Group (co)homologies. Summary

G: a group $\qquad \mathrm{Mod}_G$ = the category of left $\overset{G}{\mathbb{Z}[G]}$ modules

1. $M \in \mathrm{Mod}_G \rightsquigarrow \begin{cases} H_i(G,M) = \mathrm{Tor}_i^{\mathbb{Z}[G]}(M, \mathbb{Z}) \\ H^i(G,M) = \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, M) \end{cases}$

trivial G-module $\forall i \geq 0$

turned into a right G-module via $G \xrightarrow{\sim} G^{opp}$, $\sigma \longmapsto \sigma^{-1}$

$$0 \to I_G \longrightarrow \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \qquad \varepsilon: \sum_{\sigma\in G} n_\sigma [\sigma] \longmapsto \sum_{\sigma\in G} n_\sigma$$

augmentation ideal

Explicit formula, from the bar resolution

$$0 \leftarrow \mathbb{Z} \xleftarrow{\varepsilon} C_0(G) \xleftarrow{\partial_1} C_1(G) \xleftarrow{\partial_2} C_2(G) \leftarrow \cdots \leftarrow C_{n-1}(G) \xleftarrow{\partial_n} C_n(G) \leftarrow \cdots$$

$$C_{n-1}(G) = \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G^{n-1}] \qquad C_n(G) = \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G^n]$$

$\mathbb{Z}[G]$-module structure from "the first factor" $\mathbb{Z}[G]$ $\longrightarrow$

$$\sigma_0 \cdot [\sigma_1, \dots, \sigma_n] \xmapsto{\partial_n} \sigma_0 \cdot \sigma_1 [\sigma_2, \dots, \sigma_n] + \sum_{i=1}^{n-1} (-1)^i \sigma_0 [\sigma_1, \dots, \sigma_i\sigma_{i+1}, \dots, \sigma_n] + (-1)^n \sigma_0 [\sigma_1, \sigma_2, \dots, \sigma_{n-1}]$$

$$\rightsquigarrow \begin{cases} H_i(G,M) = H_i\left(\cdots \to C_n(G) \otimes_G M \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} C_1(G) \otimes_G M \xrightarrow{\partial_1} C_0(G) \otimes_G M \to 0\right) \quad \forall i \geq 0 \\ \qquad C_n(G) \otimes_G M = C_n(G) \otimes_{\mathbb{Z}} M / \langle \sigma x \otimes \sigma m - x \otimes m \mid \sigma \in G, x \in C_n(G), m \in M \rangle \\ H^i(G,M) = H^i\left(0 \to \mathrm{Hom}_{\mathbb{Z}[G]}(C_0(G), M) \xrightarrow{d^1} \cdots \to \mathrm{Hom}_{\mathbb{Z}[G]}(C_n(G), M) \xrightarrow{d^n} \cdots\right) \end{cases}$$

$$\mathrm{Hom}_{\mathbb{Z}[G]}(C_n(G), M) = \mathrm{Maps}(G^n, M)$$

$H_0(G,M) = M_G =: M/I_G \cdot M$, $\qquad H_1(G,\mathbb{Z}) = G^{ab} = G/(G,G)$

G-coinvariants

$H_1(G,M) = G^{ab} \otimes_{\mathbb{Z}} M$ if G operates trivially on M

G-invariants

$H^0(G,M) = M^G$, $\qquad H^1(G,M) = \mathrm{Hom}_{grp}(G,M)$ if G operates trivially on M

In particular, $H^1(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{grp}(G^{ab}, \mathbb{Q}/\mathbb{Z}) =:$ Pontryagin dual of G^{ab}

$H^2(G,M)$ = equivalence classes of group extensions $1 \to M \to \tilde{G} \to G \to 1$ such that (M + the conjugation action) = the given G-module structure on M

(Recall that

$$0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$$
$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

$\Longrightarrow$ $H_i(G, \mathbb{Z}[G]) = 0 \ \forall i \geq 1 \rightsquigarrow H_1(G,\mathbb{Z}) \cong H_0(G, I_G) = I_G/I_G^2$

Assume $|G| = g < \infty$. Then $H_i(G,\mathbb{Q}) = 0 \ \forall i \geq 1$, $H^i(G,\mathbb{Q}) = 0 \ \forall i \geq 1$.

$H^i(G, \mathbb{Z}[G]) = 0 \ \forall i \geq 1$,

$H_1(G, \mathbb{Q}/\mathbb{Z}) = 0$, $H_1(G, \mathbb{Q}) = 0$, $H^i(G, \mathbb{Z}[G]) = 0 \ \forall i \geq 1$

$H^1(G, \mathbb{Z}) = 0$, $H^2(G, I_G) = 0$,

$H^1(G, I_G) \cong \mathbb{Z}/g\mathbb{Z}$

2. Tate cohomology groups : Assume $|G| = g < \infty$

Let $C_{-r-1}(G) \overset{\text{def}}{=} \mathrm{Hom}_{\mathbb{Z}}(C_r(G), \mathbb{Z})$ $\forall r \geq 0$, with left G-action by

$$(\sigma.\underset{\in C_{-r-1}(G)}{\lambda})(x) = \lambda(\sigma^{-1}x) \quad \forall x \in C_r(G)$$

"complete resolution"

$$\begin{tikzcd}
\cdots & C_{-3}(G) \arrow[l, "\partial_{-3}"'] & C_{-2}(G) \arrow[l, "\partial_{-2}=(\partial_2)^\vee"'] & C_{-1}(G) \arrow[l, "\partial_{-1}=(\partial_1)^\vee"'] & C_0(G) \arrow[l, "\partial_0"'] \arrow[dl, "\varepsilon"] & C_1(G) \arrow[l, "\partial_1"'] & C_2(G) \arrow[l, "\partial_2"'] & \cdots \arrow[l] \\
 & & & \mathbb{Z} \arrow[u, "\varepsilon^\vee"] & & & &
\end{tikzcd}$$

$$\begin{tikzcd}
\partial_0 : C_0(G) \arrow[r] & C_{-1}(G) & \\
\mathbb{Z}[G] \arrow[u, equal] & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], \mathbb{Z}) \arrow[u, equal] \arrow[r, "\sim"] & \mathrm{Maps}(G, \mathbb{Z}) \\
{[1]} \arrow[u, phantom, "\cup"] \arrow[r, mapsto] & \sum_{\sigma\in G} \underline{\delta_\sigma} \arrow[r, leftrightarrow] & \text{the const. function } 1 \\
 & \delta_\sigma\Big(\sum_{\tau\in G} n_\tau[\tau]\Big) = n_\sigma \arrow[u] &
\end{tikzcd}$$

$\hat{H}^i(G, M)$ $\forall i \in \mathbb{Z}$

$\|$ def

$$\begin{tikzcd}
H^i\Big(\cdots \arrow[r] & \overset{\deg=-3}{C_2(G) \otimes_G M} \arrow[r, "\overset{\partial_2}{\overset{\shortparallel}{d^{-3}}}"] & \overset{\deg=-2}{C_1(G) \otimes_G M} \arrow[r, "\overset{\partial_1}{\overset{\shortparallel}{d^{-2}}}"] & \overset{\deg=-1}{C_0(G) \otimes_G M} \arrow[r, "d^0"] \arrow[d] & \mathrm{Hom}_G(C_0(G), M) \arrow[r, "d^1"] & \mathrm{Hom}_G(C_1(G), M) \arrow[r, "d^2"] & \cdots\Big) \\
 & & & \mathbb{Z} \otimes_G M \arrow[r] & \mathrm{Hom}_G(\mathbb{Z}, M) \arrow[u] & & \\
 & & & M_G \arrow[u, equal] \arrow[r, "N_G"] & M^G \arrow[u, equal] & &
\end{tikzcd}$$

$N_G = \sum_{\sigma\in G} \sigma$ (called either norm or trace)

$$\hat{H}^i(G, M) = \begin{cases} H^i(G, M) & \text{if } i \geq 1 \\ H_{-i-1}(G, M) & \text{if } i \leq -2 \end{cases}$$

$$\hat{H}^{-1}(G, M) = \mathrm{Ker}\left(M_G \xrightarrow{N_G} M^G\right) = M[N_G] / I_G \cdot M$$

$$\hat{H}^0(G, M) = \mathrm{Coker}\left(M_G \xrightarrow{N_G} M^G\right) = M^G / N_G \cdot M$$

Properties : If M is a projective $\mathbb{Z}[G]$-module, then $\hat{H}^i(G, M) = 0$ $\forall i \in \mathbb{Z}$

Special case : $G \cong \mathbb{Z}/n\mathbb{Z} = \langle\sigma\rangle$, $n \geq 2$. Let $N = 1 + \sigma + \cdots + \sigma^{n-1}$

⇝ a periodic complete resolution

$$\begin{tikzcd}
\cdots & \overset{\deg=-2}{\mathbb{Z}[G]} \arrow[l, "N"'] & \overset{\deg=-1}{\mathbb{Z}[G]} \arrow[l, "{[\sigma]-1}"'] & \overset{\deg=0}{\mathbb{Z}[G]} \arrow[l, "N"'] \arrow[d] & \mathbb{Z}[G] \arrow[l, "{[\sigma]-1}"'] & \mathbb{Z}[G] \arrow[l, "N"'] & \mathbb{Z}[G] \arrow[l, "{[\sigma]-1}"'] & \cdots \arrow[l, "N"'] \\
 & & \mathbb{Z} \arrow[u] \arrow[r, equal] & \mathbb{Z} & & & &
\end{tikzcd}$$

$$\Rightarrow \hat{H}^i(G, M) \cong \hat{H}^{i+2}(G, M) \quad \forall i \in \mathbb{Z}$$

$$\begin{cases} \hat{H}^{\text{even}}(G, M) \cong M^G / N_G \cdot M \\ \hat{H}^{\text{odd}}(G, M) \cong M[N_G] / ([\sigma]-1) \cdot M \end{cases}$$

Definition (Herbrand quotient) $G \cong \mathbb{Z}/n\mathbb{Z}$, $G \cong \mathbb{Z}/n\mathbb{Z}$

$$h_{0/1}(M) = \frac{\# \hat{H}^0(G,M)}{\# \hat{H}^1(G,M)}$$ if both $\hat{H}^0(G,M)$ and $\hat{H}^1(G,M)$ are finite

Proposition $G \cong \mathbb{Z}/n\mathbb{Z}$,

(a) If $0 \to M' \to M \to M'' \to 0$ exact in Mod_G, then $h_{0/1}(M) = h_{0/1}(M') \cdot h_{0/1}(M'')$, and if 2 of the 3 terms are defined, so is the third.

(b) If $\# M < \infty$, then $h_{0/1}(M) = 1$

$$\left(\begin{array}{l} 0 \to M^G \longrightarrow M \xrightarrow{\sigma - 1} M \longrightarrow M_G \longrightarrow 0 \\ 0 \to \hat{H}^{-1}(G,M) \longrightarrow M_G \xrightarrow{N_G} M^G \longrightarrow \hat{H}^0(G,M) \longrightarrow 0 \end{array}\right)$$

3. Change of groups

3.1 $\lambda: H \longrightarrow G$ group homomorphism, M: left G-module ⇝ $\mathrm{Res}_\lambda M$: left H-module

h induces a map of chain complexes $\lambda_\bullet: C_\bullet(H) \longrightarrow C_\bullet(G)$ $\quad x_0\otimes[x_1,\dots,x_n] \mapsto \lambda(x_0)\cdot[\lambda(x_1),\dots,\lambda(x_n)]$ $\forall\, x_0, x_1,\dots,x_n \in H$

(a) $C_\bullet(H)\otimes_H M \xrightarrow{\lambda_\bullet\otimes_H \mathrm{id}_M} C_\bullet(G)\otimes_H M \longrightarrow C_\bullet(G)\otimes_G M$ induces

$$H_i(\lambda_*): H_i(H, \overset{\mathrm{Res}_\lambda M}{M}) \longrightarrow H_i(G,M)$$

(b) $\mathrm{Hom}_G(C_\bullet(G), M) \longrightarrow \mathrm{Hom}_H(C_\bullet(G), M) \xrightarrow{\mathrm{Hom}(\lambda_\bullet, M)} \mathrm{Hom}_H(C_\bullet(H), M)$ induces

$$H^i(\lambda^*): H^i(G,M) \longrightarrow H^i(H, \overset{\mathrm{Res}_\lambda M}{M})$$

Both are morphisms of δ-functors on Mod_G = the category of all left G-modules

⇝ $H_\bullet(\lambda_*)$ is determined by $H_0(\lambda_*): M_H \longrightarrow M_G$

$H^\bullet(\lambda^*)$ is determined by $H^0(\lambda^*): M^G \longrightarrow M^H$

3.2 When $H \leqslant G$, $N \in \mathrm{ob}(\mathrm{Mod}_H)$, have

$$C_\bullet(G)\otimes_G \underbrace{\left(\mathbb{Z}[G]\otimes_{\mathbb{Z}[H]} N\right)}_{\overset{\|}{\mathrm{ind}_H^G N}} \cong \underbrace{C_\bullet(G)}_{\overset{\|}{\mathrm{Res}_H^G(C_\bullet(G))}} \otimes_{\mathbb{Z}[H]} N \qquad \leftarrow \text{a free resolution of } \mathbb{Z} \text{ in } \mathrm{Mod}_H$$

$$\mathrm{Hom}^G\Big(C_\bullet(G), \underbrace{\mathrm{Ind}_H^G N}_{\overset{\|}{\{f: G\to N \,\mid\, f(hx) = h\cdot f(x)\ \forall h\in H,\ \forall x\in G\}}}\Big) \cong \mathrm{Hom}^H\left(\mathrm{Res}_H^G C_\bullet(G), N\right)$$

$$\Longrightarrow \begin{cases} H_i(G, \mathrm{ind}_H^G N) \cong H_i(H,N) \\ H^i(G, \mathrm{Ind}_H^G N) \cong H^i(H,N) \end{cases} \quad \forall i\geq 0 \quad \text{"Shipiro's Lemma"}$$

$M\in\mathrm{Mod}_G$ ⇝

$$\begin{cases} H_\bullet((H\hookrightarrow G)_*, M) \doteq H_\bullet\left[C_\bullet(G)\otimes_{\mathbb{Z}[G]}\Big(\overbrace{\left(\mathbb{Z}[G]\otimes_{\mathbb{Z}[H]} M\right)}^{\overset{\mathrm{Ind}_H^G \mathrm{Res}_H^G M}{\|}} \xrightarrow[\sigma\otimes_H m \;\longmapsto\; \sigma\cdot m]{} M\Big)\right] \\ H^\bullet((H\hookrightarrow G)^*, M) \doteq H^\bullet\left[\mathrm{Hom}_G\Big(C_\bullet(G), M \xrightarrow[m \;\longmapsto\; (x\mapsto x\cdot m)]{} \mathrm{Ind}_H^G \mathrm{Res}_H^G M\Big)\right] \end{cases}$$

3.3. If $H \leqslant G$ and $[G:H] = a < \infty$, have transfer/corestriction maps

$M\in\mathrm{Mod}_G$ $\quad H_i(G,M) \xrightarrow{\mathrm{Ver}} H_i(H,M)$ $\quad$ and similarly if $|G|<\infty$ $\quad \hat{H}^i(G,M) \xleftarrow{\mathrm{Ver}} \hat{H}^i(H,M)$

$\quad H^i(G,M) \xleftarrow{\mathrm{Ver}} H^i(H,M)$

Defining properties of transfer

For $H^\bullet$: $\quad H^0(H,M) = M^H \xrightarrow{N_{G/H}} M^G = H^0(G,M)$

$$m \longmapsto \sum_{x\in G/H} x\cdot m$$

For $H_\bullet$: $\quad H_0(G,M) = M_G \xrightarrow{N_{H\backslash G}} M_H = H_0(H,M)$

$$m \bmod I_G\cdot M \longmapsto \sum_{x\in H\backslash G} x\cdot m \bmod I_H M \quad \swarrow \text{Well-defined in } M_I$$

(x_i) system of representatives of $H\backslash G$

$\sigma \in G$. Write $x_i \cdot \sigma = h_{\pi(i)} \cdot x_{\pi(i)}$

$\uparrow$ permutation of $H\backslash G$, depending on σ

In $\mathbb{Z}[G]$, have

$$\sum_i x_i \cdot (\sigma - 1) = \sum_i h_{\pi(i)} x_{\pi(i)} - \sum_i x_i = \sum_i (h_i - 1)\cdot\sigma \in I_H \cdot \mathbb{Z}[G]$$

$$\begin{tikzcd} H^i(G,M) \arrow[r, "Res"] \arrow[rr, bend right, "{[G:H]}"'] & H^i(H,M) \arrow[r, "Ver"] & H^i(G,M) \end{tikzcd}$$
Check H^0

Have

$$\begin{tikzcd} H_i(G,M) \arrow[r, "Ver"] \arrow[rr, bend right, "{[G:H]}"'] & H_i(H,M) \arrow[r, "(H\hookrightarrow G)_*"] & H_i(G,M) \end{tikzcd}$$
check H_0

Explicit formula for a quasi-isom of complexes in Mod_H : in Assignment 13

$$Res^G_H\, C.(G) \xrightarrow{q.\ isom} C.(H)$$

3.3. Restriction – inflation sequence (for a normal subgroup)

$N \trianglelefteq G \qquad M \in Mod_G$

$$0 \to H^i(G/N, M^N) \xrightarrow{Inf} H^i(G,M) \xrightarrow{Res} H^i(N,M)$$

$$0 \leftarrow H_i(G/N, M_N) \longleftarrow H_i(G,M) \longleftarrow H_i(N,M)$$

are exact when $i=1$,

(direct computation)

and for $i = q$ if $H^j(G,M) = 0$

$H_j(G,M) = 0$ for $1 \le j \le q-1$

(either by dimension shifting

or use Hochschild-Serre s. seq.

$$E_2^{ij} = H^i(G/N, H^j(N,M)) \Longrightarrow H^{i+j}(G,M)$$)

4. Cup product

Characterization by functorial properties G: finite group
M, N ∈ Mod_G

$$\begin{tikzcd} \hat{H}^i(G,M) \times \hat{H}^j(G,N) \arrow[r] & \hat{H}^{i+j}(G, M \otimes_{\mathbb{Z}} N) \\ (a\ ,\ b) \arrow[r, mapsto] & a \cdot b \end{tikzcd}$$

(alternative notation: $a \cup b$)

(i) functorial in M and N

(ii) When $i=j=0$, it is induced by the natural map

$$M^G \otimes_{\mathbb{Z}} N^G \longrightarrow (M \otimes_{\mathbb{Z}} N)^G$$

(iii) If $0 \to M' \to M \to M'' \to 0$ is exact and $0 \to M' \otimes_{\mathbb{Z}} N \to M \otimes_{\mathbb{Z}} N \to M'' \otimes_{\mathbb{Z}} N \to 0$ is also exact, then $\forall a'' \in \hat{H}^i(G, M'')$, $\forall b \in \hat{H}^j(G,N)$, we have

$$\underbrace{(\delta a'')}_{\substack{\Uparrow \\ \hat{H}^{i+1}(G,M')}} \cdot b = \delta \underbrace{(a'' \cdot b)}_{\substack{\Uparrow \\ \hat{H}^{i+j}(G, M'' \otimes_{\mathbb{Z}} N)}} \in \hat{H}^{i+j+1}(G, M' \otimes_{\mathbb{Z}} N)$$

(iv) If $0 \to N' \to N \to N'' \to 0$ is exact and

$0 \to M \otimes_{\mathbb{Z}} N' \to M \otimes_{\mathbb{Z}} N \to M \otimes_{\mathbb{Z}} N'' \to 0$ is also exact, $a \in \hat{H}^i(G,M)$
$b \in \hat{H}^j(G\ N'')$

then
$$(-1)^i \cdot a \cdot \underbrace{\delta b''}_{\substack{\Uparrow \\ \hat{H}^{j+1}(G,N')}} = \delta \underbrace{(a \cdot b'')}_{\substack{\Uparrow \\ \hat{H}^{i+j}(G, M \otimes_{\mathbb{Z}} N'')}} \in \hat{H}^{i+j+1}(G, M \otimes_{\mathbb{Z}} N')$$

Remark Have explicit chain map
(a co-pairing)
which induces the cup product

$$C_\bullet(G) \xrightarrow{\ \Phi\ } \underbrace{C_\bullet(G) \otimes_{\mathbb{Z}} C_\bullet(G)}_{\text{graded tensor product}}$$
$\uparrow$

[Note By defⁿ, Φ is a collection of maps $C_{i+j}(G) \xrightarrow{\varphi_{i,j}} C_i(G) \otimes_{\mathbb{Z}} C_j(G)$, $i,j \in \mathbb{Z}$

(It is not a map from the $\mathbb{Z}$-module $C_\bullet(G)$ to the module $C_\bullet(G) \otimes_{\mathbb{Z}} C_\bullet(G)$)]

April 2, 2021

5. Cohomologically trivial modules for finite groups — Theorems of Tate and Nakayama

Recall

Prop. 5.1 (p-groups) Let G be a finite p-group. Let M be a G-module

(a) Suppose that $p\cdot M=0$. If $\hat{H}^{q_0}(G,M)=0$ for some $q_0\in\mathbb{Z}$. Then M is a free $\mathbb{F}_p[G]$-module, and $\hat{H}^q(K,M)=0$ for all $q\in\mathbb{Z}$ and every subgroup $K\leq G$.

(b) Suppose that M is torsion-free, and $\hat{H}^{q_0}(G,M)=0=\hat{H}^{q_0+1}(G,M)=0$ for some $q_0\in\mathbb{Z}$. Then

(b1) $\hat{H}^q(K,M)=0 \quad \forall q\in\mathbb{Z},\ \forall K\leq M$

(b2) M/pM is a free $\mathbb{F}_p[G]$-module

Will be applied next → * (b3) Assume that M is a free $\mathbb{Z}$-module
$\forall$ torsion-free G-module N, we have $\hat{H}^q(K,\mathrm{Hom}_{\mathbb{Z}}(M,N))=0$ $\forall q\in\mathbb{Z},\forall K\leq G$

Will see in thm 5.2 that: (b4) $\exists$ a projective resolution $0\to P_1\to P_0\to M\to 0$ of length=1 of M in Mod_G

Pf. (a) dimension shift $\rightsquigarrow$ $\exists$ a G-module N s.t. $p\cdot N=0$ and
(Using induced modules, killed by p, from $\{1\}\leq G$)
$\hat{H}^{n-q_0-2}(G,N)\cong H^n(G,M)=0 \quad \forall n\in\mathbb{Z}$.

Assumption: $\hat{H}^{-2}(G,N)=H_1(G,N)=0$

Let $L\xrightarrow{\alpha}N$ be a G-linear surjection s.t. $\begin{cases} L=\mathbb{F}_p[G]\underset{\mathbb{F}_p}{\otimes}(\text{an } \mathbb{F}_p\text{-v.sp}) \\ L/I_GL\xrightarrow[\alpha]{\sim}N/I_GN. \end{cases}$ (So L is induced from the trivial subgrp.)

$0\to Q\to L\xrightarrow{\alpha}N\to 0$ + long exact sequence $\Rightarrow$ $\underbrace{H_0(G,Q)}=Q/I_GQ=0$, i.e. $I_G\cdot Q=Q$

This assumption implies $H_0(G,Q)=0$, $Q=\mathrm{Ker}(L\xrightarrow{\alpha}N)$

← Note: $\hat{H}_{-1}(G,Q)=0$ would not be enough here.

Easy Fact / Exer: $\exists\, m_0\in\mathbb{N}$ s.t. $(\mathbb{F}_p\otimes I_G)^{m_0}=0$. (E.g. induction on $|G|$.)
This is an analog of Nakayama's Lemma.

$\rightsquigarrow$ $Q=I_GQ=I_G^2Q=\cdots=I_G^{m_0}Q=0$, i.e. $L\xrightarrow{\sim}N$ $\Rightarrow \hat{H}^q(G,M)=0\ \forall q\in\mathbb{Z}$
$\Rightarrow M$ is a free $\mathbb{F}_p[G]$-module.

(b) $0\to M\xrightarrow{p}M\to M/pM\to 0 \rightsquigarrow$ Assumption $\Rightarrow \hat{H}^{q_0}(G,M/pM)=0$, hence M/pM is a free $\mathbb{F}_p[G]$-module.

Assume now that $\underline{M \text{ is a free } \mathbb{Z}\text{-module}}$. Consider $0\to N\xrightarrow{p}N\to N/pN\to 0$ exact

$$\Longrightarrow \quad 0\to \mathrm{Hom}_{\mathbb{Z}}(M,N)\xrightarrow{p}\mathrm{Hom}_{\mathbb{Z}}(M,N)\to \mathrm{Hom}_{\mathbb{Z}}(M,N/pN)\to 0$$

$$\mathrm{Hom}_{\mathbb{Z}}(M,N/pN)\ \cong\ \mathrm{Hom}_{\mathbb{F}_p}(M/pM,N/pN)$$

Key observation: is a free $\mathbb{F}_p[G]$-module

$$M/pM=\bigoplus_{i\in I}\mathbb{F}_p[G]\cdot e_i \Rightarrow \quad \mathrm{Hom}_{\mathbb{F}_p}(M/pM,N/pN) = \bigoplus_{i\in I}\mathbb{F}_p[G]\cdot\mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_p\cdot e_i, N/pN)$$

$$\Rightarrow \quad \hat{H}^q(K,\mathrm{Hom}_{\mathbb{Z}}(M,N))\ \underset{\sim}{\xrightarrow{p}}\ H^q(K,\mathrm{Hom}_{\mathbb{Z}}(M,N))\ \forall q\in\mathbb{Z},\ \forall K\leq G$$

↑ killed by $|G|\in p^{\mathbb{N}}$

QED.

(Nakayama 1957)

Thm 5.2 G : a finite group, M : a G-module.

a Sylow p-subgroup of G

Assume that $\forall$prime $p \mid \#G$, $\exists\, q(p) \in \mathbb{Z}$ s.t. $\hat{H}^{q(p)}(G_p, M) = 0 = \hat{H}^{q(p)+1}(G_p, M)$

(a) $\hat{H}^q(K, M) = 0 \quad \forall q \in \mathbb{Z}, \ \forall K \leq G$

(b) $\exists$ a G-linear projective resolution $0 \to P_1 \to P_0 \to M \to 0$ of length $= 1$

(c) If M is a free $\mathbb{Z}$-module, then M is a projective $\mathbb{Z}[G]$-module

($\Leftrightarrow$ M is a projective $\mathbb{Z}$-module)

Pf. of (c) : Pick a short exact sequence $0 \to Q \to \underset{\overset{\uparrow}{\text{free}}}{L} \to M \to 0$ in Mod_G.

Main ingredient: 5.1 (b3)

Consider the exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(M, Q) \to \mathrm{Hom}_{\mathbb{Z}}(M, L) \to \mathrm{Hom}_{\mathbb{Z}}(M, M) \to 0$$

$\because$ M is a free $\mathbb{Z}$-module

$$H^1(G_p, \mathrm{Hom}_{\mathbb{Z}}(M, Q)) = 0 \quad \forall p \mid \# G$$

Note: Not every torsion free $\mathbb{Z}$-module is projective: every projective $\mathbb{Z}$-module is free; $\mathbb{Q}$ is a torsion free and flat $\mathbb{Z}$-module but not a free $\mathbb{Z}$-module.

$\Longrightarrow H^1(G, \mathrm{Hom}_{\mathbb{Z}}(M, Q)) = 0 \quad \because \ \mathrm{Ker}\left(H^1(G, \mathrm{Hom}_{\mathbb{Z}}(M, Q)) \xrightarrow{\text{restriction}} H^1(G_p, \mathrm{Hom}_{\mathbb{Z}}(M, Q))\right)$

$= H^1(G, \mathrm{Hom}_{\mathbb{Z}}(M, Q))\big[[G : G_p]\big]$

$= \{ h \in H^1(G, \mathrm{Hom}_{\mathbb{Z}}(M, Q)) \mid \underset{=0}{[G : G_p] h} \}$

$\Longrightarrow$ M is a direct summand of L, hence projective

(a)+(b): Pick a short exact sequence $0 \to R \to L \to M \to 0$ in Mod_G.

R is a free $\mathbb{Z}$-module, and $\hat{H}^{q(p)+1}(G_p, R) = \hat{H}^{q(p)+2}(G_p, R) \quad \forall p \mid \# G$

Every submodule of a free module over a PID is free.

$\leadsto$ R is a projective $\mathbb{Z}[G]$-module by (c).

QED.

Cor. 5.3 G : a finite group, B, C : G-modules. $f : B \to C$ G-linear

(mapping cone construction) Suppose that $\forall$prime $p \mid \#(G)$, $\exists\, q(p) \in \mathbb{Z}$ s.t.

$$f_q^* : \hat{H}^q(G_p, B) \longrightarrow \hat{H}^q(G_p, C) \text{ is surjective for } q = q(p),$$

Sylow p-subgrp of G

bijective for $q = q(p)+1$ and injective for $q = q(p)+2$. Then $\forall q \in \mathbb{Z}$ and $\forall K \leq G$,

$$f_q^* : \hat{H}^q(K, B) \xrightarrow{\sim} \hat{H}^q(K, C)$$

$\mathrm{Ind}_{\{1\}}^G \mathrm{Res}_{\{1\}}^G B$

The left G-action here comes from the right translation on G, i.e. the G-module is

Pf : $0 \longrightarrow B \xrightarrow{\tilde{f}} C \oplus \overbrace{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], B)} \longrightarrow D \to 0$ short exact

$b \longmapsto \left(f(b),\ \varphi_b = (x \longmapsto x \cdot b)_{\forall x \in G}\right)$

Assumption $\Longrightarrow \hat{H}^{q(p)}(G_p, D) = 0 = \hat{H}^{q(p)}(G_p, D) \quad \forall p \mid \# D$

$\xRightarrow{\text{Thm 5.2}} \hat{H}^q(K, D) = 0 \quad \forall q \in \mathbb{Z} \Longrightarrow \hat{H}^q(K, B) \underset{\sim}{\xrightarrow{f_q^*}} \hat{H}^q(K, C)$

Prop. 5.4 $A, B, C \in \mathrm{Mod}_G$, $\varphi : A \underset{\mathbb{Z}}{\otimes} B \longrightarrow C$ G-linear

Given $\alpha \in \hat{H}^q(G, A)$. Assume $\forall$ prime $p \mid \#G$, $\exists\, n(p) \in \mathbb{Z}$ s.t the map

$$\begin{tikzcd} \hat{H}^n(G_p, B) \arrow[r] & \hat{H}^{n+q}(G_p, C) \\ \beta \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & \alpha \cup \beta \end{tikzcd}$$

$\leftarrow$ Sylow p-subgroup of G

is surjective for $n = n(p)$, bijective for $n = n(p)+1$ and injective for $n = n(p)+2$. Then for every $K \leq G$ and every $n \in \mathbb{Z}$, the map

$$\begin{tikzcd} H^n(K, B) \arrow[r] & \hat{H}^{n+q}(K, C) \\ \beta \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & \mathrm{Res}^G_K(\alpha) \cup \beta \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

is an isomorphism.

(formal argument)

Pf. The case $q = 0$ follows from Cor. 5.3.

shift dimension with diagrams

$0 \to A' \to \mathbb{Z}[G] \underset{\mathbb{Z}}{\otimes} A \to A \to 0$ + $0 \to C' \to \mathbb{Z}[G] \underset{\mathbb{Z}}{\otimes} C \to C \to 0$ + $A' \underset{\mathbb{Z}}{\otimes} B \to C'$

and

$0 \to A \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A) \to A'' \to 0$, $0 \to C \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], C) \to C'' \to 0$ + $A'' \underset{\mathbb{Z}}{\otimes} B \to C''$.

a special case of 5.4.

Theorem 5.5 (Tate 1952) Let M be a G-module, and let $\alpha \in H^2(G, M)$

Assume (i) $H^1(G_p, M) = 0$

(ii) $H^2(G_p, M) = \mathbb{Z} \cdot \mathrm{Res}_{G \geq G_p}(\alpha) \cong \mathbb{Z}/\#G_p \cdot \mathbb{Z}$ $\forall$ prime $p \mid \#G$.

Then

$$\begin{tikzcd} \hat{H}^n(K, \mathbb{Z}) \arrow[r, "\sim"] & \hat{H}^{n+2}(K, M) \\ \beta \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & \mathrm{Res}_{G \geq K}(\alpha) \cup \beta \end{tikzcd}$$

$\forall n$, $\forall K \leq G$.

Immediate Corollary of Prop. 5.4, with $A = M$, $B = \mathbb{Z}$, $n(p) = -1$ $\forall p$

Note that $H^1(G_p, \mathbb{Z}) = 0$, and the injectivity of $H^1(G_p, \mathbb{Z}) \longrightarrow H^3(G_p, M)$ holds trivially.

The case important for application is the isomorphism

$$K^{ab} = \hat{H}^{-2}(K, \mathbb{Z}) \xrightarrow{\sim} \hat{H}^0(K, M) = M^K / N_K \cdot M$$

Motivation: Class field theory

$K = G$ assumptions on $n = -1, 0$

$\Rightarrow$ $n = 1$

$$\begin{tikzcd} H^1(G, \mathbb{Z}) \arrow[r, "\alpha \cup \cdot", "\cong"'] \arrow[d, equal] & H^3(G, M) \arrow[d, equal] \\ 0 & 0 \end{tikzcd}$$

case $n = -2$

$$\begin{tikzcd}
\hat{H}^{-2}(G,\mathbb{Z}) \arrow[r, "\alpha\cup\cdot", "\sim"'] \arrow[d, phantom, "\parallel"] & \hat{H}^{0}(G,M) \arrow[d, phantom, "\parallel"] \\
H_1(G,\mathbb{Z}) \arrow[d, phantom, "\cong"] & M^G/N_G(M) \\
G^{ab} = G/(G,C) &
\end{tikzcd}$$

Remark: In class field theory, $G = \mathrm{Gal}(\bar{k}/k)$, $M = \overbrace{\mathbb{A}^\times_{\bar{k}}}/\bar{k}^\times$ = the idele class group of k

$k^\times \cong$ ← discrete, but not cocompact

← locally compact comm. group

$[k:\mathbb{Q}]<\infty$

comm. locally compact

$k \subseteq \mathbb{A}_k$: topological ring

↖ discrete compact subring of $\mathbb{A}_k$

The assumptions in Thm 5.5 are satisfied.

"concrete"

or $\mathrm{Gal}(\underbrace{E/k}_{\text{finite Galois}})$.

(Kernel = neutral component of $\mathbb{A}_k^\times/k^\times$

"almost an isom". $\mathrm{Gal}(\bar{k}/k)^{ab}$

$$\rightsquigarrow \mathrm{rec}_k : \mathbb{A}_k^\times/k^\times \twoheadrightarrow \mathrm{Gal}(k^{ab}/k)$$

$\mathrm{Gal}(k^{ab}/k) \cong \mathrm{Gal}(\bar{k}/k)^{ab}$

Artin's reciprocity law map

Going back to : Hilbert 90 and Kummer theory

$$1 \to \mu_n \to E^\times \xrightarrow{[n]} E \qquad x \mapsto x^n$$

$n \cdot 1_k \in k^\times$

$$G = \mathrm{Gal}(k^{sep}/k) = \varprojlim_{E/k \text{ finite Galois}} \mathrm{Gal}(E/k) \qquad 1 \to \mu_n \to (k^{sep})^\times \xrightarrow{[n]} (k^{sep})^\times \to 1$$

$x \mapsto x^n$

E/k finite Galois

$$\begin{tikzcd}
1 \arrow[r] & \mu_n(k) \arrow[r] & k^\times \arrow[r, "{[n]}"] & k^\times \arrow[r] & H^1(G;\mu_n) \arrow[r] & H^1(G,(k^\times)^{sep}) \arrow[d, equal] \\
 & & & & & \varinjlim\limits_{E/k \text{ finite Galois}} H^1(Gal(E/k),E^\times) \arrow[d, equal] \\
 & & & & \text{"Hilbert thm 90"} \arrow[ur] & 0
\end{tikzcd}$$

$$\rightsquigarrow \quad H^1(Gal(k^{sep}/k),\mu_n)$$

$$= k^\times/(k^\times)^n$$

If $\mu_n(k^{sep}) = \mu_n(k)$, then $\mu_n \cong \mathbb{Z}/n\mathbb{Z}$

<u>and</u>: $H^1(Gal(k^{sep}/k),\mu_n) \cong H^1(Gal(k^{sep}/k),\mathbb{Z}/n)$

$$\begin{tikzcd}
H^1(Gal(k^{sep}/k),\mu_n) \arrow[r, "\cong"] \arrow[d, equal] & H^1(Gal(k^{sep}/k),\mathbb{Z}/n) \arrow[d, equal] \\
(k^\times)/(k^\times)^n & Hom(G;\mathbb{Z}/n\mathbb{Z})
\end{tikzcd}$$

<u>Kummer theory</u>