Group (co)homologies. Summary

M_1, M_2 are left G-modules

$M_1 \otimes_G M_2 = M_1 \otimes_{\mathbb{Z}} M_2 / (\sigma x_1 \otimes \sigma x_2 - x_1 \otimes x_2)$ $\sigma \in G$, $x_1 \in M_1$, $x_2 \in M_2$

$= (M_1 \otimes_{\mathbb{Z}} M_2)_G$

G: a group Mod_G = the category of left $\overset{G}{\mathbb{Z}[G]}$ modules

1\. $M \in \mathrm{Mod}_G \rightsquigarrow \begin{cases} H_i(G, M) = \mathrm{Tor}_i^{\mathbb{Z}[G]}(M, \mathbb{Z}) \\ H^i(G, M) = \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, M) \end{cases}$

trivial G-module $\forall i \geq 0$

turned into a right G-module via $G \xrightarrow{\sim} G^{opp}$, $\sigma \mapsto \sigma^{-1}$

$\longleftarrow$ δ-functor on Mod_G

$H^0(G, M) = M^G$

$0 \to I_G \longrightarrow \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ $\varepsilon: \sum_{\sigma \in G} n_\sigma [\sigma] \longmapsto \sum_{\sigma \in G} n_\sigma$

augmentation ideal

Explicit formula, from the bar resolution

$$0 \leftarrow \mathbb{Z} \xleftarrow{\varepsilon} C_0(G) \xleftarrow{\partial_1} C_1(G) \xleftarrow{\partial_2} C_2(G) \leftarrow \cdots \leftarrow C_{n-1}(G) \xleftarrow{\partial_n} C_n(G) \leftarrow \cdots$$

$\mathbb{Z}[G]$-module structure from "the first factor" $\mathbb{Z}[G]$ $\longrightarrow$ $C_{n-1}(G) = \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G^{n-1}]$, $C_n(G) = \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G^n]$

$$\sigma_0 \cdot \sigma_1 [\sigma_2, \dots, \sigma_n] + \sum_{i=1}^{n-1} (-1)^i \sigma_0 [\sigma_1, \dots, \sigma_i \sigma_{i+1}, \dots, \sigma_n] + (-1)^n \sigma_0 [\sigma_1, \sigma_2, \dots, \sigma_{n-1}] \xleftarrow{\partial_n} \sigma_0 \cdot [\sigma_1, \dots, \sigma_n]$$

$$\rightsquigarrow \begin{cases} H_i(G, M) = H_i\left(\cdots \to C_n(G) \otimes_G M \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} C_1(G) \otimes_G M \xrightarrow{\partial_1} C_0(G) \otimes_G M \to 0\right) \quad \forall i \geq 0 \\ C_n(G) \otimes_G M = C_n(G) \otimes_{\mathbb{Z}} M / \langle \sigma x \otimes \sigma m - x \otimes m \mid \sigma \in G, x \in C_n(G), m \in M \rangle \\ H^i(G, M) = H^i\left(0 \to \mathrm{Hom}_{\mathbb{Z}[G]}(C_0(G), M) \xrightarrow{d^1} \cdots \to \mathrm{Hom}_{\mathbb{Z}[G]}(C_n(G), M) \xrightarrow{d^n} \cdots\right) \end{cases}$$

$\mathrm{Hom}_{\mathbb{Z}[G]}(C_n(G), M) = \mathrm{Maps}(G^n, M)$

$H_0(G, M) = M_G = M / I_G \cdot M$, $H_1(G, \mathbb{Z}) = G^{ab} = G/(G, G)$

G-coinvariants

$H_1(G, M) = G^{ab} \otimes_{\mathbb{Z}} M$ if G operates trivially on M

G-invariants

$H^0(G, M) = M^G$, $H^1(G, M) = \mathrm{Hom}_{grp}(G, M)$ if G operates trivially on M

In particular, $H^1(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{grp}(G^{ab}, \mathbb{Q}/\mathbb{Z}) =:$ Pontryjagin dual of G^{ab}

$H^2(G, M) =$ equivalence classes of group extensions $1 \to M \to \tilde{G} \to G \to 1$ such that (M + the conjugation action) = the given G-module structure on M

using the standard complex

(Recall that $\longleftrightarrow$ classes $0 \to M \to X_2 \to X_1 \to \mathbb{Z} \to 0$ of 2-extensions (for modules over a ring)

$\bigoplus_{\substack{\sigma \in G \\ \neq 1}} \mathbb{Z} \cdot ([\sigma] - [1])$

$\sum n_\sigma [\sigma] \longmapsto \sum n_\sigma$

$0 \to I_G \to \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$

$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$

$\Longrightarrow$ $H_2(G, \mathbb{Q}/\mathbb{Z}) \cong H_1(G, \mathbb{Z}) \cong H_0(G, I_G) = I_G / I_G^2$

$\boxed{\text{Assume } |G| = g < \infty}$ $\Longrightarrow$ $H^i(G, \mathbb{Q}) = 0$ $\forall i \geq 1$

$H^i(G, \mathbb{Q}) = 0$

$H_1(G, \mathbb{Q}/\mathbb{Z}) = 0$, $H_1(G, \mathbb{Q}) = 0$, $H^i(G, \mathbb{Z}[G]) = 0$ $\forall i \geq 1$

$H^1(G, \mathbb{Z}) = 0$, $H^2(G, I_G) = 0$,

$H^1(G, I_G) \cong \mathbb{Z}/g\mathbb{Z}$

$$0 \to H_2(G, \mathbb{Q}/\mathbb{Z}) \underset{\cong}{\xrightarrow{\partial}} H_1(G, \mathbb{Z}) \to 0$$

$H_2(G, \mathbb{Q})$ $H_1(G, \mathbb{Q})$

2. Tate cohomology groups : Assume $|G| = g < \infty$

Let $C_{-r-1}(G) \overset{def}{=} \mathrm{Hom}_{\mathbb{Z}}(C_r(G), \mathbb{Z})$ $\forall r \geq 0$, with $\overset{left}{G}$-action by

$$(\sigma . \underbrace{\lambda}_{\in C_{-r-1}(G)})(x) = \lambda(\sigma^{-1}x) \quad \forall x \in C_r(G)$$

"complete resolution"

$$\begin{tikzcd}
\cdots & C_{-3}(G) \arrow[l, "\partial_{-3}"'] & C_{-2}(G) \arrow[l, "\partial_{-2} = (\partial_2)^\vee"'] & C_{-1}(G) \arrow[l, "\partial_{-1} = (\partial_1)^\vee"'] & C_0(G) \arrow[l, "\partial_0"'] \arrow[dl, "\varepsilon"] & C_1(G) \arrow[l, "\partial_1"'] & C_2(G) \arrow[l, "\partial_2"'] & \cdots \arrow[l] \\
 & & & \mathbb{Z} \arrow[u, "\varepsilon^\vee"] \arrow[dl] & & & & \\
 & & 0 & & 0 \arrow[ul] & & &
\end{tikzcd}$$

$$\begin{tikzcd}
\partial_0 : C_0(G) \arrow[r] \arrow[d, equal] & C_{-1}(G) \arrow[d, equal] & \\
\mathbb{Z}[G] & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], \mathbb{Z}) \arrow[r, "\sim"] & \mathrm{Maps}(G, \mathbb{Z}) \\
{[1]} \arrow[r, mapsto] & \sum_{\sigma \in G} \delta_\sigma \arrow[r, leftrightarrow] & \text{the const. function } 1
\end{tikzcd}$$

$\delta_\sigma(\sum_{\tau \in G} n_\tau [\tau]) = n_\sigma$

$\hat{H}^i(G, M)$ $\quad \forall i \in \mathbb{Z}$

$\|$ def

$$H^i\left(\begin{tikzcd}
\cdots \arrow[r] & \overset{deg=-3}{C_2(G) \otimes_G M} \arrow[r, "\overset{\overset{\partial_2}{\shortparallel}}{d^{-3}}"] & \overset{deg=-2}{C_1(G) \otimes_G M} \arrow[r, "\overset{\overset{\partial_1}{\shortparallel}}{d^{-2}}"] & \overset{deg=-1}{C_0(G) \otimes_G M} \arrow[r, "d^0"] \arrow[d] & \mathrm{Hom}_G(C_0(G), M) \arrow[r, "d^1"] & \mathrm{Hom}_G(C_1(G), M) \arrow[r, "d^2"] & \cdots \\
 & & & \mathbb{Z} \otimes_G M \arrow[r] \arrow[d, equal] & \mathrm{Hom}_G(\mathbb{Z}, M) \arrow[u] \arrow[d, equal] & & \\
 & & & M_G \arrow[r, "N_G"] & M^G & &
\end{tikzcd}\right)$$

$N_G = \sum_{\sigma \in G} \sigma$ (called either norm or trace)

$$\hat{H}^i(G, M) = \begin{cases} H^i(G, M) & \text{if } i \geq 1 \\ H_{-i-1}(G, M) & \text{if } i \leq -2 \end{cases}$$

$N_G \cdot ([\sigma] - 1) = 0 \quad \forall \sigma \in G$

$$M[N_G] = \mathrm{Ker}\left(\begin{tikzcd} M \arrow[r, "N_G \cdot"] & M \end{tikzcd}\right) \qquad x \longmapsto N_G \cdot x$$

$\hat{H}^{-1}(G, M) = \mathrm{Ker}\left(M_G \xrightarrow{N_G} M^G\right) = M[N_G] / I_G \cdot M$ a sub of $H_0(G, M)$

$\hat{H}^0(G, M) = \mathrm{Coker}\left(M_G \xrightarrow{N_G} M^G\right) = M^G / \underbrace{N_G \cdot M}_{\text{all norms}}$ a quotient of $H^0(G, M)$

Properties : If M is a projective $\mathbb{Z}[G]$-module, then $\hat{H}^i(G, M) = 0$ $\forall i \in \mathbb{Z}$

Special case : $G \cong \mathbb{Z}/n\mathbb{Z} = \langle \sigma \rangle$, $n \geq 2$. Let $N = 1 + \sigma + \cdots + \sigma^{n-1}$

⇝ a periodic complete resolution

$$\begin{tikzcd}
\cdots & \overset{deg=-2}{\mathbb{Z}[G]} \arrow[l, "N"'] & \overset{deg=-1}{\mathbb{Z}[G]} \arrow[l, "{[\sigma]-1}"'] & \overset{deg=0}{\mathbb{Z}[G]} \arrow[l, "N"'] \arrow[d] & \mathbb{Z}[G] \arrow[l, "{[\sigma]-1}"'] \arrow[d, "\cong" description] & \mathbb{Z}[G] \arrow[l, "N"'] & \mathbb{Z}[G] \arrow[l, "{[\sigma]-1}"'] & \cdots \arrow[l, "N"'] \\
 & & \mathbb{Z} \arrow[u] \arrow[r, equal] & \mathbb{Z} & \mathbb{Z}[x]/(x^n - 1) & & &
\end{tikzcd}$$

$x^n - 1 = (x - 1)(x^{n-1} + \cdots + x + 1)$

$\Rightarrow \hat{H}^i(G, M) \cong \hat{H}^{i+2}(G, M)$ $\forall i \in \mathbb{Z}$

$$\begin{cases} \hat{H}^{even}(G, M) \cong M^G / N_G \cdot M \\ \hat{H}^{odd}(G, M) \cong M[N_G] / ([\sigma] - 1) \cdot M \end{cases}$$

3. Change of groups

3.1 $\lambda : H \longrightarrow G$ group homomorphism, M: left G-module $\rightsquigarrow$ $\mathrm{Res}_\lambda M$: left H-module

λ induces a map of chain complexes $\lambda_\bullet : C_\bullet(H) \longrightarrow C_\bullet(G)$ $\quad x_0\otimes[x_1,\dots,x_n] \mapsto \lambda(x_0)\cdot[\lambda(x_1),\dots,\lambda(x_n)]$ $\forall\, x_0, x_1, \dots, x_n \in H$

(a) $$C_\bullet(H) \otimes_H \overset{\mathrm{Res}_\lambda M}{M} \xrightarrow{\lambda_\bullet \otimes_H \mathrm{id}_M} C_\bullet(G) \otimes_H \overset{\uparrow}{M} \longrightarrow C_\bullet(G)\otimes_G M \quad \text{induces}$$

$$H_i(\lambda_*) : H_i(H, \overset{\mathrm{Res}_\lambda M}{M}) \longrightarrow H_i(G, M)$$

(b) $$\mathrm{Hom}_G(C_\bullet(G), M) \longrightarrow \mathrm{Hom}_H(C_\bullet(G), M) \xrightarrow{\mathrm{Hom}(\lambda_\bullet, M)} \mathrm{Hom}_H(C_\bullet(H), M) \quad \text{induces}$$

$$H^i(\lambda^*) : H^i(G, M) \longrightarrow H^i(H, \overset{\mathrm{Res}_\lambda M}{M})$$

Both are morphisms of δ-functors on Mod_G = the category of all left G-modules

$\rightsquigarrow$ $H_\bullet(\lambda_*)$ is determined by $H_0(\lambda_*) : M_H \longrightarrow M_G$

$H^\bullet(\lambda^*)$ is determined by $H^0(\lambda^*) : M^G \longrightarrow M^H$

3.2 When $H \leq G$, $N \in \mathrm{ob}(\mathrm{Mod}_H)$, have

$$C_\bullet(G) \otimes_G \underbrace{(\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N)}_{\mathrm{ind}_H^G N} \cong \underbrace{C_\bullet(G)}_{\mathrm{Res}_H^G(C_\bullet(G))} \otimes_{\mathbb{Z}[H]} N$$

$\mathrm{Res}_H^G(C_\bullet(G))$ ← a free resolution of $\mathbb{Z}$ in Mod_H

Frobenius reciprocity $\begin{cases} \mathrm{Hom}_G(\mathrm{ind}_H^G N, \overset{\text{in } M_G}{M}) \cong \mathrm{Hom}_H(N, \mathrm{Res}_H^G M) \\ \mathrm{Hom}_G(M, \mathrm{Ind}_H^G N) = \mathrm{Hom}_H(\mathrm{Res}\, M, N) \end{cases}$

$$\mathrm{Hom}^G(C_\bullet(G), \underbrace{\mathrm{Ind}_H^G N}_{\{f : G \to N \mid f(hx) = h\cdot f(x)\ \forall h\in H,\ \forall x \in G\}}) \cong \mathrm{Hom}^H(\mathrm{Res}_H^G C_\bullet(G), N)$$

$\{f : G \to N \mid f(hx) = h\cdot f(x)\ \forall h\in H,\ \forall x \in G\} \cong \mathrm{ind}_H^G N$ if $[G:H] < \infty$

$$\Longrightarrow \begin{cases} H_i(G, \mathrm{ind}_H^G N) \cong H_i(H, N) \\ H^i(G, \mathrm{Ind}_H^G N) \cong H^i(H, N) \end{cases}$$ "Shipiro's Lemma"

$M \in \mathrm{Mod}_G$ $\rightsquigarrow$
$$\begin{cases} H_\bullet((H \hookrightarrow G)_*, M) \doteq H_\bullet\left[C_\bullet(G) \otimes_{\mathbb{Z}[G]} \left(\overbrace{(\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M)}^{\mathrm{Ind}_H^G \mathrm{Res}_H^G M} \longrightarrow M \right) \right] \quad \sigma \otimes_H m \longmapsto \sigma\cdot m \\ H^\bullet((H \hookrightarrow G)^*, M) \doteq H^\bullet\left[\mathrm{Hom}_G\left(C_\bullet(G), M \longrightarrow \mathrm{Ind}_H^G \mathrm{Res}_H^G M\right)\right] \quad m \longmapsto (x \mapsto x\cdot m) \end{cases}$$

3.3. If $H \leq G$ and $[G:H] = a < \infty$, have transfer/corestriction maps

$M \in \mathrm{Mod}_G$
$$H_i(G, M) \xrightarrow{\mathrm{Ver}} H_i(H, M)$$
$$H^i(G, M) \xleftarrow{\mathrm{Ver}} H^i(H, M)$$
and similarly if $|G| < \infty$ $\quad \hat{H}^i(G, M) \xleftarrow{\mathrm{Ver}} \hat{H}^i(H, M)$

Defining properties of transfer

For $H^\bullet$: $$H^0(H, M) = M^H \xrightarrow{N_{G/H}} M^G = H^0(G, M)$$
$$m \longmapsto \sum_{x \in G/H} x\cdot m$$

For $H_\bullet$: $$H_0(G, M) = M_G \xrightarrow{N'_{G/H}} M_H = H_0(H, M)$$
$$m \bmod I_G M \longmapsto \sum_{x\in G/H} x^{-1}\cdot m \bmod I_H M$$

Well-defined in M_I

(x_i) system of representatives of $G/H \rightsquigarrow (x_i^{-1})$ is a syst. of repr. of $H\backslash G$

$\sigma \in G$. Write $x_i^{-1}\cdot\sigma = h_{\pi(i)}\cdot x_{\pi(i)}^{-1}$

↖ permutation of G/H, depending on σ

In $\mathbb{Z}[G]$, have

$$\sum_i x_i\cdot(\sigma-1) = \sum_i h_{\pi(i)}\, x_{\pi(i)}^{-1} - \sum_i x_i^{-1} = \sum_i (h_i-1)\cdot\sigma \in I_H\cdot\mathbb{Z}[G]$$

$$\begin{tikzcd} H^i(G,M) \arrow[r, "Res"] \arrow[rr, bend right, "\times a"'] & H^i(H,M) \arrow[r, "Ver"] & H^i(G,M) \end{tikzcd}$$

Check H^0

Have

$$\begin{tikzcd} H_i(G,M) \arrow[r, "Ver"] & H_i(H,M) \arrow[r, "(H\hookrightarrow G)_*"] & H_i(G,M) \end{tikzcd}$$

check H_0

Explicit formula for a quasi-isom of complexes in Mod_H : in Assignment 13.

$$\mathrm{Res}_H^G\, C.(G) \xrightarrow{\text{q. isom}} C.(H)$$

3.3. Restriction – inflation sequence

$N \trianglelefteq G \qquad M \in \mathrm{Mod}_G$

$$0 \to H^i(G/N, M^N) \xrightarrow{Inf} H^i(G,M) \xrightarrow{Res} H^i(N,M)$$

$$0 \leftarrow H_i(G/N, M_N) \leftarrow H_i(G,M) \leftarrow H_i(N,M)$$

are exact when $i=1$,

(direct computation)

and for $i=q$ if $H^j(G,M)=0$, $H_j(G,M)=0$ for $1\le j\le q-1$

either by dimension shifting

or use Hochschild-Serre s. seq.

$$E_2^{ij} = H^i(G/N, H^j(N,M)) \Longrightarrow H^{i+j}(G,M)$$