real forms of $\mathbb{P}^1_{\mathbb{C}}$: schemes X over $\mathbb{R}$ such that $X \times_{\mathbb{R}} \mathbb{C} \cong \overline{X}$. One is $\mathbb{P}^1_{\mathbb{R}}$ which was drawn in Example 1. The other is represented by the conic:

$$X = V(X_0^2 + X_1^2 + X_2^2) \subset \mathbb{P}^2_{\mathbb{R}}.$$

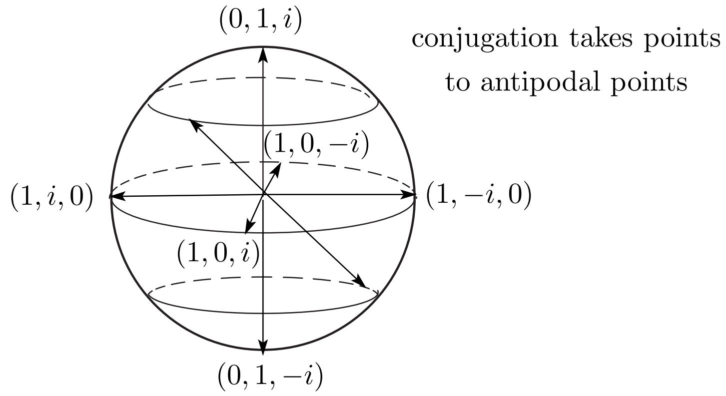

FIGURE IV.8. $X = V(X_0^2 + X_1^2 + X_2^2) \subset \mathbb{P}^2_{\mathbb{R}}$

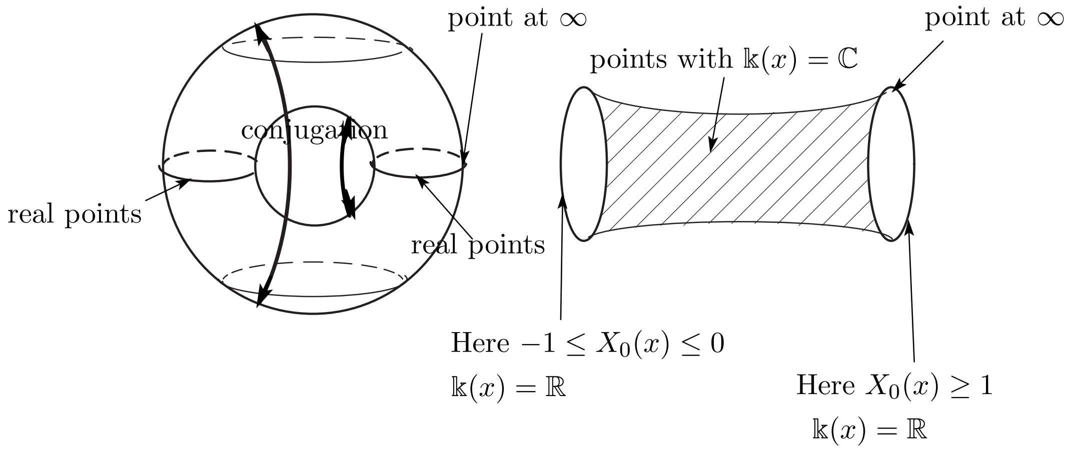

FIGURE IV.9. $X = V(X_1^2 X_2 - X_0(X_0^2 - X_2^2)) \subset \mathbb{P}^2_{\mathbb{R}}$

Then $\overline{X}$ is the same conic over $\mathbb{C}$ and, projecting from any closed point $x \in \overline{X}$, we find as in Part I [**76**] an isomorphism between $\overline{X}$ and $\mathbb{P}^1_{\mathbb{C}}$. Since X has in fact *no* $\mathbb{R}$-rational points at all ($\forall (a_0, a_1, a_2)$, $a_0^2 + a_1^2 + a_2^2 > 0$!) we cannot find a projection $\overline{X} \to \mathbb{P}^1_{\mathbb{C}}$ defined over $\mathbb{R}$. The picture is as in Figure IV.8, so X is homeomorphic in the classical topology to the real projective plane $S^2/(\text{antipodal map})$ and for all its closed points $x \in X$, $\Bbbk(x) = \mathbb{C}$.

EXAMPLE. 3. Let X be the curve $X_1^2 = X_0(X_0^2 - 1)$ in $\mathbb{P}^2_{\mathbb{R}}$. One can work out the picture by thinking of X as a double covering of the X_0-line gotten by considering the two values $\pm\sqrt{X_0(X_0^2 - 1)}$. We leave the details to the reader. One finds the picture in Figure IV.9.

EXAMPLE. 4. To see how $\overline{X}$ may be reducible when X is irreducible, look at the affine curve

$$X_0^2 + X_1^2 = 0$$

in $\mathbb{A}^2_{\mathbb{R}}$. Then $\overline{X}$ is given by:

$$(X_0 + iX_1)(X_0 - iX_1) = 0$$

and the picture is as in Figure IV.10. If $U = X \setminus \{(0,0)\}$, then U is actually already a variety over $\mathbb{C}$ via

$$p\colon U \longrightarrow \operatorname{Spec} \mathbb{C}, \quad p^*(a + ib) = a + \frac{X_1}{X_0} \cdot b$$

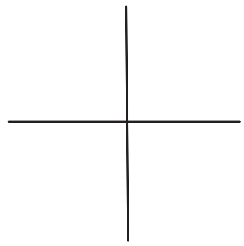

FIGURE IV.10. $X = V(X_0^2 + X_1^2) \subset \mathbb{A}^2_{\mathbb{R}}$

and in fact,

$$\left(\mathbb{R}[X_0, X_1]/(X_0^2 + X_1^2)\right)\left[\frac{1}{X_0}\right] \cong \mathbb{R}\left[\frac{X_1}{X_0}, X_0, X_0^{-1}\right] \Big/ \left(\left(\frac{X_1}{X_0}\right)^2 + 1\right)$$
$$\cong \mathbb{C}[X_0, X_0^{-1}]$$

so $U \cong \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$:

To go deeper into the theory of one-dimensional varieties over $\mathbb{R}$, see Alling-Greenleaf [**12**].

To illustrate how X may be reduced and yet have hidden nilpotents, we must look in characteristic p.

EXAMPLE. 5. Let k be an imperfect field, and consider the hypersurface $X \subset \mathbb{P}^n_k$ defined by

$$a_0 X_0^p + \cdots + a_n X_n^p = 0, \quad a_i \in k.$$

In $\overline{k}$, each a_i will be a p-th power, say $a_i = b_i^p$, so $\overline{X} \subset \mathbb{P}^n_{\overline{k}}$ is defined by

$$(b_0 X_0 + \cdots + b_n X_n)^p = 0.$$

Thus $\overline{X}$ is a "p-fold hyperplane" and the function $\sum b_i X_i / X_0$ is nilpotent and non-zero. However, provided that at least one ratio $a_i/a_j \notin k^p$, then $\sum a_i X_i^p$ is *irreducible* over k, hence X is a k-variety: Put another way, the hyperplane $L : \sum b_i X_i = 0$ in $\mathbb{P}^n_{\overline{k}}$ is "defined over k" as a set in the sense that it is Gal-invariant, hence is set-theoretically $p^{-1}(p(L))$ using $p \colon \mathbb{P}^n_{\overline{k}} \to \mathbb{P}^n_k$; but it is not "defined over k" as a subscheme of $\mathbb{P}^n_k$ unless $b_i/b_j = (a_i/a_j)^{1/p} \in k$ all i, j.

Before leaving this subject, I would like to indicate briefly the main ideas of *Descent theory* which arise when you pursue deeply the relations between X and $\overline{X}$.

(I.) If you look at Theorem 2.9, (ii) as expressing when a quasi-coherent sheaf of ideals $\overline{\mathcal{I}} \subset \mathcal{O}_{\overline{X}}$ is defined over k, it is natural to generalize it to arbitrary quasi-coherent sheaves of *modules*. The result is (assuming k perfect): given a quasi-coherent sheaf $\overline{\mathcal{F}}$ of $\mathcal{O}_{\overline{X}}$-modules, plus an action of $\mathrm{Gal}(\overline{k}/k)$ on $\overline{\mathcal{F}}$ compatible with its action on $\mathcal{O}_{\overline{X}}$, i.e., $\forall \sigma \in \mathrm{Gal}$, $U \subset \overline{X}$ isomorphisms

$$\sigma^U_{\mathcal{F}} \colon \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{F}}(\sigma_X^{-1}(U))$$

such that

$$\sigma_X^*(a) \cdot \sigma^U_{\mathcal{F}}(b) = \sigma^U_{\mathcal{F}}(a \cdot b), \quad a \in \mathcal{O}_{\overline{X}}(U), b \in \overline{\mathcal{F}}(U)$$
$$(\sigma\tau)^U_{\mathcal{F}} = \sigma_{\mathcal{F}}^{\tau_X^{-1}(U)} \circ \tau^U_{\mathcal{F}}, \quad \sigma, \tau \in \mathrm{Gal}$$

and commuting with restrictions, then there is one, and up to canonical isomorphism, only one quasi-coherent $\mathcal{F}$ on X such that (i) $\overline{\mathcal{F}} \cong \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{\overline{X}}$ and (ii) the Gal-action on $\overline{\mathcal{F}}$ goes over via this isomorphism to the Gal-action $\sigma_{\overline{\mathcal{F}}}(b \otimes a) = b \otimes \sigma_X^* a$ on $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{\overline{X}}$. More precisely, there is an *equivalence of categories* between the category of pairs $(\overline{\mathcal{F}}, \sigma^U_{\mathcal{F}})$ of quasi-coherent sheaves on $\overline{X}$ plus Gal-action and the category of quasi-coherent $\mathcal{F}$ on X.

(II.) The whole set-up in fact generalizes to a much bigger class of morphisms than $p\colon \overline{X} \to X$:

DEFINITION 2.10. Given a morphism $f\colon X \to Y$, a quasi-coherent sheaf $\mathcal{F}$ on X is flat[2] over Y if for every $x \in X$, $\mathcal{F}_x$ is a flat $\mathcal{O}_{f(x),Y}$-module. f itself is flat if $\mathcal{O}_X$ is flat. f is faithfully flat if f is flat and surjective.

Grothendieck has then proven that for any faithfully flat quasi-compact $f\colon X \to Y$, there is an equivalence of categories between:

a) the category of quasi-coherent sheaves $\mathcal{G}$ on Y,
b) the category of pairs $(\mathcal{F}, \alpha)$, $\mathcal{F}$ a quasi-coherent sheaf on X and α being "descent data", i.e., an isomorphism on $X \times_Y X$:

$$\alpha\colon p_1^*\mathcal{F} \xrightarrow{\approx} p_2^*\mathcal{F}$$

satisfying a suitable associativity condition on $X \times_Y X \times_Y X$ and restricting to the identity on the diagonal $\Delta\colon X \to X \times_Y X$.

In the special case $f = p$, k perfect, it turns out that descent data α is just another way of describing Galois actions. A good reference is Grothendieck's SGA1 [**4**, Exposé VIII][3].

(III.) The final and most interesting step of all is the problem: given $\overline{X}$ over $\overline{k}$, classify the set of all possible X's over k plus $\overline{k}$-isomorphisms $X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k} \cong \overline{X}$, up to isomorphism over k. Such an X is called a *form of* $\overline{X}$ *over* k and to find an X is called *descending* $\overline{X}$ *to* k. If k is perfect, then by Exercise 1 below ??? it is easy to see that each form of $\overline{X}$ over k is determined up to k-isomorphism by the Galois action $\{\sigma_X \mid \sigma \in \operatorname{Gal}(\overline{k}/k)\}$ on $\overline{X}$ that it induces. What is much harder, and is only true under restrictive hypotheses (such as $\overline{X}$ affine or $\overline{X}$ quasi-projective with Gal also acting on its ample sheaf $\overline{L}$, cf. Chapter III) is that every action of $\operatorname{Gal}(\overline{k}/k)$ is an *effective descent data*, i.e., comes from a descended form X of $\overline{X}$ over k. For a discussion of these matters, cf. Serre [**91**, Chapter V, §4, No. 20, pp. 102–104]. All sorts of beautiful results are known about k-forms: for instance, there is a canonical bijection between the set of k-forms of $\mathbb{P}^n_{\overline{k}}$ and the set of central simple k-algebras of rank n^2 (cf. Serre [**93**, Chapter X, §6, p. 160]).

3. The frobenius morphism

The most remarkable example of the theory of Galois actions is the case:

$$k = \mathbb{F}_q, \quad \text{the finite field with } q \text{ elements, } q = p^\nu$$

$$\overline{k} = \bigcup_{n=1}^{\infty} \mathbb{F}_{q^n}$$

$$\operatorname{Gal}(\overline{k}/k) = \text{pro-finite cyclic group generated by the frobenius}$$

$$\mathbf{f}\colon \overline{k} \to \overline{k}, \quad \mathbf{f}(x) = x^q.$$

$\mathbf{f}$ is the only automorphism of a field that is given by a polynomial! This has amazing consequences:

[2] We will discuss the meaning of this concept shortly: see §4.

[3] (Added in publication) See also FAG [**3**].

Definition 3.1. If X is any scheme in prime characteristic p, i.e., $p = 0$ in $\mathcal{O}_X$, define a morphism

$$\phi_X \colon X \longrightarrow X$$

by

a) set-theoretically, $\phi_X = $ identity,
b) $\forall U$ and $\forall a \in \mathcal{O}_X(U)$, define $\phi_X^* a = a^p$.

Definition 3.2. If X is a scheme of finite type over $k = \mathbb{F}_q$, $\overline{X} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$, then:

i) Note that $\mathbf{f}_k \colon \operatorname{Spec} \overline{k} \to \operatorname{Spec} \overline{k}$ (in the notation at the beginning of §2) is the automorphism $(\phi_{\operatorname{Spec} \overline{k}})^{-\nu}$, hence the conjugation $\mathbf{f}_X \colon \overline{X} \to \overline{X}$ defined in Definition 2.1 above is

$$1_X \times (\phi_{\operatorname{Spec} \overline{k}})^{-\nu}.$$

We write this now $\mathbf{f}_X^{\mathrm{arith}} \colon \overline{X} \to \overline{X}$.

ii) Set-theoretically identical with this morphism will be a $\overline{k}$-morphism called the geometric frobenius

$$\begin{aligned} \mathbf{f}_X^{\mathrm{geom}} &= \phi_{\overline{X}}^{\nu} \circ (1_X \times \phi_{\operatorname{Spec} \overline{k}}^{-\nu}) \\ &= \phi_X^{\nu} \times 1_{\operatorname{Spec} \overline{k}} \colon \overline{X} \to \overline{X}. \end{aligned}$$

In other words, there are two morphisms both giving the right conjugation map: an automorphism $\mathbf{f}_X^{\mathrm{arith}}$ which does not preserve scalars, and a $\overline{k}$-morphism $\mathbf{f}_X^{\mathrm{geom}}$ which however is not an automorphism. For instance, look at the case $X = \mathbb{A}_k^n$. All morphisms $\mathbb{A}_{\overline{k}}^n \to \mathbb{A}_{\overline{k}}^n$ are described by their actions on $\overline{k}[X_1, \ldots, X_n]$ and we find:

$$\phi_{\mathbb{A}^n}^* \begin{cases} X_i & \longmapsto & X_i^p \\ a & \longmapsto & a^p \end{cases}$$

$$\left(\mathbf{f}_{\mathbb{A}^n}^{\mathrm{arith}}\right)^* \begin{cases} X_i & \longmapsto & X_i \\ a & \longmapsto & a^{q^{-1}} \end{cases}, \quad \text{an automorphism of } \overline{k}[X_1, \ldots, X_n]$$

$$\left(\mathbf{f}_{\mathbb{A}^n}^{\mathrm{geom}}\right)^* \begin{cases} X_i & \longmapsto & X_i^q \\ a & \longmapsto & a \end{cases}, \quad \text{a } \overline{k}\text{-homomorphism of } \overline{k}[X_1, \ldots, X_n] \text{ into itself,}$$

where $a \in \overline{k}$. This means that completely unlike other conjugations, the graph of $\mathbf{f}_X = \mathbf{f}_X^{\mathrm{geom}}$ is closed in $\overline{X} \times_{\operatorname{Spec} \overline{k}} \overline{X}$. Corollary 2.8 gives us a very interesting expansion of the zeta-function of X in terms of the number of certain points on $\overline{X}$:

For every $\nu \geq 1$, we say that a closed point $x \in \overline{X}$ is *defined over* $\mathbb{F}_{q^\nu}$ if any of the equivalent conditions hold:

i) In $\mathbb{F}_q$-coordinates, x corresponds to a point of $(\mathbb{F}_{q^\nu})^n \subset \mathbb{A}_{\overline{k}}^n$,
ii) $\mathbb{k}(p(x)) \subset \mathbb{F}_{q^\nu}$ ($p \colon \overline{X} \to X$ is the projection in Theorem 2.3),
iii) $\mathbf{f}_X^{\nu}(x) = x$, i.e., x a fixed point of the morphism $\mathbf{f}_X^{\nu}$.

(Apply Corollary 2.8 to $\overline{k} \supset \mathbb{F}_{q^\nu}$ and to $\overline{X} \to (X \times_{\mathbb{F}_q} \mathbb{F}_{q^\nu})$.) The set of all these points we call $\overline{X}(\mathbb{F}_{q^\nu})$. Then if

$$N_\nu = \#\overline{X}(\mathbb{F}_{q^\nu}),$$

I claim that formally:

3.3.

$$\zeta_X(s) = Z_X(q^{-s}),$$

where $Z_X(t)$ is given by

$$\frac{dZ_X}{Z_X} = \left(\sum_{\nu=1}^{\infty} N_\nu \cdot t^{\nu-1}\right) dt$$
$$Z_X(0) = 1.$$

PROOF. If M_ν = number of $x \in X$ with $\Bbbk(x) \cong \mathbb{F}_{q^\nu}$, then each such point splits in $\overline{X}$ into ν distinct closed points which are in $\overline{X}(\mathbb{F}_{q^\mu})$ if $\nu \mid \mu$. Thus

$$N_\mu = \sum_{\nu|\mu} \nu \cdot M_\nu.$$

By definition:

$$\zeta_X(s) = \prod_{\nu=1}^{\infty} \left(1 - \frac{1}{q^{\nu s}}\right)^{-M_\nu}$$

so if we set

$$Z_X(t) = \prod_{\nu=1}^{\infty} (1-t^\nu)^{-M_\nu}$$

then $\zeta_X(s) = Z_X(q^{-s})$. Moreover

$$\begin{aligned}\frac{dZ_X}{Z_X} = d(\log Z_X) &= \sum_{\nu=1}^{\infty}(-M_\nu) \cdot \frac{-\nu t^{\nu-1}}{1-t^\nu} \cdot dt \\ &= \frac{1}{t}\sum_{\nu=1}^{\infty} \nu M_\nu (t^\nu + t^{2\nu} + t^{3\nu} + \cdots)dt \\ &= \frac{1}{t}\sum_{\mu=1}^{\infty} N_\mu \cdot t^\mu \cdot dt.\end{aligned}$$

□

As an example, if $X = \mathbb{A}^n_{\mathbb{F}_q}$, then

$$N_\nu = \#\left((\mathbb{F}_{q^\nu})^n\right) = q^{\nu n}$$

hence

$$Z_{\left(\mathbb{A}^n_{\mathbb{F}_q}\right)}(t) = \frac{1}{1-q^n t}.$$

Therefore by (3.3)

$$\zeta_{\left(\mathbb{A}^n_{\mathbb{F}_q}\right)}(s) = \frac{1}{1-q^{n-s}}$$

and

$$\zeta_{\mathbb{A}^n_{\mathbb{Z}}}(s) = \prod_p \left(\frac{1}{1-p^{n-s}}\right) = \zeta_0(s-n)$$

if $\zeta_0(s)$ is Riemann's zeta function

$$\zeta_0(s) = \prod_p \left(\frac{1}{1-p^{-1}}\right) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

From this, an easy consequence is:

PROPOSITION 3.4. *For all schemes X of finite type over $\mathbb{Z}$, $\zeta_X(s)$ converges if* $\mathrm{Re}(s) \gg 0$.

Idea of proof. First reduce to the case X affine and then by an affine embedding, reduce to the case of $\mathbb{A}^n$ using the fact that the Dirichlet series (1.3) for ζ_X has positive coefficients majorized by those for $\zeta_{\mathbb{A}^n}$. $\square$

If X is a scheme over a field $\mathbb{F}_q$ again, a celebrated theorem of Dwork [**35**] states that Z_X is a rational function! If we then expand it in terms of its zeros and poles:

$$Z_X(t) = \frac{\prod_{i=1}^{N}(1-\alpha_i t)}{\prod_{i=1}^{M}(1-\beta_i t)}, \quad \alpha_i, \beta_i \in \mathbb{C}$$

it follows immediately that

$$\frac{dZ_X}{Z_X} = \sum_{\nu=1}^{\infty}\left(\sum_{i=1}^{M}\beta_i^\nu - \sum_{i=1}^{N}\alpha_i^\nu\right)t^{\nu-1}dt$$

and hence:

$$N_\nu = \sum_{i=1}^{M}\beta_i^\nu - \sum_{i=1}^{N}\alpha_i^\nu.$$

It seems most astonishing that the numbers N_ν of rational points should be such an elementary sequence! Even more remarkably, Deligne [**33**] has proven Weil's conjecture that for every i,

$$|\alpha_i|,\ |\beta_i| \in \{1, q^{1/2}, q, q^{3/2}, \ldots, q^{\dim X}\}.$$

I would like to give one very simple application of the fact that the frobenius $\mathbf{f}_X = \mathbf{f}_X^{\mathrm{geom}}$ has a graph:

Proposition 3.5. *Let X be an $\mathbb{F}_q$-variety such that $\overline{X} \cong \mathbb{P}^1_{\overline{k}}$. Then X has at least one $\mathbb{F}_q$-rational point.*[4]

Proof. If X has not $\mathbb{F}_q$-rational points, then $\mathbf{f}_X \colon \overline{X} \to \overline{X}$ has no fixed points. Let $\Gamma \subset \overline{X} \times_{\overline{k}} \overline{X}$ be the graph of $\mathbf{f}_X^{\mathrm{geom}}$. Then $\Gamma \cap \Delta = \emptyset$, $\Delta =$ diagonal. But now $\overline{X} \times_{\overline{k}} \overline{X} \cong \mathbb{P}^1_{\overline{k}} \times_{\overline{k}} \mathbb{P}^1_{\overline{k}}$ and via the Segre embedding this is isomorphic to a quadric in $\mathbb{P}^3_{\overline{k}}$. In fact, if $X_0,\ X_1$ (resp. $Y_0,\ Y_1$) are homogeneous coordinates, then

$$s \colon \mathbb{P}^1_{\overline{k}} \times_{\overline{k}} \mathbb{P}^1_{\overline{k}} \longrightarrow \mathbb{P}^3_{\overline{k}}$$

via

$$\begin{array}{ccccc} (X_0, X_1) \times (Y_0, Y_1) \longrightarrow (X_0Y_0, & X_0Y_1, & X_1Y_0, & X_1Y_1). \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad \| & \| & \| & \| \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad Z_0 & Z_1 & Z_2 & Z_3 \end{array}$$

The image of s is the quadric $Q = V(Z_0Z_3 - Z_1Z_2)$. But the point is that $s(\Delta) = Q \cap V(Z_1 - Z_2)$, so s maps $\mathbb{P}^1_{\overline{k}} \times_{\overline{k}} \mathbb{P}^1_{\overline{k}} \setminus \Delta$ isomorphically onto an *affine* variety $W = Q \cap [\mathbb{P}^3_{\overline{k}} \setminus V(Z_1 - Z_2)]$. So if $\Gamma \cap \Delta = \emptyset$, we get a closed immersion

$$\mathbb{P}^1_{\overline{k}} \cong \Gamma \longrightarrow W$$

[4] We will see in Corollary VIII.1.8 that this implies $X \cong \mathbb{P}^1_k$ too. See Corollary VI.2.4 for a generalization of Proposition 3.5 to $\mathbb{P}^n$ over finite fields.

of a complete variety in an affine one. But quite generally a morphism of a complete variety Γ to an affine variety W must be a constant map. If not, choose any function $a \in \Gamma(\mathcal{O}_W)$ which is not constant on the image of Γ and consider the composition

$$\Gamma \longrightarrow W \xrightarrow{a} \mathbb{A}^1_k \subset \mathbb{P}^1_k.$$

Γ complete $\implies$ image closed $\implies$ image is one point or the whole $\mathbb{P}^1_k$. Since a is not constant on the image of Γ, the first is impossible and since $\infty \notin$ image, the second is impossible. □

There are many other classes of varieties X which always have at least one rational point over a finite field $\mathbb{F}_q$: for instance, a theorem of Lang says that this is the case for any homogeneous space: cf. Theorem VI.2.1 and Corollary VI.2.5.

4. Flatness and specialization

In this section I would like to study morphisms $f\colon X \to S$ of finite type by considering them as families $\{f^{-1}(s)\}$ of schemes of finite type over fields, parametrized by the points s of a "base space" S. In particular, the most important case in many applications and for many proofs is when $S = \operatorname{Spec} R$, R a valuation ring. Our main goal is to explain how the concept "f is flat", defined via commutative algebra (cf. Definition 2.10), means intuitively that the fibres $f^{-1}(s)$ are varying "continuously".

We recall that flatness of a module M over a ring R is usually defined by the exactness property:

Definition 4.1. M is a flat R-module if for all exact sequences

$$N_1 \longrightarrow N_2 \longrightarrow N_3$$

of R-modules,

$$M \otimes_R N_1 \longrightarrow M \otimes_R N_2 \longrightarrow M \otimes_R N_3$$

is exact.

By a simple analysis it is then checked that this very general property is in fact implied by the special cases where the exact sequence is taken to be

$$0 \longrightarrow \mathfrak{a} \longrightarrow R$$

($\mathfrak{a}$ an ideal in R), in which case it reads:

For all ideals $\mathfrak{a} \subset R$,

$$\mathfrak{a} \otimes_R M \longrightarrow M$$

is injective.

For basic facts concerning flatness, we refer the reader to Bourbaki [**26**, Chapter 1]. We list a few of these facts that we will use frequently, with some indication of proofs:

Proposition 4.2. *If M is presented in an exact sequence*

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow M \longrightarrow 0$$

where N_2 is flat over R (e.g., N_2 is a free R-module), then M is flat over R if and only if

$$N_1/\mathfrak{a}N_1 \longrightarrow N_2/\mathfrak{a}N_2$$

is injective for all ideals $\mathfrak{a} \subset R$.

This is seen by "chasing" the diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & \text{Kernel?} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & \mathfrak{a}\otimes_R N_1 & \longrightarrow & N_1 & \longrightarrow & N_1/\mathfrak{a}N_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{a}\otimes_R N_2 & \longrightarrow & N_2 & \longrightarrow & N_2/\mathfrak{a}N_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\text{Kernel?} & \longrightarrow & \mathfrak{a}\otimes_R M & \longrightarrow & M & \longrightarrow & M/\mathfrak{a}M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

To link flatness of stalks of sheaves with flatness of the module of sections over an affine open set, we need:

Proposition 4.3. *If M is a B-module and B is an A-algebra via $i\colon A \to B$, then:*

$$
\begin{aligned}
M \text{ is flat over } A &\iff \forall \mathfrak{p} \text{ prime ideals in } A,\ M\otimes_A A_{\mathfrak{p}} \text{ is } A_{\mathfrak{p}}\text{-flat.}\\
&\iff \forall \mathfrak{p} \text{ prime ideals in } B, \text{ if } \mathfrak{p}_0 = i^{-1}(\mathfrak{p}), \text{ then } M_{\mathfrak{p}} \text{ is } A_{\mathfrak{p}_0}\text{-flat.}\\
&\iff \text{The sheaf } \widetilde{M} \text{ on } \operatorname{Spec} B \text{ is flat over } \operatorname{Spec} A \text{ (Definition 2.10).}
\end{aligned}
$$

Proposition 4.4.

a) *If M is an A-module and B is an A-algebra, then*

$$M \text{ flat over } A \Longrightarrow M\otimes_A B \text{ flat over } B.$$

b) *If $\mathcal{F}$ is a quasi-coherent sheaf on X and we consider a fibre product diagram:*

$$
\begin{tikzcd}
X\times_Y Z \arrow[r, "p"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
Z \arrow[r, "q"] & Y
\end{tikzcd}
$$

then (cf. Definition 2.10)

$$\mathcal{F} \text{ flat over } Y \Longrightarrow p^*\mathcal{F} \text{ flat over } Z.$$

c) *Especially*

$$f \text{ flat} \Longrightarrow g \text{ flat.}$$

Proposition 4.5.

a) *M flat over A $\Longrightarrow$ for all non-zero divisors $a \in A$, $M \xrightarrow{a} M$ is injective.*
b) *If A is a principal ideal domain or valuation ring, the converse is true.*

The point of (a) is that $A \xrightarrow{a} A$ injective implies $M \xrightarrow{a} M$ injective.

Proposition 4.6.

a) *If M is a B-module and B is an A-algebra, where A, B are noetherian and M is finitely generated then*

$$
\begin{aligned}
M \text{ flat over } A \Longrightarrow\ & \forall \mathfrak{p}\subset B, \text{ an associated primes of } M,\\
& \mathfrak{p}\cap A \text{ is an associated prime of } A.
\end{aligned}
$$

b) *$f\colon X \to Y$ a morphism of noetherian schemes, $\mathcal{F}$ a coherent sheaf on X, then*

$$\mathcal{F} \text{ flat over } \mathcal{O}_Y \Longrightarrow f(\operatorname{Ass}(\mathcal{F})) \subset \operatorname{Ass}(\mathcal{O}_Y).$$

Especially, f flat, $\eta \in X$ a generic point implies $f(\eta) \in Y$ is a generic point.

In fact, if $\mathfrak{p}_0 \notin \mathrm{Ass}(A)$, then there exists an element $a \in \mathfrak{p}_0 A_{\mathfrak{p}_0}$ which is a non-zero divisor. Then multiplication by a is injective in $M_{\mathfrak{p}}$, hence $\mathfrak{p} \notin \mathrm{Ass}(M)$.

Proposition 4.7. *If $\mathcal{F}$ is a coherent sheaf on a scheme X, then*

$$\mathcal{F} \text{ flat over } \mathcal{O}_X \iff \mathcal{F} \text{ locally free.}$$

Proof. For each $x \in X$, there is a neighborhood U of x and a presentation

$$\mathcal{O}_U^s \xrightarrow{\alpha} \mathcal{O}_U^r \xrightarrow{\beta} \mathcal{F}|_U \longrightarrow 0.$$

Factor this through:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_U^r \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

We may assume that r is minimal, i.e., β induces an isomorphism

$$\Bbbk(x)^r \xrightarrow{\overline{\beta}} \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x.$$

By flatness of $\mathcal{F}_x$ over $\mathcal{O}_{x,X}$,

$$0 \longrightarrow \mathcal{K}_x/\mathfrak{m}_x\mathcal{K}_x \longrightarrow \Bbbk(x)^r \longrightarrow \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x \longrightarrow 0$$

is exact. Therefore $\mathcal{K}_x/\mathfrak{m}_x\mathcal{K}_x = (0)$ and $\mathcal{K}$ is trivial in a neighborhood of x by Proposition I.5.5 (Nakayama). $\square$

Another important general result is that a large class of morphisms are at least flat over an open dense subset of the image scheme:

Theorem 4.8 (Theorem of generic flatness). *Let $f\colon X \to Y$ be a morphism of finite type between two irreducible reduced noetherian schemes, with $f(\eta_X) = \eta_Y$. Then there is a non-empty open $U \subset Y$ such that $\mathrm{res}\, f\colon f^{-1}U \to U$ is flat and surjective.*

Proof. We can obviously replace Y by an affine open piece, and then covering X by affines $V_1, \ldots, V_k$, if $\mathrm{res}\, f\colon V_i \cap f^{-1}U_i \to U_i$ is flat, then $\mathrm{res}\, f\colon f^{-1}(\bigcap U_i) \to \bigcap U_i$ is flat. So we may assume $X = \mathrm{Spec}\, B$, $Y = \mathrm{Spec}\, A$, and we quote the very pretty lemma of Grothendieck. $\square$

Lemma 4.9 (SGA 1 [**4**, Exposé IV, Lemme 6.7, p. 102]). *Let A be a noetherian integral domain, B a finitely generated A-algebra, M a finitely generated B-module. Then there exists a non-zero $f \in A$ such that M_f is a free (hence flat) A_f-module.*

Proof of Lemma 4.9. [5] Let K be the quotient field of A, so that $B \otimes_A K$ is a finitely generated K-algebra and $M \otimes_A K$ is a finitely generated module over it. Let n be the dimension of the support of this module and argue by induction on n. If $n < 0$, i.e., $M \otimes_A K = (0)$, then taking a finite set of generators of M over B, one sees that there exists an $f \in A$ which annihilates these generators, and hence M, so that $M_f = (0)$ and we are through. Suppose $n \geq 0$. One knows that the B-module M has a composition series whose successive quotients are isomorphic to modules $B/\mathfrak{p}_i$, $\mathfrak{p}_i \subset B$ prime ideals. Since an extension of free modules is free, one is reduced to the case where M itself has the form $B/\mathfrak{p}$, or even is identical to B, B being an integral domain. Applying Noether's normalization lemma (Zariski-Samuel [**109**, vol. I, Chapter V, §4, Theorem 8, p. 266] This reference is for infinite field K.) to the K-algebra $B \otimes_A K$, one sees easily that there exists a non-zero $f \in A$ such that B_f is integral over a subring $A_f[t_1, \ldots, t_n]$, where the t_i are indeterminates. Therefore one can already assume B integral over $C = A[t_1, \ldots, t_n]$, so that it is a finitely generated torsion-free C-module. If m is its rank, there exists therefore an exact sequence of C-modules:

$$0 \longrightarrow C^m \longrightarrow B \longrightarrow M' \longrightarrow 0$$

[5] Reproduced verbatim.

where M' is a torsion C-module. It follows that the dimension of the support of the $C \otimes_A K$-module $M' \otimes_A K$ is strictly less than the dimension n of $C \otimes_A K$. By induction, it follows that, localizing by a suitable $f \in A$, one can assume M' is a free A-module. On the other hand C^m is a free A-module. Therefore B is a free A-module. $\square$

In order to get at what I consider the intuitive content of "flat", we need first a deeper fact:

PROPOSITION 4.10. *Let A be a local ring with maximal ideal $\mathfrak{m}$, and let $B = A[X_1, \ldots, X_n]_{\mathfrak{p}}$ where $\mathfrak{p} \cap A = \mathfrak{m}$. Let*

$$K \xrightarrow{u} L \xrightarrow{v} M$$

be finitely generated free B-modules and B-homomorphisms such that $v \circ u = 0$. If

$$K/\mathfrak{m}K \longrightarrow L/\mathfrak{m}L \longrightarrow M/\mathfrak{m}M$$

is exact, then

$$K \longrightarrow L \longrightarrow M$$

is exact and $M/v(L)$ is flat over A.

PROOF. Express u and v by matrices of elements of B and let A_0 be the subring of A generated over $\mathbb{Z}$ by the coefficients of these polynomials. Let $A_1 = (A_0)_{\mathfrak{m} \cap A_0}$. Then A_1 is a *noetherian* local ring and if $B_1 = A_1[X_1, \ldots, X_n]_{\mathfrak{p} \cap A_1[X]}$, we may define a diagram

$$K_1 \xrightarrow{u_1} L_1 \xrightarrow{v_1} M_1$$

over B_1 such that $K \to L \to M$ arises from it by $\otimes_{B_1} B$ or equivalently by $\otimes_{A_1} A$ and then localizing at $\mathfrak{p}$. Let $\mathfrak{m}_1 = \mathfrak{m} \cap A_1$, $k_1 = A_1/\mathfrak{m}_1$, $k = A/\mathfrak{m}$. Then

$$K_1/\mathfrak{m}_1K_1 \longrightarrow L_1/\mathfrak{m}_1L_1 \longrightarrow M_1/\mathfrak{m}_1M_1$$

is exact because $K/\mathfrak{m}K \to L/\mathfrak{m}L \to M/\mathfrak{m}M$ is exact and arises from it by $\otimes_{k_1} k$ (i.e., a non-exact sequence of k_1-vector spaces remains non-exact after $\otimes_{k_1} k$). Now if we prove the lemma for A_1, B_1, K_1, L_1 and M_1, it follows for A, B, K, L and M. In fact $M_1/v_1(L_1)$ flat over A_1 implies

$$M/v(L) \cong [(M_1/v_1(L_1)) \otimes_{A_1} A]_S \quad (S = \text{multiplicative system } A[X] \setminus \mathfrak{p})$$

is flat over A; and from the exact sequences:

$$K_1 \longrightarrow L_1 \longrightarrow v_1(L_1) \longrightarrow 0$$

$$0 \longrightarrow v_1(L_1) \longrightarrow M_1 \longrightarrow M_1/v_1(L_1) \longrightarrow 0$$

we deduce by $\otimes_{A_1} A$ and by localizing with respect to S that

$$K \longrightarrow L \longrightarrow (v_1(L_1) \otimes_{A_1} A)_S \longrightarrow 0$$

$$0 \longrightarrow (v_1(L_1) \otimes_{A_1} A)_S \longrightarrow M \longrightarrow M/\left(v_1(L_1) \otimes_{A_1} A\right)_S \longrightarrow 0$$

are exact, (using again $M_1/v_1(L_1)$ flat over A_1!), hence $K \to L \to M$ is exact. This reduces the lemma to the case A noetherian. In this case, we use the fact that B noetherian local with $\mathfrak{m} \subset$ maximal ideal of B implies

$$\bigcap_{n=1}^{\infty} \mathfrak{m}^n \cdot P = (0)$$

for any finitely generated B-module P (cf. Zariski-Samuel [**109**, vol. I, Chapter IV, Appendix, p. 253]). In particular

$$\bigcap_{n=1}^{\infty} \mathfrak{m}^n \cdot (L/u(K)) = (0)$$

or

$$\bigcap_{n=1}^{\infty} (\mathfrak{m}^n L + u(K)) = u(K).$$

So if $x \in \operatorname{Ker}(v)$ and $x \notin \operatorname{Image}(u)$ we can find an n such that $x \in \mathfrak{m}^n \cdot L + u(K)$, but $x \notin \mathfrak{m}^{n+1} \cdot L + u(K)$. Let $x = y + u(z)$, $y \in \mathfrak{m}^n \cdot L$, $z \in K$. The (u, v)-sequence induces by $\otimes\, \mathfrak{m}^n/\mathfrak{m}^{n+1}$ a new sequence:

$$\begin{array}{ccccc}
\mathfrak{m}^n K/\mathfrak{m}^{n+1}K & \xrightarrow{\quad u_n \quad} & \mathfrak{m}^n L/\mathfrak{m}^{n+1}L & \xrightarrow{\quad v_n \quad} & \mathfrak{m}^n M/\mathfrak{m}^{n+1}M \\
\sim\| & & \sim\| & & \sim\| \\
(\mathfrak{m}^n/\mathfrak{m}^{n+1}) \otimes_k K/\mathfrak{m}K & \longrightarrow & (\mathfrak{m}^n/\mathfrak{m}^{n+1}) \otimes_k L/\mathfrak{m}L & \longrightarrow & (\mathfrak{m}^n/\mathfrak{m}^{n+1}) \otimes_k M/\mathfrak{m}M.
\end{array}$$

The bottom sequence is exact by hypothesis. On the other hand y maps to an element $\overline{y} \in \mathfrak{m}^n L/\mathfrak{m}^{n+1}L$ such that $v_n(\overline{y}) = 0$. Therefore $\overline{y} \in \operatorname{Image} u_n$, i.e., $y \in u(\mathfrak{m}^n K) + \mathfrak{m}^{n+1}L$, hence $x \in u(K) + \mathfrak{m}^{n+1}L$ — contradiction. This proves that the (u, v)-sequence is exact. Next, if $\mathfrak{a} \subset A$ is any ideal, the same argument applies to the sequence:

$$(*) \qquad K/\mathfrak{a} \cdot K \longrightarrow L/\mathfrak{a} \cdot L \longrightarrow M/\mathfrak{a} \cdot M$$

of $B/\mathfrak{a} \cdot B$-modules. Therefore all these sequences are exact. But from the exact sequences:

$$K \longrightarrow L \longrightarrow L/u(K) \longrightarrow 0$$

$$0 \longrightarrow L/u(K) \longrightarrow M \longrightarrow M/v(L) \longrightarrow 0,$$

we get in any case exact sequences:

$$(**) \quad \begin{array}{l} K/\mathfrak{a} \cdot K \longrightarrow L/\mathfrak{a} \cdot L \longrightarrow (L/u(K)) \otimes_A A/\mathfrak{a} \longrightarrow 0 \\[1ex] (L/u(K)) \otimes_A A/\mathfrak{a} \longrightarrow M/\mathfrak{a} \cdot M \longrightarrow (M/v(L)) \otimes_A A/\mathfrak{a} \longrightarrow 0 \end{array}$$

so the exactness of $(*)$ implies that $(**)$ is exact with (0) on the left too, i.e., by Proposition 4.2 $M/v(L)$ is flat over A. $\square$

COROLLARY 4.11. *Let A be a local ring with maximal ideal $\mathfrak{m}$, and let $B = A[X_1, \ldots, X_n]_{\mathfrak{p}}$ where $\mathfrak{p} \cap A = \mathfrak{m}$. Let $f_1, \ldots, f_k \in B$. Then*

$$B/(f_1, \ldots, f_k) \text{ is a flat } A\text{-algebra} \iff \begin{bmatrix} \forall \text{ syzygies } \sum_{i=1}^k \overline{g}_i \overline{f}_i = 0 \text{ in } B/\mathfrak{m}B, \\ \exists \text{ syzygy } \sum_{i=1}^k g_i f_i = 0 \text{ in } B \\ \text{with } g_i \text{ lifting } \overline{g}_i. \end{bmatrix}$$

PROOF. $\Longleftarrow$: Since $B/\mathfrak{m}B$ is noetherian, the module of syzygies over $B/\mathfrak{m}B$ is finitely generated: let

$$\sum \overline{g}_{i,l} \overline{f}_i = 0, \qquad 1 \le l \le N$$

be a basis, and lift these to syzygies

$$\sum g_{i,l} f_i = 0.$$

Define homomorphisms:

$$B^N \xrightarrow{u} B^k \xrightarrow{v} B$$

$$u(a_1, \ldots, a_N) = \left(\sum g_{1,l} a_l, \ldots, \sum g_{k,l} a_l\right)$$

$$v(a_1, \ldots, a_k) = \sum a_i f_i.$$

Then $v \circ u = 0$ and by construction

$$(B/\mathfrak{m}B)^N \xrightarrow{\overline{u}} (B/\mathfrak{m}B)^k \xrightarrow{\overline{v}} B/\mathfrak{m}B$$

is exact. Therefore $B/v(B^k) = B/(f_1, \ldots, f_k)$ is A-flat by Proposition 4.10.

$\Longrightarrow$: Define v as above and call its kernel Syz, the module of syzygies so that we get:

$$0 \longrightarrow \mathrm{Syz} \longrightarrow B^k \xrightarrow{v} B \longrightarrow B/(f_1, \ldots, f_k) \longrightarrow 0.$$

Split this into two sequences:

$$0 \longrightarrow \mathrm{Syz} \longrightarrow B^k \longrightarrow (f_1, \ldots, f_k) \longrightarrow 0$$

$$0 \longrightarrow (f_1, \ldots, f_k) \longrightarrow B \longrightarrow B/(f_1, \ldots, f_k) \longrightarrow 0.$$

By the flatness of $B/(f_1, \ldots, f_k)$, these give:

$$\mathrm{Syz}/\mathfrak{m} \cdot \mathrm{Syz} \longrightarrow (B/\mathfrak{m}B)^k \longrightarrow (f_1, \ldots, f_k) \otimes_B B/\mathfrak{m}B \longrightarrow 0$$

$$0 \longrightarrow (f_1, \ldots, f_k) \otimes_B B/\mathfrak{m}B \longrightarrow B/\mathfrak{m}B \longrightarrow B/(\mathfrak{m}B + (f_1, \ldots, f_k)) \longrightarrow 0,$$

hence

$$\mathrm{Syz}/\mathfrak{m} \cdot \mathrm{Syz} \longrightarrow (B/\mathfrak{m}B)^k \xrightarrow{\overline{v}} B/\mathfrak{m}B \longrightarrow B/(\mathfrak{m}B + (f_1, \ldots, f_k)) \longrightarrow 0$$

is exact. Since $\mathrm{Ker}\,\overline{v}$ = syzygies in $B/\mathfrak{m}B$, this shows that all syzygies in $B/\mathfrak{m}B$ lift to Syz. □

Putting it succinctly, *flatness means that syzygies for the fibres lift to syzygies for the whole scheme and hence restrict to syzygies for the other fibres*: certainly a reasonable continuity property.

The simplest case is when R is a valuation ring. We give this a name:

DEFINITION 4.12. Let R be a valuation ring, and let η (resp. o) be the generic (resp. closed) point of $\mathrm{Spec}\,R$. Let $f \colon X \to \mathrm{Spec}\,R$ be a flat morphism of finite type (by Proposition 4.5, this means: $\mathcal{O}_X$ is a sheaf of torsion-free R-modules). Then we say that the closed fibre X_o of f is a specialization over R of the generic fibre X_η.

Note that the flatness of f is equivalent to saying that X_η is scheme-theoretically dense in X (Proposition II.3.11). In fact, if you start with any X of finite type over R, then define a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ by:

$$\mathcal{I}(U) = \mathrm{Ker}\left(\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X_\eta}(U \cap X_\eta)\right).$$

Then as in Proposition II.3.11 it follows easily that $\mathcal{I}$ is quasi-coherent and for all U affine, $\mathcal{I}(U)$ is just the ideal of R-torsion elements in $\mathcal{O}_X(U)$. If $\mathcal{O}_X/\mathcal{I}$ is the structure sheaf of the subscheme $\widetilde{X} \subset X$, then

a) $\widetilde{X}_\eta = X_\eta$

b) $\widetilde{X}_o$ is a specialization of X_η.

To give some examples of specializations, consider:

EXAMPLE. 1.) If X is reduced and irreducible, with its generic point over η, then X_o is always a specialization of X_η.

EXAMPLE. 2.) Denote by M the maximal ideal of R with the residue field $k = R/M$. The quotient field of R is denoted by K. If $f(X_1, \ldots, X_n)$ is a polynomial with coefficients in R and $\overline{f}$ is the same polynomial mod M, i.e., with coefficients in k, then the affine scheme $V(\overline{f}) \subset \mathbb{A}^n_k$ is a specialization of $V(f) \subset \mathbb{A}^n_K$ *provided that* $\overline{f} \not\equiv 0$. In fact, let $X = V(f) \subset \mathbb{A}^n_R$ and note that $R[X_1, \ldots, X_n]/(f)$ is torsion-free.

EXAMPLE. 3.) If X is anything of finite type over R, and $Y_\eta \subset X_\eta$ is any closed subscheme, there is a unique closed subscheme $Y \subset X$ with generic fibre Y_η such that Y_o is a specialization of Y_η. (Proof similar to discussion above.) It can be quite fascinating to see how this "comes out", i.e., given Y_η, guess what Y_o will be:

EXAMPLE. 4.) In $\mathbb{A}^2_K$ with coordinates x, y, let Y_η be the union of the three distinct points

$$(0,0), \quad (0,\alpha), \quad (\alpha,0), \qquad \alpha \in M, \;\; \alpha \neq 0.$$

Look at the ideal:

$$I = \operatorname{Ker}\left(R[x,y] \xrightarrow{\phi} K \oplus K \oplus K\right)$$

where $\phi(f) = (f(0,0), f(0,\alpha), f(\alpha,0))$. I is generated by

$$xy, \quad x(x-\alpha), \quad y(y-\alpha),$$

hence reducing these mod M, we find

$$Y_o = \operatorname{Spec} k[x,y]/(x^2, xy, y^2)$$

the origin with "multiplicity 3". For other triples of points, what Y_o's can occur?

EXAMPLE. 5.) (Hironaka) Take two skew lines in $\mathbb{A}^3_K$:

$$\begin{aligned} &l_1 \text{ defined by } x = y = 0 \\ &l_2 \text{ defined by } z = 0, \; x = \alpha, \quad (\alpha \in M, \alpha \neq 0). \end{aligned}$$

Let $Y_\eta = l_1 \cup l_2$. To find Y_o, first compute:

$$I = \operatorname{Ker}\left(R[x,y,z] \longrightarrow \Gamma(\mathcal{O}_{l_1}) \oplus \Gamma(\mathcal{O}_{l_2})\right).$$

One finds $I = (xz, yz, x(x-\alpha), y(x-\alpha))$. Reducing mod M, we find

$$X_o = \operatorname{Spec} k[x,y,z]/(xz, yz, x^2, xy).$$

Now

$$\sqrt{(x^2, xy, xz, yz)} = (x, yz)$$

which is the ideal of the two lines $l_1' = V(x,y)$ and $l_2' = V(x,z)$ which are the limits of l_1 and l_2 individually. But since

$$(x^2, xy, xz, yz) = (x, yz) \cap (x,y,z)^2$$

it follows that X_o has an embedded component where the two lines cross. The picture is as in Figure IV.11.

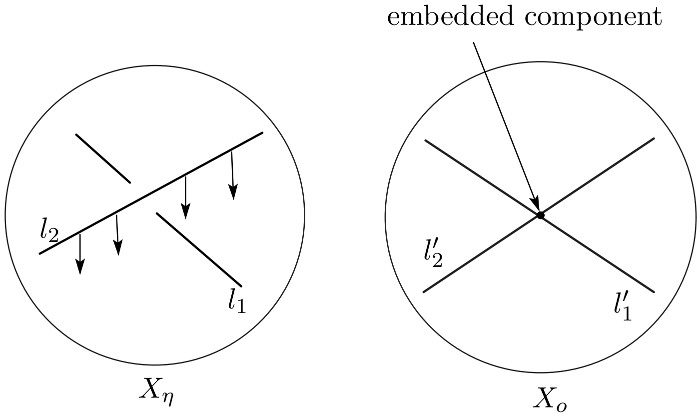

FIGURE IV.11. Specialization of two skew lines

5. Dimension of fibres of a morphism

We would like to prove some general results on the dimensions of the fibres of a morphism. We begin with the case of a specialization:

THEOREM 5.1 (Dimension Theorem). *Let R be a valuation ring with quotient field K, residue field $k = R/M$, let $S = \operatorname{Spec} R$, and let X be a reduced, irreducible scheme of finite type over S with generic point over η. Then for every component W of X_o:*

$$\dim X_\eta = \dim W$$

$$i.e., \qquad \operatorname{tr.deg}_K \mathbb{R}(X) = \operatorname{tr.deg}_k \mathbb{R}(W).$$

PROOF. First of all, we may as well replace X by an affine open subset meeting W and not meeting any other components of X_o. This reduces us to the case where $X = \operatorname{Spec} A$ and X_o is irreducible (hence $\sqrt{M \cdot A}$ prime).

Next, the inequality $\dim X_o \leq \dim X_\eta$ is really simple: because if $r = \dim X_o$, then there exist $t_1, \ldots, t_r \in A$ such that $\overline{t}_1, \ldots, \overline{t}_r \in A/\sqrt{M \cdot A}$ are independent transcendentals over k. But if the t_i are dependent over K, let

$$\sum c_\alpha t^\alpha = 0$$

be a relation. Multiplying through by a suitable constant, since R is a valuation ring, we may assume $c_\alpha \in R$ and not all c_α are in M. Then $\sum \overline{c}_\alpha \overline{t}^\alpha = 0$ in $A/\sqrt{M \cdot A}$ is a non-trivial relation over k.

To get started in the other inequality, we will use:

LEMMA 5.2. *Let $k \subset K$ be any two fields and let X be a k-variety. Then if $X \times_{\operatorname{Spec} k} \operatorname{Spec} K = X_1 \cup \cdots \cup X_t$,*

$$\dim X = \dim X_i, \quad 1 \leq i \leq t.$$

PROOF OF LEMMA 5.2. If K is an algebraic extension of k, this follows from Theorem 2.4 by going up to $\overline{k}$ and down again. If K is purely transcendental over k, let $K = k(\ldots, t_\alpha, \ldots)$. Then if A is any integral domain containing k with quotient field L,

$$A \otimes_k K = (A[\ldots, t_\alpha, \ldots] \text{ localized with respect to } K \setminus (0))$$

is another integral domain and it contains K and has quotient field $L(\ldots, t_\alpha, \ldots)$. It follows that in this case $X \times_{\operatorname{Spec} k} \operatorname{Spec} K$ is reduced and irreducible and

$$\mathbb{R}(X \times_{\operatorname{Spec} k} \operatorname{Spec} K) = \mathbb{R}(X)(\ldots, t_\alpha, \ldots).$$

Therefore

$$\begin{aligned}\dim\left(X \times_{\operatorname{Spec} k} \operatorname{Spec} K\right) &= \operatorname{tr.deg}_{k(\dots,t_\alpha,\dots)} \mathbb{R}(X)(\dots,t_\alpha,\dots)\\ &= \operatorname{tr.deg}_k \mathbb{R}(X)\\ &= \dim X.\end{aligned}$$

Putting the two cases together, we get the general result. □

Lemma 5.3. *Let R be any local integral domain (neither noetherian nor a valuation ring!), $S = \operatorname{Spec} R$, $\eta, o \in S$ as above. Let X be reduced and irreducible and let $\pi\colon X \to S$ be of finite type. Assume π has a section $\sigma\colon S \to X$. Then $\dim X_o = 0 \Longrightarrow \dim X_\eta = 0$.*

Proof of Lemma 5.3. We can replace X by an affine neighborhood of $\sigma(o)$ and so reduce to the case $X = \operatorname{Spec} A$ for simplicity. On the ring level, we get

$$\begin{tikzcd} R \arrow[r, "\pi^*"'] & A \arrow[l, bend right, "\sigma^*"'] \end{tikzcd}$$

hence

$$A = R \oplus I, \quad \text{where } I = \operatorname{Ker} \sigma^*.$$

Consider the sequence of subschemes

$$\begin{tikzcd} Y_n = \operatorname{Spec} A/I^n \arrow[dr] & \subset & X \arrow[dl] \\ & S. & \end{tikzcd}$$

If $x_1, \dots, x_m$ generate A as a ring over R, let $x_i = a_i + y_i$, $a_i \in R$, $y_i \in I$. Then $y_1^{r_1} \cdots y_m^{r_m}$ with $0 \le \sum r_i < n$ generate A/I^n as a module over R. Being finitely generated over R at all, it follows by Nakayama's lemma that if $z_1, \dots, z_n \in A/I^n$ generate $(A/I^n) \otimes_R (R/M)$ over R/M, they generate A/I^n over R. Thus

$$(*) \qquad \begin{aligned}\dim_k (A/I^n) \otimes_R k &= (\text{minimal number of generators of } A/I^n)\\ &\ge \dim_K (A/I^n) \otimes_R K.\end{aligned}$$

Now given any 0-dimensional scheme Z of finite type over a field L, then Z consists in a finite set of points $\{P_1, \dots, P_t\}$, and the local rings $\mathcal{O}_{P_i,Z}$ are artinian. Then in fact $Z \cong \operatorname{Spec}\left(\prod_{i=1}^t \mathcal{O}_{P_i,Z}\right)$, hence is affine and a natural measure of its "size" is

$$\deg_L Z \underset{\text{def}}{=} \dim_L \Gamma(\mathcal{O}_Z).$$

In this language, $(*)$ says $\deg_k (Y_n)_o \ge \deg_K (Y_n)_\eta$. But $(Y_n)_o \subset X_o$ and X_o is itself 0-dimensional, so

$$\deg_k X_o \ge \deg_k (Y_n)_o \ge \deg_K (Y_n)_\eta.$$

This bound shows that $(Y_n)_\eta = (Y_{n+1})_\eta$ if $n \gg 0$. On the other hand, $(Y_n)_\eta$ is the subscheme of X_η consisting of the single point $x = \sigma(\eta)$ and defined by the ideal $\mathfrak{m}^n_{x,X_\eta}$. Thus $\mathfrak{m}^n_{x,X_\eta} = \mathfrak{m}^{n+1}_{x,X_\eta}$ if $n \gg 0$ and since $\mathcal{O}_{x,X_\eta}$ is noetherian, this means $\mathfrak{m}^n_{x,X_\eta} = (0)$ if $n \gg 0$. Thus $\mathcal{O}_{x,X_\eta}$ is in fact finite-dimensional over K, hence

$$\begin{aligned}\dim X_\eta &= \operatorname{tr.deg}_K \mathbb{R}(X_\eta)\\ &= \operatorname{tr.deg}_K \mathcal{O}_{x,X_\eta}\\ &= 0.\end{aligned}$$

□

Lemma 5.4. *Lemma 5.3 still holds even if a section doesn't exist.*

Proof of Lemma 5.4. Choose $x \in X_o$, let $S' = \operatorname{Spec} \mathcal{O}_{x,X}$, and consider

$$\begin{tikzcd} X \times_S S' \arrow[d] \\ S' \arrow[u, bend left, "\sigma"] \end{tikzcd}$$

where $\sigma = (i, 1_{S'})$, $i \colon \operatorname{Spec} \mathcal{O}_{x,X} \to X$ being the canonical inclusion. Let X' be an irreducible component of $X \times_S S'$ containing $\sigma(S')$ with its reduced structure. Then

$$\begin{aligned} \dim X_o = 0 &\Longrightarrow \dim X'_o = 0 \text{ by Lemma 5.2} \\ &\Longrightarrow \dim X'_\eta = 0 \text{ by Lemma 5.3} \\ &\Longrightarrow \dim X_\eta = 0 \text{ by Lemma 5.2.} \end{aligned}$$

$\square$

Going back to Theorem 5.1, we have now proven that $\dim X_o = 0 \Longleftrightarrow \dim X_\eta = 0$. Suppose instead that both dimensions are positive. Choose $t \in A$ such that $\bar{t} \in A/\sqrt{M \cdot A}$ is transcendental over k and let

$$R' = (R[t] \text{ localized with respect to } \mathcal{S} = R[t] \setminus M \cdot R[t])\,.$$

This is a new valuation ring with quotient field $K(t)$ and residue field $k(t)$ and π factors:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[rdd, "\pi"'] & \operatorname{Spec} A \arrow[d] & \operatorname{Spec} A_{\mathcal{S}} = X' \arrow[l, hook'] \arrow[d, "\pi'"'] \\ & \operatorname{Spec} R[t] \arrow[d] & \operatorname{Spec} R' \arrow[l, hook'] \\ & \operatorname{Spec} R & \end{tikzcd}$$

Since t is transcendental in both A_K and $A/\sqrt{M \cdot A}$, π takes the generic points of both X_o and X_η into the subset $\operatorname{Spec} R'$ of $\operatorname{Spec} R[t]$, i.e., they lie in X'. Now $A_{\mathcal{S}}$ being merely a localization of A, X' has the same stalks as X. Therefore $\mathbb{R}(X'_\eta) = \mathbb{R}(X)$ and $\mathbb{R}(X'_o) = \mathbb{R}(X_o)$ and considering X' over $S' = \operatorname{Spec} R'$:

$$\begin{aligned} \dim X'_\eta &= \operatorname{tr.deg}_{K(t)} \mathbb{R}(X) \\ &= \operatorname{tr.deg}_K \mathbb{R}(X) - 1 \\ &= \dim X_\eta - 1 \\ \dim X'_o &= \operatorname{tr.deg}_{k(t)} \mathbb{R}(X_o) \\ &= \operatorname{tr.deg}_k \mathbb{R}(X_o) - 1 \\ &= \dim X_o - 1. \end{aligned}$$

Making an induction on $\min(\dim X_\eta, \dim X_o)$, this last step completes the proof. $\square$

The dimension theorem (Theorem 5.1) has lots of consequences: first of all it has the following generalization to general morphisms of finite type:

Corollary 5.5. *Let $f \colon X \to Y$ be a morphism of finite type between two irreducible reduced schemes with f(generic point η_X) = generic point η_Y. Then for all $y \in Y$ and all components W of $f^{-1}(y)$:*

$$\dim W \geq \operatorname{tr.deg}_{\mathbb{R}(Y)} \mathbb{R}(X) = \dim f^{-1}(\eta_Y).$$

If f is flat over Y, equality holds.

Proof. We may as well assume $f^{-1}(y)$ is irreducible as otherwise we can replace X by an open subset to achieve this. Let $w \in f^{-1}(y)$. Choose a valuation ring R:

$$\mathcal{O}_{w,X} \subset R \subset \mathbb{R}(X)$$

with

$$\mathfrak{m}_{w,X} \subset \text{maximal ideal } M \text{ of } R.$$

Now form the fibre product:

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l] \arrow[d, "f'"] \\ Y & \operatorname{Spec} R. \arrow[l] \end{tikzcd}$$

Note that f' has a section $\sigma\colon \operatorname{Spec} R \to X'$ induced by the canonical map $\operatorname{Spec} R \to \operatorname{Spec} \mathcal{O}_{w,X} \to X$ (as in Lemma 5.4).

Let X'_i be the components of X' with reduced structure, let

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l] \arrow[d, "f'"'] & \supset & X'' \\ Y & \operatorname{Spec} R. \arrow[l] \arrow[urr, "\sigma"'] & & \end{tikzcd}$$

Break up X'_η into its irreducible components and let their closures in X' with reduced structure be written $X^{(1)}, \dots, X^{(n)}$. One of these, say $X^{(1)}$ contains the image of the section σ. Let $o, \eta \in \operatorname{Spec} R$ be its closed and generic points: the various fibres are related by:

$$\begin{aligned} X^{(1)}_\eta &= \text{component of } X'_\eta, & X'_\eta &= f^{-1}(\eta_Y) \times_{\operatorname{Spec}\mathbb{R}(Y)} \operatorname{Spec} K \\ X^{(1)}_o &\subset X'_o, & X'_o &= f^{-1}(y) \times_{\operatorname{Spec}\mathbb{R}(Y)} \operatorname{Spec} R/M. \end{aligned}$$

Then:

$$\begin{aligned} \dim f^{-1}(y) &= \dim(\text{all components of } X'_o), \quad \text{by Lemma 5.2} \\ &\geq \dim(\text{any component of } X^{(1)}_o) \\ &= \dim X^{(1)}_\eta, \quad \text{by Theorem 5.1} \\ &= \dim f^{-1}(\eta_Y), \quad \text{by Lemma 5.2.} \end{aligned}$$

Now if X is flat over Y, then X' is flat over $\operatorname{Spec} R$, hence

$$X' = X^{(1)} \cup \cdots \cup X^{(n)}$$

(otherwise, let $U \subset X'$ be an open affine disjoint from $\bigcup X^{(i)}$ and if $U = \operatorname{Spec} A$, then $\operatorname{Spec}(A \otimes_R K) = U_\eta = \emptyset$, so A is a torsion R-module contradicting flatness). Therefore

$$X'_o = X^{(1)}_o \cup \cdots \cup X^{(n)}_o$$

hence for at least one i, $X^{(i)}_o =$ a component of X'_o and

$$\begin{aligned} \dim f^{-1}(y) &= \dim X^{(i)}_o \\ &= \dim X^{(i)}_\eta \\ &= \dim f^{-1}(\eta_Y). \end{aligned}$$

□

Combining Corollary 5.5 and Theorem 4.8, we get:

Corollary 5.6. *Let $f\colon X \to Y$ be as in Theorem* 5.1. *Then there is an integer n and a non-empty open $U \subset Y$ such that for all $y \in U$ and all components W of $f^{-1}Y$, $\dim W = n$.*

Combining these results and the methods of Part I [**76**, (3.16)], we deduce:

Corollary 5.7. *Let $f\colon X \to Y$ be any morphism of finite type with Y noetherian. Then the function*

$$x \longmapsto \max\{\dim W \mid W \textit{ a component of } f^{-1}(f(x)) \textit{ containing } x\}$$

is upper semi-continuous.

Another consequence of Theorem 5.1 is that we generalize Part I [**76**, (3.14)] to varieties over any ground field k:

Corollary 5.8. *Let k be a field and X a k-variety. If $t \in \Gamma(\mathcal{O}_X)$ and*

$$V(t) = \{x \in X \mid t(x) = 0\} \subsetneqq X$$

then for every component W of $V(t)$,

$$\dim W = \dim X - 1.$$

Proof. Let t define a morphism:

$$T\colon X \longrightarrow \mathbb{A}^1_k.$$

Then either T(generic point) $=$ generic point, or T(generic point) $=$ closed point a. In the second case $a \neq 0$ and $v(t) = \emptyset$ so there is nothing to prove. In the first case, $R = \mathcal{O}_{0,\mathbb{A}^1}$ is a valuation ring and making a base change:

$$\begin{tikzcd} X \arrow[d, "T"'] & X' \arrow[l] \arrow[d, "\pi"] \\ \mathbb{A}^1_k & \operatorname{Spec} R \arrow[l] \end{tikzcd}$$

we are in the situation of the dimension theorem. Now $\mathbb{R}(X) = \mathbb{R}(X')$, so

$$\begin{aligned} \dim\big(X'_\eta \text{ over quotient field of } R\big) &= \operatorname{tr.deg}_{k(t)} \mathbb{R}(X) \\ &= \operatorname{tr.deg}_k \mathbb{R}(X) - 1 \\ &= \dim X - 1 \end{aligned}$$

while W is a component of $T^{-1}(0)$, hence of $\pi^{-1}(0)$ and satisfies:

$$\begin{aligned} \dim(W \text{ over residue field of } R) &= \operatorname{tr.deg}_k \mathbb{R}(W) \\ &= \dim W. \end{aligned}$$

□

(Note that we have not used Krull's principal ideal theorem (Zariski-Samuel [**109**, vol. I, Chapter IV, §14, Theorem 29, p. 238]) in this proof.)

Up to this point, we have defined and discussed dimension only for varieties over various fields. There is a natural concept of dimension for arbitrary schemes which by virtue of the above corollary agrees with our definition for varieties:

DEFINITION 5.9. If X is a scheme, then

$$\dim X = \left\{ \begin{array}{l} \text{largest integer } n \text{ such that there exists a chain} \\ \text{of irreducible closed subsets: } \emptyset \neq Z_0 \underset{\neq}{\subsetneq} Z_1 \underset{\neq}{\subsetneq} \cdots \underset{\neq}{\subsetneq} Z_n \subset X. \end{array} \right\}$$

If $Z \subset X$ is an irreducible closed set with generic point z, then

$$\begin{array}{l} \operatorname{codim}_X Z \\ \text{or} \\ \operatorname{codim}_X z \end{array} = \left\{ \begin{array}{l} \text{largest integer } n \text{ such that there exists a chain} \\ \text{of irreducible closed subsets: } Z = Z_0 \underset{\neq}{\subsetneq} Z_1 \underset{\neq}{\subsetneq} \cdots \underset{\neq}{\subsetneq} Z_n \subset X \end{array} \right\}.$$

From the definition, one sees immediately that $\forall Z$ irreducible, closed:

$$\dim Z + \operatorname{codim}_X Z \leq \dim X.$$

But "$<$" can hold even for such spaces as $\operatorname{Spec} R$, R local noetherian integral domain! This pathology makes rather a mess of general dimension theory. The definition ties up with dimension in local ring theory as follows: if $Z \subset X$ is closed and irreducible, and $z \in Z$ is its generic point, then there is a bijection between closed irreducible $Z' \supset Z$ and prime ideals $\mathfrak{p} \subset \mathcal{O}_{z,X}$. Therefore:

$$\operatorname{codim}_X Z = \operatorname{Krull}\dim \mathcal{O}_{z,X}$$

where the Krull dim of a local ring is the maximal length of a chain of prime ideals: cf. Zariski-Samuel [**109**, vol. II, p. 288], or Atiyah-MacDonald [**19**, Chapter 11]. Moreover, in this language, Krull's principal ideal theorem (Zariski-Samuel [**109**, vol. I, Chapter IV, §14, Theorem 29, p. 238]) states:

If X is noetherian reduced and irreducible, $f \in \Gamma(\mathcal{O}_X)$, $f \neq 0$, then for all components W of $V(f)$,

$$\operatorname{codim}_X W = 1,$$

which generalizes Corollary 5.8.

COROLLARY 5.10. *Let k be a field and X a k-variety. Then the two definitions of dimension agree. More precisely, for every maximal chain*

$$\emptyset \neq Z_0 \underset{\neq}{\subsetneq} Z_1 \underset{\neq}{\subsetneq} \cdots \underset{\neq}{\subsetneq} Z_n = X$$

we have:

$$\operatorname{tr.deg}_k \mathbb{R}(Z_i) = i, \quad 0 \leq i \leq n.$$

In particular, X is "catenary", meaning that any two maximal chains have the same length. Therefore for all $Z \subset X$ closed irreducible, with generic point z:

$$\dim Z + \operatorname{codim}_X Z = \dim X$$

or

$$\operatorname{tr.deg}_k \Bbbk(z) + \operatorname{Krull}\dim \mathcal{O}_{z,X} = \dim X.$$

PROOF. Note on the one hand that a minimal irreducible closed subset Z_0 is just a closed point $Z_0 = \{z_0\}$, hence $\mathbb{R}(Z_0) = \Bbbk(z_0)$ is algebraic over k by Corollary II.2.11. On the other hand, a maximal proper closed irreducible subset $Z \underset{\neq}{\subsetneq} X$ can be analyzed by Corollary 5.8. Let $U \subset X$ be an affine open set meeting Z and let $f \in \mathcal{O}_X(U)$ be a function 0 on $Z \cap U$ and 1 at some closed point $z' \in U \setminus (U \cap Z)$. Then

$$Z \cap U \subset V(f) \underset{\neq}{\subsetneq} U,$$

hence $Z \cap U \subset W \subsetneqq U$, some component W of $V(f)$, hence $Z \subset \overline{W} \subsetneqq X$. By maximality of Z, $Z = \overline{W}$, hence

$$\begin{aligned}\operatorname{tr.deg}_k \mathbb{R}(Z) &= \operatorname{tr.deg}_k \mathbb{R}(W)\\ &= \operatorname{tr.deg}_k \mathbb{R}(X) - 1, \quad \text{by Corollary 5.8.}\end{aligned}$$

These two observations prove Corollary 5.10. □

COROLLARY 5.11. *Let R be a Dedekind domain with an infinite number of prime ideals and quotient field K and let $\pi \colon X \to \operatorname{Spec} R$ be a reduced and irreducible scheme of finite type over R with $\pi(\eta_X) = \eta_R$, the generic point of $\operatorname{Spec} R$. Then every maximal chain looks like:*

$$\emptyset \neq Z_0 \subsetneqq Z_1 \subsetneqq \cdots \subsetneqq Z_r \subsetneqq Y_{r+1} \subsetneqq \cdots \subsetneqq Y_{n+1} = X$$

where

a) $Z_r \subset \pi^{-1}(a)$ *for some closed point* $a \in \operatorname{Spec} R$ *and* $\operatorname{tr.deg}_{\Bbbk(a)} \mathbb{R}(Z_i) = i$, $0 \leq i \leq r$
b) $\pi(Y_{r+1}) \ni \eta_R$ *and* $\operatorname{tr.deg}_K \mathbb{R}(Y_{i+1}) = i$, $r \leq i \leq n$.

In particular $n = \operatorname{tr.deg}_K \mathbb{R}(X)$, *$X$ is catenary and*

$$\dim X = 1 + \operatorname{tr.deg}_K \mathbb{R}(X).$$

PROOF. This goes just like Corollary 5.10. By Chevalley's Nullstellensatz (Theorem II.2.9) a closed point $Z_0 = \{z_0\}$ of X lies over a closed point a of $\operatorname{Spec} R$ and $\Bbbk(z_0)$ is algebraic over $\Bbbk(a)$. And maximal proper closed irreducible $Z \subsetneqq X$ fall into two cases:
Case i): $Z_\eta \neq \emptyset$, so $Z_\eta \subsetneqq X_\eta$ is a maximal closed irreducible subset and so $\operatorname{tr.deg}_K \mathbb{R}(Z) = \operatorname{tr.deg}_K \mathbb{R}(X) - 1$;
Case ii): $Z_\eta = \emptyset$, so $Z \subset \pi^{-1}(a)$ in which case Z must be a component of $\pi^{-1}(a)$. Then by the Dimension Theorem (Theorem 5.1), $\operatorname{tr.deg}_{\Bbbk(a)} \mathbb{R}(Z) = \operatorname{tr.deg}_K \mathbb{R}(X)$. □

An important link between flatness and dimension theory is given by:

PROPOSITION 5.12. *Let $f \colon X \to Y$ be a flat morphism of noetherian schemes and let $x \in X$, $y = f(x)$. Then:*

i) $\operatorname{Spec} \mathcal{O}_{x,X} \to \operatorname{Spec} \mathcal{O}_{y,Y}$ *is surjective,*
ii) $\operatorname{codim}_X(x) \geq \operatorname{codim}_Y(y)$.

Moreover if f is of finite type, then

iii) *for all open sets $U \subset X$, $f(U)$ is open in Y.*

The proof is straightforward using the fact that for all $Z \subset Y$

$$\operatorname{res} f \colon f^{-1}(Z) \longrightarrow Z$$

is still flat, and applying Theorem II.2.9 and Proposition 4.6.

6. Hensel's lemma

The most important situation for specialization is when the base ring R is a *complete* discrete valuation ring, such as $\mathbb{Z}_p$ or $k[[t]]$. One of the main reasons why this case is special is that Helsel's lemma holds. This "lemma" has many variants but we would like to put it as geometrically as possible:

Lemma 6.1. (Hensel's lemma)[6]. *Let R be a complete local noetherian ring, $S = \operatorname{Spec} R$ and $\pi\colon X \to S$ a morphism of finite type. Suppose we have a decomposition of the closed fibre:*

$$X_o = Y_o \cup Z_o, \qquad \begin{array}{l} Y_o,\ Z_o \textit{ open, disjoint} \\ Y_o = \{y\} \textit{ a single point} \end{array}$$

Then we can decompose the whole scheme X:

$$X = Y \cup Z, \qquad \begin{array}{l} Y,\ Z \textit{ open disjoint} \\ Y = \operatorname{Spec} B, \textit{ finite and integral over } R \end{array}$$

so that Y_o = closed fibre of Y, Z_o = closed fibre of Z.

Proof. Let $U \subset X$ be an affine open subset such that $U \cap X_o = \{y\}$. Let $U = \operatorname{Spec} B$, and consider the ideal

$$N = \bigcap_{n=1}^{\infty} M^n \cdot B, \quad \text{where } M = \text{maximal ideal of } R.$$

Now $\mathcal{O}_{y,X}$ is a localization $B_{\mathfrak{p}}$ of B and since $M \cdot B_{\mathfrak{p}} \subset \mathfrak{p} \cdot B_{\mathfrak{p}}$, by Krull's theorem (cf. Zariski-Samuel [**109**, vol. I, Chapter IV, §7, p. 216]):

$$N \cdot B_{\mathfrak{p}} \subset \bigcap_{n=1}^{\infty} M^n \cdot B_{\mathfrak{p}} \subset \bigcap_{n=1}^{\infty} (\mathfrak{p} B_{\mathfrak{p}})^n = (0).$$

Therefore, $\exists f \in B \setminus \mathfrak{p}$ such that $f \cdot N = (0)$. Now replace B by its localization B_f and U by U_f. Using this smaller neighborhood of Y, we can assume $\bigcap_{n=1}^{\infty} M^n \cdot B = (0)$. Now recall the algebraic fact:

> *If B is an R-module such that:*
> a) $\bigcap_{n=1}^{\infty} M^n \cdot B = (0)$
> b) *$B/M \cdot B$ is finite-dimensional over R/M,*
> *then B is a finitely generated R-module* (Zariski-Samuel [**109**, vol. II, Chapter VIII, §3, Theorem 7, p. 259]).

Since $\operatorname{Spec} B/M \cdot B = U_o = Y_o$ consists in one point, $\dim_{R/M} B/MB < +\infty$ and (a) and (b) hold. Therefore B is integrally dependent on R, and by Proposition II.6.5, $\operatorname{res} \pi\colon U \to S$ is a *proper* morphism. It follows that the inclusion i:

$$\begin{tikzcd} U \arrow[rr, hook, "i"] \arrow[dr] & & X \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

is proper, hence $U = \text{Image}(i)$ is closed in X. Therefore if we set $Y = U$, $Z = X \setminus U$, we have the required decomposition. $\square$

Note that in fact, since B is integrally dependent on R, all its maximal ideals contract to $M \subset R$; since $\operatorname{Spec} B$ has only one point, namely y, over the closed point $[M] \in \operatorname{Spec} R$, this means that B has only one maximal ideal, i.e., B is local. Therefore:

$$B = \mathcal{O}_{y,X}.$$

[6]The lemma is also true whenever Y_o is proper over S: cf. EGA [**1**, Chapter III].

Corollary 6.2 (Classical Helsel's lemma). *Let R be a complete local noetherian ring with maximal ideal M and residue field $k = R/M$. Let $f(T)$ be a monic polynomial over R and let $\overline{f}$ be the reduced polynomial over k. Factor $\overline{f}$:*

$$\overline{f} = \prod_{i=1}^{n} g_i^{r_i}$$

where g_i are distinct, irreducible and monic. Then f factors:

$$f = \prod_{i=1}^{n} f_i$$

with $\overline{f}_i = g_i^{r_i}$.

Proof. Apply Hensel's Lemma 6.1 to $X = \operatorname{Spec} R[T]/(f(T))$.

Then X_o consists in n points $[(g_i)] \in \mathbb{A}^1_k$, hence X decomposes into n disjoint pieces:

$$X = \bigcup_{i=1}^{n} X_i$$
$$X_i = \operatorname{Spec} R[T]/\mathfrak{a}_i$$
$$(X_i)_o = \operatorname{Spec} k[T]/(g_i^{r_i}).$$

If $d_i = \deg(g_i^{r_i})$, then $1, T, \ldots, T^{d_i-1}$ generate the R-module $(R[T]/\mathfrak{a}_i) \otimes_R k \cong k[T]/(g_i^{r_i})$, hence by Nakayama's lemma, they generate $R[T]/\mathfrak{a}_i$. Therefore $T^{d_i} \in \sum_{j=1}^{d_i-1} RT^j$ in $R[T]/\mathfrak{a}_i$, or $\mathfrak{a}_i$ contains a monic polynomial f_i of degree d_i. Then

a) $\overline{f}_i \in (g_i^{r_i})$, and since both are monic of the same degree, $\overline{f}_i = g_i^{r_i}$,
b) $\prod f_i$ is everywhere zero on X, so $\prod f_i \in (f)$, and since both are monic of the same degree, $\prod f_i = f$.

It follows easily that $\mathfrak{a}_i = (f_i)$ too, so that the decomposition of X into components and of f into factors are really equivalent! $\square$

Corollary 6.3. *Let R, M, k, $S = \operatorname{Spec} R$ be as before. Then for all finite separable field extension $k \subset L$, there is a unique flat morphism $\pi\colon X_L \to S$ of finite type such that*

$$(*) \qquad \begin{array}{l} (X_L)_o \textit{ is reduced and consists in one point } x \\ \mathbb{k}(x) = L,\ X_L \textit{ connected.} \end{array}$$

In fact for all $p\colon Z \to S$ of finite type and α where:

$$(**) \qquad \begin{array}{l} Z_o = \textit{one point } z,\ Z \textit{ connected} \\ \alpha\colon L \hookrightarrow \mathbb{k}(z) \textit{ is } k\textit{-homomorphism,} \end{array}$$

there exists a unique S-morphism

$$f\colon Z \longrightarrow X_L$$

such that $f(z) = x$ and $f^*\colon \mathbb{k}(x) \to \mathbb{k}(z)$ is equal to α.

Proof. To construct X_L, write $L = k[X]/(\overline{f}(X))$, lift $\overline{f}$ to a polynomial f of the same degree over R and set $X_L = \operatorname{Spec} R[X]/(f(X))$. We prove next that any X_L flat over S with property $(*)$ has the universal property of Corollary 6.3 for all $p\colon Z \to S$ satisfying $(**)$. This will prove, in particular, that any two such X_L's are canonically isomorphic.

Consider

$$p_2\colon X_L \times_S Z \longrightarrow Z.$$

α induces a section $\overline{\sigma}$ of p_2 over $\{z\}$.

$$\begin{tikzcd}
\operatorname{Spec} L \times_S \operatorname{Spec} \Bbbk(z) \arrow[r, hook] & X_L \times_S Z \arrow[d] \\
\operatorname{Spec} \Bbbk(z) = \{z\} \arrow[u, "(\operatorname{Spec}\alpha{,}1)"] \arrow[r, hook] & Z.
\end{tikzcd}$$

By Hensel's Lemma 6.1, $Z = \operatorname{Spec} R'$, R' a finite local R-algebra, hence Hensel's Lemma 6.1 applies with S replaced by Z too: e.g., to p_2. It follows:

$$\begin{aligned}
X_L \times_S Z &= W_1 \cup W_2 \qquad \text{(disjoint)} \\
W_1 \cap p_2^{-1}(z) &= \operatorname{Image} \overline{\sigma} \\
W_1 &= \operatorname{Spec} R'', \ R'' \text{ a finite local } R'\text{-algebra.}
\end{aligned}$$

But p_2 is flat so R'' is flat over R', hence free (since R' is local and R'' is a finite R'-module). By assumption

$$\begin{aligned}
(X_L)_o &= \operatorname{Spec} L, \\
\text{so} \quad p_2^{-1}(z) &= \operatorname{Spec}(L \otimes_k \Bbbk(z)).
\end{aligned}$$

Now L separable over k implies that $L \otimes_k \Bbbk(z)$ is a separable $\Bbbk(z)$-algebra — in particular it has no nilpotents. Thus:

$$p_2^{-1}(z) \cap W_1 \cong \operatorname{Spec} \Bbbk(z)$$

hence $R'' \otimes_{R'} \Bbbk(z) \cong \Bbbk(z)$ and $R'' \otimes_{R'} \Bbbk(z)$ has one generator. Therefore R'' is free over R' with one generator, i.e., $W_1 \cong Z$. This means that $\overline{\sigma}$ extends uniquely to a section σ of p_2:

$$\begin{tikzcd}
 & X_L \times_S Z \arrow[d, "p_2"'] \\
\operatorname{Spec} \Bbbk(z) \arrow[ur, "\overline{\sigma}"] \arrow[r, hook] & Z \arrow[u, bend right, "\sigma"']
\end{tikzcd}$$

and $f = p_1 \circ \sigma$ has the required properties. □

Corollary 6.4. *Let R be a complete discrete valuation ring, $S = \operatorname{Spec} R$, $\pi\colon X \to S$ a morphism of finite type with X reduced and irreducible. Then:*

$$X_\eta = \textit{one point} \Longrightarrow X_o = \textit{zero or one point.}$$

This corollary allows us to define a very important map, the *specialization map* (to be used in §V.3):

Definition 6.5. Let X be of finite type over R: Let

$$\begin{aligned}
\operatorname{Max}(X_\eta) &= \text{set of closed points of } X_\eta \\
\operatorname{Max}(X_o) &= \text{set of closed points of } X_o.
\end{aligned}$$

Let

$$\operatorname{Max}(X_\eta)^\circ = \text{set of } x \in \operatorname{Max}(X_\eta) \text{ such that } x \text{ is not closed in } X.$$

Let

$$\operatorname{sp}\colon \operatorname{Max}(X_\eta)^\circ \longrightarrow \operatorname{Max}(X_o)$$

be the map

$$x \longmapsto \overline{\{x\}} \cap X_o$$

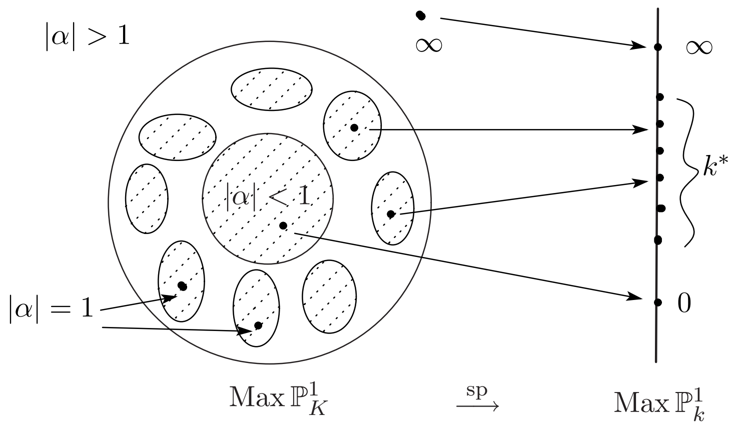

FIGURE IV.12. Specialization map for $\mathbb{P}^1_R$

(apply Corollary 6.4 to $\overline{\{x\}}$ with reduced structure; hence $\#(\overline{\{x\}} \cap X_o) = 0$ or 1). Note that if X is *proper* over S, then $\operatorname{Max}(X_\eta) = \operatorname{Max}(X_\eta)^\circ$ since $\pi(\overline{\{x\}})$ must be closed in S, hence $\overline{\{x\}} \cap X_o \neq \emptyset$.

The spaces $\operatorname{Max}(X_\eta)^\circ$ are the building blocks for the theory of "rigid analytic spaces" over K — cf. Tate [**101**].

EXAMPLE. $X = \mathbb{A}^1_R$. Then

$$\begin{aligned} \operatorname{Max}(\mathbb{A}^1_K) &= \text{set of conjugacy classes of algebraic elements over } K \\ \operatorname{Max}(\mathbb{A}^1_K)^\circ &= \text{those algebraic elements which are integral over } R \\ \operatorname{Max}(\mathbb{A}^1_k) &= \text{set of conjugacy classes of algebraic elements over } k \end{aligned}$$

and sp is the map:

if $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ is the irreducible equation for x, then $\operatorname{sp} x$ is a root of the equation $x^n + \overline{a}_1 x^{n-1} + \cdots + \overline{a}_n = 0$, $\overline{a}_i = (a_i \bmod M)$.

More succinctly, R defines an absolute value

$$x \longmapsto |x|$$

on K making X into a complete topological field, via

$$|u \cdot \pi^n| = c^{-n}, \qquad \begin{array}{l}(\text{some fixed } c \in \mathbb{R},\ c > 1 \\ \text{all } u \in R^*,\ \pi = \text{generator of } M).\end{array}$$

Then this absolute value extends to $\overline{K}$ and $\operatorname{Max}(\mathbb{A}^1_K)^\circ$ is the unit disc:

$$\{x \text{ up to conjugacy } \mid |x| \leq 1\}.$$

On the other hand, if $X = \mathbb{P}^1_R$, then $\operatorname{Max}(\mathbb{P}^1_K)$ consists in $\{\infty\}$ plus $\operatorname{Max}(\mathbb{A}^1_K)$. And now since $\mathbb{P}^1_R$ is proper over S, sp is defined on the whole set $\operatorname{Max}(\mathbb{P}^1_K)$. It extends the above sp on $\operatorname{Max}(\mathbb{A}^1_K)^\circ$, and carries ∞ as well as the whole set

$$\operatorname{Max}(\mathbb{A}^1_K) \setminus \operatorname{Max}(\mathbb{A}^1_K)^\circ = \{x \text{ up to conjugacy } \mid |x| > 1\}$$

to ∞ in $\operatorname{Max}(\mathbb{P}^1_k)$. It looks like Figure IV.12

PROPOSITION 6.6. *The map*

$$\operatorname{sp}\colon \operatorname{Max}(X_\eta)^\circ \longrightarrow \operatorname{Max}(X_o)$$

is surjective.

The proof goes by induction on $\dim X_o$. If $X_o = 0$, use Hensel's Lemma 6.1. If $x \in \mathrm{Max}(X_o)$ and $\dim X_o \geq 1$, choose $f \in \mathfrak{m}_{x,X}$ with $f \not\equiv 0$ on any component of X_o. Consider the subscheme $V(f)$ in a suitable neighborhood of x and apply Krull's principal ideal theorem (Zariski-Samuel [**109**, vol. I, Chapter IV, §14, Theorem 29, p. 238]). We leave the details to the reader.

CHAPTER V

Singular vs. non-singular

1. Regularity

The purpose of this section is to translate some well known commutative algebra into the language of schemes — as general references, see Zariski-Samuel [**109**, vol. I, Chapter IV and vol. II, Chapter VIII] and Atiyah-MacDonald [**19**, Chapter 11]. Consider:

a) $\mathcal{O}$ = local ring
b) $\mathfrak{m}$ = its maximal ideal
c) $k = \mathcal{O}/\mathfrak{m}$
d) $\mathfrak{m}/\mathfrak{m}^2$, a vector space over k
e) $\mathrm{gr}(\mathcal{O}) = \bigoplus_{n=1}^{\infty} \mathfrak{m}^n/\mathfrak{m}^{n+1}$, a graded k-algebra generated over k by $\mathfrak{m}/\mathfrak{m}^2$.

Lemma 1.1 (Easy lemma). *If* $\bigcap_{n=1}^{\infty} \mathfrak{m}^n = (0)$, *then* $\mathrm{gr}(\mathcal{O})$ *integral domain* $\Longrightarrow$ $\mathcal{O}$ *integral domain.*

Proof. If not, say $x, y \in \mathcal{O}$, $xy = 0$, $x \neq 0$, $y \neq 0$. Then $x \in \mathfrak{m}^l \setminus \mathfrak{m}^{l+1}$, $y \in \mathfrak{m}^{l'} \setminus \mathfrak{m}^{l'+1}$ for some l, l'; let $\overline{x} \in \mathfrak{m}^l/\mathfrak{m}^{l+1}$, $\overline{y} \in \mathfrak{m}^{l'}/\mathfrak{m}^{l'+1}$ be their images. Then $\overline{x} \cdot \overline{y} = 0$. □

f) Krull dim $\mathcal{O}$ = length n of the longest chain of prime ideals:

$$\mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_n = \mathfrak{m}$$

g) If $\mathcal{O}$ is noetherian, then recall that

$$\begin{aligned} \dim \mathcal{O} &= \text{least } n \text{ such that } \exists x_1, \dots, x_n \in \mathfrak{m},\ \ \mathfrak{m} = \sqrt{(x_1, \dots, x_n)} \\ \text{OR} &= \text{degree of Hilbert-Samuel polynomial } P \text{ defined by} \\ &\qquad P(n) = l(\mathcal{O}/\mathfrak{m}^n), \quad n \gg 0. \qquad (l \text{ denotes the length.}) \end{aligned}$$

Definition 1.2. Note that by (g)[1], $\dim_k \mathfrak{m}/\mathfrak{m}^2 \geq \dim \mathcal{O}$. Then $\mathcal{O}$ is regular if it is noetherian and equivalently,

$$\mathrm{gr}(\mathcal{O}) = \text{symmetric algebra generated by } \mathfrak{m}/\mathfrak{m}^2$$

OR

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim \mathcal{O}.$$

Note that

$$\mathcal{O} \text{ regular} \Longrightarrow \mathcal{O} \text{ integral domain}$$

by the Easy Lemma 1.1.

Definition 1.3. Let X be a scheme, $x \in X$. Then

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \underset{\text{def}}{=} \text{Zariski-cotangent space at } x, \text{ denoted } T^*_{x,X}$$

$$\mathrm{Hom}(\mathfrak{m}_x/\mathfrak{m}_x^2, \Bbbk(x)) \underset{\text{def}}{=} \text{Zariski-tangent space at } x, \text{ denoted } T_{x,X}.$$

[1]Since if $x_1, \dots, x_n \in \mathfrak{m}$ span $\mathfrak{m}/\mathfrak{m}^2$ over k, then by Nakayama's lemma, they generate $\mathfrak{m}$ as an ideal, hence $\dim \mathcal{O} \leq n$.

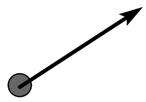

FIGURE V.1. Disembodied tangent vector

Note that we can embed $T_{x,X}$ as the set of $\Bbbk(x)$-rational points in an affine space over $\Bbbk(x)$:

$$\mathbb{T}_{x,X} = \operatorname{Spec}(\operatorname{Symm}^*(\mathfrak{m}_x/\mathfrak{m}_x^2)) \underset{\text{non-canonically}}{\cong} \mathbb{A}^n_{\Bbbk(x)}$$

if $n = \dim_{\Bbbk(x)} \mathfrak{m}_x/\mathfrak{m}_x^2$ and $\operatorname{Symm}^* =$ symmetric algebra.

In some cases, the tangent space at a point $x \in X$ has a pretty functorial definition: Suppose X is a scheme over a field k and x is a k-rational point. Then

$$T_{x,X} \cong \left\{ \begin{array}{c} \text{set of all morphisms } \tau \text{ such that} \\ \begin{tikzcd} \operatorname{Spec} k[\epsilon]/(\epsilon^2) \arrow[rr, "\tau"] \arrow[dr] & & X \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd} \\ \text{commutes and Image}\,\tau = \{x\} \end{array} \right\}.$$

In fact, by Proposition I.3.10, the set of such τ is isomorphic to the set of local k-algebra homomorphisms:

$$\tau^* \colon \mathcal{O}_{x,X} \longrightarrow k[\epsilon]/(\epsilon^2).$$

Then $\tau^*(\mathfrak{m}_{x,X}) \subset k \cdot \epsilon$ and $\tau^*(\mathfrak{m}_{x,X}^2) = (0)$. Since $\mathcal{O}_{x,X}$ is a local k-algebra with residue field k:

$$\mathcal{O}_{x,X}/\mathfrak{m}_{x,X}^2 \cong k \oplus \mathfrak{m}_{x,X}/\mathfrak{m}_{x,X}^2,$$

hence τ^* is given by a k-linear map

$$\operatorname{res} \tau^* \colon \mathfrak{m}_{x,X}/\mathfrak{m}_{x,X}^2 \longrightarrow k \cdot \epsilon$$

and any such map defines a τ^*. But the set of such maps is $T_{x,X}$. Because of this result, one often visualizes $\operatorname{Spec} k[\epsilon]/\epsilon^2$ as a sort of disembodied tangent vector as in Figure V.1.

Given a morphism $f \colon X \to Y$, let $x \in X$ and $y = f(x)$. Then f induces maps on the Zariski tangent and cotangent spaces:

i) $f^* \colon \mathcal{O}_{y,Y} \to \mathcal{O}_{x,X}$ induces a homomorphism of $\Bbbk(x)$-vector spaces:

$$df_x^* \colon \left(\mathfrak{m}_{y,Y}/\mathfrak{m}_{y,Y}^2\right) \otimes_{\Bbbk(y)} \Bbbk(x) \longrightarrow \mathfrak{m}_{x,X}/\mathfrak{m}_{x,X}^2$$

ii) Dualizing, this gives a morphism

$$df_x \colon T_{x,X} \longrightarrow T_{y,Y} \widehat{\otimes}_{\Bbbk(y)} \Bbbk(x)$$

(where $\widehat{\ }$ on $\otimes$ comes in only in case $\mathfrak{m}_{y,Y}/\mathfrak{m}_{y,Y}^2$ is infinite dimesional! — in which case $T_{y,Y}$ has a natural linear topology, and one must complete $T_{y,Y} \otimes_{\Bbbk(y)} \Bbbk(x)$, etc.)

DEFINITION 1.4. The tangent cone to X at x is $\mathbb{TC}_{x,X} = \operatorname{Spec}(\operatorname{gr}(\mathcal{O}_{x,X}))$. Since $\operatorname{gr}(\mathcal{O}_{x,X})$ is a quotient of the symmetric algebra $\operatorname{Symm}(\mathfrak{m}_x/\mathfrak{m}_x^2)$, we get a closed immersion:

$$\mathbb{TC}_{x,X} \subset \mathbb{T}_{x,X}.$$

DEFINITION 1.5. x is a regular point of X if $\mathcal{O}_{x,X}$ is a regular local ring, i.e., if $\mathbb{TC}_{x,X} = \mathbb{T}_{x,X}$. X is regular if it is locally noetherian and all its points are regular.

We will see in §4 below that a complex projective variety X is regular at a point x if and only if it is non-singular at x as defined in Part I [**76**, Chapter I]. Thus the concept of regularity can be viewed as a generalization to arbitrary schemes of the concept of non-singularity (but **n.b.** the remarks in §4 below on Sard's lemma and the examples). Many of the concepts introduced in Part I [**76**] for non-singular varieties go over to general regular schemes. For instance, a basic theorem in commutative algebra is that a regular local ring is a UFD (cf. Zariski-Samuel [**109**, vol. II, Appendix 7]; or Kaplansky [**58**, §4-2]). As we saw in §III.6, this means that we have a classical theory of divisors on a regular scheme, i.e.,

X regular $\Longrightarrow$

$$\left\{\begin{array}{l}\text{Group of Cartier divisors}\\ \mathrm{Div}(X)\text{ on }X\end{array}\right\}\cong\left\{\begin{array}{l}\text{Group of cycles formed from irreducible}\\ \text{codimension one closed subsets}\end{array}\right\}.$$

More generally, it is on a regular scheme X that there is a good intersection theory of cycles whatever their codimension. Recall that a closed irreducible subset $Z \subset X$ is said to have codimension r if the local ring $\mathcal{O}_{\eta_Z,X}$ at its generic point η_Z has Krull dimension r: hence if $z \in Z$ is any point, the prime ideal

$$\mathfrak{p}(Z) \subset \mathcal{O}_{z,X}$$

defining Z has *height* r (i.e., since, by definition, height$(\mathfrak{p}(Z)) =$ length of greatest chain of prime ideals:

$$(0) \subset \mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_h = \mathfrak{p}(Z),$$

which equals the Krull dimension of $(\mathcal{O}_{z,X})_{\mathfrak{p}(Z)} \cong \mathcal{O}_{\eta_Z,X}$). Then another basic theorem in commutative algebra is:

1.6.

Algebraic form	*If $\mathcal{O}$ is a regular local ring, $\mathfrak{p}_1, \mathfrak{p}_2 \subset \mathcal{O}$ are prime ideals, and $\mathfrak{p}'$ is a minimal prime ideal containing $\mathfrak{p}_1 + \mathfrak{p}_2$, then* $\mathrm{height}(\mathfrak{p}') \leq \mathrm{height}(\mathfrak{p}_1) + \mathrm{height}(\mathfrak{p}_2)$ *(Serre* [**89**, p. V-18]).

Geometrically, this means:

1.7.

Geometric form	*If X is a regular scheme, and $Z_1, Z_2 \subset X$ are irreducible closed subsets, then for every component W of $Z_1 \cap Z_2$:* $\mathrm{codim}\, W \leq \mathrm{codim}\, Z_1 + \mathrm{codim}\, Z_2$.

Moreover, when equality holds, there is a natural concept of the *intersection multiplicity* of Z_1 and Z_2 along W: see Serre [**89**, Chapter V]. This is defined using the functors Tor_i and allows one to define an associative, commutative, distributive product between cycles which intersect properly (i.e., with no components of too high dimension). (See also §VII.5.) There is, however, one big difficulty in this theory. One of the key methods used in Part I [**76**] in our discussion of intersections in the classical case of X over $\mathrm{Spec}\,\mathbb{C}$ is the "reduction to the diagonal Δ": instead of intersecting Z_1, Z_2 in X, we formed the intersection of $Z_1 \times_{\mathrm{Spec}\,\mathbb{C}} Z_2$ and Δ in $X \times_{\mathrm{Spec}\,\mathbb{C}} X$, and used the fact that Δ is a local complete intersection in $X \times_{\mathrm{Spec}\,\mathbb{C}} X$. This reduction works equally well for a regular variety X over any algebraically closed field k, and can be extended to all equicharacteristic X, but fails for regular schemes like $\mathbb{A}^n_{\mathbb{Z}}$ with mixed characteristic local rings (residue field of characteristic p, quotient field of characteristic 0). The problem is that the product

$$\mathbb{A}^{2n}_{\mathbb{Z}} = \mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \mathbb{A}^n_{\mathbb{Z}}$$

has dimension $2n+1$ which is less than $2(\dim \mathbb{A}^n_{\mathbb{Z}}) = 2n+2$: for instance, at the point $P \in \mathbb{A}^n_{\mathbb{Z}}$ where $X_1 = \cdots = X_n = 0$ over $[p] \in \operatorname{Spec}\mathbb{Z}$, the cotangent space to $\mathbb{A}^n_{\mathbb{Z}}$ has a basis

$$dX_1, \ldots, dX_n, dp.$$

And at the point $(P, P) \in \mathbb{A}^{2n}_{\mathbb{Z}}$, if we let X_i and Y_i be coordinates in the two factors,

$$dX_1, \ldots, dX_n, dY_1, \ldots, dY_n, dp$$

is a basis of the cotangent space. Thus it is not like a product in the arithmetic direction. One finds, e.g., that $Z_1, Z_2 \subset \mathbb{A}^n_{\mathbb{Z}}$ may intersect properly, while $Z_1 \times_{\operatorname{Spec}\mathbb{Z}} Z_2$, $\Delta \subset \mathbb{A}^{2n}_{\mathbb{Z}}$ don't; that Z_1, Z_2 may be regular while $Z_1 \times_{\operatorname{Spec}\mathbb{Z}} Z_2$ is not. Nonetheless, Serre managed to show that intersection theory works except for one property: it is still unknown whether the intersection multiplicity $i(Z_1, Z_2; W)$ is always positive!

For intersection theory on non-singular varieties of arbitrary characteristic, see Samuel [**83**]. A basic fact from commutative algebra that makes it work is the following:

Proposition 1.8. *Let R be a regular local ring of dimension r, with maximal ideal $\mathfrak{m}$, residue field k and quotient field K. Let M be a finitely generated R-module. Then there is a Hilbert-Samuel polynomial $P(t)$ of degree at most r such that*

$$P(n) = l(M/\mathfrak{m}^n M) \quad \text{if } n \gg 0. \qquad (l \text{ denotes the length.})$$

Let

$$P(t) = e\frac{t^r}{r!} + \text{lower terms.}$$

Then

$$e = \dim_K(M \otimes_R K).$$

Proof left to the reader.

2. Kähler differential

Again we begin with algebra: let B be an A-algebra:

2.1.

$$\begin{aligned} \Omega_{B/A} \underset{def}{=} & \text{ free } B\text{-module on elements } db, \text{ for all } b \in B, \\ & \text{modulo the relations:} \\ & \quad d(b_1 + b_2) = db_1 + db_2 \\ & \quad d(b_1 b_2) = b_1 \cdot db_2 + b_2 \cdot db_1, \\ & \quad d(a) = 0, \text{ for all } a \in A. \end{aligned}$$

In other words, the map

$$d\colon B \longrightarrow \Omega_{B/A}$$

is an A-derivation and $(\Omega_{B/A}, d)$ is universal — i.e., for all B-module M and all maps

$$D\colon B \longrightarrow M$$

such that

$$D(b_1 + b_2) = Db_1 + Db_2$$
$$D(b_1 b_2) = b_1 \cdot Db_2 + b_2 \cdot Db_1$$
$$Da = 0, \text{ all } a \in A,$$

there is a unique B-module homomrophism $\phi\colon \Omega_{B/A} \to M$ such that $D = \phi \circ d$.

PROPOSITION 2.2. *If*

$$I = \operatorname{Ker}(B \otimes_A B \ni b_1 \otimes b_2 \longmapsto b_1 b_2 \in B),$$

then I/I^2 is a $(B \otimes_A B)/I$-module, i.e., a B-module, and

$$\Omega_{B/A} \cong I/I^2 \qquad (\textit{as } B\textit{-module}).$$

d goes over to the map

$$\begin{aligned} B &\longrightarrow I \\ b &\longmapsto 1 \otimes b - b \otimes 1. \end{aligned}$$

PROOF. I) check that $b \mapsto b \otimes 1 - 1 \otimes b$ is an A-derivation from B to I/I^2. Therefore it extends to a B-module homomorphism $\Omega_{B/A} \to I/I^2$.

II) Define a ring $E = B \oplus \Omega_{B/A}$, where B acts on $\Omega_{B/A}$ through the module action and the product of two elements of $\Omega_{B/A}$ is always 0. Define an A-bilinear map $B \times B \to E$ by $(b_1, b_2) \mapsto (b_1 b_2, b_1 db_2)$. By the universal mapping property of $\otimes$, it factors

$$B \times B \longrightarrow B \otimes_A B \xrightarrow{\phi} E$$

and it follows immediately that $\phi(I) \subset \Omega_{B/A}$. Therefore $\phi(I^2) = (0)$ and ϕ gives $\overline{\phi}\colon I/I^2 \to \Omega_{B/A}$.

III) These maps are easily seen to be inverse to each other. $\square$

Some easy properties of Ω are:

2.3. *If B and C are A-algebras, then:*

$$\Omega_{(B \otimes_A C)/C} \cong \Omega_{B/A} \otimes_A C.$$

2.4. *If $\mathfrak{a} \subset B$ is an ideal then there is a natural map*

$$\begin{aligned} \mathfrak{a}/\mathfrak{a}^2 &\longrightarrow \Omega_{B/A} \otimes_B (B/\mathfrak{a}) \\ a &\longmapsto da \otimes 1 \end{aligned}$$

and the cokernel is isomorphic to $\Omega_{(B/\mathfrak{a})/A}$.

2.5. *If B is an A-algebra and C is a B-algebra, then there is a natural exact sequence*

$$\Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B} \longrightarrow 0.$$

EXAMPLE. 1: Let $A = k$, $B = k[X_1, \ldots, X_n]$. Then $\Omega_{B/A}$ is a free B-module with generators $dX_1, \ldots, dX_n$, and

$$dg = \sum_{i=1}^{n} \frac{\partial g}{\partial X_i} \cdot dX_i, \quad \text{all } g \in B.$$

More generally, if

$$B = k[X_1, \ldots, X_n]/(f_1, \ldots, f_m),$$

then $\Omega_{B/A}$ is generated, as B-module, by $dX_1, \ldots, dX_n$, but with relations:

$$df_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial X_j} \cdot dX_j = 0.$$

EXAMPLE. 2: What happens when A and B are fields, i.e., $\Omega_{K/k} = ?$. The dual K-vector space $\mathrm{Hom}_K(\Omega_{K/k}, K)$ is precisely the vector space $\mathrm{Der}_k(K,K)$ of k-derivations from K to K. Then it is well known:

a) $\mathrm{Der}_k(K,K) = (0) \Longleftrightarrow K/k$ is separable algebraic.
b) If $\{f_\alpha\}_{\alpha \in S}$ is a transcendence basis of K over k and K is separable over $k(\ldots, f_\alpha, \ldots)$, then a k-derivation D can have any values on the f_α and is determined by its values on the f_α's.
c) If characteristic $k = p$, then any k-derivation D kills $k \cdot K^p$. If $p^s = [K : k \cdot K^p]$ and we write $K = kK^p(b_1^{1/p}, \ldots, b_s^{1/p})$, $(b_i \in k \cdot K^p)$, and $a_i = b_i^{1/p}$, then a k-derivation D can have any values on the a_i and is determined by its values on the a_i's.

We conclude:

a′) $\Omega_{K/k} = (0) \Longleftrightarrow K/k$ is separable algebraic.
(More generally, if R is a finitely generated k-algebra, then it is not hard to show that

$$\Omega_{R/k} = (0) \Longleftrightarrow R \text{ is a direct sum of separable algebraic field extensions.)}$$

b′) If K is finitely generated and separable over k, then $\forall f_1, \ldots, f_n \in K$,

$$\begin{bmatrix} df_1, \ldots, df_n \text{ are} \\ \text{a basis of } \Omega_{K/k} \end{bmatrix} \Longleftrightarrow \begin{bmatrix} f_1, \ldots, f_n \text{ are a separating transcendence} \\ \text{basis of } K \text{ over } k \end{bmatrix}.$$

c′) If K is finitely generated over k and $\mathrm{char}(K) = p$ and $p^s = [K : k{\cdot}K^p]$, then $\forall f_1, \ldots, f_s \in K$,

$$\begin{bmatrix} f_1, \ldots, f_s \text{ are a } p\text{-basis of } K \text{ over } k, \\ \text{i.e., } K = k \cdot K^p(f_1, \ldots, f_s) \end{bmatrix} \Longleftrightarrow \begin{bmatrix} df_1, \ldots, df_s \text{ are a} \\ \text{basis of } \Omega_{K/k} \end{bmatrix}.$$

It follows easily too that if $f_1, \ldots, f_s$ are a p-basis then $\mathrm{Der}_{k(f_1,\ldots,f_s)}(K,K) = (0)$, hence K is separable algebraic over $k(f_1, \ldots, f_s)$. Thus

$$s \geq \mathrm{tr.deg}_k K$$

with equality if and only if K is separable over k.

For details here, cf. for example, Zariski-Samuel [**109**, vol. I, Chapter 2, §17].

EXAMPLE. 3: Let $A = k$, $B = k[X,Y]/(XY)$. Then by Example 1, dX and dY generate $\Omega_{B/A}$ with the one relation $XdY + YdX = 0$.

Consider the element $\omega = XdY = -YdX$. Then $X\omega = Y\omega = 0$, so the submodule M generated by ω is $k\omega$, a one-dimensional k-space. On the other hand, in Ω/M we have $XdY = YdX = 0$, so $\Omega/M \cong B \cdot dX \oplus B \cdot dY$. Note that $B \cdot dX \cong \Omega_{B_X/k}$, where $B_X = B/(Y) \cong k[X]$; likewise, $B \cdot dY \cong \Omega_{B_Y/k}$. That is, the module of differentials on $\mathrm{Spec}\, B$ (which looks like that in Figure V.2) is the module of differentials on the horizontal and vertical lines separately extended by a torsion module. (One can even check that the extension is non-trivial, i.e., does not split.)

All this is easy to globalize. Let $f \colon X \to Y$ be any morphism. The closed immersion

$$\Delta \colon X \longrightarrow X \times_Y X$$

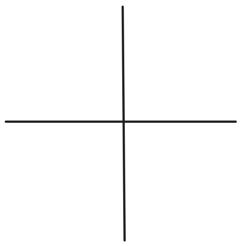

FIGURE V.2. Crossing lines

"globalizes" the multiplication homomorphism $\delta\colon B \otimes_A B \to B$. Let $\mathcal{I}$ be the quasi-coherent $\mathcal{O}_{X\times_Y X}$-ideal defining the closed subscheme $\Delta(X)$. Then $\mathcal{I}^2$ is also a quasi-coherent $\mathcal{O}_{X\times_Y X}$-ideal and $\mathcal{I}/\mathcal{I}^2$ is a quasi-coherent $\mathcal{O}_{X\times_Y X}$-module. It is also a module over $\mathcal{O}_{X\times_Y X}/\mathcal{I}$, which is $\mathcal{O}_{\Delta(X)}$ extended by zero. As all its stalks off $\Delta(X)$ are 0, $\mathcal{I}/\mathcal{I}^2$ is actually a sheaf of $(\Delta(X), \mathcal{O}_{\Delta(X)})$-modules, quasi-coherent in virtue of the nearly tautologous:

LEMMA 2.6. *If $S \subset T$ are a scheme and a closed subscheme, and if $\mathcal{F}$ is an $\mathcal{O}_S$-module, then $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_S$-module on S if and only if $\mathcal{F}$, extended by (0) on $T \setminus S$, is a quasi-coherent $\mathcal{O}_T$-module on T.*

DEFINITION 2.7. $\Omega_{X/Y}$ is the quasi-coherent $\mathcal{O}_X$-module obtained by carrying $\mathcal{I}/\mathcal{I}^2$ back to X by the isomorphism $\Delta\colon X \xrightarrow{\sim} \Delta(X)$.

Clearly, for all $U = \operatorname{Spec}(B) \subset X$ and $V = \operatorname{Spec}(A) \subset Y$ such that $f(U) \subset V$, the restriction of $\Omega_{X/Y}$ to U is just $\widetilde{\Omega_{B/A}}$. Therefore we have globalized our affine construction.

The following properties are easy to check:

2.8. *The stalks of $\Omega_{X/Y}$ are given by:*

$$(\Omega_{X/Y})_x = (\Omega_{\mathcal{O}_{x,X}/\mathcal{O}_{y,Y}})^{\sim} \qquad (\textit{if } y = f(x)).$$

2.9.

$$\Omega_{(X\times_S Y)/Y} \cong \Omega_{X/S} \otimes_{\mathcal{O}_S} \mathcal{O}_Y.$$

2.10. *If $Z \subset X$ is a closed subscheme defined by the sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, then $\exists$ a canonical map:*

$$(*) \qquad \begin{aligned} \mathcal{I}/\mathcal{I}^2 &\longrightarrow \Omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{O}_Z \\ a &\longmapsto da \otimes 1 \end{aligned}$$

and the cokernel is isomorphic to $\Omega_{Z/Y}$.

2.11. *If X is of finite type over Y, then $\Omega_{X/Y}$ is finitely generated.*

2.12. *If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then*

$$\mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{F}) \cong \{\textit{sheaf of derivations from } \mathcal{O}_X \textit{ to } \mathcal{F} \textit{ over } \mathcal{O}_Y\}.$$

(2.10) allows us to compare the Zariski-cotangent space at $x \in X$ and $\Omega_{X/Y}$. In fact, if you let $Z = \overline{\{x\}}$ with reduced structure, and look at the stalks of $(*)$ at x, you get the $\Bbbk(x)$-linear homomorphism:

$$\begin{aligned} \mathfrak{m}_x/\mathfrak{m}_x^2 &\longrightarrow (\Omega_{X/Y})_x \otimes_{\mathcal{O}_{x,X}} \Bbbk(x) \\ a &\longmapsto da \otimes 1 \end{aligned}$$

and the cokernel is

$$(\Omega_{Z/Y})_x \cong \Omega_{\Bbbk(x)/\mathcal{O}_{y,Y}} \cong \Omega_{\Bbbk(x)/\Bbbk(y)}.$$

Moreover $\mathfrak{m}_y \cdot \mathcal{O}_x$ is in the kernel since $da = 0$, $\forall a \in \mathcal{O}_y$. Now in reasonably geometric cases such as when X and Y are of finite type over an algebraically closed k, and x and y are closed points, then $\Bbbk(x) = \Bbbk(y) = k$, so $\Omega_{\Bbbk(x)/\Bbbk(y)} = (0)$; and it turns out that the induced map

$$T^*_{x,f^{-1}(y)} \cong \mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_y \cdot \mathcal{O}_x) \longrightarrow (\Omega_{X/Y})_x \otimes_{\mathcal{O}_x} \Bbbk(x)$$

is injective too, i.e., the quasi-coherent sheaf $\Omega_{X/Y}$ essentially results from glueing together the separate vector spaces $\mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_y \cdot \mathcal{O}_x)$ — which are nothing but the cotangent spaces to the fibres $f^{-1}(y)$ at various points x.

To prove this and see what happens in nasty cases, first define:

Definition 2.13 (Grothendieck). If $K \supset k$ are two fields, let

$$\Upsilon_{K/k} = \operatorname{Ker}(\Omega_{k/\mathbb{Z}} \otimes_k K \longrightarrow \Omega_{K/\mathbb{Z}})$$

called the module of imperfection.

This is a K-vector space and its dual is

$$\{\text{space of derivations } D\colon k \to K\}/\{\text{restrictions of derivations } D\colon K \to K\}$$

which is well known to be 0 iff K is separable over k (cf. Zariski-Samuel [**109**, vol. I, Chapter II, §17, Theorem 42, p. 128]).

Theorem 2.14. *For all $f\colon X \to Y$ and all $x \in X$, if $y = f(x)$, there is a canonical 4-term exact sequence:*

$$\Upsilon_{\Bbbk(x)/\Bbbk(y)} \longrightarrow T^*_{x,f^{-1}(y)} \longrightarrow \Omega_{X/Y} \otimes_{\mathcal{O}_x} \Bbbk(x) \longrightarrow \Omega_{\Bbbk(x)/\Bbbk(y)} \longrightarrow 0.$$

Proof. By (2.9), none of the terms change if we make a base change:

$$\begin{tikzcd} X \arrow[d, "f"'] & f^{-1}(y) \arrow[l] \arrow[d] \\ Y & \operatorname{Spec} \Bbbk(y). \arrow[l] \end{tikzcd}$$

Therefore we may assume $Y = \operatorname{Spec} k$, $k = \Bbbk(y)$ a field. But now $(\Omega_{X/Y})_x = \Omega_{\mathcal{O}_{x,X}/k}$ and note that if $R = \mathcal{O}_{x,X}/\mathfrak{m}_x^2$

$$\Omega_{\mathcal{O}_{x,X}/k} \otimes \Bbbk(x) \cong \Omega_{R/k} \otimes \Bbbk(x)$$

(by (2.4) applied with $\mathfrak{a} = \mathfrak{m}_x^2$). We are reduced to the really elementary:

Lemma. *Let R be a local k-algebra, with maximal ideal M, residue field $K = R/M$. Assume $M^2 = (0)$. There is a canonical exact sequence:*

$$\Upsilon_{K/k} \longrightarrow M \longrightarrow \Omega_{R/M} \otimes_R K \longrightarrow \Omega_{K/k} \longrightarrow 0.$$

Proof of Lemma. By (2.4) we have an exact sequence:

$$M \xrightarrow{\alpha} \Omega_{R/\mathbb{Z}} \otimes_R K \longrightarrow \Omega_{K/\mathbb{Z}} \longrightarrow 0.$$

Now by Cohen's structure theorem (Zariski-Samuel [**109**, vol. II, Chapter VIII, §12, Theorem 27, p. 304]), as a ring (but not necessarily as k-algebra), $R \cong K \oplus M$. Using such a direct sum decomposition, it follows that the projection of R onto M is a derivation of R into the $K = R/M$-module M, hence it factors:

$$\begin{tikzcd} R \arrow[rr, "\text{projection}"] \arrow[dr, "d"'] & & M \\ & \Omega_{R/\mathbb{Z}} \otimes_R K \arrow[ur, "\beta"'] & \end{tikzcd}$$

It is easy to see that $\beta \circ \alpha = 1_M$ and this proves that α is injective! Now the homomorphism $k \to R$ gives rise to an exact sequence $\Omega_{k/\mathbb{Z}} \otimes_k R \to \Omega_{R/\mathbb{Z}} \to \Omega_{R/k} \to 0$, hence to:

$$\begin{tikzcd} & 0 \arrow[d] & & \\ & M \arrow[d] & & \\ \Omega_{k/\mathbb{Z}} \otimes_k K \arrow[r] & \Omega_{R/\mathbb{Z}} \otimes_R K \arrow[r] \arrow[d] & \Omega_{R/K} \otimes_k K \arrow[r] & 0 \\ & \Omega_{K/\mathbb{Z}} \arrow[d] & & \\ & 0 & & \end{tikzcd}$$

It follows from this diagram that there is a natural map from $\mathrm{Ker}(\Omega_{k/\mathbb{Z}} \otimes_k K \to \Omega_{K/\mathbb{Z}})$, i.e., $\Upsilon_{K/k}$, to M and that the image is $\mathrm{Ker}(M \to \Omega_{R/K} \otimes_k K)$. This plus (2.4) proves the lemma. □

□

Corollary 2.15. *If* $\Bbbk(x)$ *is separable algebraic over* $\Bbbk(y)$*, then*

$$\mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_y \cdot \mathcal{O}_x) \longrightarrow \Omega_{X/Y} \otimes_{\mathcal{O}_x} \Bbbk(x)$$

is an isomorphism.

Example. 4: A typical case where inseparability enters is:

$$\begin{array}{ll} Y = \operatorname{Spec} k, & k \text{ imperfect and } a \in k \setminus k^p \\ X = \mathbb{A}^1_k, & x = \text{point corresponding to prime ideal } (t^p - a) \subset k[t] \\ & \text{i.e., } x = \text{point with coordinate } a^{1/p}. \end{array}$$

Then

$$\begin{aligned} \Bbbk(x) &= k(a^{1/p}) \\ \mathfrak{m}_x/\mathfrak{m}_x^2 &= (\text{free rank one } \Bbbk(x)\text{-module generated by } t^p - a) \\ \Omega_{X/Y} \otimes_{\mathcal{O}_x} \Bbbk(x) &= (\text{free rank one } \Bbbk(x)\text{-module generated by } dt) \end{aligned}$$

and the map works out:

$$\begin{aligned} \mathfrak{m}_x/\mathfrak{m}_x^2 &\longrightarrow \in \Omega_{X/Y} \otimes_{\mathcal{O}_x} \Bbbk(x) \\ t^p - a &\longmapsto \tfrac{d}{dt}(t^p - a) \cdot dt = 0 \end{aligned}$$

hence is 0.

An interesting example of the global construction of Ω is given by the projective bundles introduced in Chapter III:

Example. 5: Let S be a scheme and let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_S$-modules. Recall that we constructed $\pi\colon \mathbb{P}_S(\mathcal{E}) \to S$ by $\mathbb{P}_S(\mathcal{E}) = \mathcal{P}roj_S(\mathrm{Symm}^{\cdot}\mathcal{E})$. Let $\mathcal{K}$ be the kernel of the canonical homomorphism α:

$$0 \longrightarrow \mathcal{K} \longrightarrow \pi^*\mathcal{E} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \longrightarrow 0.$$

Then I claim:

2.16.
$$\Omega_{\mathbb{P}(\mathcal{E})/S} \cong \mathcal{K}(-1) = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1), \mathcal{K}).$$

We will prove this locally when $S = \operatorname{Spec} R$ is affine and $\mathcal{E}$ is free, leaving to the reader to check that the isomorphism is independent of the choice of basis hence globalizes. Assume

$$\mathcal{E} = \bigoplus_{i=0}^{n} \mathcal{O}_S \cdot t_i.$$

Let

$$\begin{aligned} U_i &= \text{open subset } \mathbb{P}_S(\mathcal{E})_{t_i} \\ &\cong \operatorname{Spec} R\left[\frac{t_0}{t_i}, \dots, \frac{t_n}{t_i}\right]. \end{aligned}$$

To avoid confusion, introduce an alias e_i for t_i in

$$\pi^*\mathcal{E} = \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}(\mathcal{E})} \cdot e_i$$

leaving the t_i to denote the induced global sections of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Then

$$\alpha(e_i) = t_i, \qquad 0 \le i \le n$$

and $\operatorname{Ker}(\alpha)$ has a basis on U_j:

$$e_i - \frac{t_i}{t_j} e_j, \qquad 0 \le i \le n,\ i \ne j.$$

Therefore $\mathcal{K}(-1)$ has a basis on U_j:

$$\frac{t_j \otimes e_i - t_i \otimes e_j}{t_j^2}, \qquad 0 \le i \le n,\ i \ne j.$$

On the other hand

$$\Omega_{\mathbb{P}(\mathcal{E})/S}|_{U_j} = \bigoplus_{i=0}^{n} \mathcal{O}_{U_j} \cdot d\left(\frac{t_i}{t_j}\right).$$

Define $\beta\colon \Omega_{\mathbb{P}(\mathcal{E})/S}|_{U_j} \to \mathcal{K}(-1)|_{U_j}$ by

$$\beta\left(d\left(\frac{t_i}{t_j}\right)\right) = \frac{t_j \otimes e_i - t_i \otimes e_j}{t_j^2}.$$

Heuristically, if we expand

$$d\left(\frac{t_i}{t_j}\right) = \frac{t_j dt_i - t_i dt_j}{t_j^2}$$

then β is given by the simple formula

$$\beta(dt_i) = e_i$$

which makes it clear why the definition of β is independent of the choice of basis.

Remark. $\boxed{\text{Added}}$ (cf. Example I.8.9) For a locally free $\mathcal{O}_S$-module $\mathcal{E}$ and a positive integer r, let $\pi\colon \operatorname{Grass}^r(\mathcal{E}) \to S$ be the Grassmannian scheme over S, whose Z-valued points for each S-scheme Z are in one-to-one correspondence with the $\mathcal{O}_Z$-locally free quotients $\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{E} \to \mathcal{G} \to 0$ of rank r. Let $\alpha\colon \pi^*\mathcal{E} \to \mathcal{Q} \to 0$ be the universal quotient with $\mathcal{Q}$ a locally free $\mathcal{O}_{\operatorname{Grass}^r(\mathcal{E})}$-module of rank r. Let $\mathcal{K} = \operatorname{Ker}(\alpha)$ so that we have an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \pi^*\mathcal{E} \xrightarrow{\alpha} \mathcal{Q} \longrightarrow 0.$$

Then generalizing the case $r = 1$ in (2.16) above, we have

$$\Omega_{\mathrm{Grass}^r(\mathcal{E})/S} = \mathcal{H}om_{\mathcal{O}_{\mathrm{Grass}^r(\mathcal{E})/S}}(\mathcal{Q}, \mathcal{K}).$$

Exercise. Added

- For simplicity, let $S = \mathrm{Spec}(k)$ with a field k. For a finite dimensional k-vector space E, consider the Grassmannian scheme $\mathrm{Grass}^r(E)$ over k. Let

  $$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_{\mathrm{Grass}^r(E)} \otimes_k E \xrightarrow{\alpha} \mathcal{Q} \longrightarrow 0$$

  be the universal exact sequence on $\mathrm{Grass}^r(E)$. A k-rational point $x \in \mathrm{Grass}^r(E)$ corresponds to an exact sequence of k-vector spaces

  $$0 \longrightarrow \mathcal{K}(x) \longrightarrow E \longrightarrow \mathcal{Q}(x) \longrightarrow 0,$$

  where $\mathcal{K}(x)$ and $\mathcal{Q}(x)$ are the fibres at x of $\mathcal{K}$ and $\mathcal{Q}$, respectively. Using the description of the tangent space in terms of $k[\epsilon]/(\epsilon^2)$ in §1, show

  $$T_{x,\mathrm{Grass}^r(E)} = \mathrm{Hom}_k(\mathcal{K}(x), \mathcal{Q}(x)),$$

  hence

  $$T^*_{x,\mathrm{Grass}^r(E)} = \mathrm{Hom}_k(\mathcal{Q}(x), \mathcal{K}(x)).$$

3. Smooth morphisms

DEFINITION 3.1. First of all, the canonical morphism:

$$\begin{array}{c} X = \mathrm{Spec}\, R[X_1, \ldots, X_{n+r}]/(f_1, \ldots, f_r) \\ \Big\downarrow f \\ Y = \mathrm{Spec}\, R \end{array}$$

is called smooth of relative dimension n at a point $x \in X$ whenever the Jacobian matrix evaluated at x:

$$\left(\frac{\partial f_i}{\partial X_j}(x)\right)_{\substack{1 \le i \le r \\ 1 \le j \le n+r}}$$

has maximal rank, i.e., r. Secondly, an arbitrary morphism $f\colon X \to Y$ is smooth of relative dimension n at a point $x \in X$ if there exist affine open neighborhoods $U \subset X$, $V \subset Y$ of x and y such that $f(U) \subset V$ and $\exists$ a diagram:

$$\begin{tikzcd} U \arrow[r, hook, "\text{open immersion}"'] \arrow[d, "\mathrm{res}\, f"'] & \mathrm{Spec}\, R[X_1, \ldots, X_{n+r}]/(f_1, \ldots, f_r) \arrow[d, "g"] \\ V \arrow[r, hook, "\text{open immersion}"'] & \mathrm{Spec}\, R \end{tikzcd}$$

with g of above type, i.e., $\mathrm{rk}((\partial f_i/\partial X_j)(x)) = r$. f is smooth of relative dimension n if this holds for all $x \in X$. f is étale if it is smooth of relative dimension 0.

This very concrete definition has lots of easy consequences:

PROPOSITION 3.2. *If $f\colon X \to Y$ is smooth at $x \in X$, then it is smooth in a neighborhood U of x.*

PROOF. If in some affine $U \subset X$ where f is presented as above, δ is the $r \times r$-minor of $(\partial f_i/\partial X_j)$ which is non-zero at x, then f is smooth in the distinguished open subset U_δ of U. $\square$

Proposition 3.3. *If $f\colon X \to Y$ is smooth of relative dimension n, then for all $Y' \to Y$, the canonical morphism*

$$X \times_Y Y' \longrightarrow Y'$$

is smooth of relative dimension n. In particular,

i) *for all $y \in Y$ the fibre $f^{-1}(y)$ is smooth of relative dimension n over $\Bbbk(y)$,*
ii) *if $Y = \operatorname{Spec} k$, $Y' = \operatorname{Spec} \overline{k}$, $\overline{k}$ an algebraic closure of k, then*

$$X \text{ smooth over } k \Longrightarrow \overline{X} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k} \text{ smooth over } \overline{k}.$$

Proof. Obvious. $\square$

Proposition 3.4. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are smooth morphisms at $x \in X$ and $y = f(x) \in Y$ respectively, then $g \circ f\colon X \to Z$ is smooth at x.*

Proof. Obvious. $\square$

Proposition 3.5. *A morphism $f\colon X \to Y$ is smooth of relative dimension n at x if and only if it factors in a neighborhood U of x:*

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, phantom, "\cap"] & Y \times \mathbb{A}^n \arrow[r, "p_1"] & Y \\ X \arrow[urr, "f"'] & & \end{tikzcd}$$

where g is étale.

Proof. "if" follows from the last result. As for "only if", it suffices to consider the case $X = \operatorname{Spec} R[X_1, \dots, X_{n+r}]/(f_1, \dots, f_r)$, $Y = \operatorname{Spec} R$. Say $\det\left((\partial f_i/\partial X_{n+j})\right)_{1 \le i,j \le r} \neq 0$. Let the homomorphism

$$R[X_1, \dots, X_n] \longrightarrow R[X_1, \dots, X_{n+r}]/(f_1, \dots, f_r)$$

define g. Then g is étale near x and $f = p_1 \circ g$. $\square$

Proposition 3.6. *If $f\colon X \to Y$ is smooth of relative dimension n at $x \in X$, then $\exists$ a neighborhood U of x such that $\Omega_{X/Y}|_U = \mathcal{O}_X^n|_U$. Especially, if f is étale, then $\Omega_{X/Y}|_U = (0)$.*

Proof. It suffices to show that if $S = R[X_1, \dots, X_{n+r}]/(f_1, \dots, f_r)$ and $\delta = \det(\partial f_i/\partial X_j)_{1 \le i,j \le r}$, then $(\Omega_{S/R}) \otimes_S S_\delta$ is a free S_δ-module of rank n. But $\Omega_{S/R}$ is generated over S by $dX_1, \dots, dX_{n+r}$ with relations $\sum_{j=1}^{n+r} (\partial f_i/\partial X_j) dX_j = 0$, $1 \le i \le r$. Writing these relations

$$\sum_{j=1}^{r} \frac{\partial f_i}{\partial X_j} dX_j = -\sum_{j=r+1}^{n+r} \frac{\partial f_i}{\partial X_j} dX_j$$

and letting $(\xi_{ij})_{1 \le i,j \le r} \in M_r(S_\delta)$ be the inverse of the matrix $(\partial f_i/\partial X_j)_{1 \le i,j \le r}$, it follows that in $(\Omega_{S/R}) \otimes_S S_\delta$,

$$dX_l = -\sum_{i=1}^{r} \sum_{j=r+1}^{r+n} \xi_{li} \cdot \frac{\partial f_i}{\partial X_j} \cdot dX_j, \qquad 1 \le l \le r$$

and that these are the only relations among the dX_i's. Therefore $dX_{r+1}, \dots, dX_{r+n}$ are a free basis of $(\Omega_{S/R}) \otimes_S S_\delta$. $\square$

Definition 3.7. If $f\colon X \to Y$ is smooth, let $\Theta_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X)$, called the relative tangent sheaf of X over Y. Note that it is locally free and if $x \in X$, $y = f(x)$ and $\Bbbk(x)$ is separable algebraic over $\Bbbk(y)$, then

$$(\Theta_{X/Y})_x \otimes \Bbbk(x) \cong T_{x, f^{-1}(y)}, \quad \text{the Zariski tangent space to the fibre.}$$

Moreover, by (2.12), $\Theta_{X/Y}$ is isomorphic to the sheaf $\mathcal{D}er_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ of derivations from $\mathcal{O}_X$ to itself killing $\mathcal{O}_Y$.

Note moreover that according to the proof of Proposition 3.6, X can be covered by affine open sets U in which there are functions $X_1, \ldots, X_n$ such that:

1) any differential $\omega \in \Omega_{X/Y}(U)$ can be uniquely expanded

$$\omega = \sum_{i=1}^{n} a_i \cdot dX_i, \qquad a_i \in \mathcal{O}_X(U),$$

2) any derivation $D \in \Theta_{X/Y}(U)$ can be uniquely expanded

$$D = \sum_{i=1}^{n} a_i \cdot \frac{\partial}{\partial X_i}, \qquad a_i \in \mathcal{O}_X(U)$$

($\partial/\partial X_i$ dual to dX_i).

When $Y = \operatorname{Spec}\mathbb{C}$, it is easy at this point to identify the sheaves $\Omega_{X/\mathbb{C}}$ and $\Theta_{X/\mathbb{C}}$ with the sheaves of holomorphic differential forms and holomorphic vector fields on X with "polynomial coefficients"; or alternatively, with the sheaves of polynomial sections of the cotangent vector bundle and tangent vector bundle to X. We will discuss this in §VIII.3.

I would like to examine next the relationship between the local rings $\mathcal{O}_{x,X}$ and $\mathcal{O}_{y,Y}$ when there is smooth morphism $f\colon X \to Y$ with $f(x) = y$. When there is no residue field extension, the completions of these rings are related in the simplest possible way:

PROPOSITION 3.8. *If $f\colon X \to Y$ is smooth of relative dimension n at x and if the natural map:*

$$\Bbbk(y) \xrightarrow{\approx} \Bbbk(x), \qquad \textit{where } y = f(x)$$

is an isomorphism, then the formal completions are related by:

$$\widehat{\mathcal{O}}_{x,X} \cong \widehat{\mathcal{O}}_{y,Y}[[t_1, \ldots, t_n]].$$

PROOF. The problem being local, we may assume

$$\begin{aligned} X &= \operatorname{Spec} R[X_1, \ldots, X_{n+r}]/(f_1, \ldots, f_r) \\ Y &= \operatorname{Spec} R, \quad R \text{ local ring, } y = \text{closed point of } Y, \\ &\text{with } \det\left(\frac{\partial f_i}{\partial X_j}(x)\right)_{1\le i,j\le r} \neq 0. \end{aligned}$$

Now if $x = [\mathfrak{p}]$, $\mathfrak{p} \subset R[X_1, \ldots, X_{n+r}]$, then we have inclusions:

$$\Bbbk(y) = R/(R \cap \mathfrak{p}) \subset R[X_1, \ldots, X_{n+r}]/\mathfrak{p} \subset \Bbbk(x).$$

Since all these are equal, $\exists a_1, \ldots, a_{n+r} \in R$ such that $X_i - a_i \in \mathfrak{p}$; more succinctly, x is the point over $y \in Y$ where $X_1 = a_i, \ldots, X_{n+r} = a_{n+r}$. Then $\mathfrak{p} \supset (\mathfrak{p} \cap R + (X_1 - a_1, \ldots, X_{n+r} - a_{n+r}))$ and in fact equality must hold because the ideal on the right is already maximal. Now we may as well change coordinates replacing $X_i - a_i$ by X_i so that x is at the origin, i.e., $\mathfrak{p} = \mathfrak{p} \cap R + (X_1, \ldots, X_{n+r})$. Now if $Z = Y \times \mathbb{A}^{n+r}$, we have

$$\begin{aligned} \mathcal{O}_{x,X} &\cong \mathcal{O}_{x,Z}/(f_1, \ldots, f_r), \\ \mathcal{O}_{x,Z} &= \text{localization of } \mathcal{O}_{y,Y}[X_1, \ldots, X_{n+r}] \text{ at the} \\ &\qquad \text{maximal ideal } \mathfrak{m}_{y,Y} + (X_1, \ldots, X_{n+r}), \end{aligned}$$

hence

$$\widehat{\mathcal{O}}_{x,X} \cong \widehat{\mathcal{O}}_{x,Z}/(f_1,\dots,f_r),$$
$$\widehat{\mathcal{O}}_{x,Z} \cong \widehat{\mathcal{O}}_{y,Y}[[X_1,\dots,X_{n+r}]].$$

Using the hypothesis that f is smooth at x, everything now follows (with $R = \widehat{\mathcal{O}}_{y,Y}[[X_1,\dots,X_n]]$, $Y_i = X_{n+i}$) from:

THEOREM 3.9 (Formal Implicit Function Theorem). *Let R be a ring complete in the $\mathfrak{a}$-adic topology for some ideal $\mathfrak{a} \subset R$. Suppose $f_1,\dots,f_r \in R[[Y_1,\dots,Y_r]]$ satisfy*

a) $f_i(0) \in \mathfrak{a}$
b) $\det(\partial f_i/\partial Y_j)(0) \in R^*$.

Then there are unique elements $g_i \in \mathfrak{a}$, $1 \le i \le r$, such that

a) *$Y_i - g_i \in$ ideal generated by $f_1,\dots,f_r$ in $R[[Y]]$*
b) *$f_i(g_1,\dots,g_r) = 0$, $1 \le i \le r$;*

equivalently, (a) *and* (b) *say that the following maps are well-defined isomorphisms inverse to each other:*

$$R \underset{\substack{\textit{substitution}\\ h(Y)\mapsto h(g)}}{\overset{\textit{inclusion}}{\rightleftarrows}} R[[Y_1,\dots,Y_r]]/(f_1,\dots,f_r).$$

PROOF OF THEOREM 3.9. The matrix $(\partial f_i/\partial Y_j)(0)$ is invertible in $M_r(R)$, so changing coordinates by its inverse, we may assume

$$f_i = a_i + Y_i + (\text{terms of degree } \ge 2 \text{ in } Y\text{'s}).$$

Then making induction on r, it is enough to show $\exists g(Y_1,\dots,Y_{r-1})$ so that:

$$R[[Y_1,\dots,Y_{r-1}]] \underset{\substack{\text{substitution}\\ \text{of } g \text{ for } Y_r}}{\overset{\text{canonical map}}{\rightleftarrows}} R[[Y_1,\dots,Y_r]]/(f_r)$$

are well-defined inverse isomorphisms. Letting $R' = R[[Y_1,\dots,Y_{r-1}]]$, $\mathfrak{a}' = \mathfrak{a}\cdot R + (Y_1,\dots,Y_{r-1})$, we reduce the proof to the case $r = 1$! We then have merely the linear case of the Weierstrass Preparation Theorem: $f(0) \in \mathfrak{a}$, $f'(0) = 1$, then $\exists$ a unit $u \in R[[Y]]$ and $a \in \mathfrak{a}$ such that $f(Y) = u(Y)\cdot(Y-a)$. This is proven easily by successive approximations:

$$a_1 = 0$$
$$a_{n+1} = a_n - f(a_n)$$
$$a = \lim_{n\to\infty} a_n.$$

One checks by induction that $f(a_n) \in \mathfrak{a}^n$, hence $f(a) = 0$. Making the substitution $Z = Y - a$, $g(Z) = f(Y+a)$ has no constant terms, so $g(Z) = Z\cdot\widetilde{g}(Z)$, so $f(Y) = g(Y-a) = (Y-a)\cdot\widetilde{g}(Y-a)$. Let $u(Y) = \widetilde{g}(Y-a)$. Since $\widetilde{g}(a) = f'(a) \equiv f'(0) \pmod{\mathfrak{a}}$, $u(0) \in R^*$, hence $u \in R[[Y]]^*$. □

□

Unfortunately, there is no such simple structure theorem for $\widehat{\mathcal{O}}_{x,X}$ as $\widehat{\mathcal{O}}_{y,Y}$-algebra in general. If $\Bbbk(x)$ is separable algebraic over $\Bbbk(y)$, then one can still say something: let

$$\widetilde{\mathcal{O}} = \text{the unique finite free } \widehat{\mathcal{O}}_{y,Y}\text{-algebra with } \widetilde{\mathcal{O}}/\mathfrak{m}_{y,Y}\widetilde{\mathcal{O}} \cong \Bbbk(x)$$

as defined in Corollary IV.6.3. Note that $\operatorname{Spec}\widetilde{\mathcal{O}}$ is, in fact étale over $\operatorname{Spec}\mathcal{O}_{y,Y}$: if $\mathbb{k}(x) \cong \mathbb{k}(y)[T]/(\overline{f}(T))$ and f lifts $\overline{f}$ and has the same degree, then

$$\widetilde{\mathcal{O}} \cong \widehat{\mathcal{O}}_{y,Y}[T]/(f(T))$$

and

$$\Big(\text{Image in } \widetilde{\mathcal{O}}/\mathfrak{m}_{y,Y}\widetilde{\mathcal{O}} \text{ of } f'(T)\Big) = \overline{f}'(T) \not\equiv 0$$

since $\mathbb{k}(x)$ is separable over $\mathbb{k}(y)$. Then it can be proven that

$$\widehat{\mathcal{O}}_{x,X} \cong \widetilde{\mathcal{O}}[[t_1, \ldots, t_n]].$$

If X is étale over Y, this follows directly from the universal property Corollary IV.6.3 of $\widetilde{\mathcal{O}}$. In general, choose the lift f of $\overline{f}$ to have coefficients in $\mathcal{O}_{y,Y}$ and replacing Y by a neighborhood of y, we get a diagram:

$$\begin{tikzcd} X \arrow[d, "f"'] & \operatorname{Spec}\mathcal{O}_X[T]/(f(T)) = X' \arrow[l, "q"'] \arrow[d, "f'"] \\ Y & \operatorname{Spec}\mathcal{O}_Y[T]/(f(T)) = Y' \arrow[l, "p"] \end{tikzcd}$$

There is one point $y' \in Y'$ over $y \in Y$ and $\mathbb{k}(y') \cong \mathbb{k}(x)$; then we get a point $x' \in X'$ over x and y' as the image of

$$\operatorname{Spec}\mathbb{k}(x) \longrightarrow \operatorname{Spec}\big(\mathbb{k}(x) \otimes_{\mathbb{k}(y)} \mathbb{k}(y')\big) \longrightarrow X \times_Y Y' = X'.$$

Applying Proposition 3.8 to the smooth f' and the étale q, we find:

$$\widehat{\mathcal{O}}_{x,X} \cong \widehat{\mathcal{O}}_{x',X'} \cong \widehat{\mathcal{O}}_{y',Y'}[[t_1, \ldots, t_n]] \cong \widetilde{\mathcal{O}}[[t_1, \ldots, t_n]].$$

At any point of a smooth morphism, there is a simple structure theorem for $\operatorname{gr}\mathcal{O}_{x,X}$ as $\operatorname{gr}\mathcal{O}_{y,Y}$-algebra, hence for $\mathbb{TC}_{X,x}$ as a scheme over $\mathbb{TC}_{Y,y}$:

PROPOSITION 3.10. *If $f\colon X \to Y$ is smooth at x of relative dimension n and $y = f(x)$, then $\operatorname{gr}\mathcal{O}_x$ is a polynomial ring in n variables over $\operatorname{gr}(\mathcal{O}_y) \otimes_{\mathbb{k}} \mathbb{k}(x)$ — more precisely, $\exists t_1, \ldots, t_n \in \mathfrak{m}_x/\mathfrak{m}_x^2$ such that*

$$\mathfrak{m}_x^\nu/\mathfrak{m}_x^{\nu+1} \cong \bigoplus_{l=0}^{\nu} \bigoplus_{\left(\substack{\text{multi-indices} \\ \alpha,\ |\alpha|=\nu-l}\right)} \Big(\mathfrak{m}_y^l/\mathfrak{m}_y^{l+1} \otimes_{\mathbb{k}(y)} \mathbb{k}(x)\Big) \cdot t^\alpha$$

Thus

$$\mathbb{TC}_{X,x} \cong \mathbb{TC}_{y,Y} \times_{\operatorname{Spec}\mathbb{k}(y)} \mathbb{A}^n_{\mathbb{k}(x)}.$$

PROOF. There are two cases to consider: adding a new variable and dividing by a new equation. The first is:

LEMMA 3.11. *Let $x \in Y \times \mathbb{A}^1$, let t be the variable in $\mathbb{A}^1$ and let $y = p_1(x) \in Y$. Note that $p_1^{-1}(y) \cong \mathbb{A}^1_{\mathbb{k}(y)}$. Either:*

1) *x is the generic point of $\mathbb{A}^1_{\mathbb{k}(y)}$ in which case $\mathfrak{m}_x = \mathfrak{m}_y \cdot \mathcal{O}_x$, $\mathbb{k}(x) \cong \mathbb{k}(y)(t)$ and*

$$(\mathfrak{m}_y^\nu/\mathfrak{m}_y^{\nu+1}) \otimes_{\mathbb{k}(y)} \mathbb{k}(x) \xrightarrow{\approx} \mathfrak{m}_x^\nu/\mathfrak{m}_x^{\nu+1}$$

is an isomorphism,

2) *x is a closed point of $\mathbb{A}^1_{\Bbbk(y)}$ in which case $\exists$ a monic polynomial $f(t)$ such that $\mathfrak{m}_x = \mathfrak{m}_y \cdot \mathcal{O}_x + f \cdot \mathcal{O}_x$, $\Bbbk(x) \cong \Bbbk(y)[t]/(\overline{f})$, and*

$$\bigoplus_{l=0}^{\nu} \Big((\mathfrak{m}_y^l/\mathfrak{m}_y^{l+1}) \otimes_{\Bbbk(y)} \Bbbk(x) \Big) \cdot f_1^{\nu-l} \xrightarrow{\approx} \mathfrak{m}_x^\nu/\mathfrak{m}_x^{\nu+1}$$

is an isomorphism (here f_1 = image of f in $\mathfrak{m}_x/\mathfrak{m}_x^2$).

Proof of Lemma 3.11. In the first case,

$$\mathcal{O}_x = \text{localization of } \mathcal{O}_y[t] \text{ with respect to prime ideal } \mathfrak{m}_y \cdot \mathcal{O}_y[t].$$

Then $\mathfrak{m}_x$ is generated by $\mathfrak{m}_y \cdot \mathcal{O}_y[t]$, hence by $\mathfrak{m}_y$, and:

$$\begin{aligned} \mathfrak{m}_x^\nu/\mathfrak{m}_x^{\nu+1} &\cong \Big((\mathfrak{m}_y \cdot \mathcal{O}_y[t])^\nu \,/\, (\mathfrak{m}_y \cdot \mathcal{O}_y[t])^{\nu+1} \Big) \otimes_{\mathcal{O}_y[t]} \mathcal{O}_x \\ &\cong \big((\mathfrak{m}_y^\nu/\mathfrak{m}_y^{\nu+1}) \otimes_{\Bbbk(y)} \Bbbk(y)[t] \big) \otimes_{\mathcal{O}_y[t]} \mathcal{O}_x \\ &\cong \big(\mathfrak{m}_y^\nu/\mathfrak{m}_y^{\nu+1} \big) \otimes_{\Bbbk(y)} \big(\Bbbk(y)[t] \otimes_{\mathcal{O}_y[t]} \mathcal{O}_x \big) \\ &\cong \big(\mathfrak{m}_y^\nu/\mathfrak{m}_y^{\nu+1} \big) \otimes_{\Bbbk(y)} \Bbbk(y)(t) \end{aligned}$$

Taking $\nu = 0$, this shows that $\Bbbk(x) \cong \Bbbk(y)(t)$ and putting this back in the general case, we get what we want.

In the second case,

$$\begin{aligned} \mathcal{O}_x = \ & \text{localization of } \mathcal{O}_y[t] \text{ with respect to maximal ideal } \mathfrak{p} \\ & \text{where } \mathfrak{p} = \text{inverse image of principal ideal } (\overline{f}) \subset \Bbbk(y)[t], \\ & \overline{f} \text{ monic and irreducible of some degree } d. \end{aligned}$$

Lift $\overline{f}$ to a monic $f \in \mathcal{O}_y[t]$. Then $\mathfrak{p} = \mathfrak{m}_y \cdot \mathcal{O}_y[t] + f \cdot \mathcal{O}_y[t]$, hence $\mathfrak{m}_x = \mathfrak{p} \cdot \mathcal{O}_x = \mathfrak{m}_y \cdot \mathcal{O}_x + f \cdot \mathcal{O}_x$. Now since $\mathfrak{p}$ is maximal, $\mathcal{O}_y[t]/\mathfrak{p}^{\nu+1} \xrightarrow{\approx} \mathcal{O}_x/\mathfrak{m}_x^{\nu+1}$ for all ν, hence $\mathfrak{p}^\nu/\mathfrak{p}^{\nu+1} \xrightarrow{\approx} \mathfrak{m}_x^\nu/\mathfrak{m}_x^{\nu+1}$. On the other hand, $\mathcal{O}_y[t]/(f^{\nu+1})$ is a free $\mathcal{O}_y$-module with basis:

$$1, t, \ldots, t^{d-1}, f, ft, \ldots, ft^{d-1}, \ldots, f^\nu, f^\nu t, \ldots, f^\nu t^{d-1}.$$

In terms of this basis:

$$\mathfrak{p}^m/(f^{\nu+1}) = \bigoplus_{l=0}^{m} \bigoplus_{i=1}^{d-1} \mathfrak{m}_y^l \cdot f^{m-l} \cdot t^i,$$

hence

$$\mathfrak{p}^\nu/\mathfrak{p}^{\nu+1} \cong \bigoplus_{l=0}^{\nu} \bigoplus_{i=0}^{d-1} \Big(\mathfrak{m}_y^l/\mathfrak{m}_y^{l+1} \Big) \cdot f_1^{\nu-l} \cdot t^i.$$

Now $\Bbbk(x) \cong \Bbbk(y)[t]/(\overline{f}) = \bigoplus_{i=0}^{d-1} \Bbbk(x) \cdot t^i$, so in this direct sum decomposition,

$$\bigoplus_{i=0}^{d-1} \Big(\mathfrak{m}_y^l/\mathfrak{m}_y^{l+1} \Big) \cdot f_1^{\nu-l} \cdot t^i = \Big(\mathfrak{m}_y^l/\mathfrak{m}_y^{l+1} \Big) \otimes_{\Bbbk(y)} \Bbbk(x) \cdot f_1^{\nu-l}$$

and (2) follows. □

By induction, Proposition 3.10 follows for the case $X = Y \times \mathbb{A}^n$, $f = p_1$. Now every smooth morphism is locally of the form:

$$\begin{tikzcd} X = V(f_1, \ldots, f_r) \arrow[dr, "f"'] & \subset & Y \times \mathbb{A}^{n+r}\text{: call this scheme } Z \arrow[dl, "p_1"] \\ & Y & \end{tikzcd}$$

Consider the homomorphism:

$$\mathfrak{m}_{x,Z}/\left(\mathfrak{m}_{x,Z}^2 + \mathfrak{m}_y \cdot \mathcal{O}_{x,Z}\right) \longrightarrow \Omega_{Z/Y} \otimes_{\mathcal{O}_Z} \mathbb{k}(x).$$

$\Omega_{Z/Y}$ is a free $\mathcal{O}_Z$-module with basis $dX_1, \ldots, dX_{n+r}$ and the canonical map takes:

$$f \bmod \mathfrak{m}_{x,Z}^2 \longrightarrow \sum_{j=1}^{n+r} \frac{\partial f_i}{\partial X_j} \cdot dX_j.$$

By smoothness, the images of the f_i in $\Omega_{Z/Y} \otimes \mathbb{k}(x)$ are independent over $\mathbb{k}(x)$, hence the f_i in $\mathfrak{m}_{x,Z}/\left(\mathfrak{m}_{x,Z}^2 + \mathfrak{m}_y \cdot \mathcal{O}_{x,Z}\right)$ are independent over $\mathbb{k}(x)$. Proposition 3.10 now follows by induction on r using:

LEMMA 3.12. *Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a local homomorphism of local rings such that $\operatorname{gr} \mathcal{O}_2$ is a polynomial ring in r variables over $\operatorname{gr} \mathcal{O}_1$. Let $f \in \mathfrak{m}_2$ have non-zero image in $\mathfrak{m}_2/\left(\mathfrak{m}_2^2 + \mathfrak{m}_x \cdot \mathcal{O}_2\right)$. Then*

$$\operatorname{gr}(\mathcal{O}_2/f \cdot \mathcal{O}_2) \cong \operatorname{gr}(\mathcal{O}_2)/f_1 \cdot \operatorname{gr}(\mathcal{O}_2) \qquad (f_1 = \textit{image of } f \textit{ in } \mathfrak{m}_2/\mathfrak{m}_2^2)$$

and is a polynomial ring in $r-1$ variables over $\operatorname{gr} \mathcal{O}_1$.

PROOF OF LEMMA 3.12. By induction, $\operatorname{gr}(\mathcal{O}_2/f \cdot \mathcal{O}_2)$ is the quotient of $\operatorname{gr} \mathcal{O}_2$ by the leading forms of all elements $f \cdot g$ of $f \cdot \mathcal{O}_2$. If $g \in \mathfrak{m}_2^l \setminus \mathfrak{m}_2^{l+1}$, its leading form $\overline{g}$ is in $\mathfrak{m}_2^l/\mathfrak{m}_2^{l+1}$. The hypothesis on f means that f_1 can be taken as one of the variables in the presentation of $\operatorname{gr} \mathcal{O}_2$ as a polynomial ring, hence f_1 is a non-zero-divisor in $\operatorname{gr} \mathcal{O}_2$. Therefore $f_1 \cdot \overline{g} \neq 0$, i.e., $f \cdot g \notin \mathfrak{m}_2^{l+2}$ and the leading form of $f \cdot g$ is equal to $f_1 \cdot \overline{g}$. Thus $\operatorname{gr}(\mathcal{O}_2/f \cdot \mathcal{O}_2) \cong (\operatorname{gr} \mathcal{O}_2)/f_1 \cdot \operatorname{gr} \mathcal{O}_2$ as required. □

□

COROLLARY 3.13. *If $f \colon X \to Y$ is smooth at x of relative dimension n and $y = f(x)$, then*

$$df_x^* \colon T_{y,Y}^* \otimes_{\mathbb{k}(y)} \mathbb{k}(x) \longrightarrow T_{x,X}^* \quad \textit{is injective,}$$

hence $df_x \colon T_{x,X} \to T_{y,Y} \widehat{\otimes}_{\mathbb{k}(y)} \mathbb{k}(x)$ is surjective.

COROLLARY 3.14. *If $f \colon X \to Y$ is smooth at x and $y = f(x)$ is a regular point of Y, then x is a regular point of X.*

COROLLARY 3.15. *If a K-variety X is smooth of relative dimension n over K at some point $x \in X$, then $n = \dim X$.*

PROOF. Apply Proposition 3.10 to the generic point $\eta \in X$. □

COROLLARY 3.16. *If $f \colon X \to Y$ is smooth of relative dimension n, then its fibres $f^{-1}(y)$ are reduced and all components are n-dimensional.*

PROOF. Combine Lemma 1.1, Proposition 3.3 and Corollary 3.14. □

COROLLARY 3.17. *If $f \colon X \to \operatorname{Spec} k$ is smooth at $x \in X$ and we write:*

$$\mathcal{O}_{x,X} = k[X_1, \ldots, X_{n+r}]_{\mathfrak{p}}/(f_1, \ldots, f_r)$$

as usual, then the module of syzygies

$$\sum_{i=1}^{r} g_i f_i = 0, \qquad g_i \in k[X]_{\mathfrak{p}}$$

is generated by the trivial ones:

$$(f_j) \cdot f_i + (-f_i) \cdot f_j = 0, \qquad 1 \leq i < j \leq r.$$

Proof. Let $B = k[X_1, \ldots, X_{n+r}]_{\mathfrak{p}}$ and $K = B/\mathfrak{p} \cdot B$. We have seen in the proof of Proposition 3.10 that $\operatorname{gr} B$ is a graded polynomial ring over K in which $\overline{f}_1, \ldots, \overline{f}_r \in \mathfrak{p}B/(\mathfrak{p}B)^2$ are independent linear elements. We apply:

Lemma 3.18. *Let A be any ring. Over $A[T_1, \ldots, T_r]$, the module of syzygies*

$$\sum_{i=1}^{r} g_i T_i = 0, \qquad g_i \in A[T]$$

is generated by the trivial ones:

$$(T_j) \cdot T_i + (-T_i) \cdot T_j = 0, \qquad 1 \leq i < j \leq r.$$

(Proof is a direct calculation which we leave to the reader.)

Therefore we know the syzygies in $\operatorname{gr} B$! Now let Syz be the module of all syzygies:

$$0 \longrightarrow \operatorname{Syz} \longrightarrow B^r \xrightarrow{v} B$$

$$v(a_1, \ldots, a_r) = \sum a_i f_i$$

and let Triv be the submodule of Syz generated by the "trivial" ones. Now

$$\bigcap_{\nu=1}^{\infty} \mathfrak{p}^{\nu}(B^r / \operatorname{Triv}) = (0)$$

so

$$\operatorname{Triv} = \bigcap_{\nu=1}^{\infty} \left((\mathfrak{p}^{\nu} B)^r + \operatorname{Triv}\right).$$

Therefore if $\operatorname{Syz} \supsetneqq \operatorname{Triv}$, we can find a syzygy $(g_1, \ldots, g_r)$ with $g_i \in \mathfrak{p}^{\nu} B$ such that for no trivial syzygy $(h_1, \ldots, h_r)$ are all $g_i + h_i \in \mathfrak{p}^{\nu+1} B$. Let $\overline{g}_i =$ image of g_i in $\mathfrak{p}^{\nu} B / \mathfrak{p}^{\nu+1} B$. Then

$$\sum \overline{g}_i \overline{f}_i = 0$$

is a syzygy in $\operatorname{gr} B$. By Lemma 3.18,

$$(\overline{g}_1, \ldots, \overline{g}_r) = \sum_{1 \leq i < j \leq r} \overline{a}_{ij} (0, \ldots, \overset{\substack{i\text{-th} \\ \text{place}}}{\overline{f}_j}, \ldots, \overset{\substack{j\text{-th} \\ \text{place}}}{-\overline{f}_i}, \ldots, 0).$$

Lifting the $\overline{a}_{ij}$ to B, this gives a contradiction. □

Combining Corollary 3.17 with Proposition IV.4.10 now shows (See Proposition VII.5.7 for a strengthening.):

Proposition 3.19. *Let $f\colon X \to Y$ be a smooth morphism. Then f is flat and for every $x \in X$, if*

$$\mathcal{O}_{x,X} = \mathcal{O}_{y,Y}[X_1, \ldots, X_{n+r}]_{\mathfrak{p}} / (f_1, \ldots, f_r)$$

as usual, then the module of syzygies:

$$\sum_{i=1}^{r} g_i f_i = 0, \qquad g_i \in \mathcal{O}_{y,Y}[X]_{\mathfrak{p}}$$

is generated by the trivial ones.

Proof. Let $A = \mathcal{O}_{y,Y}$, $B = \mathcal{O}_{y,Y}[X]_{\mathfrak{p}}$ and apply Proposition IV.4.10 to the sequence:

$$\begin{aligned}
&B^{r(r-1)/2} \xrightarrow{u} B^r \xrightarrow{v} B \\
&u(\ldots, a_{ij}, \ldots) = (\ldots, -\sum_{l<i} f_l \cdot a_{li} + \sum_{i<l} f_l \cdot a_{il}, \ldots) \\
&v(a_1, \ldots, a_r) = \sum a_i f_i.
\end{aligned}$$

By Corollary 3.17, it is exact after $\otimes_{\mathcal{O}_{y,Y}} \Bbbk(y)$ so it is exact as it stands and $\operatorname{Coker} v$ is A-flat. $\square$

In fact, it can be shown[2] that if $f\colon X \to Y$ is any morphism which can be expressed locally as

$$\operatorname{Spec} A[X_1, \ldots, X_{n+r}]/(f_1, \ldots, f_r) \longrightarrow \operatorname{Spec} A$$

where all fibres have dimension n, then f has the two properties of Proposition 3.19, i.e., f is flat and the syzygies among the f_i are trivial. Such a morphism f is called a *relative local complete intersection.* The property of the syzygies being generated by the trivial ones is an important one in homological algebra; in particular when it holds, it implies that one can explicitly *resolve* $B/(f_1, \ldots, f_r)$ as B-module, i.e., give all higher order syzygies as well: we will prove this later — §VII.5.

An interesting link can be made between the concept of smoothness and the theory of schemes over *complete* discrete valuation rings (§IV.6). In fact, let R be a complete discrete valuation ring, $S = \operatorname{Spec} R$, $k = R/M$, $K =$ fraction field of R. Let

$$f\colon X \longrightarrow S$$

be a smooth morphism of relative dimension n. Consider the specialization:

$$\mathrm{sp}\colon \operatorname{Max}(X_\eta)^\circ \longrightarrow \operatorname{Max}(X_o)$$

introduced in §IV.6. Let $x \in X_o$ be a k-rational point. Then the smoothness of f allows one to construct *analytic coordinates* on X near x, so that

$$\begin{aligned}
\mathrm{sp}^{-1}(x) &\cong \text{open } n\text{-dimensional polycylinder in } \mathbb{A}^n_K \\
\text{i.e.} \quad &\cong \{x \in \operatorname{Max}(\mathbb{A}^n_K) \mid |p_i(x)| < 1, \text{ all } i\}\,.
\end{aligned}$$

4. Criteria for smoothness

In this section, we will present four important criteria for the smoothness of a morphism f. The first concerns when a variety X over a field k is smooth over $\operatorname{Spec} k$. But it holds equally well for any reduced and irreducible scheme X of finite type over a *regular* scheme Y:

Criterion 4.1. *Let Y be a regular irreducible scheme and $f\colon X \to Y$ a morphism of finite type. Assume X is reduced and irreducible and that $f(\eta_X) = \eta_Y$. Let $r = \operatorname{tr.deg}_{\mathbb{R}(Y)} \mathbb{R}(X)$. Then $\forall x \in X$*

a) $\dim_{\Bbbk(x)} \left(\Omega_{X/Y} \otimes_{\mathcal{O}_x} \Bbbk(x)\right) \geq r$
b) *equality holds if and only if f is smooth at x in which case the relative dimension must be r and $\Omega_{X/Y} \cong \mathcal{O}_X^r$ in a neighborhood of x.*

---

[2]One need only generalize Corollary 3.17 and this follows from the Cohen-Macaulay property of $k[X_1, \ldots, X_n]$: cf. Zariski-Samuel [**109**, vol. II, Appendix 6].

Proof. Let $\eta \in X$ be its generic point. Then

$$(\Omega_{X/Y})_\eta \cong \Omega_{\mathbb{R}(X)/\mathbb{R}(Y)}.$$

This $\mathbb{R}(X)$-vector space is dual to the vector space of $\mathbb{R}(Y)$-derivations from $\mathbb{R}(X)$ into itself. But by Example 2 in §2, the dimension of this space is $\geq \operatorname{tr.deg}_{\mathbb{R}(Y)} \mathbb{R}(X)$. Now since f is of finite type, $\Omega_{X/Y}$ is a finitely generated $\mathcal{O}_X$-module, hence by Proposition I.5.5 (Nakayama), $\forall x \in X$

$$\dim_{\Bbbk(x)} \big(\Omega_{X/Y} \otimes \Bbbk(x)\big) \geq \dim_{\mathbb{R}(X)} (\Omega_{X/Y})_\eta \geq \operatorname{tr.deg}_{\mathbb{R}(Y)} \mathbb{R}(X) = r.$$

Now if f is smooth at any $x \in X$, it is smooth at η and then by Corollary 3.15 its relative dimension must be r, hence $\Omega_{X/Y} \cong \mathcal{O}_X^r$ near x, hence

$$\dim_{\Bbbk(x)} \big(\Omega_{X/Y} \otimes \Bbbk(x)\big) = r.$$

Now assume conversely that $r = \dim_{\Bbbk(x)} \big(\Omega_{X/Y} \otimes \Bbbk(x)\big)$. To prove f is smooth at x, we replace X and Y by affine neighborhoods of x and y, so we have:

$$\begin{aligned} X &= \operatorname{Spec} R[X_1, \ldots, X_n]/(f_1, \ldots, f_l) \\ Y &= \operatorname{Spec} R. \end{aligned}$$

Then

$$\begin{aligned} &\Omega_{X/Y} \\ &\quad\cong \bigoplus_{i=1}^n \mathcal{O}_X \cdot dX_i \Bigg/ \left(\text{modulo relations } \sum_{j=1}^n \frac{\partial f_i}{\partial X_j} \cdot dX_j = 0, \ \ 1 \leq i \leq l\right) \end{aligned}$$

hence

$$\begin{aligned} &\Omega_{X/Y} \otimes \Bbbk(x) \\ &\quad\cong \bigoplus_{i=1}^n \Bbbk(x) \cdot dX_i \Bigg/ \left(\text{modulo relations } \sum_{j=1}^n \frac{\partial f_i}{\partial X_j}(x) \cdot dX_j = 0, \ \ 1 \leq i \leq l\right). \end{aligned}$$

The matrix $(\partial f_i / \partial X_j)$ is known as the Jacobian matrix for the above presentation of X. It follows that

$$\dim_{\Bbbk(x)} \big(\Omega_{X/Y} \otimes \Bbbk(x)\big) = n - \operatorname{rk} \left(\frac{\partial f_i}{\partial X_j}(x)\right).$$

Therefore in our case $(\partial f_i / \partial X_j(x))$ has rank $n - r$. Pick out $f_{i_1}, \ldots, f_{i_{n-r}}$ such that

$$\operatorname{rk} \left(\frac{\partial f_{i_l}}{\partial X_j}(x)\right) = n - r$$

and hence define

$$\widetilde{X} = \operatorname{Spec} R[X_1, \ldots, X_n]/(f_{i_1}, \ldots, f_{i_{n-r}}).$$

Then we get a diagram

$$\begin{tikzcd} X \arrow[rr, hook] \arrow[dr, "f"'] & & \widetilde{X} \arrow[dl, "\widetilde{f}"] \\ & Y & \end{tikzcd}$$

and find that $\widetilde{f}$ is smooth of relative dimension r at x. But then by Corollary 3.14, $\mathcal{O}_{x,\widetilde{X}}$ is a regular local ring. In particular it is an integral domain and $\widetilde{X}$ has a unique component $\widetilde{X}_\circ$ containing x. By Corollary 3.15 applied to the generic point of $\widetilde{X}_\circ$,

$$r = \operatorname{tr.deg}_{\mathbb{R}(Y)} \mathbb{R}(\widetilde{X}_\circ).$$

In other words, both $\mathcal{O}_{x,\widetilde{X}}$ and its quotient $\mathcal{O}_{x,X} = \mathcal{O}_{x,\widetilde{X}}/(\text{other } f_i\text{'s})$ are integral domains of the same transcendence degree over $\mathbb{k}(Y)$! This is only possible if they are equal (cf. Part I [**76**, Proposition (1.14)]). So $\mathcal{O}_{x,X} = \mathcal{O}_{x,\widetilde{X}}$, hence $X = \widetilde{X}$ in a neighborhood of x and X is smooth over Y at x. □

Corollary 4.2 (Jacobian Criterion for Smoothness). *If in the situation of Criterion* 4.1, $Y = \operatorname{Spec} R$, $X = \operatorname{Spec} R[X_1, \dots, X_n]/(f_1, \dots, f_l)$, *then*

$$f \text{ is smooth at } x \Longleftrightarrow \operatorname{rk}\left(\frac{\partial f_i}{\partial X_j}(x)\right) = n - r.$$

Corollary 4.3. *In the situation of Criterion* 4.1,

$$\left[\begin{array}{l}\exists x \in X \text{ such that} \\ f \text{ is smooth (resp. étale) at } x\end{array}\right] \Longleftrightarrow \left[\begin{array}{l}\mathbb{k}(X) \text{ is separable} \\ \text{(resp. separable algebraic)} \\ \text{over } \mathbb{k}(Y)\end{array}\right].$$

Proof. If f is smooth somewhere, it is smooth at η; and the criterion at η is:

$$\dim(\text{vector space of } \mathbb{k}(Y)\text{-derivations of } \mathbb{k}(X) \text{ to } \mathbb{k}(X)) = \operatorname{tr.deg}_{\mathbb{k}(Y)} \mathbb{k}(X).$$

By Example 2 in §2, this is equivalent to $\mathbb{k}(X)$ being separable over $\mathbb{k}(Y)$. □

Corollary 4.4. *If $f\colon X \to Y$ is étale, then for all $y \in Y$, the fibre $f^{-1}(y)$ is a finite set of reduced points each of which is* $\operatorname{Spec} K$, *K separable algebraic over* $\mathbb{k}(y)$.

Proof. Proposition 3.3 and Corollary 4.3. □

Corollary 4.5. *In the situation of Criterion* 4.1 *if $x \in X$, $y = f(x)$, then f is smooth over Y at x if and only if the fibre $f^{-1}(y)$ is smooth of relative dimension r over* $\operatorname{Spec} \mathbb{k}(y)$ *at x (n.b. one must assume the two r's are the same, i.e.,* $\dim f^{-1}(y) = \operatorname{tr.deg}_{\mathbb{k}(Y)} \mathbb{k}(X)$*).*

A slightly more general version of Criterion 4.1 is sometimes useful:

Criterion. 4.1$^+$ *Let Y be a regular irreducible scheme and let $f\colon X \to Y$ be a morphism of finite type. Let*

$$X = X_1 \cup \cdots \cup X_t$$

be the components of X and assume $f(\eta_{X_i}) = \eta_Y$, $1 \le i \le t$. Let

$$r = \min_{1 \le i \le t}\Bigl(\operatorname{tr.deg}_{\mathbb{k}(Y)} \mathbb{k}(X_{i,\mathrm{red}})\Bigr).$$

Then for all $x \in X$:

a) $\dim_{\mathbb{k}(x)} \Omega_{X/Y} \otimes_{\mathcal{O}_x} \mathbb{k}(x) \ge r$
b) *equality holds if and only if f is smooth of relative dimension r at x.*

In some cases, we can give a criterion for smoothness via Zariski tangent spaces (as in the theory of differential geometry):

Criterion 4.6. *Let $f\colon X \to Y$ be as in the previous criterion. Assume further that $\mathbb{k}(x)$ is separable over $\mathbb{k}(y)$. Then*

$$f \text{ is smooth at } x \Longleftrightarrow \left[\begin{array}{l} x \text{ is a regular point of } X \text{ and} \\ df_x\colon T_{x,X} \to T_{y,Y} \otimes_{\mathbb{k}(y)} \mathbb{k}(x) \text{ is surjective}\end{array}\right].$$

Proof. "$\Longrightarrow$" was proven in Corollaries 3.13 and 3.14. To go backwards, use the lemma:

LEMMA 4.7. *Let X be a noetherian scheme and $X' \subset X$ a closed subscheme. Suppose $x \in X'$ is a point which is simultaneously regular on both X and X' and suppose $r = \dim \mathcal{O}_{x,X} - \dim \mathcal{O}_{x,X'}$. Then $\exists$ a neighborhood $U \subset X$ of x and $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that the ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ defining X' is given by*

$$\mathcal{I}|_U = \sum_{i=1}^{r} f_i \cdot \mathcal{O}_X$$

and moreover $\overline{f}_1, \ldots, \overline{f}_r \in \mathfrak{m}_{x,X}/\mathfrak{m}_{x,X}^2$ are independent over $\Bbbk(x)$.

PROOF OF LEMMA 4.7. We know $\mathcal{O}_{x,X'} \cong \mathcal{O}_{x,X}/\mathcal{I}_x$, hence

$$\mathrm{gr}(\mathcal{O}_{x,X'}) \cong \mathrm{gr}(\mathcal{O}_{x,X}) / (\text{ideal generated by leading forms of elements of } \mathcal{I}_x) .$$

Both "gr" are graded polynomial rings, the former in $m+r$ variables, the latter in m variables for some m. This is only possible if the ideal of leading forms is generated by r independent linear forms $\overline{f}_1, \ldots, \overline{f}_r$. Lift these to $f'_1, \ldots, f'_r \in \mathcal{I}_x$, hence to $f_1, \ldots, f_r \in \mathcal{I}(U)$ for some open $U \subset X$. New $\sum f_i \cdot \mathcal{O}_{x,X} \subset \mathcal{I}_x$ so we get three rings:

$$\mathcal{O}_{x,X} \xrightarrow{\alpha} \mathcal{O}_{x,X}/\textstyle\sum f_i \cdot \mathcal{O}_{x,X} \xrightarrow{\beta} \mathcal{O}_{x,X}/\mathcal{I}_x = \mathcal{O}_{x,X'}.$$

These induces:

$$\mathrm{gr}(\mathcal{O}_{x,X}) \xrightarrow{\mathrm{gr}(\alpha)} \mathrm{gr}\left(\mathcal{O}_{x,X}/\textstyle\sum f_i \cdot \mathcal{O}_{x,X}\right) \xrightarrow{\mathrm{gr}(\beta)} \mathrm{gr}(\mathcal{O}_{x,X'}).$$

But by construction, $\mathrm{Ker}(\mathrm{gr}(\alpha) \circ \mathrm{gr}(\alpha)) \subset \mathrm{Ker}(\mathrm{gr}(\alpha))$, so $\mathrm{gr}(\beta)$ is an isomorphism. Then β is an isomorphism too, hence $\mathcal{I}_x = \sum f_i \cdot \mathcal{O}_{x,X}$. Now because X is noetherian, the two sheaves $\mathcal{I}|_U$ and $\sum f_i \cdot \mathcal{O}_X|_U$ are both finitely generated and have the same stalks at x: hence they are equal in some open $U' \subset U$. □

Now whenever $f \colon X \to Y$ is a morphism of finite type, Y is noetherian, $x \in X$ is a regular point and $y = f(x) \in Y$ is a regular point, factor f locally:

$$\begin{tikzcd} X = V(f_1, \ldots, f_l) \arrow[r, hook] \arrow[dr, "f"'] & Y \times \mathbb{A}^n = Z \arrow[d, "p_1"] \\ & Y \end{tikzcd}$$

and note that $\mathcal{O}_{x,X} \cong \mathcal{O}_{x,Z}/(f_1, \ldots, f_l)$ where $\mathcal{O}_{x,X}$ and $\mathcal{O}_{x,Z}$ are both regular. It follows from Lemma 4.7 that in some neighborhood of x, $X = V(f_1, \ldots, f_s)$ where $\overline{f}_1, \ldots, \overline{f}_s \in \mathfrak{m}_{x,Z}/\mathfrak{m}_{x,Z}^2$ are independent. Now df_x surjective means dually that

$$\begin{array}{ccc} (\mathfrak{m}_y/\mathfrak{m}_y^2) \otimes_{\Bbbk(y)} \Bbbk(x) & \longrightarrow & \mathfrak{m}_{x,X}/\mathfrak{m}_{x,X}^2 \\ & & \| \wr \\ & & \mathfrak{m}_{x,Z} \Big/ \left(\mathfrak{m}_{x,Z}^2 + \sum_{i=1}^{s} \overline{f}_i \cdot \Bbbk(x) \right) \end{array}$$

is injective. This implies that $\overline{f}_1, \ldots, \overline{f}_s$ are also independent in $\mathfrak{m}_{x,Z}/(\mathfrak{m}_{x,Z}^2 + \mathfrak{m}_y \cdot \mathcal{O}_{x,Z})$. Since $\Bbbk(x)$ is separable over $\Bbbk(Y)$, $\Upsilon_{\Bbbk(x)/\Bbbk(y)} = (0)$, hence

$$\mathfrak{m}_{x,Z}/(\mathfrak{m}_{x,Z}^2 + \mathfrak{m}_y \cdot \mathcal{O}_{x,Z}) \longrightarrow \Omega_{Z/Y} \otimes_{\mathcal{O}_Z} \Bbbk(x)$$

is injective. Therefore finally $df_1, \ldots, df_s \in \Omega_{Z/Y} \otimes_{\mathcal{O}_Z} \Bbbk(x)$ are independent, which is precisely the condition that $V(f_1, \ldots, f_s)$ is smooth over Y at x. □

The most important case for these results is when $Y = \operatorname{Spec} k$, X a k-variety. There are then in fact two natural notions of "non-singularity" for a point $x \in X$.

a) x a regular point,

b) $X \to \operatorname{Spec} k$ smooth at x.

Our results show that they almost coincide! In fact:

$$x \text{ a regular point} \Longleftarrow x \text{ a smooth point,} \qquad \text{by Corollary 3.14}$$

and if $\Bbbk(x)$ is separable over k, then:

$$x \text{ a regular point} \Longleftrightarrow x \text{ a smooth point,} \qquad \text{by Criterion 4.6.}$$

But by the Jacobian Criterion 4.2, if $\overline{k} =$ algebraic closure of k, and $\overline{X} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ and $\overline{x} \in \overline{X}$ lies over x, then

$$x \text{ smooth on } X \Longleftrightarrow \overline{x} \text{ smooth on } \overline{X}.$$

Putting this together:

$$\begin{aligned} x \text{ regular on } X \Longleftarrow\ & x \text{ smooth on } X \\ \Longleftrightarrow\ & \overline{x} \text{ smooth on } \overline{X} \\ \Longleftrightarrow\ & \overline{x} \text{ regular on } \overline{X}. \end{aligned}$$

The pathological situation where these are not all equivalent occurs only over an imperfect field k and is quite interesting. It stems from the geometric fact that over an algebraically closed ground field in characteristic p, Sard's lemma fails abysmally:

Example. Let k be an algebraically closed field of characteristic $p \neq 0$. There exist morphisms $f\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ such that every fibre $f^{-1}(x)$ (x closed point) is singular.

a) $f\colon \mathbb{A}^1_k \to \mathbb{A}^1_k$ given by $f(a) = a^p$. Then if $b \in \mathbb{A}^1_k$ is a closed point and $b = a^p$, the scheme-theoretic fibre is:

$$\begin{aligned} f^{-1}(b) &= \operatorname{Spec} k[X]/(X^p - b) \\ &= \operatorname{Spec} k[X]/(X-a)^p \\ &\cong \operatorname{Spec} k[X']/(X'^p), \qquad (\text{if } X' = X - a) \end{aligned}$$

none of which are reduced. Similarly, the differential

$$df\colon T_{a,\mathbb{A}^1} \longrightarrow T_{a^p,\mathbb{A}^1}$$

is everywhere 0 and f is nowhere étale.

b) $f\colon \mathbb{A}^2_k \to \mathbb{A}^1_k$ given by $f(a,b) = a^2 - b^p$. Then if $d \in \mathbb{A}^1_k$ is a closed point and $d = c^p$, the scheme-theoretic fibre is:

$$\begin{aligned} f^{-1}(d) &= \operatorname{Spec} k[X,Y]/(X^2 - Y^p - d) \\ &= \operatorname{Spec} k[X,Y]/(X^2 - (Y+c)^p) \\ &\cong \operatorname{Spec} k[X,Y']/(X^2 - Y'^p), \qquad \text{if } Y' = Y + c. \end{aligned}$$

Thus the fibre $f^{-1}(d)$ is again a k-variety, in fact a plane curve, but with a singularity at $X = Y' = 0$ as in Figure V.3:

c) Now if t is the coordinate on $\mathbb{A}^1_k$, then $\mathbb{R}(\mathbb{A}^1_k) = k(t)$: a non-perfect field of characteristic p. Consider the *generic fibre* $f^{-1}(\eta)$ of the previous example. It is a 1-dimensional $k(t)$-variety equal to:

$$\begin{aligned} \mathbb{A}^2_k \times_{\mathbb{A}^1_k} \operatorname{Spec} k(t) &= \operatorname{Spec} k[X,Y] \otimes_{k[t]} k(t) \\ &= \operatorname{Spec} k(t)[X,Y]/(X^2 - Y^p - t), \end{aligned}$$

i.e., it is the plane curve $X^2 = Y^p + t$. But now $t \notin k(t)^p$, so this curve is not isomorphic over $k(t)$ to $X^2 = (Y')^p$. In fact, $k[X,Y] \otimes_{k[t]} k(t)$ is a localization of $k[X,Y]$, so the

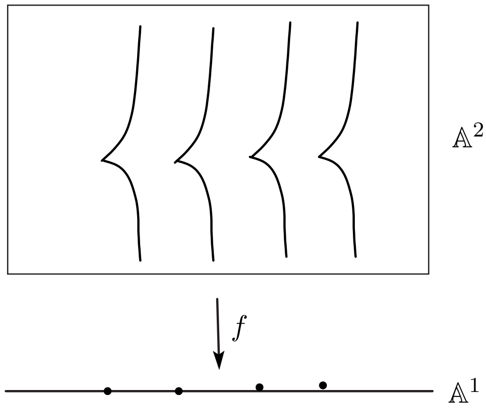

FIGURE V.3. Every fibre is singular.

local rings of $f^{-1}(\eta)$ are all local rings of $\mathbb{A}_k^2$ too, hence they are all regular, i.e., $f^{-1}(\eta)$ is a regular scheme! But the Jacobian matrix of the defining equations of this curve is:

$$\frac{\partial}{\partial X}(X^2 - Y^p - t) = 2X$$
$$\frac{\partial}{\partial Y}(X^2 - Y^p - t) = 0$$

so all 1×1-minors vanish at the point $x = V(X, Y^p + t) \in X$. Thus X is not smooth over $k(t)$ at x.

The third and fourth criteria for smoothness are more general and do not assume that the base scheme Y is regular.

CRITERION 4.8. *Consider a finitely presented morphism $f\colon X \to Y$. Take a point $x \in X$ and let $y = f(x)$. Then*

$$f \text{ is smooth at } x \iff \left[\begin{array}{l} f \text{ is flat at } x \text{ and the fibre} \\ f^{-1}(y) \text{ is smooth over } \Bbbk(y) \text{ at } x. \end{array}\right]$$

PROOF. $\Longrightarrow$ was proven in Propositions 3.3 and 3.19. To prove the converse, we may assume $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} A[X_1, \ldots, X_n]/(f_1, \ldots, f_r)$. Then let $x = [\mathfrak{p}]$, where $\mathfrak{p}$ is a prime ideal in $A[X_1, \ldots, X_n]$ and let $\mathfrak{q} = \mathfrak{p} \cap A$ and $k =$ (quotient field of $A/\mathfrak{q}) \cong \Bbbk(y)$. Note that the fibre $f^{-1}(y)$ equals

$$\operatorname{Spec} k[X_1, \ldots, X_n]/(\overline{f}_1, \ldots, \overline{f}_r).$$

If s is the dimension of $f^{-1}(y)$ at x, it follows that

$$\operatorname{rk}\left(\frac{\partial \overline{f}_i}{\partial X_j}(x)\right) = n - s.$$

Thus $n - s \le r$ and renumbering, we may assume that:

$$\det_{1 \le i,j \le n-s}\left(\frac{\partial \overline{f}_i}{\partial X_j}(x)\right) \ne 0.$$

Consider the diagram:

$$\begin{tikzcd}[column sep=small] X = \operatorname{Spec} A[X]/(f_1, \ldots, f_r) \arrow[dr, "f"', "\text{(flat at } x)"] & \subset & \operatorname{Spec} A[X]/(f_1, \ldots, f_{n-s}) = X' \arrow[dl, "f'", "\text{(smooth at } x)"'] \\ & \operatorname{Spec} A & \end{tikzcd}$$

Then the fibres: $f^{-1}(y) \subset (f')^{-1}(y)$ over y are both smooth of dimension s at x, hence they are equal in a neighborhood of x. I claim that in fact X and X' are equal in a neighborhood of x, hence f is smooth at x. To prove this, it suffices to show

$$(f_1, \dots, f_r) \cdot A[X]_{\mathfrak{p}} = (f_1, \dots, f_{n-s}) \cdot A[X]_{\mathfrak{p}}$$

or, by Nakayama's lemma, to show

$$\frac{(f_1, \dots, f_r) \cdot A[X]_{\mathfrak{p}}}{(f_1, \dots, f_{n-s}) \cdot A[X]_{\mathfrak{p}}} \otimes_{A_{\mathfrak{q}}} k = (0).$$

But consider the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \dfrac{(f_1, \dots, f_r) \cdot A[X]_{\mathfrak{p}}}{(f_1, \dots, f_{n-s}) \cdot A[X]_{\mathfrak{p}}} \arrow[r] & \dfrac{A[X]_{\mathfrak{p}}}{(f_1, \dots, f_{n-s}) \cdot A[X]_{\mathfrak{p}}} \arrow[r] \arrow[d, equal, "\sim"'] & \dfrac{A[X]_{\mathfrak{p}}}{(f_1, \dots, f_r) \cdot A[X]_{\mathfrak{p}}} \arrow[r] \arrow[d, equal, "\sim"'] & 0. \\ & & \mathcal{O}_{x,X'} & \mathcal{O}_{x,X} & \end{tikzcd}$$

The last ring is flat over A, so

$$\begin{tikzcd} 0 \arrow[r] & \dfrac{(f_1, \dots, f_r) \cdot A[X]_{\mathfrak{p}}}{(f_1, \dots, f_{n-s}) \cdot A[X]_{\mathfrak{p}}} \otimes_{A_{\mathfrak{q}}} k \arrow[r] & \dfrac{A[X]_{\mathfrak{p}}}{(f_1, \dots, f_{n-s}) \cdot A[X]_{\mathfrak{p}}} \otimes_{A_{\mathfrak{q}}} k \arrow[r] \arrow[d, equal, "\sim"'] & \dfrac{A[X]_{\mathfrak{p}}}{(f_1, \dots, f_r) \cdot A[X]_{\mathfrak{p}}} \otimes_{A_{\mathfrak{q}}} k \arrow[r] \arrow[d, equal, "\sim"'] & 0 \\ & & \mathcal{O}_{x,(f')^{-1}(y)} & \mathcal{O}_{x,f^{-1}(y)} & \end{tikzcd}$$

is exact. But $\mathcal{O}_{x,(f')^{-1}(y)} \xrightarrow{\approx} \mathcal{O}_{x,f^{-1}(y)}$, so the module on the left is (0). $\square$

Corollary 4.9. *Let $f \colon X \to Y$ be a finitely presented morphism. Then for all $x \in X$, $y = f(x)$,*

$$f \text{ is étale at } x \iff \left[\begin{array}{l} f \text{ is flat at } x, \text{ the fibre } f^{-1}(y) \text{ is reduced} \\ \text{at } x \text{ and } \Bbbk(x) \text{ is separable algebraic over } \Bbbk(y). \end{array} \right]$$

The last criterion is a very elegant idea due to Grothendieck. It is an infinitesimal criterion involving A-valued points of X and Y when A is an artin local ring. We want to consider a *lifting* for such point described by the diagram:

$$\begin{tikzcd} \operatorname{Spec} A/I \arrow[r, "\psi_0"] \arrow[d, phantom, "\cap"] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r, "\phi_1"'] \arrow[ur, dashed, "\psi_1"] & Y \end{tikzcd}$$

This means that we have an A-valued point ϕ_1 of Y and a lifting ψ_0 of the induced (A/I)-valued point (I is any ideal in A). Then the problem is to lift ϕ_1 to an A-valued point ψ_1 of X extending ψ_0. The criterion states:

Criterion 4.10. *Let $f \colon X \to Y$ be any morphism of finite type where Y is a noetherian scheme. Then f is smooth if and only if:*

> *For all artin local rings A, ideals $I \subset A$, and all A-valued points ϕ_1 of Y and (A/I)-valued points ψ_0 of X such that:*
>
> $$f \circ \psi_0 = \textit{restriction of } \phi_1 \textit{ to } \operatorname{Spec} A/I$$
>
> *there is an A-valued point ψ_1 of X such that*
>
> $$\begin{aligned} f \circ \psi_1 &= \phi_1 \\ \psi_0 &= \textit{restriction of } \psi_1 \textit{ to } \operatorname{Spec} A/I. \end{aligned}$$

(See diagram.)

$f\colon X \to Y$ satisfying the lifting property in Criterion 4.10 is said to be *formally smooth* in EGA [**1**, Chapter IV, §17]. This criterion plays crucial roles in deformation theory (cf. §VIII.5).

PROOF. Suppose first that f is smooth and ψ_0, ϕ_1 are given. Look at the induced morphism f_1:

$$\begin{tikzcd} & X_1 = X \times_Y \operatorname{Spec} A \arrow[d, "f_1"] \\ \operatorname{Spec} A/I \arrow[ur, "\psi_0'"] & \subset \quad \operatorname{Spec} A \end{tikzcd}$$

which is smooth by Proposition 3.3. Then ψ_0 defines a *section* ψ_0' of f_1 over the subscheme $\operatorname{Spec} A/I$ of the base which we must extend to a section of f_1 over the whole of $\operatorname{Spec} A$. Let $y \in \operatorname{Spec} A$ be its point and let $x \in X_1$ be the image of ψ_0'. Then $\Bbbk(x) = \Bbbk(y)$, so by Proposition 3.8

$$\widehat{\mathcal{O}}_{x,X_1} \cong A[[t_1, \dots, t_n]].$$

If the section ψ_0' is given by

$$(\psi_0')^*(t_i) = \overline{a}_i \in A/I,$$

choose $a_i \in A$ over $\overline{a}_i$. Then define a section ψ_i' of f_1 by

$$(\psi_1')^*(t_i) = a_i.$$

Now suppose f satisfies the lifting criterion. Choose $x \in X$. We will verify the definition of smoothness directly, i.e., find a local presentation of f near x as

$$\operatorname{Spec} R[X_1, \dots, X_n]/(f_1, \dots, f_l) \longrightarrow \operatorname{Spec} R$$

where $\det(\partial f_i/\partial X_j) \neq 0$. To start, let f be presented locally by

$$\operatorname{Spec} R[X_1, \dots, X_n]/I \longrightarrow \operatorname{Spec} R$$

and let

$$r = \dim_{\Bbbk(x)} \left(\Omega_{X/Y} \otimes \Bbbk(x)\right).$$

We may replace X by $\operatorname{Spec} R[X_1, \dots, X_n]/I$ and Y by $\operatorname{Spec} R$ if we wish. Since $\Omega_{X/Y} \otimes \Bbbk(x)$ is generated by $dX_1, \dots, dX_n$ with relations $df = 0$, $f \in I$, we can choose $f_1, \dots, f_{n-r} \in I$ such that

$$\Omega_{X/Y} \otimes \Bbbk(x) \cong \left(\bigoplus_{i=1}^{n} \Bbbk(x) \cdot dX_i\right) \Big/ \langle df_1, \dots, df_{n-r}\rangle$$

and in particular

$$\det\left(\frac{\partial f_i}{\partial X_j}(x)\right) \neq 0.$$

This allows us to *factor* f locally through a smooth morphism:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"'] & X_1 = \operatorname{Spec} R[X_1, \dots, X_n]/(f_1, \dots, f_{n-r}) \arrow[dl, "f_1"] \\ Y & \end{tikzcd}$$

where f_1 is smooth at x and

$$\Omega_{X_1/Y} \otimes \Bbbk(x) \longrightarrow \Omega_{X/Y} \otimes \Bbbk(x) \tag{4.11}$$

is an isomorphism.

We now apply the lifting property to the artin local rings $A_\nu = \mathcal{O}_{x,X_1}/\mathfrak{m}^\nu_{x,X_1}$ and the ideals $I_\nu = I \cap \mathfrak{m}^{\nu-1}_{x,X_1} + \mathfrak{m}^\nu_{x,X_1}$. We want to define by induction on ν morphisms r_ν:

$$\begin{tikzcd}
\operatorname{Spec} \mathcal{O}_{x,X}/\mathfrak{m}^\nu_{x,X} \arrow[rr, hook] \arrow[d, phantom, "\cap"] & & X \arrow[d, "f"] \\
\operatorname{Spec} \mathcal{O}_{x,X_1}/\mathfrak{m}^\nu_{x,X_1} \arrow[r, hook] \arrow[urr, dashed, "r_\nu"] & X_1 \arrow[r, "f_1"'] & Y
\end{tikzcd}$$

which extend each other. Given r_ν, r_ν plus the canonical map

$$\operatorname{Spec} \mathcal{O}_{x,X_1}/(I + \mathfrak{m}^{\nu+1}_{x,X_1}) = \operatorname{Spec} \mathcal{O}_{x,X}/\mathfrak{m}^{\nu+1}_{x,X} \hookrightarrow X$$

induce a map

$$\operatorname{Spec} \mathcal{O}_{x,X_1}/(I \cap \mathfrak{m}^\nu_{x,X_1} + \mathfrak{m}^{\nu+1}_{x,X_1}) \longrightarrow X.$$

(This is because $\mathcal{O}_{x,X_1}/(I \cap \mathfrak{m}^\nu_{x,X_1} + \mathfrak{m}^{\nu+1}_{x,X_1})$ can be identified with the subring of $(\mathcal{O}_{x,X_1}/(I + \mathfrak{m}^{\nu+1}_{x,X_1})) \oplus \mathcal{O}_{x,X_1}/\mathfrak{m}^\nu_{x,X_1}$ of pairs both members of which have the same image in $\mathcal{O}_{x,X_1}/(I+\mathfrak{m}^\nu_{x,X_1})$.) Apply the lifting property to find $r_{\nu+1}$. Now the whole family $\{r_\nu\}$ defines a morphism r:

$$\begin{tikzcd}
 & \operatorname{Spec} \widehat{\mathcal{O}}_{x,X_1} \arrow[dl, "r"'] \arrow[d, phantom, "\cap"] \\
X \arrow[r, hook] \arrow[dr] & X_1 \arrow[d] \\
 & Y
\end{tikzcd}$$

which is in effect a retraction of a formal neighborhood of X in X_1 onto X, all over Y. Ring-theoretically, this means

$$\widehat{\mathcal{O}}_{x,X_1} \cong \widehat{\mathcal{O}}_{x,X} \oplus J$$

and where the R-algebra structure of $\widehat{\mathcal{O}}_{x,X_1}$ is given by the R-algebra structure of $\widehat{\mathcal{O}}_{x,X}$. It follows that

$$\Omega_{X_1/Y} \otimes \widehat{\mathcal{O}}_{x,X} \cong (\Omega_{X/Y} \otimes \widehat{\mathcal{O}}_{x,X}) \oplus (J/J^2).$$

But, then applying (4.11), we find

$$(J/J^2) \otimes \Bbbk(x) = (0),$$

hence by Nakayama's lemma, $J = (0)$. Thus $\widehat{\mathcal{O}}_{x,X_1} \cong \widehat{\mathcal{O}}_{x,X}$, hence $\mathcal{O}_{x,X_1} \cong \mathcal{O}_{x,X}$ and $X \cong X_1$ in a neighborhood of x. □

5. Normality

Recall that in §III.6 we defined a scheme X to be *normal* if its local rings $\mathcal{O}_{x,X}$ are integral domains integrally closed in their quotient field. In particular, if $X = \operatorname{Spec} R$ is affine and integral, then

$$\begin{aligned} X \text{ is normal} &\iff R_\mathfrak{p} \text{ integrally closed in } \mathbb{R}(X),\ \forall \mathfrak{p} \\ &\iff R \text{ integrally closed in } \mathbb{R}(X) \end{aligned}$$

(using the facts (i) that a localization of an integrally closed domain is integrally closed and (ii) $R = \bigcap_\mathfrak{p} R_\mathfrak{p}$.) An important fact is that regular schemes are normal. This can be proven either using the fact that regular local rings are UFD's (cf. Zariski-Samuel [**109**, vol. II, Appendix 7]; or Kaplansky [**58**, §4-2]) and that all UFD's are integrally closed (Zariski-Samuel [**109**, vol. I, p. 261]); or one can argue directly that for a noetherian local ring $\mathcal{O}$, $\operatorname{gr} \mathcal{O}$ integrally closed $\Longrightarrow$ $\mathcal{O}$ integrally closed (Zariski-Samuel [**109**, vol. II, p. 250]). As we saw in §III.6, normality for noetherian rings is really the union of two distinct properties, each interesting in its own right. We wish to globalize this. First we must find how to express globally the condition:

$$R = \bigcap_{\mathfrak{p} \text{ non-zero minimal prime}} R_{\mathfrak{p}}.$$

Proposition-Definition 5.1. *Let X be a noetherian scheme with no embedded components and let $x \in X$ be a point of codimension at least 2. Say $\eta_1, \ldots, \eta_n$ are the generic points of the components of X containing x. The following are equivalent:*

a) *$\forall$ neighborhoods U of x, and $f \in \mathcal{O}_X\left(U \setminus (\overline{\{x\}} \cap U)\right)$, there is a neighborhood $U' \subset U$ of x such that f extends to $f' \in \mathcal{O}_X(U')$.*

a$'$)

$$\mathcal{O}_{x,X} = \bigcap_{\substack{y \in X \text{ with} \\ x \in \overline{\{y\}} \\ x \neq y}} \mathcal{O}_{y,X}$$

(all these rings being subrings of the total quotient ring $\bigoplus_{i=1}^n \mathcal{O}_{\eta_i,X}$).

b) *$\forall f \in \mathfrak{m}_{x,X}$ with $f(\eta_i) \neq 0$ all i, x is not an embedded point of the subscheme $V(f)$ defined near x.*

b$'$) *$\exists f \in \mathfrak{m}_{x,X}$ with $f(\eta_i) \neq 0$ all i, and x not an embedded point of $V(f)$.*

Points with these properties we call proper points; others are called improper[3]. If all points are proper, X is said to have Property S2.

Proof. It is easy to see (a) $\iff$ (a$'$), and (b) $\implies$ (b$'$) is obvious. To see (b$'$) $\implies$ (a), take

$$g \in \mathcal{O}_X\left(U \setminus (\overline{\{x\}} \cap U)\right), \qquad U \text{ affine}$$

and let $f \in \mathfrak{m}_{x,X}$ be such that $V(f)$ has no embedded components. Then the distinguished open set U_f of U is inside $U \setminus (\overline{\{x\}} \cap U)$, hence we can write:

$$g = g_1/f^m, \quad g_1 \in \mathcal{O}_X(U).$$

We now prove by induction on l that $g_1/f^l \in \mathcal{O}_{x,X}$, starting with $l = 0$ where we know it, and ending at $l = m$ where it proves that $g \in \mathcal{O}_{x,X}$, hence $g \in \mathcal{O}_X(U')$ some $U' \subset U$. Namely, if $l < m$, and $h = g_1/f^l \in \mathcal{O}_{x,X}$, consider the function $\overline{h}$ induced by h on $V(f)$ in a neighborhood of x. Since $h = f^{m-l} \cdot g$, it follows that $\overline{h} = 0$ on $V(f) \setminus (\overline{\{x\}} \cap V(f))$, i.e., $\operatorname{Supp} \overline{h} \subset \overline{\{x\}} \cap V(f)$. Since x is not an embedded component of $V(f)$, $\overline{h} = 0$ at x too, i.e., $g_1/f^{l+1} = h/f \in \mathcal{O}_{x,X}$.

To see (a$'$) $\implies$ (b), suppose $f \in \mathfrak{m}_{x,X}$, $f(\eta_i) \neq 0$ and suppose $g \in \mathcal{O}_{x,X}$ restricts to a function $\overline{g}$ on $V(f)$ whose support is contained in $\overline{\{x\}} \cap V(f)$. Then for all $y \in X$ with $x \in \overline{\{x\}}$, $x \neq y$, $\overline{g}$ is 0 in $\mathcal{O}_{y,V(f)}$, i.e., $g \in f \cdot \mathcal{O}_{y,X}$. Then

$$g/f \in \bigcap_{\substack{y \in X \\ x \in \overline{\{y\}} \\ x \neq y}} \mathcal{O}_{y,X} = \mathcal{O}_{x,X},$$

hence $\overline{g} = 0$. $\square$

Criterion 5.2 (Basic criterion for normality (Krull-Serre)). *Let X be a reduced noetherian scheme. Then*

$$X \text{ is normal} \iff \begin{cases} \text{a) } \forall x \in X \text{ of codimension } 1,\ X \text{ is regular at } x \\ \text{b) } X \text{ has Property S2.} \end{cases}$$

[3] This is not standard terminology; it is suggested by an old Italian usage: cf. Semple-Roth [**86**, Chapter 13, §6.4].

In particular (a) and (b) imply that the components of X are disjoint.

Proof. If X is affine and irreducible, say $X = \operatorname{Spec} R$, then Property S2, in form (a′), implies immediately:

$$\forall \mathfrak{p} \text{ prime ideal in } R: \qquad R_{\mathfrak{p}} = \bigcap_{\substack{\mathfrak{q} \text{ non-zero minimal prime} \\ \mathfrak{q} \subset \mathfrak{p}}} R_{\mathfrak{q}}.$$

Since

$$R = \bigcap_{\text{all } \mathfrak{p}} R_{\mathfrak{p}},$$

the criterion reduces to Krull's result (Theorem III.6.5). Everything in the criterion being local, it remains to prove (a) + (b) $\implies$ all components of X are disjoint. Let

$$S = \{x \in X \mid x \text{ is in at least two components of } X\},$$

and let x be some generic point of S. Then $\mathcal{O}_{x,X}$ is not a domain so by (a), $\operatorname{codim} x \geq 2$. Then consider the function e which is 1 on one of the components through x, 0 on all the others. Clearly

$$e \in \bigcap_{\substack{y \in X \\ x \in \overline{\{y\}} \\ x \neq y}} \mathcal{O}_{y,X}, \qquad e \notin \mathcal{O}_{x,X}$$

which contradicts S2. Thus $S = \emptyset$. □

Here is an example of how this criterion is used:

Proposition 5.3. *Assume X is a regular irreducible scheme and $Y \subsetneq X$ is a reduced and irreducible codimension 1 subscheme. Then Y has Property S2.*

Proof. Let $y \in Y$ be a point of codimension ≥ 2 and let $f \in \mathcal{O}_{y,X}$ be a local equation for Y. Take any $g \in \mathfrak{m}_{y,Y} \setminus f\mathcal{O}_{y,X} \setminus \mathfrak{m}_{y,X}^2$. Let $\overline{g}$ be the image of g in $\mathcal{O}_{y,Y}$, let Z be the subscheme of X defined by $g = 0$ near y, and let $\overline{f}$ be the image of f in $\mathcal{O}_{y,Z}$. Then

$$\begin{aligned} y \text{ is a proper point of } Y &\iff \{y\} \text{ not embedded component of } V(\overline{g}) \subset Y \\ &\iff \{y\} \text{ not embedded component of } V(f,g) \subset X \\ &\iff \{y\} \text{ not embedded component of } V(\overline{f}) \subset Z \\ &\iff y \text{ is a proper point of } Z. \end{aligned}$$

But $\mathcal{O}_{y,Z} = \mathcal{O}_{y,X}/g \cdot \mathcal{O}_{y,X}$ is regular (since $g \notin \mathfrak{m}_{y,X}^2$), hence Z is normal at y hence every point is proper. □

Corollary 5.4. *If X is regular, irreducible, $Y \subsetneq X$ is reduced irreducible of codimension 1, then if Y itself is regular at all points of codimension 1, Y is normal.*

Another application of the basic criterion is:

Proposition 5.5. *Let $f: Y \to X$ be a smooth morphism, where X is a normal noetherian scheme. Then Y is normal (and locally noetherian).*

Proof. As X is the disjoint union of its components, we can replace X by one of these and so assume X irreducible with generic point η. Note that since $\mathcal{O}_{\eta,X} =$ the field $\mathbb{R}(X)$, the local rings of any $y \in f^{-1}(\eta)$ on the fibre $f^{-1}(\eta)$ and on Y are the same.

a) Y is reduced: in fact f flat implies

$$f(\mathrm{Ass}(\mathcal{O}_Y)) \subset \mathrm{Ass}(\mathcal{O}_X) = \{\eta\}.$$

For for all $y \in \mathrm{Ass}(\mathcal{O}_Y)$,

$$\mathcal{O}_{y,Y} = \mathcal{O}_{y,f^{-1}(\eta)}$$

is an integral domain, since $f^{-1}(\eta)$ is smooth over $\mathrm{Spec}\,\mathbb{R}(X)$, hence is regular.

b) If $y \in Y$ has codimension ≤ 1, then by Corollary IV.5.10, $f(y)$ has codimension 0 or 1, hence X is regular at $f(y)$. Since f is smooth, Y is regular at y by Corollary 3.14.

c) If $y \in Y$ has codimension > 1, we seek some $g \in \mathcal{O}_{y,Y}$ with $g(y) = 0$, $g \not\equiv 0$ on any component of Y through y, and such that $V(g)$ has no embedded components through y. There are two cases:

c$_1$)

$$\begin{aligned} f(y) = \eta \Longrightarrow & \mathcal{O}_{y,Y} = \mathcal{O}_{y,f^{-1}(\eta)} \text{ regular, hence normal} \\ \Longrightarrow & \text{any } g \in \mathfrak{m}_{y,Y},\ g \neq 0 \text{ has this property} \\ & \text{by the Basic Criterion 5.2.} \end{aligned}$$

c$_2$) $f(y) = x$ has codimension ≥ 1 in X. But then since X is normal, there is a $g \in \Gamma(U_x, \mathcal{O}_X)$, U_x some neighborhood of x, such that $g(x) = 0$, $g(\eta) \neq 0$ and $V(g)$ has no embedded components. Then $f^*(g) \in \Gamma(f^{-1}U_x, \mathcal{O}_Y)$ is not zero at any generic points of Y while $f^*(g)(y) = 0$. Moreover,

$$V(f^*(g)) \cong V(g) \times_X Y,$$

so $V(f^*(g))$ is smooth over $V(g)$. We get:

$$\begin{aligned} x \in \mathrm{Ass}(V(f^*(g))) \Longrightarrow & f(x) \in \mathrm{Ass}\, V(g) \\ \Longrightarrow & f(x) = \text{generic point of } V(g) \\ \Longrightarrow & \text{codimension of } f(x) \text{ is } 1 \\ \Longrightarrow & X \text{ regular at } f(x) \\ \Longrightarrow & \mathcal{O}_{y,Y} \text{ regular, hence normal} \\ \Longrightarrow & V(f^*(g)) \text{ has no embedded} \\ & \text{components through } y. \end{aligned}$$

□

In particular this shows that a smooth scheme over a normal scheme is locally irreducible and if one looks back at the proof of Criterion 4.1 for smoothness, one sees that it now extends *verbatim* to the case where the image scheme is merely assumed normal, i.e., (as generalized in Criterion 4.1$^+$):

CRITERION 5.6. *Let X be an irreducible normal noetherian scheme and $f\colon Y \to X$ a morphism of finite type. Assume all components Y_i of Y dominate X and let*

$$r = \min \mathrm{tr.deg}_{\mathbb{R}(X)} \mathbb{R}(Y_{i,\mathrm{red}}).$$

Then $\forall y \in Y$

a) $\dim_{\Bbbk(y)} \Omega_{Y/X} \otimes_{\mathcal{O}_x} \Bbbk(y) \geq r$

b) *equality holds if and only if f is smooth at y of relative dimension r.*

EXAMPLE. The simplest way to get non-normal schemes is to start with any old scheme and "collapse" the tangent space at a point or "identify" two distinct points. To be precise, let

$$X = \operatorname{Spec} R$$

be a k-variety.

a) If $x = [\mathfrak{m}]$ is a k-rational point, so that $R \cong k + \mathfrak{m}$, consider

$$X_0 = \operatorname{Spec}(k + \mathfrak{m}^2).$$

The natural morphism:

$$\pi\colon X \longrightarrow X_0$$

is easily seen to be bijective, but if $f \in \mathfrak{m} \setminus \mathfrak{m}^2$, the f is integrally dependent on $k + \mathfrak{m}^2$, but $\notin k + \mathfrak{m}^2$. So X_0 is not normal.

b) If $x_i = [\mathfrak{m}_i]$, $i = 1, 2$ are two k-rational points, let

$$\begin{aligned} R_0 &= \{f \in R \mid f(x_1) = f(x_2)\} \\ &= k + \mathfrak{m}_1 \cap \mathfrak{m}_2 \\ X_0 &= \operatorname{Spec} R_0. \end{aligned}$$

The natural morphism

$$\pi\colon X \longrightarrow X_0$$

is bijective except that x_1, x_2 have the same image. Moreover, if $f \in R$, then f satisfies the equation:

$$(X - f(x_1))(X - f(x_2)) = a, \quad \text{where } a = (f - f(x_1))(f - f(x_2)) \in R_0.$$

So X_0 is not normal. Moreover, one can check that $\Omega_{X/X_0} = (0)$ but π is not étale in this case so this morphism illustrates the fact that Criterion 4.1^+ does not extend to non-normal Y's.

One of the major reasons why normal varieties play a big role in algebraic geometry is that all varieties can be "normalized", i.e., there is a canonical process modifying them only slightly leading to a normal variety. If there were a similar easy canonical process leading from a general variety to a regular one, life would be much simpler!

PROPOSITION-DEFINITION 5.7. *Let X be a reduced and irreducible scheme. Let $L \supset \mathbb{R}(X)$ be a finite algebraic extension. Then there is a unique quasi-coherent sheaf of $\mathcal{O}_X$-algebra:*

$$\mathcal{O}_X \subset \mathcal{A} \subset \textit{constant sheaf } L$$

such that for all affine U:

$$\mathcal{A}(U) = \textit{integral closure of } \mathcal{O}_X(U) \textit{ in } L.$$

We set

$$\begin{aligned} X_L &= \mathcal{S}\!\textit{pec}_X(\mathcal{A}) \\ &\underset{def}{=} \textit{union of affines } \operatorname{Spec}\mathcal{A}(U), \\ &\qquad \textit{as } U \textit{ runs over affines in } X, \end{aligned}$$

and call this the normalization of X in L. In particular, if $L = \mathbb{R}(X)$, we call this the normalization of X. X_L is normal and irreducible with function field L.

To see that this works, use (I.5.9), and check that if $U = \operatorname{Spec} R$ is an affine in X and U_f is a distinguished open set, then $\mathcal{A}(U_f) = \mathcal{A}(U) \otimes_R R_f$. This is obvious.

Note for instance that in the two examples above, normalization just undoes the clutching or identification: X is the normalization of X_0.

Sadly, normalization is seriously flawed as a tool by the very unfortunate fact that even for some of the nicest schemes X you could imagine — e.g., regular affine and 1-dimensional — there are cases where X_L is not of finite type over X. This situation has been intensively studied, above all by Nagata (cf. his book [**78**] and Matsumura [**69**, Chapter 12]). We have no space to describe the rather beautiful pathology that he revealed and the way he "explained" it. Suffices it to recall that:

5.8.

- *X noetherian normal L separable over $\mathbb{R}(X) \Longrightarrow X_L$ of finite type over X.*
- *X itself of finite type over a field $\Longrightarrow X_L$ of finite type over X (cf. Zariski-Samuel* [**109**, vol. I, Chapter V, §4]*).*
- *X itself of finite type over $\mathbb{Z} \Longrightarrow X_L$ of finite type over X (cf. Nagata* [**78**, (37.5)]*).*

We conclude with a few miscellaneous remarks on normalization. The schemes $\operatorname{Proj} R$ can be readily normalized by taking the integral closure of R:

Proposition 5.9. *Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a graded integral domain and let*

$$\begin{aligned} K_0 &= \text{field of elements } f/g,\ f, g \in R_n \text{ for some } n,\ g \neq 0 \\ &= \mathbb{R}(\operatorname{Proj} R). \end{aligned}$$

Then if $t =$ any fixed element of R_1, the quotient field of R is isomorphic to $K_0(t)$. Let $L_0 \supset K_0$ be a finite algebraic extension and let

$$S = \text{integral closure of } R \text{ in } L_0(t).$$

Then S is graded and $\operatorname{Proj} S$ *is the normalization of* $\operatorname{Proj} R$ *in L_0.*

Proof. Left to the reader. □

An interesting relation between normalization and associated points is given by:

Proposition 5.10. *Let X be a reduced and irreducible noetherian scheme and let*

$$\pi\colon \widetilde{X} \longrightarrow X, \qquad \widetilde{X} = \mathcal{S}pec_X(\mathcal{A})$$

be its normalization. Assume π is of finite type hence $\mathcal{A}$ is coherent. Then for all $y \in X$ of codimension at least 2:

$$y \text{ is an improper point} \Longleftrightarrow y \in \operatorname{Ass}(\mathcal{A}/\mathcal{O}_X).$$

The proof is easy using the fact that every point of $\widetilde{X}$ is proper.

One case in which normalization does make a scheme regular is when its dimension is one. This can be used to prove:

Proposition 5.11. *Let k be a field, $K \supset k$ a finitely generated extension of transcendence degree 1. Then there is one and (up to isomorphism) only one regular complete k-variety X with function field K, and it is projective over k.*

PROOF. Let $R^0 \subset K$ be a finitely generated k-algebra with quotient field K, let $X^0 = \operatorname{Spec} R^0$ and embed X^0 in $\mathbb{A}^n_k$ for some n using generators of R^0. Let $\overline{X^0}$ be the closure of X^0 in $\mathbb{P}^n_k$ and write it as $\operatorname{Proj} R'$. Let R'' be the integral closure of R' in its quotient field. Then by Proposition 5.9, $X'' = \operatorname{Proj} R''$ is normal. Since it has dimension 1, it is regular and has the properties required. Uniqueness is easy using Proposition II.4.8, and the fact that the local rings of of the closed points of X'' are valuation rings, hence *maximal* proper subrings of K. $\square$

6. Zariski's Main Theorem

A second major reason why normality is important is that Zariski's Main Theorem holds for general normal schemes. To understand this in its natural context, first consider the classical case: $k = \mathbb{C}$, X a k-variety, and x is a closed point of X. Then we have the following two sets of properties:

N1) X *formally normal* at x, i.e., $\widehat{\mathcal{O}}_{x,X}$ an integrally closed domain.

N2) X *analytically normal* at x, i.e., $\mathcal{O}_{x,X,\mathrm{an}}$, the ring of germs of holomorphic functions at x, is an integrally closed domain.

N3) X *normal* at x.

N4) *Zariski's Main Theorem* holds at x, i.e., $\forall f\colon Z \to X$, f birational and of finite type with $f^{-1}(x)$ finite, then $\exists U \subset X$ Zariski-open with $x \in U$ and

$$\operatorname{res} f\colon f^{-1}U \longrightarrow U$$

an isomorphism.

U1) X *formally unibranch* at x, i.e., $\operatorname{Spec}\left(\widehat{\mathcal{O}}_{x,X}\right)$ irreducible.

U2) X *analytically unibranch* at x, i.e., $\operatorname{Spec}(\mathcal{O}_{x,X,\mathrm{an}})$ irreducible, or equivalently, the germ of analytic space defined by X at x is irreducible.

U3) X *unibranch* at x, i.e., if $X' =$ normalization of X in $\mathbb{R}(X)$, $\pi\colon X' \to X$ the canonical morphism, then $\pi^{-1}(x) =$ one point.

U4) X *topologically unibranch* at x — cf. Part I [**76**, (3.9)].

U5) The *Connectedness Theorem* holds at x, i.e., $\forall f\colon Z \to X$, f proper, Z integral, $f(\eta_Z) = \eta_X$ and $\exists U \subset X$ Zariski-open with $f^{-1}(y)$ connected for all $y \in U$, then $f^{-1}(x)$ is connected too.

6.1. *I claim:*

i) all properties N are equivalent,
ii) all properties U are equivalent,
iii) $\mathrm{N} \Longrightarrow \mathrm{U}$.

Modulo two steps for which we refer the reader to Zariski-Samuel [**109**] and Gunning-Rossi [**48**], this is proven as follows:

N1 $\Longleftrightarrow$ N2 $\Longleftrightarrow$ N3**:** We have inclusions:

$$\mathcal{O}_{x,X} \subset \mathcal{O}_{x,X,\mathrm{an}} \subset \widehat{\mathcal{O}}_{x,X}$$

and

$$\mathcal{O}_{x,X,\mathrm{an}} \cap \mathbb{R}(X) = \mathcal{O}_{x,X}$$

$$\widehat{\mathcal{O}}_{x,X} \cap \begin{pmatrix} \text{total quotient} \\ \text{ring of } \mathcal{O}_{x,X,\mathrm{an}} \end{pmatrix} = \mathcal{O}_{x,X,\mathrm{an}}$$

(This follows from the fact that if $f, g \in \mathcal{O}$, $\mathcal{O}$ noetherian local, then $f|g$ in $\mathcal{O}$ iff $f|g$ in $\widehat{\mathcal{O}}$: cf. Part I [**76**, §1D].) Therefore the implications

$$\begin{aligned}\widehat{\mathcal{O}}_{x,X} \text{ integrally closed domain} &\Longrightarrow \mathcal{O}_{x,X,\mathrm{an}} \text{ integrally closed domain}\\ &\Longrightarrow \mathcal{O}_{x,X} \text{ integrally closed domain}\end{aligned}$$

are obvious. The fact:

$$\mathcal{O}_{x,X} \text{ integrally closed domain} \Longrightarrow \widehat{\mathcal{O}}_{x,X} \text{ integrally closed domain}$$

is a deep Theorem of Zariski (cf. Zariski-Samuel [**109**, vol. II, p. 320]). He proved this for all points x on k-varieties X, for all perfect fields k. It was later generalized by Nagata to schemes X of finite type over any field k or over $\mathbb{Z}$ (cf. Nagata [**78**, (37.5)]). Although this step appears quite deep, note that if we strengthen the hypothesis and assume $\mathcal{O}_{x,X}$ actually regular, then since regularity is a property of $\mathrm{gr}(\mathcal{O}_{x,X})$ and $\mathrm{gr}(\mathcal{O}_{x,X}) \cong \mathrm{gr}(\widehat{\mathcal{O}}_{x,X})$, it follows very simply that $\widehat{\mathcal{O}}_{x,X}$ is also regular, hence is an integrally closed domain!

N1 $\Longrightarrow$ U1**:** Obvious.

U1 $\Longrightarrow$ U2**:** Obvious because

$$\mathcal{O}_{x,X,\mathrm{an}}/\sqrt{(0)} \subset \widehat{\mathcal{O}}_{x,X}/\sqrt{(0)},$$

so if the latter is a domain, so is the former.

U2 $\Longrightarrow$ U4**:** See Gunning-Rossi [**48**, p. 115].

U4 $\Longrightarrow$ U5**:** This was proven in Part I [**76**, (3.24)] for projective morphisms f. The proof generalizes to any proper f.

U5 $\Longrightarrow$ U3**:** Let $\pi\colon X' \to X$ be the normalization of X in $\mathbb{R}(X)$. π is of finite type by (5.8), hence it is proper by Proposition II.6.5. π is birational, hence an isomorphism over some non-empty $U \subset X$. Therefore U5 applies to π and $\pi^{-1}(x)$ is connected. But since $X' = \operatorname{Spec}\mathcal{A}$, $\mathcal{A}$ coherent, $\pi^{-1}(x) = \operatorname{Spec}(\mathcal{A}_x/\mathfrak{m}_x \cdot \mathcal{A}_x)$ and $\mathcal{A}_x/\mathfrak{m}\mathcal{A}_x$ is finite-dimensional over $\mathbb{C}$; thus $\pi^{-1}(x)$ is a finite set too, hence it consists in one point.

U3 $\Longrightarrow$ U1**:** Let $\mathcal{O}'_{x,X}$ be the integral closure of $\mathcal{O}_{x,X}$ in $\mathbb{R}(X)$: it is a local ring and a finite $\mathcal{O}_{x,X}$-module. By flatness of $\widehat{\mathcal{O}}_{x,X}$ over $\mathcal{O}_{x,X}$, we find

$$\widehat{\mathcal{O}}_{x,X} \subset \mathcal{O}'_{x,X} \otimes_{\mathcal{O}_{x,X}} \widehat{\mathcal{O}}_{x,X}$$

and by finiteness of $\mathcal{O}'_{x,X}$,

$$\mathcal{O}'_{x,X} \otimes_{\mathcal{O}_{x,X}} \widehat{\mathcal{O}}_{x,X} \cong \text{completion } \widehat{\mathcal{O}'}_{x,X} \text{ of } \mathcal{O}'_{x,X} \text{ in its } \mathfrak{m}\text{-adic topology.}$$

By N3 $\Longrightarrow$ N1, $\widehat{\mathcal{O}'}_{x,X}$ is a domain, so therefore $\widehat{\mathcal{O}}_{x,X}$ is a domain and U1 is proven.

N3 $\Longrightarrow$ N4**:** (Zariski's Main Theorem) We use the fact already proven that N3 $\Longrightarrow$ N1 $\Longrightarrow$ U1 $\Longrightarrow$ U5 and prove N3+U5 $\Longrightarrow$ N4. This is quite easy using Chow's lemma (Theorem II.6.3). Let $f\colon Z \to X$ be a birational morphism of finite type with $f^{-1}(x)$ finite. Then we can find a diagram:

$$\begin{tikzcd} Z' \arrow[r, hook, "\substack{\text{open}\\ \text{dense}}"'] \arrow[d, "g"'] & \overline{Z'} \arrow[r, hook] & \mathbb{P}^n \times X \arrow[ddll, "p_2"] \\ Z \arrow[d, "f"'] & & \\ X & & \end{tikzcd}$$

where g is proper and birational, $\overline{Z'} =$ closure of Z' in $\mathbb{P}^n \times X$ with reduced structure. Now if we write $f^{-1}(x) = \{y_1, \dots, y_t\}$, then since f is of finite type, each y_i is open in $f^{-1}(x)$ and proper over $\mathbb{C}$. Then if $Y_i = g^{-1}(y_i)$, each Y_i is open in $(f \circ g)^{-1}(x)$ and proper over $\mathbb{C}$. Let $h =$ restriction of p_2 to $\overline{Z'}$. Then $(f \circ g)^{-1}(x)$ is open in $h^{-1}(x)$, hence each Y_i is open in $h^{-1}(x)$. But being proper over $\mathbb{C}$, Y_i must also be closed in $h^{-1}(x)$:

$$h^{-1}(x) = Y_1 \cup \cdots \cup Y_t \cup \big(h^{-1}(x) \setminus (f \circ g)^{-1}(x)\big)$$

is a decomposition of $h^{-1}(x)$ into open and closed pieces. So the Connectedness Theorem implies $t = 1$ and $x \notin h(\overline{Z'} \setminus Z')$. But h is proper so $h(\overline{Z'} \setminus Z')$ is closed in X. Replacing X by $X \setminus h(\overline{Z'} \setminus Z')$, we can therefore assume $Z' = \overline{Z'}$, i.e., Z' is proper over X. It follows that Z is proper over X, and $f^{-1}(x) =$ one point y.

Next replacing X by a smaller neighborhood U of x and Z by $f^{-1}(U)$, we can assume Z and X are affine: to see this, let V be any affine neighborhood of y. Since f is proper, $f(Z \setminus V)$ is closed. Let U be an affine neighborhood of x contained in $X \setminus f(Z \setminus V)$. Then $f^{-1}(U) \subset V$ and $f^{-1}(U)$ is affine by Proposition II.4.5.

Now if $X = \operatorname{Spec} R$, $Z = \operatorname{Spec} R[x_1, \dots, x_n]$, where $x_i \in \mathbb{R}(X)$, consider the morphism $[x_i]\colon Z \to \mathbb{A}^1_{\mathbb{C}} \subset \mathbb{P}^1_{\mathbb{C}}$. This induces

$$([x_i], f)\colon Z \longrightarrow \mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} X$$

which is proper since f is proper. Let Γ_i be its image. Then Γ_i is closed and $(\infty, x) \notin \Gamma_i$. Therefore there is some expression:

$$\begin{aligned} g(t) = & a_m t^m + a_{m-1} t^{m-1} + \cdots + a_0 \\ & a_i \in \mathcal{O}_{x,X} \\ & t = \text{coordinate on } \mathbb{P}^1_{\mathbb{C}} \\ & g \equiv 0 \text{ on } \Gamma_i \\ & \bar{t}^m \cdot g \neq 0 \text{ at } (\infty, x). \end{aligned}$$

Thus $a_m \notin \mathfrak{m}_{x,X}$, and x_i, as an element of $\mathbb{R}(X)$, satisfies $g(x_i) = 0$. In other words, x_i is integrally dependent on $\mathcal{O}_{x,X}$. So $x_i \in \mathcal{O}_{x,X}$, hence $x_i \in \mathcal{O}_X(U_i)$ for some neighborhood U_i of x. It follows that f is an isomorphism over $U_1 \cap \cdots \cap U_n$.

N4 $\Longrightarrow$ N3**:** Let $\pi\colon X' \to X$ be the normalization of X in $\mathbb{R}(X)$ and apply Zariski's Main Theorem with $f = \pi$.

Now consider the same situation for general integral noetherian[4] schemes. N2, U2 and U4 do not make sense, but N1, N3, N4, U1, U3 and U5 do.

We need modify U5 however to read:

$\widetilde{\text{U5}}$) The *Connectedness Theorem* holds at x, i.e., $\forall f\colon Z \to X$, f proper, Z integral, $f(\eta_Z) = \eta_X$ and the *generic geometric fibre* of f connected (i.e., if $\Omega =$ an algebraic closure of $\mathbb{R}(X)$, then via the canonical

$$i\colon \operatorname{Spec} \Omega \longrightarrow X,$$

$Z \times_X \operatorname{Spec} \Omega$ should be connected), then $f^{-1}(x)$ is connected too.

[4] N3 $\Longrightarrow$ N4 is proved even for non-noetherian X in EGA [**1**, Chapter IV, (8.12.10)].

6.2. Then Zariski (for k-varieties) and Grothendieck (in general) have shown:

$$\begin{tikzcd} \mathrm{N1} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathrm{N3} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \mathrm{N4} \\ \mathrm{U1} \arrow[r, Rightarrow] & \mathrm{U3} \arrow[r, Leftrightarrow] & \widetilde{\mathrm{U5}} \end{tikzcd}$$

but Nagata [**78**, Appendix A1] has given counterexamples to N3 $\Longrightarrow$ N1, U3 $\Longrightarrow$ U1. To prove these implications, first note that N1 $\Longrightarrow$ U1 and N3 $\Longrightarrow$ U3 are obvious; that N1 $\Longrightarrow$ N3 is proven just as above. Moreover, N4 $\Longrightarrow$ N3 and $\widetilde{\mathrm{U5}}$ $\Longrightarrow$ U3 are proven as above, except that since the normalization $\pi\colon X' \to X$ may not be of finite type, N4 and $\widetilde{\mathrm{U5}}$ should be applied to partial normalizations, i.e., $\operatorname{Spec} R[a_1, \ldots, a_n] \to \operatorname{Spec} R$, a_i integrally dependent on R. Moreover, N3 + $\widetilde{\mathrm{U5}}$ $\Longrightarrow$ N4 is proven as above. Therefore it remains to prove U1 $\Longrightarrow$ U3 and U3 $\Longrightarrow$ $\widetilde{\mathrm{U5}}$.

U1 $\Longrightarrow$ U3**:** This is an application of Hensel's lemma (Lemma IV.6.1). If $\pi^{-1}(x)$ has more than one point, it is easy to see that we can find an element $a \in \mathbb{R}(X)$ integrally dependent on $\mathcal{O}_{x,X}$ such that already in the morphism:

$$\widetilde{\pi}\colon \operatorname{Spec} \mathcal{O}_{x,X}[a] \longrightarrow \operatorname{Spec} \mathcal{O}_{x,X}$$

$\pi^{-1}(x)$ consists in more than one point. Consider the three rings:

$$\mathcal{O}_{x,X} \subset \mathcal{O}_{x,X}[a] \subset \mathbb{R}(X).$$

Tensoring with $\widehat{\mathcal{O}}_{x,X}$, we get:

$$\widehat{\mathcal{O}}_{x,X} \subset \widehat{\mathcal{O}}_{x,X} \otimes_{\mathcal{O}_{x,X}} \mathcal{O}_{x,X}[a] \subset \widehat{\mathcal{O}}_{x,X} \otimes_{\mathcal{O}_{x,X}} \mathbb{R}(X).$$

Dividing all three rings by their nilpotents, we get

$$\widehat{\mathcal{O}}_{x,X}/\sqrt{(0)} \subset \left(\widehat{\mathcal{O}}_{x,X} \otimes_{\mathcal{O}_{x,X}} \mathcal{O}_{x,X}[a]\right)/\sqrt{(0)} \subset \left(\widehat{\mathcal{O}}_{x,X} \otimes \mathbb{R}(X)\right)/\sqrt{(0)}.$$

By U1, $\widehat{\mathcal{O}}_{x,X}/\sqrt{(0)}$ is a domain, and since $\mathbb{R}(X)$ is a localization of $\mathcal{O}_{x,X}$, $\left(\widehat{\mathcal{O}}_{x,X} \otimes \mathbb{R}(X)\right)/\sqrt{(0)}$ is a localization of $\widehat{\mathcal{O}}_{x,X}/\sqrt{(0)}$, i.e.,

$$\left(\widehat{\mathcal{O}}_{x,X} \otimes \mathbb{R}(X)\right)/\sqrt{(0)} \subset \text{quotient field of } \widehat{\mathcal{O}}_{x,X}/\sqrt{(0)}.$$

This implies that $\left(\widehat{\mathcal{O}}_{x,X} \otimes \mathcal{O}_{x,X}[a]\right)/\sqrt{(0)}$ is a domain hence $\operatorname{Spec}(\widehat{\mathcal{O}}_{x,X} \otimes \mathcal{O}_{x,X}[a])$ is irreducible. Now look at

$$\widehat{\pi}\colon \operatorname{Spec}(\widehat{\mathcal{O}}_{x,X} \otimes \mathcal{O}_{x,X}[a]) \longrightarrow \operatorname{Spec} \widehat{\mathcal{O}}_{x,X}.$$

But since $\widehat{\pi}^{-1}(\text{closed point}) \cong \widetilde{\pi}^{-1}(x)$, which has more than one point, by Hensel's lemma (Lemma IV.6.1), $\operatorname{Spec}(\widehat{\mathcal{O}}_{x,X} \otimes \mathcal{O}_{x,X}[a])$ is not irreducible!

U3 $\Longrightarrow$ $\widetilde{\mathrm{U5}}$**:** (i.e., Unibranch implies the Connectedness Theorem.) We follow Zariski's idea (cf. Zariski [**108**]) and deduce this as an application of the fundamental theorem of "holomorphic functions" (cf. [**108**, Chapter VIII]. See also "GFGA" in §VIII.2.):

6.3 (Fundamental theorem of "holomorphic functions"). $\forall f\colon Z \to X$ *proper,* X *noetherian, then* $f_*\mathcal{O}_Z$ *is a coherent sheaf of* $\mathcal{O}_X$*-algebras and for all* $x \in X$

$$\varprojlim_{\nu} (f_*\mathcal{O}_Z)_x/\mathfrak{m}_x^{\nu} \cdot (f_*\mathcal{O}_Z)_x \cong \varprojlim_{\nu} \Gamma\left(f^{-1}(x), \mathcal{O}_Z/\mathfrak{m}_x^{\nu} \cdot \mathcal{O}_Z\right).$$

To apply this to the situation of $\widetilde{\mathrm{U5}}$, suppose $f^{-1}(x) = W_1 \cup W_2$, W_i open disjoint. Then define idempotents:

$$e_\nu \in \Gamma(f^{-1}(x), \mathcal{O}_Z/\mathfrak{m}_x^\nu \cdot \mathcal{O}_Z)$$
$$e_\nu = 0 \text{ on } W_1, \quad e_\nu = 1 \text{ on } W_2.$$

These define an element $\widehat{e}$ in the limit: approximating this with and element $e \in (f_*\mathcal{O}_Z)_x \bmod \mathfrak{m}_x \cdot (f_*\mathcal{O}_Z)_x$, it follows that $e = 0$ on W_1, $e = 1$ on W_2. Let e extend to a section of $f_*\mathcal{O}_Z$ in an affine neighborhood $U = \operatorname{Spec} R$ of x.

Next, for all open $U \subset X$,

$$f_*\mathcal{O}_Z(U) \underset{\text{def}}{=} \Gamma(f^{-1}(U), \mathcal{O}_Z) \subset \Gamma(f^{-1}(\eta_X), \mathcal{O}_{f^{-1}(\eta_X)}).$$

The generic fibre $f^{-1}(\eta_X)$ of f is a complete variety over the field $\mathbb{R}(X)$, hence

$$L = \Gamma(f^{-1}(\eta_X), \mathcal{O}_{f^{-1}(\eta_X)})$$

is a field, finite and algebraic over $\mathbb{R}(X)$. Applying the theory of §IV.2, $f^{-1}(\eta_X)$ is also a variety over L and passing to the algebraic closure $\overline{\mathbb{R}(X)}$ of $\mathbb{R}(X)$, we find that the geometric scheme:

$$\overline{f^{-1}(\eta_X)} = f^{-1}(\eta_X) \times_{\operatorname{Spec} \mathbb{R}(X)} \operatorname{Spec} \overline{\mathbb{R}(X)} \longrightarrow \operatorname{Spec} \overline{\mathbb{R}(X)}$$

in fact lies over $\operatorname{Spec}(L \otimes_{\mathbb{R}(X)} \overline{\mathbb{R}(X)})$. All points of the latter are conjugate, so $\overline{f^{-1}(\eta_X)}$ maps onto $\operatorname{Spec}(L \otimes_{\mathbb{R}(X)} \overline{\mathbb{R}(x)})$. By assumption $\overline{f^{-1}(\eta_X)}$ is connected, hence $\operatorname{Spec}(L \otimes_{\mathbb{R}(X)} \overline{\mathbb{R}(X)})$ consists in one point, hence L is purely inseparable over $\mathbb{R}(X)$. So we may assume $L^{p^l} \subset \mathbb{R}(X)$. In particular $e^{p^l} \in \mathbb{R}(X)$.

Since $f_*\mathcal{O}_Z(U)$ is a finite R-module, e^{p^l} is integrally dependent on R too. Let R' be the integral closure of R in $\mathbb{R}(X)$ and we can factor the restriction of f to $f^{-1}(U)$ via the function e^{p^l}:

$$\begin{tikzcd}
Z \arrow[dd, "f"'] & \supset & f^{-1}U \arrow[dd, "\operatorname{res} f"'] \arrow[dr] & & \operatorname{Spec} R' \arrow[dl, "g"'] \\
& & & \operatorname{Spec} R[e^{p^l}] \arrow[d, "f'"'] & \\
X & \supset & U \arrow[r, equal] & \operatorname{Spec} R &
\end{tikzcd}$$

Since e^{p^l} takes on values 0 and 1 on $f^{-1}(x)$, it follows that $(f')^{-1}(x)$ consists in at least two points! But R' integral over $R[e^{p^l}]$ so g is surjective by the going-up theorem (Zariski-Samuel [**109**, vol. I, Chapter V, §2, Theorem 3, p. 257]).

An elementary proof that N1 $\Longrightarrow$ N4 can be given along the lines of the proof that U1 $\Longrightarrow$ U3. We sketch this: Given $f \colon Z \to X$ as in N4, form the diagram:

$$\begin{tikzcd}
Z \arrow[d, "f"'] & \left(\operatorname{Spec} \widehat{\mathcal{O}}_{x,X}\right) \times_X Z = Z' \arrow[l] \arrow[d, "f'"'] \\
X & \operatorname{Spec} \widehat{\mathcal{O}}_{x,X} = X' \arrow[l]
\end{tikzcd}$$

Decompose Z' via Hensel's lemma (Lemma IV.6.1). Then it follows that Z'_{red} has a component Z'' which projects by a finite birational morphism to X'. It can be shown next that $Z'' \xrightarrow{\sim} X'$, hence f' has a section. Using

$$\widehat{\mathcal{O}}_{x,X} \cap \mathbb{R}(X) = \mathcal{O}_{x,X},$$

it follows easily that f is a local isomorphism.

There is yet another statement that Grothendieck calls "Zariski's Main Theorem" which generalizes the statement we have used so far. This is the result:

Theorem 6.4 (Zariski-Grothendieck "Main Theorem"). *Let X be any quasi-compact scheme and suppose*

$$f\colon Z \longrightarrow X$$

is a morphism of finite type with finite fibres. Then there exists a factorization of f:

$$Z \xhookrightarrow{\ i\ } \mathcal{S}pec_X\, \mathcal{A} \xrightarrow{\ \pi\ } X$$

where i is an open immersion and $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras such that for all affine $U \subset X$, $\mathcal{A}(U)$ is finitely generated and integral over $\mathcal{O}_X(U)$.

The proof can be found in EGA: (in [**1**, Chapter III, (4.4.3)] for X noetherian f quasi-projective; in [**1**, Chapter IV, (8.12.6)] for f of finite presentation; in [**1**, Chapter IV, (18.12.13)] in the general case!) We will not use this result in this book. Theorem 6.4 has the following important corollaries which we will prove and use (for X noetherian):

Corollary 6.5. *Let $f\colon Z \to X$ be a morphism. Then the following are quivalent:*

a) *f is proper with finite fibres,*
b) *f is finite* (Definition II.6.6), *i.e., the sheaf $\mathcal{A} = f_*\mathcal{O}_Z$ is quasi-coherent, for all $U \subset X$ affine $\mathcal{A}(U)$ is finitely generated as algebra and integral over $\mathcal{O}_X(U)$, and the natural morphism $Z \to \mathcal{S}pec_X(\mathcal{A})$ is an isomorphism.*

Proof using Theorem 6.4. (b) $\Longrightarrow$ (a) is elementary: use Proposition II.6.5. As for (a) $\Longrightarrow$ (b), everything is local over X so we may assume $X = \operatorname{Spec} R$. Then by Theorem 6.4 f factors:

$$Z \hookrightarrow \operatorname{Spec} B \longrightarrow \operatorname{Spec} R.$$

Since Z is proper over $\operatorname{Spec} R$, the image of Z in $\operatorname{Spec} B$ is closed as well as open, hence $Z \cong \operatorname{Spec} B/\mathfrak{a}$ for some ideal $\mathfrak{a}$. Then $f_*\mathcal{O}_Z \cong \widetilde{B/\mathfrak{a}} \cong \operatorname{Spec} f_*\mathcal{O}_Z$. $\square$

Corollary 6.6 (Characterization of normalizations). *Let X be an integral scheme, Z a normal, integral scheme and $f\colon Z \to X$ a proper surjective morphism with finite fibres. Then $\mathbb{R}(Z)$ is a finite algebraic extension of $\mathbb{R}(X)$ and X is isomorphic to the normalization of X in $\mathbb{R}(Z)$.*

Proof. Straightforward. $\square$

Corollary 6.7. *Let X be a normal noetherian scheme, $f\colon Z \to X$ a proper étale morphism with Z connected. Then Z is isomorphic to the normalization of X in some finite separable field extension $L \supset \mathbb{R}(X)$.*

Proof. This reduces to Corollary 6.6 because of Proposition 5.5. $\square$

Independent proof of Corollary 6.5 when X is noetherian. Assume $f\colon Z \longrightarrow X$ given, proper with finite fibres. Let $\mathcal{A} = f_*\mathcal{O}_Z$. Then by the fundamental theorem of "holomorphic functions" (6.3), $\mathcal{A}$ is an $\mathcal{O}_X$-module of finite type, hence $\mathcal{A}(U)$ is finitely generated as algebra and integral over $\mathcal{O}_X(U)$ for all affine U. Let $Y = \mathcal{S}pec_X\, \mathcal{A}$ so that we have a factorization:

$$\begin{tikzcd} Z \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

Note that Y is noetherian, h is proper with finite fibres and now $h_*\mathcal{O}_Z \cong \mathcal{O}_Y$. We claim that under these hypotheses, h is an isomorphism, which will prove Corollary 6.5. First of all, h is surjective: in fact h proper implies $h(Z)$ closed and if $h(Z) \subsetneqq Y$, then $h_*\mathcal{O}_Z$ would be annihilated by some power of the ideal of $h(Z)$, hence would not be isomorphic to $\mathcal{O}_Y$. Secondly, h is injective: if $h^{-1}(y)$ consisted in more than one point, we argue as in the proof that U3 $\Longrightarrow \widetilde{\text{U5}}$ and find a non-trivial idempotent in

$$\varprojlim_{\nu}(h_*\mathcal{O}_Z)_y/\mathfrak{m}_y^\nu \cdot (h_*\mathcal{O}_Z)_y.$$

But since $h_*\mathcal{O}_Z \cong \mathcal{O}_Y$, this is just the completion $\widehat{\mathcal{O}}_{y,Y}$ which is a local ring. The only idempotent in local rings are 0 and 1 so this is a contradiction. Thus h is bijective and closed, hence it is a homeomorphism. Since $h_*\mathcal{O}_Z \cong \mathcal{O}_Y$, h even sets up an isomorphism of the ringed space $(Z, \mathcal{O}_Z)$ with $(Y, \mathcal{O}_Y)$, i.e., $Z \cong Y$ as schemes. □

7. Multiplicities following Weil

We can generalize to the case of schemes the concept of multiplicity of a point for a finite morphism introduced for complex varieties by topological means in Part I [**76**, (3.12), (4.19)]:

Definition 7.1 (à la Weil). Let X be a noetherian integral scheme, $x \in X$ a formally unibranch point. Let $f\colon Y \to X$ be a morphism of finite type and let y be an isolated point of $f^{-1}(x)$. Then we define $\operatorname{mult}_y(f)$ as follows: Let $R = \widehat{\mathcal{O}}_{x,X}/\sqrt{(0)}$: By assumption this is an integral domain. Let $K =$ quotient field of R. Form the fibre product:

$$\begin{array}{ccc} Y & \longleftarrow & Y' \\ {\scriptstyle f}\downarrow & & \downarrow \\ X & \longleftarrow & \operatorname{Spec} R \end{array}$$

Let $y' \in Y'$ be the unique point over y. By Hensel's lemma (Lemma IV.6.1):

$$\begin{aligned} Y' &= Y_1' \cup Y_2' \qquad \text{(disjoint)} \\ Y_1' &= \operatorname{Spec} \mathcal{O}_{y',Y'}, \text{ being finite over } \operatorname{Spec} R. \end{aligned}$$

Define

$$\operatorname{mult}_y f = \dim_K \left(\mathcal{O}_{y',Y'} \otimes_R K\right).$$

If we write down all the schemes that this interesting definition suggests, we get the diagram in Figure V.4 which needs to be pondered (we let $N = \sqrt{(0)}$ in $\widehat{\mathcal{O}}_{x,X}$): This shows that to get $\operatorname{mult}_y f$, we take the generic fibre of f, extend it to the bigger ground field $K \supset \mathbb{R}(X)$, split this K-scheme into two disjoint pieces in some sense by specializing from η_X to x, and then measure the size of one of these pieces!

A few comments on this definition:

7.3. $[\Bbbk(y) : \Bbbk(x)]_s$ *divides* $\operatorname{mult}_y f$, *hence we write*

$$\operatorname{mult}_y(f) = [\Bbbk(y) : \Bbbk(x)]_s \cdot \operatorname{mult}_y^\circ(f).$$

Proof. Let $L \subset \Bbbk(y)$ be the subfield of elements separable over $\Bbbk(x)$ and let $\widetilde{\mathcal{O}}$ be the finite étale extension of $\widehat{\mathcal{O}}_{x,X}$ with residue field L, as in Corollary IV.6.3 (see also §3 of the present chapter). Then by Corollary IV.6.3, $\mathcal{O}_{y',Y'}$ is an $\widetilde{\mathcal{O}}$-algebra, hence if $\widetilde{K}$ is the quotient field of

(7.2)

$$
\begin{tikzcd}
 & \operatorname{Spec}\widehat{\mathcal{O}}_{y,Y} \arrow[dl] \arrow[ddd] & \operatorname{Spec}\left(\widehat{\mathcal{O}}_{y,Y}/N\cdot\widehat{\mathcal{O}}_{y,Y}\right) \arrow[l, hook'] \arrow[d, "\approx"'] & & \\
 & & \operatorname{Spec}(\mathcal{O}_{y',Y'})\cup Y_2' \arrow[d, equal] & & \\
Y \arrow[d, "f"'] & & Y\times_X\operatorname{Spec}R \arrow[ll] \arrow[d] & & \operatorname{Spec}(\mathcal{O}_{y',Y'}\otimes_R K)\cup\cdots \\
X & \operatorname{Spec}\widehat{\mathcal{O}}_{x,X} \arrow[l] & \operatorname{Spec}\widehat{\mathcal{O}}_{x,X}/N=R \arrow[l, hook'] & Y\times_X\operatorname{Spec}K \arrow[ul] \arrow[d] \arrow[ur, equal] \arrow[dr, equal] & \\
 & \operatorname{Spec}\mathbb{R}(X)=\{\eta_X\} \arrow[ul] & & \operatorname{Spec}K \arrow[ll] \arrow[ul] & f^{-1}(\eta_X)\times_{\operatorname{Spec}\mathbb{R}(X)}\operatorname{Spec}K
\end{tikzcd}
$$

FIGURE V.4

$\widetilde{\mathcal{O}}$, $\mathcal{O}_{y',Y'}\otimes_R K$ is a vector space over $\widetilde{K}$. Therefore $[\widetilde{K}:K]|\operatorname{mult}_y f$. But

$$
\begin{aligned}
[\widetilde{K}:K] &= \text{rank of } \widetilde{\mathcal{O}} \text{ as free } \widehat{\mathcal{O}}_{x,X}\text{-module}\\
&= [L:\mathbb{k}(x)]\\
&= [\mathbb{k}(y):\mathbb{k}(x)]_s.
\end{aligned}
$$

$\square$

7.4. $\operatorname{mult}_y f\ge 1$ *if and only if* Y *has a component* Y_1 *through* y *dominating* X *(i.e.,* $\eta_{Y_1}\mapsto\eta_X$*).*

PROOF. If Y has no such component, there will be some non-zero $a\in\mathcal{O}_{x,X}$ such that $f^*a=0$ in $\mathcal{O}_{y,Y}$. Therefore $f^*a=0$ in $\mathcal{O}_{y',Y'}$ and $\mathcal{O}_{y',Y'}\otimes_R K=(0)$. To prove the converse, use generic flatness (Theorem IV.4.8): there is a non-zero $a\in\mathcal{O}_{x,X}$ such that the localization $(\mathcal{O}_{y,Y})_a$ is flat over $(\mathcal{O}_{x,X})_a$. Making the base change, it follows that Y_1' is flat over $\operatorname{Spec}R$ over the open set R_a. But then

$$
\begin{aligned}
\operatorname{mult}_y f=0 &\Longrightarrow \mathcal{O}_{y',Y'}\otimes_R R_a=(0)\\
&\Longrightarrow a^l=0 \text{ in } \mathcal{O}_{y',Y'} \text{ for some } l\\
&\Longrightarrow a^l=0 \text{ in } \widehat{\mathcal{O}}_{y,Y}/N\cdot\widehat{\mathcal{O}}_{y,Y} \text{ (see diagram in Figure V.4)}\\
&\Longrightarrow a^m=0 \text{ in } \widehat{\mathcal{O}}_{y,Y} \text{ for some } m\\
&\Longrightarrow a^m=0 \text{ in } \mathcal{O}_{y,Y}\\
&\Longrightarrow \text{no component of } Y \text{ through } y \text{ dominates } X.
\end{aligned}
$$

$\square$

7.5. *Assume* X *is formally normal at* x *and that all associated points of* Y *lie over* η_X*. Then*

$$\operatorname{mult}_y^\circ f=1 \textit{ if and only if } f \textit{ is étale at } y.$$

PROOF. If f is étale, then f is flat, hence $Y_1'\to\operatorname{Spec}R$ is flat, hence $\mathcal{O}_{y',Y'}$ is a free R-module of some rank n. But on the one hand,

$$n=\dim_K\mathcal{O}_{y',Y'}\otimes_R K=\operatorname{mult}_y f$$

and on the other hand:

$$n=\dim_{\mathbb{k}(x)}\mathcal{O}_{y',Y'}\otimes_R\mathbb{k}(x)=\dim_{\mathbb{k}(x)}\mathcal{O}_{y,f^{-1}(x)}.$$

But $f^{-1}(x)$ is zero-dimensional and reduced at y because f is étale, hence $\mathcal{O}_{y,f^{-1}(x)} = \Bbbk(y)$, hence $n = [\Bbbk(y) : \Bbbk(x)]$. But f étale also implies $\Bbbk(y)$ separable over $\Bbbk(x)$, so $\mathrm{mult}_y^\circ f = 1$. Conversely, if $\mathrm{mult}_y^\circ f = 1$, then using the notation of the proof of (7.3), $\mathcal{O}_{y',Y'} \otimes_R K \cong \widetilde{K}$. Now $\widetilde{\mathcal{O}}$ is étale over $\widehat{\mathcal{O}}_{x,X}$ which we have assumed is an integrally closed domain. Therefore $\widetilde{\mathcal{O}}$ is an integrally closed domain. But if $\mathfrak{a} = \{a \in \mathcal{O}_{y',Y'} \mid a \cdot b = 0 \text{ for some } b \in R,\ b \neq 0\}$, then $\mathcal{O}_{y',Y'}/\mathfrak{a}$ is an $\widetilde{\mathcal{O}}$-algebra, integrally dependent on $\widetilde{\mathcal{O}}$ and contained in $\mathcal{O}_{y',Y'} \otimes_R K = \widetilde{K}$. Thus $\mathcal{O}_{y',Y'}/\mathfrak{a} = \widetilde{\mathcal{O}}$. Using generic flatness of f as in (7.4), we find $a \in \mathcal{O}_{x,X}$ such that $(\mathcal{O}_{y',Y'})_a$ is flat over R_a. Since this means $(\mathcal{O}_{y',Y'})_a$ is torsion-free as R_a-module, $\mathfrak{a}_a = (0)$ or $a^l \cdot \mathfrak{a} = (0)$, some l. But now by hypothesis $a \neq 0$ at any associated point of Y so

$$\mathcal{O}_{y,Y} \xrightarrow{a} \mathcal{O}_{y,Y}$$

is injective. Since $Y \times_X \operatorname{Spec} R$ is flat over Y,

$$\mathcal{O}_{y',Y'} \xrightarrow{a} \mathcal{O}_{y',Y'}$$

is injective too. Therefore $\mathfrak{a} = (0)$, and $\mathcal{O}_{y',Y'} \cong \widetilde{\mathcal{O}}$. Therefore

$$\begin{aligned}(\Omega_{Y/X})_y \otimes_{\mathcal{O}_{y,Y}} \Bbbk(y) &\cong (\Omega_{Y_1'/\operatorname{Spec} R}) \otimes_{\mathcal{O}_{y',Y'}} \Bbbk(y) \\ &\cong (\Omega_{\operatorname{Spec}\widetilde{\mathcal{O}}/\operatorname{Spec} R}) \otimes_{\widetilde{\mathcal{O}}} L \\ &= (0)\end{aligned}$$

so Y is étale over X at y by Criterion 4.1^+. □

The most famous result about multiplicities is the formula $n = \sum e_i f_i$ (cf. Zariski-Samuel [**109**, vol. I, p. 287]). In our language, the result is:

THEOREM 7.6. *Let $f\colon Y \to X$ be a finite surjective morphism between integral schemes, and assume X formally irreducible at x. Then if $f^{-1}(x) = \{y_1, \ldots, y_t\}$:*

$$[\mathbb{R}(Y) : \mathbb{R}(X)] = \sum_{i=1}^{t} \mathrm{mult}_y^\circ(f) \cdot [\Bbbk(y) : \Bbbk(x)]_s.$$

PROOF. This follows immediately from the big diagram in Figure V.4: in fact,

$$Y \times_X \operatorname{Spec} R = \bigcup_{i=1}^{t} Y_i' \qquad \text{(disjoint)}$$

where Y_i' has one closed point y_i' lying over $y_i \in Y$. Then

$$\operatorname{Spec}(\mathbb{R}(Y) \otimes_{\mathbb{R}(X)} K) = f^{-1}(\eta_X) \times_{\operatorname{Spec}\mathbb{R}(X)} \operatorname{Spec} K = \bigcup_{i=1}^{t} \operatorname{Spec}(\mathcal{O}_{y_i',Y_i'} \otimes_R K),$$

hence

$$\mathbb{R}(Y) \otimes_{\mathbb{R}(X)} K \cong \bigoplus_{i=1}^{t} \left[\mathcal{O}_{y_i',Y_i'} \otimes_R K\right].$$

Therefore

$$\begin{aligned}[\mathbb{R}(Y) : \mathbb{R}(X)] &= \dim_K \mathbb{R}(Y) \otimes_{\mathbb{R}(X)} K \\ &= \sum_{i=1}^{t} \dim_K \left(\mathcal{O}_{y_i',Y_i'} \otimes_R K\right) \\ &= \sum_{i=1}^{t} \mathrm{mult}_{y_i} f.\end{aligned}$$

□

$\boxed{\textbf{Exercise}}$.

- When x is a regular point of X, use $\boxed{\text{Exercise 1, §4A}}$ with $R = \widehat{\mathcal{O}}_{x,X}$ to prove that
$$\mathrm{mult}_y(f) = e(\mathfrak{m}_{x,X} \cdot \mathcal{O}_{y,Y}; \mathcal{O}_{y,Y}).$$
Use this to give a second proof of the equality of the "results" of Part I and Part II in case X is non-singular at x.
- In the definition of $\mathrm{mult}_y(f)$, say $\widetilde{X}$ is any intermediate integral scheme:
$$X \longleftarrow \widetilde{X} \longleftarrow \mathrm{Spec}\, R$$
$$\eta_X \longleftarrow\!\shortmid\, \eta_{\widetilde{X}} \longleftarrow\!\shortmid\, [(0)]$$
such that the decomposition of Y' is induced by a decomposition already over $\widetilde{X}$:
$$Y \times_X \widetilde{X} = \widetilde{Y}_1 \cup \widetilde{Y}_2.$$
Let $\widetilde{y} =$ image of y' in $\widetilde{Y}_1$, $\widetilde{x} =$ image of x' in $\widetilde{X}$ and $\widetilde{K} = \mathbb{R}(\widetilde{X})$. Then show
$$\mathrm{mult}_y\, f = \dim_{\widetilde{K}} \left(\mathcal{O}_{\widetilde{y},\widetilde{Y}_1} \otimes_{\mathcal{O}_{\widetilde{x},\widetilde{X}}} \widetilde{K} \right).$$
Now if X is of finite type over $\mathbb{C}$, take
$$\widetilde{X} = \mathrm{Spec}\, \mathcal{O}_{x,X,\mathrm{an}}.$$
Using the fact that $\mathcal{O}_{y,Y,\mathrm{an}}$ is a finite $\mathcal{O}_{x,X,\mathrm{an}}$-module, show that $Y \times_X \widetilde{X}$ as above decomposes and that $\widetilde{Y}_1 = \mathrm{Spec}\, \mathcal{O}_{y,Y,\mathrm{an}}$. Deduce that the multiplicity of (7.1) is equal to the multiplicity of Part I [**76**, (4.19)].

CHAPTER VI

Group schemes, etc.

1. Group schemes $\boxed{\textbf{Split and modified}}$

Definition 1.1. Let $f\colon G \to S$ be an S-scheme. Then G is a group scheme over S if we are given three S-morphisms:

$$\begin{array}{ll} \mu\colon G \times_S G \longrightarrow G & (\text{“multiplication”}) \\ \iota\colon G \longrightarrow G & (\text{“inverse”}) \\ \epsilon\colon S \longrightarrow G & (\text{“identity”}) \end{array}$$

such that the following diagrams commute:

a) (“associativity”)

$$\begin{tikzcd}
G \times_S (G \times_S G) \arrow[r, "1_G \times \mu"] \arrow[d, equal, "\wr"'] & G \times_S G \arrow[dr, "\mu"] & \\
& & G \\
(G \times_S G) \times_S G \arrow[r, "\mu \times 1_G"] & G \times_S G \arrow[ur, "\mu"'] &
\end{tikzcd}$$

b) (“left and right identity laws”)

$$\begin{tikzcd}
G \times_S S \arrow[r, "1_G \times \epsilon"] \arrow[d, equal, "\wr"'] & G \times_S G \arrow[dr, "\mu"] & \\
G \arrow[rr, "1_G"] & & G \\
S \times_S G \arrow[r, "\epsilon \times 1_G"'] \arrow[u, equal, "\wr"] & G \times_S G \arrow[ur, "\mu"'] &
\end{tikzcd}$$

c) (“left and right inverse laws”)

$$\begin{tikzcd}
& G \times_S G \arrow[r, "1_G \times \iota"] & G \times_S G \arrow[dr, "\mu"] & \\
G \arrow[ur, "\Delta"] \arrow[rr, "f"] \arrow[dr, "\Delta"'] & & S \arrow[r, "\epsilon"] & G \\
& G \times_S G \arrow[r, "\iota \times 1_G"] & G \times_S G \arrow[ur, "\mu"'] &
\end{tikzcd}$$

To relate this to the usual idea of a group, let $p\colon T \to S$ be any scheme over S and consider $\mathrm{Hom}_S(T, G)$, the set of T-valued points of G over S! Then:

a$'$) via μ, get a law of composition in $\mathrm{Hom}_S(T, G)$:
$\forall f, g \in \mathrm{Hom}_S(T, G)$, define $f \cdot g$ to be the composition:

$$T \xrightarrow{(f,g)} G \times_S G \xrightarrow{\mu} G$$

(this is associative by virtue of (a)),

b$'$) via ϵ, get a distinguished element $\epsilon \circ p \in \mathrm{Hom}_S(T, G)$ which is a two-sided identity for this law of composition by virtue of (b), c′) via ι, get a map $f \mapsto f^{-1}$ of $\mathrm{Hom}_S(T,G)$, $f^{-1} = \iota \circ f$, which is a two-sided inverse for this law of composition by virtue of (c).

Summarizing, (μ, ϵ, ι) make $\mathrm{Hom}_S(T,G)$ into an ordinary group for every T over S: For instance, if $S = \operatorname{Spec} k$, then the set of k-rational points of G is a group, and if k is algebraically closed and G is of finite type over k, this means that the set of closed points of G is a group. If you think about it, this is really what one should expect: for instance suppose you want to consider $\mathbb{A}^n_k$ as a group via vector addition. If $\mathbb{A}^n_k = \operatorname{Spec} k[X_1, \ldots, X_n]$, then for any two k-valued points P', P'' their sum is defined by:

$$X_i(P' + P'') = X_i(P') + X_i(P'');$$

thus if $\mu(P', P'') = P' + P''$, then the pull-back of the function X_i is computed via:

$$\begin{aligned}
\mu^*(X_i) &= X_i(\mu(P', P'')) \\
&= X_i(P') + X_i(P'') \\
&= (X_i \circ p_1)(P', P'') + (X_i \circ p_2)(P', P'') \\
&= (p_1^* X_i + p_2^* X_i)(P', P'').
\end{aligned}$$

Thus the law of composition:

$$\begin{tikzcd}
\mathbb{A}^n_k \times_{\operatorname{Spec} k} \mathbb{A}^n_k \arrow[r, "\mu"] \arrow[d, equal] & \mathbb{A}^n_k \arrow[d, equal] \\
\operatorname{Spec} k[p_1^* X_1, \ldots, p_1^* X_n, p_2^* X_1, \ldots, p_2^* X_n] & \operatorname{Spec} k[X_1, \ldots, X_n]
\end{tikzcd}$$

is defined by $\mu^* X_i = p_1^* X_i + p_2^* X_i$. Similarly, define ι and ϵ via $\iota^* X_i = -X_i$ and $\epsilon^* X_i = 0$. Now if $\eta \in \mathbb{A}^n_k$ is the generic point, then to try to add η to itself, one would choose a point $\zeta \in \mathbb{A}^n_k \times \mathbb{A}^n_k$ such that $p_1(\zeta) = p_2(\zeta) = \eta$ and define $\eta + \eta$ to be $\mu(\zeta)$. But, taking $n = 1$ for instance, then one could take

$$\zeta = \begin{cases} \text{generic point of } \mathbb{A}^1_k \times \mathbb{A}^1_k \\ \text{or} \\ \text{generic point of line } p_1^* X = -p_2^* X + a, \qquad (a \in k). \end{cases}$$

In the first case, one sees that $\mu(\zeta) =$ generic point of $\mathbb{A}^1_k$, and in the second case, $\mu(\zeta) =$ (the point $X = a$)! The moral is that $\eta + \eta$ is not well-defined.

Replace the following by more general accounts on group schemes?

Another standard group scheme is: define

$$\begin{aligned}
\mathrm{GL}_{n,k} &= \operatorname{Spec}\left(k[X_{11}, \ldots, X_{nn}][\frac{1}{\det(X_{ij})}] \right) \\
\mu^*(X_{ij}) &= \sum_{k=1}^{n} p_1^* X_{ik} \cdot p_2^* X_{kj} \\
\epsilon^*(X_{ij}) &= \delta_{ij} \\
\iota^*(X_{ij}) &= (-1)^{i+j} \cdot ((j,i)\text{-th minor of } (X_{ij})) \, \frac{1}{\det(X_{i'j'})}.
\end{aligned}$$

More elegantly, all the group schemes $\mathrm{GL}_{n,k}$ (resp. $\mathbb{A}^n_k$) over various base schemes $\operatorname{Spec} k$ are "induced" from one single group scheme $\mathrm{GL}_{n,\mathbb{Z}}$ (resp. $\mathbb{A}^n_{\mathbb{Z}}$) over $\operatorname{Spec} \mathbb{Z}$. One checks readily that if $f \colon G \to S$ is a group scheme over S, and $p \colon T \to S$

is any morphism, then $p_2\colon G\times_S T\to T$ is a group scheme over T in a canonical way. And one can define "universal" general linear and affine group scheme by:

$$\mathrm{GL}_{n,\mathbb{Z}} = \mathrm{Spec}\left(\mathbb{Z}[X_{11},\ldots,X_{nn}][\frac{1}{\det(X_{ij})}]\right)$$
$$\mathbb{A}^n_{\mathbb{Z}} = \mathrm{Spec}\,\mathbb{Z}[X_1,\ldots,X_n]$$
$$\mu^*,\epsilon^*,\iota^* \text{ given by the same formulae as before.}$$

(Added in publication)

In terms of the way we defined S-schemes as representable functors $(\mathrm{Sch}/S)^\circ\to(\mathrm{Sets})$ in §I.8, we can formulate group schemes over S as follows:

Denote by (Groups) the category consisting of groups and homomorphisms of groups. Then group schemes G over S are exactly those S-schemes such that the functors h_G they represent are *group functors*, that is, factor through the functor (Groups) $\to$ (Sets) (that sends a group to its underlying set and a homomorphism to the underlying map)

$$h_G\colon (\mathrm{Sch}/S)^\circ \longrightarrow (\mathrm{Groups}) \longrightarrow (\mathrm{Sets}).$$

Here are some examples:

EXAMPLE 1.2. (cf. Example I.8.4) $G_{a,S} = \mathcal{S}pec_S(\mathcal{O}_S[T])$ is a commutative group scheme over S with the additive group

$$\mathrm{Hom}_S(Z,G_{a,S}) = \Gamma(Z,\mathcal{O}_Z) \quad \text{for} \quad Z\in(\mathrm{Sch}/S)$$

and with an obvious homomorphism $f^*\colon\Gamma(Z,\mathcal{O}_Z)\to\Gamma(Z',\mathcal{O}_{Z'})$ for every S-morphism $f\colon Z'\to Z$.

More generally, we have:

EXAMPLE 1.3. (cf. Example I.8.5) Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module on S. Then the relatively affine S-scheme

$$\mathcal{S}pec_S(\mathrm{Symm}(\mathcal{F})),$$

where $\mathrm{Symm}(\mathcal{F})$ is the symmetric algebra of $\mathcal{F}$ over $\mathcal{O}_S$, represents the additive group functor G defined as follows:

$$G(Z) = \mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{O}_Z\otimes_{\mathcal{O}_S}\mathcal{F},\mathcal{O}_Z) \quad \text{for } Z\in(\mathrm{Sch}/S)$$

with the obvious homomorphism

$$G(f)=f^*:\mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{F}_Z,\mathcal{O}_Z)\to\mathrm{Hom}_{\mathcal{O}_{Z'}}(\mathcal{O}_{Z'}\otimes_{\mathcal{O}_S}\mathcal{F},\mathcal{O}_{Z'})=\mathrm{Hom}_{\mathcal{O}_{Z'}}(f^*(\mathcal{O}_Z\otimes_{\mathcal{O}_S}\mathcal{F}),f^*\mathcal{O}_Z)$$

for $f\in\mathrm{Hom}_S(Z',Z)$.

Similarly to Example 1.2, we have:

EXAMPLE 1.4. (cf. Example I.8.6) $G_{m,S} = \mathcal{S}pec_S(\mathcal{O}_S[T,T^{-1}])$ is a commutative group scheme over S with the multiplicative group

$$\mathrm{Hom}_S(Z,G_{m,S}) = \Gamma(Z,\mathcal{O}_Z)^* \quad \text{for } Z\in(\mathrm{Sch}/S),$$

where the asterisk denotes the set of invertible elements, with the obvious homomorphism

$$f^*\colon\Gamma(Z,\mathcal{O}_Z)^*\to\Gamma(Z,\mathcal{O}_{Z'})^*$$

for each $f\in\mathrm{Hom}_S(Z',Z)$.

More generally:

EXAMPLE 1.5. (cf. Example I.8.7) Let n be a positive integer.

$$\mathrm{GL}_{n,S} = \mathcal{S}pec_S\left(\mathcal{O}_S\left[T_{11},\ldots,T_{nn},\frac{1}{\det(T)}\right]\right),$$

where $T = (T_{ij})$ is the $n \times n$-matrix with indeterminates T_{ij} as entries, is a relatively affine S-group scheme representing the multiplicative group functor

$$\mathrm{Hom}_S(Z, \mathrm{GL}_{n,S}) = \mathrm{GL}_n(\Gamma(Z, \mathcal{O}_Z)), \qquad \text{for } Z \in (\mathrm{Sch}/S),$$

the set of $n \times n$-matrices with entries in $\Gamma(Z, \mathcal{O}_Z)$, with obvious homomorphisms corresponding to S-morphisms. Clearly, $G_{m,S} = \mathrm{GL}_{1,S}$.

Even more generally, we have (cf. EGA [**1**, Chapter I, revised, Proposition (9.6.4)]):

EXAMPLE 1.6. (cf. Example I.8.8) Let $\mathcal{E}$ be a locally free $\mathcal{O}_S$-module of finite rank (cf. Definition 5.3). The group multiplicative functor G defined by

$$G(Z) = \mathrm{Aut}_{\mathcal{O}_Z}(\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{E}) \quad \text{for } Z \in (\mathrm{Sch}/S)$$

with obvious homomorphisms corresponding to S-morphisms is represented by a relatively affine group S-scheme $\mathrm{GL}(\mathcal{E})$. Example 1.5 is a special case with

$$\mathrm{GL}_{n,S} = \mathrm{GL}(\mathcal{O}_S^{\oplus n}).$$

EXAMPLE 1.7. For a positive integer n and a scheme S, the "multiplicative group of n-th roots of unity" $\mu_{n,S}$ is the multiplicative group scheme over S defined by

$$\mu_{n,S}(Z) = \{\zeta \in \Gamma(Z, \mathcal{O}_Z)^* \mid \zeta^n = 1\}, \qquad \forall Z \in (\mathrm{Sch}/S)$$

with obvious homomorphisms corresponding to S-morphisms $Z' \to Z$. It is represented by the S-scheme

$$\mu_{n,S} = \mathcal{S}pec_S(\mathcal{O}_S[t]/(t^n - 1)).$$

EXAMPLE 1.8. Let S be a scheme of prime characteristic p (that is, $p = 0$ in $\mathcal{O}_S$, e.g., $S = \mathrm{Spec}(k)$ for a field k of characteristic $p > 0$). $\alpha_{p,S}$ is an additive group scheme over S defined by

$$\alpha_{p,S}(Z) = \{\xi \in \Gamma(Z, \mathcal{O}_Z) \mid \xi^p = 0\}, \qquad \forall Z \in (\mathrm{Sch}/S)$$

with obvious homomorphisms corresponding to S-morphisms $Z' \to Z$. It is represented by the S-scheme

$$\alpha_{p,S} = \mathcal{S}pec_S(\mathcal{O}_S[t]/(t^p)).$$

For $\nu \geq 2$, we can define $\alpha_{p^\nu,S}$ similarly.

EXAMPLE 1.9. The relative Picard functor in Example I.8.12 is the commutative group functor

$$\mathrm{Pic}_{X/S}\colon (\mathrm{Sch}/S)^\circ \to (\mathrm{Groups})$$

defined by

$$\mathrm{Pic}_{X/S}(Z) = \mathrm{Coker}[\varphi^*\colon \mathrm{Pic}(Z) \longrightarrow \mathrm{Pic}(X \times_S Z)] \quad \text{for each } S\text{-scheme } \varphi\colon Z \to S$$

and the homomorphism $f^*\colon \mathrm{Pic}_{X/S}(Z) \to \mathrm{Pic}_{X/S}(Z')$ induced by the inverse image by each S-morphism $f\colon Z' \to Z$. The "sheafified" version of the relative Picard functor $\mathrm{Pic}_{X/S}$ when representable thus gives rise to a commutative group scheme over S called the *relative Picard scheme* of X/S. The reader is again referred to FGA [**2**, exposés 232, 236] as well as Kleiman's account on the interesting history (before and after FGA [**2**]) in FAG [**3**, Chapter 9]. See also Bosch, Lütkebohmert and Raynaud [**25**].

$S = \mathrm{Spec}(k)$, with a field k. Murre [**77**] on the criterion of representability of commutative group functors over fields.

$\mathrm{Lie}(\mathrm{Pic}_{X/k}) = H^1(X, \mathcal{O}_X)$.

Mumford [**73**]

Example 1.10. X scheme over S. $G\colon (\mathrm{Sch}/S)^\circ \to (\mathrm{Groups})$ defined by

$$G(Z) = \mathrm{Aut}_Z(X \times_S Z), \quad \text{for} \quad Z \in (\mathrm{Sch}/S)$$

and an obvious homomorphism $G(Z) \to G(Z')$ induced by the base extension by each S-morphism $f\colon Z' \to Z$. FGA [**2**, exposés 195, 221].

The *automorphism group scheme* $\mathrm{Aut}_S(X)$ over S if representable.

$X = \mathbb{P}^n_S$, $\mathrm{Aut}_S(\mathbb{P}^n_S) = \mathrm{PGL}_{n+1,S}$ (cf. Mumford [**72**, Chapter 0, §5, p.20])

$$\mathbb{P}^n_S = \mathcal{P}roj_S(\mathcal{O}_S[X_0, \dots, X_n]) = \mathbb{P}^n_{\mathbb{Z}} \times S, \qquad \mathrm{PGL}_{n+1,S} = \mathrm{PGL}_{n+1,\mathbb{Z}} \times S$$

$$\mathrm{PGL}_{n+1} = \mathrm{PGL}_{n+1,\mathbb{Z}} \text{ open subset of } \mathrm{Proj}(\mathbb{Z}[A_{00}, \dots, A_{nn}]) \text{ with } \det(A_{ij}) \neq 0.$$

Matsusaka

$S = \mathrm{Spec}(k)$ with a field k. Matsumura-Oort [**70**] on the criterion of representability of group functors over fields generalizing the commutative case dealt with by Murre [**77**].

$\mathrm{Lie}(\mathrm{Aut}_k(X)) = H^0(X, \Theta_X)$ the tangent space of $\mathrm{Aut}_k(X)$ at id_X.

Theorem 1.11 (Cartier [**27**]). *Any group scheme G of finite type over a field k of characteristic 0 is smooth, hence, in particular, reduced.*

Proof. We reproduce the proof in [**74**, Chapter III, §11, Theorem, p.101]. Denote by $e \in G$ the image of the identity morphism $\epsilon\colon \mathrm{Spec}(k) \to G$. Obviously e is a k-rational point, that is, $\Bbbk(e) = k$. For simplicity, we denote

$$\mathcal{O} = \mathcal{O}_{e,G}, \qquad \mathfrak{m} = \mathfrak{m}_{e,G}.$$

By what we saw in §V.4, it suffices to show that $\mathcal{O}$ is a regular local ring, since the argument works for the base extension $G \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,\overline{k}$ to the algebraic closure $\overline{k}$ and the translation by $\mathrm{Spec}(\overline{k})$-valued points of G are isomorphisms sending e to the other closed points of $G \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,\overline{k}$.

Choose $x_1, \dots, x_n \in \mathfrak{m}$ so that their images form a k-basis of $\mathfrak{m}/\mathfrak{m}^2$. Thus we obtain a continuous surjective k-algebra homomorphism from the formal power series ring to the completion of $\mathcal{O}$:

$$\alpha\colon k[[t_1, \dots, t_n]] \longrightarrow \widehat{\mathcal{O}}, \qquad \alpha(t_i) = x_i.$$

As we show immediately after this proof (cf. Proposition-Definition 1.12), the map

$$\mathrm{Der}_k(\mathcal{O}) \longrightarrow \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k) = T_{e,G}$$

sending a local vector field $D \in \mathrm{Der}_k(\mathcal{O})$ at e to the tangent vector of G at e sending $f \in \mathfrak{m}$ to $(Df)(e)$ is surjective. Hence we can choose $D_1, \dots, D_n \in \mathrm{Der}_k(\mathcal{O})$ such that

$$D_i(x_j) = \delta_{ij}.$$

The D_i's obviously induce derivations of the completion $\widehat{\mathcal{O}}$ so that we get the Taylor expansion map (k is of characteristic 0!)

$$\begin{aligned} \beta\colon \widehat{\mathcal{O}} &\longrightarrow k[[t_1, \dots, t_n]] \\ f &\longmapsto \sum_{\substack{\nu_1, \dots, \nu_n \\ \nu_i \geq 0}} \frac{(D_1^{\nu_1} \cdots D_n^{\nu_n} f)(e)}{\nu_1! \cdots \nu_n!} t_1^{\nu_1} \cdots t_n^{\nu_n}, \end{aligned}$$

which is a continuous k-algebra homomorphism. β is surjective since $\beta(x_i) \equiv t_i \mod (t_1, \dots, t_n)^2$. Consequently, $\beta \circ \alpha$ is a surjective k-algebra homomorphism of $k[[t_1, \dots, t_n]]$ onto itself, hence is an automorphism. Thus α is injective as well so that

$$\alpha\colon k[[t_1, \dots, t_n]] \xrightarrow{\sim} \widehat{\mathcal{O}},$$

and $\widehat{\mathcal{O}}$ is regular, hence so is $\mathcal{O}$. □

In general, let G be a scheme over a field k, and e a k-rational point of G. Denote by $\operatorname{Der}_k(\mathcal{O}_G)$ the space of global k-derivations of $\mathcal{O}_G$ into itself, that is, the space of *vector fields* on G.

Introduce the k-algebra of "dual numbers"

$$\Lambda = k[\delta]/(\delta^2) = k \oplus k\delta.$$

Then the vector fields $D \in \operatorname{Der}_k(\mathcal{O}_G)$ are in one-to-one correspondence with the Λ-algebra automorphisms

$$\widetilde{D}\colon \mathcal{O}_G \otimes_k \Lambda \xrightarrow{\sim} \mathcal{O}_G \otimes_k \Lambda$$

inducing the identity automorphism modulo δ by

$$\widetilde{D}(a + b\delta) = a + ((Da) + b)\delta, \qquad a, b \in \mathcal{O}_G.$$

Likewise, the tangent vectors $t \in \operatorname{Der}_k(\mathcal{O}_{e,G}, k)$ of G at e are in one-to-one correspondence with the Λ-algebra homomorphisms

$$\widetilde{t}\colon \mathcal{O}_{e,G} \otimes_k \Lambda \longrightarrow \Lambda$$

inducing the canonical surjection $\mathcal{O}_{e,G} \to k \mod \mathfrak{m}_{e,G}$ by

$$\widetilde{t}(a + b\delta) = a(e) + (t(a) + b(e))\delta, \qquad a, b \in \mathcal{O}_{e,G}.$$

Proposition-Definition 1.12. *Let G be a group scheme over a field k. A vector field $D \in \operatorname{Der}_k(\mathcal{O}_G)$ is said to be* left invariant *if*

$$\begin{CD}
\mathcal{O}_G @>{D}>> \mathcal{O}_G \\
@V{\mu^*}VV @VV{\mu^*}V \\
\mathcal{O}_{G \times_k G} @>{1 \otimes_k D}>> \mathcal{O}_{G \times_k G}
\end{CD}$$

is a commutative diagram. The k-vector space $\operatorname{Lie}(G)$ *of left invariant vector fields on G is called the* Lie algebra *of G. We have a natural isomorphism of k-vector spaces*

$$\operatorname{Lie}(G) \xrightarrow{\sim} T_{e,G}.$$

Proof. Let

$$\widetilde{D}\colon \mathcal{O}_G \otimes_k \Lambda \xrightarrow{\sim} \mathcal{O}_G \otimes_k \Lambda$$

be the the Λ-algebra automorphism corresponding to a vector field $D \in \operatorname{Der}_k(\mathcal{O}_G)$. Then the left invariance of D is equivalent to the commutativity of the following diagram

$$(*) \qquad \begin{CD}
G \times_k G \times_k \operatorname{Spec}\Lambda @>{1_G \times \widetilde{D}}>> G \times_k G \times_k \operatorname{Spec}\Lambda \\
@V{\mu \times 1_\Lambda}VV @VV{\mu \times 1_\Lambda}V \\
G \times_k \operatorname{Spec}\Lambda @>>{\widetilde{D}}> G \times_k \operatorname{Spec}\Lambda,
\end{CD}$$

where we use the same symbol $\widetilde{D}$ for the $(\operatorname{Spec}\Lambda)$-automorphism $G \times_k \operatorname{Spec}\Lambda \xrightarrow{\sim} G \times_k \operatorname{Spec}\Lambda$ induced by $\widetilde{D}\colon \mathcal{O}_G \otimes_k \Lambda \xrightarrow{\sim} \mathcal{O}_G \otimes_k \Lambda$, etc.

If we denote

$$D' = p_1 \circ \widetilde{D}\colon G \times_k \operatorname{Spec}\Lambda \xrightarrow{\widetilde{D}} G \times_k \operatorname{Spec}\Lambda \xrightarrow{p_1} G,$$

then the commutativity of the diagram $(*)$ is equivalent to

$$D'(x \cdot y, l) = x \cdot D'(y, l), \qquad \forall x, y \in G(Z), \quad \forall l \in (\operatorname{Spec}\Lambda)(Z) \quad (Z\text{-valued points})$$

for any k-scheme Z, or equivalently,

$$D'(x, l) = x \cdot D'(\epsilon, l), \qquad \forall x \in G(Z), \quad \forall l \in (\operatorname{Spec}\Lambda)(Z)$$

for any k-scheme Z. If we denote

$$\widetilde{t} = p_1 \circ \widetilde{D} \circ (\epsilon, 1_\Lambda)\colon \operatorname{Spec}\Lambda \longrightarrow G \times_k \operatorname{Spec}\Lambda \xrightarrow{\widetilde{D}} G \times_k \operatorname{Spec}\Lambda \xrightarrow{p_1} G,$$

then $\widetilde{D}$ is the right multiplication by $\widetilde{t} \in G(\operatorname{Spec}\Lambda)$. Thus the Λ-valued points $\widetilde{t}$ of G are in one-to-one correspondence with the automorphisms $\widetilde{D}$ of $G \times_k \operatorname{Spec}\Lambda$ over $\operatorname{Spec}\Lambda$ such that the diagram $(*)$ commutes by the correspondence

$$p_1 \circ \widetilde{D} \circ (\epsilon, 1_\Lambda) = \widetilde{t}.$$

Thus the left invariant vector fields $D \in \operatorname{Der}_k(\mathcal{O}_G)$ are in one-to-one correspondence with the tangent vectors

$$t \in \operatorname{Der}_k(\mathcal{O}_{e,G}, k) = T_{e,G}.$$

□

Remark. When $S = \operatorname{Spec}(k)$ with a field k of characteristic $p > 0$, the additive group scheme

$$\alpha_{p^\nu,S} = \operatorname{Spec}(k[t]/(t^{p^\nu}))$$

is not reduced with only one point! If n is divisible by p, the "multiplicative group of roots of unity" $\mu_{n,S}$ is not reduced either. Indeed, if $n = p^\nu \times n'$ with n' not divisible by p, then

$$\mu_{n,S} = \operatorname{Spec}(k[t]/(t^n - 1)) = \operatorname{Spec}(k[t]/(t^{n'} - 1)^{p^\nu}).$$

Definition 1.13. An S-morphism $f\colon H \to G$ is a *homomorphism* of group schemes over S if the map

$$f(Z)\colon H(Z) \longrightarrow G(Z), \qquad \forall Z \in (\mathrm{Sch}/S)$$

is a group homomorphism. The *kernel* $\operatorname{Ker}(f)$ is then defined as the group functor

$$\operatorname{Ker}(f)(Z) = \operatorname{Ker}(f(Z)\colon H(Z) \longrightarrow G(Z)), \qquad \forall Z \in (\mathrm{Sch}/S)$$

with obvious homomorphisms corresponding to S-morphisms $Z' \to Z$.

Obviously, $\operatorname{Ker}(f)$ is a group scheme over S represented by the fibre product

$$\begin{tikzcd} \operatorname{Ker}(f) \arrow[r] \arrow[d] & S \arrow[d, "\epsilon_G"] \\ H \arrow[r, "f"'] & G, \end{tikzcd}$$

where ϵ_G is the identity morphism for G.

Example 1.14. If G is a commutative group scheme over S with the group law written additively, the morphism $n\mathrm{id}_G$ for any positive integer defined by

$$G(Z) \ni \xi \longmapsto n\mathrm{id}_G(\xi) = n\xi = \underbrace{\xi + \cdots + \xi}_{n \text{ times}} \in G(Z), \qquad \forall Z \in (\mathrm{Sch}/S)$$

is obviously a homomorphism of group schemes over S. Very often we denote ${}_nG = \mathrm{Ker}(n\mathrm{id}_G)$. For example

$$\mu_{n,S} = {}_nG_{m,S}.$$

There is an important homomorphism peculiar to characteristic $p > 0$.

Definition 1.15. Let S be a scheme of prime characteristic p (that is, $p = 0$ in $\mathcal{O}_S$, e.g., $S = \mathrm{Spec}(k)$ with a field k of characteristic $p > 0$). As in Definition IV.3.1 denote by

$$\phi_S\colon S \longrightarrow S$$

the morphism that is set-theoretically the identity map while $\phi_S^*(a) = a^p$ for all open $U \subset S$ and for all $a \in \Gamma(U, \mathcal{O}_S)$. For any S-group scheme $\pi\colon G \to S$, we have a commutative diagram

$$\begin{tikzcd} G \arrow[r, "\phi_G"] \arrow[d, "\pi"'] & G \arrow[d, "\pi"] \\ S \arrow[r, "\phi_S"'] & S, \end{tikzcd}$$

hence a homomorphism of S-group schemes called the *Frobenius* homomorphism

$$F\colon G \longrightarrow G^{(p)} = G^{(p/S)} := (S, \phi_S) \times_S G,$$

where (S, ϕ_S) denotes the S-scheme $\phi_S\colon S \to S$. We define the iterated Frobenius homomorphism

$$F^\nu\colon G \to G^{(p^\nu)} = G^{(p^\nu/S)}$$

similarly.

Example 1.16. We have

$$\alpha_{p,S} = \mathrm{Ker}(F\colon G_{a,S} \longrightarrow G_{a,S}^{(p)}).$$

Example 1.17. An S-group scheme $\pi\colon X \to S$ is called an *abelian scheme* if π is smooth and proper with geometric fibres connected. X turns out to be commutative (at lease when S is noetherian). (cf. Mumford [**72**, Corollary 6.6, p.117])

When $S = \mathrm{Spec}(k)$ with a field k, an abelian scheme X over S is called an *abelian variety* over k. Thus X is a geometrically connected group scheme proper and smooth over k. In this case, the commutativity is shown in two different ways in Mumford [**74**, pp.41 and 44]. X is also shown to be *divisible*, that is, $n\mathrm{id}_X$ is surjective for any positive integer n.

When $k = \mathbb{C}$, the set $X(\mathbb{C})$ of $\mathbb{C}$-valued points of an abelian variety X over $\mathbb{C}$ turns out to be a complex torus.

Example 1.18. An *algebraic group* G is a *smooth* group scheme of finite type over a field k. An algebraic group G over k is *affine* if and only if it can be realized as a *linear group*, that is, as a closed subgroup of a general linear group $\mathrm{GL}_{n,k}$.

Definition 1.19. Suppose $\phi\colon H \to G$ is a homomorphism of S-group schemes. A pair $(G/H, \pi)$ of an S-scheme G/H and an S-morphism $\pi\colon G \to G/H$ is said to be the *quotient* of G by H, if it is universal for all pairs (Y, f) of an S-scheme Y and an S-morphism $f\colon G \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} G \times_S H \arrow[r, "\mu_G \circ (1_G \times_S \phi)"] \arrow[d, "p_1"'] & G \arrow[d, "f"] \\ G \arrow[r, "f"'] & Y, \end{tikzcd}$$

that is, there exists a unique S-morphism $f'\colon G/H \to Y$ such that $f = f' \circ \pi$. If H is a *normal* S-subgroup scheme of G with ϕ the canonical monomorphism so that $H(Z)$ is a normal subgroup of $G(Z)$ for any $Z \in (\mathrm{Sch}/S)$, then G/H inherits a unique structure of S-group scheme such that $\pi\colon G \to G/H$ is an S-homomorphism with $\mathrm{Ker}(\pi) = H$. In this case G/H is called the *quotient* group scheme.

$\boxed{\text{The rest of this section is yet to be worked out.}}$

We certainly need conditions for the existence of G/H.

- FGA [**2**, exposé 212, Corollaries 7.3 and 7.4] shows the existence in the case where S is the spectrum of an artinian ring (in particular, a field): Suppose G is of finite type and flat over S and that H is an S-subgroup scheme of G with H flat over S. Then G/H exists with $\pi\colon G \to G/H$ flat and surjective. Moreover, the quotient is shown to commute with base changes $S' \to S$.
- Demazure-Gabriel [**34**, Chapter III, §3] and SGA3 [**6**, exposés $\mathrm{VI_A}$ and $\mathrm{VI_B}$] deal with the quotient in terms of the "sheafification" of the contravariant functor
$$(\mathrm{Sch}/S) \ni Z \longmapsto G(Z)/H(Z) \in (\mathrm{Sets}).$$
- (cf. Borel [**23**, Chapter II, Theorem 6.8]) If G is an algebraic group over a field k and H is a closed algebraic subgroup over k, then G/H exists (Weil 1955 and Rosenlicht 1956) and is a smooth quasi-projective (cf. Definition II.5.8) algebraic variety over k (Chow 1957). See Raynaud [**81**] for the corresponding results in the case of more general base schemes S.

Action $G \times X \to X$ in Definition 2.2.3.

Example 1.20. $\mathrm{PGL}_{n+1} = \mathrm{GL}_{n+1}/G_m$ where $G_m \subset \mathrm{GL}_{n+1}$ is the normal subgroup scheme of "invertible scalar matrices".

Theorem 1.21 (Chevally 1953). (See Rosenlicht [**82**, Theorem 16] and Chevalley [**28**]. A "modern" proof can be found in Conrad [**32**].) *A geometrically connected algebraic group G has a geometrically connected closed affine normal subgroup L such that G/L is an abelian variety. Such L is unique and contains all other geometrically connected closed affine subgroups of G.*

Theorem 1.22 (Chevalley). (cf. Demazure-Gabriel [**34**, Chapter III, §3.5] and SGA3 [**6**, $\mathrm{VI_B}$, Theorem 11.17, p.408]) *If G is an affine algebraic group and H is a closed normal algebraic subgroup, then G/H is an affine algebraic group.*

Example 1.23. Borel subgroups (cf. Borel [**23**])
Classification of semi-simple (affine) algebraic groups (cf. Chevalley [**30**].)
Generalizations in SGA3 [**6**], Demazure-Gabriel [**34**].

2. Lang's theorem $\boxed{\text{split}}$

We can combine the geometric frobenius morphism (Definition IV.3.2) with ideas of smoothness to give a very pretty result due to Lang [**66**].

Theorem 2.1 (Lang). *Let $k = \mathbb{F}_q$, $\overline{k}$ = an algebraic closure of k.*

a) *Let G be a connected reduced group scheme of finite type over* $\mathrm{Spec}\, k$ *and let* $\overline{G} = G \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \overline{k}$. *Then $\overline{G}$ will be regular (smooth over $\overline{k}$) and irreducible.*

b) *Let*

$$\mathbf{f}_G = \mathbf{f}_G^{\text{geom}} \colon \overline{G} \longrightarrow \overline{G}$$

be the geometric frobenius morphism (cf. Definition IV.3.2*). Define a $\overline{k}$-morphism $\psi\colon \overline{G} \to \overline{G}$ on closed points by*

$$x \longmapsto \psi(x) = x \cdot \mathbf{f}_G(x)^{-1}$$

and in general by the composition:

$$\psi\colon \overline{G} \xrightarrow{\Delta} \overline{G} \times_{\operatorname{Spec} \overline{k}} \overline{G} \xrightarrow{(1_{\overline{G}} \times (\iota \circ \mathbf{f}_G))} \overline{G} \times_{\operatorname{Spec} \overline{k}} \overline{G} \xrightarrow{\mu} \overline{G}.$$

Then ψ is finite étale and surjective.

c) *Moreover the group $G(k)$ of k-rational points of G is finite and if we let each $a \in G(k)$ act on $\overline{G}$ by right translation R_a, then*
   1) $\forall a \in G(k)$, $\psi \circ R_a = \psi$
   2) $\forall x, y \in \overline{G}$, $\psi(x) = \psi(y) \iff \exists a \in G(k)$ *such that* $x = R_a(y)$.

Proof. According to Theorem IV.2.4, $\overline{G}$ is reduced because $\mathbb{F}_q$ is perfect. Therefore the set of regular (smooth over $\overline{k}$) points $U \subset \overline{G}$ is dense (cf. Jacobian criterion in Corollary V.4.2). But if $x, y \in \overline{G}$ are any two closed points, right translation by $x^{-1} \cdot y$ is an automorphism of $\overline{G}$ taking x to y. So if $x \in U$, then $y \in U$ too. Therefore U contains every closed point, hence $U = \overline{G}$. But then the components of $\overline{G}$ are disjoint. Now the identity point $e = \text{Image}(\epsilon)$ is a k-rational point of $\overline{G}$, hence it is $\text{Gal}(\overline{k}/k)$-invariant. Therefore the component $\overline{G}_\circ$ of $\overline{G}$ containing e as well as $\overline{G} \setminus \overline{G}_\circ$ are Gal-invariant open sets. By Theorem IV.2.3, this implies that G is disconnected too, unless $\overline{G} = \overline{G}_\circ$. This proves (a).

Next note that $\mathbf{f}_G\colon \overline{G} \to \overline{G}$ is a homomorphism of $\overline{k}$-group schemes, i.e.,

$$\begin{array}{ccc}
\overline{G} \times_{\operatorname{Spec} \overline{k}} \overline{G} & \xrightarrow{\mu} & \overline{G} \\
{\scriptstyle \mathbf{f}_G \times \mathbf{f}_G} \big\downarrow & & \big\downarrow {\scriptstyle \mathbf{f}_G} \\
\overline{G} \times_{\operatorname{Spec} \overline{k}} \overline{G} & \xrightarrow{\mu} & \overline{G}
\end{array}$$

commutes. This is because if you write $\overline{G} = G \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$, then μ equals $\mu' \times 1_{\overline{k}}$ where $\mu'\colon G \times_{\operatorname{Spec} k} G \to G$ is multiplication for G; but by definition $\mathbf{f}_G = \phi_G^\nu \times 1_{\overline{k}}$ (if $q = p^\nu$) and for any morphism $g\colon X \to Y$ in characteristic p, $\phi_X \circ g = g \circ \phi_X$ (cf. Definition IV.3.1). Then for all closed points $x \in \overline{G}$, $a \in G(k)$

$$\begin{aligned}
\psi \circ R_a(x) &= \psi(x \cdot a) \\
&= x \cdot a \cdot \mathbf{f}_G(x \cdot a)^{-1} \\
&= x \cdot a \cdot \mathbf{f}_G(a)^{-1} \cdot \mathbf{f}_G(x)^{-1} \\
&= x \cdot a \cdot a^{-1} \cdot \mathbf{f}_G(x)^{-1} \\
&= \psi(x)
\end{aligned}$$

and for all closed points $x, y \in \overline{G}$:

$$\begin{aligned}
\psi(x) = \psi(y) &\iff x \cdot \mathbf{f}_G(x)^{-1} = y \cdot \mathbf{f}_G(y)^{-1} \\
&\iff y^{-1} \cdot x = \mathbf{f}_G(y)^{-1} \cdot \mathbf{f}_G(x) \\
&\iff y^{-1} \cdot x = \mathbf{f}_G(y^{-1} \cdot x) \\
&\iff y^{-1} \cdot x \text{ is } \mathrm{Gal}(\overline{k}/k)\text{-invariant} \\
&\iff y^{-1} \cdot x = a \in G(k) \\
&\iff x = R_a(y) \text{ for some } a \in G(k).
\end{aligned}$$

But now for any scheme X of finite type over k, $X(k)$ is finite. The last result shows that the two closed subsets of $\overline{G} \times_{\operatorname{Spec} \overline{k}} \overline{G}$, namely

$$\bigcup_{a \in G(k)} (\text{Graph of } R_a) \text{ and the fibre product:} \qquad \begin{tikzcd} & \overline{G} \times_{\overline{G}} \overline{G} \arrow[dl] \arrow[dr] & \\ \overline{G} \arrow[dr, "\psi"'] & & \overline{G} \arrow[dl, "\psi"] \\ & \overline{G} & \end{tikzcd}$$

have the same closed points. Therefore these sets are equal. This proves (c).

Now we come to the main point — (b). We prove first that ψ is étale using Criterion V.4.6: $\forall x \in \overline{G}$ closed, $d\psi_x \colon T_{x,\overline{G}} \to T_{\psi(x),\overline{G}}$ is an isomorphism. We use:

Lemma 2.2. *If X is a scheme over $k = \mathbb{F}_q$ and $\overline{X} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$, then the $\overline{k}$-morphism $\mathbf{f}_X = \mathbf{f}_X^{\mathrm{geom}} \colon \overline{X} \to \overline{X}$ induces the zero map*

$$\mathbf{f}_X^* \colon \Omega_{\overline{X}/\overline{k}} \longrightarrow \Omega_{\overline{X}/\overline{k}}.$$

Proof of Lemma 2.2. We may as well assume X affine, say $= \operatorname{Spec} R$. Then $\overline{X} = \operatorname{Spec}(R \otimes_k \overline{k})$ and $\mathbf{f}_X$ is induced by the homomorphism

$$\begin{aligned}
R \otimes_k \overline{k} &\longrightarrow R \otimes_k \overline{k} \\
\sum a_i \otimes b_i &\longmapsto \sum a_i^q \otimes b_i.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbf{f}_X^* \left(d(\textstyle\sum a_i \otimes b_i) \right) &= d(\textstyle\sum a_i^q \otimes b_i) \\
&= \textstyle\sum d(a_i^q) \otimes b_i + \sum a_i^q \otimes db_i \\
&= 0.
\end{aligned}$$

□

By Chapter V, this means that for all closed points of $x \in \overline{G}$,

$$(d\mathbf{f}_X)_x \colon T_{x,\overline{G}} \longrightarrow T_{\mathbf{f}_X(x),\overline{G}}$$

is zero. To compute $d\psi_x \colon T_{x,\overline{G}} \to T_{\psi(x),\overline{G}}$, use the identification of $T_{x,\overline{G}}$ with the set of $\overline{k}[\epsilon]$-valued points $t \colon \operatorname{Spec} \overline{k}[\epsilon] \to \overline{G}$ of $\overline{G}$ with $\mathrm{Image}(t) = \{x\}$ (cf. §V.1). In terms of this identification, if $t \in T_{x,\overline{G}}$, then $d\psi_x(t)$ is nothing but $\psi \circ t$. Hence using the group law in the set of $\overline{k}[\epsilon]$-valued points of $\overline{G}$:

$$d\psi_x(t) = t \cdot \mathbf{f}_X(t)^{-1}.$$

But if O_y is the 0 tangent vector at y, i.e.,

$$\operatorname{Spec} \overline{k}[\epsilon] \longrightarrow \operatorname{Spec} \overline{k} \longrightarrow \overline{G} \text{ with image } y,$$

then Lemma 2.2 showed that $\mathbf{f}_X(t) = O_{\mathbf{f}(x)}$, hence

$$d\psi_x(t) = t \cdot O_{\mathbf{f}_X(x)^{-1}}, \qquad \forall t \in T_{x,X}.$$

The map $t \mapsto t \cdot O_{\mathbf{f}_X(x)}$ is then an inverse to $d\psi_x$ so $d\psi_x$ is an isomorphism.

Next, f is surjective. In fact for all closed points $a \in \overline{G}$ we can introduce a new morphism $\psi^{(a)}$ given on closed points by:

$$\psi^{(a)}(x) = x \cdot a \cdot \mathbf{f}_X(x)^{-1}.$$

The same argument given for ψ also shows that $\psi^{(a)}$ is étale. Therefore $\psi^{(a)}$ is flat and by Ex. ?????, Image$(\psi^{(a)})$ is open. Therefore

$$\mathrm{Image}(\psi) \cap \mathrm{Image}(\psi^{(a)}) \neq \emptyset,$$

i.e., $\exists$ closed points $b_1, b_2 \in \overline{G}$ such that

$$b_1 \cdot \mathbf{f}_X(b_1)^{-1} = b_2 \cdot a \cdot \mathbf{f}_X(b_2)^{-1}.$$

Then one calculates immediately that $\psi(b_2^{-1} \cdot b_1) = a$.

Finally ψ is finite: by Ex. ?????, $\exists$ a non-empty open $U \subset \overline{G}$ such that $\mathrm{res}\,\psi \colon \psi^{-1}U \to U$ is finite. But if L_a is left translation by a, then for all closed points $a \in \overline{G}$, consider the commutative diagram:

$$\begin{tikzcd} \overline{G} \arrow[r, "L_a", "\approx"'] \arrow[d, "\psi"'] & \overline{G} \arrow[d, "\psi"] \\ \overline{G} \arrow[r, "L_a \circ R_{\mathbf{f}(a)^{-1}}", "\approx"'] & \overline{G} \end{tikzcd}$$

It follows that $\mathrm{res}\,\psi$ is finite from $L_a(\psi^{-1}U)$ to $L_a(R_{\mathbf{f}(a)^{-1}}(U))$ too. Since $\overline{G}$ is covered by the open sets $L_a(\psi^{-1}U)$, ψ is everywhere finite. $\square$

For example, applied to $\mathbb{A}^1_k$, the theorem gives the Artin-Schreier homomorphism:

$$\begin{aligned} \psi \colon \mathbb{A}^1_k &\longrightarrow \mathbb{A}^1_k \\ \psi(x) &= x - x^q \\ \mathrm{Ker}\,\psi &= \mathbb{F}_q \subset \mathbb{A}^1_k. \end{aligned}$$

On $\mathrm{GL}_1(\overline{k})$, ψ is the homomorphism

$$\begin{aligned} \psi(x) &= x^{1-q} \\ \mathrm{Ker}\,\psi &= \mathbb{F}_q^* \subset \mathrm{GL}_1(\overline{k}), \end{aligned}$$

while on $\mathrm{GL}_2(\overline{k})$, ψ is given by:

$$\begin{aligned} \psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a^q & b^q \\ c^q & d^q \end{pmatrix}^{-1} \\ &= \frac{1}{(ad-bc)^q} \begin{pmatrix} ad^q - bc^q & -ab^q + ba^q \\ cd^q - dc^q & -cb^q + da^q \end{pmatrix}. \end{aligned}$$

Lang invented this theorem because of its remarkable application to homogeneous spaces for G over k. We need another definition to explain this:

Definition 2.3. Let $f \colon G \to S$ plus (μ, ϵ, ι) be a group scheme and let $p \colon X \to S$ be any scheme over S. Then an action of G on X is an S-morphism:

$$\sigma \colon G \times_S X \longrightarrow X$$

such that the following diagrams commute:

a) ("associativity")

$$\begin{tikzcd} (G\times_S G)\times_S X \arrow[r, "\mu\times 1_X"] \arrow[d, equal, "\sim"'] & G\times_S X \arrow[dr, "\sigma"] & \\ & & X \\ G\times_S (G\times_S X) \arrow[r, "1_G\times\sigma"'] & G\times_S X \arrow[ur, "\sigma"'] & \end{tikzcd}$$

b) ("identity acts by identity")

$$\begin{tikzcd} S\times_S X \arrow[r, "\epsilon\times 1_X"] \arrow[d, equal, "\sim"'] & G\times_S X \arrow[d, "\sigma"] \\ X \arrow[r, "1_X"] & X \end{tikzcd}$$

Corollary 2.4. *Let G be a connected reduced group scheme of finite type over $k=\mathbb{F}_q$ and let X be a scheme of finite type over k on which G acts via σ. Let Σ be a set of subschemes of $\overline{X}=X\times_{\operatorname{Spec} k}\operatorname{Spec}\overline{k}$ such that:*

a) *$\forall Z\in\Sigma$, $a\in\overline{G}$ closed, $\sigma(a,Z)\in\Sigma$ and $\forall Z_1,Z_2\in\Sigma$, $\exists a\in\overline{G}$ closed such that $\sigma(a)(Z_1)=Z_2$*[1]
b) *if $\mathbf{f}_X^{\text{arith}}\colon\overline{X}\to\overline{X}$ is the frobenius automorphism (cf. Definition* IV.3.2*), then $\forall Z\in\Sigma$, $\mathbf{f}_X^{\text{arith}}(Z)\in\Sigma$.*

Then Σ contains at least one subscheme Z of the form $Z'\times_{\operatorname{Spec} k}\operatorname{Spec}\overline{k}$, Z' a subscheme of X.

Proof. Start with any $Z\in\Sigma$ and combine (a) and (b) to write

$$\mathbf{f}_X^{\text{arith}}(Z)=\sigma(a)(Z),\quad a\in\overline{G}\text{ closed.}$$

By Lang's theorem (Theorem 2.1),

$$a^{-1}=b\cdot\mathbf{f}_G(b)^{-1},\quad b\in\overline{G}\text{ closed.}$$

Now on closed points, $\mathbf{f}_G^{\text{geom}}=\mathbf{f}_G^{\text{arith}}$, so we deduce

$$\mathbf{f}_X^{\text{arith}}(Z)=\sigma(\mathbf{f}_G^{\text{arith}}b)(\sigma(b^{-1})(Z)),$$

hence since σ is defined over k:

$$\begin{aligned}\mathbf{f}_X^{\text{arith}}\left(\sigma(b^{-1})(Z)\right)&=\sigma(\mathbf{f}_X^{\text{arith}}b^{-1})(\mathbf{f}_X^{\text{arith}}(Z))\\&=\sigma(b^{-1})(Z).\end{aligned}$$

Therefore $\sigma(b^{-1})(Z)\in\Sigma$ and is invariant under $\operatorname{Gal}(\overline{k}/k)$. So by Theorem IV.2.9, $\sigma(b^{-1})(Z)=Z'\times_{\operatorname{Spec} k}\operatorname{Spec}\overline{k}$ for some subscheme Z' of X. □

Corollary 2.5. *Let G, X be as above. Assume the group of closed points of $\overline{G}$ acts transitively on the set of closed points of $\overline{X}$. Then $X(k)\neq\emptyset$.*

Proof. Apply Corollary 2.4 with $\Sigma=$ the closed points of $\overline{X}$. □

If X is a smooth quadric hypersurface in $\mathbb{P}_k^n$, or a smooth cubie curve in $\mathbb{P}_k^2$, it can be shown that such a G always exists, hence X has a k-rational point! For some conics in $\mathbb{P}_k^2$, the next corollary tells us more:

[1] $\sigma(a)$ is short for the automorphism of $\overline{X}$:

$$\overline{X}=\operatorname{Spec}\overline{k}\times_{\operatorname{Spec}\overline{k}}\overline{X}\xrightarrow{\{a\}\times\overline{X}}\overline{G}\times_{\operatorname{Spec}\overline{k}}\overline{X}\xrightarrow{\sigma}\overline{X}.$$

COROLLARY 2.6. *Let Y be a scheme of finite type over k such that*

$$\overline{Y} \cong \mathbb{P}^n_{\overline{k}} \text{ over } \overline{k}.$$

Then

$$Y \cong \mathbb{P}^n_k \text{ over } k.$$

PROOF. Take the X in Corollary 2.4 to be $Y \times_{\operatorname{Spec} k} \mathbb{P}^n_k$. Let Σ be the set of graphs of $\overline{k}$-isomorphisms from $\mathbb{P}^n_{\overline{k}}$ to $\overline{Y}$. Let $G = \mathrm{GL}_{n+1,k}$ and let G act on X by acting trivially on Y and acting in the usual fashion on $\mathbb{P}^n_k$ (one should check that this action is a morphism). Recall that every $\overline{k}$-automorphism of $\mathbb{P}^n_{\overline{k}}$ is induced by the action of some $g \in \mathrm{GL}_{n+1}(\overline{k}) =$ the closed points of G (Ex. 3 ?????, Example 1.1.10): this shows that the closed points of $\overline{G}$ act transitively on Σ. It follows that the graph Γ_f of some $f\colon \mathbb{P}^n_{\overline{k}} \xrightarrow{\approx} \overline{Y}$ is defined over k, hence $f = f' \times 1_{\overline{k}}$, where f' is a k-isomorphism of $\mathbb{P}^n_k$ and Y (compare Ex. 1 ?????). $\square$

REMARK. See Proposition IV.3.5 and Corollary VIII.1.8 for $\mathbb{P}^1$ over finite fields.

3. Belyi's three point theorem

(Added in publication)

The following result is due to Belyi [**21**], [**22**]:

THEOREM 3.1 (Belyi's three point theorem). *Let C be an irreducible curve proper smooth curve over $\mathbb{C}$. Then C is defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers (that is, $C = C_0 \times_{\operatorname{Spec}(\overline{\mathbb{Q}})} \operatorname{Spec}(\mathbb{C})$ for some C_0 over $\overline{\mathbb{Q}}$) if and only if it can be represented as a covering of the projective line $\mathbb{P}^1_{\mathbb{C}}$ branched only at $0,\ 1,\ \infty$.*

Let C and C' be irreducible curves proper and smooth over an algebraically closed field k, and $f\colon C \to C'$ a finite surjective separable morphism. The *ramification locus* of f is the finite set of closed points of C at which f is not étale, and coincides with $\operatorname{Supp}(\Omega_{C/C'})$ by Definition V.3.1 and Criterion V.4.1.

$$\Delta(f) = f(\operatorname{Supp}(\Omega_{C/C'}))$$

is called the *branch locus* of f.

REMARK. (Added in Publication) This result is closely related to “dessins d'enfants” introduced by Grothendieck [**43**]. See, for instance, Luminy Proceedings [**44**].

PROOF OF THE “ONLY IF” PART OF THEOREM 3.1. We show that if C is an irreducible curve proper and smooth over $\overline{\mathbb{Q}}$, then there exists a finite surjective morphism $f\colon C \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ such that $\Delta(f)(\overline{\mathbb{Q}}) \subset \{0, 1, \infty\}$.

Since the function field $\mathbb{R}(C)$ is an extension of $\overline{\mathbb{Q}}$ of transcendence degree 1, choose $f_0 \in \mathbb{R}(C) \setminus \overline{\mathbb{Q}}$, which gives a finite surjective morphism

$$f_0\colon C \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}}.$$

Without loss of generality, we may assume $\Delta(f_0)(\overline{\mathbb{Q}}) \subset \mathbb{P}^1(\overline{\mathbb{Q}})$ contains ∞.

We now show the existence of a non-constant polynomial $g(t) \in \mathbb{Q}[t]$ such that the composite morphism

$$g \circ f_0\colon C \xrightarrow{f_0} \mathbb{P}^1_{\overline{\mathbb{Q}}} \xrightarrow{g} \mathbb{P}^1_{\overline{\mathbb{Q}}}$$

satisfies $\Delta(g \circ f_0)(\overline{\mathbb{Q}}) \subset \mathbb{P}^1(\mathbb{Q})$ by induction on

$$\delta(f_0) = \sum_{y \in \Delta(f_0)(\overline{\mathbb{Q}})} ([\Bbbk(y) : \mathbb{Q}] - 1).$$

There is nothing to prove if $\delta(f_0) = 0$. If $\delta(f_0) > 0$, choose $y_1 \in \Delta(f_0)(\overline{\mathbb{Q}})$ with $[\Bbbk(y_1) : \mathbb{Q}] > 1$. Let $g_1(t)$ be the minimal polynomial over $\mathbb{Q}$ of y_1. We have $\delta(g_1 \circ f_0) < \delta(f_0)$, since $g_1(y_1) = 0$.

Thus it suffices to show the following:

LEMMA 3.2 (Belyi). *If $f_1 : \mathbb{P}^1_{\overline{\mathbb{Q}}} \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ is a finite surjective morphism with $\Delta(f_1)(\overline{\mathbb{Q}}) \subset \mathbb{P}^1(\mathbb{Q})$, then there exists a finite surjective morphism*

$$h : \mathbb{P}^1_{\mathbb{Q}} \longrightarrow \mathbb{P}^1_{\mathbb{Q}}$$

such that $\Delta(h \circ f_1)(\overline{\mathbb{Q}}) \subset \{0, 1, \infty\}$.

THE FIRST PROOF OF LEMMA 3.2. We prove the existence of $h(t) \in \mathbb{Q}[t]$ by induction on the cardinality $\#\Delta(f_1)(\overline{\mathbb{Q}})$.

If $\#\Delta(f_1)(\overline{\mathbb{Q}}) \leq 3$, we choose h to be a linear fractional transformation with coefficients in $\mathbb{Q}$ that sends $\Delta(f_1)(\overline{\mathbb{Q}})$ to $\{0, 1, \infty\}$.

If $\#\Delta(f_1)(\overline{\mathbb{Q}}) > 3$, we may choose a suitable linear fractional transformation with coefficients in $\mathbb{Q}$ and assume that

$$\Delta(f_1)(\overline{\mathbb{Q}}) \supset \{0, 1, \frac{n}{m+n}, \infty\}$$

for positive integers m, n. Let

$$h(t) = \frac{(m+n)^{m+n}}{m^m n^n} t^m (1-t)^n \in \mathbb{Q}[t],$$

which gives a morphism $h : \mathbb{P}^1_{\mathbb{Q}} \longrightarrow \mathbb{P}^1_{\mathbb{Q}}$ with

$$\begin{aligned} h(0) &= 0 \\ h(1) &= 0 \\ h(\frac{n}{m+n}) &= 1. \end{aligned}$$

Thus we have $\#\Delta(h \circ f_1)(\overline{\mathbb{Q}}) < \#\Delta(f_1)(\overline{\mathbb{Q}})$. □

THE SECOND PROOF OF LEMMA 3.2. By linear fractional transformation with coefficients in $\mathbb{Q}$ we may assume

$$\Delta(f_1)(\overline{\mathbb{Q}}) = \{\lambda_1, \ldots, \lambda_n, \infty\} \subset \mathbb{P}^1(\mathbb{Q})$$

with $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}$ such that

$$0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n, \qquad \gcd(\lambda_2, \ldots, \lambda_n) = 1.$$

Denote the Vandermonde determinant by

$$w = W(\lambda_1, \ldots, \lambda_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{vmatrix} = \prod_{j>l} (\lambda_j - \lambda_l).$$

Similarly, denote the Vandermonde determinant for each $i = 1, \ldots, n$ by

$$w_i = (-1)^{n-i} W(\lambda_1, \ldots, \widehat{\lambda_i}, \ldots, \lambda_n) = (-1)^{n-i} \prod_{\substack{j>l \\ j,l \neq i}} (\lambda_j - \lambda_l),$$

where $\widehat{\lambda_i}$ means λ_i deleted. It is easy to check that

$$\sum_{i=1}^{n} \frac{w_i}{t-\lambda_i} = \frac{w}{\prod_{i=1}^{n}(t-\lambda_i)}$$
$$\sum_{i=1}^{n} w_i = 0$$
$$\sum_{i=1}^{n} \lambda_i^{n-1} w_i = w.$$

Let $r_i = w_i/\gcd(w_1,\dots,w_n) \in \mathbb{Z}$ and

$$h(t) = \prod_{i=1}^{n}(t-\lambda_i)^{r_i} \in \mathbb{Q}(t).$$

Note that $\sum_{i=1}^{n} r_i = 0$. Since

$$\frac{h'(t)}{h(t)} = \sum_{i=1}^{n} \frac{r_i}{t-\lambda_i} = \frac{w/\gcd(w_1,\dots,w_n)}{\prod_{i=1}^{n}(t-\lambda_i)},$$

the ramification locus of $h\colon \mathbb{P}^1_{\mathbb{Q}} \longrightarrow \mathbb{P}^1_{\mathbb{Q}}$ is contained in $\{\lambda_1,\dots,\lambda_n,\infty\}$, while $\Delta(h)(\overline{\mathbb{Q}}) \subset \{0,1,\infty\}$. We see that

$$\begin{aligned} h(\lambda_i) &= 0, \qquad n-i \text{ even} \\ h(\lambda_i) &= \infty, \qquad n-i \text{ odd} \\ h(\infty) &= 1 \\ \Delta(h)(\overline{\mathbb{Q}}) &= \{0,1,\infty\}. \end{aligned}$$

Then the composite

$$\mathbb{P}^1_{\mathbb{Q}} \xrightarrow{f_1} \mathbb{P}^1_{\mathbb{Q}} \xrightarrow{h} \mathbb{P}^1_{\mathbb{Q}}$$

has the property $\Delta(h\circ f_1)(\overline{\mathbb{Q}}) \subset \{0,1,\infty\}$. □

□

Proof of the "if" part of Theorem 3.1. We show that if $g'\colon C \to \mathbb{P}^1_{\mathbb{C}}$ is a finite covering with $\Delta(g')(\mathbb{C}) \subset \{0,1,\infty\}$, then there exists a curve C_0 over $\overline{\mathbb{Q}}$ such that $C = C_0 \times_{\mathrm{Spec}(\mathcal{Q})} \mathrm{Spec}(\mathbb{C}) = (C_0)_{\mathbb{C}}$.

We basically follow the arguments in notes provided by Carlos Simpson.

Here is what we are going to do: We construct a "deformation" $f\colon X \to S$ of C parametrized by an irreducible affine smooth variety S over $\overline{\mathbb{Q}}$. Then the fibre over a $\overline{\mathbb{Q}}$-rational point $s_0 \in S$ turns out to be C_0 we are looking for.

Since C is projective (cf. Proposition V.5.11), we have a closed immersion $C \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$. In view of the covering $g'\colon C \to \mathbb{P}^1_{\mathbb{C}}$ and the Segre embedding (cf. Example I.8.11 and Proposition II.1.2), we have closed immersions

$$C \hookrightarrow \mathbb{P}^N_{\mathbb{C}} \times_{\mathrm{Spec}(\mathbb{C})} \mathbb{P}^1_{\mathbb{C}} \hookrightarrow \mathbb{P}^{2N+1}_{\mathbb{C}}.$$

Using an idea similar to that in the proof of Proposition IV.1.4, we have a subring $R \subset \mathbb{C}$ generated over $\overline{\mathbb{Q}}$ by the coefficients of the finite number of homogeneous equations defining C as well as $\mathbb{P}^N_{\mathbb{C}} \times_{\mathrm{Spec}(\mathbb{C})} \mathbb{P}^1_{\mathbb{C}}$ in $\mathbb{P}^{2N+1}_{\mathbb{C}}$ and a scheme X of finite type over R with closed immersions

$$X \hookrightarrow \mathbb{P}^N_R \times_{\mathrm{Spec}(R)} \mathbb{P}^1_R \hookrightarrow \mathbb{P}^{2N+1}_R$$

such that the base extension by $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(R)$ gives rise to

$$C \hookrightarrow \mathbb{P}^N_{\mathbb{C}} \times_{\mathrm{Spec}(\mathbb{C})} \mathbb{P}^1_{\mathbb{C}} \hookrightarrow \mathbb{P}^{2N+1}_{\mathbb{C}}.$$

$S = \mathrm{Spec}(R)$ is an integral scheme of finite type over $\overline{\mathbb{Q}}$. Replacing S by a suitable non-empty affine open subset, we may assume S to be smooth over $\overline{\mathbb{Q}}$. Moreover, S is endowed with a fixed $\mathbb{C}$-valued point $\mathrm{Spec}(\mathbb{C}) \to S$. Denote the structure morphism of X by $f\colon X \to S$. By construction, we have a factorization

$$f\colon X \xrightarrow{g} \mathbb{P}^1_S \longrightarrow S$$

with f and g projective. Moreover, the base extension by $\mathrm{Spec}(\mathbb{C}) \to S$ gives rise to

$$C = X_{\mathbb{C}} \xrightarrow{g'} \mathbb{P}^1_{\mathbb{C}} \to \mathrm{Spec}(\mathbb{C}).$$

We now show the following:

LEMMA 3.3. *There exists a non-empty affine open subset $S_0 \subset S$ such that the restriction*

$$f\colon X_0 = f^{-1}(S_0) \xrightarrow{g} \mathbb{P}^1_{S_0} \longrightarrow S_0$$

to S_0 satisfies the following conditions:

i) *X_0 is integral.*
ii) *$f\colon X_0 \to S_0$ is surjective and smooth of relative dimension* 1.
iii) *$g\colon X_0 \to \mathbb{P}^1_{S_0}$ is étale outside $\{0,1,\infty\} \times S_0$. We may further assume*

$$f(X_0 \setminus (\{0,1,\infty\} \times S_0)) = S_0.$$

PROOF OF LEMMA 3.3.

Proof of (i): Let K be the function field of S so that K is the field of fractions of R and is a subfield of $\mathbb{C}$. The fibre of f over the generic point η_S of S is $f^{-1}(\eta_S) = X_K$, whose base extension by $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(K)$ gives rise to the original curve $C = X_{\mathbb{C}}$. Hence X_K is integral.

Let $X = \bigcup_i X_i$ be the irreducible decomposition with the generic point η_i of X_i for each i. Let U_i's be mutually disjoint neighborhoods of η_i. Since $X_K = f^{-1}(\eta_S)$ is irreducible, at most one U_i intersects $f^{-1}(\eta_S)$. If none of the η_i's were in $f^{-1}(\eta_S)$, then for each i we would have $f(\eta_i) \neq \eta_S$ so that $\eta_S \notin \overline{f(\eta_i)}$ and $f^{-1}(\eta_S) \cap f^{-1}(\overline{f(\eta_i)}) = \emptyset$. Since closed $f^{-1}(\overline{f(\eta_i)})$ contains X_i, we would have $f^{-1}(\eta_S) \cap X_i = \emptyset$ for all i, a contradiction. Thus there exists exactly one i such that $\eta_i \in f^{-1}(\eta_S)$. Hence $f^{-1}(\eta_S) \subset X_i$ and $\eta_S \notin f(X \setminus X_i)$. By Chevalley's Nullstellensatz (cf. Theorem II.2.9) $f(X \setminus X_i)$ is constructible. Thus there exists an open neighborhood S_0 of η_S with $S_0 \cap f(X \setminus X_i) = \emptyset$. Hence $f^{-1}(S_0) \cap (X \setminus X_i) = \emptyset$ so that $f^{-1}(S_0) \subset X_i$ is irreducible. Obviously, we may replace S_0 by a non-empty affine open subset.

Let us replace S and X by this S_0 and $f^{-1}(S_0)$, respectively so that we may now assume X to be irreducible.

We next show that there exists a non-empty affine open subset $\mathrm{Spec}(R_t) \subset S = \mathrm{Spec}(R)$ for some $t \in R$ such that $f^{-1}(\mathrm{Spec}(R_t))$ is reduced. Indeed, let $X = \bigcup_i \mathrm{Spec}(A_i)$ be a finite affine open covering. Since $X_K = f^{-1}(\eta_S)$ is reduced, $A_i \otimes_R K$ is reduced for all i. Obviously, there exists a non-zero divisor $t_i \in R$ such that $A_i \otimes_R R_{t_i}$ is reduced. Letting $t = \prod_i t_i$, we see that $X \times_S \mathrm{Spec}(R_t) = f^{-1}(\mathrm{Spec}(R_t))$ is reduced.

Proof of (ii): Let us replace S and X by $\mathrm{Spec}(R_t)$ and $f^{-1}(\mathrm{Spec}(R_t))$ in (i), respectively so that we may assume X to be integral with the generic point η_X of X mapped by f to η_S.

Since $X_{\mathbb{C}} = C$ is smooth of relative dimension 1 over $\mathbb{C}$, so is X_K smooth of relative dimension 1 over K. By what we saw in §V.3, the stalks of $\Omega_{X/S}$ at points $x \in f^{-1}(\eta_S)$ are locally free of rank 1. Thus we find an open neighborhood U of $f^{-1}(\eta_S)$ such that $f\colon U \to S$ is smooth of relative dimension 1. Since f is projective, $f(X \setminus U)$ is closed and does not contain η_S. Hence

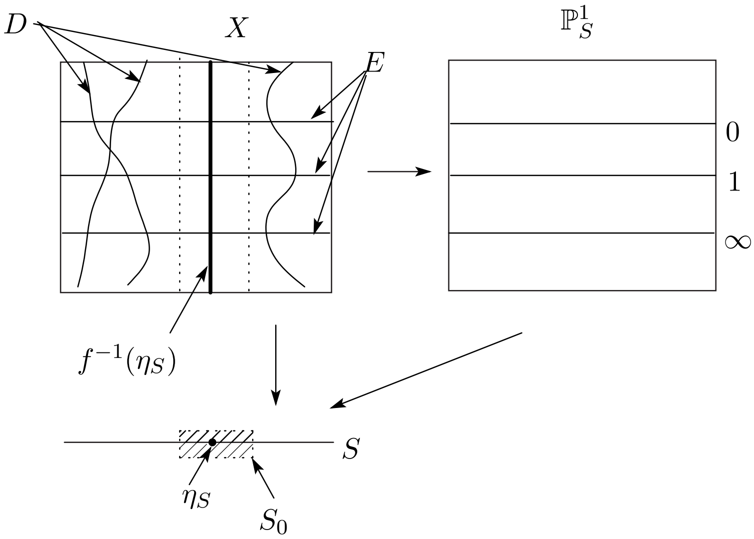

FIGURE VI.1

$S_0 = S \setminus f(X \setminus U)$ is an open neighborhood of η_S such that $f^{-1}(S_0) \to S_0$ is smooth of relative dimension 1.

Replacing S and X by this S_0 and $f^{-1}(S_0)$, respectively, we may thus assume $f\colon X \to S$ to be smooth of relative dimension 1.

Proof of (iii): The base extension of $g\colon X \to \mathbb{P}^1_S$ by $\mathrm{Spec}(\mathbb{C}) \to S$ is $g'\colon C = X_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$, which is étale outside $\{0, 1, \infty\}$. Hence the base extension

$$g_K\colon X_K = f^{-1}(\eta_S) \to \mathbb{P}^1_K$$

by $\mathrm{Spec}(K) \to S$ is étale outside $\{0, 1, \infty\}$. By what we say in §V.3, we have

$$\mathrm{Supp}(\Omega_{X/\mathbb{P}^1_S}) \cap f^{-1}(\eta_S) \subset \{0, 1, \infty\} \times \{\eta_S\}.$$

Denote

$$E = g^{-1}(\{0, 1, \infty\} \times S)$$
$$D = \text{the closure of } \left(\mathrm{Supp}(\Omega_{X/\mathbb{P}^1_S}) \setminus E\right).$$

Thus $\eta_S \notin f(D \setminus E)$, which is constructible again by Chevalley's Nullstellensatz. Hence $\eta_S \notin \overline{f(D \setminus E)} = f(D)$, and there exists an open affine neighborhood of η_S such that $S_0 \cap f(D) = \emptyset$. Consequently, $f^{-1}(S_0) \cap D = \emptyset$ so that $g\colon f^{-1}(S_0) \to \mathbb{P}^1_{S_0}$ is étale outside $\{0, 1, \infty\} \times S_0$, and

$$g\colon f^{-1}(S_0) \setminus E \longrightarrow \mathbb{P}^1_{S_0}$$

is étale.

We may thus replace S and X by this S_0 and $f^{-1}(S_0)$, respectively. If $f(X \setminus E) \neq S$, then since $f(X \setminus E)$ contains η_S and is constructible again by Chevalley's Nullstellensatz, there exists an affine open neighborhood S_0 of η_S such that for $X_0 = f^{-1}(S_0)$, we have $f(X_0 \setminus E) = S_0$. Thus we are in the situation as in Figure VI.1. □

To continue the proof of the "if" part of Theorem 3.1, we denote X_0 and S_0 obtained in Lemma 3.3 by X and S, respectively.

Choose a closed point $s_0 \in S$. Obviously, we have $\Bbbk(s_0) = \overline{\mathbb{Q}}$. Thus $C_0 = f^{-1}(s_0)$ is an irreducible projective smooth curve over $\overline{\mathbb{Q}}$. We now show

$$C \cong C_0 \times_{\mathrm{Spec}(\overline{\mathbb{Q}})} \mathrm{Spec}(\mathbb{C}) \quad \text{as algebraic curves,}$$

which would finish the proof of the "if" part of Theorem 3.1.

The base change by $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\overline{\mathbb{Q}})$ of what we obtained in Lemma 3.3 gives rise to

$$f_{\mathbb{C}}\colon X_{\mathbb{C}} \xrightarrow{g_{\mathbb{C}}} \mathbb{P}^1_{\mathbb{C}} \times_{\mathrm{Spec}(\mathbb{C})} S_{\mathbb{C}} \longrightarrow S_{\mathbb{C}}.$$

We also have two $\mathbb{C}$-valued points of $S_{\mathbb{C}}$:

$$\begin{aligned} t_0\colon \mathrm{Spec}(\mathbb{C}) &\longrightarrow S_{\mathbb{C}} \quad \text{induced by } \Bbbk(s_0) = \overline{\mathbb{Q}} \hookrightarrow \mathbb{C} \\ t_1\colon \mathrm{Spec}(\mathbb{C}) &\longrightarrow S_{\mathbb{C}} \quad \text{induced by } \Bbbk(\eta_S) = K \hookrightarrow \mathbb{C} \end{aligned}$$

so that

$$\begin{aligned} (f_{\mathbb{C}})^{-1}(t_0) &= C_0 \times_{\mathrm{Spec}(\overline{\mathbb{Q}})} \mathrm{Spec}(\mathbb{C}) \\ (f_{\mathbb{C}})^{-1}(t_1) &= C. \end{aligned}$$

As we explain later in §VIII.2, let us consider the associated complex analytic spaces and holomorphic maps. For simplicity, we denote

$$M = X_{\mathbb{C}}^{\mathrm{an}}, \quad T = S_{\mathbb{C}}^{\mathrm{an}}, \quad \mathbb{P}^1(\mathbb{C}) = (\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}}, \quad \varphi = f_{\mathbb{C}}^{\mathrm{an}}, \quad \psi = g_{\mathbb{C}}^{\mathrm{an}}.$$

Thus we have

$$\varphi\colon M \xrightarrow{\psi} \mathbb{P}^1(\mathbb{C}) \times T \longrightarrow T,$$

where M and T are connected complex manifolds, $\varphi\colon M \to T$ is a proper smooth holomorphic map of relative dimension 1, $\psi\colon M \to \mathbb{P}^1(\mathbb{C}) \times T$ is a finite covering unramified outside $\{0, 1, \infty\} \times T$. We can regard t_0 and t_1 as points of T so that

$$\begin{aligned} \varphi^{-1}(t_0) &= \Big(C_0 \times_{\mathrm{Spec}(\overline{\mathbb{Q}})} \mathrm{Spec}(\mathbb{C})\Big)^{\mathrm{an}} \\ \varphi^{-1}(t_1) &= C^{\mathrm{an}}. \end{aligned}$$

Lemma 3.4. *For any pair of points $t, t' \in T$, one has*

$$\varphi^{-1}(t) \cong \varphi^{-1}(t') \qquad \text{as complex manifolds.}$$

As a consequence of this lemma, one has

$$(C_0 \times_{\mathrm{Spec}(\overline{\mathbb{Q}})} \mathrm{Spec}(\mathbb{C}))^{\mathrm{an}} = \varphi^{-1}(t_0) \cong \varphi^{-1}(t_1) = C^{\mathrm{an}}.$$

In view of a GAGA result given as Corollary VIII.2.11 later, we have

$$C_0 \times_{\mathrm{Spec}(\overline{\mathbb{Q}})\,\mathrm{Spec}(\mathbb{C})} \cong C \qquad \text{as algebraic curves.}$$

□

Proof of Lemma 3.4. For simplicity, denote

$$P^\circ = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}, \qquad M^\circ = \psi^{-1}(P^\circ \times T)$$

so that the restriction of ψ to M° induces a finite surjective unramifield covering

$$\psi^\circ\colon M^\circ \longrightarrow P^\circ \times T.$$

For each $t \in T$, let

$$\psi_t^\circ\colon \varphi^{-1}(t) \cap M^\circ \longrightarrow P^\circ \times \{t\} \xrightarrow{\sim} P^\circ$$

be the restriction of ψ° to the fibre over t.

We claim that for any pair of points $t, t' \in T$, there exists a homeomorphism

$$h\colon \varphi^{-1}(t) \cap M^\circ \longrightarrow \varphi^{-1}(t') \cap M^\circ$$

such that the diagram

$$\begin{tikzcd} \varphi^{-1}(t) \cap M^\circ \arrow[rr, "h"] \arrow[dr, "\psi_t^\circ"'] & & \varphi^{-1}(t') \cap M^\circ \arrow[dl, "\psi_{t'}^\circ"] \\ & P^\circ & \end{tikzcd}$$

is commutative.

Before proving this claim, let us continue the proof of Lemma 3.4. Since ψ_t° and $\psi_{t'}^\circ$ are finite unramifield coverings of P°, they are local analytic isomorphisms. Hence

$$h\colon \varphi^{-1}(t) \cap M^\circ \xrightarrow{\sim} \varphi^{-1}(t') \cap M^\circ$$

is necessarily an analytic isomorphism. Examining h on a disjoint open disc at each point of the finite ramification loci $\varphi^{-1}(t)\setminus(\varphi^{-1}(t)\cap M^\circ)$ and $\varphi^{-1}(t')\setminus(\varphi^{-1}(t')\cap M^\circ)$, we see by the Riemann Extension Theorem that h extends to a unique analytic isomorphism $h\colon \varphi^{-1}(t) \xrightarrow{\sim} \varphi^{-1}(t)$.

It remains to prove the above claim. Since T is path-connected, it suffices to show the claim for t' in a contractible open neighborhood (e.g., open ball) U of t. Denote by $\psi_U^\circ\colon \varphi^{-1}(U)\cap M^\circ \to P^\circ \times U$ the restriction of ψ°. Thus we have a commutative diagram

$$\begin{tikzcd} \varphi^{-1}(t) \cap M^\circ \arrow[rr, hook] \arrow[d, "\psi_t^\circ"'] & & \varphi^{-1}(U) \cap M^\circ \arrow[d, "\psi_U^\circ"] \\ P^\circ \arrow[r, "\sim"] & P^\circ \times \{t\} \arrow[r, hook] & P^\circ \times U. \end{tikzcd}$$

The finite surjective unramified covering ψ_U° corresponds, in terms of the fundamental groups, to a subgroup

$$\pi_1(\varphi^{-1}(U) \cap M^\circ) \subset \pi_1(P^\circ \times U)$$

of finite index. The restriction of this covering to the covering ψ_t° along the fibre corresponds to a subgroup

$$\pi_1(\varphi^{-1}(t) \cap M^\circ) \subset \pi_1(P^\circ).$$

Since U is assumed to be contractible, the restriction to the fibre induces isomorphisms

$$\begin{array}{ccc} \pi_1(\varphi(U) \cap M^\circ) & \subset & \pi_1(P^\circ \times U) \\ \wr\downarrow & & \downarrow\wr \\ \pi_1(\varphi^{-1}(t) \cap M^\circ) & \subset & \pi_1(P^\circ), \end{array}$$

hence a commutative diagram

$$\begin{tikzcd} (\varphi^{-1}(t) \cap M^\circ) \times U \arrow[rr, "\text{homeo}"] \arrow[dr, "p_1"'] & & \varphi^{-1}(U) \cap M^\circ \arrow[dl, "\psi_U^\circ"] \\ & U. & \end{tikzcd}$$

□

4. Fulton's proof of connectedness of $\mathfrak{M}_g$ $\boxed{\textbf{No manuscript}}$