Math 602 Midterm Exam, Fall 2006

Part I. For each of the following, either give an example, or show that none exists.

1. (10 pts) A non-cyclic abelian group with 42 elements.

2. (10 pts) A group of order 16 which that has no subgroup of order 4.

3. (10 pts) A solvable group which is not nilpotent.

4. (10 pts) A non-split extension of groups, i.e. a surjective homomorphism of groups $\pi : G \to Q$ such that there is no homomorphism $\alpha : Q \to G$ with $\pi \circ \alpha = \mathrm{Id}_Q$.

5. (10 pts) A surjective $\mathbb{R}$-linear homomorphism $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 \to \mathbb{R}^3$.

6. (10 pts) A ring R, a left R-module N and a short exact sequence of left R-modules

$$0 \to M' \to M \to M'' \to 0$$

such that the induced sequence of abelian groups

$$0 \to \mathrm{Hom}_R(M'', N) \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M', N) \to 0$$

is not exact.

7. (10 pts) A ring R, a left R-module N and a short exact sequence of left R-modules

$$0 \to M' \to M \to M'' \to 0$$

such that the induced sequence of abelian groups

$$0 \to M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0$$

is not exact.

8. (15 pts) A prime ideal $I \subset \mathbb{Q}[\mathbb{Z}/4\mathbb{Z}] =: R$ such that $\dim_{\mathbb{Q}}(R/I) \geq 2$.

9. (15 pts) A maximal ideal in $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$.

Part II.

10. (20 pts) (a) Determine the cardinality of the groups $\mathrm{Aut}_{\mathrm{grp}}((\mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/7\mathbb{Z}))$.
(b) Is the group $\mathrm{Aut}_{\mathrm{grp}}((\mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/7\mathbb{Z}))$ cyclic? (Either prove that it is cyclic, or prove that it is not cyclic.)

11. (15 pts) Let M be the $\mathbb{Z}$-module $(\mathbb{Z}/10\mathbb{Z}) \oplus (\mathbb{Z}/25\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$. Determine the $\mathbb{Z}$-modules $\bigwedge_{\mathbb{Z}}^3 M$.

11a. (15 extra credit pts) Find the structure of the $\mathbb{Z}$-module $\bigwedge_{\mathbb{Z}}^2 M$.

12. (20 extra credit pts) Is there a finitely generated non-trivial $\mathbb{Z}$-module M such that there exist a natural number $N_0 \in \mathbb{N}$ such that $\mathrm{S}_{\mathbb{Z}}^i M = (0)$ for all $i \geq N_0$? (Either give an example, or show that none exists.)