Group (co)homologies. Summary

G: a group $\qquad \mathrm{Mod}_G$ = the category of left $\overset{G}{\mathbb{Z}[G]}$ modules

1. $M \in \mathrm{Mod}_G \rightsquigarrow \begin{cases} H_i(G,M) = \mathrm{Tor}_i^{\mathbb{Z}[G]}(M, \mathbb{Z}) & \forall i \geq 0 \\ H^i(G,M) = \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, M) \end{cases}$

trivial G-module

turned into a right G-module via $G \xrightarrow{\sim} G^{opp}$, $\sigma \mapsto \sigma^{-1}$

$$0 \to I_G \longrightarrow \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \qquad \varepsilon : \sum_{\sigma\in G} n_\sigma [\sigma] \longmapsto \sum_{\sigma \in G} n_\sigma$$

augmentation ideal

Explicit formula, from the bar resolution

$$\begin{tikzcd} 0 & \mathbb{Z} \arrow[l] & C_0(G) \arrow[l, "\varepsilon"'] & C_1(G) \arrow[l, "\partial_1"'] & C_2(G) \arrow[l, "\partial_2"'] & \cdots \arrow[l] & C_{n-1}(G) \arrow[l] \arrow[d, equal] & C_n(G) \arrow[l, "\partial_n"'] \arrow[d, equal] & \cdots \arrow[l] \\ & & & & & & \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G^{n-1}] & \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G^n] & \end{tikzcd}$$

$\mathbb{Z}[G]$-module structure from "the first factor" $\mathbb{Z}[G]$ ⟶

$$\sigma_0 \cdot \sigma_1[\sigma_2, \dots, \sigma_n] + \sum_{i=1}^{n-1} (-1)^i \sigma_0[\sigma_1, \dots, \sigma_i\sigma_{i+1}, \dots, \sigma_n] + (-1)^n \sigma_0[\sigma_1, \sigma_2, \dots, \sigma_{n-1}] \xleftarrow{\partial_n} \sigma_0 \cdot [\sigma_1, \dots, \sigma_n]$$

$$\rightsquigarrow \begin{cases} H_i(G,M) = H_i(\cdots \to C_n(G) \otimes_G M \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} C_1(G) \otimes_G M \xrightarrow{\partial_1} C_0(G) \otimes_G M \to 0) \quad \forall i \geq 0 \\ \qquad C_n(G) \otimes_G M = C_n(G) \otimes_{\mathbb{Z}} M / \langle \sigma x \otimes \sigma m - x \otimes m \mid \sigma \in G, x \in C_n(G), m \in M \rangle \\ H^i(G,M) = H^i(0 \to \mathrm{Hom}_{\mathbb{Z}[G]}(C_0(G), M) \xrightarrow{d^1} \cdots \to \underset{\overset{\parallel}{\mathrm{Maps}(G^n, M)}}{\mathrm{Hom}_{\mathbb{Z}[G]}(C_n(G), M)} \xrightarrow{d^n} \cdots) \end{cases}$$

$H_0(G,M) = M_G = M/I_G \cdot M$, $\qquad H_1(G, \mathbb{Z}) = G^{ab} = G/(G,G)$

G-coinvariants $\qquad H_1(G,M) = G^{ab} \otimes_{\mathbb{Z}} M$ if G operates trivially on M

G-invariants

$H^0(G,M) = M^G$, $\qquad H^1(G,M) = \mathrm{Hom}_{grp}(G,M)$ if G operates trivially on M

In particular, $H^1(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{grp}(G^{ab}, \mathbb{Q}/\mathbb{Z}) =:$ Pontryjagin dual of G^{ab}

$H^2(G,M)$ = equivalence classes of group extensions $1 \to M \to \tilde{G} \to G \to 1$ such that $(M$ + the conjugation action$)$ = the given G-module structure on M

(Recall that

$$\begin{matrix} 0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0 \\ 0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \end{matrix} \Longrightarrow$$

$H_i(G, \mathbb{Z}[G]) = 0\ \forall i \geq 1 \rightsquigarrow H_1(G, \mathbb{Z}) \cong H_0(G, I_G) = I_G/I_G^2$

Assume $|G| = g < \infty$. Then $H_i(G, \mathbb{Q}) = 0\ \forall i \geq 1$, $H^i(G, \mathbb{Q}) = 0\ \forall i \geq 1$.

$H^i(G, \mathbb{Z}[G]) = 0\ \forall i \geq 1$,

$H_1(G, \mathbb{Q}/\mathbb{Z}) = 0$, $H_1(G, \mathbb{Q}) = 0$, $H^i(G, \mathbb{Z}[G]) = 0\ \forall i \geq 1$

$H^1(G, \mathbb{Z}) = 0$, $H^2(G, I_G) = 0$,

$H^1(G, I_G) \cong \mathbb{Z}/g\mathbb{Z}$

2. Tate cohomology groups: Assume $|G| = g < \infty$

Let $C_{-r-1}(G) \overset{\text{def}}{=} \mathrm{Hom}_{\mathbb{Z}}(C_r(G), \mathbb{Z})$ $\forall r \geq 0$, with $\overset{\text{left}}{G}$-action by $(\sigma.\underbrace{\lambda}_{\in C_{-r-1}(G)})(x) = \lambda(\sigma^{-1}x)$ $\forall x \in C_r(G)$

"complete resolution"

$$\begin{tikzcd}
\cdots & C_{-3}(G) \arrow[l, "\partial_{-3}"'] & C_{-2}(G) \arrow[l, "\partial_{-2}=(\partial_2)^\vee"'] & C_{-1}(G) \arrow[l, "\partial_{-1}=(\partial_1)^\vee"'] & C_0(G) \arrow[l, "\partial_0"'] \arrow[dl, "\varepsilon"] & C_1(G) \arrow[l, "\partial_1"'] & C_2(G) \arrow[l, "\partial_2"'] & \cdots \arrow[l] \\
 & & & \mathbb{Z} \arrow[u, "\varepsilon^\vee"] & & & &
\end{tikzcd}$$

$$\begin{tikzcd}
\partial_0 : & C_0(G) \arrow[r] \arrow[d, equal] & C_{-1}(G) \arrow[d, equal] & \\
 & \mathbb{Z}[G] & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], \mathbb{Z}) \arrow[r, "\sim"] & \mathrm{Maps}(G, \mathbb{Z}) \\
 & {[1]} \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & \sum_{\sigma \in G} \delta_\sigma \arrow[r, leftrightarrow] & \text{the const. function } 1 \\
 & & \delta_\sigma\left(\sum_{\tau \in G} n_\tau[\tau]\right) = n_\sigma \arrow[u] &
\end{tikzcd}$$

$\hat{H}^i(G, M)$ $\forall i \in \mathbb{Z}$

$$\begin{tikzcd}
\hat{H}^i(G, M) \arrow[d, equal, "\text{def}"] & & & & & & \\
H^i\Big(\cdots \arrow[r] & \overset{\deg=-3}{C_2(G) \otimes_G M} \arrow[r, "\overset{\overset{\partial_2}{\|}}{d^{-3}}"] & \overset{\deg=-2}{C_1(G) \otimes_G M} \arrow[r, "\overset{\overset{\partial_1}{\|}}{d^{-2}}"] & \overset{\deg=-1}{C_0(G) \otimes_G M} \arrow[r, "d^0"] \arrow[d] & \mathrm{Hom}_G(C_0(G), M) \arrow[r, "d^1"] & \mathrm{Hom}_G(C_1(G), M) \arrow[r, "d^2"] & \cdots\Big) \\
 & & & \mathbb{Z} \otimes_G M \arrow[r] \arrow[d, equal] & \mathrm{Hom}_G(\mathbb{Z}, M) \arrow[u] \arrow[d, equal] & & \\
 & & & M_G \arrow[r, "N_G"] & M^G & &
\end{tikzcd}$$

$N_G = \sum_{\sigma \in G} \sigma$ (called either norm or trace)

$$\hat{H}^i(G, M) = \begin{cases} H^i(G, M) & \text{if } i \geq 1 \\ H_{-i-1}(G, M) & \text{if } i \leq -2 \end{cases}$$

$$\hat{H}^{-1}(G, M) = \mathrm{Ker}\left(M_G \xrightarrow{N_G} M^G\right) = M[N_G] / I_G M$$

$$\hat{H}^0(G, M) = \mathrm{Coker}\left(M_G \xrightarrow{N_G} M^G\right) = M^G / N_G \cdot M$$

Properties: If M is a projective $\mathbb{Z}[G]$-module, then $\hat{H}^i(G, M) = 0$ $\forall i \in \mathbb{Z}$

Special case: $G \cong \mathbb{Z}/n\mathbb{Z} = \langle \sigma \rangle$, $n \geq 2$. Let $N = 1 + \sigma + \dots + \sigma^{n-1}$

$\rightsquigarrow$ a periodic complete resolution

$$\begin{tikzcd}
\cdots & \overset{\deg=-2}{\mathbb{Z}[G]} \arrow[l, "N"'] & \overset{\deg=-1}{\mathbb{Z}[G]} \arrow[l, "{[\sigma]-1}"'] & \overset{\deg=0}{\mathbb{Z}[G]} \arrow[l, "N"'] \arrow[d] & \mathbb{Z}[G] \arrow[l, "{[\sigma]-1}"'] & \mathbb{Z}[G] \arrow[l, "N"'] & \mathbb{Z}[G] \arrow[l, "{[\sigma]-1}"'] & \cdots \arrow[l, "N"'] \\
 & & \mathbb{Z} \arrow[u] \arrow[r, equal] & \mathbb{Z} & & & &
\end{tikzcd}$$

$\Rightarrow$ $\hat{H}^i(G, M) \cong \hat{H}^{i+2}(G, M)$ $\forall i \in \mathbb{Z}$

$$\begin{cases} \hat{H}^{\text{even}}(G, M) \cong M^G / N_G \cdot M \\ \hat{H}^{\text{odd}}(G, M) \cong M[N_G] / ([\sigma]-1) \cdot M \end{cases}$$

Definition (Herbrand quotient) $G \cong \mathbb{Z}/n\mathbb{Z}$, $G \cong \mathbb{Z}/n\mathbb{Z}$

$$h_{0/1}(M) = \frac{\# \hat{H}^0(G,M)}{\# \hat{H}^1(G,M)}$$ if both $\hat{H}^0(G,M)$ and $\hat{H}^1(G,M)$ are finite

Proposition $G \cong \mathbb{Z}/n\mathbb{Z}$,

(a) If $0 \to M' \to M \to M'' \to 0$ exact in Mod_G, then $h_{0/1}(M) = h_{0/1}(M') \cdot h_{0/1}(M'')$, and if 2 of the 3 terms are defined, so is the third.

(b) If $\# M < \infty$, then $h_{0/1}(M) = 1$

$$\left(\begin{tikzcd}
0 \arrow[r] & M^G \arrow[r] & M \arrow[r, "\sigma-1"] & M \arrow[r] & M_G \arrow[r] & 0 \\
0 \arrow[r] & \hat{H}^{-1}(G,M) \arrow[r] & M_G \arrow[r, "N_G"] & M^G \arrow[r] & \hat{H}^0(G,M) \arrow[r] & 0
\end{tikzcd}\right)$$

3. Change of groups

3.1 $\lambda : H \longrightarrow G$ group homomorphism, M: left G-module ⇝ $\mathrm{Res}_\lambda M$: left H-module

h induces a map of chain complexes $\lambda_\cdot : C_\cdot(H) \longrightarrow C_\cdot(G)$ $\quad x_0\otimes[x_1,\dots,x_n] \mapsto \lambda(x_0)\cdot[\lambda(x_1),\dots,\lambda(x_n)]$ $\forall\, x_0, x_1, \dots, x_n \in H$

(a) $C_\cdot(H)\otimes_H M \xrightarrow{\lambda_\cdot\otimes_H \mathrm{id}_M} C_\cdot(G)\otimes_H M \longrightarrow C_\cdot(G)\otimes_G M$ induces

$$H_i(\lambda_*) : H_i(H, \overset{\mathrm{Res}_\lambda M}{M}) \longrightarrow H_i(G, M)$$

(b) $\mathrm{Hom}_G(C_\cdot(G), M) \longrightarrow \mathrm{Hom}_H(C_\cdot(G), M) \xrightarrow{\mathrm{Hom}(\lambda_\cdot\,,M)} \mathrm{Hom}_H(C_\cdot(H), M)$ induces

$$H^i(\lambda^*) : H^i(G, M) \longrightarrow H^i(H, \overset{\mathrm{Res}_\lambda M}{M})$$

Both are morphisms of δ-functors on Mod_G = the category of all left G-modules

⇝ $H_\cdot(\lambda_*)$ is determined by $H_0(\lambda_*) : M_H \longrightarrow M_G$

$H^\cdot(\lambda^*)$ is determined by $H^0(\lambda^*) : M^G \longrightarrow M^H$

3.2 When $H \leqslant G$, $N \in \mathrm{ob}(\mathrm{Mod}_H)$, have

$$C_\cdot(G) \otimes_G \underbrace{\left(\mathbb{Z}[G]\otimes_{\mathbb{Z}[H]} N\right)}_{\overset{\|}{\mathrm{ind}_H^G N}} \cong \underbrace{C_\cdot(G)}_{\overset{\|}{\mathrm{Res}_H^G(C_\cdot(G))} \leftarrow \text{ a free resolution of } \mathbb{Z} \text{ in } \mathrm{Mod}_H} \otimes_{\mathbb{Z}[H]} N$$

$$\mathrm{Hom}^G\Big(C_\cdot(G), \underbrace{\mathrm{Ind}_H^G N}_{\overset{\|}{\{f : G\to N \mid f(hx) = h\cdot f(x)\ \forall h\in H, \forall x\in G\}}}\Big) \cong \mathrm{Hom}^H\left(\mathrm{Res}_H^G C_\cdot(G), N\right)$$

$$\Longrightarrow \begin{cases} H_i(G, \mathrm{ind}_H^G N) \cong H_i(H, N) \\ H^i(G, \mathrm{Ind}_H^G N) \cong H^i(H, N) \end{cases} \forall_{i\geq 0} \quad \text{"Shipiro's Lemma"}$$

$M \in \mathrm{Mod}_G$ ⇝

$$\begin{cases} H_\cdot((H\hookrightarrow G)_*, M) \doteq H_\cdot\Big[C_\cdot(G)\otimes_{\mathbb{Z}[G]}\Big(\overbrace{\left(\mathbb{Z}[G]\otimes_{\mathbb{Z}[H]} M\right)}^{\mathrm{Ind}_H^G \mathrm{Res}_H^G M \atop \|} \longrightarrow M\Big)\Big] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \sigma\otimes_H m \longmapsto \sigma\cdot m \\ H^\cdot((H\hookrightarrow G)^*, M) \doteq H^\cdot\Big[\mathrm{Hom}_G\Big(C_\cdot(G), M \longrightarrow \mathrm{Ind}_H^G \mathrm{Res}_H^G M\Big)\Big] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad m \longmapsto (x\mapsto x\cdot m) \end{cases}$$

3.3. If $H \leqslant G$ and $[G:H] = a < \infty$, have transfer/corestriction maps

$M \in \mathrm{Mod}_G$

$$H_i(G, M) \xrightarrow{\mathrm{Ver}} H_i(H, M)$$
$$H^i(G, M) \xleftarrow{\mathrm{Ver}} H^i(H, M)$$

and similarly if $|G| < \infty$ $\quad \hat{H}^i(G, M) \xleftarrow{\mathrm{Ver}} \hat{H}^i(H, M)$

Defining properties of transfer

For $H^\cdot$: $\quad H^0(H, M) = M^H \xrightarrow{N_{G/H}} M^G = H^0(G, M)$

$$m \longmapsto \sum_{x\in G/H} x\cdot m$$

For $H_\cdot$: $\quad H_0(G, M) = M_G \xrightarrow{N_{H\backslash G}} M_H = H_0(H, M)$

$$m \bmod I_G M \longmapsto \sum_{x\in H\backslash G} x\cdot m \bmod I_H M \;\leftarrow\; \text{Well-defined in } M_I$$

(x_i) system of representatives of $H\backslash G$

$\sigma \in G$. Write $x_i \cdot \sigma = h_{\pi(i)} \cdot x_{\pi(i)}$

$\uparrow$ permutation of $H\backslash G$, depending on σ

In $\mathbb{Z}[G]$, have

$$\sum_i x_i \cdot (\sigma - 1) = \sum_i h_{\pi(i)} x_{\pi(i)} - \sum_i x_i = \sum_i (h_i - 1)\cdot \sigma \in I_H \cdot \mathbb{Z}[G]$$

$$\begin{tikzcd} H^i(G,M) \arrow[r, "\mathrm{Res}"] \arrow[rr, bend right, "{[G:H]}"'] & H^i(H,M) \arrow[r, "\mathrm{Ver}"] & H^i(G,M) \end{tikzcd}$$ Check H^0

Have

$$\begin{tikzcd} H_i(G,M) \arrow[r, "\mathrm{Ver}"] \arrow[rr, bend right, "{[G:H]}"'] & H_i(H,M) \arrow[r, "(H \hookrightarrow G)_*"] & H_i(G,M) \end{tikzcd}$$ check H_0

Explicit formula for a quasi-isom of complexes in Mod_H: in Assignment 13.

$$\mathrm{Res}^G_H \, C.(G) \xrightarrow{\text{q. isom}} C.(H)$$

3.3. Restriction – inflation sequence (for a normal subgroup)

$N \trianglelefteq G \qquad M \in \mathrm{Mod}_G$

$$0 \to H^i(G/N, M^N) \xrightarrow{\mathrm{Inf}} H^i(G,M) \xrightarrow{\mathrm{Res}} H^i(N,M)$$

$$0 \leftarrow H_i(G/N, M_N) \longleftarrow H_i(G,M) \longleftarrow H_i(N,M)$$

are exact when $i=1$, (direct computation)

and for $i = q$ if $H^j(G,M) = 0$, $H_j(G,M) = 0$ for $1 \leq j \leq q-1$

(either by dimension shifting or use Hochschild-Serre s. seq.

$$E_2^{ij} = H^i(G/N, H^j(N,M)) \Longrightarrow H^{i+j}(G,M)$$)

March 29, 2021

First, review + further comments on cup product

4. Cup product

characterization by functorial properties G: finite group $M, N \in \mathrm{Mod}_G$

$$\hat{H}^i(G, M) \times \hat{H}^j(G, N) \longrightarrow \hat{H}^{i+j}(G, M \underset{\mathbb{Z}}{\otimes} N)$$
$$(a\ ,\ b) \longmapsto a \cdot b$$

(alternative notation: $a \cup b$)

(i) functorial in M and N

(ii) When $i = j = 0$, it is induced by the natural map
$$M^G \underset{\mathbb{Z}}{\otimes} N^G \longrightarrow (M \underset{\mathbb{Z}}{\otimes} N)^G$$

(iii) If $0 \to M' \to M \to M'' \to 0$ is exact and $0 \to M' \underset{\mathbb{Z}}{\otimes} N \to M \underset{\mathbb{Z}}{\otimes} N \to M'' \underset{\mathbb{Z}}{\otimes} N \to 0$ is also exact, then $\forall a'' \in \hat{H}^i(G, M'')$, ${}^\forall b \in \hat{H}^j(G, N)$, we have
$$\underbrace{(\delta a'')}_{\substack{\pitchfork \\ \hat{H}^{i+1}(G, M')}} \cdot b = \delta\underbrace{(a'' \cdot b)}_{\substack{\pitchfork \\ \hat{H}^{i+j}(G, M'' \underset{\mathbb{Z}}{\otimes} N)}} \in \hat{H}^{i+j+1}(G, M' \underset{\mathbb{Z}}{\otimes} N)$$

(iv) If $0 \to N' \to N \to N'' \to 0$ is exact and $0 \to M \underset{\mathbb{Z}}{\otimes} N' \to M \underset{\mathbb{Z}}{\otimes} N \to M \underset{\mathbb{Z}}{\otimes} N'' \to 0$ is also exact, $a \in \hat{H}^i(G, M)$, $b \in \hat{H}^j(G\ N'')$

then
$$(-1)^i \cdot a \cdot \underbrace{\delta b''}_{\substack{\pitchfork \\ \hat{H}^{j+1}(G, N')}} = \delta\underbrace{(a \cdot b'')}_{\substack{\pitchfork \\ \hat{H}^{i+j}(G, M \underset{\mathbb{Z}}{\otimes} N'')}} \in \hat{H}^{i+j+1}(G, M \underset{\mathbb{Z}}{\otimes} N')$$

<u>Remark</u> Have explicit chain map (a co-pairing) which induces the cup product
$$C_\bullet(G) \xrightarrow{\ \Phi\ } \underbrace{C_\bullet(G) \underset{\mathbb{Z}}{\otimes} C_\bullet(G)}_{\text{graded tensor product}}$$

[Note By def^n, Φ is a collection of maps $C_{i+j}(G) \xrightarrow{\varphi_{i,j}} C_i(G) \underset{\mathbb{Z}}{\otimes} C_j(G)$, $i, j \in \mathbb{Z}$

(It is not a map from the $\mathbb{Z}$-module $C_\bullet(G)$ to the module $C_\bullet(G) \underset{\mathbb{Z}}{\otimes} C_\bullet(G)$)]

Ref: Cartan–Eilenberg, Homological Algebra.

Grothendieck "Tohoku"

5. Cohomologically trivial modules for finite groups — Theorems of Tate and Nakayama

Prop. 5.1 Nakayama (p-groups) Let G be a finite p-group. Let M be a G-module

(a) Suppose that $p\cdot M = 0$. If $\hat{H}^{q_0}(G, M)=0$ for some $q_0 \in \mathbb{Z}$. Then M is a free $\mathbb{F}_p[G]$-module, and $\hat{H}^q(K, M)=0$ for all $q \in \mathbb{Z}$ and every subgroup $K \leq G$. (i.e. M is cohomologicall trivial)

(b) Suppose that M is torsion-free, over $\mathbb{Z}$ and $\hat{H}^{q_0}(G,M)=0=\hat{H}^{q_0+1}(G,M)=0$ for some $q_0 \in \mathbb{Z}$. Then

(b1) $\hat{H}^q(K, M)=0$ $\forall q \in \mathbb{Z}$, $\forall K \leq \cancel{M}\, G$

(b2) M/pM is a free $\mathbb{F}_p[G]$-module

(b3) $\forall$ M: $\mathbb{Z}$-free torsion-free G-module N, we have $\hat{H}^q(K, \mathrm{Hom}_{\mathbb{Z}}(M,N))=0$ $\forall q \in \mathbb{Z}, \forall K \leq G$

Will see in thm 5.2 that: (b4) $\exists$ a projective resolution $0 \to P_1 \to P_0 \to M \to 0$ of length=1 of M in Mod_G

Pf. (a) dimension shift $\rightsquigarrow$ $\exists$ a G-module N s.t. $p\cdot N=0$ and $\hat{H}^{n-q_0-2}(G, N) \cong H^n(G, M)=0$ $\forall n \in \mathbb{Z}$.

$I_G \otimes \mathbb{F}_p$ is nilpotent

Assumption: $H_1(G, N) = \hat{H}^{-2}(G, N)=0$

Let $L \overset{\alpha}{\twoheadrightarrow} N$ be a G-linear surjection s.t. $\underbrace{L}_{\text{free } \mathbb{F}_p[G]\text{-module}}/I_G L \xrightarrow[\alpha]{\sim} N/I_G N$.

$0 \to Q \to L \xrightarrow{\alpha} N \to 0 \Rightarrow \underline{Q/I_G Q = (0)}$, i.e. $I_G \cdot Q = Q$

exer

Easy Fact/Exer: $\exists\, m_0 \in \mathbb{N}$ s.t. $\mathbb{F}_p \otimes I_G^{m_0}=0$. (E.g. induction on $|G|$.)

$\rightsquigarrow$ $Q = I_G Q = I_G^2 Q = \cdots = I_G^{m_0} Q = 0$, i.e. $L \xrightarrow{\sim} N$

(b) $0 \to M \xrightarrow{p} M \to M/pM \to 0 \rightsquigarrow$ Assumption $\Rightarrow \hat{H}^{q_0}(G, M/pM)=0$, M/pM is a free $\mathbb{F}_p[G]$-module.. $0 \to N \xrightarrow{p} N \to N/pN \to 0$ exact

$$\rightsquigarrow\ 0 \to \mathrm{Hom}_{\mathbb{Z}}(M,N) \xrightarrow{p} \mathrm{Hom}_{\mathbb{Z}}(M,N) \to \mathrm{Hom}_{\mathbb{Z}}(M, N/pN) \to 0$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|\wr$

Key observation: is a free $\mathbb{F}_p[G]$-module $\longrightarrow$ $\mathrm{Hom}_{\mathbb{F}_p}(M/pM, N/pN)$

$$M/pM = \bigoplus_{i \in I} \mathbb{F}_p[G]\cdot e_i \Rightarrow \bigoplus_{i \in I} \mathbb{F}_p[G]\cdot \mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_p \cdot e_i, N/pN)$$

$$\Rightarrow \quad \hat{H}^q(K, \overset{\mathrm{Hom}_{\mathbb{Z}}(M,N)}{\cancel{M}}) \xrightarrow[\sim]{p} H^q(K, \overset{\mathrm{Hom}_{\mathbb{Z}}(M,N)}{\cancel{M}}) \quad \forall q \in \mathbb{Z}, \forall K \leq G$$

killed by $|G| \in p^{\mathbb{N}}$

a group killed by $|K| \in p^{\mathbb{N}}$

QED.

$\hat{H}^j(G,L)=0 \quad \forall j\in\mathbb{Z}$

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] & L \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

e.g. L: free $\mathbb{F}[G]$-module

induced from $\{1\}$ | $\mathbb{Z}[G]\otimes_{\mathbb{Z}}$ (a $\mathbb{F}_p$-v. space)

$$\begin{tikzcd} \hat{H}^i(G,L) \arrow[r] \arrow[d, phantom, "\parallel" rotate=90] & \underline{\hat{H}^{\boxed{i}}(G,M)} \arrow[r] \arrow[r, bend left, squiggly] & \underline{\hat{H}^{\boxed{i+1}}(G,N)} \arrow[r] & \hat{H}^{i+1}(G,L) \arrow[d, phantom, "\parallel" rotate=90] \\ 0 & & & 0 \end{tikzcd}$$

dually,

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & \underbrace{\mathrm{Ind}_{\{1\}}^{G}(\ \ )}_{} \arrow[r] & N' \arrow[r] & 0 \end{tikzcd}$$

(Nakayama 1957)

Thm 5.2 G : a finite group, M : a G-module

a Sylow p-subgroup of G

Assume that $\forall$prime $p \mid \#G$, $\exists\, q(p) \in \mathbb{Z}$ s.t. $\hat{H}^{q(p)}(G_p, M) = 0 = \hat{H}^{q(p)+1}(G_p, M)$

(a) $\hat{H}^q(K, M) = 0 \quad \forall q \in \mathbb{Z}, \ \forall K \leq G$

(b) $\exists$ a G-linear projective resolution $0 \to P_1 \to P_0 \to M \to 0$ of length $=1$.

(c) If M is a free $\mathbb{Z}$-module, then M is a projective $\mathbb{Z}[G]$-module

($\Leftrightarrow$ M is a projective $\mathbb{Z}$-module)

Pf. of (c) : Pick a short exact sequence $0 \to Q \to \underset{\substack{\uparrow \\ \text{free}}}{L} \to M \to 0$ in Mod_G.

(rmk. Want to split $0 \to Q \to L \to M \to 0$)

Consider the exact sequence $0 \to \mathrm{Hom}_{\mathbb{Z}}(M, Q) \to \mathrm{Hom}_{\mathbb{Z}}(M, L) \to \mathrm{Hom}_{\mathbb{Z}}(M, M) \to 0$

$\because$ M is a free $\mathbb{Z}$-module

Note: Not every torsion free $\mathbb{Z}$-module is projective: every projective $\mathbb{Z}$-module is free; $\mathbb{Q}$ is a torsion free and flat $\mathbb{Z}$-module but not a free $\mathbb{Z}$-module.

$H^1(G_p, \mathrm{Hom}_{\mathbb{Z}}(M, Q)) = 0 \quad \forall p \mid \#G$ by 5.1 (b)

$\Longrightarrow H^1(G, \mathrm{Hom}_{\mathbb{Z}}(M, Q)) = 0 \quad \because \ \mathrm{Ker}\left(H^1(G, \mathrm{Hom}_{\mathbb{Z}}(M,Q)) \xrightarrow{\text{restriction}} H^1(G_p, \mathrm{Hom}_{\mathbb{Z}}(M,Q))\right)$

$= H^1(G, \mathrm{Hom}_{\mathbb{Z}}(M,Q))\,\big[[G:G_p]\big]$

$= \{ h \in H^1(G, \mathrm{Hom}_{\mathbb{Z}}(M,Q)) \mid \underset{=0}{[G:G_p]h} \}$

$\Longrightarrow$ M is a direct summand of L, hence projective

Proofs of (a) + (b): Pick a short exact sequence $0 \to R \to L \to M \to 0$ in Mod_G.

R is a free $\mathbb{Z}$-module, and $\hat{H}^{q(p)+1}(G_p, R) = \hat{H}^{q(p)+2}(G_p, R) \quad \forall p \mid \#G$

Every submodule of a free module over a PID is free.

$\rightsquigarrow$ R is a projective $\mathbb{Z}[G]$-module by (c).

QED.

Cor. 5.3 G : a finite group, B, C : G-modules. $f : B \to C$ G-linear

(mapping cone construction) Suppose that $\forall$prime $p \mid \#(G)$, $\exists\, q(p) \in \mathbb{Z}$ s.t.

$f_q^* : \hat{H}^q(G_p, B) \to \hat{H}^q(G_p, C)$ is surjective for $q = q(p)$,

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bijective for $q = q(p)+1$ and injective for $q = q(p)+2$. Then $\forall q \in \mathbb{Z}$ and $\forall K \leq G$,

$$f_q^* : \hat{H}^q(K, B) \xrightarrow{\sim} \hat{H}^q(K, C)$$

Pf: $0 \longrightarrow B \longrightarrow C \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], B) \longrightarrow D \to 0$ short exact

$b \longmapsto \left(f(b),\ \varphi_b = (x \longmapsto x \cdot b)_{\forall x \in G}\right)$

Assumption $\Longrightarrow \hat{H}^{q(p)}(G_p, D) = 0 = \hat{H}^{q(p)}(G_p, D) \quad \forall p \mid \# D$

$\xRightarrow{\text{Thm 5.2}} \hat{H}^q(K, D) = 0 \quad \forall q \in \mathbb{Z} \Longrightarrow \hat{H}^q(K, B) \xrightarrow[\sim]{f_q^*} \hat{H}^q(K, C)$

Prop. 5.4 $A, B, C \in \mathrm{Mod}_G$, $\varphi: A \otimes_{\mathbb{Z}} B \longrightarrow C$ G-linear

Given $\alpha \in \hat{H}^q(G, A)$. Assume $\forall$ prime $p \mid \#G$, $\exists n(p) \in \mathbb{Z}$ s.t. the map

$$\begin{tikzcd}
\hat{H}^n(G_p, B) \arrow[r] & \hat{H}^{n+q}(G_p, C) \\
\beta \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \alpha \cup \beta
\end{tikzcd}$$

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is surjective for $n = n(p)$, bijective for $n = n(p)+1$ and injective for $n = n(p)+2$. Then for every $K \leq G$ and every $n \in \mathbb{Z}$, the map

$$\begin{tikzcd}
H^n(K, B) \arrow[r] & \hat{H}^{n+q}(K, C) \\
\beta \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \mathrm{Res}^G_K(\alpha) \cup \beta \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

is an isomorphism.

Pf. The case $q=0$ follows from Cor. 5.3.

shift dimension with diagrams

$$0 \to A' \to \mathbb{Z}[G] \otimes_{\mathbb{Z}} A \to A \to 0 \quad + \quad 0 \to C' \to \mathbb{Z}[G] \otimes_{\mathbb{Z}} C \to C \to 0 \quad + \quad A' \otimes_{\mathbb{Z}} B \to C'$$

and

$$0 \to A \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A) \to A'' \to 0, \quad 0 \to C \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], C) \to C'' \to 0 \quad + \quad A'' \otimes_{\mathbb{Z}} B \to C''.$$

Theorem 5.5 (Tate 1952) Let M be a G-module, and let $\alpha \in H^2(G, M)$

Assume (i) $H^1(G_p, M) = 0$

(ii) $H^2(G_p, M) = \mathbb{Z} \cdot \mathrm{Res}_{G \geq G_p}(\alpha) \cong \mathbb{Z}/\#G_p \cdot \mathbb{Z}$ $\quad \forall$ prime $p \mid \#G$.

Then

$$\begin{tikzcd}
\hat{H}^n(K, \mathbb{Z}) \arrow[r, "\sim"] & \hat{H}^{n+2}(K, M) \\
\beta \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \mathrm{Res}_{G \geq K}(\alpha) \cup \beta
\end{tikzcd} \qquad \forall n, \ \forall K \leq G.$$

Immediate Corollary of Prop. 5.4, with $A = M$, $B = \mathbb{Z}$, $n(p) = -1$ $\forall p$

Note that $H^1(G_p, \mathbb{Z}) = 0$, and the injectivity of $H^1(G_p, \mathbb{Z}) \longrightarrow H^3(G_p, M)$ holds trivially.

The case important for application is the isomorphism

$$K^{ab} = \hat{H}^{-2}(K, \mathbb{Z}) \xrightarrow{\sim} \hat{H}^0(K, M) = M^K / N_K \cdot M$$