Recall $\quad G$: a finite group

$\quad\quad\quad\; \overset{\vee\!|}{H}$: subgroup

(W, ρ) : linear repr. of H over k ⇐ a field

$\rightsquigarrow \quad \mathrm{Ind}_H^G(W, \rho) \longleftrightarrow \underbrace{k[G] \otimes_{k[H]} W}$

a $k[G]$-module, i.e. a k-linear repr. of G

Frobenius reciprocity

For any G-repr. (V, ψ)

$$\mathrm{Hom}_G(\underset{\underset{k[G]\otimes_H W}{\shortparallel}}{\mathrm{Ind}_H^G(W)}, V) \xrightarrow{\sim} \mathrm{Hom}_H(W, V)$$

$$\Longrightarrow \quad (\mathrm{Ind}_H^G(\rho), \psi)_G = (\rho, \mathrm{Res}_H^G(\psi))$$

Frobenius reciprocity.

Example / Illustration (of Frobenius reciprocity)

Defⁿ/Notation: Let A be a finite cyclic group. Define

(i) $\theta_A : A \to \mathbb{Z} \subseteq \mathbb{C}$ by

$$\theta_A(x) = \begin{cases} \# A & \text{if } x \text{ generates } A \\ 0 & \text{otherwise} \end{cases}$$

(i.e. $\theta_A = (\# A) \cdot$ (char. function of generators))

(In particular $\theta_A = 1$ if $A = \{1\}$, the trivial group)

(ii) $\lambda_A : A \to \mathbb{Z}$ by

Euler's ϕ-function

$$\mathrm{reg}_G(x) = \begin{cases} \# G & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \end{cases}$$

$$\lambda_A := \varphi(\# A) \cdot \mathrm{reg}_A - \theta_A$$

<u>Lemma 1</u>: Let $A \neq \{1\}$ be a finite cyclic group. Then

$\lambda_A = \mathbb{Z}_{\geq 0}$ - linear combination of irreducible non-trivial characters of A

$\iff$ ($(\lambda_A, \psi) \in \mathbb{Z}_{\geq 0}$ $\forall$ non-trivial irred. character ψ of A

and $(\lambda_A, \mathbb{1}) = 0$)

Pf: $(\lambda_A, \mathbb{1}) = \frac{1}{\# A} \sum_{a \in A} \overbrace{\lambda_A(a)}^{=\mathrm{reg}}$

$$= \underbrace{\frac{1}{\# A}\left(\sum_{a \in A} \varphi(\#(A))\,\mathrm{reg}(a)\right)}_{\varphi(\# A)} - \underbrace{\frac{1}{\# A}\sum_{a \in A} \theta_A(a)}_{\phi(\# A)}$$

ψ: non-trivial 1-dim'l char. of A

$$(\lambda_A, \psi) = \underbrace{(\varphi(\# A)\cdot \mathrm{reg}_A, \psi)}_{\varphi(\# A) \cdot (\mathbb{1}_{\{1\}}, \mathrm{Res}^A_{\{1\}} \psi)_A \,=\, \varphi(\# A) \cdot 1} - \underbrace{(\theta, \psi)}_{\underbrace{\sum_{a \in A,\ \langle a \rangle = A} \psi(a)}_{\in \mathbb{Z}}}$$

$A \cong \mathbb{Z}/n\mathbb{Z}$

∵ $\sum_{b \in (\mathbb{Z}/n\mathbb{Z})^\times} \zeta^b \in \mathbb{Z}$ $\forall \zeta \in \mu_n$

$$= \varphi(\#A) \sum_{\substack{a \in A \\ \langle a \rangle = A}} \left(1 - \psi(a)\right) .$$

$$= \varphi(\#A) \cdot \sum_{\substack{a \in A \\ \langle a \rangle = A}} \underbrace{\mathrm{Re}\left(1 - \psi(a)\right)}_{\substack{\not\geq \\ 0}} \in \mathbb{R}_{\not\geq 0}$$

QED.

Lemma 2 (Brauer) Let G be a finite group. Then

$$u_G := \mathrm{reg}_G - \mathbb{1}_G = \frac{1}{\# G} \cdot \sum_{\substack{A \subseteq G \\ \text{cyclic}}} \mathrm{Ind}_A^G(\lambda_A)$$

← defined before, the character of a repr. of A

$$\sum_{\substack{\chi \neq 1 \\ \chi\text{: irred char.}}} \chi(1) \cdot \chi$$

Consequently $(\# G) \cdot u_G$ is a sum of induced characters from non-trivial characters of cyclic subgroups.

Note: This lemma is a key ingredient in the Brauer–Siegel theorem on the asymptotic behavior of

$$\underbrace{R_K}_{\text{regulator}} \cdot \underbrace{h_K}_{\text{class nber of } K} \quad \text{as } \mathrm{disc}(K) \to \infty$$

K: a number field

$$\log(R_K \cdot h_K) / \tfrac{1}{2} \log |\mathrm{disc}_K| \to 1$$

as $\mathrm{disc}(K) \to \infty$.

$[K:\mathbb{Q}]$ bounded.

Proof Must show

$$\Big(\sum_{\substack{A\subseteq G\\ \text{cyclic}}} \mathrm{Ind}_A^G(\lambda_A),\ \chi\Big) = \#G\cdot(u_G,\chi)$$

$\forall$ irred character χ of G

$$\#G\cdot(u_G,\chi) = \#G\cdot\big[(\mathrm{reg}_G,\chi) - (1,\chi)\big]$$

$$= \#G\cdot\chi(1) - \sum_{x\in G}\chi(x)$$

Recall

$$\lambda_A = \varphi(\#A)\cdot \mathrm{reg}_A - \theta_A$$

$$\Big(\sum_{\substack{A\leq G\\ A \text{ cyclic}}} \mathrm{Ind}_A^G(\lambda_A),\ \chi\Big) = \sum_{\substack{A\leq G\\ \text{cyclic}}} (\lambda_A, \mathrm{Res}_A^G(\chi))$$

$$= \sum_{\substack{A\leq G\\ \text{cyclic}}} \varphi(\#A)\cdot \underbrace{(\mathrm{reg}_A, \chi|_A)}_{=\chi(1)} - \underbrace{\sum_{\substack{A\leq G\\ \text{cyclic}}} \underbrace{(\theta_A, \chi|_A)}_{=\sum_{\substack{a\in A\\ \langle a\rangle = A}} \chi(a)}}_{=\sum_{x\in G}\chi(x)}$$

Q.E.D.

Notation: Let $R(G)$ = the Grothendieck group associated to the category of all finite dim$^{\underline{l}}$ $\mathbb{C}$-repr. of G.

$$= \sum_{\substack{\chi:\text{ irred.}\\ \text{characters of } G}} \mathbb{Z}\cdot\chi$$

$R(G)$ has a natural structure as a commutative ring:

From $V_1 \underset{\mathbb{C}}{\otimes} V_2$ $\quad (V_1,\rho_1), (V_2,\rho_2)$ finite dim$^{\underline{l}}$ rep of G

$$\Rightarrow \quad ch(V_1 \otimes V_2, \rho_1 \otimes \rho_2) = ch(V_1,\rho_1) \cdot ch(V_2,\rho_2)$$

<u>Question</u>: A: cyclic subgroup of G

$$R(G) \underset{\mathrm{Ind}_A^G}{\overset{\mathrm{Res}_A^G}{\rightleftarrows}} R(A)$$

Compare $R(G)$ with $\overbrace{\sum_{\substack{A \leq G \\ \text{cyclic}}} \mathrm{Im}\left(\mathrm{Ind}_A^G\right)}^{\text{an ideal of } R(G)} =: \mathcal{I}$

<u>Exercise</u>: $\mathrm{Ind}_H^G(W,\rho) \otimes (V,\psi)$

(W,ρ) repr. of H $\qquad \cong \qquad \mathrm{Ind}_H^G\left((W,\rho) \otimes \mathrm{Res}_H^G(V,\psi)\right)$

(V,ψ) repr. of G $\qquad$ as repr. of G

<u>Ans</u> (Corollary of Lemma 1 + Lemma 2):

<u>Artin's Theorem</u> $\quad [R(G) : \mathcal{I}] < \infty$!

$$\therefore \quad \#G \cdot \underbrace{u_G}_{\substack{\| \\ (\mathrm{reg}_G - \mathbb{1})}} \in \mathcal{I} \qquad \mathrm{reg}_G \in \mathcal{I}$$

$$\Rightarrow \quad \#G \cdot \mathbb{1} \in \mathcal{I}$$

an ideal of $R(G)$