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{ Semisimplicity: semisimple/simple rings and modules
{ central simple algebras

References

follows the approach of Bourkaki, Chap.s 8, 9.

1. Serge Lang. Algebra, the chapters on semisimplicity

2. Herstein, Noncommutative rings — central simple algebras
(Carus Math Monograph MAA)

structural theory of non-commutative rings.
esp. Jacobson radicals

Also Notes

Recall: Let R be a ring. A (left) R-module M is semi-simple if one (hence all) of the following equiv. conditions holds)

(i) M is a direct sum of simple R-submodules

(ii) M is a sum of simple R-submodules

(iii) Every R-submodule $N \subseteq M$ is a direct summand, i.e. $\exists$ a R-submodule $N' \subseteq M$ s.t. $N \cap N' = (0)$, $\underbrace{N + N' = M}_{M = N \oplus N'}$

Equivalence: Exercise (Use Zorn's Lemma)

An example of a non-semisimple module:?

- $\mathbb{Z}$ as a $\mathbb{Z}$-module ← not semi-simple
- k: a field $R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a,b,d \in k \right\} \subseteq M_2(k)$

$M = k^2$ (column vectors)

The only R-submodules of M are: (0), $M = k^2$, $\overset{N}{\overset{\shortparallel}{\left\{ \begin{pmatrix} u \\ 0 \end{pmatrix} : u \in k \right\}}}$

The only simple submodule of M is N!

- $k[\mathbb{Z}/p\mathbb{Z}]$ p: a prime number k: a field of char. p.

R

Exer Show that R is <u>NOT</u> a simple R-module

Exer Lemma M_1, M_2 simple R-modules, $M_1 \not\cong M_2$ $\Rightarrow$ $\mathrm{Hom}_R(M_1, M_2) = 0$

Lemma M: simple R-module.

Then $\mathrm{End}_R(M)$ is a <u>division ring</u>!

(Exer.) $1_M \neq 0_M$ <u>i.e.</u> every $\alpha \in \mathrm{End}_R(M)$ is an R-linear autom.
$\alpha \neq 0$

Lemma Let D be a division ring,
D "is" a left D-module.

Then i) D is a simple left D-module

ii) $\mathrm{End}_D(D) \cong D^{opp}$

suppose $\alpha \in \mathrm{End}_D(D)$ $\alpha(1) \in D$

$\leadsto$ $\alpha(x) = \alpha(x \cdot 1)$ $\forall x \in D$

$= x \cdot \alpha(1)$

i.e. $\alpha = r_u$ for some $u \in D$ ($u = \alpha(1)$)

$\alpha \circ \beta(x)$

$= \alpha(x \cdot \beta(1)) = x \cdot \beta(1) \cdot \alpha(1)$

$\alpha \leadsto \alpha(1)$ $\beta \leadsto \beta(1)$ $\alpha \circ \beta \leadsto \beta(1) \cdot \alpha(1)$

Example: Assume D: a division algebra. $n \in \mathbb{N}_{\geq 1}$

$\leadsto$ $D^{\oplus n} = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in D \right\}$ a left D-module.

? a semi-simple left D-module

<u>Yes!</u> <u>Of course</u>

$$\mathrm{End}_D(\underbrace{D^{\oplus n}}_{D^n}) \cong M_n(\underbrace{D^{opp}}_{= \mathrm{End}_D(D)})$$

Example Let R be a ring. $M = M_1 \oplus \cdots \oplus M_m$

$M_i = N_i^{\oplus n_i}$ for $i = 1, \dots, m$

N_i : a simple R-module

$n_i \geq 1$

and $N_i \not\cong N_j$ $\forall i \neq j$. Let $\mathrm{End}_R(N_i) = D_i$ ← a division ring.

$$\mathrm{End}_R(M) \underset{\substack{\uparrow \\ i\neq j \quad \mathrm{Hom}_R(M_i, M_j) = (0)}}{\cong} \mathrm{End}_R(M_1) \times \cdots \times \mathrm{End}_R(M_m)$$

$$\cong M_{n_1}(D_1) \times \cdots \times M_{n_m}(D_m)$$

Suppose k is a field contained in $Z(R)$

is $k \subseteq Z(D_i)$

and $\dim_k M < \infty \implies \dim_k D_i < \infty$

Suppose moreover k is algebraically closed.

Then $D_i = k$ $\forall i$.

Example: k is a field.

G : finite group. $\#G \cdot 1_k \neq 0_k$

i.e. $\#G$ is prime to $\mathrm{char}(k)$.

Then Every $k[G]$-module is semi-simple. (Exer!)

(Recall: $\displaystyle\int_G = \underbrace{\frac{1}{\#G} \cdot \left(\sum_{\sigma \in G} \sigma \right)}_{\in\, k[G]}$

Apply to a

k-linear splitting of a given short exact sequence of $k[G]$-module to produce G-equivariant splitting)

Def: A ring R is semisimple if
R is a direct sum of simple left R-ideals.

Remark This def^n uses left R-modules. If you use right R-module, the resulting concept is actully equivalent!

Thm (Jacobson's density theorem)
Let M be a semisimple left R-module
Let $S := \mathrm{End}_R(M)$ ($\rightsquigarrow$ M has a natural structure as a left S-module)
For every element $\alpha \in \mathrm{End}_S(M)$
and $x_1, \dots, x_n \in M$,
there exists an element $z \in R$ s.t.

$$\alpha(x_1) = z \cdot x_1, \quad , \alpha(x_n) = z \cdot x_n$$

Pf: Consider the R-module $M^{\oplus n} = \overbrace{M \oplus \cdots \oplus M}^{n\text{-times}} = N$

$$\ni (x_1, \dots, x_n)$$

$\rightsquigarrow$ $\mathrm{End}_S(M)$ acts on $M^{\oplus n}$ naturally

The assertion is:

$$\alpha \cdot (x_1, \dots, x_n) \in R \cdot (x_1, \dots, x_n) \qquad \exists \underset{R\text{-submodule}}{Q \leq N}$$

$$M \oplus \cdots \oplus M = N = R \cdot (x_1, \dots, x_n) \oplus Q$$

Let $\pi : N \longrightarrow R \cdot (x_1, \dots, x_n)$ be the projection w.r.t. the above direct sum decomposition.

$$\alpha(x_1, \dots, x_n) = \alpha \circ \pi (x_1, \dots, x_n) = \pi \circ \alpha (x_1, \dots, x_n)$$
$$\subseteq R \cdot (x_1, \dots, x_n)$$

$\left[\ \underline{\text{Used}} \quad \mathrm{End}_R(M^{\oplus n}) = M_n(S)\ \right]$

α commute with S

$\Rightarrow \begin{bmatrix} \alpha & & \\ & \alpha & \\ & & \ddots & \\ & & & \alpha \end{bmatrix}$ commutes with $\underbrace{M_n(S)}_{\ni\ \pi}$

<u>QED.</u>