Math 620 Home Work Assignment 2

Problems 1–3 below aims to clarify the linear algebra involved in the definition of the Shimura reflex field and the reflex type norm. Problem 4 contains the linear algebra used in the proof of the basic congruence result for abelian varieties with complex multiplication. In the application to complex multiplication, the field E is a CM field, and Φ is a CM-type. Problem 5 is the famous example of Mumford of Hodge cycles not spanned by divisor classes.

1. Let E be a number field, and let $\Phi \subset \mathrm{Hom}_{\mathrm{alg}}(E, \overline{\mathbb{Q}})$ be a subset of embeddings of E into an algebraic closure $\overline{\mathbb{Q}_\ell}$ of $\mathbb{Q}$. Let $\underline{E^\times} = \mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_\mathrm{m})$. The characters group of $\underline{E^\times}$ is naturally the free abelian group with basis $\{\chi_\sigma : \sigma \in \mathrm{Hom}_{\mathrm{alg}}(E, \overline{\mathbb{Q}})\}$, where χ_σ is the character of $\underline{E^\times}$ whose restriction to $E^\times = \underline{E^\times}(\mathbb{Q})$ coincides with the restriction of $\sigma : E \to \mathbb{Q}$ to $E^\times$. The natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathrm{Hom}_{\mathrm{alg}}(E, \overline{\mathbb{Q}})$ is compatible with the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $X^*(\underline{E^\times})$. Let $\mu_\Phi : \mathbb{G}_\mathrm{m} \to \underline{E^\times}$ be the cocharacter of $\underline{E^\times}$ such that

$$\langle \chi_\sigma, \mu_\Phi \rangle = \begin{cases} 1 & \text{if} \quad \sigma \in \Phi \\ 0 & \text{if} \quad \sigma \notin \Phi \end{cases}$$

under the pairing between the $X^*(\underline{E^\times})$ and $X_*(\underline{E^\times})$, for every $\sigma \in \mathrm{Hom}_{\mathrm{alg}}(E, \overline{\mathbb{Q}})$.

   (a) Let $\mathrm{Stab}_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(\Phi)$ be the open subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which stabilizes the subset $\Phi \subseteq \mathrm{Hom}_{\mathrm{alg}}(E, \overline{\mathbb{Q}})$. Show that the field of definition $F(\Phi))$ of μ_Φ is the fixed subfield of $\mathrm{Stab}_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(\Phi)$ in $\overline{\mathbb{Q}}$.

   (b) Let F be a number field. Show that the $\mathbb{Q}$-torus $\mathrm{Res}_{F/\mathbb{Q}}(\underline{E^\times})$ is naturally isomorphic to

$$\underline{(E \otimes_\mathbb{Q} F)^\times} = \mathrm{Res}_{E\otimes_\mathbb{Q} F/\mathbb{Q}}(\mathbb{G}_\mathrm{m})\,.$$

   (c) Show that $X^*\big(\underline{(E \otimes_\mathbb{Q} F)^\times}\big)$ is the free abelian group with bases

$$\{\, \chi_{\sigma,\tau} : \sigma \in \mathrm{Hom}_{\mathrm{alg}}(E, \overline{\mathbb{Q}}), \tau \in \mathrm{Hom}_{\mathrm{alg}}(F, \overline{\mathbb{Q}}) \,\}\,,$$

   where $\chi_{\sigma,\tau}$ is the character of $\underline{(E \otimes_\mathbb{Q} F)^\times}$ whose restriction to the rational points $(E \otimes_\mathbb{Q} F)^\times$ of the torus $\underline{(E \otimes_\mathbb{Q} F)^\times}$ is induced by

$$\sigma \cdot \tau : E \otimes_\mathbb{Q} F \xrightarrow{\sigma\otimes\tau} \overline{\mathbb{Q}} \otimes_\mathbb{Q} \overline{\mathbb{Q}} \xrightarrow{\text{mult}} \overline{\mathbb{Q}}\,.$$

   Moreover the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $X^*\big(\underline{(E \otimes_\mathbb{Q} F)^\times}\big)$ comes from the diagonal action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathrm{Hom}_{\mathrm{alg}}(E, \overline{\mathbb{Q}}) \times \mathrm{Hom}_{\mathrm{alg}}(F, \overline{\mathbb{Q}})$.

2. Keep the notations in Problem 1. above. Let $F = F(\Phi)$, and let

$$\mathrm{Res}_{F/\mathbb{Q}}(\mu_\Phi) : \underline{F^\times} = \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_{\mathrm{m}}) \to \mathrm{Res}_{F/\mathbb{Q}}(\underline{E^\times}) = \underline{(E \otimes_{\mathbb{Q}} F)^\times}$$

be the $\mathbb{Q}$-rational homomorphism obtained by applying the functor $\mathrm{Res}_{F/\mathbb{Q}}$ to the cocharacter μ_Φ, which is defined over F. Show that the map from $X^*\big(\underline{(E \otimes_{\mathbb{Q}} F)^\times}\big)$ to $X^*\big(\underline{F^\times}\big)$ induced by $\mathrm{Res}_{F/\mathbb{Q}}(\mu_\Phi)$ is

$$\chi_{\sigma,\tau} \mapsto \begin{cases} \chi_\tau & \text{if} \quad \sigma \in \tau \cdot \Phi \\ \mathbf{1} & \text{if} \quad \sigma \notin \tau \cdot \Phi \end{cases} \qquad \begin{array}{l} \forall\, \sigma \in \mathrm{Hom}_{\mathrm{alg}}(E, \overline{\mathbb{Q}}), \\ \forall\, \tau \in \mathrm{Hom}_{\mathrm{alg}}(F, \overline{\mathbb{Q}})\,. \end{array}$$

Notice that the $\tau \cdot \Phi$ is a well-defined subset of $\mathrm{Hom}_{\mathrm{alg}}(E, \overline{\mathbb{Q}})$, since the fixer of F in $\overline{\mathbb{Q}}$ is the stabilizer subgroup of Φ in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

3. Notations as above. Consider the standard representation of $E^\times$ on E as a $\mathbb{Q}$-rational linear representation of $\underline{E^\times}$. The weights of this representation are the characters χ_τ of $\underline{E^\times}$, $\tau \in \mathrm{Hom}_{\mathrm{alg}}(E, \overline{\mathbb{Q}})$, and each weight space V_{χ_σ} is one-dimensional. The direct sum

$$\oplus_{\sigma \in \Phi} V_{\chi_\sigma}$$

is a subrepresentation of the standard representation of E, and is defined over F. In other words, there is a unique F-linear subspace $M = M(\Phi) \subset E \otimes_{\mathbb{Q}} F$, which is stable under $E^\times$, such that $M \otimes_{\mathbb{F}} \overline{\mathbb{Q}} = \oplus_{\sigma \in \Phi} V_{\chi_\sigma}$. There is a natural structure on M as an $(E \otimes_{\mathbb{Q}} F)$-module.

   (a) The $(E \otimes_{\mathbb{Q}} F)$-module M defines a $\mathbb{Q}$-rational representation of $\underline{(E \otimes_{\mathbb{Q}} F)^\times}$, of dimension $\mathrm{Card}(\Phi) \cdot [F : \mathbb{Q}]$. Show that the weights of this representation are

   $$\left\{\, \chi_{\sigma,\tau} \,|\, \sigma \in \mathrm{Hom}_{\mathrm{alg}}(E, \overline{\mathbb{Q}}),\ \tau \in \mathrm{Hom}_{\mathrm{alg}}(F, \overline{\mathbb{Q}})\,,\ \sigma \in \tau\Phi \,\right\}\,,$$

   and each weight space is one-dimensional.

   (b) We have two $\mathbb{Q}$-rational homomorphisms from $\underline{(E \otimes_{\mathbb{Q}} F)^\times}$. On the one hand, we have the composition

   $$\mathrm{N}\mu_\Phi : \underline{F^\times} = \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_{\mathrm{m}}) \xrightarrow{\mathrm{Res}_{F/\mathbb{Q}}(\mu_\Phi)} \mathrm{Res}_{F/\mathbb{Q}}(\underline{E^\times}) \xrightarrow{\mathrm{Nm}_{F/\mathbb{Q}}} \underline{E^\times}\,,$$

   where the second arrow is the $F/\mathbb{Q}$-norm for the torus $\underline{E^\times}$. On the other hand, multiplication by elements of $F^\times$ are E-linear automorphisms of M. Taking the E-linear determinant gives a $\mathbb{Q}$-rational homomorphism

   $$\det_{M,E} : \underline{F^\times} \to \underline{E^\times}$$

   of tori. Prove that these two homomorphisms are equal.

4. Let p be a prime number, and let let $\mathfrak{P}$ be a prime ideal of $\mathcal{O}_F$ lying above p. Let $M_{\mathbb{Z}}$ be a $\mathbb{Z}$-lattice in M which is stable under multiplication by $\mathcal{O}_E \times_{\mathbb{Z}} \mathcal{O}_F$. The quotient module $M_{\mathbb{Z}}/\mathfrak{P}M$ is an Artinian $\mathcal{O}_E$-module; denote by $\chi_{\mathcal{O}_E}(\chi(M/\mathfrak{P}M)$ its "Euler characteristic" as defined in *Corps Locaux*, Chap. I, §5. (See Chap. I, §5 and Chap. III, §1 for some basic properties of this Euler characteristic.) Prove the following equality

$$\chi_{\mathcal{O}_E}(M/\mathfrak{P}M) = \mathrm{N}\mu_{\Phi}(\mathfrak{P}) = \det_{M,E}(\mathfrak{P})$$

of $\mathcal{O}_E$-modules.

5. Suppose that K is a CM field with $[K:\mathbb{Q}] = 8$, such that the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ of the Galois closure L of K is isomorphic to the group of all rigid symmetries of a cube $C \in \mathbb{R}^3$, and $\mathrm{Gal}(L/K)$ corresponds to the stabilizer of a vertex of the cube C. The Galois group $\mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to $S_4 \times \mu_2$; the complex conjugation, corresponding to $(1,-1) \in S_4 \times \mu_2$, induces the antipodal map about the center of the cube C. Let Φ be the set of embeddings of K into $\mathbb{C}$ corresponding to the four vertices of a chosen face of C; Φ is a CM-type for Φ. Let (A, α, K, Φ) be an 8-dimensional abelian variety with multiplication $\alpha : K \to \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ by K of CM-type Φ.

   (a) Determine the Mumford-Tate group $\mathrm{sMT}(\mathrm{H}_1(A(\mathbb{C}), \mathbb{Q}))$.

   (b) Determine the dimension of the Hodge cycles in $\wedge^4(\mathrm{H}_1(A(\mathbb{C}), \mathbb{Q}))(-2)$ and the dimension of the linear span of products of divisor classes in $\wedge^4(\mathrm{H}_1(A(\mathbb{C}), \mathbb{Q}))(-2)$.

   (c) Determine the dimension of the largest subgroup of $\mathrm{GL}(\mathrm{H}_1(A(\mathbb{C}), \mathbb{Q}))$ which fixes all divisor classes in self-products $A \times \cdots \times A$ of A.

   (d) By varying the CM-type Φ, what are the possible dimensions of the resulting Mumford-Tate group?

6. Determine the dimension of the Mumford-Tate group of an abelian variety with complex multiplication by a CM field K such that the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ of the Galois closure of K is isomorphic to the group of all symmetries of a regular icosahedron, the stabilizer subgroup of a vertex corresponds to $\mathrm{Gal}(L/K)$, and the CM-type Φ corresponding to the subset consisting of a vertex v and the five vertices adjacent to v.

7. Let E be a totally real number field. Suppose that $h : \mathfrak{S} \to \underline{E^\times}$ is a $\mathbb{Q}$-rational homomorphism from the Serre group $\mathfrak{S}$ to $\underline{E^\times}$. Prove that h factors through the embedding $\mathbb{G}_{\mathrm{m}} \to \underline{E^\times}$.