Notes and Errata for AG2

CHAPTER I: SCHEMES AND SHEAVES: DEFINITIONS

1. Definition 3.5: $(Y, \mathcal{O}_Y)$, (not $\mathcal{O}_y$).

2. External tensor product of quasi-coherent sheaves, after Cor. 5.6 but before 5.7: Here $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G}$ stands for the *external* tensor product of quasi-coherent sheaves on X and Y; the result is a quasi-coherent sheaf on the fibre product $X \times_S Y$. It is perhaps better to use the notation $\mathcal{F} \boxtimes_{\mathcal{O}_S} \mathcal{G}$ instead, and reserve $\otimes$ for tensor product of sheaves of $\mathcal{O}_X$-modules on the *same* scheme X.

3. Definition 5.3: Here a coherent $\mathcal{O}_X$-module is defined to be an quasi-coherent $\mathcal{O}_X$-module locally of finite presentation. As noted in the footnote this definition will be used only when X is locally noetherian, in which case it suffices to require that it is locally of finite type.

CHAPTER II: EXPLORING THE WORLD OF SCHEMES

1. In the last (group of) displayed formula(e) in Example 3.8, the right hand side of the first line of the formula should be
$$\mathfrak{p} = \sqrt{\mathfrak{q}} \text{ and } \mathfrak{q} = R \cap \mathfrak{q} \cdot (R_{\mathfrak{p}})$$
(not $\mathfrak{q} = \sqrt{\mathfrak{q}}$)

2. Lines 3 and 4 of the footnote inserted in line -3 of the proof of Theorem 3.9: The sentence spanning line 3 and line 4 should be “This gives our assertion (i)” (not (ii)).

3. In the statement of Prop. 3.14, change the second sentence to: If $g \circ f$ is an immersion, then f is an immersion.

   The corresponding statement for *closed immersions* is false. As an example, let $Y = U_1 \cup U_2$ be “$\mathbb{A}^1$ with duplicated origin” as in Example 4.4, $X = U_1 = \operatorname{Spec} k[T_1]$, $Z = \operatorname{Spec} k[T]$, f be the inclusion of U_1 to X, and $g \colon X \to Z$ be the natural projection. Clearly $g \circ f$ is a closed immersion; in fact it is an isomorphism. But f is not a closed immersion—U_1 is not closed in X.

   PROOF OF PROP. 3.14: The morphism f is the composition of $\Gamma_f \colon X \to X \times_Z Y$ and $\mathrm{pr}_2 \colon X \times_Z Y \to Y$; pr_2 is an immersion since $g \circ f \colon X \to Z$ is, and we know that Γ_f is an immersion (true for every morphism).

4. In the statement of Prop. 4.5, change “assume X is separated” to “assume Y is separated”.

5. Two paragraphs before Theorem 5.9 (Elimination theory for Proj), in the opening sentence starting with “Note too that for $\mathbb{P}^n_R$, the realization of $\mathbb{P}^n_R \times_{\operatorname{Spec} R} \mathbb{P}^m_R$ as a Proj in (h) above ...”, ”(h)” should be changed to ”(i)”.

6. §6, according to the standard definition, a morphism $f \colon X \to Y$ is *proper* if it is separated, of finite type and universally closed. Here both X and Y are assumed to be separated over $\mathbb{Z}$ according to the convention in §4; this implies that f is separated.

7. Lemma 6.4: “$U_i \subset X'$, $V_i \subset X$” should read “$U_i \subset X$, $V_i \subset Y$”.

CHAPTER III. ELEMENTARY GLOBAL STUDY OF Proj$R$

1. The displayed formula in the statement of Lemma 4.6 should read

$$\varinjlim_{n} \operatorname{Hom}(R_{\geq n}, N_{\geq n}) \xrightarrow{\sim} \Gamma\widetilde{N}\,.$$

In line 16 of the proof of Lemma 4.6, the displayed formula should read

$$\operatorname{Hom}(R_{\geq n}, N_{\geq n}) \longrightarrow \Gamma\widetilde{N}\,,$$

In the statement and the proof of Lemma 4.6, the map from $\varinjlim_n \operatorname{Hom}(R_{\geq n}, N_{\geq n})$ to $\Gamma\widetilde{N}$ is assumed to be understood but not explicitly given. This map is defined as follow. Suppose that $\alpha\colon R_{\geq n} \to N_{\geq n}$ is a homomorphism of graded R-modules. The global section of $\widetilde{N}$ corresponding to α is given by $\frac{\alpha(x^I)}{(x^I)}$, where $I = (i_0, i_1, \ldots, i_r) \in \mathbb{N}^r$ is any multi-index with $|I| := i_0 + i_1 + \cdots + i_r \geq n$, and $x^I := x_0^{i_0} x_1^{i_1} \cdots x_r^{i_r}$. In other words, the restriction of this section to the affine open subset $(\mathbb{P}_A^r)_{x^I}$ is $\frac{\alpha(x^I)}{x^I}$, an element of degree 0 in the localization N_{x^I}.

CHAPTER IV. GROUND FIELDS AND BASE RINGS

1. (suggestion) In the proof of Lemma 6.1 (Hensel's Lemma), change the first line of the statement quoted from vol. II of Zariski-Samuel, p. 259, from "If B is an R-module such that" to "If B is a module over a complete local ring (R,M) such that"

CHAPTER V. SINGULAR VS. NON-SINGULAR

1. (suggested footnote) Four lines before Proposition 1.8, about Serre's conjecture on the positivity of intersection multiplicity

$$\chi(R/P, R/Q) := \sum_{i=0}^{\infty} \operatorname{length}_R \operatorname{Tor}_i^R(R/P, R/Q)\,,$$

where R is a regular local ring, P and Q are prime ideals in R such that $R/(P+Q)$ has finite length (therefore $\dim(R/P) + \dim(R/Q) \leq \dim(R)$. Serre conjectured that $\chi(R/P, R/Q) \geq 0$ (non-negativity) and $\chi(R/P, R/Q) > 0$ if and only if $\dim(R/P) + \dim(R/Q) = \dim(R)$ (positivity). Serre proved the assertions when R contains a field (the equi-characteristic case) using reduction to the diagonal. For the mixed characteristic case, the vanishing (the if part of the positivity conjecture) was proved in 1985 by Roberts and independently by Gillet-Soulé, and the non-negativity conjecture was proved by Gabber in the middle of 1990's. The positivity conjecture in the mixed characteristic case is still open.

2. (suggestion) The proof of Theorem 2.14 proves slightly more than stated and provides a 6-term exact sequence

$$0 \longrightarrow \Upsilon^{\Bbbk(x)}_{\mathscr{O}_{x,X} \otimes_{\mathscr{O}_{y,Y}} \Bbbk(y)/\Bbbk(y)} \longrightarrow \Upsilon_{\Bbbk(x)/\Bbbk(y)} \longrightarrow T^*_{x,f^{-1}(y)} \longrightarrow \Omega_{X/Y} \otimes_{\mathscr{O}_{x,X}} \Bbbk(x) \longrightarrow \Omega_{\Bbbk(x)/\Bbbk(y)} \longrightarrow 0\,,$$

where

$$\Upsilon^{\Bbbk(x)}_{\mathscr{O}_{x,X} \otimes_{\mathscr{O}_{y,Y}} \Bbbk(y)/\Bbbk(y)} := \operatorname{Ker}\Big(\Omega_{\Bbbk(y)/\mathbb{Z}} \otimes_{\Bbbk(y)} \Bbbk(x) \longrightarrow \Omega_{\mathscr{O}_{x,X} \otimes_{\mathscr{O}_{y,Y}} \Bbbk(y)/\mathbb{Z}} \otimes_{\mathscr{O}_{x,X}} \Bbbk(x)\Big)\,.$$

(Note that the $\mathscr{O}_x$ in the displayed 4-term exact sequence has been changed to $\mathscr{O}_{x,X}$.)

3. In the displayed formula in the statement of the Lemma in the proof of Theorem 2.14, change "$\Omega_{R/M} \otimes_R K$" to "$\Omega_{R/k} \otimes_R K$". (Here M is the maximal ideal of R, and R is a k-algebra.)

4. In the last displayed diagram in the proof of Theorem 2.14 (and the Lemma in the proof) the entry $\Omega_{R/K} \otimes_k K$ should be replaced by $\Omega_{R/k} \otimes_k K$. The same typo appeared in the last sentence of the proof of Theorem 2.14.

5. In the Example (after Lemma 4.7 and before Criterion 4.8), before the line of example (a), $f\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ should be replace by $f\colon \mathbb{A}^m_k \to \mathbb{A}^n_k$. (For instance example (b) is a morphism $f\colon \mathbb{A}^2_k \to \mathbb{A}^1_k$.

6. (suggestion) line -2 in the proof of Criterion 4.6: perhaps insert "by Theorem 2.14" after "is injective"?

7. In the proof of Criterion 4.10 (formal smoothness implies smoothness), there are too many X_1's: In the displayed formula before equation (4.11), X_1 denotes both one of the n variables $X_1,\dots,X_n$ *and* the scheme $\operatorname{Spec} R[X_1,\dots,X_n]/(f_1,\dots,f_{n-r})$,
   (suggestion): change the name of the variables to $T_1,\dots,T_n$.

   In the next paragraph in the proof of Criterion 4.6, I stands for an ideal in the coordinate ring of X_1, which is the image of the ideal I in the previous part of the proof.
   (suggestion): use I_1 for the image of I in the coordinate ring of X_1.

8. line -4 of the proof of Proposition-Definition 5.1: change "Then for all $y \in X$ with $x \in \overline{\{x\}}$" to "Then for all $y \in X$ with $y \in \overline{\{x\}}$"

9. The statement and proof of Proposition-Definition 5.1 use the concept of "embedded points", which does not seem to have been defined in Chapter 2, §3. (In Example 3.7, this concept is alluded to under quotations, but a definition was not given.)

   Definition. A point x of a locally noetherian scheme X is *not an embedded point* if the natural map $\mathscr{O}_{X,x} \to \Gamma(\operatorname{Spec}(\mathscr{O}_{X,x}) - \{x\})$ is injective. Equivalently, x is an embedded point of X if $\dim(\mathscr{O}_{X,x}) \geq 1$ and x is an associated point of $\mathscr{O}_{X,x}$.

10. line 2 of the proof of Proposition 5.3: $g \in \mathfrak{m}_{y,Y} \setminus f\mathscr{O}_{y,X} \setminus \mathfrak{m}^2_{y,X}$ should be changed to

$$g \in \mathfrak{m}_{y,Y} \setminus \left(f\mathscr{O}_{y,X} \cup \mathfrak{m}^2_{y,X}\right)$$

11. In the last group of displayed formulas in c_2),

    - change the first implication to

$$x \in \operatorname{Ass}(\mathscr{O}_{V(f^*(g))}) \implies f(x) \in \operatorname{Ass}(\mathscr{O}_{V(g)})$$

    - In the last implication, change "$V(f^*(g))$ has no embedded components through y" to

$$V(f^*(g)) \text{ has no embedded components through } x.$$

12. In the statement of Proposition 5.9, insert "$R_1 \neq (0)$" after "be a graded integral domain".

13. In the statement of U5), change "$\exists U \subset X$ Zariski open" to "$\exists U \subset X$ Zariski open dense"

14. (suggestion): Recall for the convenience of the readers
Definition. An irreducible algebraic variety X over $\mathbb{C}$ is *topologically unibranch* at a point $x \in X(\mathbb{C})$ if for every closed subvariety $Y \subsetneq X$ and every open subset $V \ni x$ for the classical topology, there exist a classical open neighborhood $U \ni x$ contained in V such that $U - (U \cap Y(\mathbb{C}))$ is connected for the classical topology.

15. (suggestion): (suggestion) Add the proof of the implication U4 $\Longrightarrow$ U5 for the convenience of the readers. (Note In Part I (3.24), Zariski's connectedness theorem is proved for birational correspondence for projective varieties, plus a footnote saying that the same proof works in more general situations.)

PROOF OF U4 $\Longrightarrow$ U5. The proof uses a basic fact, that the proper morphism $f\colon Z \to X$ induces a *topological proper* map $f_{\mathbb{C}}\colon Z(\mathbb{C}) \to X(\mathbb{C})$. (That means: the inverse image of any compact subsets of $X(\mathbb{C})$ is compact, which implies that the image of any closed subsets of $Z(\mathbb{C})$ is closed.)

Suppose that $f^{-1}(x)$ were the disjoint union of two non-empty close subvarieties Y_1 and Y_2. Then There exist disjoint classical open subsets $W_1 \supset Y_1(\mathbb{C})$ and $W_2 \supset Y_2(\mathbb{C})$ in $Z(\mathbb{C})$. Let

$$V_0 := X(\mathbb{C}) - f(X(\mathbb{C}) - (W_1 \cup W_2)),$$

an open neighborhood of x. Since X is topologically unibranch at x, there exists a classical open neighborhood V of x in W such that $V \cap U(\mathbb{C})$ is connected. From $V \subset V_0$ we get

$$f^{-1}(U(\mathbb{C}) \cap V) \subseteq f^{-1}(V) \subseteq W_1 \cup W_2 .$$

Since each fiber $f^{-1}(y)$ is connected for $y \in U(\mathbb{C}) \cap V$ and f is surjective, we deduce that

$$U(\mathbb{C}) \cap V \ \subseteq\ [(U(\mathbb{C}) \cap V) - f(Z(\mathbb{C}) - W_1)] \cup [(U(\mathbb{C}) \cap V) - f(Z(\mathbb{C}) - W_2)],$$

and the right hand side is a disjoint union of two open subsets of the connected open subset $U(\mathbb{C}) \cap V$ for the classical topology. So one of two open subsets is equal to $U(\mathbb{C}) \cap V$; say we have $(U(\mathbb{C}) \cap V) - f(Z(\mathbb{C}) - W_1) = U(\mathbb{C}) \cap V$. This implies that $f(W_1) \cap U(\mathbb{C}) = \emptyset$, or equivalently, $W_1 \subseteq f^{-1}(X - U)(\mathbb{C})$. That is impossible because $f^{-1}(X - U)$ is a proper subvariety of the irreducible variety Z and W_1 is an open subset of $Z(\mathbb{C})$. □

16. In the third displayed formula in proof of the implication (N3) $\Longrightarrow$ (N4) in 6.1, the formula should be

$$([x_i], f)\colon Z \longrightarrow \mathbb{P}^1_{\mathbb{C}} \times_{\operatorname{Spec}\mathbb{C}} X$$

In the last group of displayed formulas in the proof of (N3) $\Longrightarrow$ (N4) in 6.1,

- change the first "$g(t)$" to something like $p(t)$ and the second "g" to $p(t)$. (too many g's; g has been used for a morphism from Z' to Z.)
- typo: $\bar{t}^m \ \longrightarrow\ t^{-m}$

17. (suggestion) Line 2 of the statement of $\widetilde{U5}$, change "generic geometric fibre" to "geometric generic fibre"?

18. line 5 after the statement of Zariski's theorem of holomorphic functions in 6.3: change "approximating this with and element" to "approximating this with an element".

19. (suggestion) In the web of implications in 6.2, it is remarked that there are counterexamples to N3 $\Longrightarrow$ N1 and U3 $\Longrightarrow$ U1 in the Appendix of Nagata's book. It is perhaps useful to say that N3 $\Longrightarrow$ N1 and U3 $\Longrightarrow$ U1 when X is excellent, and say that (a) excellent rings include fields, $\mathbb{Z}$, complete local rings and Dedekind domains of generic characteristic 0, and (b) finitely generated algebras over excellent rings, including quotients of excellent rings, are excellent.

20. line -3 of the elementary proof of N1 $\Longrightarrow$ N4: Replace "It can be shown next that $Z'' \xrightarrow{\sim} X'$" by "This means that $Z' = \mathrm{Spec} R'$, where R' is local domain finite over the normal local domain $\widehat{\mathscr{O}}_{x,X}$ contained in the fraction field of $\widehat{\mathscr{O}}_{x,X}$. It follows that $Z'' \xrightarrow{\sim} X'$"

21. (suggestion: geometrically unibranch)
Definition. A *local* ring R is said to be *unibranch* if R_{red} is an integral domain and the integral closure of R_{red} in its fraction field is a local ring. If in addition the residue field of the integral closure of R_{red} is a purely inseparable extension of the residue field of R, then we say that R is *geometrically unibranch*. A scheme X is said to be unibranch or geometrically unibranch at a point x if the local ring $\mathscr{O}_{x,X}$ is.

Consider the following properties for a pair (X,x), where X is a noetherian integral scheme.

GU3) X is geometrically unibranch at x.

GU5) (strong form of Zariski's Connectedness Theorem) For every proper morphism $f\colon Z \to X$ with Z integral and $f(\eta_Z) = \eta_X$, if the generic fiber of f is geometrically connected, then $f^{-1}(x)$ is geometrically connected too.

Then we have the following implications.

$$\begin{array}{ccccc} \mathrm{N3} & \Longrightarrow & \mathrm{GU3} & \Longrightarrow & \mathrm{U3} \\ \Updownarrow & & \Updownarrow & & \Updownarrow \\ \mathrm{N4} & & \mathrm{GU5} & \Longrightarrow & \widetilde{\mathrm{U5}} \end{array}$$

Remark. There are alternative definitions of (geometric) unibranchedness, analogous to U2).

- (X,x) is unibranch $\Longleftrightarrow$ Spec $\left(\mathscr{O}^{\mathrm{h}}_{x,X}\right)$ is irreducible.
- (X,x) is geometrically unibranch $\Longleftrightarrow$ Spec $\left(\mathscr{O}^{\mathrm{sh}}_{x,X}\right)$ is irreducible.

Here

- $\mathscr{O}^{\mathrm{h}}_{X,x}$ is the *henselization* of $\mathscr{O}_{x,X}$, defiined as the inductive limit of all étale local algebras over $\mathscr{O}_{x,X}$ with trivial residue fields extension,
- $\mathscr{O}^{\mathrm{sh}}_{X,x}$ is the *strict henselization* of $\mathscr{O}_{x,X}$, defiined as the inductive limit of all étale local algebras over $\mathscr{O}_{x,X}$.

22. line 3 of the statement of Corollary 6.6 (Characterization of normalization), change "and X is isomorphic to the normalization of X" to "and Z is isomorphic to the normalization of X".

CHAPTER VII. THE COHOMOLOGY OF COHERENT SHEAVES

1. The line after the statement of Theorem 2.1, replace “ref” by “res”.
2. 3 lines before Proposition 3.7, about “there is a map of sheaves $f_*\mathcal{F} \to \mathcal{F}$ with respect to f”. There does not seem to be a spelled-out definition in the text about maps between sheaves with respect to a continuous map. So this can be fixed by inserting a definition in Chapter I, or replacing the above quote passage by “there is a map of sheaves $f^\bullet f_*\mathcal{F}$”.

CHAPTER VIII. APPLICATIONS OF COHOMOLOGY

1. In the first sentence of Chapter 8, change “to demonstrated” to “to demonstrate”.
2. In the appendix in §1, on residues of differentials on curves, the paragraph before Definition 1, the assertion “ϕ and ψ” induce mutually inverse isomorphisms” is problematic. The do give isomorphism between W and W', but the two isomorphism are not necessarily inverse of each other.
3. In the statement of (R6) in Theorem 3 of the appendix on residues of differentials on curves, change $V' = K' \otimes_k V$ to $V' = K' \otimes_K V$ (i.e. tensoring over K, not k).
4. (suggestion) In the proof of the beginning of Theorem 3 that $\mathrm{Tr}([f_1, g_1])$ induces a residue map $\mathrm{Res}_A^V : \Omega^1_{K/k} \longrightarrow k$, 4 lines before Lemma 4, it is probably better to choose $(fh)_1 := h_1 f_1$ (because K is commutative), delete “by the commutativity of K” in the next line, and replace the displayed formula by
$$[f_1, g_1 h_1] = [f_1 g_1, h_1] + [h_1 f_1, g_1]\,.$$
5. (suggestion) In the proof of Theorem 5 (ii), in the line before the displayed formula
$$\mathrm{Res}_{A_S}^{V_S}(fdg) = \mathrm{Res}_{A_{S\setminus S'}}^{V_{S\setminus S'}}(fdg) + \sum_{x\in S'} \mathrm{Res}_x(fdg)\,,$$
change “By (R5)” to “By (R5) and (R1)”.
6. (suggestion) At the end of the proof of (R5), insert between “by a similar argument” and ”equals $\mathrm{Res}_{A\cap B}^V(fdg) - \mathrm{Res}_B^V(fdg)$”, the clause
$$\text{using } F' := \{\, \theta \in \mathrm{End}_k(V) \,|\, \theta V \prec B,\ \theta B \prec A\cap B,\ \theta(A\cap B) \prec 0 \,\}\,.$$
7. In the proof of Theorem 6 (Serre duality for proper smooth curves over a field), Lemma 8 only treats the case when the closed point y_0 is rational over k; the argument there is not sufficient to cover the case when $[\mathbb{k}(y_0) : k] \cong 0 \pmod p$. There are two ways to get around. One way is to base change to the algebraic closure of k. One can also extend the local calculation in Lemma 8, as follows. Pick a k-morphism α from an open neighborhood U of y to $\mathbb{A}^1 = \mathrm{Spec}\, k[s]$ which is étale at y. Let $z = \alpha(y)$, $L = k(s)$. Write $\omega = h\alpha^* ds$, $h \in \mathbb{R}(X)$, and $\mathrm{ord}_y(h) = n$. We need to show that there exists an element $f_y \in K_y$ with $\mathrm{ord}_y(f_y) = -n-1$ such that $\mathrm{Res}_y(f_y h ds) \neq 0$. We know that $\mathrm{Tr}_{K_y/L_z}(A_y) = A_z$ because α is étale at y, so by Theorem 5 (iii) it suffices to exhibit an element $g_z \in L_z$ such that $\mathrm{ord}_z(g_z) = -1$ and $\mathrm{Res}_z(g_z ds) \neq 0$. Let $q(s)$ be a monic irreducible polynomial in $k[s]$ corresponding to the closed point $z \in \mathbb{A}^1_k$, and let $d = \deg(q(s))$. Then everything follows from the following formula
$$\mathrm{Res}_z\left(\left(\sum_{1\le i\le N} \frac{a_{-i}(s)}{q(s)^i} \right) \cdot ds \right) = b_{d-1}\,, \qquad a_{-1}(s) = \sum_{0\le j\le d-1} b_j \cdot s^j$$

for local residues at z of rational differentials, where each $a_{-i}(s)$ is a polynomial in $k[s]$ of degree at most $d-1$.

The last formula can be proved either by direct computation using Tate's definition, or using the residue theorem and compute

$$-\mathrm{Res}_{\infty}\left(\left(\sum_{1\leq i\leq N}\frac{a_{-i}(s)}{q(s)^i}\right)\cdot ds\right)=b_{d-1}$$

because z and ∞ are the only poles.