04/05/2021

Hilbert 90 revisited Did: finite cyclic ext^n E/k

Equivalent statement (a) $H^1(Gal(E/k), E^\times) = 0$

(b) $H^1(Gal(E/k), (E,+)) = 0$

$Gal(E/k) = \langle\sigma\rangle = \sigma^{\mathbb{Z}/n\mathbb{Z}}$

Old statement: If $x \in E^\times$ and $Nm_{E/k}\, x = 1$, then $\exists\, y \in E^\times$ s.t. $x = y^{-1} \cdot {}^\sigma y$

$$\underbrace{H^1(Gal(E/k)}_{\text{cyclic of order } n}, \underset{\substack{\uparrow \\ \text{a } Gal(E/k)\text{-module}}}{E^\times}) = \hat{H}^1(Gal(E/k), E^\times) \cong \hat{H}^{-1}(\overbrace{Gal(E/k)}^{G}, E^\times)$$

$$\| $$

$$Ker\left(E^\times_G \xrightarrow[Nm_{E/k}]{} (E^\times)^G\right) = E^\times[Nm_{E/k}] / \underbrace{I_G \cdot E^\times}_{= (\sigma-1)\cdot E^\times}$$

Without assuming that E/k is finite cyclic:

- $H^i(Gal(E/k), (E,+)) = 0 \quad \forall i \geq 1$ ∵ The $Gal(E/k)$-module $(E,+)$ is $\underset{\substack{\uparrow \\ \text{"normal basis theorem"}}}{\cong} k[G] = ind_{\{1\}}^{Gal(E/k)} \underset{\substack{\uparrow \\ \text{trivial } G\text{-module}}}{k}$

Hilbert 90, for general finite Galois extensions.

- $H^1(Gal(E/k), E^\times) = 0$

Pf Given a 1-cocycle $(a_s)_{s \in G}$ (for the standard (non-homog.) complex)

i.e. $a_s \in E^\times \quad \forall s \in G$

and $\underbrace{a_s \cdot {}^s a_t = a_{st}}_{\text{cocycle condition}} \quad \forall s,t \in G$

Consider elements of the form

$$b = \sum_{s \in G} a_s \cdot {}^s c \qquad c \in E$$

$\forall s \in G$

$${}^s b = \sum_{t \in G} {}^s(a_t \cdot {}^t c) = \sum_{t \in G} \underbrace{{}^s a_t}_{= a_s^{-1} \cdot a_{st}} \cdot {}^{st} c = a_s^{-1} \cdot b$$

If $b \neq 0$, then $b^{-1} \cdot a_s \, {}^s b = 1 \quad \forall s \in G$

$\Rightarrow (a_s)$ is a coboundary.

Suffices to show: $\exists c \in E$ s.t. $\sum_{s \in G} a_s \cdot {}^s c \neq 0$.

$\Uparrow$

linear independence of characters.

Q.E.D.

<u>Remark</u>: Generally $\underbrace{H^2(E/k, E^\times)}$ $\neq 0$.

parametrizes isom. classes of central division algebras D/k (finite dim'l) such that $D \otimes_k E \cong_{/E} M_d(E)$ for $d^2 = \dim_k(D)$

e.g. $H^2(\mathbb{C}/\mathbb{R}, \mathbb{C}^\times) \cong \mathbb{Z}/2\mathbb{Z}$

the non-trivial elt of the l.h.s.

$\longleftrightarrow$ $\mathbb{H}$ ← Hamiltonian quaternions over $\mathbb{R}$

Kummer theory:

Preliminary discussion:

<u>Recall</u>

k^{sep} = a separable closure of k

= the union of all finite separable subextensions E/k of a fixed algebraic closure k^a of k

G_k

$$G_k = \mathrm{Gal}(k^{sep}/k) = \varprojlim_{\substack{k \subseteq E \subseteq k^{sep} \\ E/k \text{ finite Galois}}} \mathrm{Gal}(E/k)$$

a profinite group (hence compact & Hausdorff)

<u>Define</u>: $$H^i(G_k, E^\times) \underset{\text{def}}{=} \varinjlim_{\substack{U \subseteq G \\ \text{open}}} H^i(G/U, (E^\times)^U)$$

a <u>discrete</u> G_k-module,

<u>i.e.</u> $\forall x \in E^\times$, $\exists$ an open subgroup $U \subseteq G_k$ (of finite index) s.t. $x \in (E^\times)^U$

Same def^{n}, for cohomology of profinite group with discrete coefficient modules

$$\rightsquigarrow \begin{cases} H^1(G_k, (k^{sep})^\times) = 0 \\ H^i(G_k, (k^{sep}, +)) = 0 \quad \forall i \geq 1 \end{cases}$$

Let $n \in \mathbb{N}_{\geq 1}$, $n \cdot 1_k \in k^\times$

Have a short exact sequence

$$1 \to \mu_n(k^{sep}) \to (k^{sep})^\times \xrightarrow{[n]} (k^{sep})^\times \to 1$$

$$\rightsquigarrow \underbrace{((k^{sep})^\times)^{G_k}}_{= k^\times} \xrightarrow{[n]} \underbrace{((k^{sep})^\times)^{G_k}}_{= k^\times} \to H^1(G_k, \mu_n) \to 0$$

i.e. $H^1(G_k, \mu_n) = k^\times / (k^\times)^n$ (Assuming only that $n \cdot 1_k \in k^\times$)

Assume furthermore $\mu_n(k^{sep}) \subseteq k^\times$

non-canonical $\cong \mathbb{Z}/n\mathbb{Z}$

then $H^1(G_k, \mu_n) \cong \mathrm{Hom}_{\substack{grp \\ cont.}}(G_k^{ab}, \mu_n)$

$G_k^{ab} = G_k / \overline{(G_k, G_k)}$

$H^1(G_k, \mu_n) = k^\times / (k^\times)^n$

$=$ the subgroup of $\underbrace{\text{the Pontryagin dual of } G_k^{ab}}_{(G_k^{ab})^\vee = \mathrm{Hom}_{cont}(G_k^{ab}, \mathbb{Q}/\mathbb{Z})}$, equal to $(G_k^{ab})^\vee[n]$.

$(\mathrm{Hom}_{cont}(G_k^{ab}, \mathbb{C}_1^\times))$

$n^{-1}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$

<u>Explicitly</u>: a subgroup $\Gamma \subseteq k^\times/(k^\times)^n$

$$\rightsquigarrow k(\sqrt[n]{a}\ ;\ a\in\Gamma) = E_\Gamma$$

an abelian extension of k, finite over k if $\#\Gamma<\infty$ s.t. $\mathrm{Gal}(E_\Gamma/k)$ is killed by n.

$$\Gamma \xrightarrow{\sim} \mathrm{Hom}_{grp}(\mathrm{Gal}(E_\Gamma/k), \mu_n)$$

$$k^\times/(k^\times)^n \ni \bar a \longmapsto \left(\sigma \longmapsto \frac{\sigma\cdot\sqrt[n]{a}}{\sqrt[n]{a}}\right), \quad a \in k^\times,\ \sigma\in\mathrm{Gal}(E_\Gamma/\ ),\ \frac{\sigma\cdot\sqrt[n]{a}}{\sqrt[n]{a}}\in\mu_n$$

indep. of the choice of repr. of $\bar a$

<u>Artin-Schreier theory</u>

E / k (algebraic) separable abelian

$p=\mathrm{char}(k)>0$

"Artin-Schreier map"

$$1\to(\mathbb{F}_p,+)\to(k^{sep},+)\xrightarrow{\wp}(k^{sep},+)\to 0$$

$$x \longmapsto x^p-x$$

$$\rightsquigarrow k/\wp k \xrightarrow{\sim} H^1(G_k,\mathbb{F}_p) = \mathrm{Hom}_{grp,cont}(G_k^{ab},\mathbb{F}_p)$$

$$\wp k = \{x^p-x \mid x\in k\}$$

<u>Get</u>: all abelian Galois extensions of k s.t. the Galois group is killed by p

$$\updownarrow$$

$$\Gamma \subseteq k/\wp k$$