Theorem 1 (Brauer) G: a finite group. Then every character of G is a $\mathbb{Z}$-linear combination of characters induced from elementary subgroups

$(\mathbb{Z}/\ell\mathbb{Z}) \times (\text{a } p\text{-group})$ ℓ: prime to p

$\Downarrow$

Theorem 2 (Brauer) G: finite group. Then every character of G is $\mathbb{Z}$-linear combination of monomial characters.

Def^n A: finite cyclic group

define $\theta_A : A \to \mathbb{Z}$ $\quad \theta_A(y) = \begin{cases} a \overset{\text{def}}{=} \#A & \text{if } \langle y \rangle = A \\ 0 & \text{if } \langle y \rangle \subsetneqq A. \end{cases}$

Lemma 1 G: finite group. Then

$$\sum_{\substack{A \subseteq G \\ \text{cyclic}}} \mathrm{Ind}_A^G(\theta_A) = \#G =: g$$

Proof by brute-force computation

$$\Big(\sum_{\substack{A \subseteq G \\ A \text{ cyclic}}} \mathrm{Ind}_A^G(\theta_A) \Big)(x) = \sum_{\substack{A \leq G \\ \text{cyclic}}} \overbrace{\sum_{\substack{yxy^{-1} \in A \\ y \in G}} 1}^{\begin{cases} \#A & \text{if } yxy^{-1} \text{ is a generator of } A \\ 0 & \text{otherwise} \end{cases}}$$

$$= \sum_{y \in G} 1$$

q.e.d.

Will use

Exer. (We saw a variant of this before.)

Remark: A finite cyclic. θ_A is a virtual character of A

an HW problem

i.e. $\forall$ irred character χ of A, $(\theta_A | \chi) \in \mathbb{Z}$.

$A = \mathbb{Z}/a\mathbb{Z}$ $\quad \chi : A \to \mathbb{C}^\times$, $m + a\mathbb{Z} \mapsto e^{2\pi\sqrt{-1}\, u m/a}$, $u \in \mathbb{Z}$ $\qquad \sum_{m \in (\mathbb{Z}/a\mathbb{Z})^\times} e^{2\pi\sqrt{-1}\, um/a} \in \mathbb{Z}$

Cor 2 Every $\mathbb{Z}$-valued class function f on a finite group G s.t. $f(x) \equiv 0 \pmod{\# G}$ $\forall x \in G$ is an $\mathcal{O}$-linear combination of characters induced from cyclic subgroups.

$$\mathcal{O} = \mathcal{O}_{\mathbb{Q}[\mu_g]} = \mathbb{Z}[e^{2\pi\sqrt{-1}/g}] \qquad g = \# G$$

Pf: $f = g \cdot f_1$ ← $\mathbb{Z}$-valued class function

$$= \Big(\sum_{\substack{A \le G \\ \text{cyclic}}} n_A \operatorname{Ind}_A^G(\theta_A) \Big) \cdot f_1$$

$$= \sum_{\substack{A \le G \\ \text{cyclic}}} n_A \cdot \underbrace{\operatorname{Ind}_A^G(\theta_A) \cdot f_1}_{\operatorname{Ind}_A^G\big(\underbrace{\theta_A \cdot \operatorname{Res}_A^G(f_1)}\big)}$$

a $\mathbb{Z}$-valued function whose values are $\equiv 0 \pmod{\# A}$

$$\underbrace{\Big(\theta_A \cdot \operatorname{Res}_A^G(f_1)\Big)}_{= h} = \sum_{\chi \in \operatorname{Hom}(A, \mathbb{C}^\times)} (h, \chi)\, \chi$$

$$(h, \chi) = \frac{1}{a} \sum_{y \in A} h(y)\, \underbrace{\overline{\chi(y)}}_{\in \mathcal{O}} \qquad \frac{1}{a} h(y) \in \mathbb{Z}$$

q.e.d.

Question: Can we replace $\mathcal{O} = \mathbb{Z}[e^{2\pi\sqrt{-1}/g}]$ by $\mathbb{Z}$ in Cor 2?

p: a prime nber. $p \mid g = \#G$

<u>Construction</u>: $G \ni x$: p-regular $p \mid \mathrm{card}(G)$ a prime divisor of $\#G$

i.e. order$(x) \not\equiv 0$ (p)

$P \leqslant Z(x)$, Sylow p-subgroup of $Z(x)$ $H = \underbrace{\langle x \rangle}_{\overset{\|}{C}} \times P \leqslant Z(x) \leqslant G$ p-elem. subgp

Let $\psi_x : C \longrightarrow \mathbb{Z}$ be $(\# \langle x \rangle) \cdot$ (the characteristic function of the singleton subset $\{x\} \subseteq C$)

$$\left(\begin{array}{l} \psi_x(y) = 0 \text{ if } y \underset{C}{\in} \neq x \\ \qquad = \#C \text{ if } y = x \end{array} \right)$$

<u>Define</u> $\tilde{\psi}_x : H \to \mathbb{Z}$, $\tilde{\psi}_x = (\pi : H \to C)^* \psi_x$

$\psi_x' := \mathrm{Ind}_H^G (\tilde{\psi}_x)$ Clearly ψ_x' is a $\mathbb{Z}$-valued class fnc

<u>Rmk</u>: ψ_x' vanishes on elements of G which are not preg.

<u>Lemma 3</u> (i) $\psi_x'(x) \not\equiv 0 \pmod p$

(ii) $\psi_x'(y) = 0$ for every p-regular element of G not conjugate to x

(Note: In fact $\forall y \in G$, let $y = y_r \cdot y_s$, where $y_r, y_s \in \langle y \rangle$ preg order $\in p^{\mathbb{N}}$

then $\begin{cases} \psi_x'(y) = 0 \text{ if } y_r \text{ is not conjugate to } x, \\ \psi_x'(y) \not\equiv 0 \pmod p \text{ if } y_r \text{ is conjugate to } x \end{cases}$)

Proof (straight-forward)

Have explained (ii). (i): easy <u>exercise</u> (direct computation)

Fix $p \mid g$, p : prime

Corollary 4 $\forall$ prime divisor $p \mid \mathrm{card}(G)$,
$\quad =: g$

(a) $\exists$ $\xi \in \underbrace{\mathbb{Z}[\mu_g]}_{\mathcal{O}} \otimes_{\mathbb{Z}} R_{p\text{-elem}}(G)$ $\leftarrow$ all $\mathbb{Z}$-linear combinations of induced characters from p-elem. subgroup

$$R_{p\text{-elem}}(G) = \sum_{\substack{H \leq G \\ p\text{-elementary}}} \mathrm{Im}\left(\mathrm{Ind}_H^G : R(H) \to R(G) \right)$$

i.e. ξ is a $\mathbb{Z}[\mu_g]$-linear combination of characters induced from p-elementary subgroups

such that ξ is $\mathbb{Z}$-valued central function on G

and $\xi(x) \not\equiv 0 \pmod p$ $\forall x \in G$

(b) Write $g = p^a \cdot m$ $\quad p^a \| g$, $m \not\equiv 0 \pmod p$

Then $\underbrace{\xi^{\varphi(p^a)}(x)}$ $- 1 \equiv 0 \pmod{p^a}$ $\forall x \in G$

Immediate from (a)

$\mathbb{Z}$-valued class function, $\in \mathcal{O} \otimes_{\mathbb{Z}} R_{p\text{-elem}}(G)$.

$\mathrm{card}\left((\mathbb{Z}/p^a\mathbb{Z})^\times\right) = \varphi(p^a)$

Consider the set $Y_{G,p}$ of all p-regular G-conjugacy classes of G

Pick a set of representatives : $m = \# Y_{G,p}$

$\{x_1, \ldots, x_m\} \subseteq G$ $\leadsto$ Have $\psi'_{x_1}, \ldots, \psi'_{x_m}$ $\mathbb{Z}$-valued class functions.

Let $\xi = \psi'_{x_1} + \cdots + \psi'_{x_m}$ $\quad$ $\psi'_{x_i}(x_j) \begin{cases} \text{prime to } p & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$\leadsto$ $\xi(x_i)$ is prime to p $\forall i = 1, \ldots, m$

i.e. the values of ξ on p-reg. elements are all prime to p.

For any element $y \in G$, consider the decomposition of $\langle y \rangle$ into a product of a cyclic p-group and a cyclic group of cardinality prime to p.

$$y = \underset{\substack{\text{has order}\\ \text{a power of } p}}{y'} \cdot \underset{\substack{\text{has order}\\ \text{prime to } p}}{y''}, \qquad y', y'' \in \langle y \rangle$$

y', y'': uniquely determined by y

$\forall$ $\mathbb{Z}$-valued class function h on G which is an $\mathcal{O}$-linear combination of characters

$h|_{\langle y \rangle}$: a $\mathbb{Z}$-valued class function on $\langle y \rangle$

$$\Rightarrow \exists\, n \in \mathbb{N}_{>0} \text{ s.t. } y^{p^n} = y''^{p^n}$$

$$\Rightarrow h(y) \equiv h(y'') \pmod{p}$$

$\xi \in \mathcal{O} \otimes_{\mathbb{Z}} R_{p\text{-elem}}(G)$ $\qquad$ $g = p^a \cdot m$ $\quad$ $p^a \| \underset{\text{card}(G)}{g}$

$$\xi^{\varphi(p^a)} - 1 \equiv 0 \pmod{p^a}$$

$$\Rightarrow \underbrace{m\cdot\left(\xi^{\varphi(p^a)} - 1\right)} \equiv 0 \ (\mathrm{mod}\ \underset{\text{Card}(G)}{g})$$

$$\Rightarrow m\left(\underbrace{\xi^{\varphi(p^a)}} - 1\right) \in \mathcal{O} \otimes_{\mathbb{Z}} R_{elem}(G)$$

$\Uparrow$

$\mathcal{O} \otimes R_{elem}(G)$ is an ideal in $\mathcal{O} \otimes R(G)$

$$\Rightarrow \underset{\text{prime}}{\forall p} \,|\, g = \#G, \ \underset{\text{prime to } p}{m} \in \mathcal{O} \otimes_{\mathbb{Z}} R_{elem}(G)$$

<u>Theorem</u> (Brauer) $\forall$ prime divisor $p \mid \mathrm{card}(G) = g = p^a \cdot m$, $\gcd(m,p)=1$

$$\underbrace{\underbrace{\mathbb{Z}[\mu_g]}_{\neq 0} \otimes_{\mathbb{Z}} \left(R(G) / R_{elem}(G)\right)}_{=\,0}$$ is killed by m.

m: prime to p

$\therefore$ It is killed by every prime number

But $\mathbb{Z}[\mu_g]$ is free $\mathbb{Z}$-module. so $R(G)/R_{elem}(G) = 0$. <u>QED</u>.

<u>Theorem'</u> $$R(G) = \sum_{\substack{p \mid g \\ p:\text{ prime}}} R_{p\text{-}elem}(G)$$

<u>Theorem''</u> Every character of G is a $\mathbb{Z}$-linear combination of characters induced from elementary subgroups. Consequently every character of G is a $\mathbb{Z}$-linear combination of $\underbrace{\text{characters induced from 1-dim}^{l}\text{ characters of subgroups}}$

"monomial" characters.