

02/19/2021

separable extⁿ, separable elements, normal extension.

Artin "Galois Theory"

k : a base field.

a large field $\rightarrow \Omega$

E/k : extension field $(k \hookrightarrow E)$ $(k + \text{other fields})$

Have defined:

$E \ni \alpha$ being algebraic / k ← Exer.

$$\left(\Rightarrow k[\alpha] = \underline{k(\alpha)} \right)$$

↑ the smallest subfield of E containing k and α .

Defⁿ: An algebraic element α in E/k is separable over k if the irreducible poly $\text{Irr}(\alpha, k; x) \in k[x]$

is separable in the sense (equiv. algebraic) that \exists an extension field

Ω of E such that

$$k[x] / (\text{Irr}(\alpha, k; x)) \xrightarrow{\sim} k[\alpha] \stackrel{||}{=} k(\alpha)$$

$x \longmapsto \alpha$

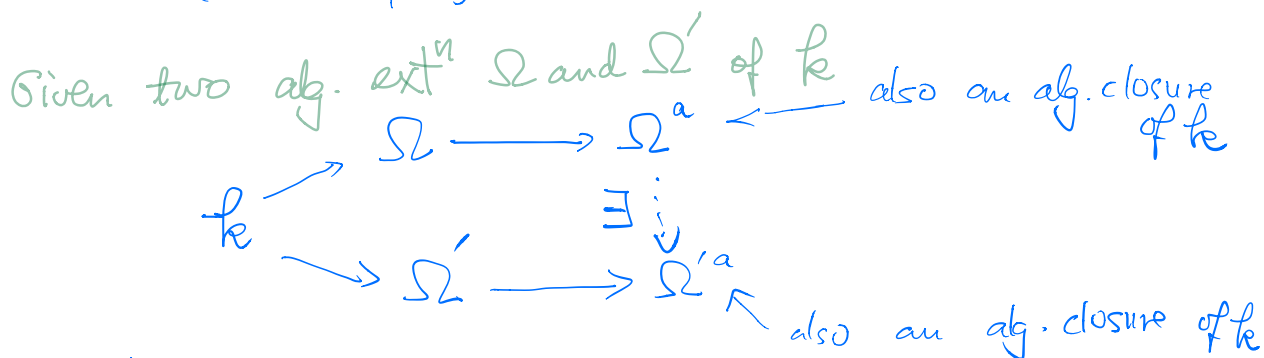
$$\text{Irr}(\alpha, k; x) = (x - \beta_1) \cdots (x - \beta_n)$$

with mutually distinct $\beta_1, \dots, \beta_n \in \Omega$.

Remark: This definition of separability for elements of $k[x]$ is indep. of the choice of the extension field Ω .

Recall Fact: For any field k , there exists an algebraic closure $\bar{k} = k^{\text{alg}} = k^a$, unique up to non-unique isom.

i.e. If Ω_1, Ω_2 are two algebraic closures of k
 \exists a k -linear isom $\Omega_1 \xrightarrow{\sim} \Omega_2$
 (not unique).



Q. How to show/prove: α/k algebraic separable
 Every element $\beta \in k(\alpha)$ is separable!

Naive/direct approach:

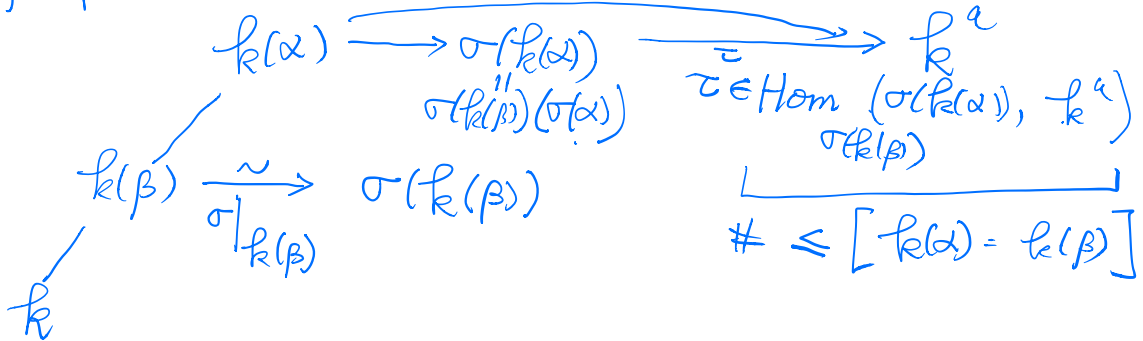
Write $\beta = h(\alpha)$ $h(x) \in k[x]$

Show the irred. poly $\text{Irr}(\beta, k; x)$ is separable \overline{k} .

(Not obvious.)

Lemma: If α is sep. alg / k , and k^a is an alg. closure of k ,
 then $\# \text{Hom}_{k, \text{ring}}(k(\alpha), k^a) = [k(\alpha) : k]$

pf of Lemma: Pick and fix an alg. closure k^a/k



\Rightarrow If $\text{Hom}_k(k(\beta), k^a) < [k(\beta) : k]$

then $\text{Hom}_k(k(\alpha) = k) < [k(\alpha) = k]$.

$$\# \text{Hom}_k(k(\alpha), k^a) = \# \text{Hom}_k(k(\beta), k^a)$$

$$\# [k(\alpha) = k(\beta)]$$

$$\# \text{Hom}_{k(\beta)}(k(\alpha), k^a)$$

$$\leq [k(\alpha) : k(\beta)]$$

$$\Rightarrow \# \text{Hom}_k(k(\beta), k^a) = [k(\beta) : k]$$

Def: A finite algebraic extⁿ E/k q.e.d.
 is separable if $\# \text{Hom}_k(E, k^a) = [E : k]$

(Note $\# \text{Hom}_k(E, k^a) \leq [E : k]$

$\forall E/k$ finite extⁿ field)

Lemma/Exer. E/k finite extⁿ of fields

E/k is separable \iff Every element of E is sep. / k

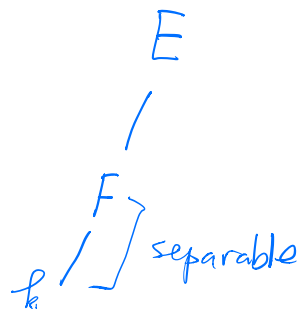
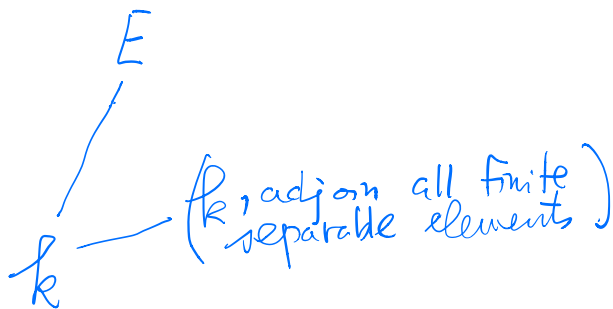
Defⁿ: An algebraic field extⁿ E/k is separable iff every element of E is sep. over k .

($\iff E/k$ is a filtered/directed union of finite separable extension fields)

Remarks

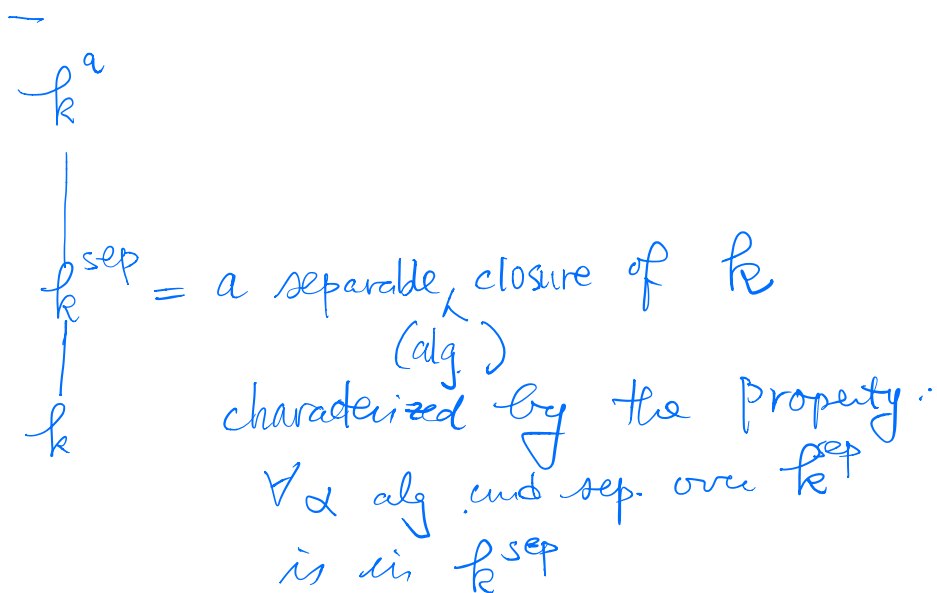
1) Every alg. extⁿ field E/k with $\text{char}(k)=0$ is separable!
(via derivations)

2) Every algebraic extⁿ field E/k contains a unique largest $\underbrace{\text{subext}^n}_{\text{separable}}$ F/k



Every element $\alpha \in E$ separable over F is an element of F .

$F \stackrel{\text{def}}{=} \overline{k}^{\text{sep}}$: the separable closure of k in E



3. Example k = an field of char $p > 0$.

$$k(t_1, \dots, t_n) = \text{frac} \frac{k[t_1, \dots, t_n]}{\text{poly ring in } n\text{-variables}/k}$$

$$k(t_1^p, \dots, t_n^p)$$

Exer!

$$\left[k(t_1, \dots, t_n) : \frac{k(t_1^p, \dots, t_n^p)}{k} \right] \stackrel{\downarrow}{=} p^n$$

the "smallest" subfield of $k(t_1, \dots, t_n)$ containing k and t_1^p, \dots, t_n^p

The separable closure of $k(t_1^p, \dots, t_n^p)$ in $k(t_1, \dots, t_n)$ is $k(t_1^p, \dots, t_n^p)$.

$$\begin{array}{c}
 \text{finite} \\
 \left[\begin{array}{c} E \\ | \\ F \\ | \\ k \end{array} \right.
 \end{array}
 \quad
 \begin{aligned}
 & \#(\text{Hom}_k(E, k^a)) \\
 &= \#(\text{Hom}_k(F, k^a)) \cdot \# \text{Hom}_F(E, k^a) \\
 & \# \text{Hom}_{k(t_1^p, \dots, t_n^p)}(k(t_1, \dots, t_n), k(t_1^p, \dots, t_n^{\text{alg}})) \\
 & \qquad \qquad \qquad = 1
 \end{aligned}$$

$$\begin{array}{c}
 \text{finite} \\
 \left[\begin{array}{c} E \\ | \\ F \\ | \\ k \end{array} \right.
 \end{array}
 = \text{the separable closure of } k \text{ in } E$$

$\text{char}(F) = p > 0$

Then E/F is purely inseparable, in the sense that the separable closure of F in E is F .

$$\iff \forall \alpha \in E \quad \exists n \in \mathbb{N}_{\geq 1} \text{ s.t. } \alpha^{p^n} \in F$$

$$\iff \exists n \in \mathbb{N}_{\geq 1} \text{ s.t. } \alpha^{p^n} \in F \quad \forall \alpha \in E$$

Next time = { inseparable ext^n
Normal ext^n
primitive elements i.e. field $\text{ext}^n E/K$
s.t. $E = K(\alpha)$
for some $\alpha \in E$
--- Galois ext^n .