Group (co)homologies. Summary

G: a group $\qquad \mathrm{Mod}_G$ = the category of left $\underset{\mathbb{Z}[G]}{G}$ modules

1. $M \in \mathrm{Mod}_G \rightsquigarrow \begin{cases} H_i(G, M) = \mathrm{Tor}_i^{\mathbb{Z}[G]}(M, \mathbb{Z}) & \forall i \geq 0 \\ H^i(G, M) = \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, M) \end{cases}$

$\mathbb{Z}$: trivial G-module

M: turned into a right G-module via $G \xrightarrow{\sim} G^{opp}$, $\sigma \longmapsto \sigma^{-1}$

$$0 \to I_G \longrightarrow \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \qquad \varepsilon : \sum_{\sigma \in G} n_\sigma [\sigma] \longmapsto \sum_{\sigma \in G} n_\sigma$$

I_G: augmentation ideal

Explicit formula, from the bar resolution

$$0 \leftarrow \mathbb{Z} \xleftarrow{\varepsilon} C_0(G) \xleftarrow{\partial_1} C_1(G) \xleftarrow{\partial_2} C_2(G) \leftarrow \cdots \leftarrow C_{n-1}(G) \xleftarrow{\partial_n} C_n(G) \leftarrow \cdots$$

$\mathbb{Z}[G]$-module structure from "the first factor" $\mathbb{Z}[G]$ $\longrightarrow$ $C_{n-1}(G) = \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G^{n-1}]$, $\quad C_n(G) = \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G^n]$

$$\sigma_0 \cdot \sigma_1 [\sigma_2, \ldots, \sigma_n] + \sum_{i=1}^{n-1} (-1)^i \sigma_0 [\sigma_1, \ldots, \sigma_i \sigma_{i+1}, \ldots, \sigma_n] + (-1)^n \sigma_0 [\sigma_1, \sigma_2, \ldots, \sigma_{n-1}] \xleftarrow{\partial_n} \sigma_0 \cdot [\sigma_1, \ldots, \sigma_n]$$

$$\rightsquigarrow \begin{cases} H_i(G, M) = H_i\left(\cdots \to C_n(G) \otimes_G M \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} C_1(G) \otimes_G M \xrightarrow{\partial_1} C_0(G) \otimes_G M \to 0\right) \quad \forall i \geq 0 \\ \qquad C_n(G) \otimes_G M = C_n(G) \otimes_{\mathbb{Z}} M / \langle \sigma x \otimes \sigma m - x \otimes m \mid \sigma \in G, x \in C_n(G), m \in M \rangle \\ H^i(G, M) = H^i\left(0 \to \mathrm{Hom}_{\mathbb{Z}[G]}(C_0(G), M) \xrightarrow{d^1} \cdots \to \mathrm{Hom}_{\mathbb{Z}[G]}(C_n(G), M) \xrightarrow{d^n} \cdots\right) \end{cases}$$

$\mathrm{Hom}_{\mathbb{Z}[G]}(C_n(G), M) = \mathrm{Maps}(G^n, M)$

$H_0(G, M) = M_G := M / I_G \cdot M$, $\quad$ (M_G: G-coinvariants) $\qquad H_1(G, \mathbb{Z}) = G^{ab} = G/(G,G)$

$H_1(G, M) = G^{ab} \otimes_{\mathbb{Z}} M$ if G operates trivially on M

$H^0(G, M) = M^G$, (M^G: G-invariants) $\qquad H^1(G, M) = \mathrm{Hom}_{grp}(G, M)$ if G operates trivially on M

In particular, $H^1(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{grp}(G^{ab}, \mathbb{Q}/\mathbb{Z}) =:$ Pontryagin dual of G^{ab}

$H^2(G, M)$ = equivalence classes of group extensions $1 \to M \to \tilde{G} \to G \to 1$ such that (M + the conjugation action) = the given G-module structure on M

(Recall that

$$\begin{aligned} 0 &\to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0 \\ 0 &\to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \end{aligned} \quad \Longrightarrow$$

$H_i(G, \mathbb{Z}[G]) = 0 \ \forall i \geq 1 \rightsquigarrow H_1(G, \mathbb{Z}) \cong H_0(G, I_G) = I_G / I_G^2$

Assume $|G| = g < \infty$. Then $H_i(G, \mathbb{Q}) = 0 \ \forall i \geq 1$, $H^i(G, \mathbb{Q}) = 0 \ \forall i \geq 1$, $H^i(G, \mathbb{Z}[G]) = 0 \ \forall i \geq 1$,

$H_1(G, \mathbb{Q}/\mathbb{Z}) = 0$, $H_1(G, \mathbb{Q}) = 0$, $H^i(G, \mathbb{Z}[G]) = 0 \ \forall i \geq 1$

$H^1(G, \mathbb{Z}) = 0$, $H^2(G, I_G) = 0$,

$H^1(G, I_G) \cong \mathbb{Z}/g\mathbb{Z}$

2. Tate cohomology groups: Assume $|G| = g < \infty$

Let $C_{-r-1}(G) \overset{\text{def}}{=} \mathrm{Hom}_{\mathbb{Z}}(C_r(G), \mathbb{Z})$ $\forall r \geq 0$, with left G-action by $(\sigma.\lambda)(x) = \lambda(\sigma^{-1}x)$ $\forall x \in C_r(G)$, $\lambda \in C_{-r-1}(G)$

"complete resolution"

$$\begin{tikzcd} \cdots & C_{-3}(G) \arrow[l, "\partial_{-3}"'] & C_{-2}(G) \arrow[l, "\partial_{-2} = (\partial_2)^\vee"'] & C_{-1}(G) \arrow[l, "\partial_{-1} = (\partial_1)^\vee"'] & C_0(G) \arrow[l, "\partial_0"'] \arrow[d, "\varepsilon"] & C_1(G) \arrow[l, "\partial_1"'] & C_2(G) \arrow[l, "\partial_2"'] & \cdots \arrow[l] \\ & & & & \mathbb{Z} \arrow[ul, "\varepsilon^\vee"] & & & \end{tikzcd}$$

$\partial_0 : C_0(G) \longrightarrow C_{-1}(G)$

$$\begin{array}{ccc} C_0(G) & \longrightarrow & C_{-1}(G) \\ \| & & \| \\ \mathbb{Z}[G] & & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], \mathbb{Z}) \xrightarrow{\sim} \mathrm{Maps}(G, \mathbb{Z}) \\ \text{\rotatebox{90}{$\in$}} & & \\ [1] & \longmapsto & \sum_{\sigma \in G} \delta_\sigma \longleftrightarrow \text{the const. function } 1 \end{array}$$

$\delta_\sigma\left(\sum_{\tau \in G} n_\tau [\tau]\right) = n_\sigma$

$$\hat{H}^i(G, M) \quad \forall i \in \mathbb{Z}$$

$\|$ def

$$H^i\left(\begin{tikzcd}[column sep=small] \cdots \arrow[r] & \underset{\deg = -3}{C_2(G) \otimes_G M} \arrow[r, "d^{-3} = \partial_2"] & \underset{\deg=-2}{C_1(G) \otimes_G M} \arrow[r, "d^{-2} = \partial_1"] & \underset{\deg=-1}{C_0(G) \otimes_G M} \arrow[r, "d^0"] \arrow[d] & \mathrm{Hom}_G(C_0(G), M) \arrow[r, "d^1"] & \mathrm{Hom}_G(C_1(G), M) \arrow[r, "d^2"] & \cdots \\ & & & \mathbb{Z} \otimes_G M \arrow[r] \arrow[d, equal] & \mathrm{Hom}_G(\mathbb{Z}, M) \arrow[u] \arrow[d, equal] & & \\ & & & M_G \arrow[r, "N_G"] & M^G & & \end{tikzcd} \right)$$

$N_G = \sum_{\sigma \in G} \sigma$ (called either norm or trace)

$$\hat{H}^i(G, M) = \begin{cases} H^i(G, M) & \text{if } i \geq 1 \\ H_{-i-1}(G, M) & \text{if } i \leq -2 \end{cases}$$

$$\hat{H}^{-1}(G, M) = \mathrm{Ker}\left(M_G \xrightarrow{N_G} M^G\right) = M[N_G]/I_G M$$

$$\hat{H}^0(G, M) = \mathrm{Coker}\left(M_G \xrightarrow{N_G} M^G\right) = M^G/N_G M$$

$\mathrm{ind}_{\{1\}}^G N \xrightarrow{\sim} \mathrm{Ind}_{\{1\}}^G N$

Properties: If M is a projective $\mathbb{Z}[G]$-module, then $\hat{H}^i(G, M) = 0$ $\forall i \in \mathbb{Z}$

Special case: $G \cong \mathbb{Z}/n\mathbb{Z} = \langle \sigma \rangle$, $n \geq 2$. Let $N = 1 + \sigma + \cdots + \sigma^{n-1}$

$\rightsquigarrow$ a periodic complete resolution

$$\begin{tikzcd} \cdots & \mathbb{Z}[G] \arrow[l, "N"'] & \underset{\deg=-2}{\mathbb{Z}[G]} \arrow[l, "{[\sigma]-1}"'] & \underset{\deg=-1}{\mathbb{Z}[G]} \arrow[l, "N"'] & \underset{\deg=0}{\mathbb{Z}[G]} \arrow[l, "{[\sigma]-1}"'] \arrow[d] & \mathbb{Z}[G] \arrow[l, "N"'] & \mathbb{Z}[G] \arrow[l, "{[\sigma]-1}"'] & \cdots \arrow[l, "N"'] \\ & & & \mathbb{Z} \arrow[u] \arrow[r, equal] & \mathbb{Z} & & & \end{tikzcd}$$

$\Rightarrow$ $\hat{H}^i(G, M) \cong \hat{H}^{i+2}(G, M)$ $\forall i \in \mathbb{Z}$

$$\begin{cases} \hat{H}^{\text{even}}(G, M) \cong M^G/N_G \cdot M \\ \hat{H}^{\text{odd}}(G, M) \cong M[N_G]/([\sigma]-1) \cdot M \end{cases}$$

Definition (Herbrand quotient) $G \cong \mathbb{Z}/n\mathbb{Z}$, $G \cong \mathbb{Z}/n\mathbb{Z}$

$$h_{0/1}(M) = \frac{\# \hat{H}^0(G,M)}{\# \hat{H}^1(G,M)} \quad \text{if both } \hat{H}^0(G,M) \text{ and } \hat{H}^1(G,M) \text{ are finite}$$

Proposition $G \cong \mathbb{Z}/n\mathbb{Z}$,

(a) If $0 \to M' \to M \to M'' \to 0$ exact in Mod_G, then $h_{0/1}(M) = h_{0/1}(M') \cdot h_{0/1}(M'')$, and if 2 of the 3 terms are defined, so is the third.

(b) If $\# M < \infty$, then $h_{0/1}(M) = 1$

$$\left(\begin{array}{l} 0 \to M^G \longrightarrow M \xrightarrow{\sigma - 1} M \longrightarrow M_G \longrightarrow 0 \\ 0 \to \hat{H}^{-1}(G,M) \longrightarrow M_G \xrightarrow{N_G} M^G \longrightarrow \hat{H}^0(G,M) \longrightarrow 0 \end{array} \right)$$

p: odd prime

$G = \mathbb{Z}/p\mathbb{Z}$

e.g. $M \subseteq N \leftarrow$ free $\mathbb{Z}$-module

$\mathbb{Z}[G] \cong \mathbb{Z}[x]/(x^p-1)$

$M \otimes_{\mathbb{Z}} \mathbb{Q} \subseteq N \otimes_{\mathbb{Z}} \mathbb{Q}$ e.g. $\mathbb{Q}[G] \cong \mathbb{Q}[x]/(x^p-1) = \mathbb{Q} \times \mathbb{Q}(\zeta_p)$, $\zeta_p = e^{2\pi i/p}$

$M' \subseteq$

s.t. $M' \otimes_{\mathbb{Z}} \mathbb{Q} = M \otimes_{\mathbb{Z}} \mathbb{Q}$

$\mathbb{Z}/p\mathbb{Z}$ acts on $\mathbb{Q}$ and $\circlearrowright \mathbb{Q}(\zeta_p)$

$$\begin{array}{ccc} M' & & M \\ & {}_{\text{finite index}}\supseteq M'' \subseteq_{\text{finite index}} & \end{array}$$

$\rightsquigarrow h_{0/1}(M') = h_{0/1}(M)$

3. Change of groups

3.1 $\lambda: H \longrightarrow G$ group homomorphism, M: left G-module $\rightsquigarrow$ $\mathrm{Res}_\lambda M$: left H-module

λ induces a map of chain complexes $\lambda_\bullet: C_\bullet(H) \longrightarrow C_\bullet(G)$ $\quad x_0 \otimes [x_1,\dots,x_n] \mapsto \lambda(x_0)\cdot[\lambda(x_1),\dots,\lambda(x_n)]$ $\forall\, x_0, x_1,\dots,x_n \in H$

(a) $C_\bullet(H) \otimes_H M \xrightarrow{\lambda_\bullet \otimes_H \mathrm{id}_M} C_\bullet(G) \otimes_H M \longrightarrow C_\bullet(G) \otimes_G M$ induces

q. isom in Mod_H $\quad \mathrm{Res}_\lambda M$

$H_i(\lambda_*): H_i(H, M) \longrightarrow H_i(G, M)$ "corestriction map"

(b) $\mathrm{Hom}_G(C_\bullet(G), M) \longrightarrow \mathrm{Hom}_H(C_\bullet(G), M) \xrightarrow{\mathrm{Hom}(\lambda_\bullet, M)} \mathrm{Hom}_H(C_\bullet(H), M)$ induces

$\mathrm{Res}_\lambda M$

$H^i(\lambda^*): H^i(G, M) \longrightarrow H^i(H, M)$ "restriction maps"

Both are morphisms of δ-functors on Mod_G = the category of all left G-modules

$\rightsquigarrow$ $H_\bullet(\lambda_*)$ is determined by $H_0(\lambda_*): M_H \longrightarrow M_G$

$*$ $H^\bullet(\lambda^*)$ is determined by $H^0(\lambda^*): M^G \longrightarrow M^H$ $\Big]$ $\longleftarrow$ degree shifting.

3.2 When $H \leqslant G$, $N \in \mathrm{ob}(\mathrm{Mod}_H)$, have $\qquad \mathrm{ind}_H^G N = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$

$$C_\bullet(G) \otimes_G \underbrace{(\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N)}_{= \mathrm{ind}_H^G N} \cong \underbrace{C_\bullet(G)}_{= \mathrm{Res}_H^G(C_\bullet(G))} \otimes_{\mathbb{Z}[H]} N$$

$\mathrm{Res}_H^G(C_\bullet(G)) \leftarrow$ a free resolution of $\mathbb{Z}$ in Mod_H

$$\mathrm{Hom}^G\big(C_\bullet(G), \underbrace{\mathrm{Ind}_H^G N}_{= \{f: G \to N \mid f(hx) = h\cdot f(x)\ \forall h \in H, \forall x \in G\}}\big) \cong \mathrm{Hom}^H\big(\mathrm{Res}_H^G C_\bullet(G), N\big)$$

$$\Longrightarrow \begin{cases} H_i(G, \mathrm{ind}_H^G N) \cong H_i(H, N) \\ H^i(G, \mathrm{Ind}_H^G N) \cong H^i(H, N) \end{cases}$$

"Shapiro's Lemma" $\Leftarrow$ immediate from properties of adjoint functors.

$M \in \mathrm{Mod}_G$ $\rightsquigarrow$

$$\begin{cases} H_\bullet((H \hookrightarrow G)_*, M) = H_\bullet\Big[C_\bullet(G) \otimes_{\mathbb{Z}[G]} \Big(\underbrace{(\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M)}_{= \mathrm{Ind}_H^G \mathrm{Res}_H^G M} \longrightarrow M\Big)\Big] \quad \sigma \otimes_H m \longmapsto \sigma\cdot m \\ H^\bullet((H \hookrightarrow G)^*, M) = H^\bullet\Big[\mathrm{Hom}_G\Big(C_\bullet(G), M \longrightarrow \mathrm{Ind}_H^G \mathrm{Res}_H^G M\Big)\Big] \quad m \longmapsto (x \mapsto x\cdot m) \end{cases}$$

3.3. If $H \leqslant G$ and $[G:H] = a < \infty$, have transfer/corestriction maps

$M \in \mathrm{Mod}_G$ $\quad H_i(G, M) \xrightarrow{\mathrm{Ver}} H_i(H, M)$ $\quad$ and similarly $\quad \hat{H}^i(G, M) \xleftarrow{\mathrm{Ver}} \hat{H}^i(H, M)$

$\qquad H^i(G, M) \xleftarrow{\mathrm{Ver}} H^i(H, M)$ $\quad$ if $|G| < \infty$

Defining properties of transfer

For $H^\bullet$: $\quad H^0(H, M) = M^H \xrightarrow{N_{G/H}} M^G = H^0(G, M)$

$$m \longmapsto \sum_{x \in G/H} x\cdot m$$

For $H_\bullet$: $\quad H_0(G, M) = M_G \xrightarrow{N_{H\backslash G}} M_H = H_0(H, M)$

$$m \bmod I_G M \longmapsto \sum_{x \in H\backslash G} x\cdot m \bmod I_H M$$

Well-defined in M_I

(x_i) system of representatives of $H\backslash G$

$\sigma \in G$. Write $x_i \cdot \sigma = h_{\pi(i)} \cdot x_{\pi(i)}$

permutation of $H\backslash G$, depending on σ

In $\mathbb{Z}[G]$, have

$$\sum_i x_i \cdot (\sigma - 1) = \sum_i h_{\pi(i)}\, x_{\pi(i)} - \sum_i x_i = \sum_i (h_i - 1)\cdot \sigma \in I_H \cdot \mathbb{Z}[G]$$

$$\begin{tikzcd} H^i(G,M) \arrow[r, "Res"] \arrow[rr, bend right, "{[G:H]}"'] & H^i(H,M) \arrow[r, "Ver"] & H^i(G,M) \end{tikzcd}$$ Check H^0

Have

$$\begin{tikzcd} H_i(G,M) \arrow[r, "Ver"] \arrow[rr, bend right, "{[G:H]}"'] & H_i(H,M) \arrow[r, "(H\hookrightarrow G)_*"] & H_i(G,M) \end{tikzcd}$$ check H_0

Explicit formula for a quasi-isom of complexes in Mod_H: in Assignment 13

$$\mathrm{Res}^G_H\, C_\cdot(G) \xrightarrow{q.\ isom} C_\cdot(H)$$

3.3. Restriction – inflation sequence (for a normal subgroup)

$N \trianglelefteq G \qquad M \in \mathrm{Mod}_G$

$$0 \to H^i(G/N, M^N) \xrightarrow{Inf} H^i(G,M) \xrightarrow{Res} H^i(N,M)$$

$$0 \leftarrow H_i(G/N, M_N) \leftarrow H_i(G,M) \leftarrow H_i(N,M)$$

are exact when $i=1$, (direct computation)

and for $i = q$ if $H^j(G,M) = 0$, $H_j(G,M) = 0$ for $1 \le j \le q-1$

(either by dimension shifting or use Hochschild-Serre s. seq.

$$E_2^{ij} = H^i(G/N, H^j(N,M)) \Longrightarrow H^{i+j}(G,M)\ )$$

4. Cup product

Characterization by functorial properties G: finite group, $M, N \in \mathrm{Mod}_G$

$$\hat{H}^i(G,M) \times \hat{H}^j(G,N) \longrightarrow \hat{H}^{i+j}(G, M\otimes_{\mathbb{Z}} N)$$
$$(a\ ,\ b) \longmapsto a\cdot b$$

(alternative notation: $a \cup b$)

(i) functorial in M and N

(ii) When $i=j=0$, it is induced by the natural map
$$M^G \otimes_{\mathbb{Z}} N^G \longrightarrow (M\otimes_{\mathbb{Z}} N)^G$$

(iii) If $0\to M'\to M\to M''\to 0$ is exact and $0\to M'\otimes_{\mathbb{Z}} N \to M\otimes_{\mathbb{Z}} N\to M''\otimes_{\mathbb{Z}} N\to 0$ is also exact, then $\forall a''\in \hat{H}^i(G,M'')$, $\forall b\in \hat{H}^j(G,N)$, we have
$$\underbrace{(\delta a'')}_{\hat{H}^{i+1}(G,M')}\cdot b = \delta\underbrace{(a''\cdot b)}_{\hat{H}^{i+j}(G,M''\otimes_{\mathbb{Z}} N)} \in \hat{H}^{i+j+1}(G, M'\otimes_{\mathbb{Z}} N)$$

(iv) If $0\to N'\to N\to N''\to 0$ is exact and $0\to M\otimes_{\mathbb{Z}} N'\to M\otimes_{\mathbb{Z}} N\to M\otimes_{\mathbb{Z}} N''\to 0$ is also exact, $a\in \hat{H}^i(G,M)$, $b\in \hat{H}^j(G, N'')$

then
$$(-1)^i\, a\cdot \underbrace{\delta b''}_{\hat{H}^{j+1}(G,N')} = \delta\underbrace{(a\cdot b'')}_{\hat{H}^{i+j}(G, M\otimes_{\mathbb{Z}} N'')} \in \hat{H}^{i+j+1}(G, M\otimes_{\mathbb{Z}} N')$$

<u>Remark</u> Have explicit chain map (a co-pairing) which induces the cup product
$$C_\cdot(G) \xrightarrow{\Phi} \underbrace{C_\cdot(G)\otimes_{\mathbb{Z}} C_\cdot(G)}_{\text{graded tensor product}}$$

[Note: By def^n, Φ is a collection of maps $C_{i+j}(G) \xrightarrow{\varphi_{i,j}} C_i(G)\otimes_{\mathbb{Z}} C_j(G)$, $i,j\in\mathbb{Z}$
(It is *not* a map from the $\mathbb{Z}$-module $C_\cdot(G)$ to the module $C_\cdot(G)\otimes_{\mathbb{Z}} C_\cdot(G)$)]

5. Cohomologically trivial modules for finite groups — Theorems of Tate and Nakayama

Prop. 5.1. (p-groups) Let G be a finite p-group Let M be a G-module

(a) Suppose that $p\cdot M=0$. If $\hat{H}^{q_0}(G,M)=0$ for some $q_0\in\mathbb{Z}$. Then M is a free $\mathbb{F}_p[G]$-module, and $\hat{H}^q(K,M)=0$ for all $q\in\mathbb{Z}$ and every subgroup $K\leq G$.

(b) Suppose that M is torsion-free, and $\hat{H}^{q_0}(G,M)=0=\hat{H}^{q_0+1}(G,M)=0$ for some $q_0\in\mathbb{Z}$. Then

(b1) $\hat{H}^q(K,M)=0\quad\forall q\in\mathbb{Z},\ \forall K\leq M$

(b2) M/pM is a free $\mathbb{F}_p[G]$-module

(b3) $\forall$ torsion-free G-module N, we have $\hat{H}^q(K,\mathrm{Hom}_{\mathbb{Z}}(M,N))=0$ $\forall q\in\mathbb{Z},\forall K\leq G$

Will see in thm 5.2 that: (b4) $\exists$ a projective resolution $0\to P_1\to P_0\to M\to 0$ of length=1 of M in Mod_G

Pf. (a) dimension shift $\rightsquigarrow$ $\exists$ a G-module N s.t. $p\cdot N=0$ and $\hat{H}^{n-q_0-2}(G,N)\cong H^n(G,M)=0\quad\forall n\in\mathbb{Z}$.

Assumption: $H_1(G,N)=0$

Let $L\overset{\alpha}{\twoheadrightarrow}N$ be a G-linear surjection s.t. $L/I_GL\xrightarrow[\sim]{\alpha}N/I_GN$.

$0\to Q\to L\xrightarrow{\alpha}N\to 0\quad\Rightarrow\quad Q/I_GQ=(0)$, i.e. $I_G\cdot Q=Q$

Easy Fact/Exer: $\exists m_0\in\mathbb{N}$ s.t. $I_G^{m_0}=0$. (E.g. induction on $|G|$.)

$\rightsquigarrow\quad Q=I_GQ=I_G^2Q=\cdots=I_G^{m_0}Q=0$, i.e. $L\xrightarrow{\sim}N$

(b) $0\to M\xrightarrow{p}M\to M/pM\to 0\rightsquigarrow$ Assumption $\Rightarrow$ $\hat{H}^{q_0}(G,M/pM)=0$,

M/pM is a free $\mathbb{F}_p[G]$-module.. $\quad 0\to N\xrightarrow{p}N\to N/pN\to 0$ exact

$\rightsquigarrow\quad 0\to\mathrm{Hom}_{\mathbb{Z}}(M,N)\xrightarrow{p}\mathrm{Hom}_{\mathbb{Z}}(M,N)\to\mathrm{Hom}_{\mathbb{Z}}(M,N/pN)\to 0$

$\mathrm{Hom}_{\mathbb{Z}}(M,N/pN)\cong\mathrm{Hom}_{\mathbb{F}_p}(M/pM,N/pN)$

Key observation: is a free $\mathbb{F}_p[G]$-module

$M/pM=\bigoplus_{i\in I}\mathbb{F}_p[G]\cdot e_i\Rightarrow\quad \mathrm{Hom}_{\mathbb{F}_p}(M/pM,N/pN)=\bigoplus_{i\in I}\mathbb{F}_p[G]\cdot\mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_p\cdot e_i,N/pN)$

$\Rightarrow\quad\hat{H}^q(K,M)\xrightarrow[\sim]{p}\hat{H}^q(K,M)\quad\forall q\in\mathbb{Z},\ \forall K\leq G$

killed by $|G|\in p^{\mathbb{N}}$

QED.

(Nakayama 1957)

<u>Thm 5.2</u> G: a finite group, M: a G-module. a Sylow p-subgroup of G

Assume that $\forall$prime $p \mid \#G$, $\exists\, q(p) \in \mathbb{Z}$ s.t. $\hat{H}^{q(p)}(G_p, M) = 0 = \hat{H}^{q(p)+1}(G, M)$

(a) $\hat{H}^q(K, M) = 0 \quad \forall q \in \mathbb{Z}, \ \forall K \leq G$

(b) $\exists$ a G-linear projective resolution $0 \to P_1 \to P_0 \to M \to 0$ of length $=1$

(c) If M is torsion free, then M is a projective $\mathbb{Z}[G]$-module

Pf. of (c): Pick a short exact sequence $0 \to Q \to \underset{\text{free}}{L} \to M \to 0$ in Mod_G.

Consider the exact sequence $0 \to \mathrm{Hom}_{\mathbb{Z}}(M, Q) \to \mathrm{Hom}_{\mathbb{Z}}(M, L) \to \mathrm{Hom}_{\mathbb{Z}}(M, M) \to 0$

$H^1(G_p, \mathrm{Hom}_{\mathbb{Z}}(M, Q)) = 0 \quad \forall p \mid \#G$

$\Longrightarrow H^1(G, \mathrm{Hom}_{\mathbb{Z}}(M,Q)) = 0 \quad \because \ \mathrm{Ker}\left(H^1(G, \mathrm{Hom}_{\mathbb{Z}}(M,Q)) \xrightarrow{\text{restriction}} H^1(G_p, \mathrm{Hom}_{\mathbb{Z}}(M,Q))\right)$

$\Longrightarrow M$ is a direct summand of L, hence projective $\quad = H^1(G, \mathrm{Hom}_{\mathbb{Z}}(M,Q))\,[[G:G_p]]$

$= \{ h \in H^1(G, \mathrm{Hom}_{\mathbb{Z}}(M,Q)) \mid \underbrace{[G:G_p]h}_{=0} \}$

(a)+(b): Pick a short exact sequence $0 \to R \to L \to M \to 0$ in Mod_G.

R is torsion free, and $\hat{H}^{q(p)+1}(G_p, R) = \hat{H}^{q(p)+2}(G_p, R) \quad \forall p \mid \#G$

$\rightsquigarrow R$ is a projective $\mathbb{Z}[G]$-module by (c).

QED.

<u>Cor. 5.3</u> G: a finite group, B, C: G-modules. $f: B \to C$ G-linear

(mapping cone construction) Suppose that $\forall$prime $p \mid \#(G)$, $\exists\, q(p) \in \mathbb{Z}$ s.t.

$f_q^*: \hat{H}^q(G_p, B) \longrightarrow \hat{H}^q(G_p, C)$ is surjective for $q = q(p)$,

Sylow p-subgrp of G

bijective for $q = q(p)+1$ and injective for $q = q(p)+2$. Then $\forall q \in \mathbb{Z}$ and $\forall K \leq G$,

$$f_q^*: \hat{H}^q(K, B) \xrightarrow{\ \sim\ } \hat{H}^q(K, C)$$

Pf: $0 \longrightarrow B \longrightarrow C \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], B) \longrightarrow D \to 0$ short exact

$b \longmapsto \left(f(b),\ \varphi_b = (x \mapsto x \cdot b)_{\forall x \in G}\right)$

Assumption $\Longrightarrow \hat{H}^{q(p)}(G_p, D) = 0 = \hat{H}^{q(p)}(G_p, D) \quad \forall p \mid \# D$

$\overset{\text{Thm 5.2}}{\Longrightarrow} \hat{H}^q(K, D) = 0 \quad \forall q \in \mathbb{Z} \Longrightarrow \hat{H}^q(K, B) \underset{\sim}{\xrightarrow{f_q^*}} \hat{H}^q(K, C)$

<u>Prop. 5.4</u> $A, B, C \in \mathrm{Mod}_G$, $\varphi: A \otimes_{\mathbb{Z}} B \longrightarrow C$ G-linear

Given $\alpha \in \hat{H}^q(G, A)$. Assume $\forall$ prime $p \mid \#G$, $\exists n(p) \in \mathbb{Z}$ s.t. the map

$$\begin{array}{ccc} \hat{H}^n(G_p, B) & \longrightarrow & \hat{H}^{n+q}(G_p, C) \\ \beta & \longmapsto & \alpha \cup \beta \end{array}$$

is surjective for $n = n(p)$, ← Sylow p-subgroup of G

bijective for $n = n(p)+1$ and injective for $n = n(p)+2$. Then for every $K \leq G$ and every $n \in \mathbb{Z}$, the map

$$\begin{array}{ccc} H^n(K, B) & \longrightarrow & \hat{H}^{n+q}(K, C) \\ \beta & \longmapsto & \mathrm{Res}^G_K(\alpha) \cup \beta \end{array}$$

is an isomorphism.

Pf. The case $q=0$ follows from Cor. 5.3.

shift dimension with diagrams

$0 \to A' \to \mathbb{Z}[G] \otimes_{\mathbb{Z}} A \to A \to 0$ + $0 \to C' \to \mathbb{Z}[G] \otimes_{\mathbb{Z}} C \to C \to 0$ + $A' \otimes_{\mathbb{Z}} B \to C'$

and

$0 \to A \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A) \to A'' \to 0$, $0 \to C \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], C) \to C'' \to 0$ + $A'' \otimes_{\mathbb{Z}} B \to C''$.

Theorem 5.5 (Tate 1952) Let M be a G-module, and let $\alpha \in H^2(G, M)$

Assume (i) $H^1(G_p, M) = 0$

(ii) $H^2(G_p, M) = \mathbb{Z} \cdot \mathrm{Res}_{G \geq G_p}(\alpha) \cong \mathbb{Z}/\#G_p \cdot \mathbb{Z}$ $\quad \forall$ prime $p \mid \#G$.

Then

$$\begin{array}{ccc} \hat{H}^n(K, \mathbb{Z}) & \xrightarrow{\sim} & \hat{H}^{n+2}(K, M) \\ \beta & \longmapsto & \mathrm{Res}_{G \geq K}(\alpha) \cup \beta \end{array} \quad \forall n, \ \forall K \leq G.$$

Immediate Corollary of Prop. 5.4, with $A = M$, $B = \mathbb{Z}$, $n(p) = -1$ $\forall p$

Note that $H^1(G_p, \mathbb{Z}) = 0$, and the injectivity of $H^1(G_p, \mathbb{Z}) \longrightarrow H^3(G_p, M)$ holds trivially.

The case important for application is the isomorphism

$$K^{ab} = \hat{H}^{-2}(K, \mathbb{Z}) \xrightarrow{\sim} \hat{H}^0(K, M) = M^K / N_K \cdot M$$