02/19/2021

separable ext^n, separable elements, normal extension

Artin "Galois Theory"

k: a base field. a large field $\to \Omega$

E/k: extension field ($k \hookrightarrow E$) $\quad \cup\!\!/ \quad$ (k + other fields)

Have defined:

$E \ni \alpha$ being algebraic $/k$

$(\Rightarrow k[\alpha] = \underbrace{k(\alpha)}_{}) \leftarrow$ Exer.

$\llcorner$ the smallest subfield of E containing k and α.

Def^n: An algebraic element α in E/k is separable over k if the irreducible poly $\mathrm{Irr}(\alpha, k; x) \in k[x]$ is separable in the sense that $\exists$ an (equiv. algebraic) extension field Ω of E such that

$k[x]/(\mathrm{Irr}(\alpha, k; x)) \xrightarrow{\sim} k[\alpha] = k(\alpha)$

$x \longmapsto \alpha$

$$\mathrm{Irr}(\alpha, k; x) = (x-\beta_1)\cdots(x-\beta_n)$$

with mutually distinct $\beta_1, \cdots, \beta_n \in \Omega$.

Remark: This definition of separability for elements of $k[x]$ is indep. of the choice of the extension field Ω.

Recall

<u>Fact</u>: For any field k, there exists <u>an</u> algebraic closure $\bar{k} = k^{alg} = k^{a}$, unique up to non-unique isom.

<u>i.e</u> If Ω_1, Ω_2 are two algebraic closures of k

$\exists$ a k-linear isom $\Omega_1 \xrightarrow{\sim} \Omega_2$

(not unique).

Given two alg. ext^n Ω and Ω' of k

$$
\begin{tikzcd}
 & \Omega \arrow[r] & \Omega^{a} \arrow[d, dotted, "\exists"'] \\
k \arrow[ur] \arrow[dr] & & \\
 & \Omega' \arrow[r] & \Omega'^{a}
\end{tikzcd}
$$

Ω^{a}: also an alg. closure of k

Ω'^{a}: also an alg. closure of k

<u>Q.</u> How to show/prove: α/k algebraic separable

Every element $\beta \in k(\alpha)$ is <u>separable</u>!

Naive/direct approch:

Write $\beta = h(\alpha)$ $\quad h(x) \in k[x]$

<u>Show</u> the irred. poly $\mathrm{Irr}(\beta, k; x)$ is separable $/k$.

(Not obvious.)

<u>Lemma</u>: If α is sep. alg $/k$. and k^{a} is an alg. closure of k,

<u>then</u> $\# \mathrm{Hom}_{k,\text{ring}}(k(\alpha), k^{a}) = [k(\alpha) : k]$

Pf: Assume $f(x) = \mathrm{Irr}(\alpha, k; x)$

$$f(x) = (x-\underset{\alpha}{\overset{\|}{\alpha_1}}) \cdots (x-\alpha_n)$$

Each α_i determines a k-linear embedding

$$\sigma_i : \underset{k[x]/(f(x))}{\overset{\|}{k(\alpha)}} \longrightarrow k^a$$

$$x \longmapsto \alpha_i$$

<u>So</u> $\# \mathrm{Hom}_k(k(\alpha), k^a) \geq n = [k(\alpha) : k]$

$\# \mathrm{Hom}_k(\underset{k[x]/(f(x))}{\overset{\|}{k(\alpha)}}, k^a) \leq n$ obvious/easy

<u>q.e.d.</u>

<u>Lemma</u> α/k sep. alg. $\beta \in k(\alpha)$

$\Rightarrow \beta/k$ is separable alg.

<u>Remark:</u> α/k is algebraic, not separable.

then $\# \mathrm{Hom}_k(\underset{k[x]/f(x)}{\overset{\|}{\underline{k(\alpha)}}}, k^a) < [k(\alpha) : k]$.

$f(x) = \prod_i (x - \alpha_i)$ $\quad \alpha_i \in k^a$

$\alpha = \underline{\alpha_1 = \alpha_2}$ $\quad \Rightarrow \# \mathrm{Hom}_k(k(\alpha), k^a)$

$\lneqq [k(\alpha) : k]$

Pf of Lemma: Pick and fix an alg. closure k^a/k

$$
\begin{tikzcd}
k(\alpha) \arrow[r] \arrow[rr, bend left, "\sigma"] & \sigma(k(\alpha)) = \sigma(k(\beta))(\sigma(\alpha)) \arrow[r, "\tau"] & k^a \\
k(\beta) \arrow[u, dash] \arrow[r, "\sim", "\sigma|_{k(\beta)}"'] & \sigma(k(\beta)) & \\
k \arrow[u, dash] & &
\end{tikzcd}
$$

$\tau \in \underbrace{\mathrm{Hom}_{\sigma(k(\beta))}(\sigma(k(\alpha)), k^a)}_{\# \leq [k(\alpha) : k(\beta)]}$

$\Rightarrow$ If $\mathrm{Hom}_k(k(\beta), k^a) < [k(\beta) : k]$

then $\mathrm{Hom}_k(k(\alpha) : k) < [k(\alpha) : k]$.

$$\underset{\overset{\|}{[k(\alpha) : k(\beta)]}}{\# \mathrm{Hom}_k(k(\alpha), k^a)} = \# \mathrm{Hom}_k(k(\beta), k^a) \cdot \underbrace{\# \mathrm{Hom}_{k(\beta)}(k(\alpha), k^a)}_{\leq [k(\alpha) : k(\beta)]}$$

$$\Rightarrow \# \mathrm{Hom}_k(k(\beta), k^a) = [k(\beta) : k]$$

q.e.d.

Def: A finite algebraic ext^n E/k is separable if $\# \mathrm{Hom}_k(E, k^a) = [E : k]$

(Note $\# \mathrm{Hom}_k(E, k^a) \leq [E : k]$ $\forall E/k$ finite ext^n field)

Lemma/Exer. E/k finite ext^n of fields

E/k is separable $\iff$ Every element of E is sep. $/k$

Def^n: An algebraic field ext^n E/k is separable iff every element of E is sep. over k.

($\iff$ E/k is a filtered/directed union of finite separable extension fields)

Remarks

1) Every alg. ext^n field E/k with $\mathrm{char}(k)=0$ is separable!
(via derivations)

2) Every algebraic ext^n field E/k contains a unique largest separable subext^n F/k

$$\begin{array}{c} E \\ | \\ k \end{array} \quad (k, \text{adjoin all finite separable elements})$$

$$\begin{array}{cl} E & \\ | & \\ F & \\ | & \text{separable} \\ k & \end{array}$$

Every element $\alpha \in E$ separable over F is an element of F.

$F \underset{\text{def}}{=:}$ the separable closure of k in E

$$\begin{array}{l} \bar{k}^{a} \\ \quad | \\ k^{sep} = \text{a separable (alg) closure of } k \\ \quad | \\ k \end{array}$$

characterized by the property: $\forall \alpha$ alg and sep. over k^{sep} is in k^{sep}

3. Example k = an field of char $p > 0$.

$$k(t_1, \dots, t_n) = \text{frac}\, \underbrace{k[t_1, \dots, t_n]}_{\text{poly ring in } n\text{-variables}/k}$$

$$\begin{array}{l} k(t_1, \dots, t_n) \\ \quad | \\ k(t_1^p, \dots, t_n^p) \end{array}$$

Exer!

$$\Big[k(t_1, \dots, t_n) : \underbrace{k(t_1^p, \dots, t_n^p)}_{\text{the smallest subfield of } k(t_1, \dots, t_n) \text{ containing } k \text{ and } t_1^p, \dots, t_n^p} \Big] = p^n$$

The separable closure of $k(t_1^p,\dots,t_n^p)$ in $k(t_1,\dots,t_n)$ is $\underline{\quad k(t_1^p,\dots,t_n^p) \quad}$.

finite

$$\begin{array}{c} E \\ | \\ F \\ | \\ k \end{array}$$

$$\#\left(\mathrm{Hom}_k(E, k^a)\right) = \#\left(\mathrm{Hom}_k(F, k^a)\right) \cdot \#\,\mathrm{Hom}_F(E, k^a)$$

$$\#\,\mathrm{Hom}_{k(t_1^p,\dots,t_n^p)}\left(k(t_1,\dots,t_n),\ k(t_1^p,\dots,t_n^p)^{alg}\right) = 1$$

finite

$$\begin{array}{l} E \\ | \\ F = \text{the separable closure of } k \text{ in } E \\ | \\ k \end{array}$$

$\mathrm{char}(F) = p > 0$

Then E/F is purely inseparable, in the sense that the separable closure of F in E is F.

$\iff \forall \alpha \in E \ \exists n \in \mathbb{N}_{\geq 1}$ s.t. $\alpha^{p^n} \in F$

$\iff \exists n \in \mathbb{N}_{\geq 1}$ s.t. $\alpha^{p^n} \in F \ \forall \alpha \in E$

Next time:

- Normal ext^n
- inseparable ext^n
- primitive elements i.e. field ext^n E/k s.t. $E = k(\alpha)$ for some $\alpha \in E$
- ... Galois ext^n