Induced repr :

$$G = SL_2(\mathbb{R})$$
$$\cup$$
$$B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; \begin{array}{l} a, d \in \mathbb{R}^\times, ad = 1 \\ b = 0 \end{array} \right\}$$

$\chi : B \longrightarrow \mathbb{C}^\times$ (What are the 1-dim$^{\underline{l}}$ characters?)

$\chi_1, \chi_2 : \mathbb{R}^\times \longrightarrow \mathbb{C}^\times$ $\qquad \psi_{\chi_1,\chi_2} : B \to \mathbb{C}^\times$

$\mathbb{R}^\times \cong \mu_2 \times \mathbb{R}$ $\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \chi_1(a)\cdot\chi_2(d)$

Define $\mathrm{Ind}_B^G(\psi_{\chi_1,\chi_2})$

Exer : How to define this ?

∞- dim$^{\underline{l}}$ ↑

as a space function on G satisfying some transformation formula w.r.t. left transl. by element of B ↖

"Most of the time", it is irreducible.

Correction

Def: A group G is <u>supersolvable</u> iff $\exists$ a sequence

$$(1) = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_n = G$$

s.t. (i) $G_i \trianglelefteq G \quad \forall i = 1, \ldots, n-1$

and (ii) G_i / G_{i-1} is <u>cyclic</u> $\quad \forall i = 1, \ldots, n-1$.

<u>Note</u>: If we change (i) to the weaker cond^n (abelian)

(i)$'$ $G_i \trianglelefteq G_{i+1} \quad \forall i = 1, \ldots, n-1$,

then (i)$'$ + (ii) $\Longleftrightarrow$ (i)$'$ + (ii)$'$ $\overset{\text{def}}{\Longleftrightarrow}$ G is <u>solvable</u>

where (ii)$'$: G_i / G_{i-1} is a finitely generated abelian group

<u>Note</u> If G is supersolvable, so is every subgroup and every quotient group of G.

<u>Lemma</u>: If G is a non-abelian supersolvable group, then $\exists$ an abelian normal subgroup $A \trianglelefteq G$ such that $A \supseteq Z(G)$ and $A/Z(G)$ is a non-trivial cyclic group.

Pf: $Z(G) \subsetneq G \quad \because G$ is not abelian

$\overline{G} = G/Z(G)$ is supersolvable

Pick $x \notin Z(G)$ s.t. $\bar{x} \in Z(\overline{G})$, $\bar{x} = x \bmod Z(G)$

Then $A := \langle Z(G), x \rangle$ fits the bill.

q.e.d.

<u>Proposition</u>: Let G be a supersolvable (finite) group. Let (V, ρ) be an irred. finite dim^l $\mathbb{C}$-repr. of G. Then $\exists$ a subgroup $H \leqslant G$ and a 1-dim^l repr. $\chi: H \to \mathbb{C}^\times$ of H such that $(V, \rho) \cong \mathrm{Ind}_H^G(\chi)$

"Every irred. repr. of a supersolvable group is <u>monomial</u>".

Pf: By induction on Card(G). May assume: G is not abelian

Given (V,ρ) irred repr. of $G \triangleright A$ abelian normal

Consider: $(V,\rho)|_A$

1) If $(V,\rho)|_A$ is isotypic, then $\forall a \in A,\ \rho(a) \in \mathbb{C}^\times \mathrm{Id}_W$ so $\rho(a) \in Z(\rho(G))$ $\forall a \in A$

$\therefore \mathrm{Ker}(\rho) \neq \{1\}$

$\Rightarrow$ OK by induction.

2) If $(V,\rho)|_A$ is not isotypic, then $(V,\rho) = \mathrm{Ind}_H^G(\psi)$ by previous discussion just recalled

QED.

Recall: We considered the following situation before.

$N \trianglelefteq G$

(W,ψ) irred. rep of G

$(W,\psi)|_N = \bigoplus W(\chi)$ ← "blocks in W"

χ: irred rep. of N

χ-isotypic component of $(W,\psi)|_N$

The conjugation action of G on N permutes the $W(\chi)$'s

Let $H = \{ y \in G \mid \chi_y : N \to GL(W(\chi)),\ n \mapsto \chi(y^{-1} n y) \}$

G operates transitively on $\{W(\chi)\}$

Unless $(W,\psi)|_N$ is isotypic, (W,ψ) is induced from a repr. of H.

Def: A subgroup $H \leq G$ ← finite is *elementary* if $\exists$ a prime number p, a cyclic group C of order prime to p, and a finite p-group P such that $H \cong C \times P$ (↑ p-reg)

Examples of p-elementary subgroups:

Pick a p-reg. element $x \in G$, $x \neq 1$: $\langle x \rangle \times$ (a Sylow p-subgroup $P \leq Z(x)$)

Exercise: $A \rtimes H \cong G$ ← finite

(A: abelian)

Analyse irreducible characters of G

(In terms of irred. characters of H.)

Note: Every irred character of $A \times H$ is of the form

$$(a, h) \mapsto \chi(a) \cdot \rho(h)$$

where χ is a one-dim' character of A and ρ is an irred character of H

G nilpotent

$\iff \exists \{1\} = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_n = G$

s.t. (i) $G_i \trianglelefteq G$ $\forall i$

(ii) $Z(G/G_i) \supseteq G_{i+1}/G_i$

Exercise: (Re-)examine character tables of several p-groups and explicitly identify each irred. char. as induced from some subgroup!

e.g. Finite Heisenberg group $\left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{F}_{p^r} \right\}$

p=2 D_4 (D_8) Q $U_n(\mathbb{F}_q) \subseteq GL_n(\mathbb{F}_q) \supseteq \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$

<u>Theorem 1</u> (Brauer) G: a finite group. Then every character of G is a $\mathbb{Z}$-linear combination of characters induced from elementary subgroups

$\Downarrow$

<u>Theorem 2</u> (Brauer) G: finite group. Then every character of G is $\mathbb{Z}$-linear combination of monomial characters.

Def^n A: finite cyclic group

define $\theta_A : A \to \mathbb{Z}$ $\theta_A(y) = \begin{cases} a \overset{\text{def}}{=} \#A & \text{if } \langle y \rangle = A \\ 0 & \text{if } \langle y \rangle \subsetneq A. \end{cases}$

<u>Lemma 1</u> G: finite group. Then

$$\sum_{\substack{A \subseteq G \\ \text{cyclic}}} \mathrm{Ind}_A^G(\theta_A) = \#G$$

Cor 2 Every $\mathbb{Z}$-valued class function f on a finite group G s.t. $f(x) \equiv 0 \pmod{\# G}$ $\forall x \in G$ is an $\mathcal{O}$-linear combination of characters induced from cyclic subgroups.

$\mathcal{O} = \mathcal{O}_{\mathbb{Q}[\mu_g]}$ $g = \# G$

<u>Construction</u>: $G \ni x$: p-regular

$P \leqslant Z(x)$, $H = \underbrace{\langle x \rangle}_{\substack{\parallel \\ C}} \times P$

Sylow p-subgroup of $Z(x)$

Let $\psi_x : \ \to \mathbb{Z}$ be the characteristic function of the element $x \in C$

<u>Define</u> $\psi : H \to \mathbb{Z}$, $\psi = (\pi : H \to C)^* \psi_x$

$\psi' := \mathrm{Ind}_H^G (\psi)$

<u>Lemma</u> (i) $\psi'(x) \not\equiv 0 \pmod{p}$

(ii) $\psi'(y) = 0$ for every p-regular element of G not conjugate to x