Exercise 10, 11/20/2005

1. Let F be a number field. Find an explicit bound, C, in terms of $|\mathrm{disc}(F/Q)|$, and r_1, r_2 for F, such that every element of the ideal class group for F is represented by an ideal $I \subset \mathcal{O}_F$ such that $|\mathcal{O}_F/I| \leq C$. Illustrate this method: Find two number fields with non-trivial class groups and compute the class groups.

In Problems 2–11, C denotes an algebraic curve over $\mathbb{F}_q$ corresponding to a global field $K \supset \mathbb{F}_p$ with $\mathbb{F}_q$ as its field of constants. Recall that a vector bundle $\mathcal{F}$ of rank m over C is a pair (E, U), where E is an m-dimensional vector space over K and $U = \prod_{v \in \Sigma_F} U_v$ is an adelic lattice in $E \otimes_K \mathbb{A}_K$. In our notation, $E = \mathcal{F}_\eta$ and $U = U(\mathcal{F})$. We defined $\deg(\mathcal{F})$ to be $\mu(U(\mathcal{F}))$, where $\mu = \mu_E$ is the canonical measure on $E \otimes_K \mathbb{A}_K$. Also, define $\mathrm{H}^0(\mathcal{F})$ and $\mathrm{H}^1(\mathcal{F})$ by $\mathrm{H}^0(\mathcal{F}) := E \cap U$, $\mathrm{H}^1(\mathcal{F}) := (E \otimes_K \mathbb{A}_K)/(E + U)$. Define $\det(\mathcal{F})$ to be the line bundle corresponding to the pair $(\bigwedge_K^m(\mathcal{F}_\eta), \bigwedge_R^m(U(\mathcal{F})))$, where $R := \prod_{v \in \Sigma_F} \mathcal{O}_v$. The direct sum of two vector bundles, the dual of a vector bundle and the tensor product of two vector bundles is defined similarly. For instance, the dual $\mathcal{F}^\vee$ of $\mathcal{F}$ defined to be the pair $\Big(\mathrm{Hom}_K(\mathcal{F}_\eta, K),\ \prod_{v \in \Sigma_F} \mathrm{Hom}_{\mathcal{O}_v}(U_v, \mathcal{O}_v)\Big)$. The canonical line bundle $\omega = \omega_C$ on C is the pair $(\Omega, \prod_{v \in \Sigma_F} \omega_v)$, where $\Omega := \mathrm{Hom}_{\mathrm{cont}, \mathbb{F}_q}(\mathbb{A}_K/K, \mathbb{F}_q)$, and ω_v is the largest $\mathcal{O}_v$-submodule in $\Omega \otimes_K K_v$ consisting of elements which are trivial on $\mathcal{O}_v$.

2. Show that $\mathrm{Hom}_{\mathrm{cont}, \mathbb{F}_q}(\mathbb{A}_K/K, \mathbb{F}_q)$ is a one-dimensional vector space, isomorphic to the vector space $\mathrm{Hom}_{\mathrm{cont}, \mathbb{F}_q}(\mathbb{A}_K/K, \mathbb{C}^\times)$ when post-composed with $\exp \circ \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$.

3. Prove that $\mathrm{Hom}_{\mathbb{F}_q}(\mathrm{H}^1(\mathcal{F}), \mathbb{F}_q)$ is naturally isomorphic to $\mathrm{H}^0(\mathcal{F}^\vee \otimes \omega_C)$.

4. Let $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2$ be vector bundles on C of degree m, m_1, m_2 respectively.

(i) Show that $\deg(\mathcal{F}^\vee) = -m$, where $\mathcal{F}^\vee$ is the dual of $\mathcal{F}$.

(ii) Show that $\deg(\mathcal{F}_1 \oplus \mathcal{F}_2) = \deg(\mathcal{F}_1) + \deg(\mathcal{F}_2)$.

(iii) Show that $\deg(\mathcal{F}_1 \otimes \mathcal{F}_2) = \deg(\mathcal{F}_1)^{m_2} + \deg(\mathcal{F}_2)^{m_1}$.

5. Show that $\deg(\mathcal{F}) = \deg(\det(\mathcal{F}))$.

6. Let $\mathcal{L}$ be a line bundle on C.

(i) Let s be a non-zero element of $\mathcal{L}_\eta$. Show that

$$\deg(\mathcal{L}) = \log_q[U(\mathcal{L}) : U(\mathcal{L}) \cap (\prod_v \mathcal{O}_v) \cdot s] - \log_q[(\prod_v \mathcal{O}_v) \cdot s : U(\mathcal{L}) \cap (\prod_v \mathcal{O}_v) \cdot s]$$

(ii) Suppose that $\deg(\mathcal{L}) < 0$. Show that $\mathrm{H}^0(\mathcal{L}) = (0)$.

(iii) Suppose that $\deg(\mathcal{L}) = 0$ and that $\mathrm{H}^0(\mathcal{L}) \neq (0)$. Prove that $\mathcal{L} \cong \mathcal{O}_C$, where $\mathcal{O}_C$ is the pair $(K, \prod_{v \in \Sigma_F} \mathcal{O}_v)$.

7. Let C be a curve of genus zero. Show that any two line bundles of the same degree are isomorphic.

8. Let K be the rational function field $\mathbb{F}_q(t)$.

(i) Compute $\mu(\mathbb{A}_K/K)$ and show that the curve corresponding to K, denoted $\mathbb{P}^1_{\mathbb{F}_q}$, has genus zero. Here μ is the canonical measure on $\mathbb{A}_K$

(ii) Compute the volume of $\mathbb{A}^\times_{K,1}/K^\times$. Here we use the Haar measure on $\mathbb{A}^\times_{K,1}$ induced by the Haar measure $\frac{\mu}{||\cdot||}$ on $\mathbb{A}^\times_K$ and the measure $\frac{dx}{x}$ on $\mathbb{R}^\times_{>0}$.

9. Classify all line bundles on $\mathbb{P}^1_{\mathbb{F}_q}$.

10.* Classify all curves of genus zero over $\mathbb{F}_q$.

11.* Classify all vector bundles on $\mathbb{P}^1_{\mathbb{F}_q}$.