Semisimplicity and radical

Jacobson density theorem

M: semisimple left R-module, $D = \mathrm{End}_R(M)$, $\phi \in \mathrm{End}_D(M)$

$\forall$ finite sequence $x_1, \dots, x_n \in M$, $\exists\, \alpha \in R$ s.t. $\alpha(x_i) = \phi(x_i)$ for $i = 1, \dots, n$

In other words, the image $\bar{R}$ of R in $\mathrm{End}_D(M)$ is a dense subring of $\mathrm{End}_D(M)$ for the "operator topology" on $\mathrm{End}_D(M)$, generated by the following family of open subsets $\left\{ U_{(a_1,\dots,a_m;\, b_1,\dots,b_m)} \right\}_{\substack{m \in \mathbb{N} \\ a_i, b_i \in M}}$ $\quad U_{(a_1,\dots,a_m;\, b_1,\dots,b_m)} = \left\{ f \in \mathrm{End}_D(M) \,\middle|\, f(a_i) = b_i \ \forall i = 1, \dots, m \right\}$

Rmk: The version usually seen is the case when M is a simple R-module. Then $D = \mathrm{End}_R(M)$ is a division ring, M is a left vector space over D, $\mathrm{End}_D(M)$ is the ring of all D-linear operators on M. The density theorem says that the image of R in $\mathrm{End}_D(M)$ is a dense subring of $\mathrm{End}_D(M)$.

Two definitions of (simple and) semisimple rings

A. Bourbaki's definition (which we will follow):

A ring R is <u>semi-simple</u> if R is a semisimple left R-module
$\ \ \ \ $ ↳ R_s in Bourbaki's notation $\qquad R_d$ right R-module

A ring R is <u>simple</u> if it is semisimple and has exactly one isomorphism class of simple left R-modules
left ideal

<u>Proposition 2</u> (consequence of this definition of semi-simplicity)

Let R be a semisimple ring (in the sense of Bourbaki)

(1) Every left R-module is semisimple — easy $\quad \{L_i\}_{i \in I}$ $\quad R = \bigoplus_i J_i$ $\quad \Rightarrow 1 = e_1 + \dots + e_m$, $e_i \in J_i$ $\quad L_i$: simple left ideals mutually non-isom $\quad J_i$ = sum of left ideals isom to L_i

(2) There exist simple left ideals $L_1, \dots, L_m$, mutually non-isomorphic, such that every simple left R-module is isomorphic to one of $\{L_1, \dots, L_m\}$

(3) For $i = 1, \dots, m$, let $\underset{J_i}{R_i} = \sum_{\substack{I:\text{ simple left} \\ \text{ideal isomorphic} \\ \text{to } L_i}} I$. Then R_i is a two-sided ideal. Moreover $R = R_1 \oplus \dots \oplus R_m$

(4) $\exists\, e_i \in R_i$, $i = 1, \dots, m$ such that $e_1 + \dots + e_m = 1$, $e_i \cdot e_j = 0$ if $i \neq j$, $e_i^2 = e_i\ \forall i$
$R_i = e_i R e_i$ is a ring, and $R \cong R_1 \times \dots \times R_m$ as a product ring

k: a field. $M = \bigoplus_{\mathbb{N}} k$

$R = $ a doubly transitive k-subalg. of $\mathrm{End}_k(M)$,
$\cap$
$\mathrm{End}_k(M)$

~~Ex~~

doubly transitive
$\underset{\text{def}}{\Longleftrightarrow}$ $\forall\, u, v \in M$, k-lin. indep
$\forall\, u', v' \in M$
$\exists\, \alpha \in R$ s.t. $\alpha(u) = u'$
$\alpha(v) = v'$

Exer: a) Such a subring exists!

b) M is a simple R-module,
and $\mathrm{End}_R(M) = k$

Examples of semisimple rings:

1) V: finite dim'l vector space / k = a field
$T \in \mathrm{End}_k(V)$. $\rightsquigarrow$ $k[T] \subseteq \mathrm{End}_k(V) \curvearrowright V$

$k[T] \cong k[X]/(\text{min. poly of } T)$, $\text{min. poly of } T = f(x)$, is semisimple

$\Updownarrow$

$f(x)$ = a product of mutually distinct monic irred. polynomials.
monic

2) G : a finite group

k : a field.

$k[G]$ is semisimple

$\Longleftrightarrow 1_k \cdot \#G \in k^\times$ $\quad \Longleftarrow$ using $\int_G$

$\xRightarrow{\text{exer}}$

When $k = \mathbb{C}$: $\quad k[G] \cong \prod_{i \in I} M_{n_i}(k)$

or, k alg. closed

$\#G \cdot 1_k \neq 0$ in k.

$$\underbrace{I}_{} \overset{\text{bij}}{\longleftrightarrow} \{\text{isom classes of irred. } k\text{-linear rep.}\}$$

(ρ_i, V_i) irred.

$n_i = \dim V_i$

$k[G] \cong \prod_{i \in I} M_{n_i}(k)$

$\rightsquigarrow$ Every simple module is $\cong$ a simple module of $\underbrace{M_{n_i}(k)}_{\text{a simple ring.}}$ for a unique $i \in I$

$M_{n_i}(k) \supseteq M_{n_i}(k) \begin{bmatrix} 0 & & 0 \\ & 0\ 1 & \\ & & 0 \end{bmatrix}$

$\underbrace{(j, j)}_{= J_j}$

k not algebraically closed. $\#G \cdot 1_k \in k^\times$

$k[G] \cong \prod_{i \in I_k} M_{n_i}(D_i)$

Example 1) D_i is a field $\supsetneq k$

$\boxed{?_?}$ 2) D_i is non-commutative

1) $k = \mathbb{Q}$, $G = \mathbb{Z}/3\mathbb{Z}$. $k[G] \cong k[X]/(X^3-1) \cong k \times \underbrace{k[X]/(X^2+X+1)}_{\mathbb{Q}(\sqrt{-3})}$

(5) Each R_i is a simple ring.

<u>Proposition</u> 3 Let R be a simple ring and let L be a simple left ideal. Let $D = \mathrm{End}_R(L)$, and $R \cong L^{\oplus r}$ as left R-modules. Then $\dim_D(L) = r$ and $R \cong \mathrm{End}_D(L) \cong M_r(D^{opp})$.

Rmk: The key is $\mathrm{End}_R(L^{\oplus r}) \cong \mathrm{End}_R(R_s) = R^{opp}$

$$\mathrm{End}_R(L^{\oplus r}) \cong M_r(D)$$

<u>Corollary</u> 4. Every semisimple ring is Artinian.

B. Ring theorists' definition of semisimple rings and radicals

(B1) The <u>radical</u> $\mathrm{Rad}(M)$ of a left R-module M is

$$\mathrm{Rad}(M) = \bigcap_{\substack{N \subsetneq M \text{ submodule} \\ M/N \text{ simple}}} N$$

(B2) The <u>radical</u> $\mathrm{Rad}(R)$ of a ring R is

$$\mathrm{Rad}(R) = \bigcap_{\substack{M: \text{simple} \\ \text{left } R\text{-module}}} \mathrm{Ann}_R(M) = \bigcap_{\substack{M: \text{semisimple} \\ \text{left } R\text{-module}}} \mathrm{Ann}_R(M)$$

Ring theorists say a ring is semisimple if its radical is 0.

Relation between the two notions

R is semisimple following Bourbaki $\iff R \cong \prod_{i=1}^{m} M_{n_i}(D_i)$ D_i = division algebra

$\Downarrow$

$\mathrm{Rad}(R) = (0) \iff \exists$ injective ring homomorphism

$\iota : R \hookrightarrow \prod_{i \in I} R_i$

s.t.

$pr_i \circ \iota : R \to R_i$ is surjective $\forall i \in I$

I: indexing set
D_i: division ring $\forall i \in I$
V_i: left D_i-modules
$R_i \subseteq \mathrm{End}_{D_i}(V_i)$
<u>doubly transitive</u> subring

By def^n, $R_i \subseteq \mathrm{End}_{D_i}(V_i)$ is doubly transitive if $\forall u, v \in V_i$ linearly indep. over V_i, $\forall u', v' \in V_i$, $\exists\ \alpha \in R_i$ s.t. $\alpha(u) = u'$, $\alpha(v) = v'$

Properties of radicals:

Lemma 5 (i) $\forall$ non-zero R-module M of finite type. $\mathrm{Rad}(M) \subsetneqq M$

(ii') $f: M \longrightarrow M''$ R-linear $\Rightarrow$ $f(\mathrm{Rad}(M)) \subseteq \mathrm{Rad}(M'')$

(iii') $N \subseteq M \Rightarrow \mathrm{Rad}(N) \subseteq \mathrm{Rad}(M)$

$\mathrm{Rad}(M/N) \supseteq \mathrm{Rad}(M) + N / N$ $\left(\Rightarrow \mathrm{Rad}(M/N) = \mathrm{Rad}(M)/N \text{ if } N \subseteq \mathrm{Rad}(M) \right)$

(iv) $\mathrm{Rad}(M) =$ the smallest R-submodule N s.t. $\mathrm{Rad}(M/N) = 0$

(v) If M is of finite type, $N \subseteq M$ R-submodule and $M = N + \mathrm{rad}(M)$, then $N = M$

(Consequence of (i) and (iii') — consider M/N; analog of Nakayama's Lemma)

(vi) $\mathrm{Rad}\left(\prod_{i \in I} M_i\right) \subseteq \prod_{i \in I} \mathrm{Rad}(M_i)$, $\mathrm{Rad}\left(\bigoplus_{i \in I} M_i\right) = \bigoplus_{i \in I} \mathrm{Rad}(M_i)$

Prop. 6 M: R-module with a finit set of generators $x_1, \dots, x_n$

$y \in M$ Then $y \in \mathrm{Rad}(M) \iff \forall a_1, \dots, a_n \in R$

$$\sum_{i=1}^{n} R \cdot (x_i + a_i y) = M$$

Lemma 7. M. R-module

$\mathrm{Rad}(M) = 0 \iff \exists$ simple R-modules M_i $i \in I$ $\swarrow$ possibly infinite

and an injective R-linear map

$$M \longrightarrow \prod_{i \in I} M_i$$