Induced representations (mostly with base field k = alg. closed of characteristic 0)

- Frobenius reciprocity
- "Mackey's formula" $H, K \leq G$ G: finite $\rho : H \to GL(W)$

$$\mathrm{Res}^G_K \, \mathrm{Ind}^G_H (W, \rho) \cong \bigoplus_{s \in K \backslash G / H} \mathrm{Ind}^K_{H_s} (W, \rho_s)$$

$H_s := K \cap sHs^{-1}$

$\rho_s : H_s \longrightarrow GL(W)$, $\rho_s(x) = \rho(s^{-1} x s)$ $\forall x \in H_s$

Consequences / Applications

- Mackey's irreducibility criterion
- Classification of all irred repr. of semi-direct products $G = A \rtimes H$, A: abelian
  (reduce to: irred. rep. of H)
- Every irred. repr. of a supersolvable group is monomial, i.e. induced from a degree-one repr. of a subgroup.

Recall: $G \geq H$ $\rho : H \longrightarrow GL(W)$ W/k: v. space

so W is a $k[H]$-module (usually $k = \mathbb{C}$)

$\rightsquigarrow$ $k[G] \otimes_{k[H]} H \longleftrightarrow$ a k-linear repr. of G

denoted by: $\mathrm{Ind}^G_H (W, \rho)$

Alternative way:

$$U := \{ f : G \longrightarrow W \mid f(hg) = \rho(h) f(g) \quad \forall h \in H, \forall g \in G \}$$

a finite dim'l vector space, $\dim = [G:H] \cdot \dim_k(W)$
with a (left) action from right translation

$$\forall x \in G, \quad x : f \longmapsto (y \mapsto f(yx) \quad \forall y \in G)$$

Exer. This repr. of G on U is isomorphic to $\mathrm{Ind}_H^G(W,\rho)$

Exercise / Question: What is the character of $\mathrm{Ind}_H^G(W,\rho)$ (expressed in terms of $\mathrm{ch}(\rho)$) ?

Pick a set of repr. of $G/H = \{x_1, \dots, x_m\}$

$$\rightsquigarrow \quad k[G] \cong \bigoplus_{x \in G/H} x\, k[H] \rightsquigarrow \mathrm{Ind}_H^G(W,\rho) = \bigoplus_{x \in G/H} x_i \cdot W$$

compatible as right $k[H]$-modules

$\forall y \in G$, have a linear endom. of

$$\tilde{\rho}(y) \circlearrowright k[G] \otimes_{k[H]} W \qquad (\text{left mult. by } y)$$

Clearly: $\tilde{\rho}(y): x_i \cdot W \longrightarrow x_j \cdot$ if $y x_i H \subseteq x_j H$

(left mult. by y induces a permutation of G/H)

$$\mathrm{ch}(\mathrm{Ind}_H^G(W,\rho)) = \mathrm{Tr}\, \underbrace{\tilde{\rho}(y)}$$

after choose a basis for each $x_i \cdot W$

$\tilde{\rho}(y)$: permutes these vector subspaces

only diagonal blocks of the matrix repr. of $\tilde{\rho}(y)$ contribute to the trace

Conclude: $\forall y \in G$,

$\mathrm{Tr}(\tilde{\rho}(y))$ = sum of the traces of the restriction of $\tilde{\rho}(y)$ to $x_i \cdot W$ s.t. $y x_i H = x_i H$

Note: $\mathrm{Tr}\, \tilde{\rho}(y)\big|_{x_i W} = \mathrm{Tr}_W\left(x_i^{-1}\tilde{\rho}(y)\, x_i\right)$

$$= \sum_{\substack{x \in G/H \\ x^{-1}yx \in H}} \underbrace{\mathrm{Tr}_W(\rho(x^{-1}yx))}_{\substack{\shortparallel \\ ch(\rho)(y)}}$$

$$ch(\mathrm{Ind}_H^G(\rho))(y) = \frac{1}{\# H} \sum_{\substack{x \in G \\ x^{-1}yx \in H}} ch(\rho)(x^{-1}yx)$$

Question 1: decomposition of $\mathrm{Ind}_H^G(W,\rho)$ into irred. repr. ?

Question 2: Given an irred repr. (V,ξ) of G, what is the multiplicity of (V,ξ) in $\mathrm{Ind}_H^G(W,\rho)$?

$\rightsquigarrow$ $\dim_k \mathrm{Hom}_G(\underset{\substack{\uparrow \\ \text{a repr. of } G}}{V}, \mathrm{Ind}_H^G(W,\rho))$

$$= \dim_k \underbrace{\mathrm{Hom}_G(\underbrace{\mathrm{Ind}_H^G(W,\rho)}_{\substack{\shortparallel \\ k[G]\otimes_{k[H]} W}}, V)}_{= \mathrm{Hom}_H(W, \mathrm{Res}_H^G V)}$$

i.e.

Frobenius reciprocity

$$\boxed{\begin{aligned}&\left(ch(V)\,\middle|\,ch(\mathrm{Ind}_H^G(W,\rho))\right)_G = \left(ch(\mathrm{Ind}_H^G(W,\rho))\,\middle|\,ch(V)\right)_G \\ &= \left(ch(W)\,\middle|\,\mathrm{Res}_H^G\, ch(V)\right)_H\end{aligned}}$$

Recall

Pairing between characters (central functions)

f, g : $\underbrace{\text{virtual characters}}_{\substack{\| \\ \mathbb{Z}\text{-linear combination} \\ \text{of characters}}}$ of G ← finite group

$\#G \cdot 1_k \neq 0$

$$\rightsquigarrow \quad (f \mid g) = \frac{1}{\#G} \sum_{x \in G} f(x) \cdot g(x^{-1})$$

formula valid for any field k s.t. $\#G \cdot 1_k \neq 0$

$$\underset{\text{if } k=\mathbb{C}}{=} \frac{1}{\#G} \sum_{x \in G} f(x) \cdot \overline{g(x)}$$

$\in \mathbb{Z}$.

$\in \mathbb{N}$ if f, g are both characters.

If k is alg. closed.

$$\left.\begin{aligned} f &= ch(V_1, \rho_1) \\ g &= ch(V_2, \rho_2)\end{aligned}\right\} \Rightarrow (f \mid g) = \begin{cases} 1 & \text{if } \rho_1 \cong \rho_2 \\ 0 & \text{if } \rho_1 \not\cong \rho_2\end{cases}$$

orthogonality relation (a consequence of Schur's Lemma)

Extremely easy example:

S_3 has 3 conjugacy classes

and 3 isom. classes of irred. repr.

$\mathbb{1}$, sgn, 2-dim^l ← realized as the tautological repr. of D_6 on $\mathbb{R}^2$

$D_6 \cong S_3$

$\cong$

$\mathrm{Ind}_N(\chi)$ ← a non-trivial character of N

N ← the cyclic subgroup of order 3

What's the character of ?

$1 \longmapsto 2$

$(12) \longmapsto 0$

$(123) \longmapsto \zeta + \zeta^{-1} = -1$

roots of $x^2+x+1=0$

$\chi : (123) \longmapsto \zeta \in \mu_3$, $\zeta \neq 1$

$S_3/N = N \sqcup (12)N$

$(12)^{-1} \cdot (123) \cdot (12) = (123)^{-1}$

	1	3	2
	1	(12)	(123)
$\mathbb{1}$	1	1	1
sgn	1	−1	1
ch(ρ)	2	0	−1

$\mathrm{Ind}_H^G(H, \mathbb{1})$

= the permutation repr. of G on $k[H\backslash G]$

$\cup\!\!/$

$\sum_{x \in G/H} [x]$

<u>Question</u> When is an induced repr. $\mathrm{Ind}_H^G(W,\rho)$ irreducible?

Clearly necessary: (W,ρ) is irred.

Is this sufficient?